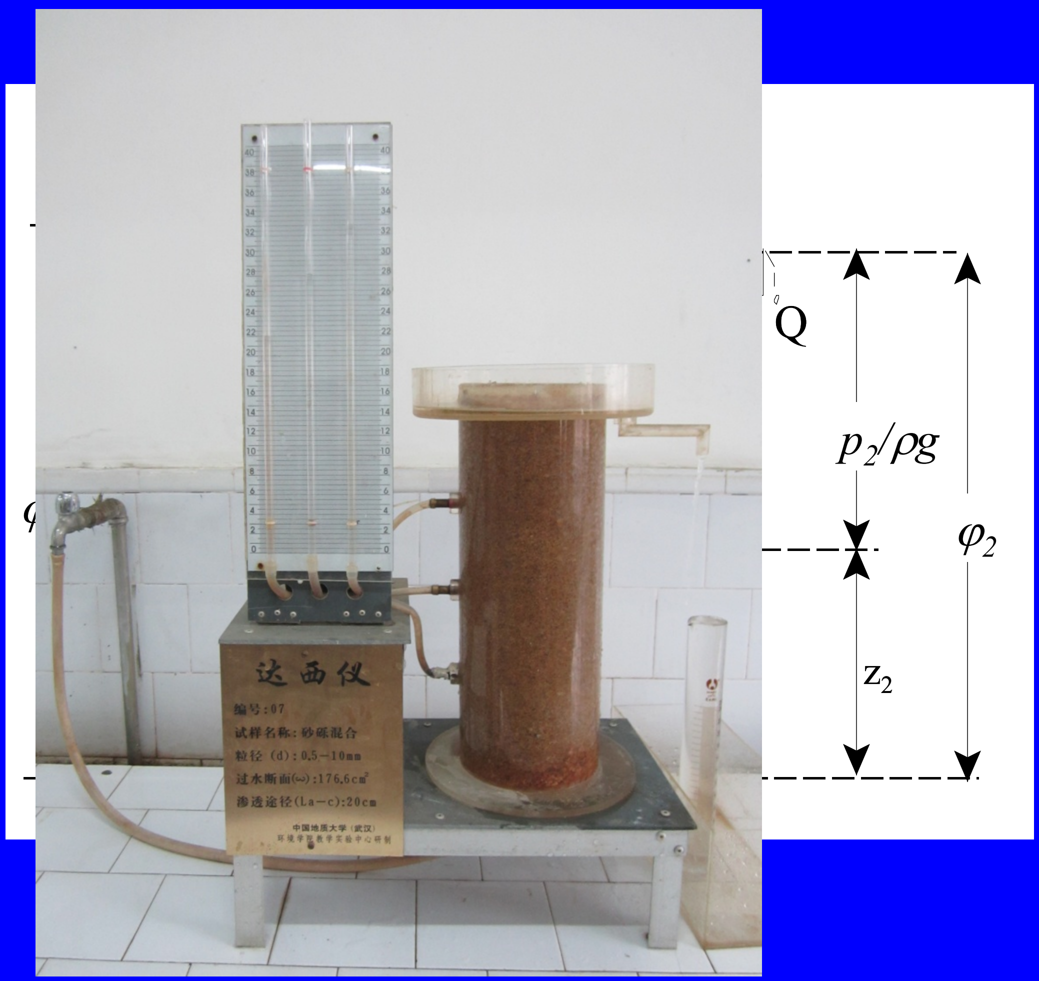
Equations of groundwater flow

- Darcy's law
- Equation of continuity
- Basic equations of steady groundwater flow
- Basic equations of transient groundwater flow

Darcy's law

- Darcy's experiment



$$Q = -K A \frac{\varphi_2 - \varphi_1}{L}$$

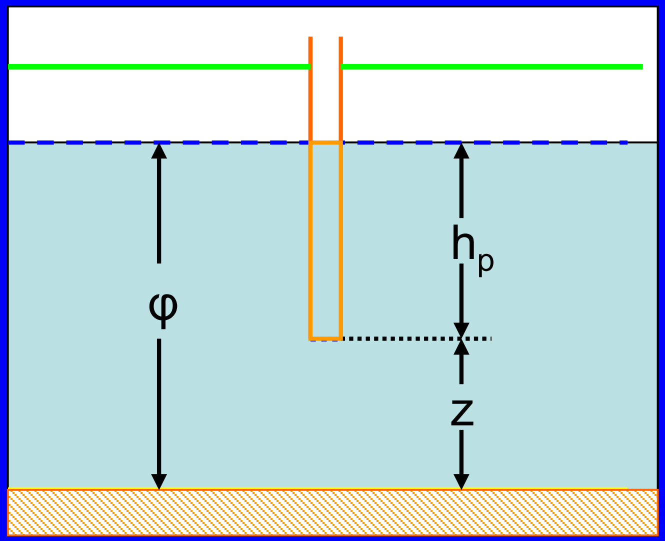
Q :	total discharge, $[L^3T^{-1}]$;
A :	cross-sectional area, $[L^2]$;
K :	coefficient of permeability, $[LT^{-1}]$;
φ_1, φ_2:	water levels in the left and right reservoirs, [L];
L:	length of the sand column (distance), [L].

