Some additional methods

- What if there are more wells?
- What if a well is close to a boundary?
- What if the system is more complex?

- Method of superposition
- Method of images
- Flow net

Method of superposition

Principle of the method

For a linear system (linear partial differential equation):

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} = 0$$

If s_1 and s_2 are solutions, the linear combination:

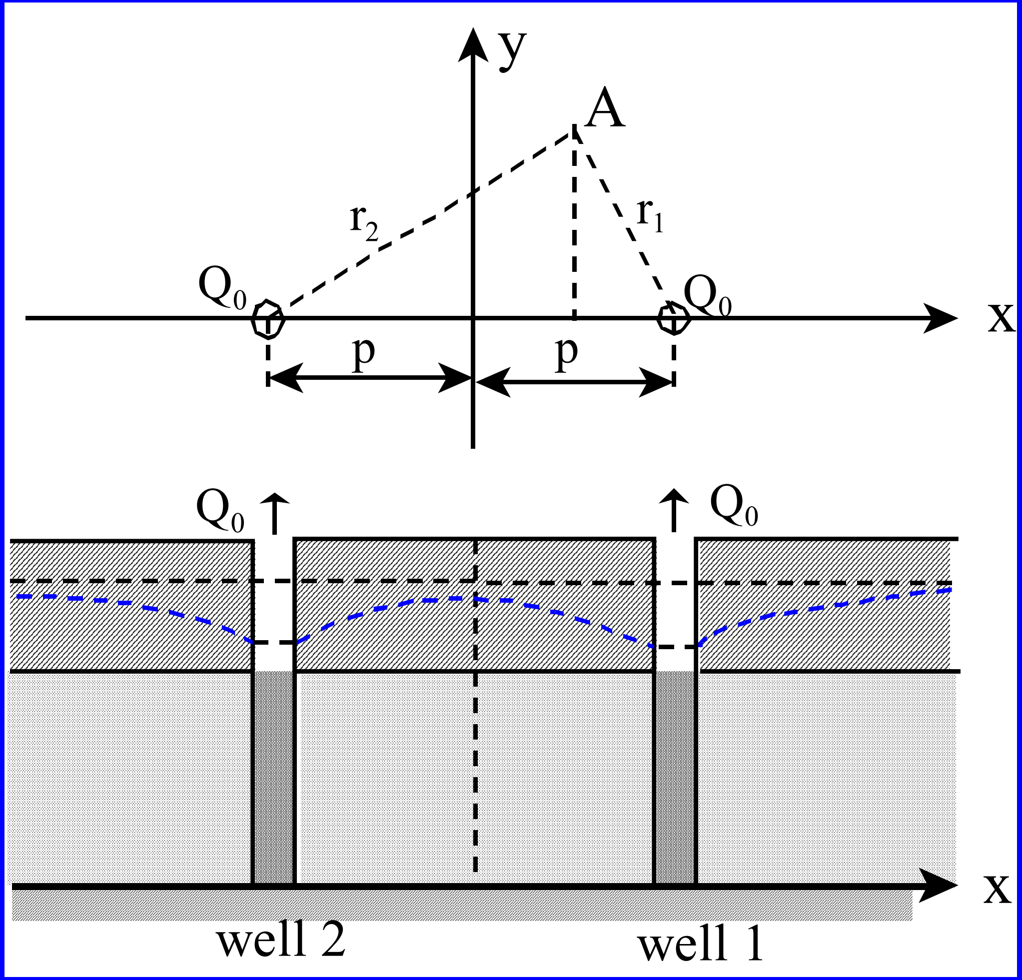
$$s = \alpha s_1 + \beta s_2$$

is also the solution.

Method of superposition

Case 1: Two pumping wells in a confined aquifer

- Conceptual hydrogeological model
  - The confined aquifer is bounded with a circular constant head boundary;
  - The aquifer is homogeneous and isotropic;
  - Two pumping wells locate at a distance of 2p with the same pumping rate.



Method of superposition

Case 1: Two pumping wells in a confined aquifer

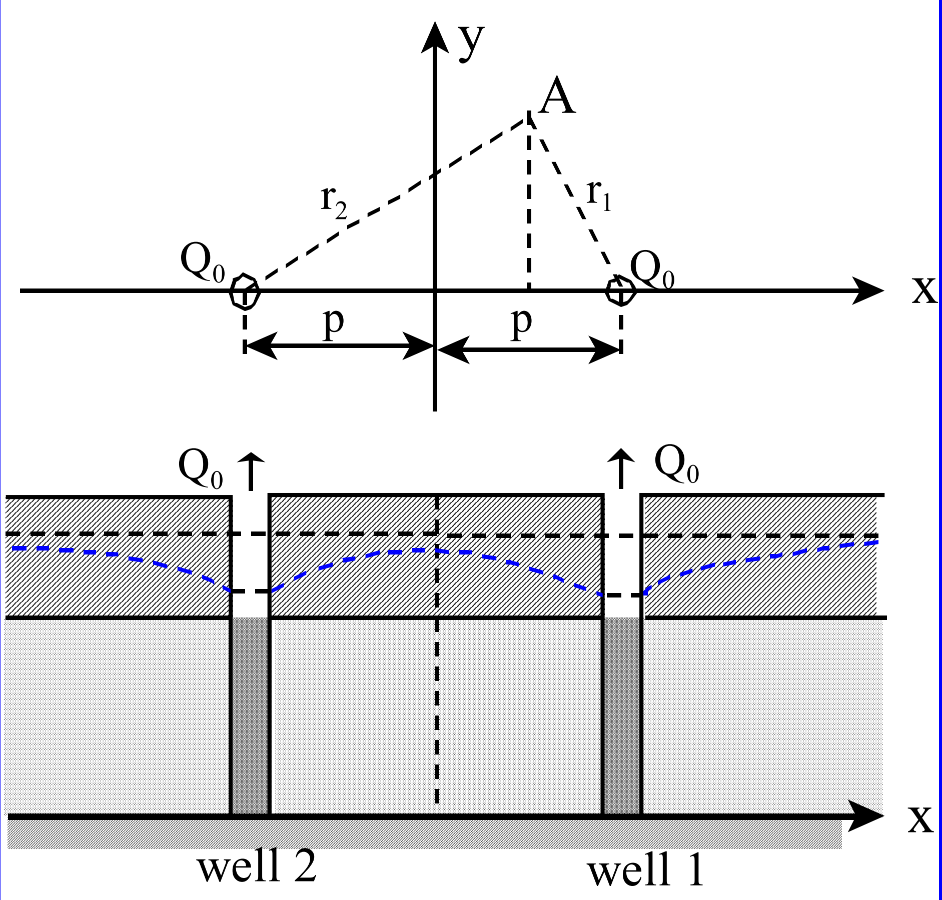
Drawdown distribution

$$s = \frac{Q_0}{2\pi T}\ln\left(\frac{R}{r}\right)$$

$$r = \sqrt{x^2 + y^2}$$

$$s_1 = \frac{Q_0}{2\pi T}\ln\left(\frac{R}{r_1}\right) \qquad r_1 = \sqrt{(x - p)^2 + y^2}$$

$$s_2 = \frac{Q_0}{2\pi T}\ln\left(\frac{R}{r_2}\right) \qquad r_2 = \sqrt{(x + p)^2 + y^2}$$



Method of superposition

Case 1: Two pumping wells in a confined aquifer

Total drawdown caused by two pumping wells

$$s = s_1 + s_2 = \frac{Q_0}{2\pi T} \ln\left(\frac{R^2}{r_1 r_2}\right)$$

$$s = \frac{Q_0}{2\pi T} \ln\left(\frac{R^2}{\sqrt{[(x-p)^2 + y^2][(x+p)^2 + y^2]}}\right)$$

Method of superposition

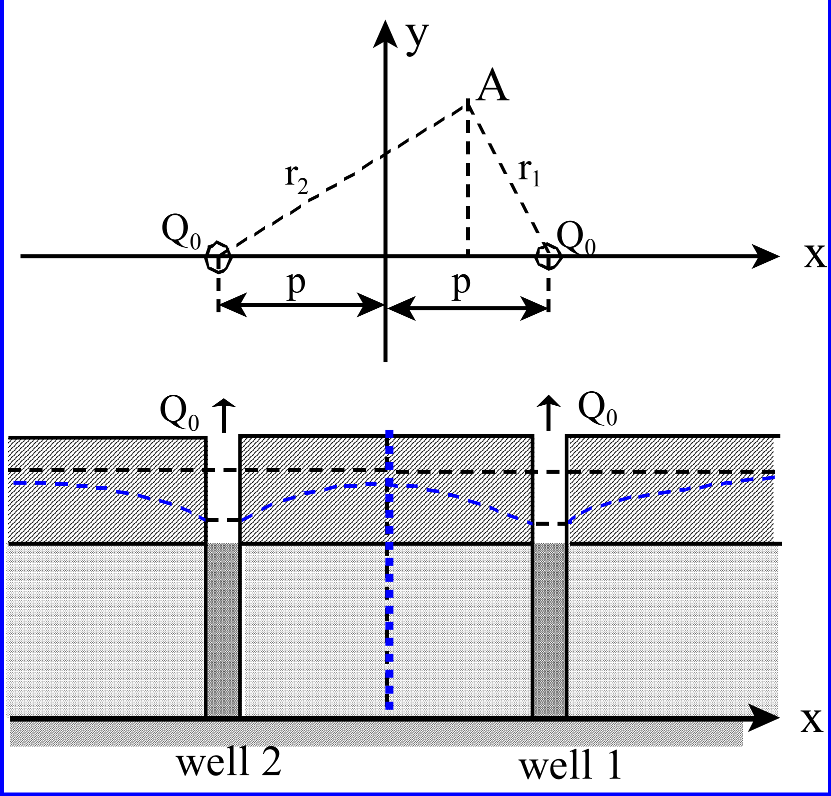
Case 1: Two pumping wells in a confined aquifer

Groundwater divide

$$\frac{\partial s}{\partial x} = -\frac{Q_0}{2\pi T}\left[\frac{x-p}{(x-p)^2+y^2}+\frac{x+p}{(x+p)^2+y^2}\right]$$

$$\frac{\partial s}{\partial x} = 0 \quad \text{when } x=0$$

A groundwater divide exists at the middle of two pumping wells along with the y axis

