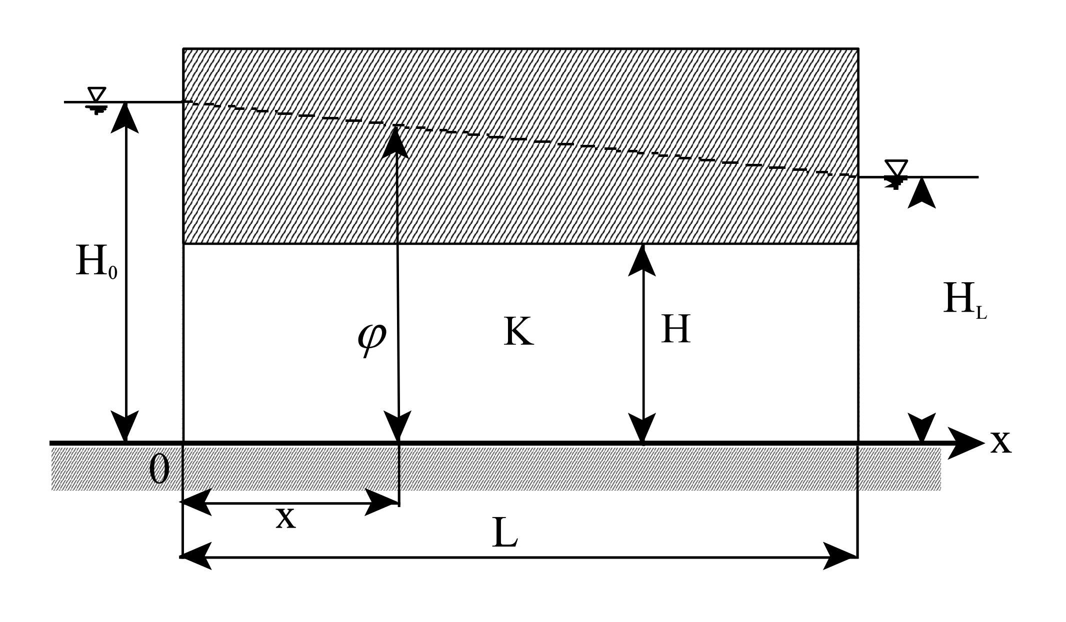
Steady groundwater flow in aquifers

- A confined aquifer
- An unconfined aquifer
- A semi-confined aquifer

- Conceptual hydrogeological model
- Mathematical model
- Analytical solution
- Analysis of the solution

Steady groundwater flow in a confined aquifer

- Conceptual hydrogeological model
  - The aquifer is confined by two impermeable layers on the top and bottom;
  - The aquifer consists of homogeneous porous medium;
  - The aquifer is of uniform thickness;
  - The aquifer is bounded with two parallel rivers on the left and right with constant river stage.



Questions:

1. $\varphi(x) = ?$
2. $Q_x = ?$
3. Residence time?

Steady groundwater flow in a confined aquifer

- Mathematical model
  - Governing equation of steady groundwater flow;
  - Boundary conditions of external influences.

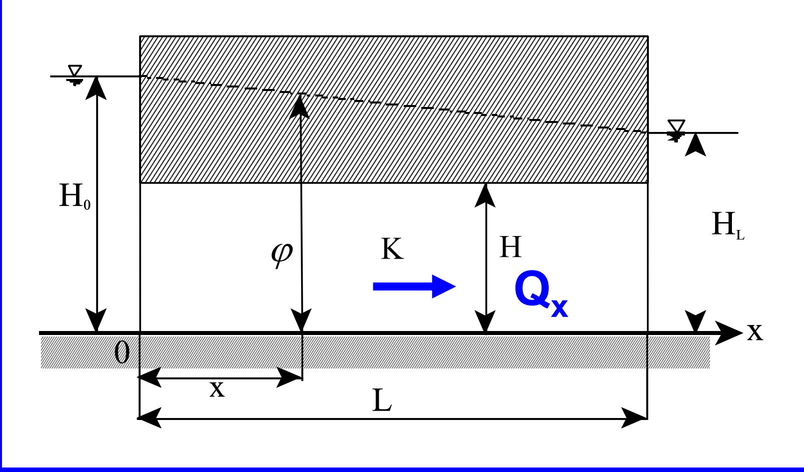
One dimensional steady groundwater flow:

$$\frac{\partial^2 \varphi}{\partial x^2} = 0$$

Boundary conditions of two rivers:

$$\varphi|_{x=0} = H_0$$

$$\varphi|_{x=L} = H_L$$



Steady groundwater flow in a confined aquifer

- Analytical solution

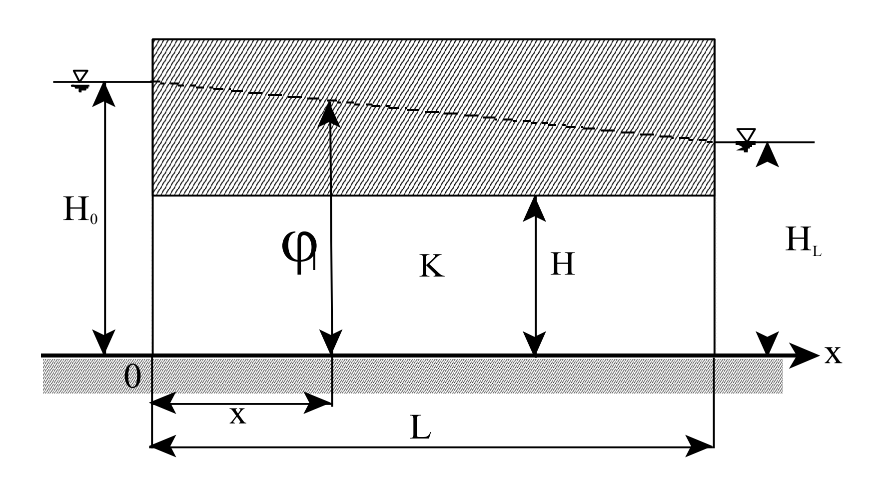
Integration of the partial differential equation:

$$\varphi = c_1 x + c_2$$

Determine integration constants using boundary conditions:

$$c_2 = H_0$$

$$c_1 = -\frac{H_0 - H_L}{L}$$



Final solution:

$$\varphi = H_0 - \frac{H_0 - H_L}{L} x$$

Groundwater head distribution does not depend on hydraulic conductivity and aquifer thickness.

Steady groundwater flow in a confined aquifer

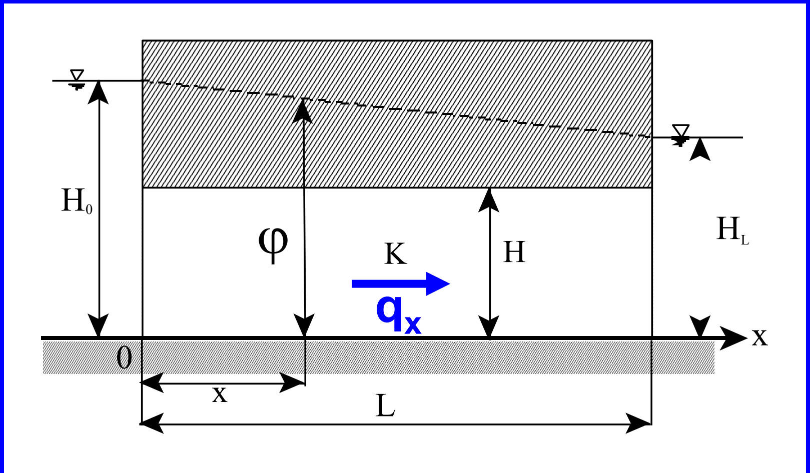
- Groundwater flow

Specific discharge:

$$q = K \frac{H_0 - H_L}{L}$$

Unit-width discharge:

$$Q = H . q = K H \frac{H_0 - H_L}{L} = T \frac{H_0 - H_L}{L}$$



The unit width discharge is the same as the inflow from the left river to the aquifer and the discharge of the aquifer to the right river. It depends on hydraulic gradient and tramsmissivity.

Steady groundwater flow in a confined aquifer

- Residence time

Specific discharge:

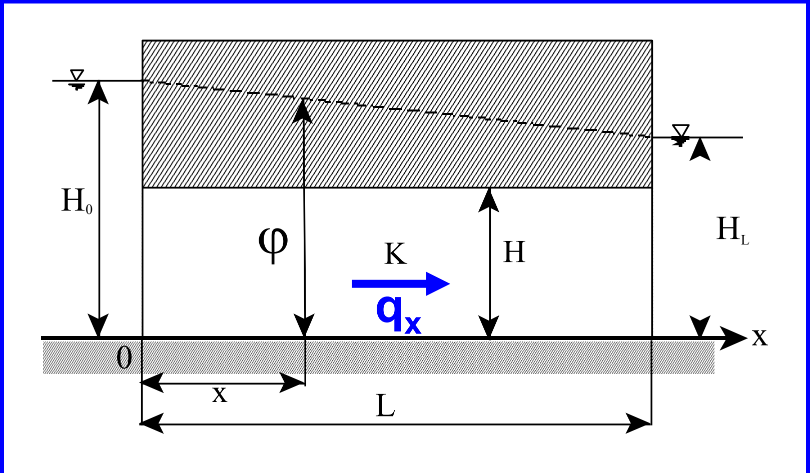
$$q = K \frac{H_0 - H_L}{L}$$

Average velocity:

$$v = \frac{q}{n_e} = \frac{K}{n_e} \frac{H_0 - H_L}{L}$$

Travel time:

$$t = \int_0^L \frac{dx}{v} = \frac{n_e L}{q} = \frac{n_e L^2}{K(H_0 - H_L)}$$

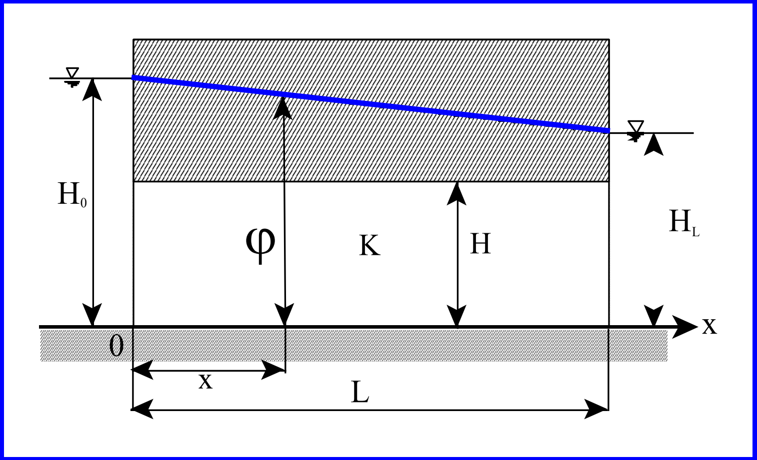


Mean residence time

$$\tau = \frac{V}{Q} = \frac{HLn_e}{T(H_0 - H_L)/L} = \frac{n_e L^2}{K(H_0 - H_L)}$$

Steady groundwater flow in a confined aquifer

- Example



Aquifers	L(m)	h_0 (m)	h_l (m)	K(m/d)	H(m)	T(m^2/d)	n	q(m/d)	Q(m^2/d)	v(m/d)	t (days)	τ (years)
Local	1000	20	15	10	20	200	0.2	0.05	1	0.25	4000	11
Mediam	10000	100	50	10	100	1000	0.2	0.05	5	0.25	40000	110
Regional	100000	400	100	10	200	2000	0.2	0.03	6	0.15	666667	1826

For a local aquifer of 1km length, the residence time is in the range of tens years. For a medium aquifer of 10km length, the residence time can be in the order of hundreds years. For regional aquifers (length > 100km), the residence time could be in thousands years.