Hydraulic head (Groundwater head)

$$\varphi = z + h_p$$

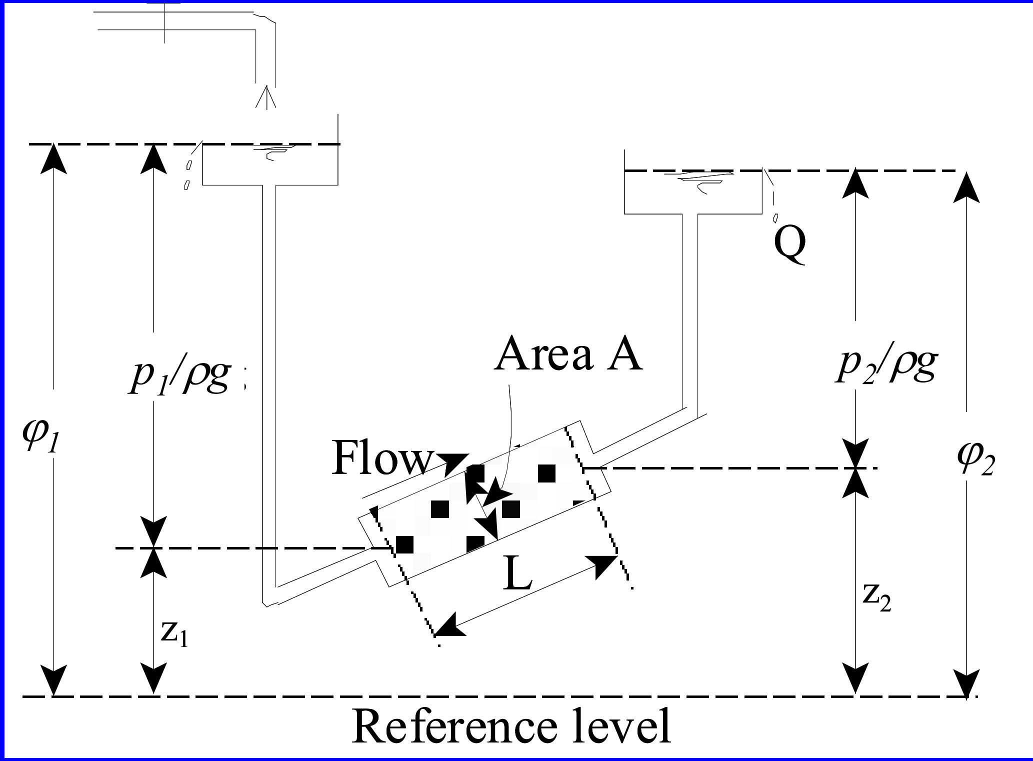
$$p = \rho g h_p$$

$$\varphi = z + \frac{p}{\rho g}$$



z : elevation of the point concerned above the reference level, [L];
p : pressure in the fluid at that point, [$ML^{-1}T^{-2}$];
ρ : density of fluid (mass per unit volume), [$M\ L^{-3}$];
g : acceleration of gravity, [LT^{-2}]. $g = 9.81$ m/sec^2.

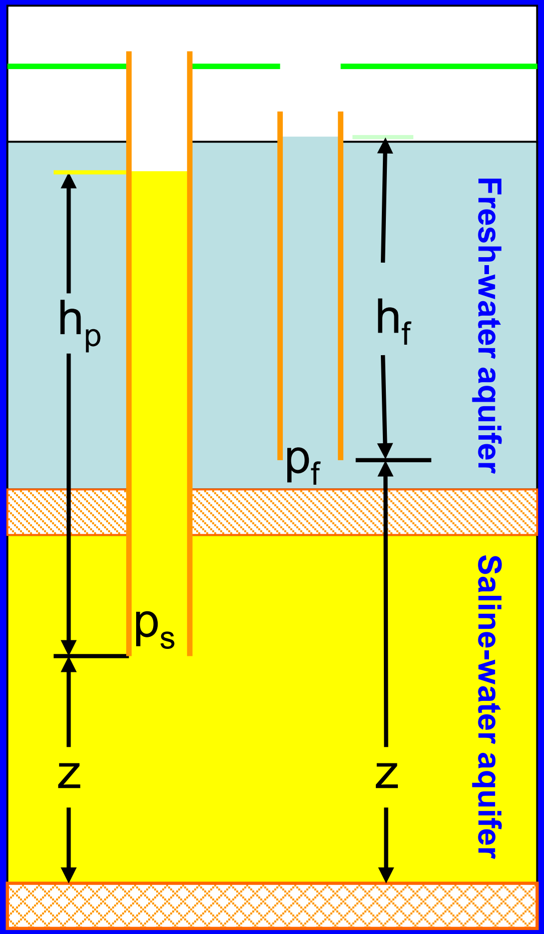
Hydraulic head (Groundwater head)



$$\varphi_1 = z_1 + \frac{p_1}{\rho g}$$

$$\varphi_2 = z_2 + \frac{p_2}{\rho g}$$

Hydraulic head in saline water



Saline-water pressure:

$$p_s = \rho_s g h_p$$

Fresh-water pressure:

$$p_f = \rho_f g h_f$$

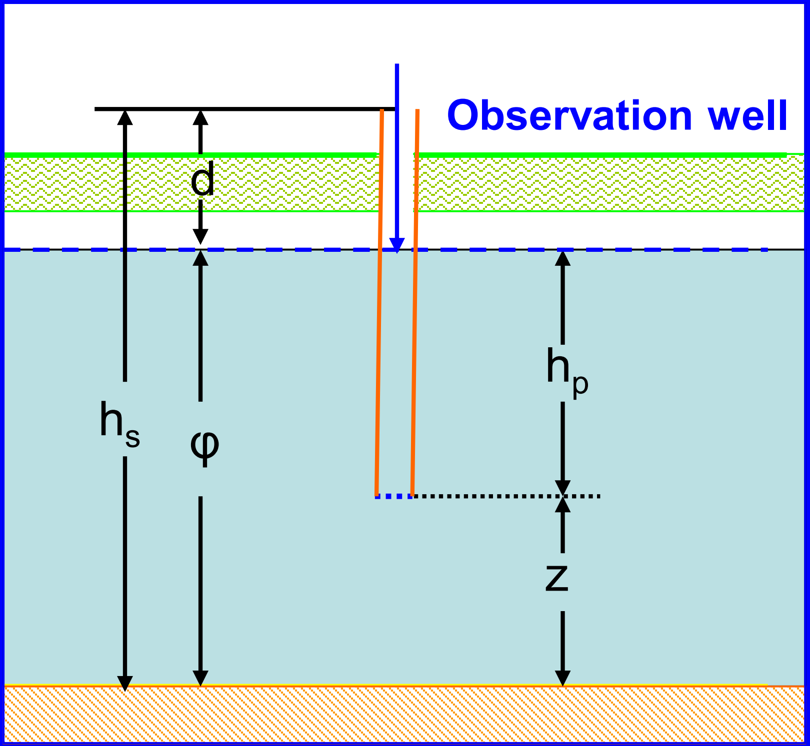
$$p_s = p_f$$

Fresh-water head:

$$h_f = \frac{\rho_s}{\rho_f} h_p$$

Measuring groundwater head

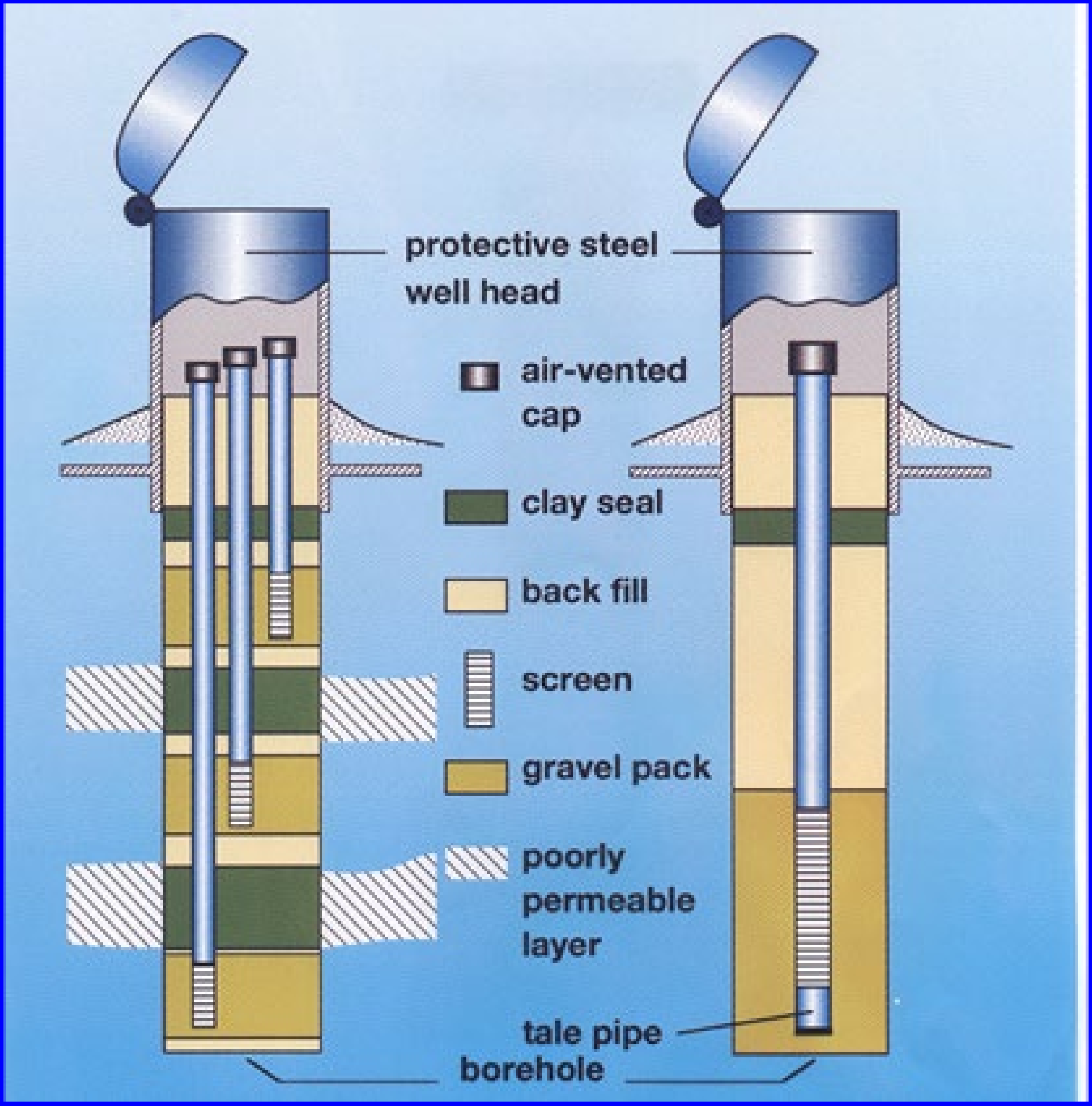
$$\varphi = h_s - d$$



d: depth to water level in the well;
h_s: elevation of the top of the observation well.