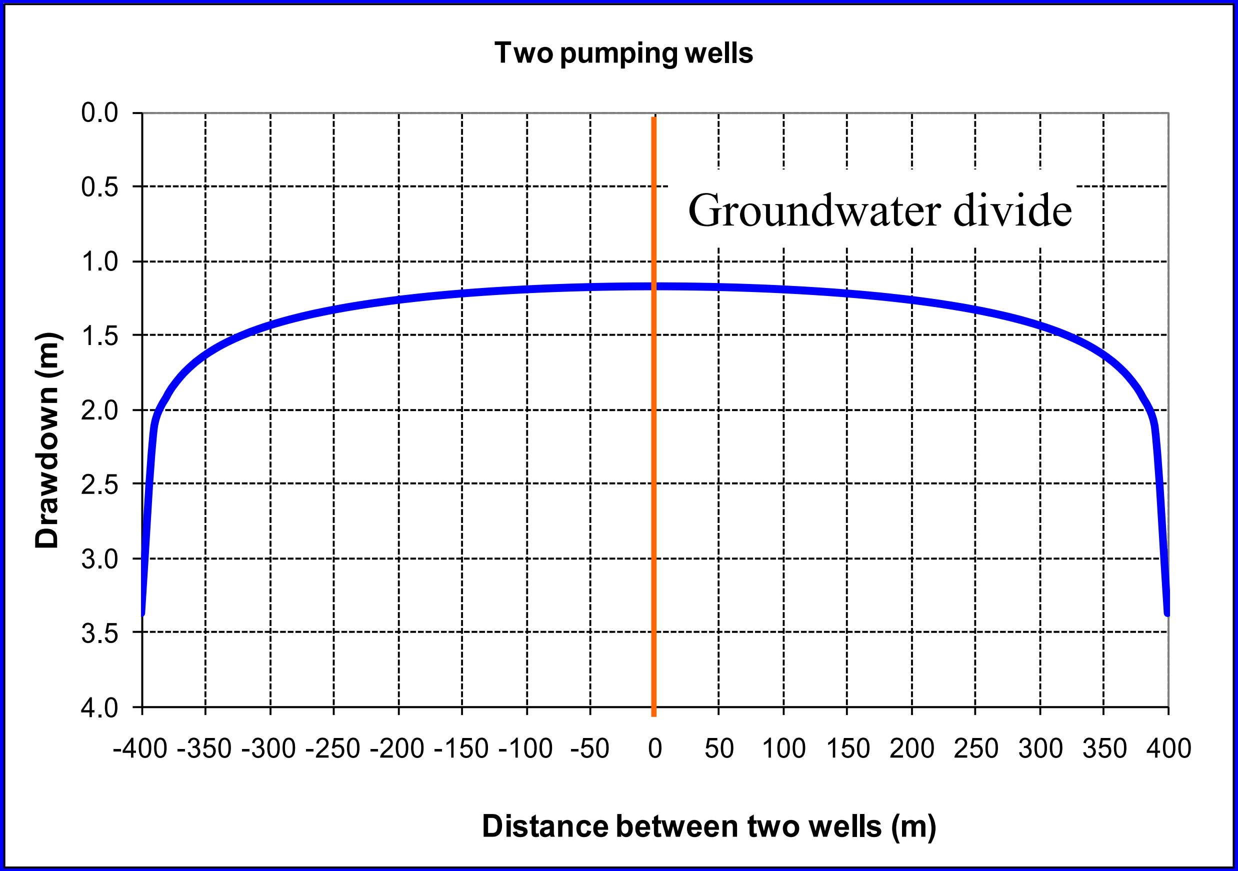


Method of superposition

Case 1: Two pumping wells in a confined aquifer

Drawdown distribution

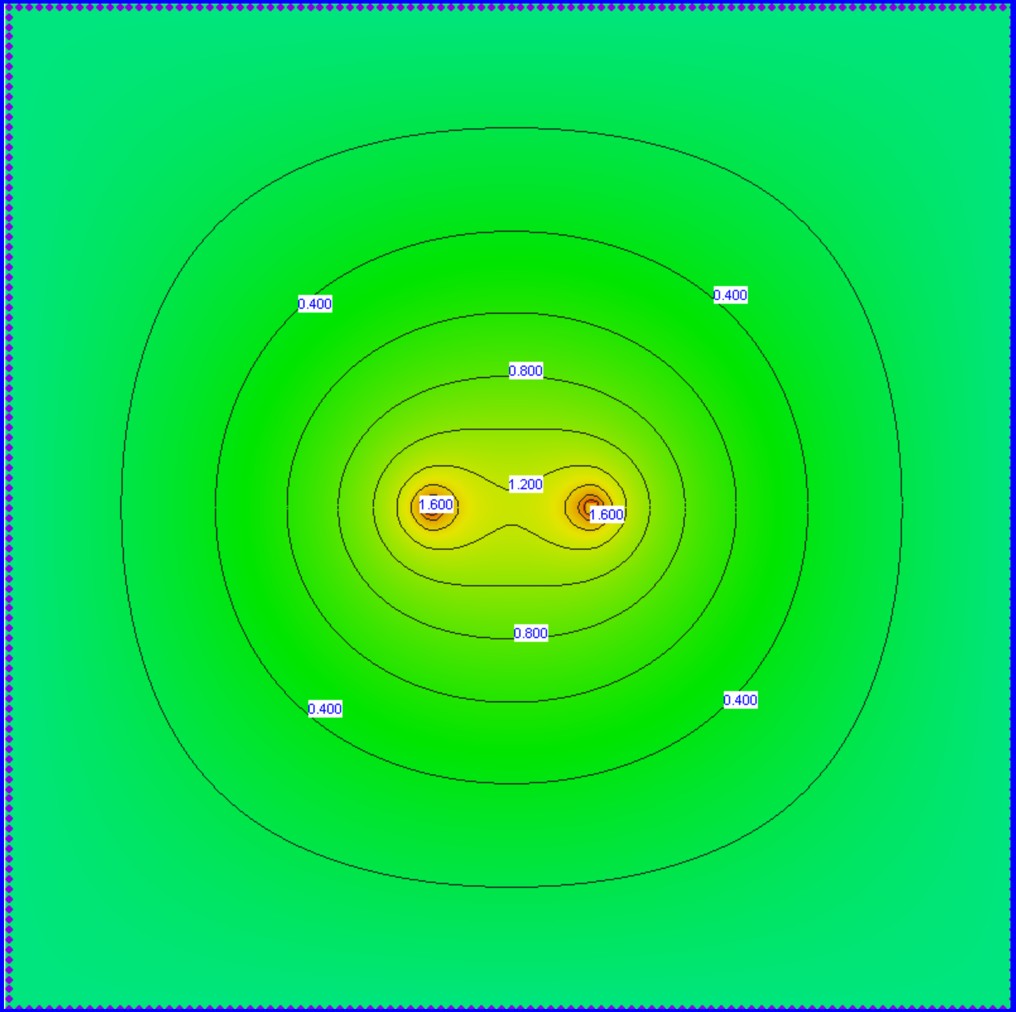
Example:
Q_0=5000 m^3/d
K=50 m/d
H=50 m
p=400 m
n=0.3
s_{max}=3.37m



Method of superposition

Case 1: Two pumping wells in a confined aquifer

Drawdown distribution simulated by a numerical model



Example:

Q_0=5000 m^3/d

K=50 m/d

H=50 m

p=400 m

n=0.3

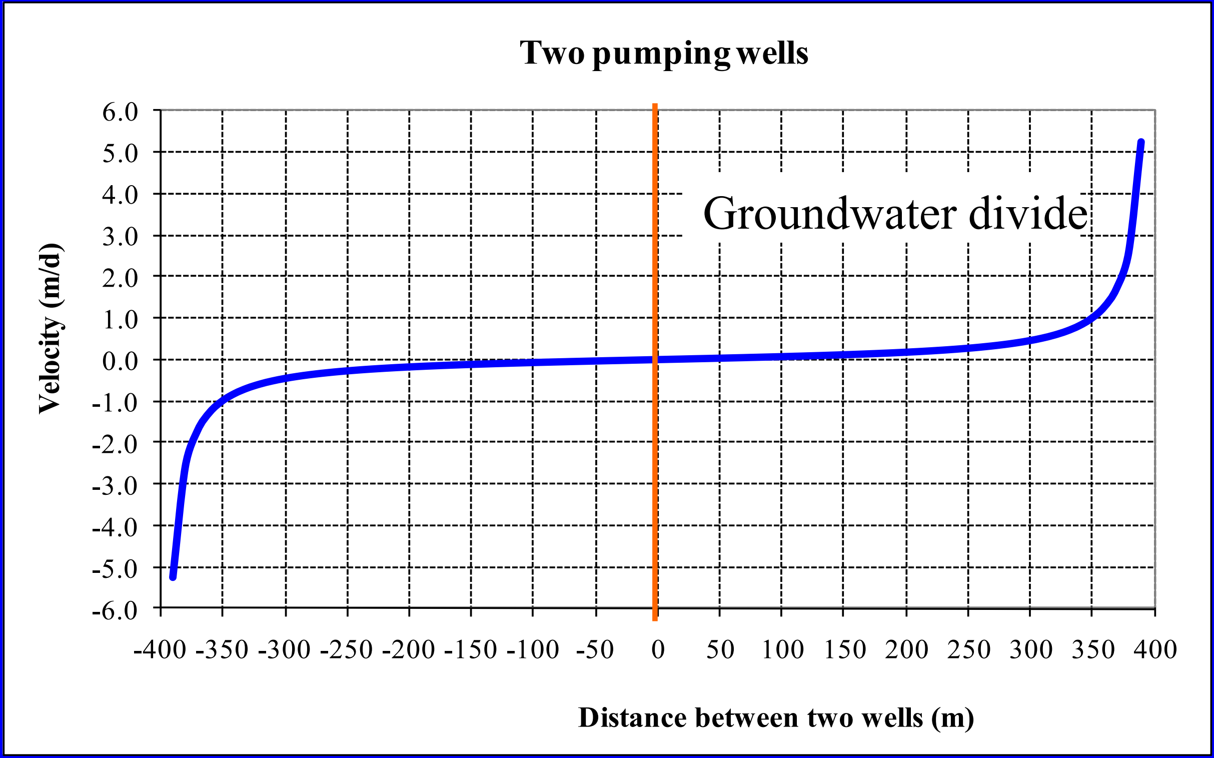
Side=5050 m

Method of superposition

Case 1: Two pumping wells in a confined aquifer

Velocity distribution $$v_x = \frac{K}{ne}\frac{\partial s}{\partial x} = -\frac{Q_0}{\pi H n_e}\frac{x}{x^2 - p^2}$$

Example:
Q_0=5000 m^3/d
K=50 m/d
H=50 m
p=400 m
n=0.3
v_{max}=265m/d



Method of superposition

Case 1: Two pumping wells in a confined aquifer

on the line that connects 2 wells togetheer

Shortest travel time from the boundary to the pumping well:

$$t = \int_R^p \frac{dx}{v_x} = \frac{\pi H n_e}{Q_0} \int_p^R \frac{x^2 - p^2}{x} dx = \frac{\pi H n_e}{Q_0} [\frac{1}{2}(R^2 - p^2) - p^2 \ln(\frac{R}{p})]$$

Example:

Q_0=5000 m^3/d, K=50 m/d, H=50 m/d, n=0.3

R=2525 m, p=400 m

t_{mim}=72 years

Method of superposition

Case 1: Two pumping wells in a confined aquifer

Simulated flow field

Example:

Q_0=5000 m^3/d

K=50 m/d

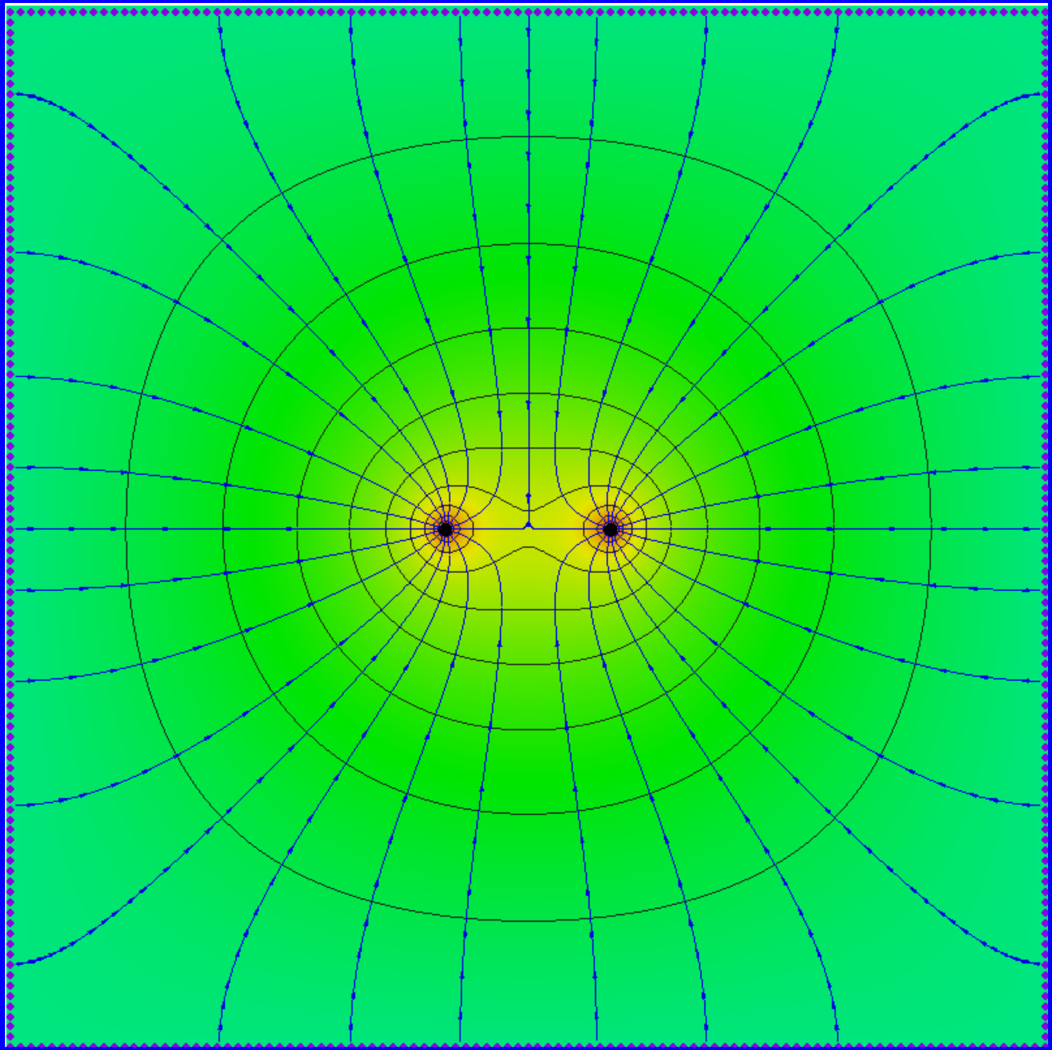
H=50 m

Side=5050 m

p=400 m

n=0.3

Travel time between two marks is 10 years



Method of superposition

Case 1: Two pumping wells in a confined aquifer

Drawdown near a well

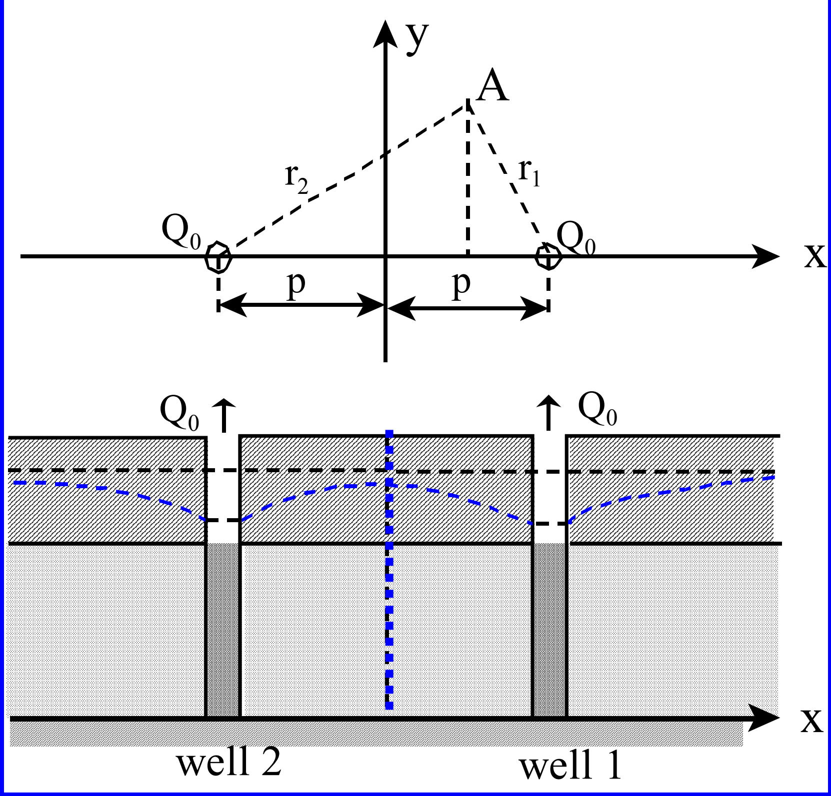
$$x = p + r\cos\theta, \qquad y = r\sin\theta, \qquad r \prec\prec p$$