Steady groundwater flow in a confined aquifer

- Example

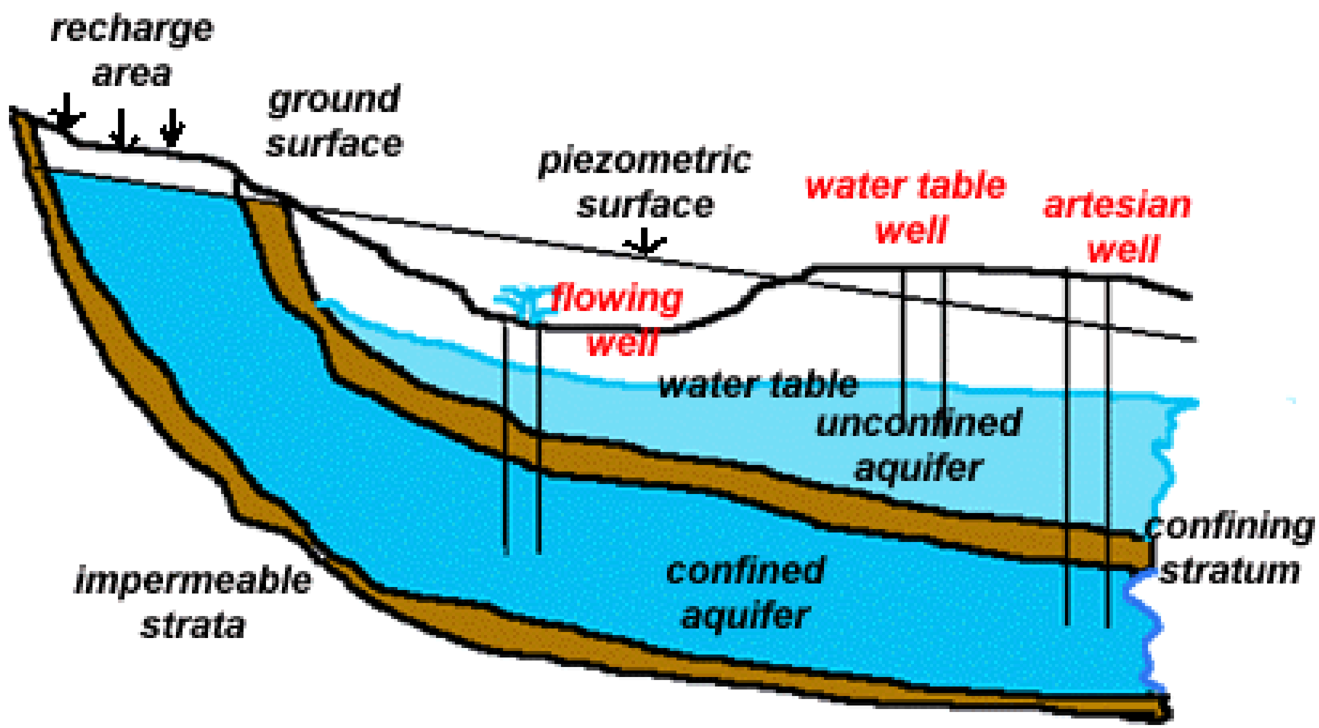


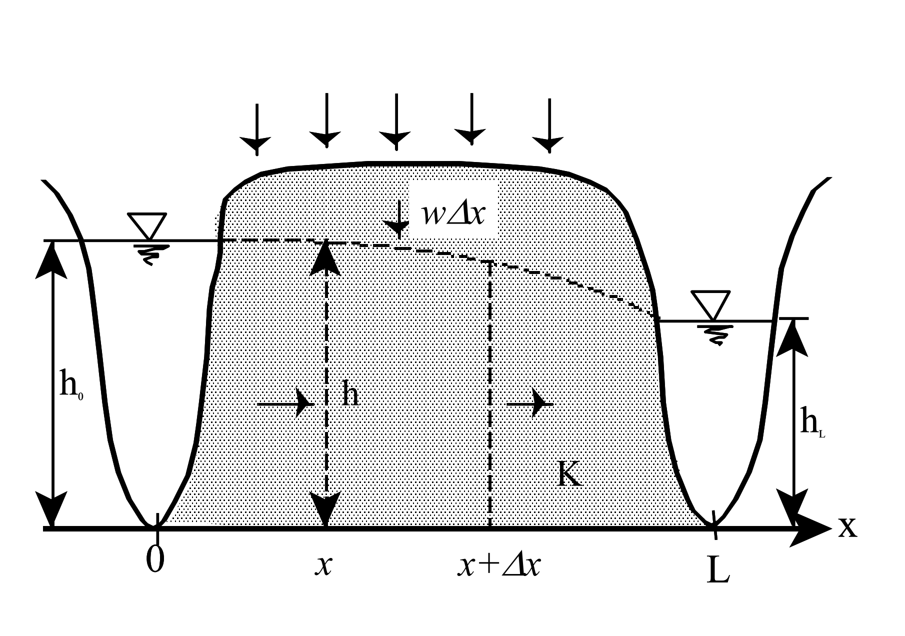
Fig. 5. Types of aquifers

Assignments:

- Describe conceptual hydrogeological model of the confined aquifer
- Use water balance and Darcy’s equation to derive the partial differential equation
- Which factors determine groundwater head distribution in a confined aquifer?
- Use analytical solution to compute specific discharge, average velocity, and travel time
- Solve the first problem of Exercise 7.2

Water balance and residence time of steady flow in an unconfined aquifer

- Conceptual hydrogeological model
  - An unconfined aquifer is above a horizontal impermeable base;
  - The porous medium is homogeneous;
  - The aquifer receives uniform recharge;
  - The aquifer is bounded by two rivers of constant stages h_0 and h_L.
  - The flow is assumed to be one-dimensional horizontal flow (Dupuit's assumption).



Questions:

1. $h(x) = ?$
2. $Q_x = ?$
3. Residence time?

Steady groundwater flow in an unconfined aquifer

- Mathematical model

Water balance equation:

$$Q_x + w\Delta x = Q_x + \Delta Q_x$$

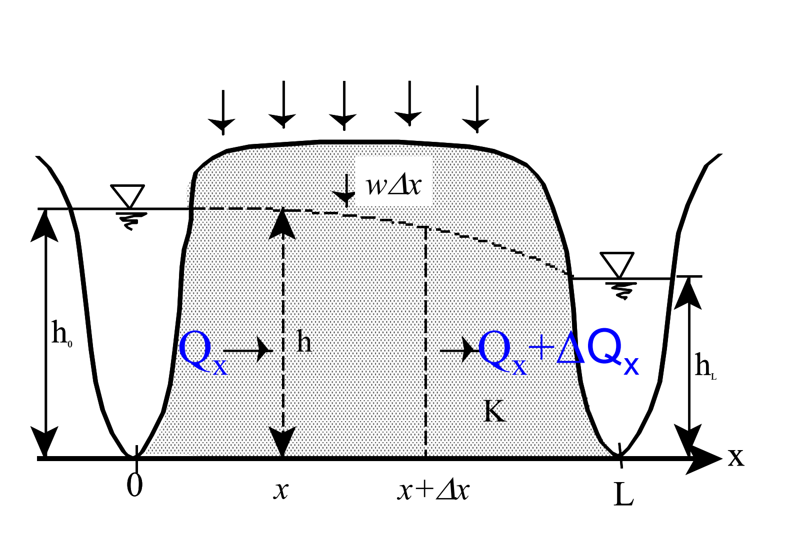
$$hq_x + w\Delta x = hq_x + \frac{\partial (hq_x)}{\partial x}\Delta x$$

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \frac{w}{K} = 0$$

Boundary conditions of two rivers:

$$h|_{x=0} = h_0$$

$$h|_{x=L} = h_L$$



Steady groundwater flow in an unconfined aquifer

- Analytical solution

Integration of the partial differential equation:

$$h^2 + \frac{w}{K} x^2 = c_1 \ x + c_2$$

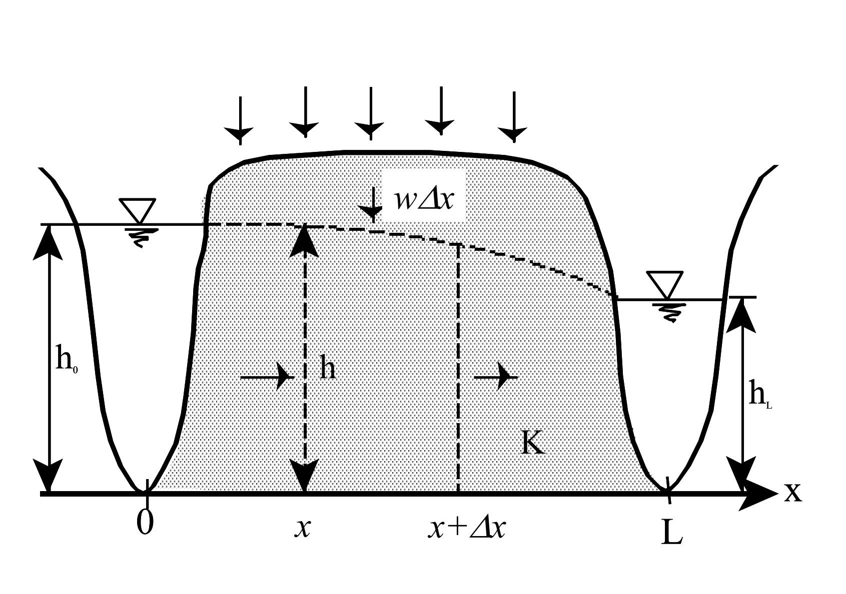
Determine integration constants using boundary conditions:

$$x = 0 \quad c_2 = h_0^{\ 2}$$

$$x = L \quad c_1 = -\frac{h_0^{\ 2} - h_L^{\ 2}}{L} + \frac{w}{K} L$$

Final solution:

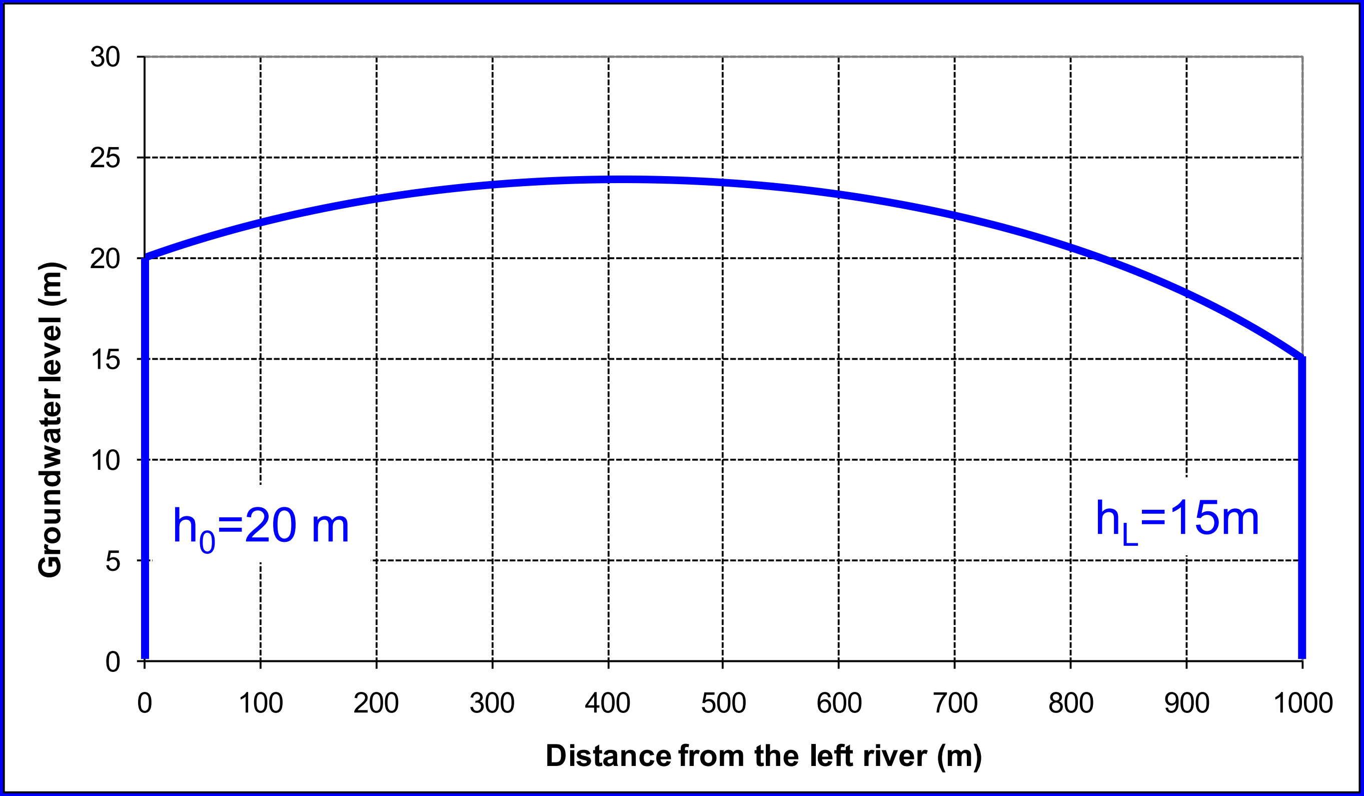
$$h^2 = h_0^{\ 2} - \left(\frac{h_0^{\ 2} - h_L^{\ 2}}{L} - \frac{wL}{K}\right)x - \frac{w}{K} x^2$$



Steady groundwater flow in an unconfined aquifer

- Groundwater level distribution

Example:
L=1000 m
K=10 m/d
w=0.01 m/d
h_0=20 m
h_L=15 m

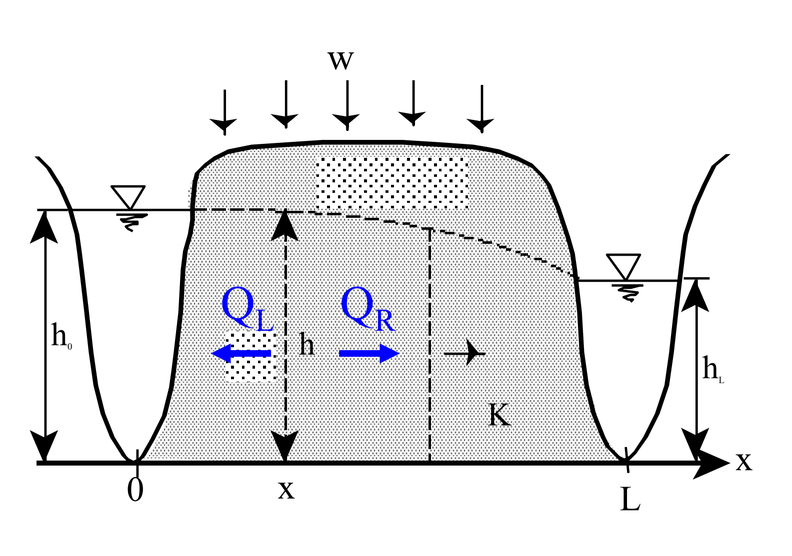


Steady groundwater flow in an unconfined aquifer

- Groundwater divide

Unit-width discharge:

$$Q_x = -Kh\frac{\partial h}{\partial x} = \frac{K}{2}\left(\frac{h_0^2 - h_L^2}{L} - \frac{wL}{K}\right) + wx = 0$$



