Mathematical Techniques for Computer Science - Notes

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1 Analytical Geometry in the Plane

Points: Given points P and Q with coordinates $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$, their distance d can be computed using pythagoras:

$$d = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$$

Vectors:

- Pair $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ can be a movement of the plane: every point $P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ is shifted to $P' = \begin{pmatrix} p_1 + v_1 \\ p_2 + v_2 \end{pmatrix}$
- Uppercase letters are used for points and lowercase with arrows for vectors.
- A vector has length $|\vec{v}| = \sqrt{(v_1)^2 + (v_2)^2}$ which is the distance each point travels under the movement described by $\vec{v}$.
- A Vector of length 1 is a Unit Vector.
- $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the null vector.
- $\vec{v}/|\vec{v}|$ is the unit vector pointing in the same direction as $\vec{v}$.
- The vector $\vec{PQ}$ that moves point P into point Q has coordinates (q1-p1, q2-p2).
- All points X that can be reached from P by following some distance along the direction of $\vec{v}$ lie on the straight line $X = P + (s \cdot v)$.
- This is a parametric representation of a line where the parameter is s.
- If given two points P and Q in the plane then this defines a straight line $X = P + (s \cdot PQ)$.
- If given two lines $X = P + (s \cdot v)$ and $Y = Q + (t \cdot w)$ their point of intersection satisfies $p1 + sv1 = q1 + tw1$ and $p2 + sv2 = q2 + tw2$. This can be solved using Gaussian Elimination.

2 Geometry in 3 Dimensions

Planes The parametric representation of a plane has the form $X = P + s \cdot \vec{v} + t \cdot \vec{w}$ where P is a point in space and $\vec{v}$ and $\vec{w}$ are vectors (neither of which are null). $\vec{w}$ must not point in the same (or opposite) direction as $\vec{v}$, otherwise it is just a line.
Three points P, Q and R which are not all on the same line determine a plane $X = P + s \cdot \vec{PQ} + t \cdot \vec{PR}$.

Intersection Tasks

- **Test whether a point lies on a line or plane** For point Q: $X = P + (s \cdot v)$ therefore $P + (s \cdot v) = Q$.
- **Find the intersection point between lines/planes** Simply set the two equations as equal to each other.

Solve using Gaussian Elimination.

Two-point description of a line Two points P and Q determine a line

$$X = P + s \cdot \vec{PQ}$$

This can be rewritten as

$$\begin{aligned} X &= P + s \cdot (Q - P) \\ &= P + s \cdot Q - s \cdot P \\ &= (1 - s) \cdot P + s \cdot Q \end{aligned}$$

This can also be done with a plane using 3 points.

Vector spaces So we have a null vector, we can add two vectors, and we can multiply vectors with a scalar. Any structure that satisfies the laws of vector algebra and carries these two operations is called a **vector space**.

The idea of a vector space is more complex than just 3-dimensional movements. Scalars can come from any field $\mathbb{F}$. We will call one theoretical field $GF(2)$. One then speaks of a "vector space over $\mathbb{F}$".

Subspaces If $\vec{v}$ is an element of some vector space then we can generate a **subspace** (a **sub-vector space**) by considering all vectors of the form $s \cdot \vec{v}$. Another subspace would be all expressions of the form $s \cdot \vec{v} + t \cdot \vec{w}$.

Another way of looking at parametric representations is that we pick a point and allow all movements from a subspace to act on this point. This leads to an **affine subspace**. In other words, lines and planes are affine subspaces of 2D/3D.

Bases If two generators point in the same direction then one of them is redundant and can be removed. A set of generators that includes a redundant generator is said to be **linearly dependent**. A set of generators that can not be made any smaller is a **basis** of the subspace. The number of elements of a basis is the **dimension** of the subspace.

If we start with some set of generators and we are not sure if any of them are redundant, we can write the vectors as rows into a matrix and run Gaussian elimination. The rows of the resulting echelon form will be a basis for the subspace. Any redundancy will show itself as rows consisting entirely of zeros.

Codes Subspaces in $GF(2)^n$ are used as linear **codes** in coding theory. This is coding to spot and correct errors in data, not programming or cryptography.