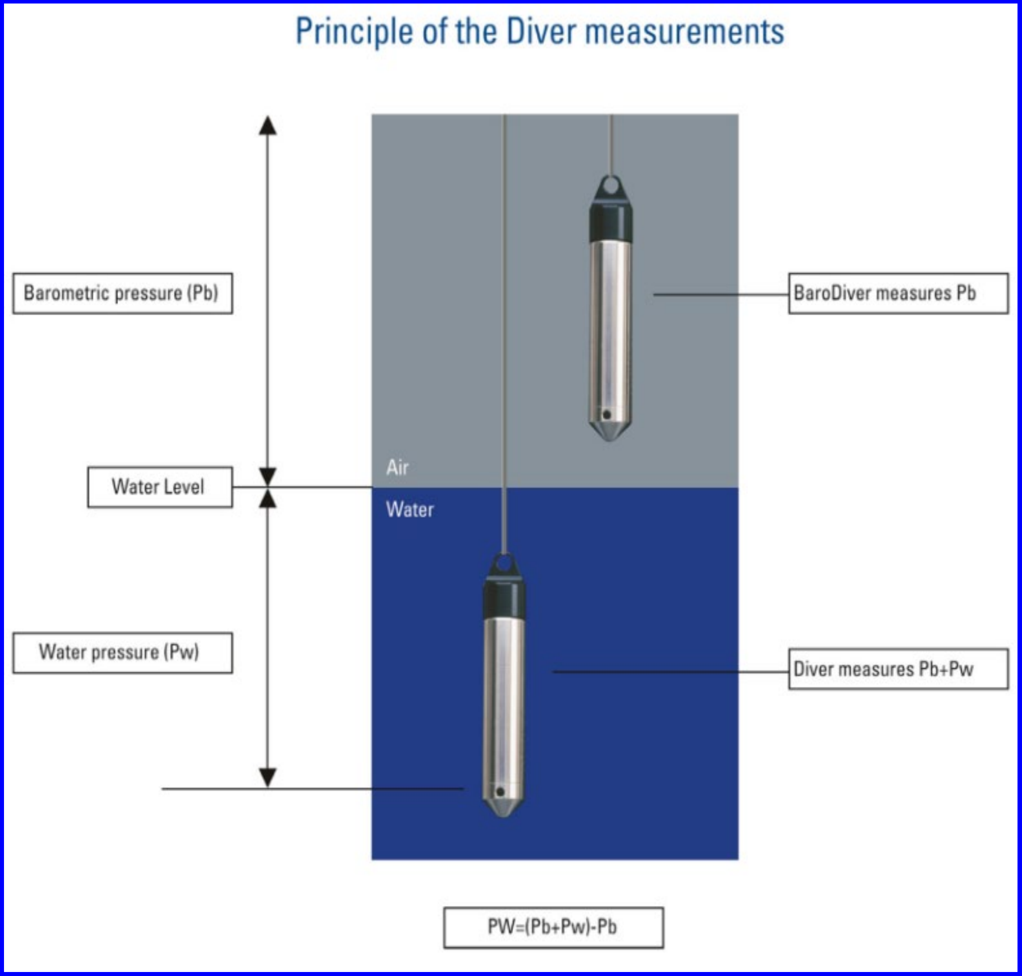
Measuring groundwater head

Water level meters/taps

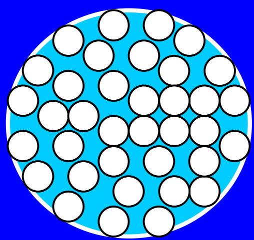


Measuring groundwater head

Automatic dataloggers: Divers



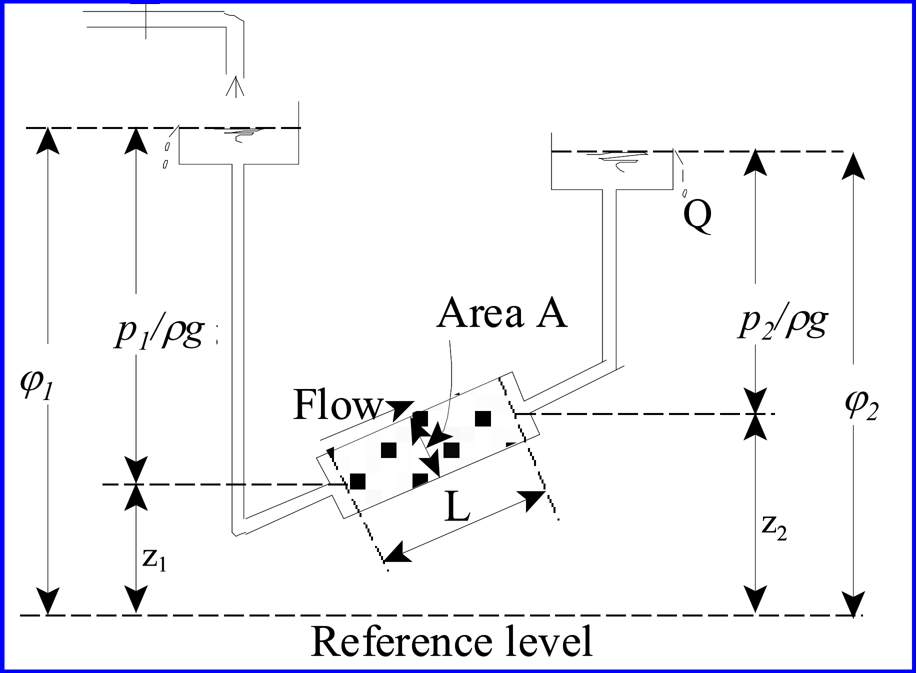
Specific discharge



A: total cross-sectional area including grains and pores

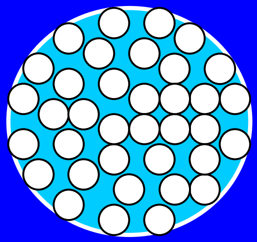
$$q = \frac{Q}{A} = -K\frac{\varphi_2 - \varphi_1}{L}$$

$$q = -K\frac{d\varphi}{dl}$$



q: specific discharge, $[LT^{-1}]$;
$d\varphi/dl$: hydraulic gradient [-].

Groundwater flow velocity



A: total cross-sectional area including grains and pores;
n*A: pore area through which groundwater flows.

$$v = \frac{Q}{n * A} = \frac{q}{n}$$

q: specific discharge, $[LT^{-1}]$;
v: flow velocity, $[LT^{-1}]$.

$$v \geq q$$

Hydraulic conductivity

K: the Darcy's proportionality constant, called hydraulic conductivity or coefficient of permeability, depends on:

- Properties of water;
- Properties of porous medium;
- Temperature.

$$K = \frac{g}{\nu}\kappa = \frac{\rho g}{\eta}\kappa$$

ν: kinematic viscosity, $[L^2T^{-1}]$
η: dynamic viscosity, $[ML^{-1}T^{-1}]$
κ: intrinsic permeability, $[L^2]$

Kozeny-Carman's formula

$$\kappa = c\, d^2 \frac{n^3}{(1-n)^2}$$

d: average diameter of particle size, [L];
n: porosity, [-];
c: constant to be determined with the experiment