Groundwater divide:

$$d = \frac{L}{2} - \frac{K}{2}\frac{h_0^2 - h_L^2}{wL}$$

$$d = \frac{L}{2} - \left(\frac{Q}{wL}\right)L$$

$$d > 0 \ if \ \frac{Q}{wL} < \frac{1}{2}$$

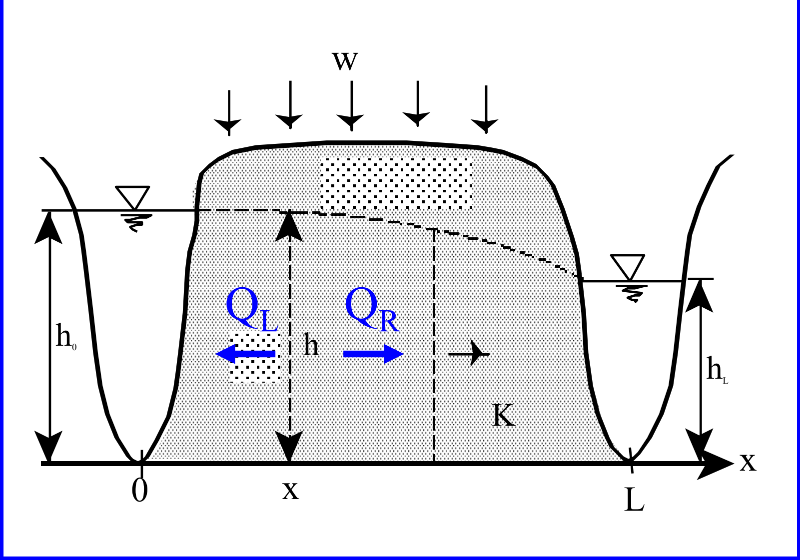
A water divide can only be formed in between two rivers when the total recharge is larger than 2 times the discharge induced by head difference of two rivers.

Steady groundwater flow in an unconfined aquifer

- Highest water level at groundwater divide

$$d = \frac{L}{2} - \frac{K}{2}\frac{h_0^2 - h_L^2}{wL}$$

$$h_{max}^2 = h_0^2 - \left(\frac{h_0^2 - h_L^2}{L} - \frac{w\,L}{K}\right)d - \frac{W}{K}d^2$$



When $h_0 = h_L$:

$$d = \frac{L}{2}$$

$$h^2{}_{max} = h_0^2 + \frac{w\,L^2}{4K}$$

Steady groundwater flow in an unconfined aquifer

- Groundwater discharges

Unit-width discharge:

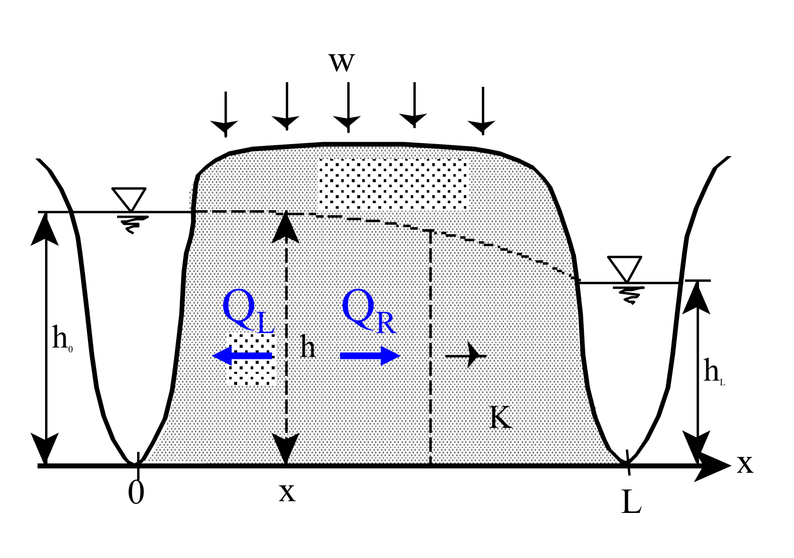
$$Q = -Kh\frac{\partial h}{\partial x} = \frac{K}{2}\left(\frac{h_0^2 - h_L^2}{L} - \frac{wL}{K}\right) + wx$$

Discharge to the left river:

$$Q_L = -\left[\frac{wL}{2} - \frac{K}{2}\frac{h_0^2 - h_L^2}{L}\right]$$

Discharge to the right river:

$$Q_R = \frac{wL}{2} + \frac{K}{2}\frac{h_0^2 - h_L^2}{L}$$



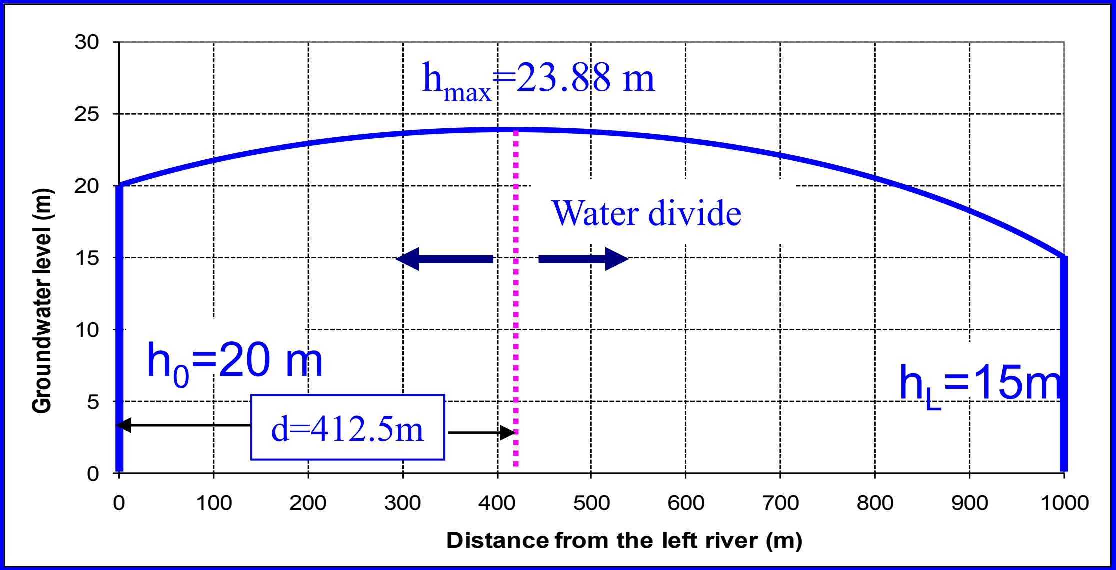
When $h_0 = h_L$:

$$|Q_L| = |Q_R| = \frac{1}{2}wL$$

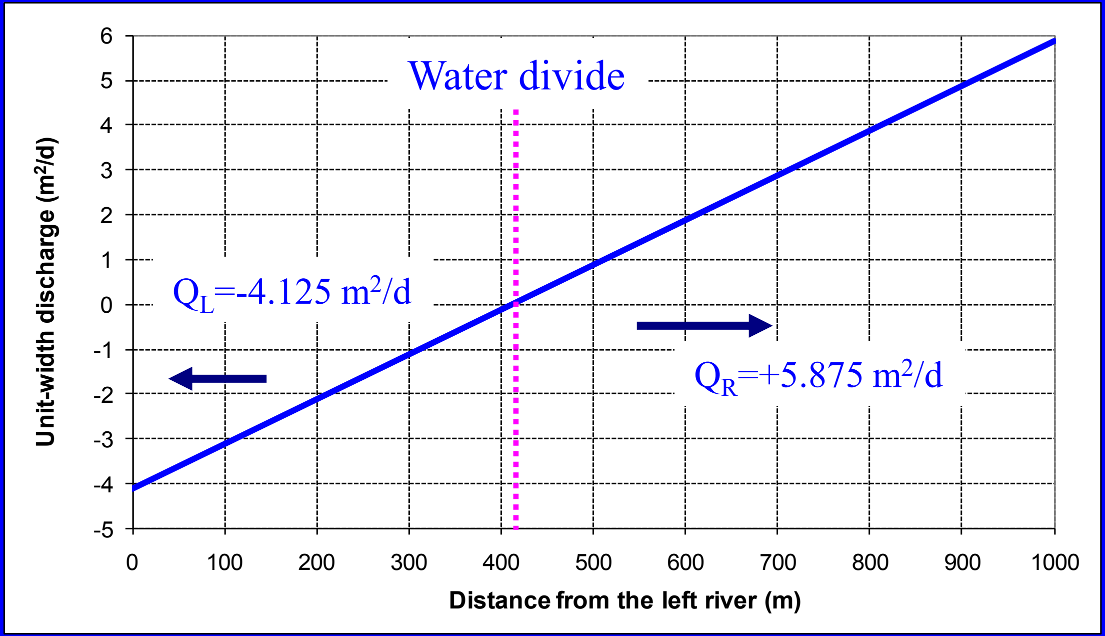
$$Q = |Q_L| + |Q_R| = wL$$

Steady groundwater flow in an unconfined aquifer

Assume:
$L=1000$ m
$K=10$ m/d
$w=0.01$ m/d
$h_0=20$ m
$h_L=15$ m



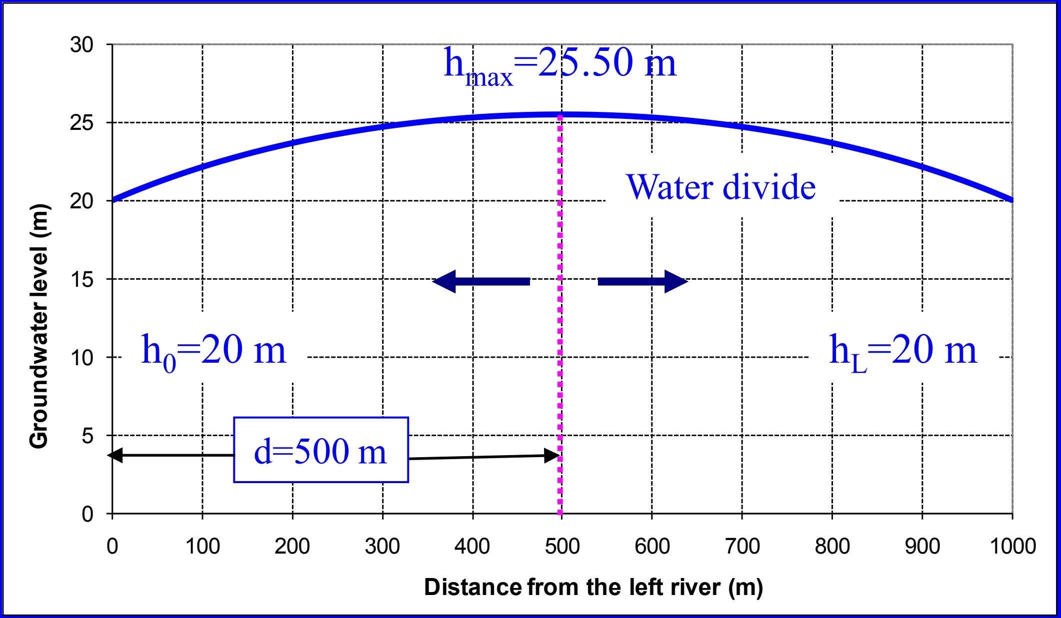
Results:
$d=412.5$ m
$h_{max}=23.88$ m
$Q_L=-4.125$ m^2/d
$Q_R=+5.875$ m^2/d
$w*L=10$ m^2/d



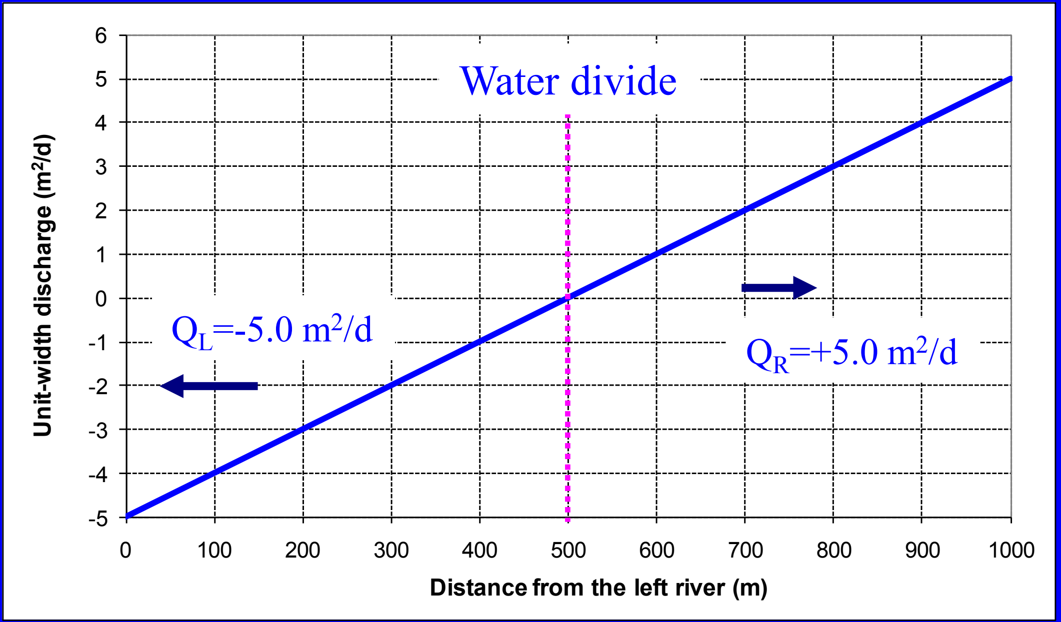
Steady groundwater flow in an unconfined aquifer