$$s = \frac{Q_0}{2\pi T}\ln\left(\frac{R^2/2p}{r}\right)$$

$$s = \frac{Q_0}{2\pi T}\ln\left(\frac{R_{eq}}{r}\right)$$

$$R_{eq} = \frac{R^2}{2p}$$

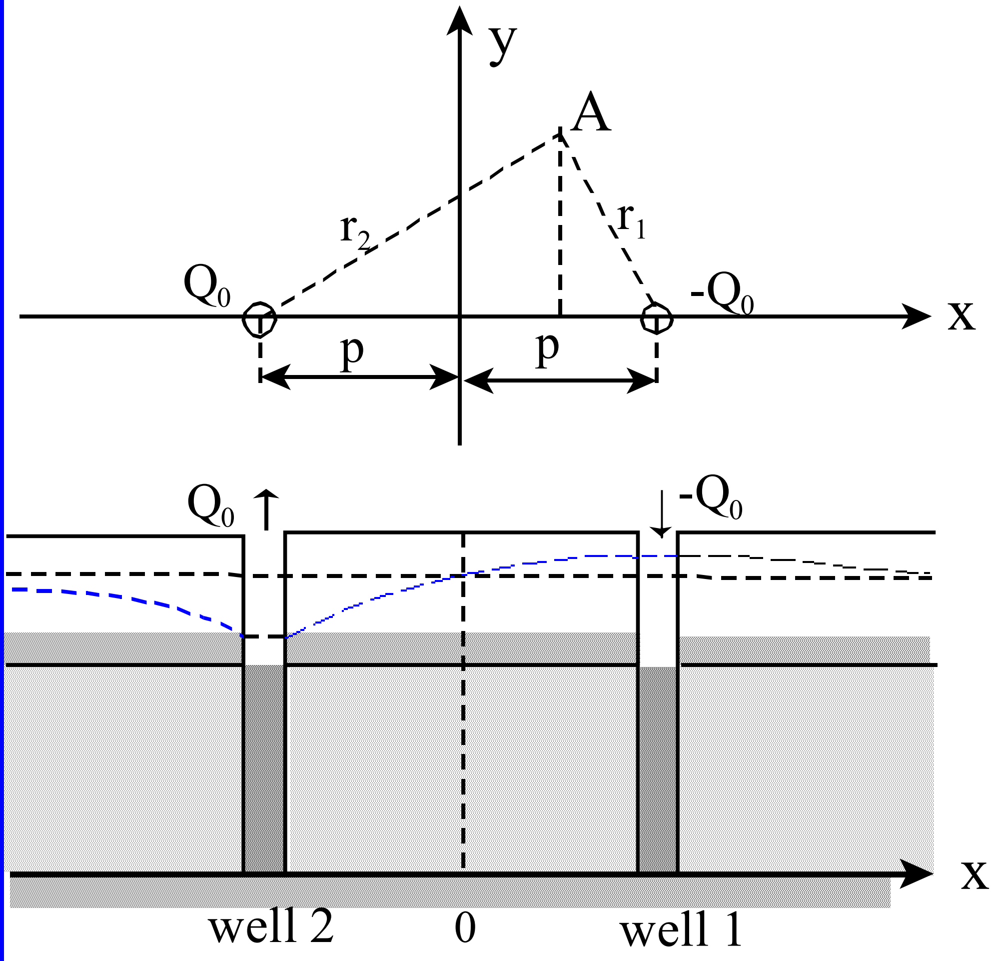
R_{eq} is the equivalent radius of influence



Method of superposition

Case 2: One pumping and one injection wells in a confined aquifer

- Conceptual hydrogeological model
  - The confined aquifer is bounded with a circular constant head boundary;
  - The aquifer is homogeneous and isotropic;
  - One pumping well and one injection well operate at a distance of 2p with the same rate.



Method of superposition

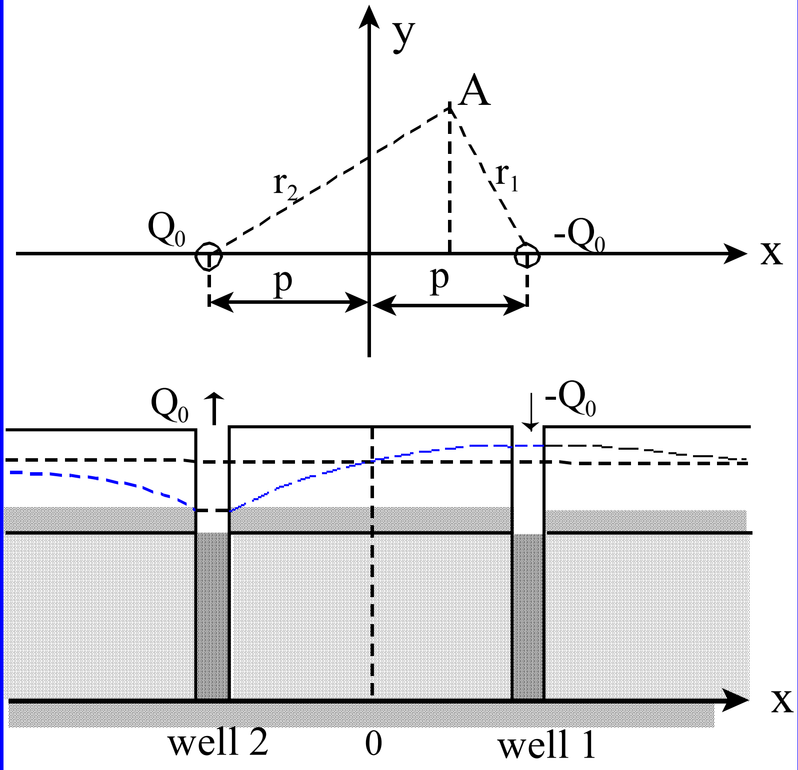
Case 2: One pumping and one injection wells in a confined aquifer

Drawdown distribution

$$s_1 = -\frac{Q_0}{2\pi T}\ln\left(\frac{R}{r_1}\right) \qquad r_1 = \sqrt{(x-p)^2 + y^2}$$

$$s_2 = \frac{Q_0}{2\pi T}\ln\left(\frac{R}{r_2}\right) \qquad r_2 = \sqrt{(x+p)^2 + y^2}$$

$$s = s_1 + s_2 = \frac{Q_0}{2\pi T}\ln\left(\sqrt{\frac{(x-p)^2 + y^2}{(x+p)^2 + y^2}}\right)$$

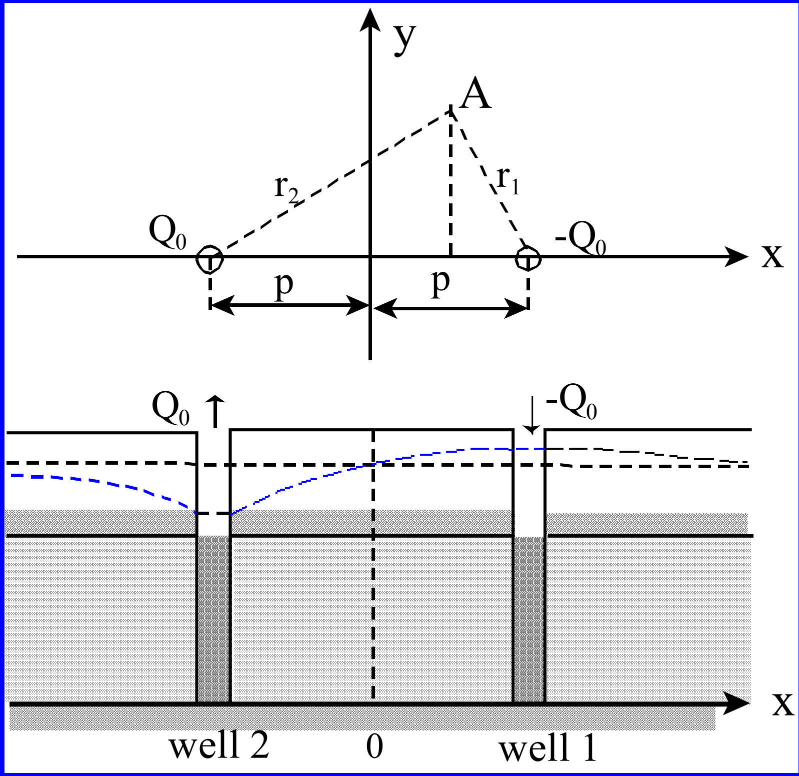


Method of superposition

Case 2: One pumping and one injection wells in a confined aquifer

Drawdown distribution

$$s = s_1 + s_2 = \frac{Q_0}{2\pi T}\ln\left(\sqrt{\frac{(x-p)^2+y^2}{(x+p)^2+y^2}}\right)$$



1) The drawdown s is independent of R;
2) At x = 0, s = 0 ;
3) Drawdown near the pumping well:

$$s = \frac{Q_0}{2\pi T}\ln\left(\frac{R_{eq}}{r}\right)$$

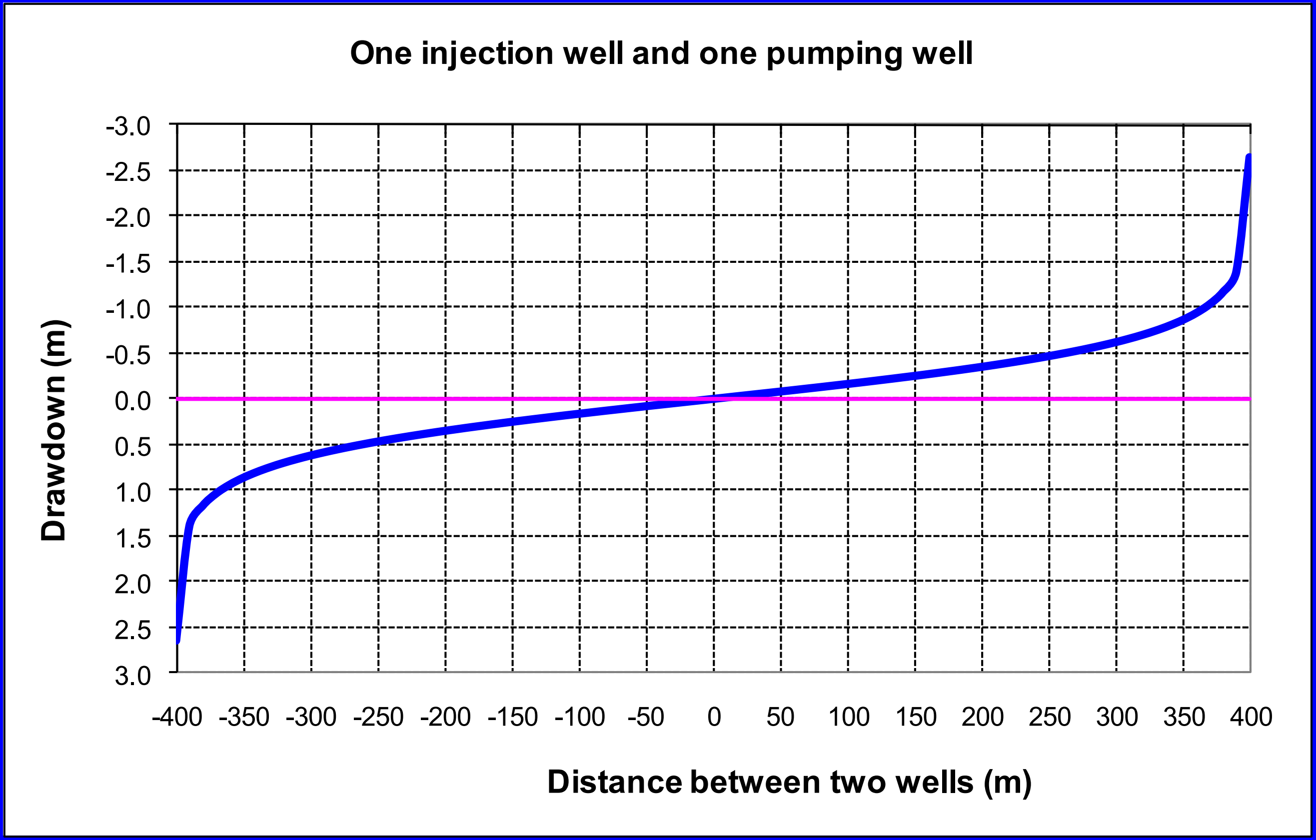
$R_{eq} = 2p$

Method of superposition

Case 2: One pumping and one injection wells in a confined aquifer

Drawdown distribution

Example:
Q_0=5000 m^3/d
K=50 m/d
H=50 m
p=400 m
n=0.3
s_{max}=2.64m



Method of superposition

Case 2: One pumping and one injection wells in a confined aquifer

Drawdown distribution simulated by a numerical model

Example:

Q_0=5000 m^3/d

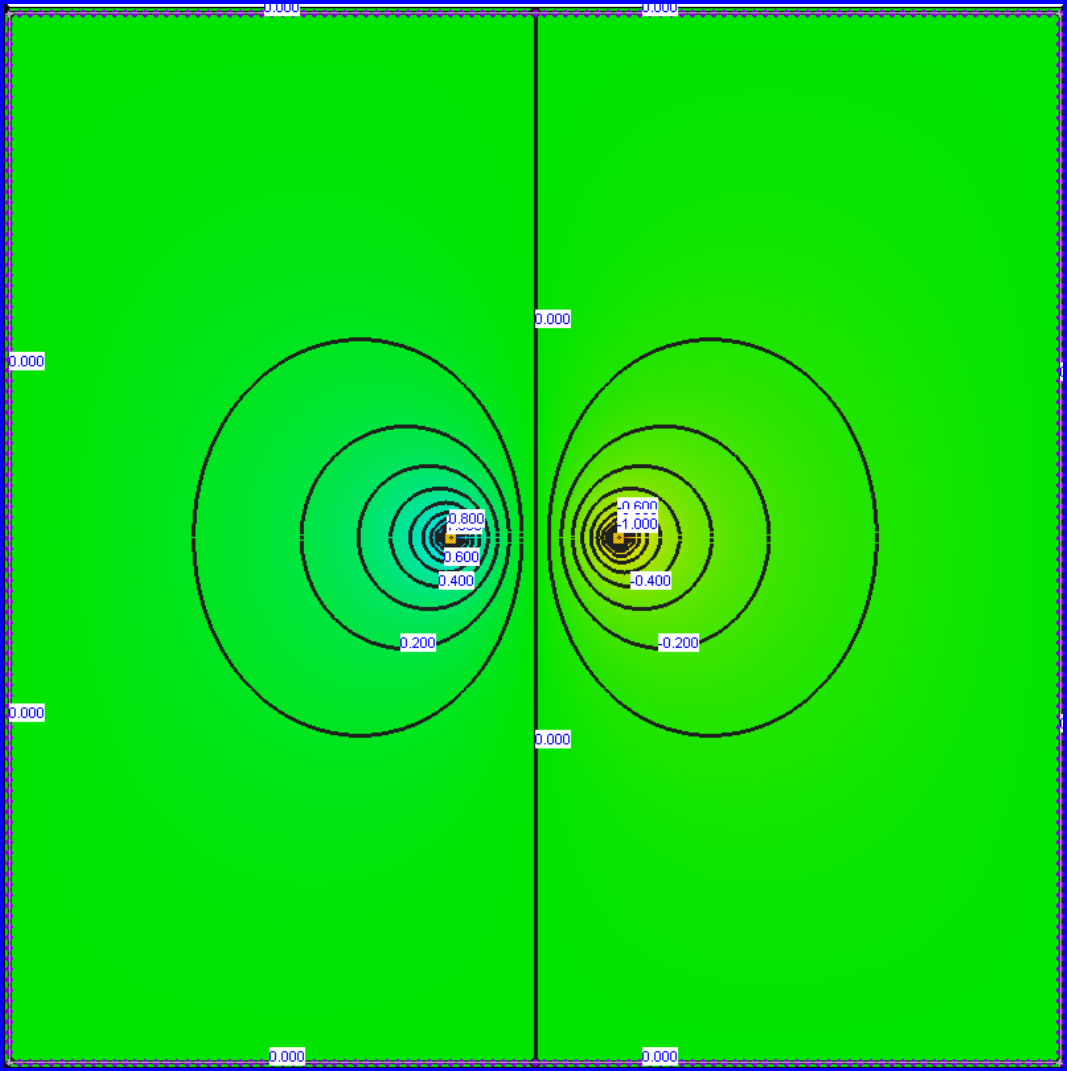
K=50 m/d

H=50 m

p=400 m

n=0.3

Side=5050 m



Method of superposition

Case 2: One pumping and one injection wells in a confined aquifer

Flow distribution between two wells

Since y = 0 along the x-axis

$$s = \frac{Q_0}{2\pi T} \ln\left(\frac{p - x}{p + x}\right) \qquad |x| < p$$

$$q_x = -K\frac{\partial \varphi}{\partial x} = K\frac{\partial s}{\partial x} = -\frac{Q_0}{2\pi H}\frac{2p}{p^2 - x^2}$$

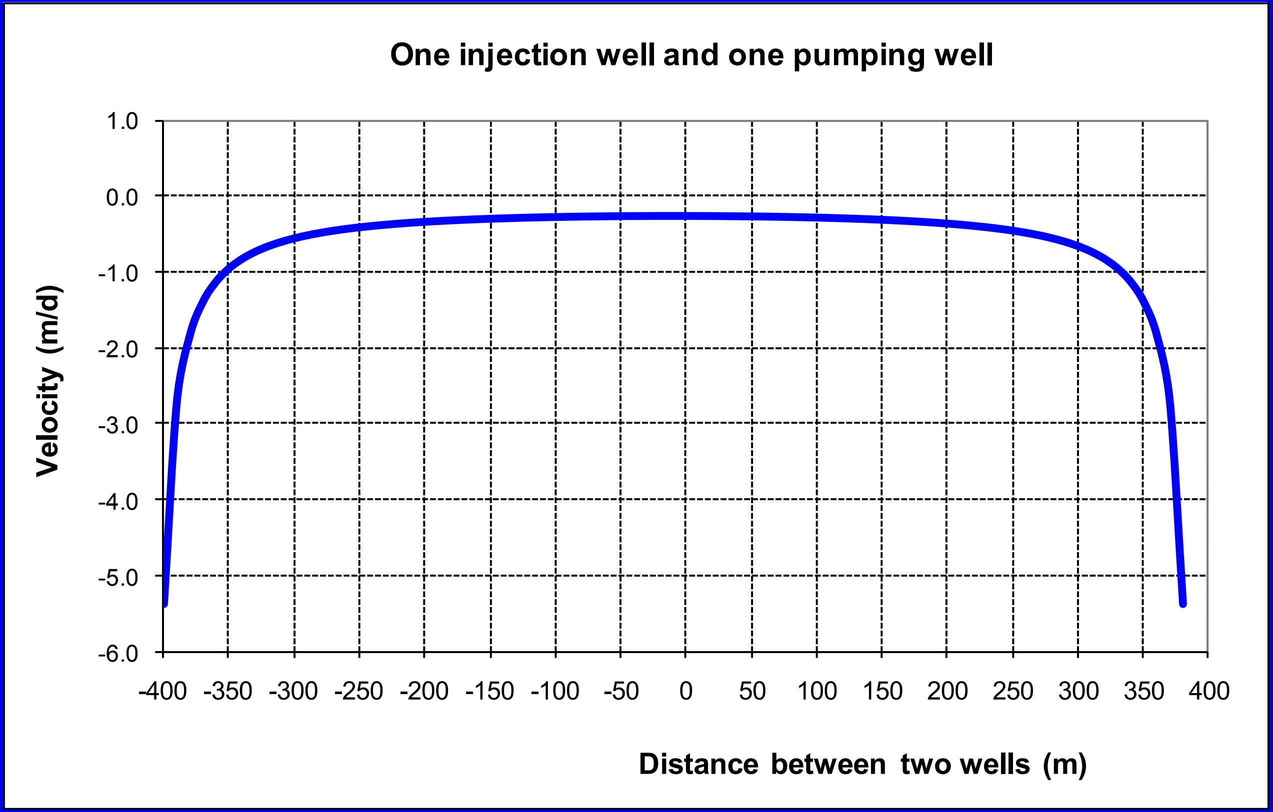
$$v_x = -\frac{Q_0}{2\pi n_e H}\frac{2p}{p^2 - x^2}$$

Method of superposition

Case 2: One pumping and one injection wells in a confined aquifer

Flow distribution between two wells

Example:
Q_0=5000 m^3/d
K=50 m/d
H=50 m
p=400 m
n=0.3
v_{max}=265m/d



Method of superposition

Case 2: One pumping and one injection wells in a confined aquifer

Minimum travel time from injection well to pumping well

$$t = \int_{p}^{-p} \frac{dx}{v_x} = \frac{\pi n_e H}{p Q_0} \int_{-p}^{p} (p^2 - x^2)\, dx$$

$$t = \frac{4\pi n_e H p^2}{3 Q_0}$$

straight line between 2 wells

60 days for T is good

Example:

$Q_0 = 5000\ m^3/d$

$K = 50\ m/d$

$H = 50\ m$

$p = 400\ m$

$n = 0.3$

t=2010 days

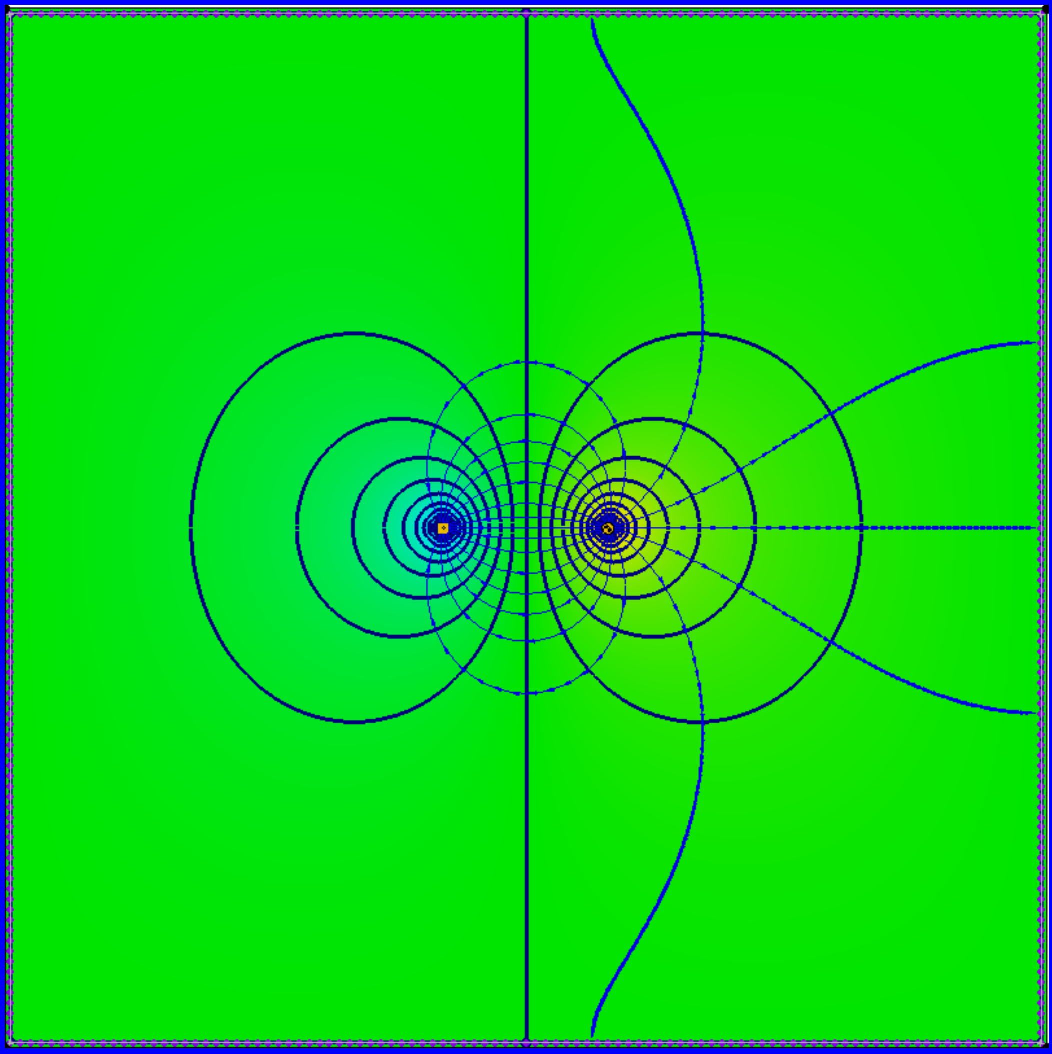
5.5 years

Method of superposition

Case 2: One pumping and one injection wells in a confined aquifer

Flow field simulated by a numerical model

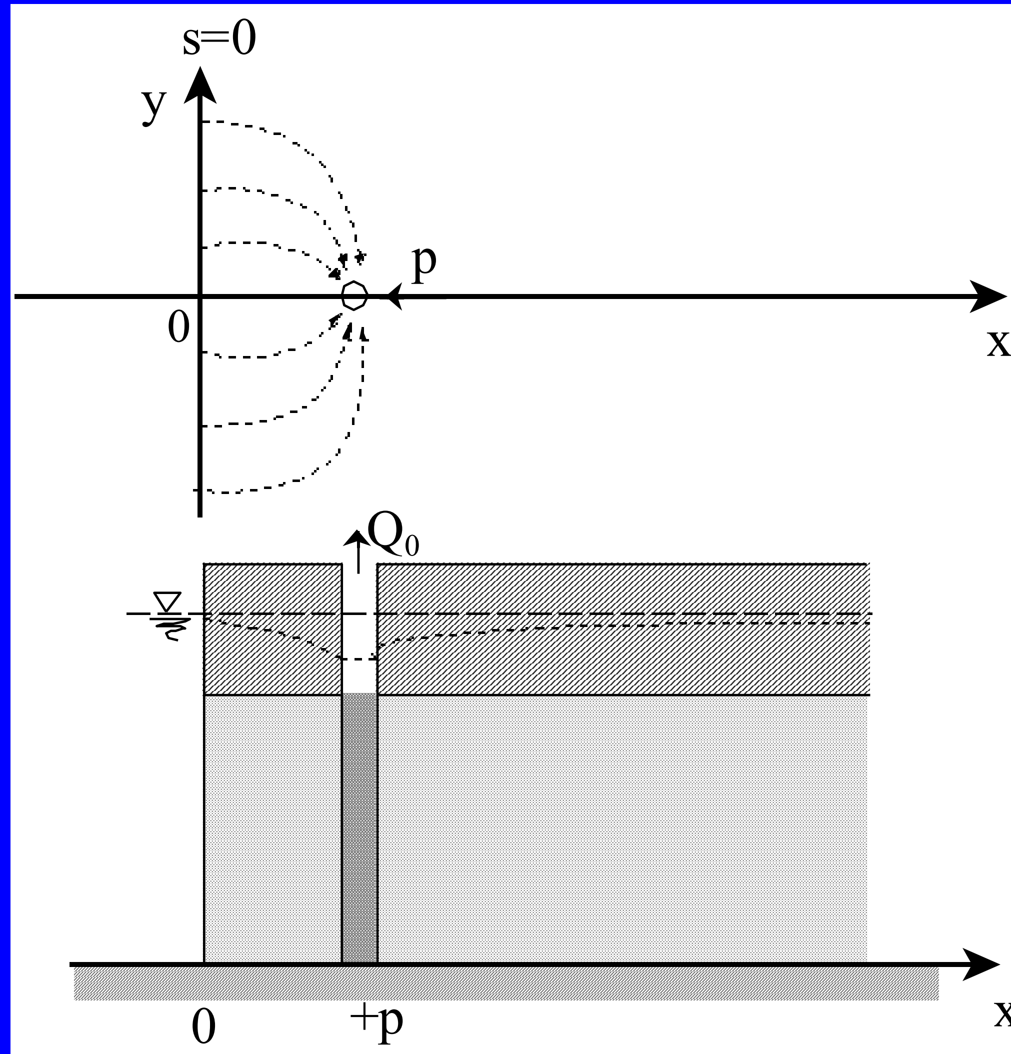
Example:
Q_0=5000 m^3/d
K=50 m/d
H=50 m
Side=5050 m
p=400 m
n=0.3
The travel time between two marks is 5 years



Method of images

Case 1: A pumping well near a constant head boundary

- Conceptual hydrogeological model
  - The confined aquifer is homogeneous and isotropic;
  - The aquifer is bounded by a straight constant head boundary on the left;
  - The aquifer extends to infinity in the right half;
  - A pumping well is located near the boundary with the constant pumping rate.



Method of images

Case 1: A pumping well near a constant head boundary

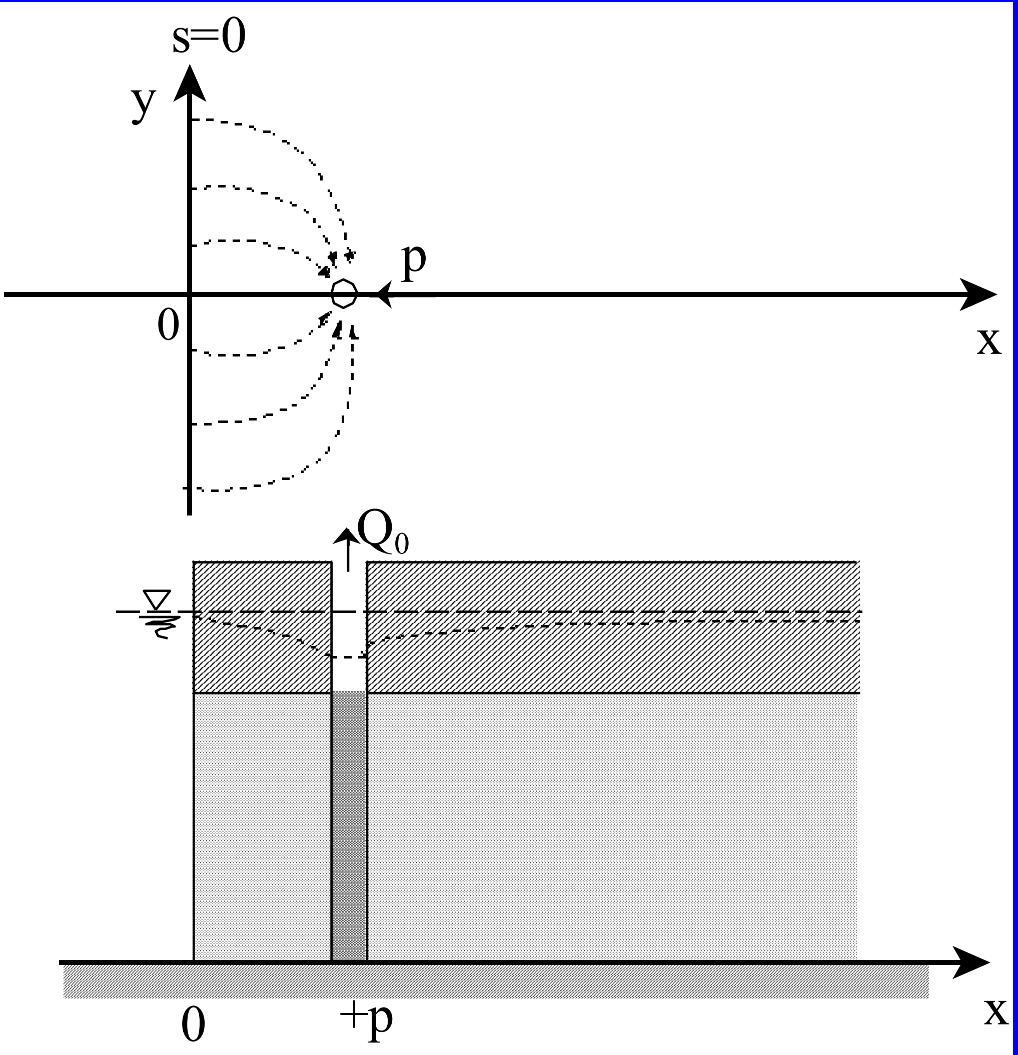
- Conditions for the solution:

  1) It must satisfy the Laplace's equation for x > 0, except in the point (x=p, y=0);

  2) s = 0 when x approaches the infinity;

  3) s = 0 when x = 0 along the y-axis;

  4) the amount of water leaves the aquifer at point (x=p, y=0) must equal pumping rate.



Method of images

Case 1: A pumping well near a constant head boundary

- Method of the image:

  1) replacing the semi-infinite aquifer with an fictitious infinite aquifer with the same hydrogeological parameters;

  2) considering the constant head boundary as a mirror;

  3) putting an imaginary injecting well at the image location of the pumping well, i.e. at point (x=-p, y=0);

  4) giving the injecting rate equal to the pumping rate;

  5) using the principle of superposition to find the solution.

Method of images

Case 1: A pumping well near a constant head boundary

Drawdown caused by the pumping well:

$$s_1 = \frac{Q_0}{2\pi T} \ln\left(\frac{R}{\sqrt{(x-p)^2 + y^2}}\right)$$

Drawdown caused by the injection well:

$$s_2 = -\frac{Q_0}{2\pi T} \ln\left(\frac{R}{\sqrt{(x+p)^2 + y^2}}\right)$$

Total drawdown:

$$s = s_1 + s_2 = \frac{Q_0}{2\pi T} \ln\left(\sqrt{\frac{(x+p)^2 + y^2}{(x-p)^2 + y^2}}\right)$$

Please verify that s satisfies the 4 conditions!

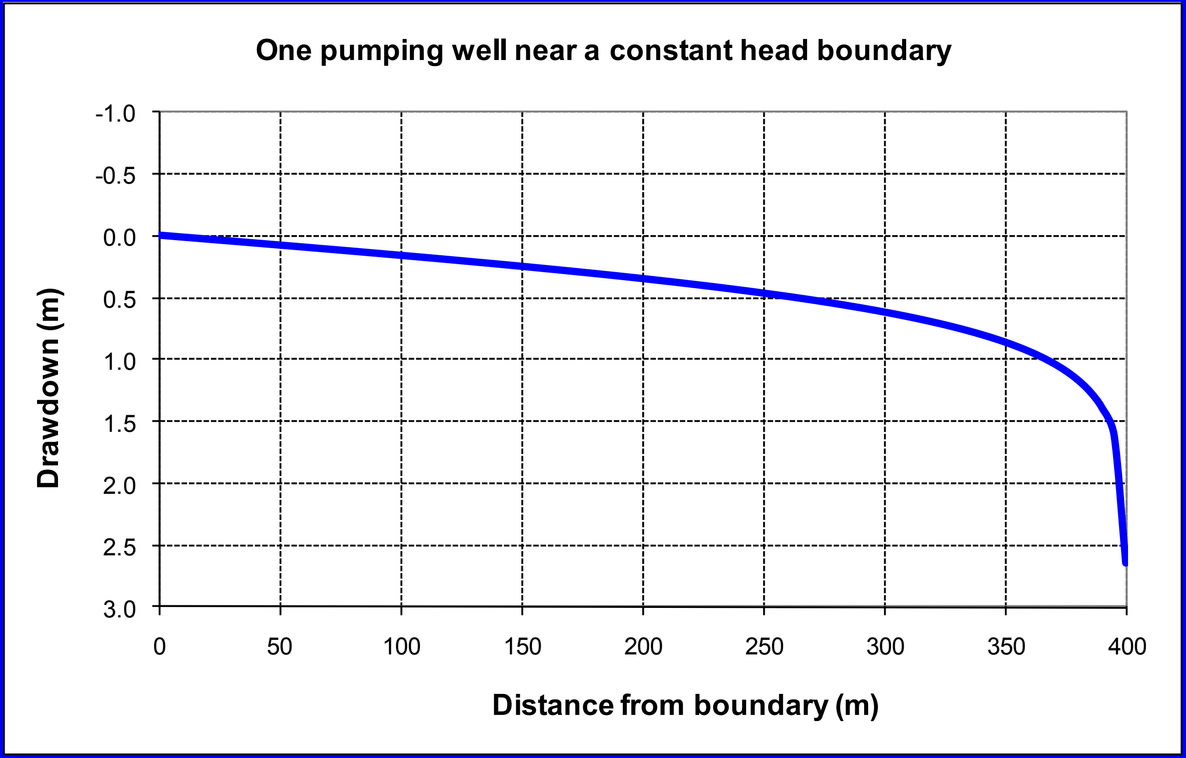
Method of images

Case 1: A pumping well near a constant head boundary

Drawdown along the x axis:

$$s = \frac{Q_0}{2\pi T} \ln \left(\frac{x+p}{p-x}\right)$$

Example:
Q_0=5000 m^3/d
K=50 m/d
H=50 m
p=400 m
n=0.3
s_{max}=2.64m



Method of images

Case 1: A pumping well near a constant head boundary

Travel time along the x axis:

$$\frac{\partial s}{\partial x} = \frac{Q_0}{2\pi T}\left[\frac{1}{x+p} + \frac{1}{p-x}\right] = \frac{Q_0}{2\pi t}\,\frac{2p}{p^2 - x^2}$$

$$q_x = K\frac{\partial s}{\partial x} = \frac{Q_0}{2\pi H}\,\frac{2p}{p^2 - x^2}$$