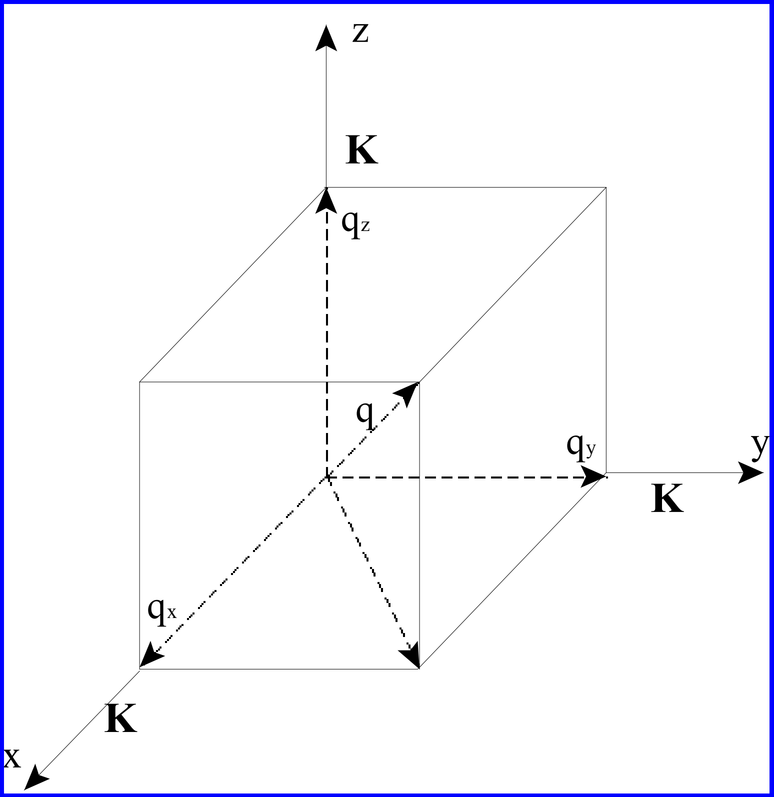
Generalization of Darcy's law

Isotropic medium: K doesn't depend on flow direction

$$q_x = -K\frac{\partial\varphi}{\partial x}$$

$$q_y = -K\frac{\partial\varphi}{\partial y}$$

$$q_z = -K\frac{\partial\varphi}{\partial z}$$



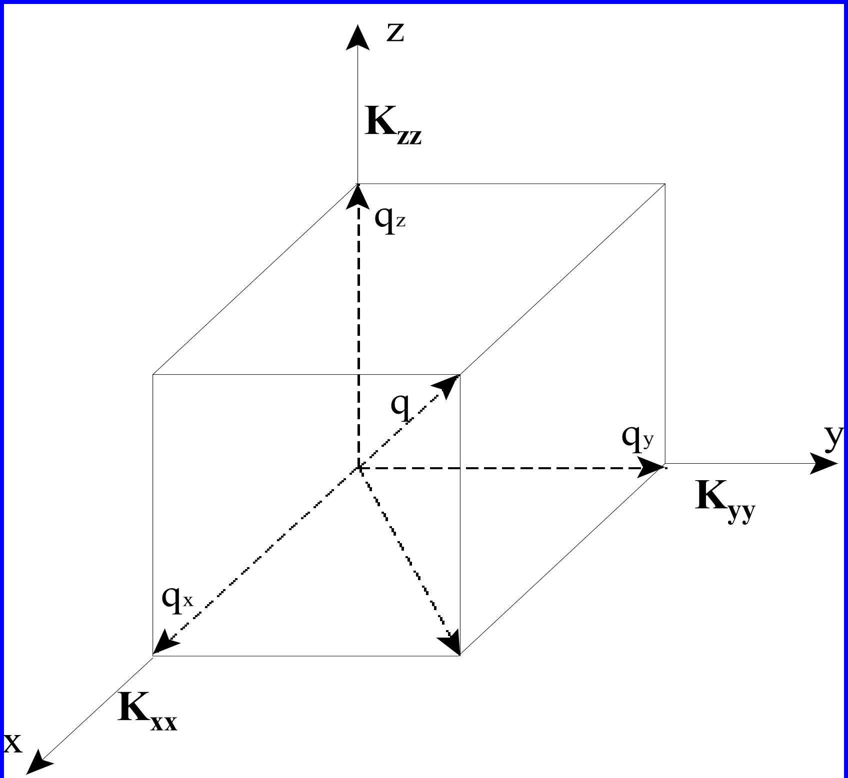
Generalization of Darcy's law

Anisotropic medium: K depends on flow direction

$$q_x = -K_{xx}\frac{\partial \varphi}{\partial x}$$

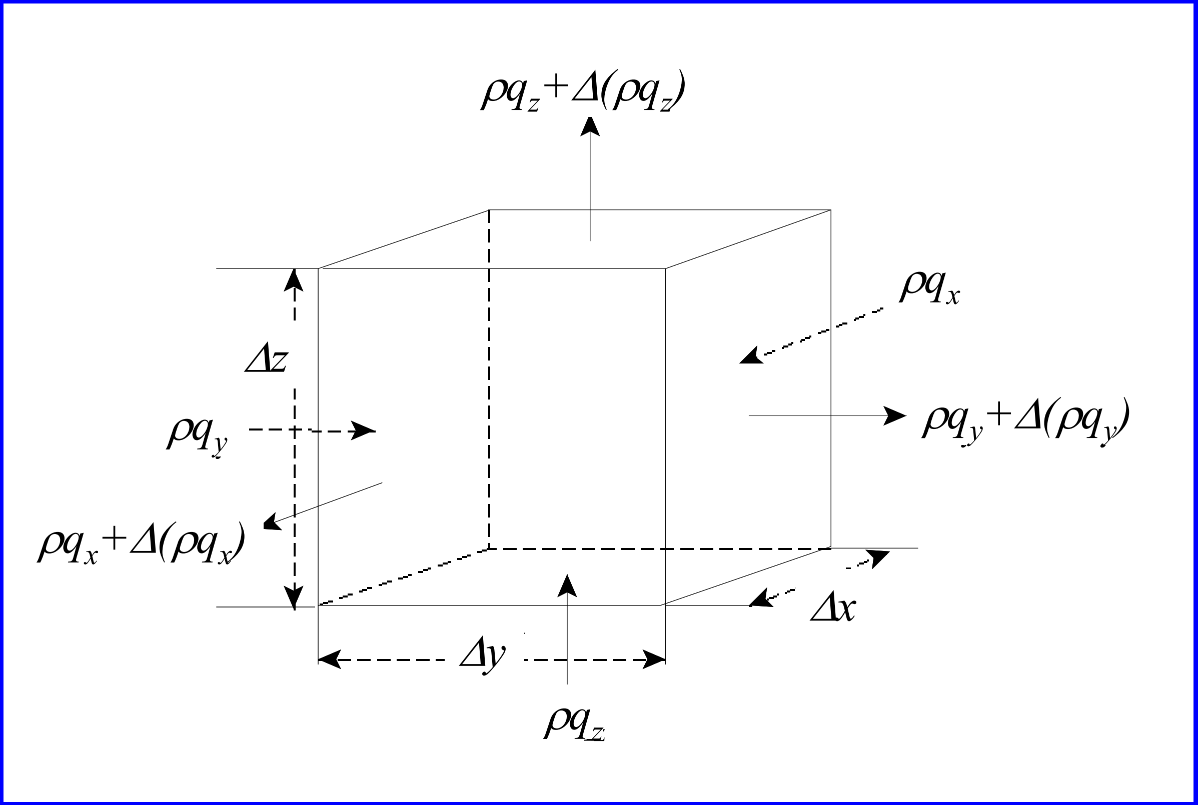
$$q_y = -K_{yy}\frac{\partial \varphi}{\partial y}$$

$$q_z = -K_{zz}\frac{\partial \varphi}{\partial z}$$



Equations of continuity