When $h_0 = h_L$

Example:
L=1000 m
K=10 m/d
w=0.01 m/d
h_0=20 m
h_L=20 m



d=500m
h_{max}=25.50 m
Q_L=-5.0 m^2/d
Q_R=+5.0 m^2/d
$w*L$=10 m^2/d



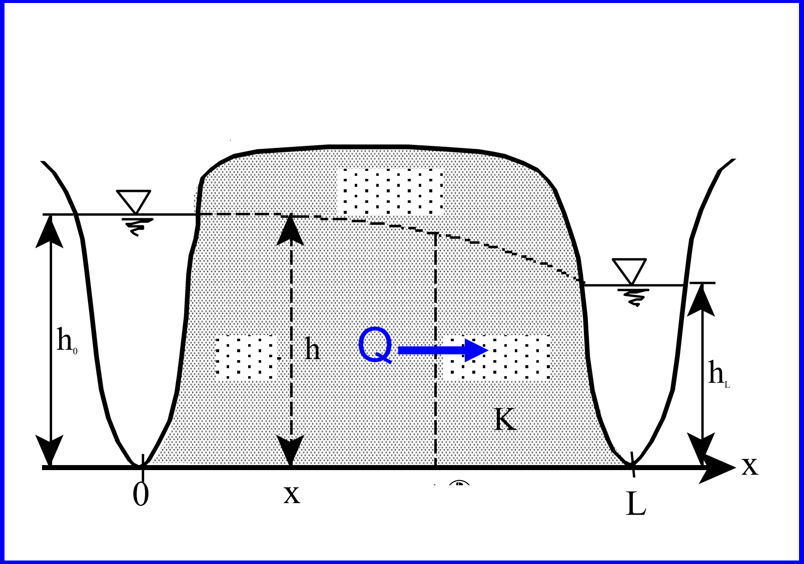
Steady groundwater flow in an unconfined aquifer

- Special case of no recharge

When there is no recharge:

Water table:

$$h^2 = h_0^2 - \frac{h_0^2 - h_L^2}{L} x$$



Unit-width discharge:

$$Q = \frac{K}{2} \frac{h_0^2 - h_L^2}{L} = K \cdot \frac{h_0 + h_L}{2} \cdot \frac{h_0 - h_L}{L}$$

Average thickness

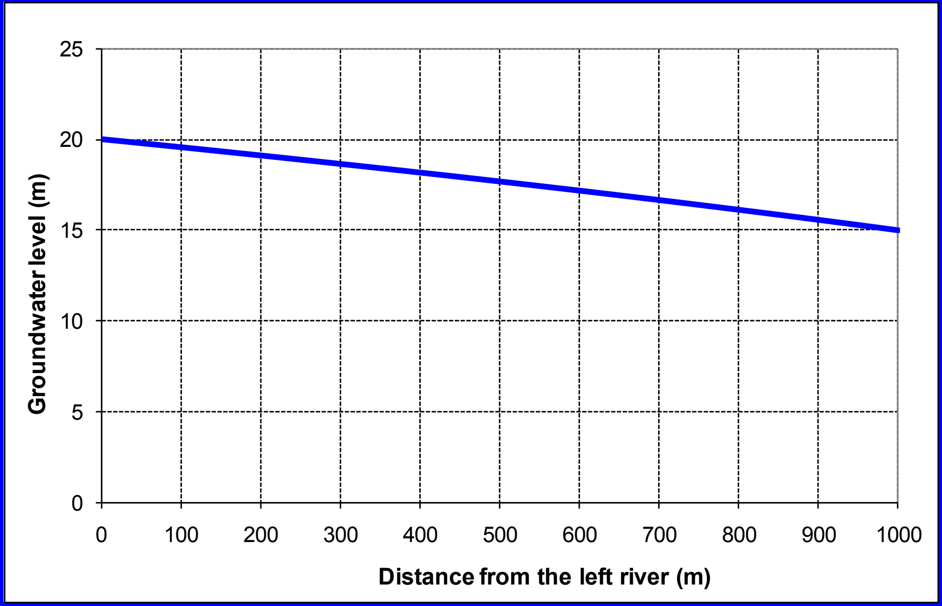
Average hydraulic gradient

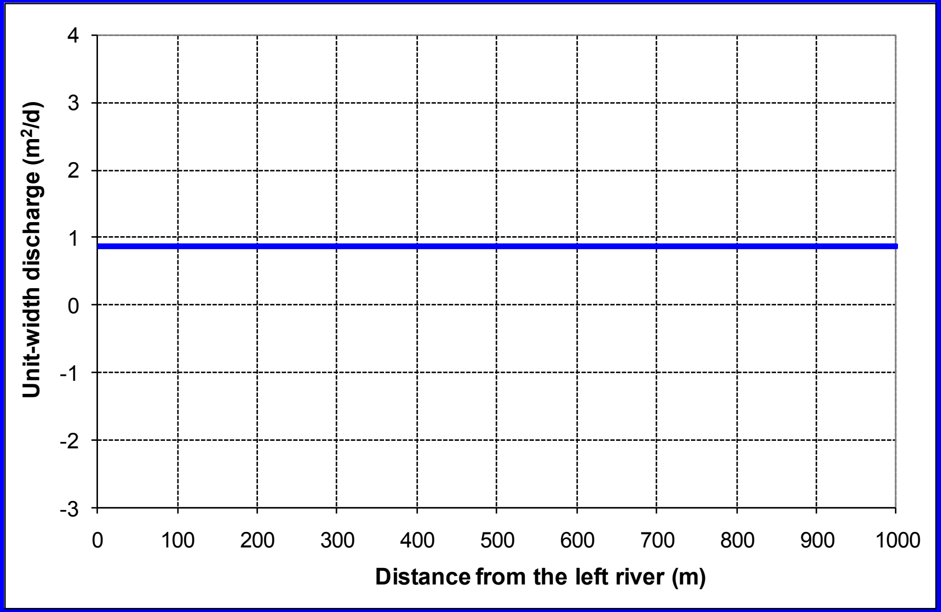
Steady groundwater flow in an unconfined aquifer

When there is no recharge

Example:
L=1000 m
K=10 m/d
w=0.0 m/d
h_0=20 m
h_L=15 m

Q_L=0.875 m^2/d
Q_R=0.875 m^2/d





Steady groundwater flow in an unconfined aquifer

Travel time when there is no recharge

$$q_x = \frac{Q}{h} \quad V_x = \frac{q_x}{n_e} = \frac{Q}{n_e}\frac{1}{h}$$

$$t = \int_0^L \frac{dx}{v_x} = \frac{n_e}{Q}\int_0^L h\,dx = \frac{n_e}{Q}\int_0^L \left[h_0^2 - \frac{h_0^2 - h_L^2}{L}x \right]^{1/2} dx$$

$$= \frac{n_e}{Q}\left\{ -\frac{2L}{3(h_0^2 - h_L^2)}\left[h_0^2 - \frac{h_0^2 - h_L^2}{L}x \right]^{\frac{3}{2}} \right\}\Bigg|_0^L$$

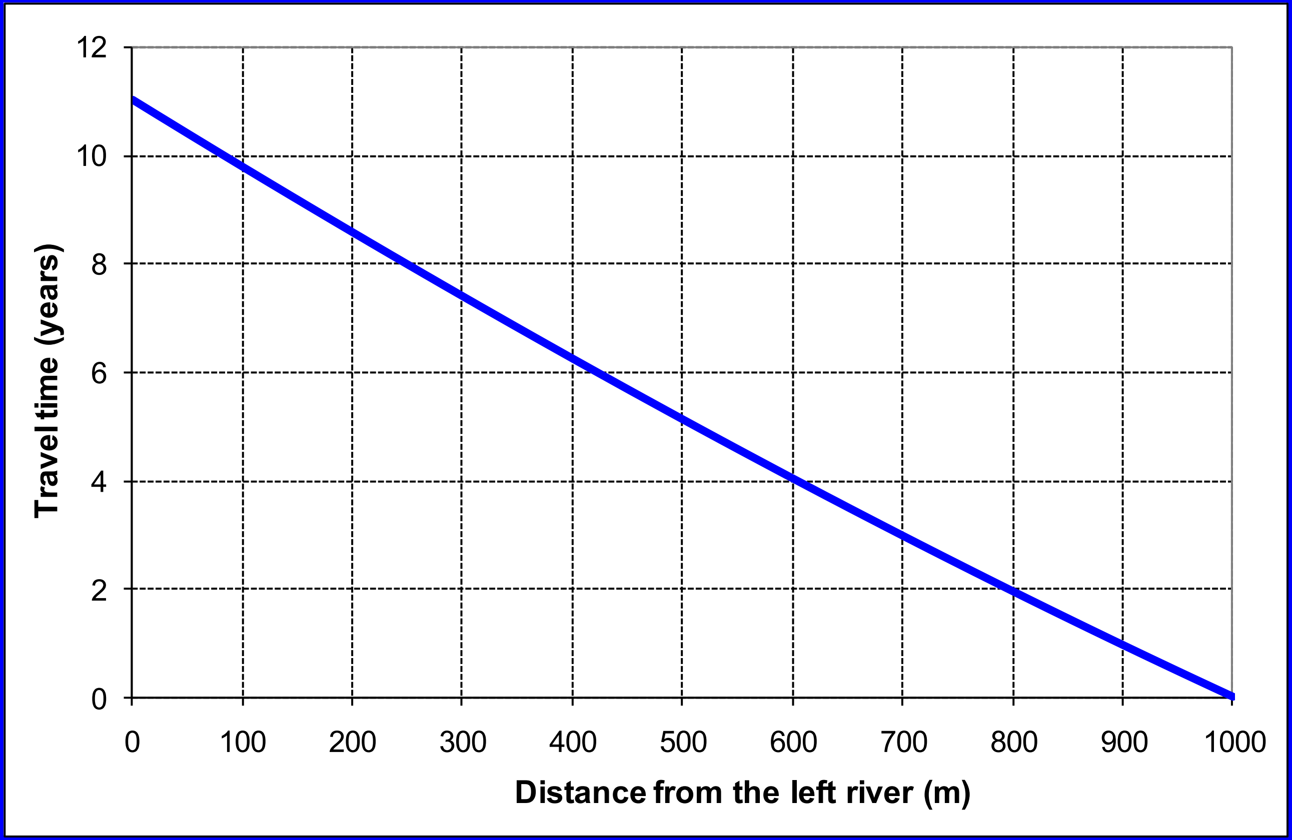
$$t = \frac{4L^2 n_e}{3K}\frac{(h_0^3 - h_L^3)}{(h_0^2 - h_L^2)^2}$$

Steady groundwater flow in an unconfined aquifer

Travel time when there is no recharge

Example:
L=1000 m
K=10 m/d
w=0.0 m/d
h_0=20 m
h_L=15 m

Q_L=0.875 m^2/d
t=4027 days
=11 years!



Steady groundwater flow in an unconfined aquifer

- Special case of same river levels

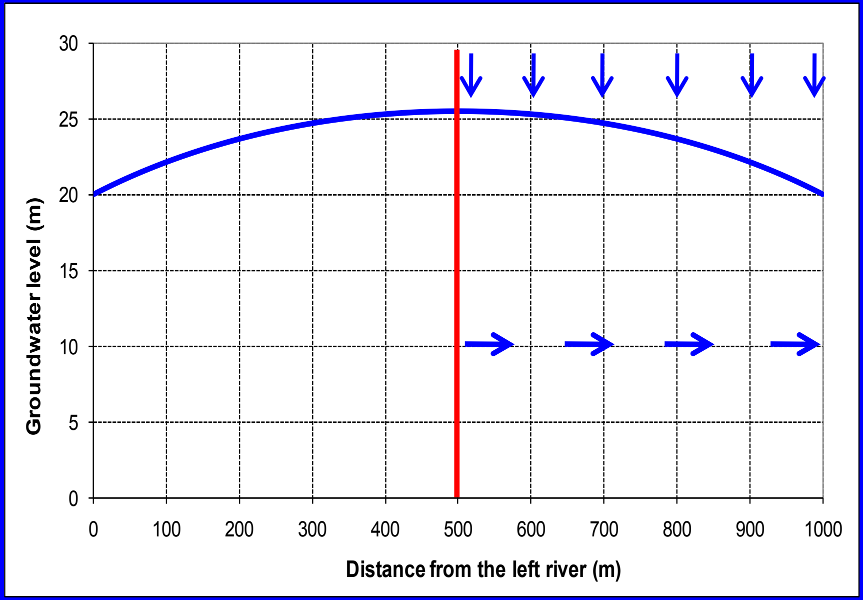
When $h_0 = h_L$

$$d = \frac{L}{2}$$

$$h^2_{max} = h_0^2 + \frac{wL^2}{4K}$$

$$h^2 = h_0^2 + \frac{wL}{K}x - \frac{w}{K}x^2$$

$$Q_x = -\frac{wL}{2} + wx$$



$$y = x - \frac{1}{2}L$$

$$h^2 = h^2_{max} - \frac{w}{K}y^2$$

$$Q_y = wy$$

Steady groundwater flow in an unconfined aquifer

Average velocity

$$v = \frac{wy}{n_e \sqrt{h_{\max}^2 - \frac{w}{K} y^2}}$$

Travel time

$$t = \int_y^{L/2} \frac{dy}{v} = \int_y^{L/2} \frac{n_e \sqrt{h_{\max}^2 - \frac{w}{K} y^2}}{wy} dy$$