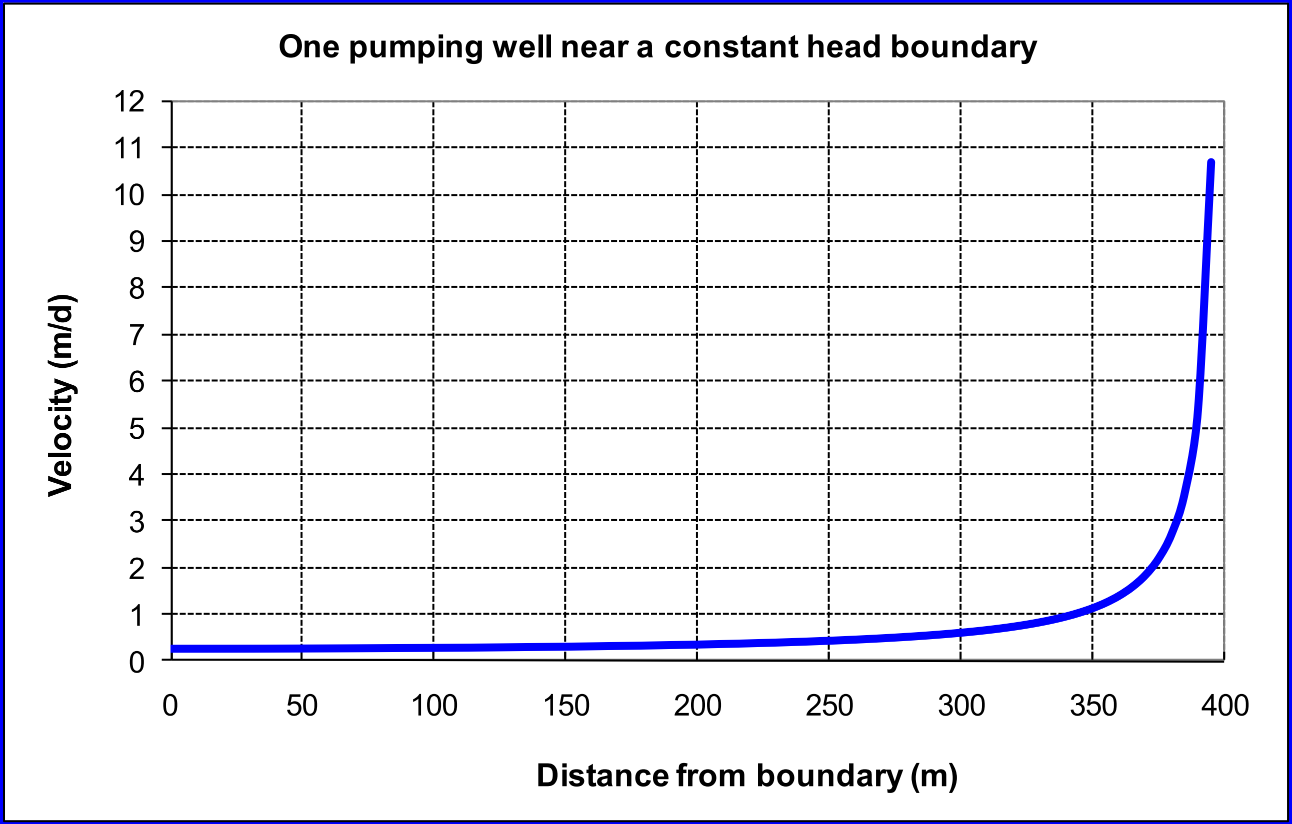
$$t = \frac{2\pi H n_e}{2pQ_0}\int_0^p (p^2 - x^2)dx = \frac{2\pi H n_e}{2pQ_0}\left[p^3 - \frac{1}{3}p^3\right] = \frac{2}{3}\,\frac{\pi n_e H p^2}{Q_0}$$

Method of images

Case 1: A pumping well near a constant head boundary

Velocity distribution along the x axis:

Example:
Q_0=5000 m^3/d
K=50 m/d
H=50 m
p=400 m
n=0.3
v_{max}=265m/d
t=1005 days
2.7.5 years



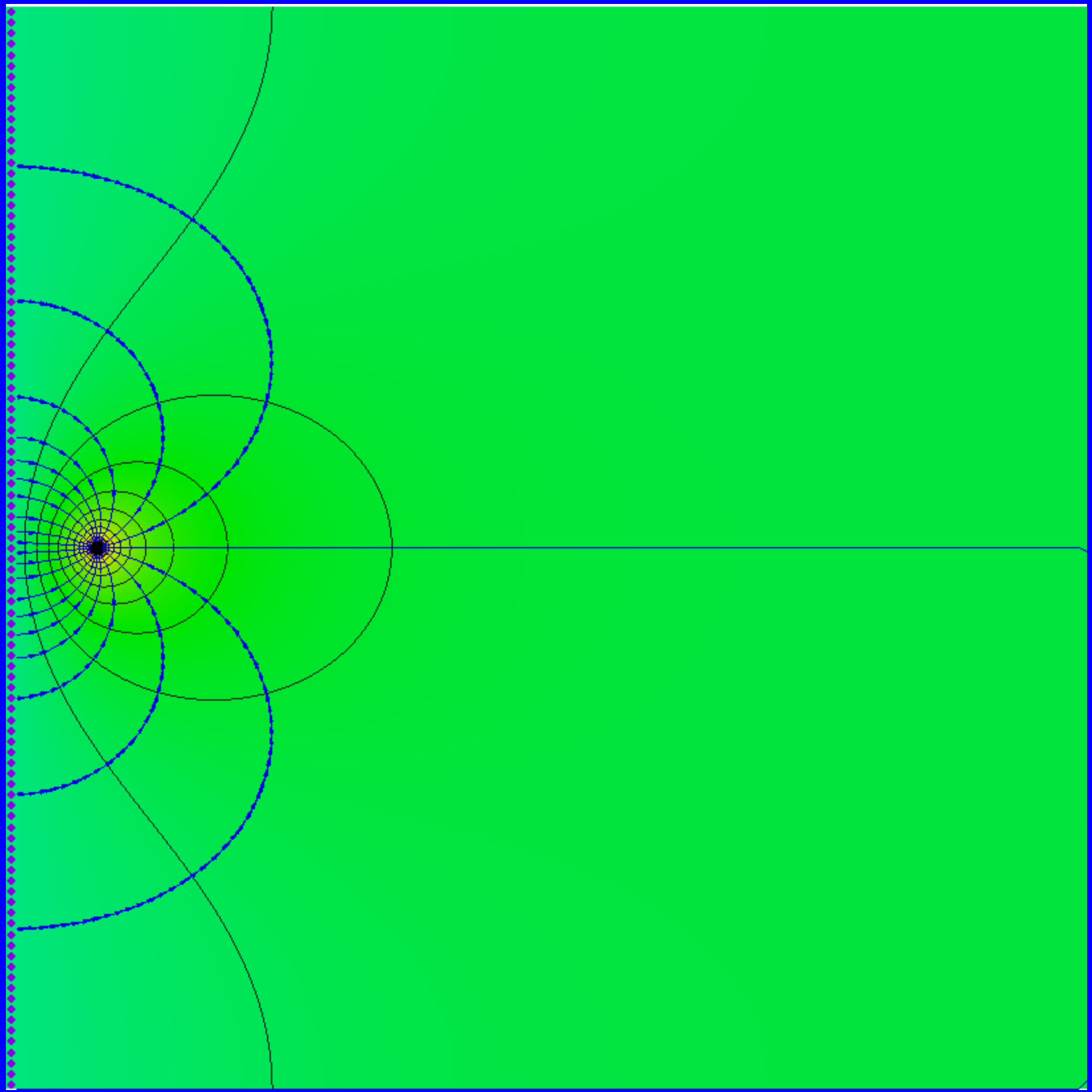
Method of images

Case 1: A pumping well near a constant head boundary

Model simulated flow field

Pumped water comes from constant head boundary

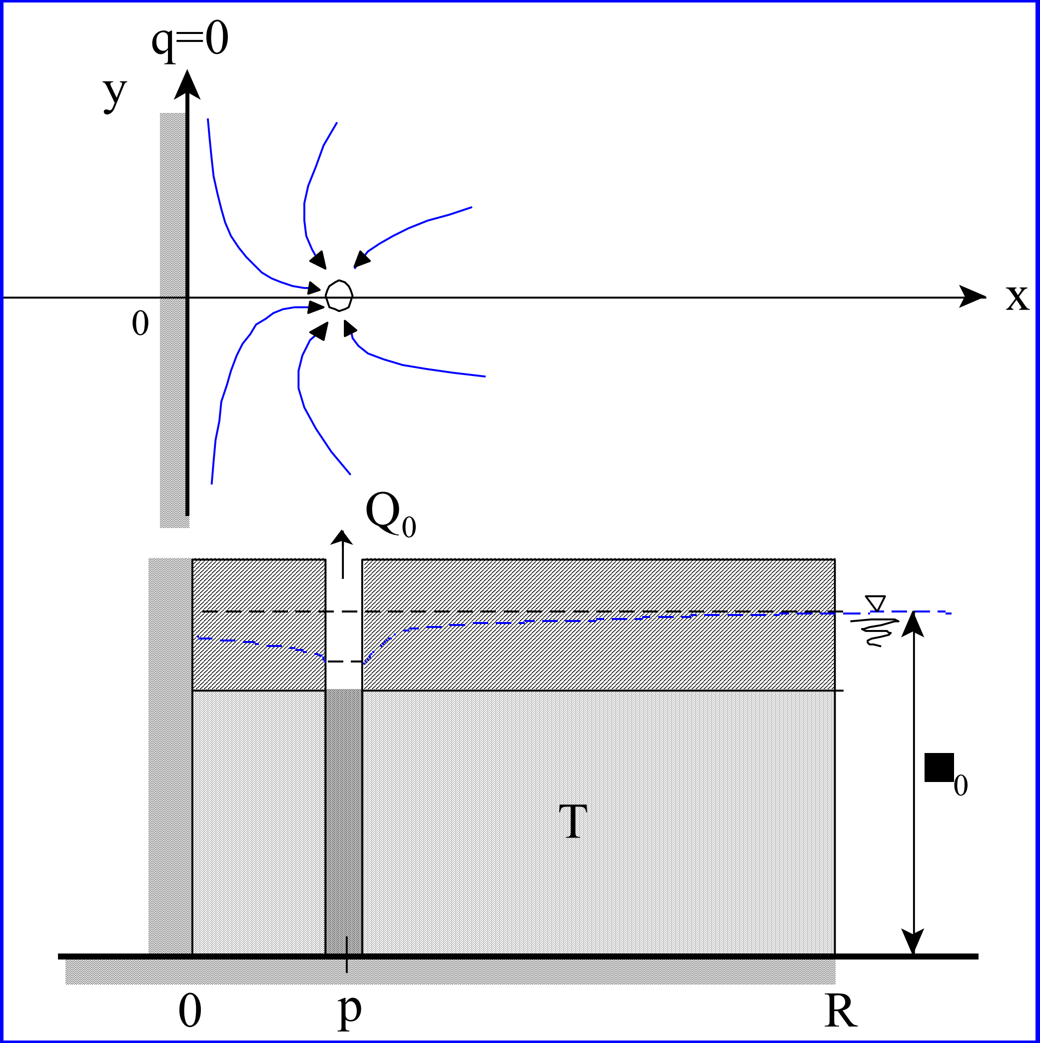
Travel time between two marks is 2 years