Conservation of mass or mass balance



Total mass in – total mass out = change of mass storage

Equations of continuity

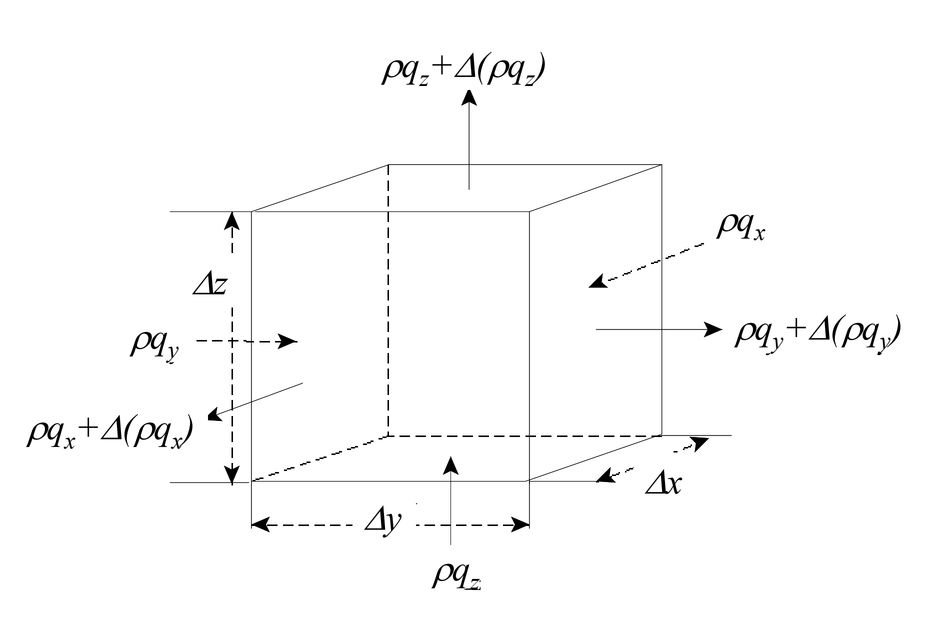
Excess mass in y direction:

Left face:

$$\rho q_y \Delta x \Delta z$$

Right face:

$$\left[\rho q_y + \frac{\partial(\rho q_y)}{\partial y}\Delta y\right]\Delta x \Delta z$$



Excess mass in y direction:

$$\rho q_y \Delta x \Delta z - \left[\rho q_y + \frac{\partial(\rho q_y)}{\partial y}\Delta y\right]\Delta x \Delta z = -\frac{\partial(\rho q_y)}{\partial y}\Delta x \Delta y \Delta z$$

Equations of continuity

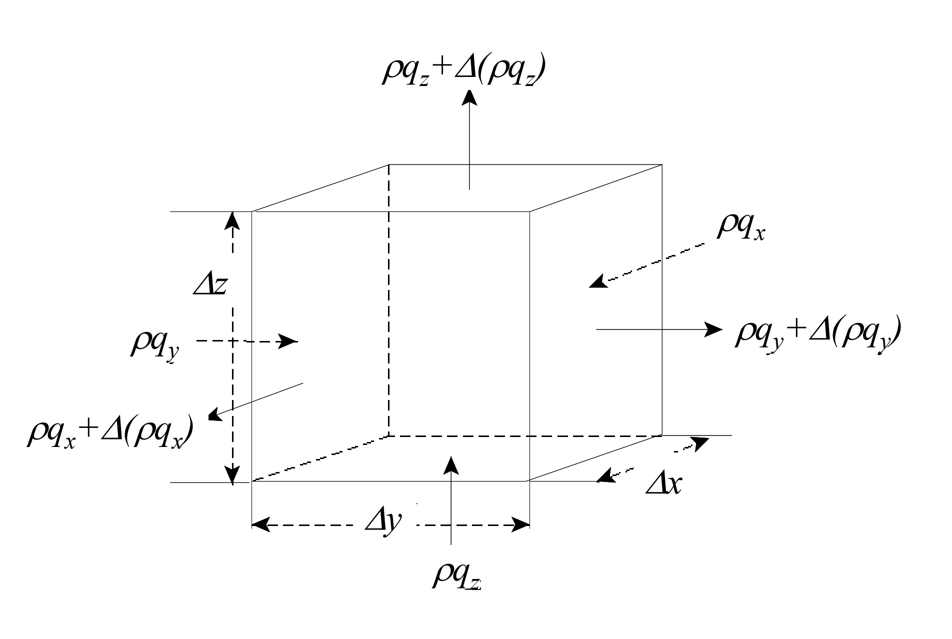
Excess mass in x direction:

Back face:

$$\rho q_x \Delta y \Delta z$$

Front face:

$$\left[\rho q_x + \frac{\partial(\rho q_x)}{\partial x}\Delta x\right]\Delta y \Delta z$$



Excess mass in y direction:

$$\rho q_x \Delta y \Delta z - \left[\rho q_x + \frac{\partial(\rho q_x)}{\partial x}\Delta x\right]\Delta y \Delta z = -\frac{\partial(\rho q_x)}{\partial x}\Delta x \Delta y \Delta z$$

Equations of continuity

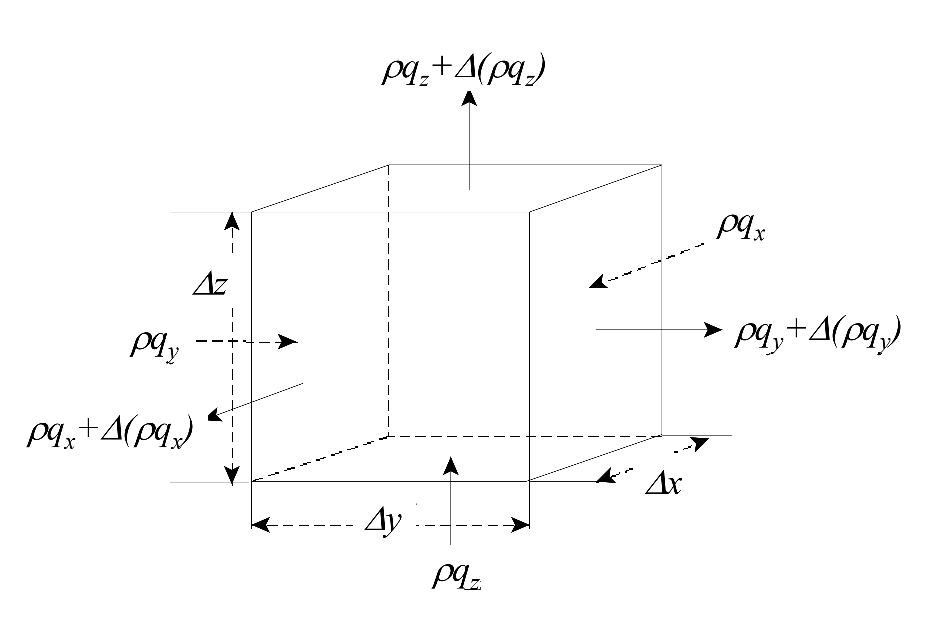
Excess mass in z direction:

Bottom face:

$$\rho q_z \Delta x \Delta y$$

Top face:

$$\left[\rho q_z + \frac{\partial(\rho q_z)}{\partial z}\Delta z\right]\Delta x \Delta y$$



Excess mass in y direction:

$$\rho q_z \Delta x \Delta y - \left[\rho q_z + \frac{\partial(\rho q_z)}{\partial z}\Delta z\right]\Delta x \Delta y = -\frac{\partial(\rho q_z)}{\partial z}\Delta x \Delta y \Delta z$$

Equations of continuity

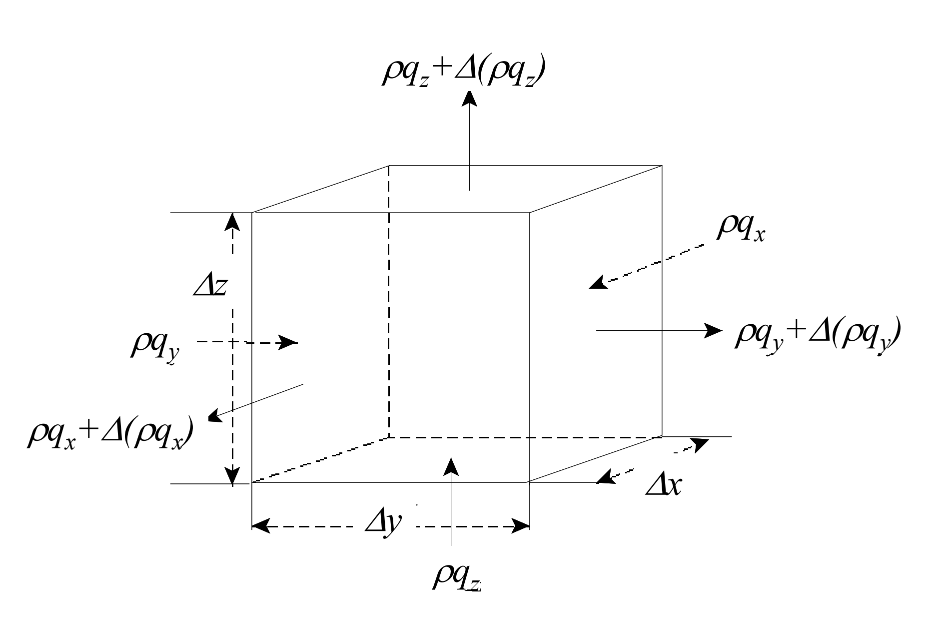
Changes of mass storage per unit time:

Total mass stored:

$$\Delta M = \rho * (\Delta x \Delta y \Delta z) * n$$

Change of the mass storage:

$$\frac{\partial \Delta M}{\partial t} = \frac{\partial [\rho * (\Delta x \Delta y \Delta z) * n]}{\partial t}$$