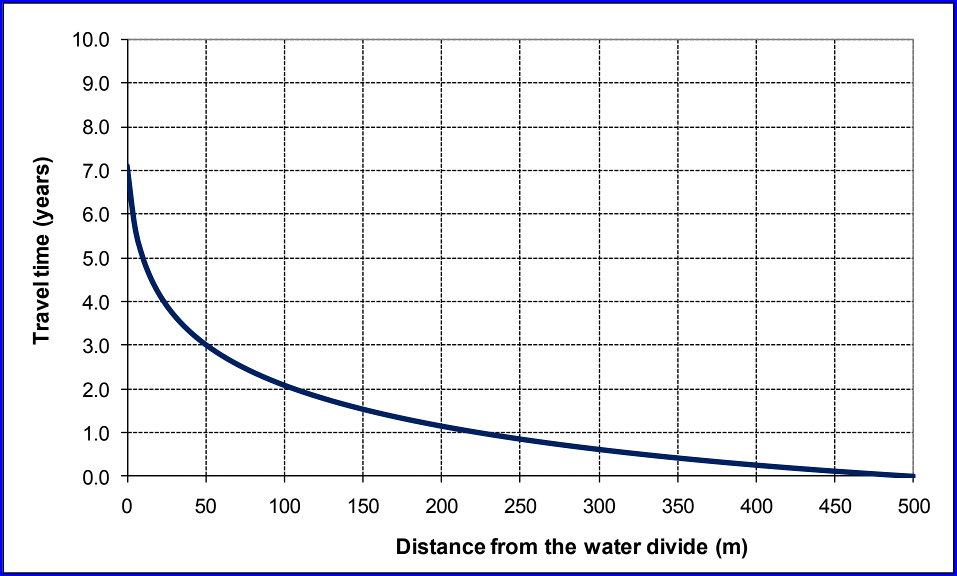
The numerical integration can be used to solve the travel time equation.

Steady groundwater flow in an unconfined aquifer

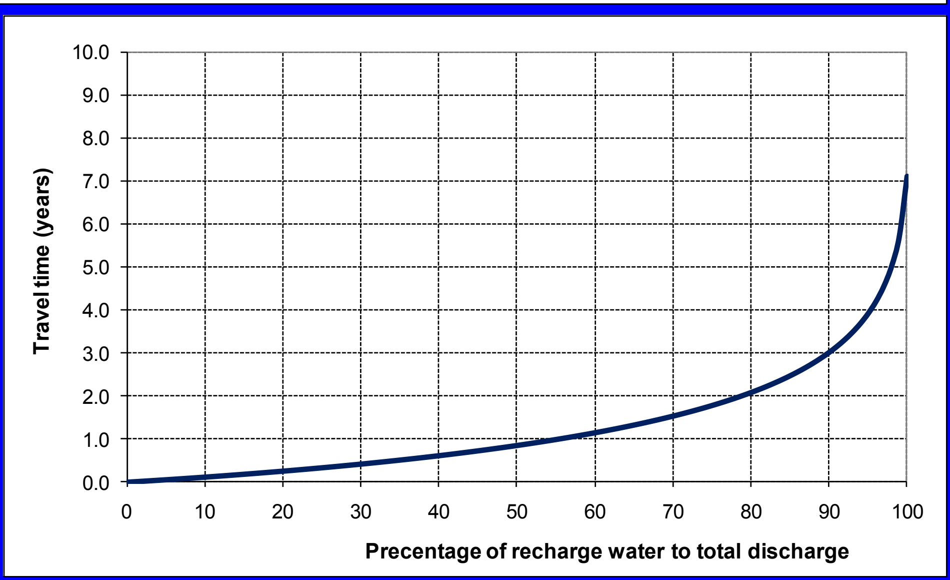
Travel time when there is recharge

Example:
$L=1000$ m
$K=10$ m/d
$w=0.01$ m/d
$h_L=20$ m
$n_e=0.2$
$Q_L=5$ m^2/d

$t_0=7.1$years travel time decreases drastically towards the discharge river .



Mean residence time
$\tau=V/Q=2375/5=475$ days=1.3 years
Arithmetic mean of travel time =474 days=1.3 years

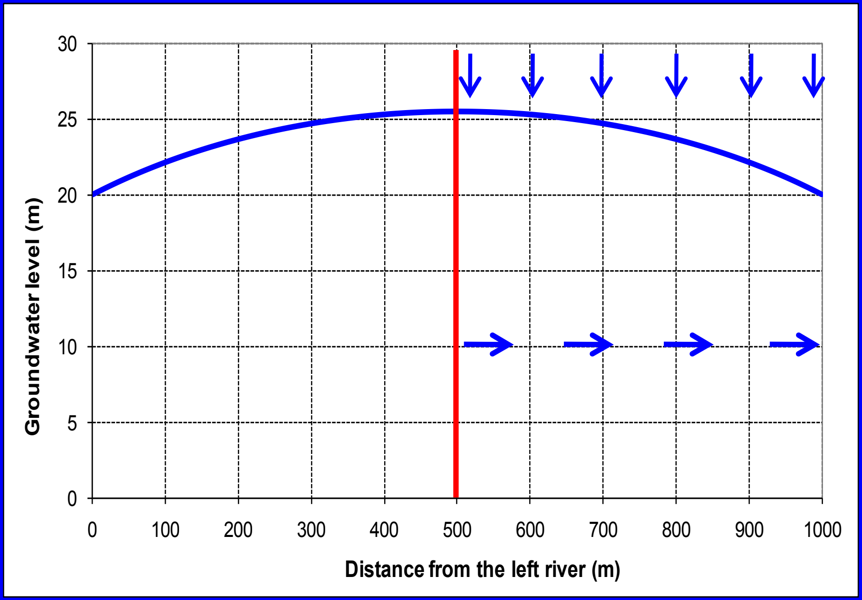


Steady groundwater flow in an unconfined aquifer

Mean residence time of some typical aquifers

Aquifers	L/2(m)	h_0 (m)	h_l (m)	K(m/d)	H(m)	w(m/d)	n	Q(m^2/d)	τ (days)	τ (years)
Local	500	20	20	10	20	0.001	0.2	0.5	4082	11.2
Mediam	5000	100	100	10	100	0.001	0.2	5	21579	59
Regional	50000	100	100	10	100	0.0005	0.2	25	118369	324

The mean residence times range from tens years to hundreds years. For a given aquifer, the mean residence time is inversely proportional to recharge rate, the smaller the recharge rate, the longer the mean residence time.

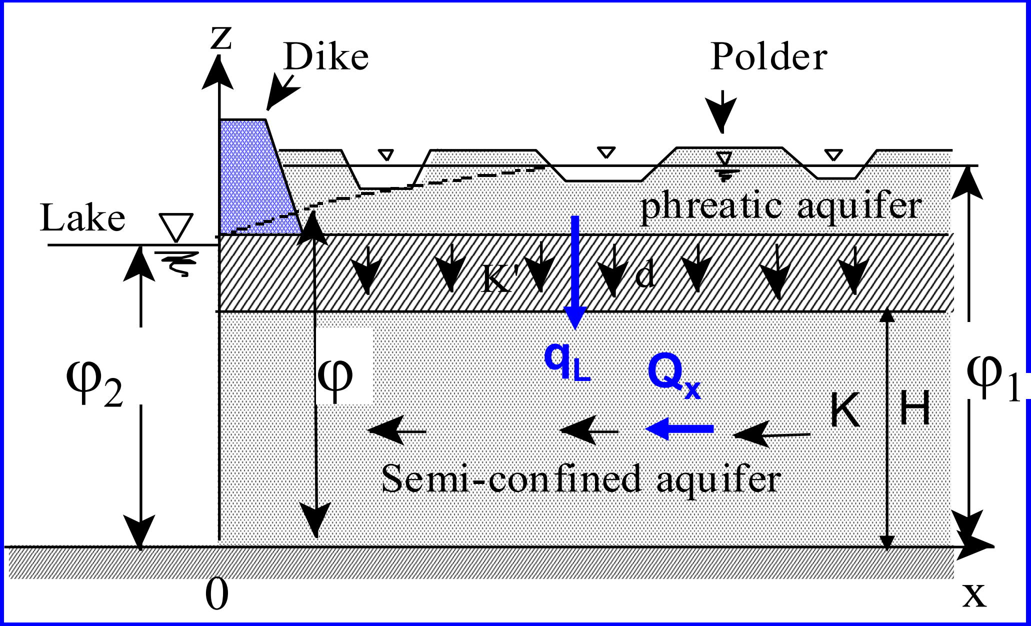


Assignments:

- Describe conceptual hydrogeological model of the unconfined aquifer
- Use water balance and Darcy's equation to derive the partial differential equation
- Which factors determine groundwater head distribution in an unconfined aquifer?
- Compare the differences with the confined aquifer
- Use analytical solution to compute specific discharge, average velocity, and travel time
- Solve the second problem of Exercise 7.2

Steady flow in a semi-confined aquifer

- Conceptual hydrogeological model
  - The semi-confined aquifer is homogeneous with uniform thickness;
  - The semi-permeable layer is homogeneous with constant vertical hydraulic conductivity and thickness;
  - The aquifer system is bounded by a lake on the left with constant level and extends infinite on the right;
  - Groundwater in the phreatic aquifer leaks to semi-confined aquifer, however, water table is assumed to keep constant.



Questions:

1. $\varphi(x) = ?$
2. $Q_x = ?$
3. Residence time?

Steady flow in a semi-confined aquifer

- Mathematical model

Water balance equation:

$$\frac{\partial}{\partial x}(H\,q_x) = q_L$$

Leakage:

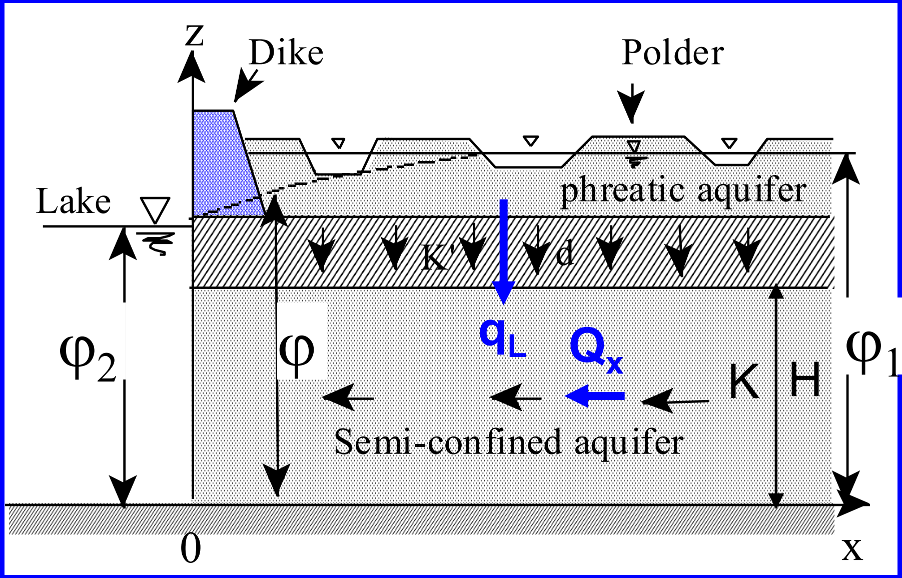
$$q_L = K'\frac{\varphi_1 - \varphi}{d} = \frac{\varphi_1 - \varphi}{c}$$

c=d/K', resistance of the semi-permeable layer [days or years]

Governing equation:

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{\varphi - \varphi_1}{\lambda^2} = 0 \qquad 0 \le x \prec \infty$$

λ^2=Tc, leakage factor (m)



Steady flow in a semi-confined aquifer

- Mathematical model

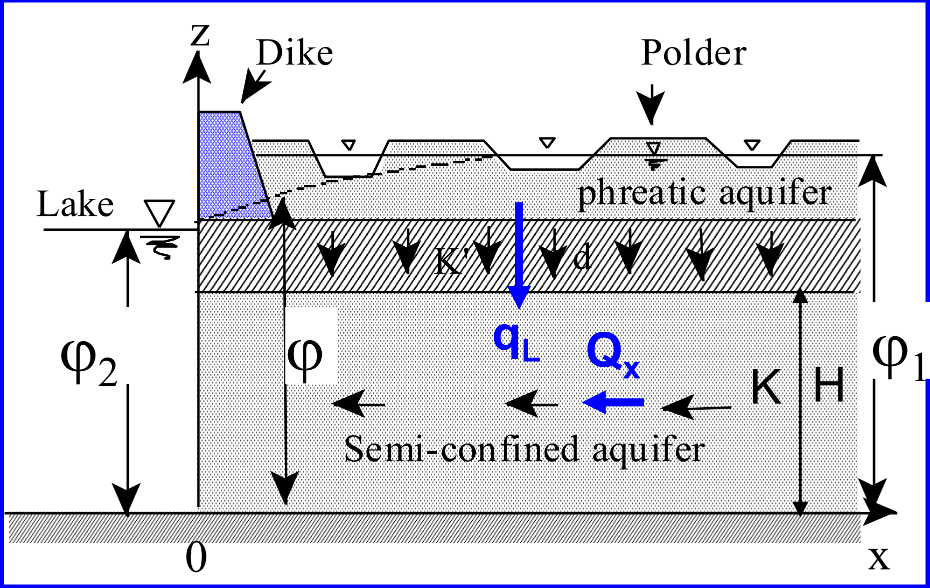
Governing equation:

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{\varphi - \varphi_1}{\lambda^2} = 0 \qquad 0 \le x \prec \infty$$

Boundary conditions:

$$\varphi|_{x=0} = \varphi_2$$

$$\varphi|_{x\to\infty} = \varphi_1$$



Steady flow in a semi-confined aquifer

- Analytical solution:

General solution:

$$\varphi - \varphi_1 = c_1 \exp(\frac{x}{\lambda}) + c_2 \exp(-\frac{x}{\lambda})$$

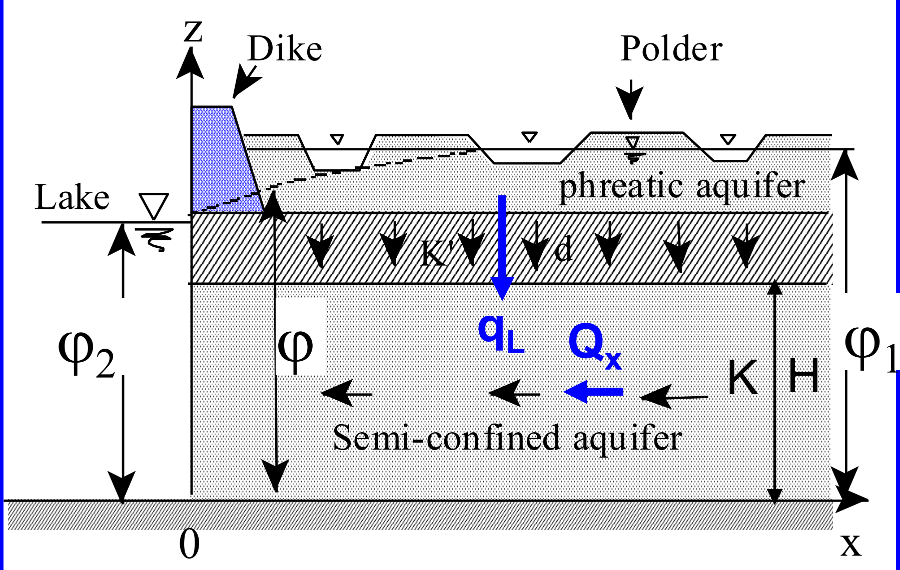
Boundary conditions:

$$c_1 = 0$$

$$c_2 = -(\varphi_1 - \varphi_2)$$

Final solution:

$$\varphi = \varphi_1 - (\varphi_1 - \varphi_2) \exp(-\frac{x}{\lambda})$$

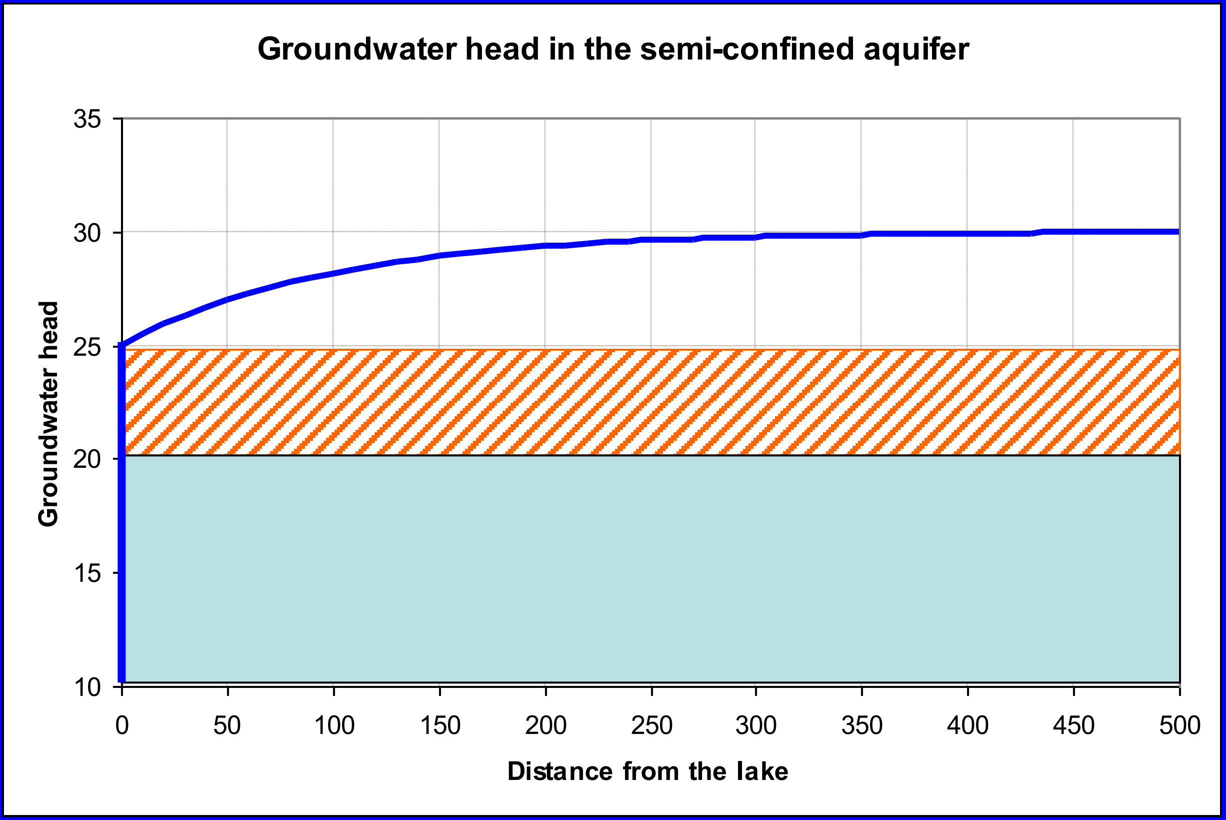


Steady flow in a semi-confined aquifer

Example:
$K=10$ m/d
$H=20$ m
$K'=0.1$ m/d
$d=5$ m
$\varphi_1=30$ m
$\varphi_2=25$ m

$T=200\ m^2/d$
$c=50$ d
$\lambda=100$ m

$$\varphi = \varphi_1 - (\varphi_1 - \varphi_2) \exp\left(-\frac{x}{\lambda}\right)$$



Steady flow in a semi-confined aquifer

- Discharge:

Specific discharge:

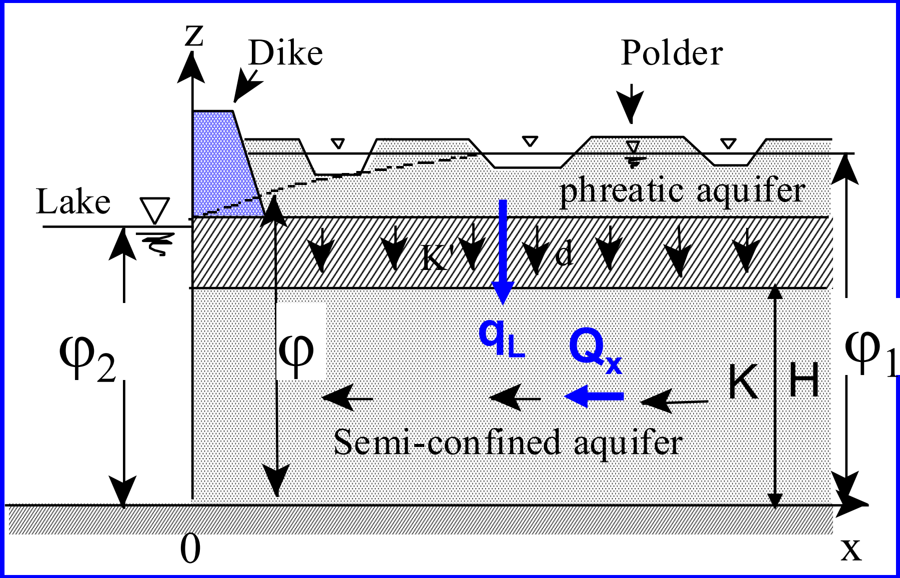
$$q_x = -K\frac{\partial\varphi}{\partial x} = -K\frac{\varphi_1 - \varphi_2}{\lambda}\exp\left(-\frac{x}{\lambda}\right)$$

Unit-width discharge:

$$Q_x = H q_x = -\frac{T(\varphi_1 - \varphi_2)}{\lambda}\exp\left(-\frac{x}{\lambda}\right)$$

Unit-width discharge to the lake:

$$Q_o = -T\cdot\frac{\varphi_1 - \varphi_2}{\lambda}$$



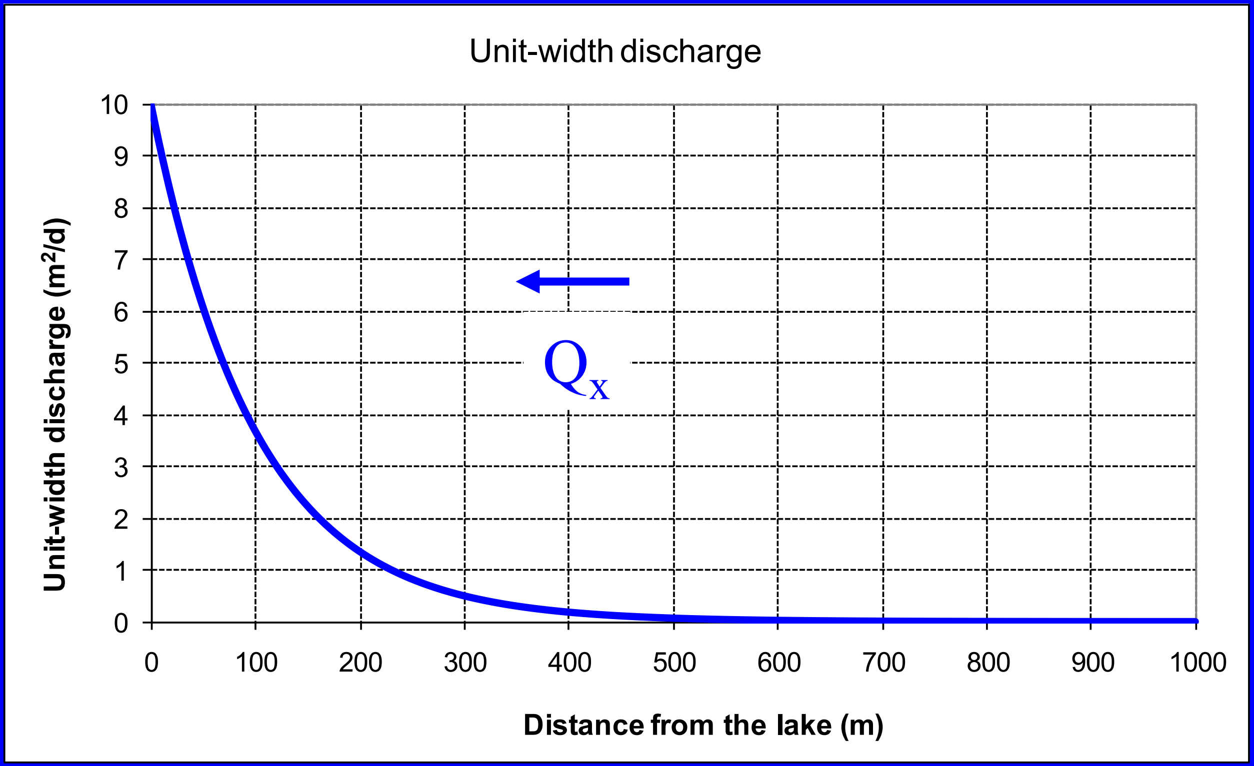
The leakage factor, λ, determines the hydraulic gradient and the influence zone of the lake.