Method of images

Case 2: A pumping well near an impermeable boundary

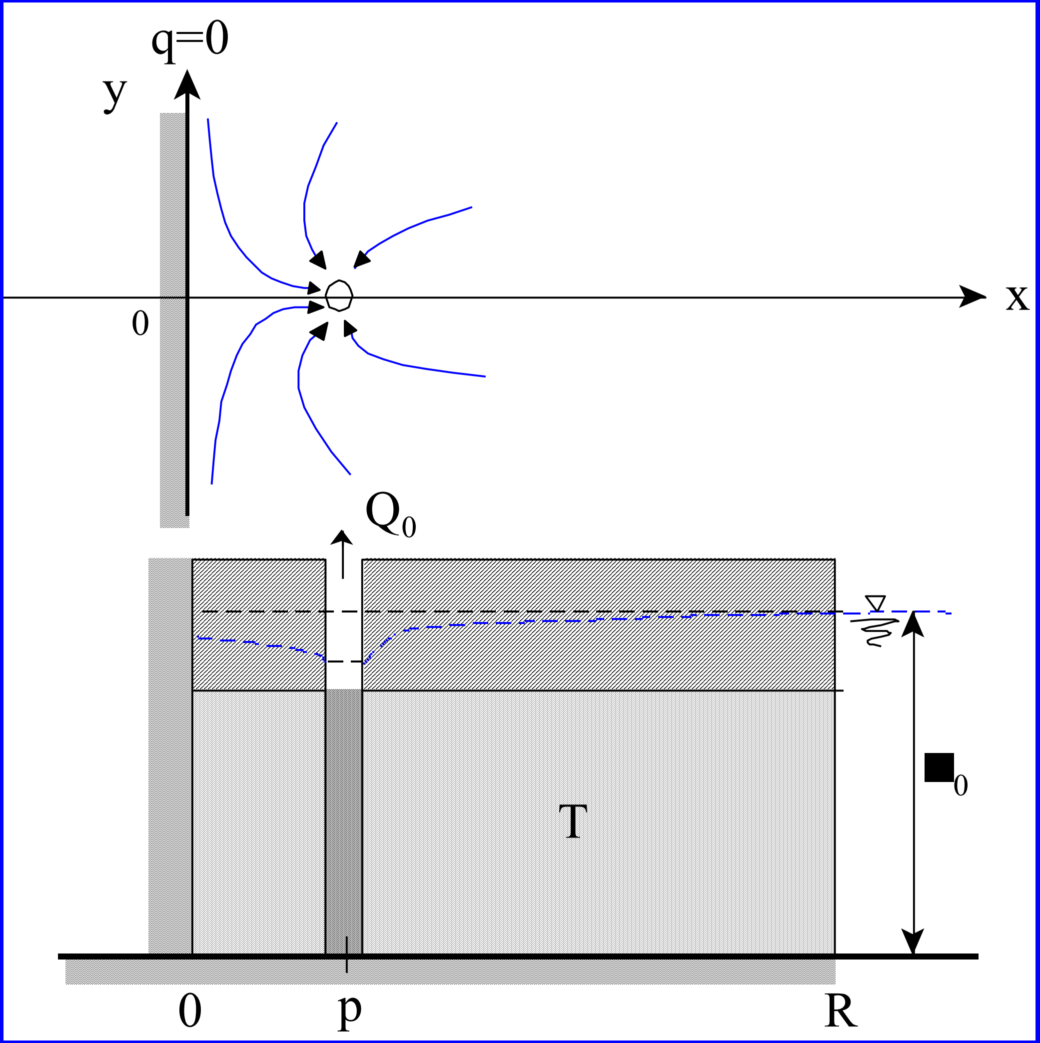
- Conceptual hydrogeological model
  - The confined aquifer is homogeneous and isotropic;
  - The half circular aquifer is bounded by a straight impermeable boundary on the left;
  - A pumping well is located near the boundary with the constant pumping rate.



Method of images

Case 2: A pumping well near an impermeable boundary

- Conditions for the solution:
  1) It must satisfy the Laplace's equation for x > 0, except in the point (x=p, y=0);
  2) s = 0 when r = R at the circular head boundary;
  3) q_x or ds/dx = 0 at x = 0 along the y-axis;
  4) The amount of water leaves the aquifer at point (x=p, y=0) must equal pumping rate.



Method of images

Case 2: A pumping well near an impermeable boundary

- Method of the image:

1) replacing the half-circular aquifer with an fictitious circular aquifer with the same hydrogeological parameters;
2) considering the impermeable boundary as a mirror;
3) putting an imaginary pumping well at the image location of the real pumping well, i.e. at point ($x=-p$, $y=0$);
4) giving the imaginary pumping rate equal to the real pumping rate;
5) using the principle of superposition to find the solution.

Method of images

Case 2: A pumping well near an impermeable boundary

Drawdown caused by the real pumping well:

$$s_1 = \frac{Q_0}{2\pi T} \ln\left(\frac{R}{\sqrt{(x-p)^2 + y^2}}\right)$$

Drawdown caused by the imaginary pumping well:

$$s_2 = \frac{Q_0}{2\pi T} \ln\left(\frac{R}{\sqrt{(x+p)^2 + y^2}}\right)$$

Total drawdown:

$$s = s_1 + s_2 = \frac{Q_0}{2\pi T} \ln\left(\frac{R^2}{\sqrt{[(x+p)^2 + y^2][(x-p)^2 + y^2]}}\right)$$

Please verify that s satisfies the 4 conditions!

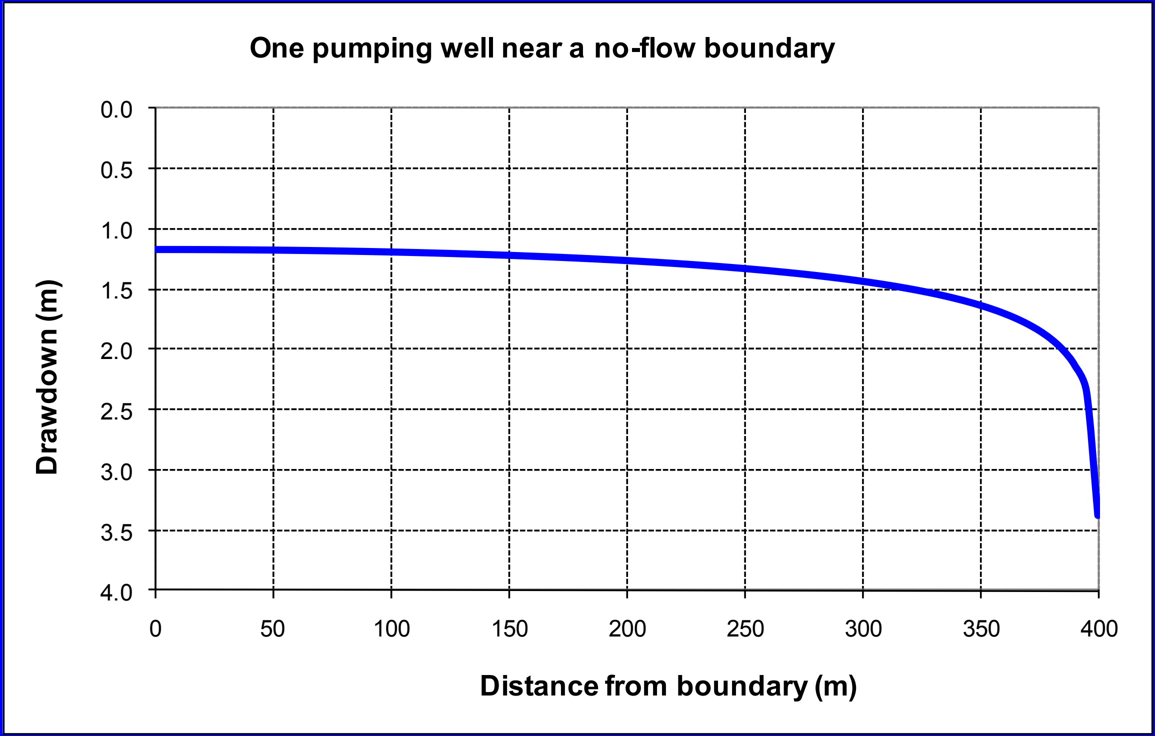
Method of images

Case 2: A pumping well near an impermeable boundary

Drawdown along x-axis:

$$s = \frac{Q_0}{2\pi T}\ln\left(\frac{R^2}{p^2 - x^2}\right)$$

Example:
Q_0=5000 m^3/d
K=50 m/d
H=50 m
p=400 m
n=0.3
s_{max}=3.37m



Method of images

Case 2: A pumping well near an impermeable boundary

Velocity along x-axis:

$$v_x = \frac{Q_0}{2\pi H n_e} \frac{2x}{p^2 - x^2}$$

Example:

Q_0=5000 m^3/d

K=50 m/d

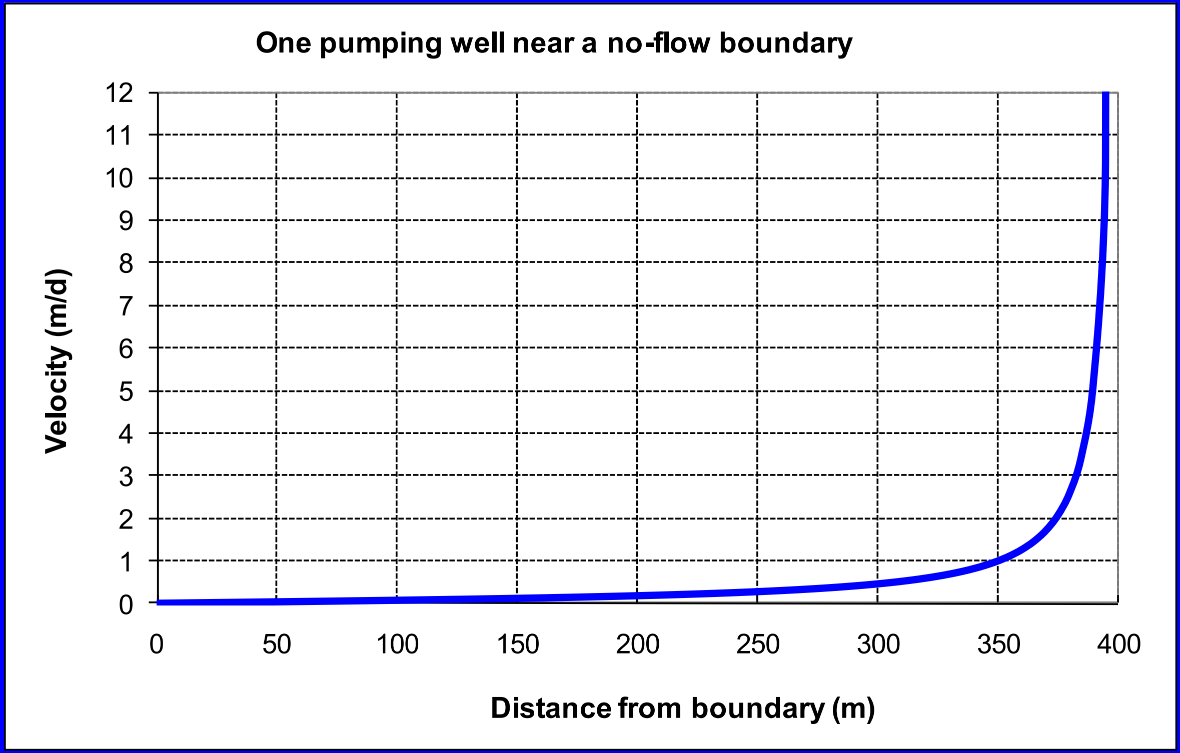
H=50 m

p=400 m

n=0.3

v_{max}=265m/d

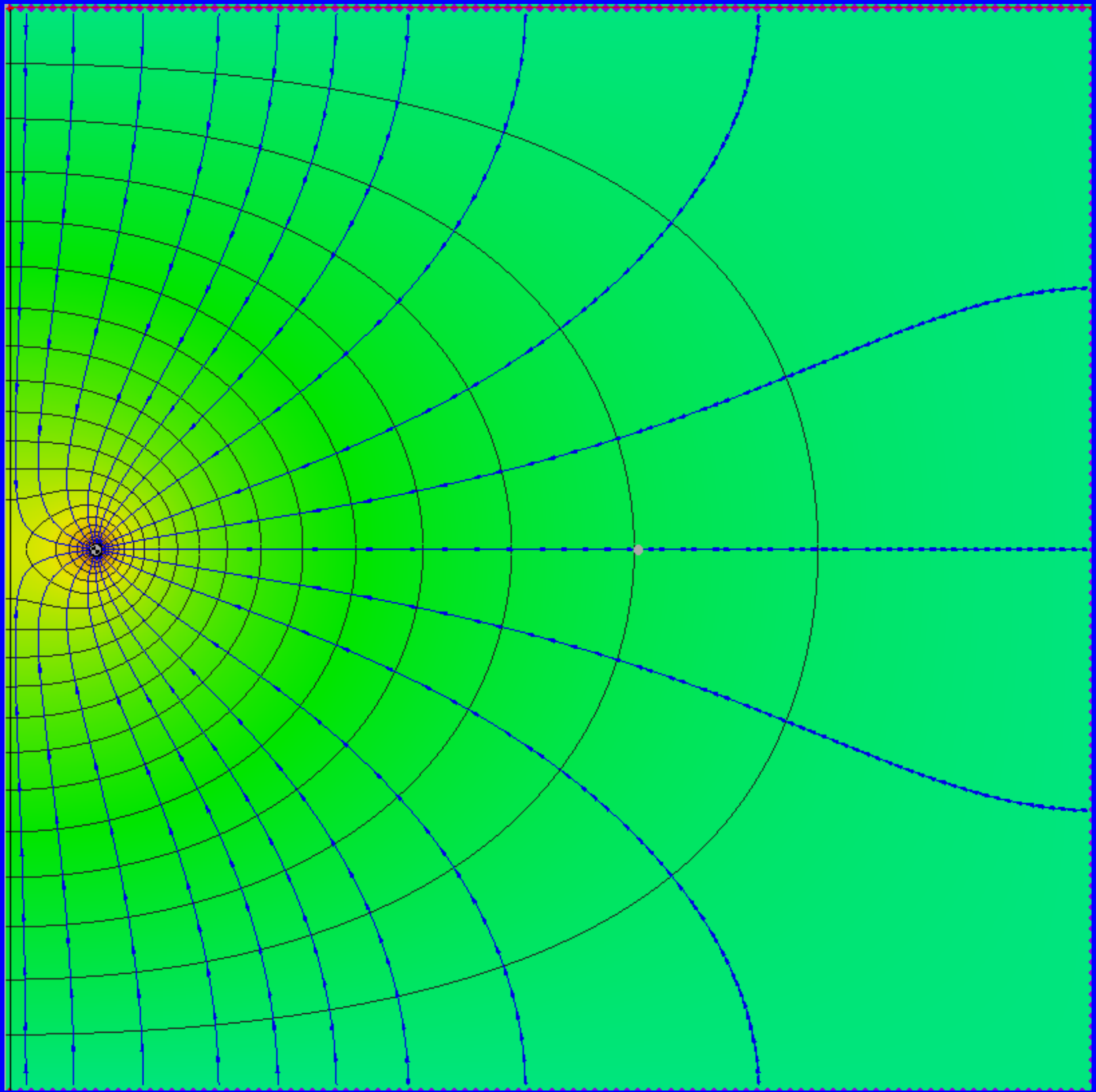
When x=0, the velocity is zero, no flow at the impermeable boundary