Equations of continuity

$$\frac{\partial}{\partial t}[\rho(\Delta x \Delta y \Delta z)n] = -[\frac{\partial(\rho q_x)}{\partial x} + \frac{\partial(\rho q_y)}{\partial y} + \frac{\partial(\rho q_z)}{\partial z}]\,\Delta x\,\Delta y\,\Delta z$$

Change of mass storage

Total mass in – total mass out

Basic equations of steady flow

Total mass in = Total mass out

No change of mass storage

$$\frac{\partial}{\partial t}[\rho(\Delta x \Delta y \Delta z)n] = 0$$

$$\frac{\partial(\rho q_x)}{\partial x} + \frac{\partial(\rho q_y)}{\partial y} + \frac{\partial(\rho q_z)}{\partial z} = 0$$

Basic equations of steady flow

If the density is a constant:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = 0$$

Water balance equation for steady state groundwater flow

Substitute Darcy's equations for specific discharges:

$$\frac{\partial}{\partial x}(K_{xx}\frac{\partial \varphi}{\partial x}) + \frac{\partial}{\partial y}(K_{yy}\frac{\partial \varphi}{\partial y}) + \frac{\partial}{\partial z}(K_{zz}\frac{\partial \varphi}{\partial z}) = 0$$

Equation for steady state groundwater flow in anisotropic and heterogeneous porous medium

Basic equations of steady flow

Equation for steady state groundwater flow in anisotropic and homogeneous porous medium

$$K_{xx}\frac{\partial^2 \varphi}{\partial x^2} + K_{yy}\frac{\partial^2 \varphi}{\partial y^2} + K_{zz}\frac{\partial^2 \varphi}{\partial z^2} = 0$$

Equation for steady state groundwater flow in isotropic and homogeneous porous medium

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

Basic equations of steady flow

Boundary conditions

First type: specified head

$$\varphi|_{\Gamma_1} = \varphi_1(x, y, z) \qquad (x, y, z) \in \Gamma_1$$

Second type: specified flow

$$K\frac{\partial \varphi}{\partial n}|_{\Gamma_2} = q(x, y, z) \qquad (x, y, z) \in \Gamma_2$$

Third type: head-dependent flow

$$q(x, y, z)|_{\Gamma_3} = K'\frac{\varphi - \varphi_0}{B'} \qquad (x, y, z) \in \Gamma_3$$

Basic equations of steady flow

Initial conditions

$$\varphi\big|_{t=0} = \varphi_0(x, y, z)$$

Initial conditions refer to the distribution of groundwater heads everywhere in the aquifer at the beginning of a reference time. It is necessary only for transient groundwater flow

Basic equations of steady flow

Mathematical model

Governing equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

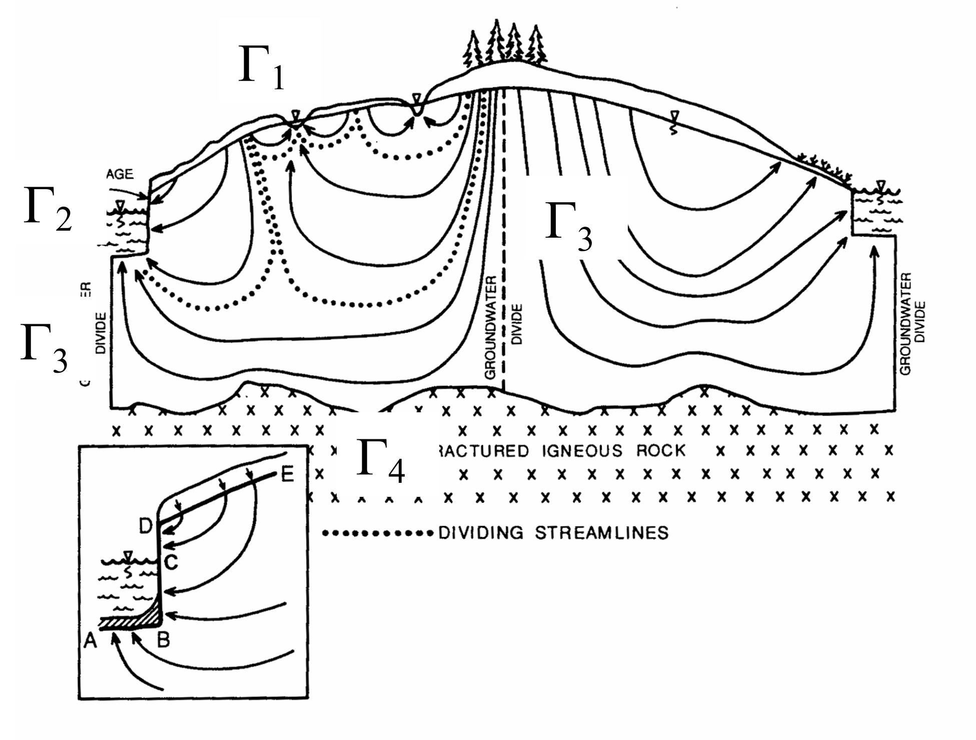
Boundary conditions

$$\varphi|_{\Gamma_1} = \varphi_1(x) \qquad x \in \Gamma_1$$

$$\varphi|_{\Gamma_2} = \varphi_2(z) \qquad z \in \Gamma_2$$

$$K\frac{\partial \varphi}{\partial x}|_{\Gamma_3} = 0 \qquad z \in \Gamma_3$$

$$K\frac{\partial \varphi}{\partial z}|_{\Gamma_4} = 0 \qquad x \in \Gamma_4$$



A unique solution of the model exits only if one of boundary conditions is specified head boundary!

What to learn in this chapter:

- What are the main conditions of Darcy's law?
- Which factors determine the groundwater flow?
- How to measure groundwater head?
- How to use Darcy's equation to compute groundwater flow?

Assignments:

- Summarize "What to learn"
- Solve Exercise 7.1