Steady flow in a semi-confined aquifer

Example:
K=10 m/d
H=20 m
K'=0.1 m/d
d=5 m
φ_1=30 m
φ_2=25 m

$$Q_x = H\,q_x = -\frac{T\,(\varphi_1 - \varphi_2)}{\lambda}\exp\left(-\frac{x}{\lambda}\right)$$

T=200 m^2/d
c=50 d
λ=100 m
Q_0=10 m^2/d

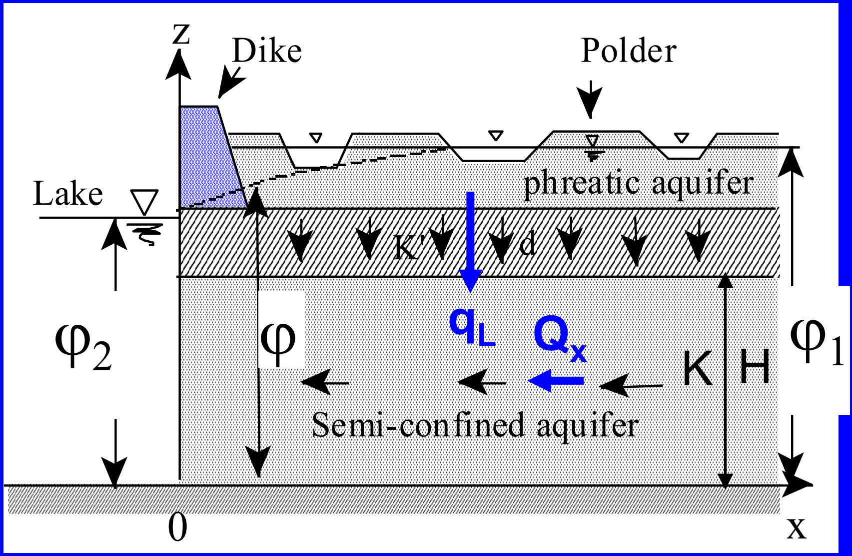


Steady flow in a semi-confined aquifer

- leakage:

Leakage rate:

$$q_L = \frac{\varphi_1 - \varphi}{c} = \frac{(\varphi_1 - \varphi_2)}{c} \exp\left(-\frac{x}{\lambda}\right)$$



Leakage at the distance x

$$Q_L = \int_0^x q_L dx = \frac{(\varphi_1 - \varphi_2)}{c} \int_0^x \exp\left(-\frac{x}{\lambda}\right) dx = \frac{\lambda(\varphi_1 - \varphi_2)}{c}\left[1 - e^{-x/\lambda}\right]$$

Total leakage:

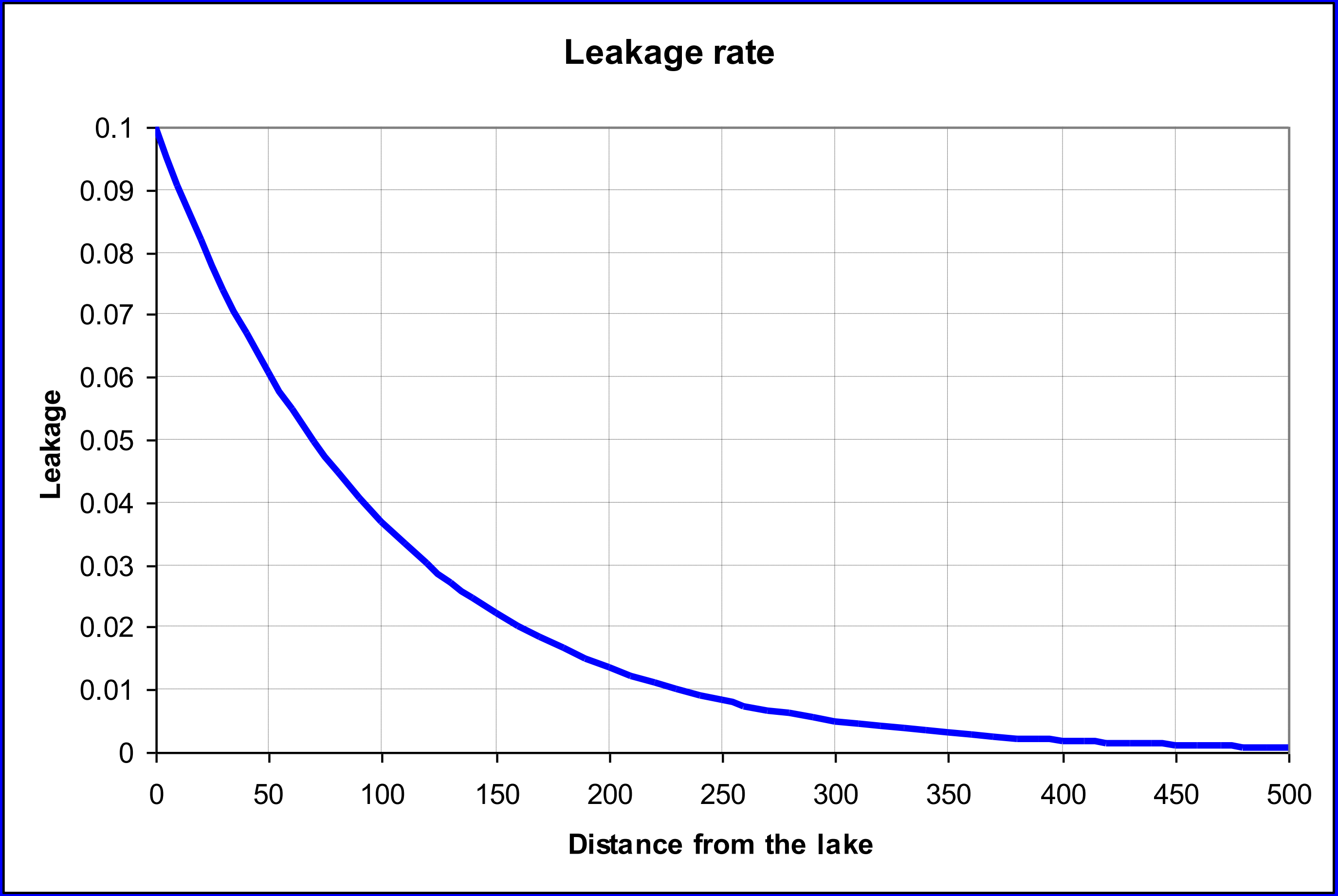
$$Q_L = \frac{\lambda(\varphi_1 - \varphi_2)}{c}$$

The total leakage equals the total discharge to the lake.

Steady flow in a semi-confined aquifer

Example:
K=10 m/d
H=20 m
K'=0.1 m/d
d=5 m
φ_1=30 m
φ_2=25 m

$$q_L = \frac{(\varphi_1 - \varphi_2)}{c} \exp\left(-\frac{x}{\lambda}\right)$$

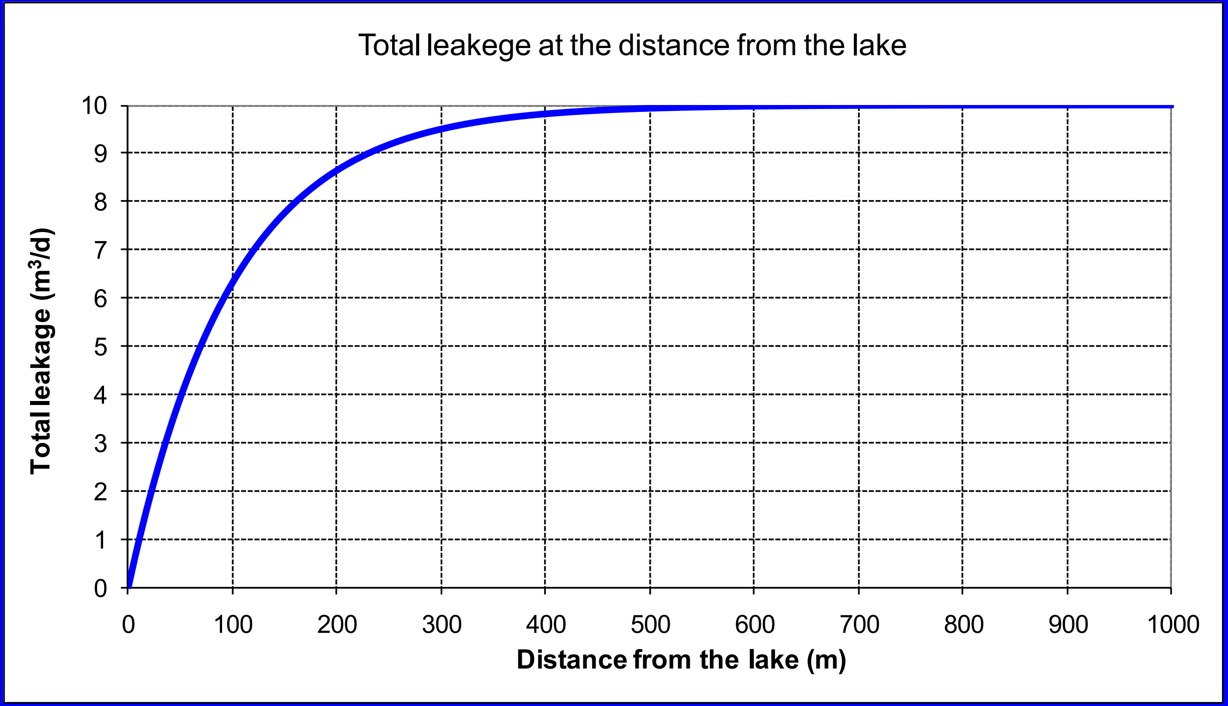


T=200 m^2/d
c=50 d
λ=100 m
q_L=0.1 m^2/d

Steady flow in a semi-confined aquifer

$$Q_L = \frac{\lambda(\varphi_1 - \varphi_2)}{c}[1 - e^{-x/\lambda}]$$

Example:
K=10 m/d
H=20 m
K'=0.1 m/d
d=5 m
φ_1=30 m
φ_2=25 m



T=200 m²/d
c=50 d
λ=100 m
Q_L=10 m²/d

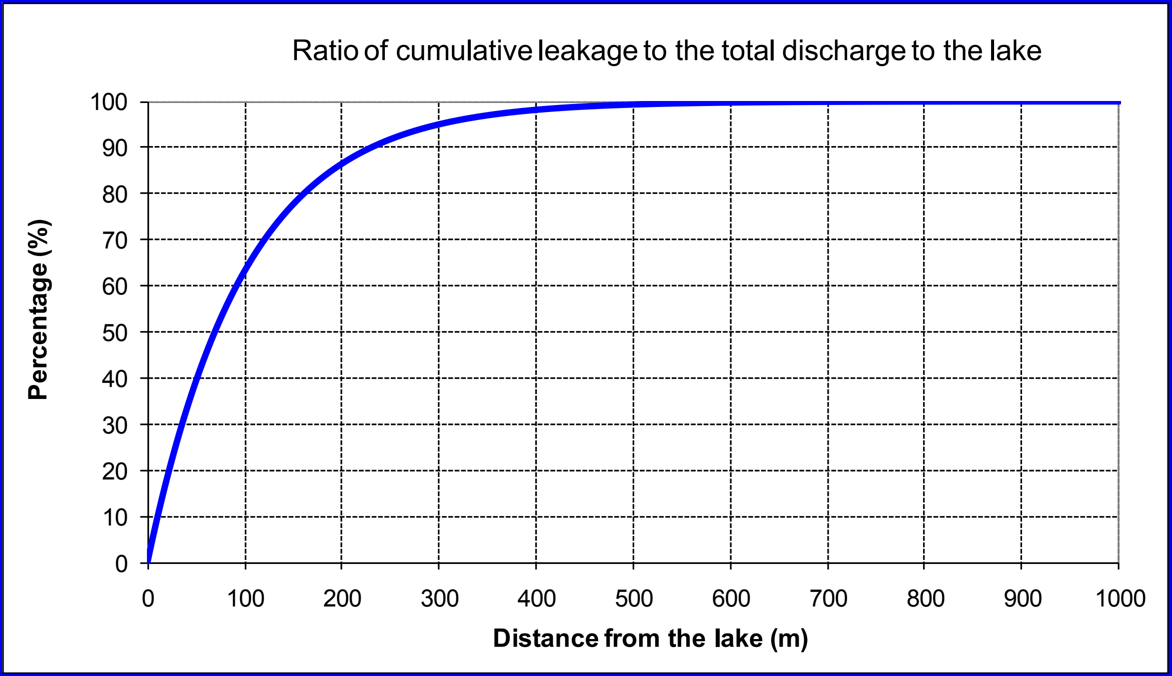
The total leakage is the same as the discharge to the lake.

Steady flow in a semi-confined aquifer

Example:
K=10 m/d
H=20 m
n=0.3
K'=0.1 m/d
d=5 m
φ_1=30 m
φ_2=25 m
c=50 d
λ=100 m

$$\frac{Q_L}{Q_0} = 1 - e^{-x/\lambda}$$

$$x = -\ln\left(1 - \frac{Q_L}{Q_0}\right)\lambda$$



x	Percentage
4λ	98.17
5λ	99.33
6λ	99.75
7λ	99.91
8λ	99.97
9λ	99.99
10λ	100.00

More leakage water comes from shorter distance
5λ can be considered as the 99% zone of influence

Steady flow in a semi-confined aquifer

- Travel time:

Velocity:

$$v_x = \frac{q_x}{n_e} = -\frac{K}{n_e}\frac{\varphi_1 - \varphi_2}{\lambda}\exp\left(-\frac{x}{\lambda}\right)$$

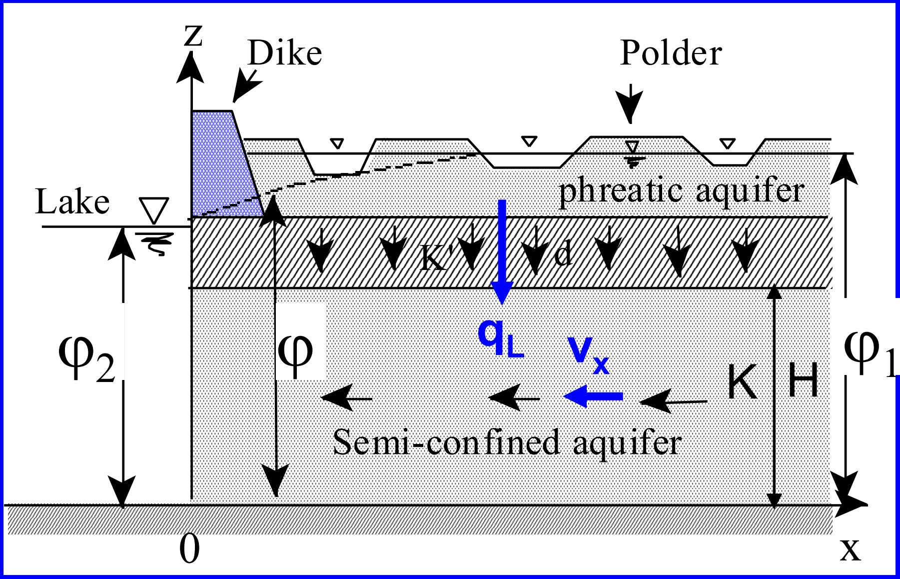
Travel time:

$$t = \int_x^0 \frac{dx}{v_x} = -\left[\frac{K}{n_e}\frac{\varphi_1 - \varphi_2}{\lambda}\right]^{-1}\int_x^0 \frac{dx}{e^{-x/\lambda}}$$

$$t = \frac{n_e}{K}\frac{\lambda^2}{\varphi_1 - \varphi_2}\left[e^{x/\lambda} - 1\right]$$

Travel time from x=5λ:

$$t_{max} \cong 148\,\frac{n_e}{K}\frac{\lambda^2}{\varphi_1 - \varphi_2}$$



Steady flow in a semi-confined aquifer

Example:

$K = 10$ m/d

$H = 20$ m

$n_e = 0.3$

$K' = 0.1$ m/d

$d = 5$ m

$\varphi_1 = 30$ m

$\varphi_2 = 25$ m

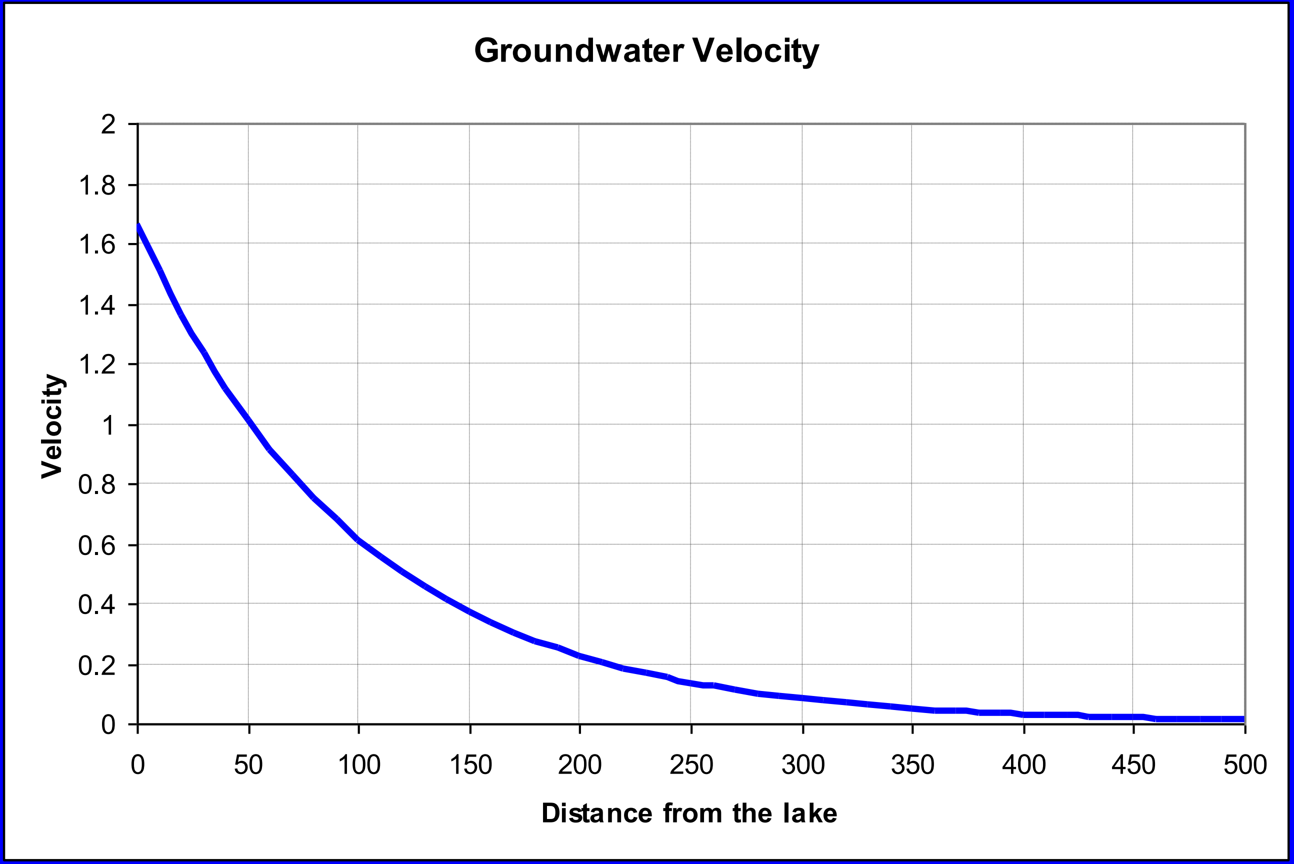
$T = 200$ m^2/d

$c = 50$ d

$\lambda = 100$ m

$v_0 = 1.7$ m/d

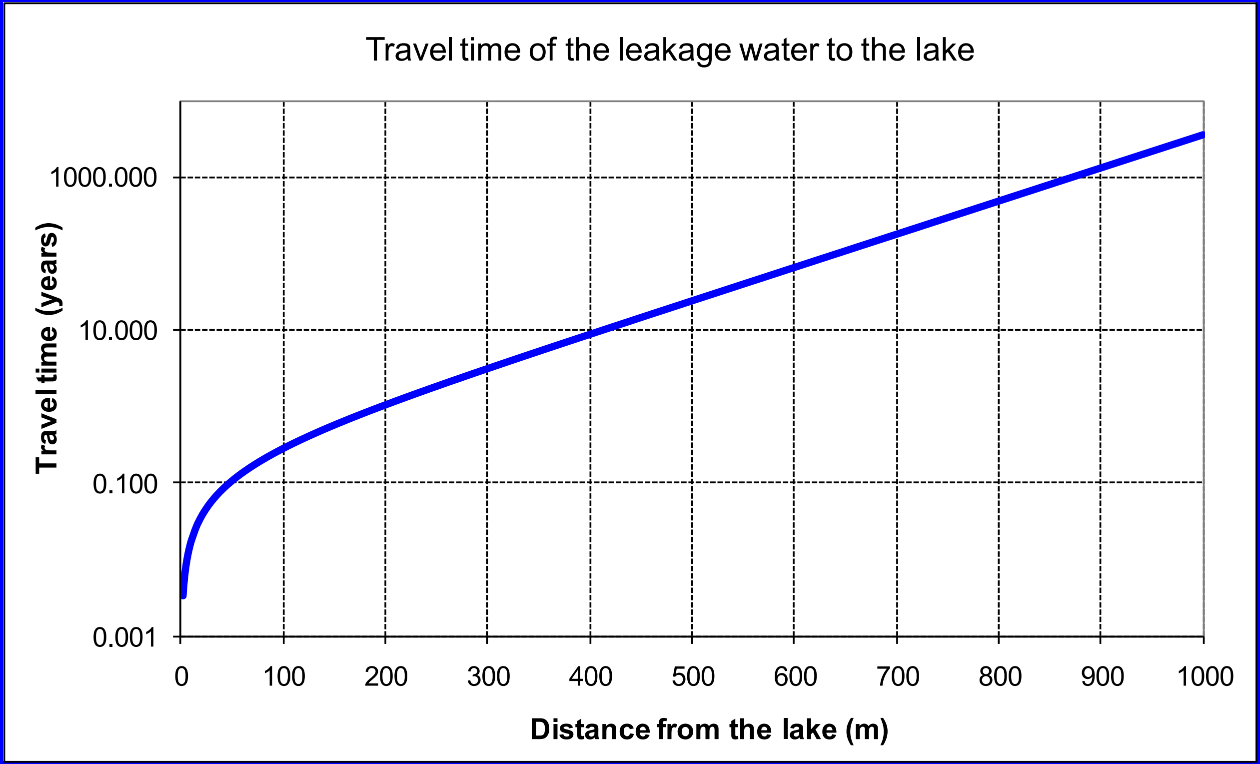
$$v_x = -\frac{K}{n_e}\frac{\varphi_1 - \varphi_2}{\lambda}\exp\left(-\frac{x}{\lambda}\right)$$



Steady flow in a semi-confined aquifer

Example:
$K=10$ m/d
$H=20$ m
$n_e=0.3$
$K'=0.1$ m/d
$d=5$ m
$\varphi_1=30$ m
$\varphi_2=25$ m

$$t = \frac{n_e}{K}\frac{\lambda^2}{\varphi_1 - \varphi_2}\left[e^{x/\lambda} - 1\right]$$



$T=200$ m^2/d
$c=50$ d
$\lambda=100$ m
$t_{99\%}=22$ years

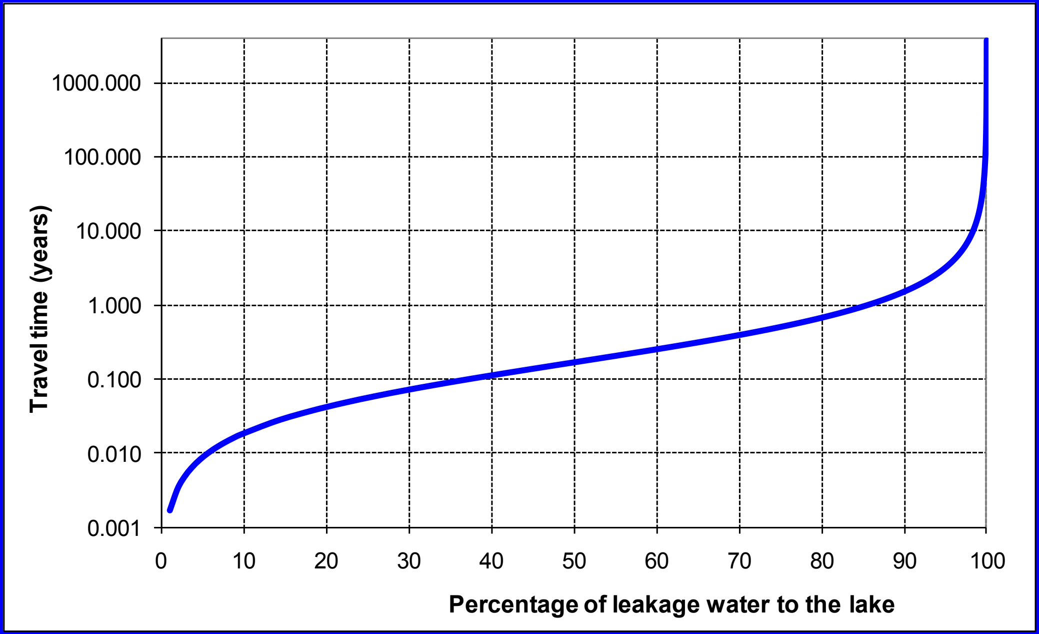
Steady flow in a semi-confined aquifer

- Mean residence time:

$$\tau = 5\frac{\lambda H n_e}{Q_0}$$

$$\tau_L = (4 + e^{-5})\frac{\lambda H n_e}{Q_0}$$

$$\tau_d = \frac{e^5 - 6}{5}\frac{\lambda H n_e}{Q_0}$$



Length of the aquifer	Percentage of leakage water to the lake (%)	Mean residence time (years)	Leakage rate weighted mean residence time (yeas)	Distance weighted mean residence time (years)	Maximum residence time (years)
4λ	98.17	0.66	0.50	2.04	9
5λ	99.33	0.82	0.66	4.68	24
6λ	99.75	0.99	0.82	10.86	66
7λ	99.91	1.15	0.99	25.56	180
8λ	99.97	1.32	1.15	61.07	490
9λ	99.99	1.48	1.32	147.82	1,332
10λ	100.00	1.64	1.48	361.90	3,621

Steady flow in a semi-confined aquifer

Mean residence time for some typical semi-confined aquifers

Aquifers	$5\lambda(m)$	φ_1 (m)	φ_2 (m)	$K(m/d)$	$H(m)$	c (days)	$\lambda(m)$	n	$Q_0(m^2/d)$	τ (days)	τ_L (years)	τ (years)
Local	500	30	25	10	20	50	100	0.3	10	240	0.66	0.82
Mediam	5000	30	25	10	20	5000	1000	0.3	1	24040	65.86	82.19
Regional	10000	30	25	10	20	20000	2000	0.3	0.5	96162	263.46	328.77

The leakage rate weighted mean residence time is smaller than the mean residence time calculated by dividing storage with the discharge to the lake.

Groundwater age in the semi-confined aquifer will be much older than the residence time. The age of groundwater in the semi-confined aquifer includes the travel time in the top unconfined aquifer, the confining layer, and the residence time in the semi-confined aquifer.

Assignments:

- Describe conceptual hydrogeological model of the semi-confined aquifer
- Use water balance and Darcy’s equation to derive the partial differential equation
- Which factors determine groundwater head distribution in a semi-confined aquifer?
- Compare the differences with the confined and unconfined aquifers
- Use analytical solution to compute specific discharge, leakage, and travel time
- Solve the third problem of Exercise 7.3