Method of images

Case 2: A pumping well near an impermeable boundary

Model simulated flow field



Pumped water comes from inflow boundary

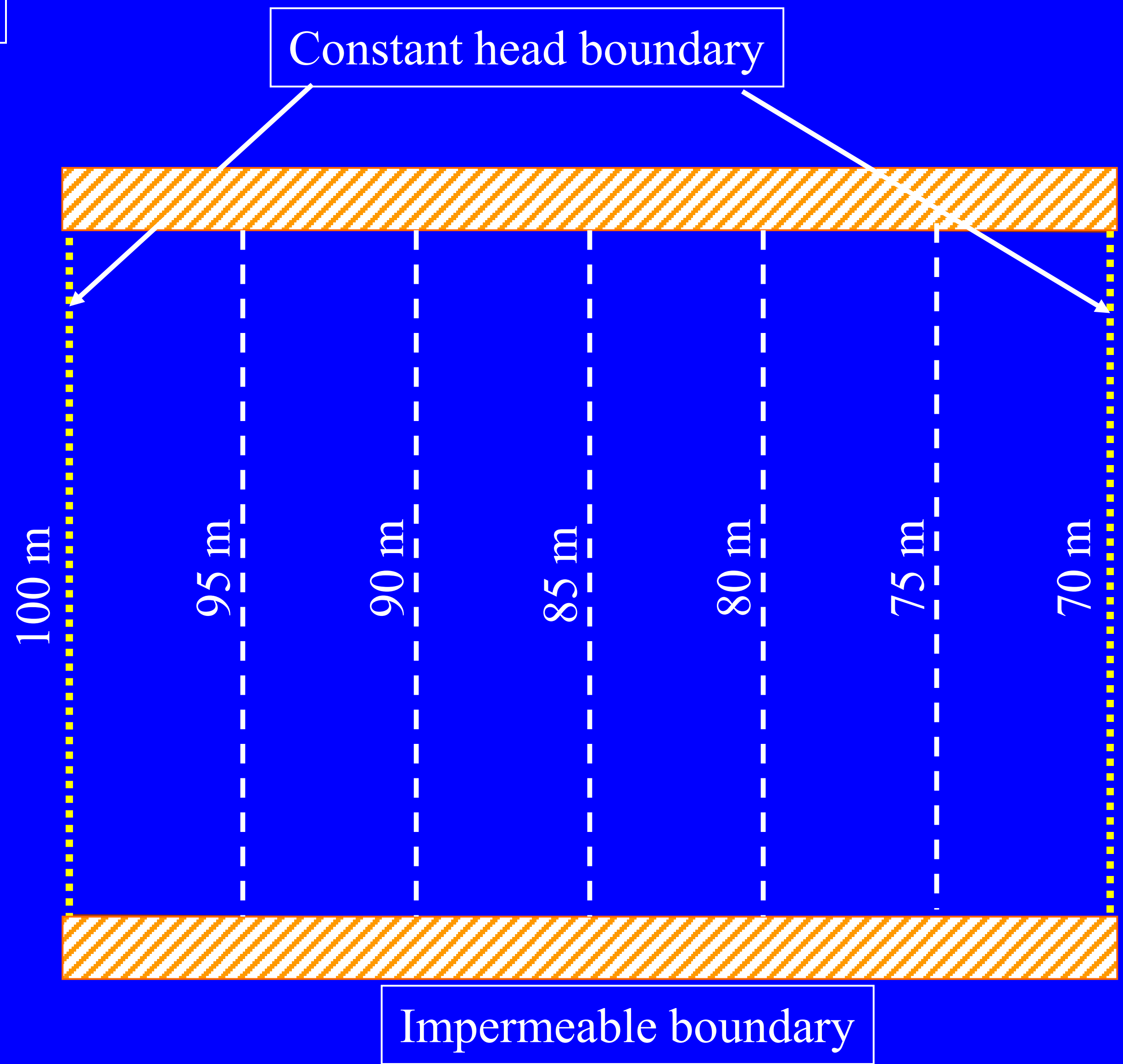
Flowline is parallel to the impermeable boundary

Travel time between marks is 10 years

Flow net

- Equipotential lines

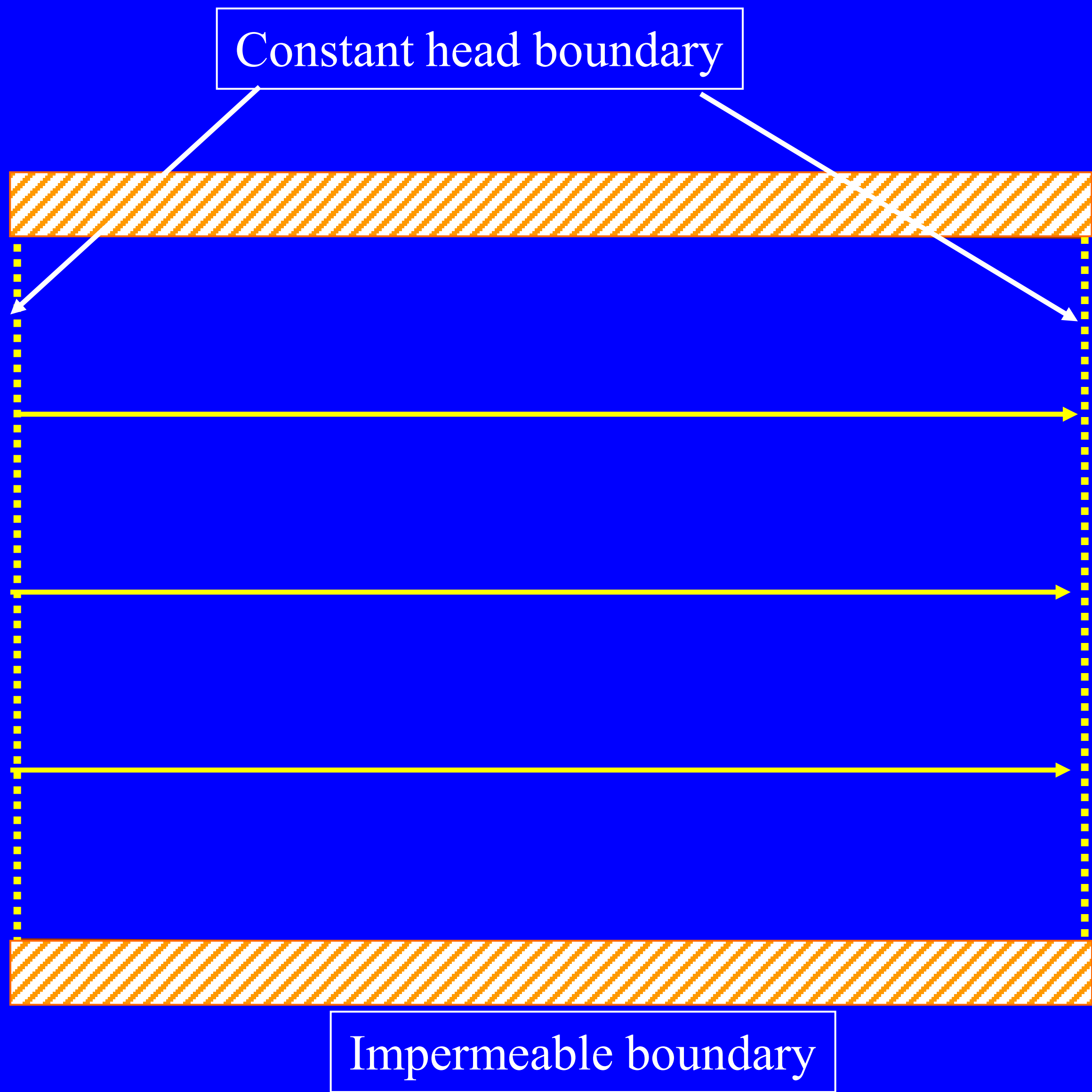
A lines of constant groundwater head is called equipotential lines. If the interval between equipotential lines is constant, equipotential lines form a contour map of groundwater head.



Flow net

- Flow lines

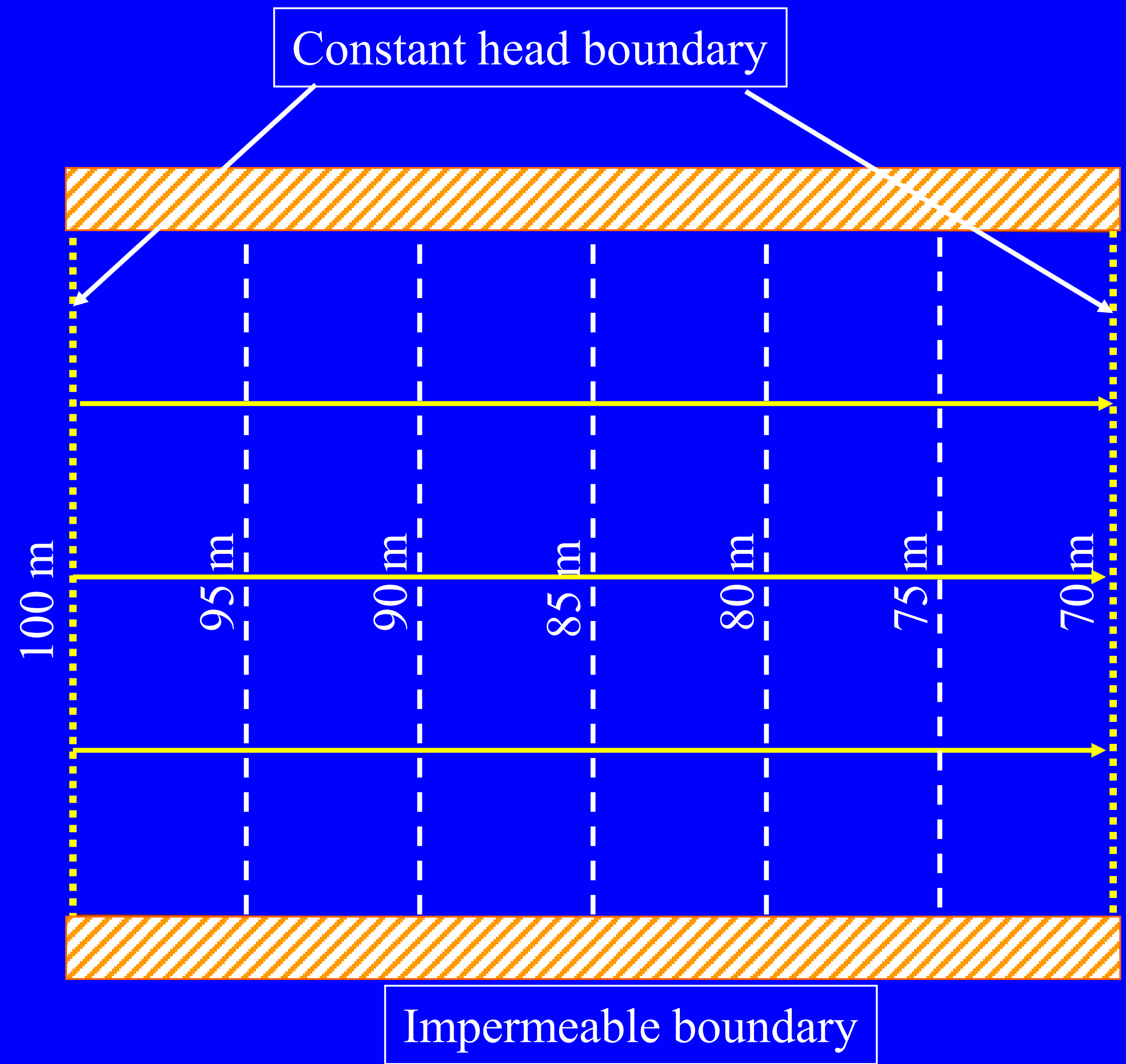
A flow line is an imaginary line that traces the path of a water particle flowing through an aquifer. In an isotropic aquifer, flow lines are parallel to impermeable boundary.



Flow net

- Flow net

In a two-dimensional flow field, a network of equipotential lines and flow lines constitutes a flow net. In an isotropic aquifer, equipotential line and flow lines will form a network of elementary squares.



Flow net

- Calculation of discharge

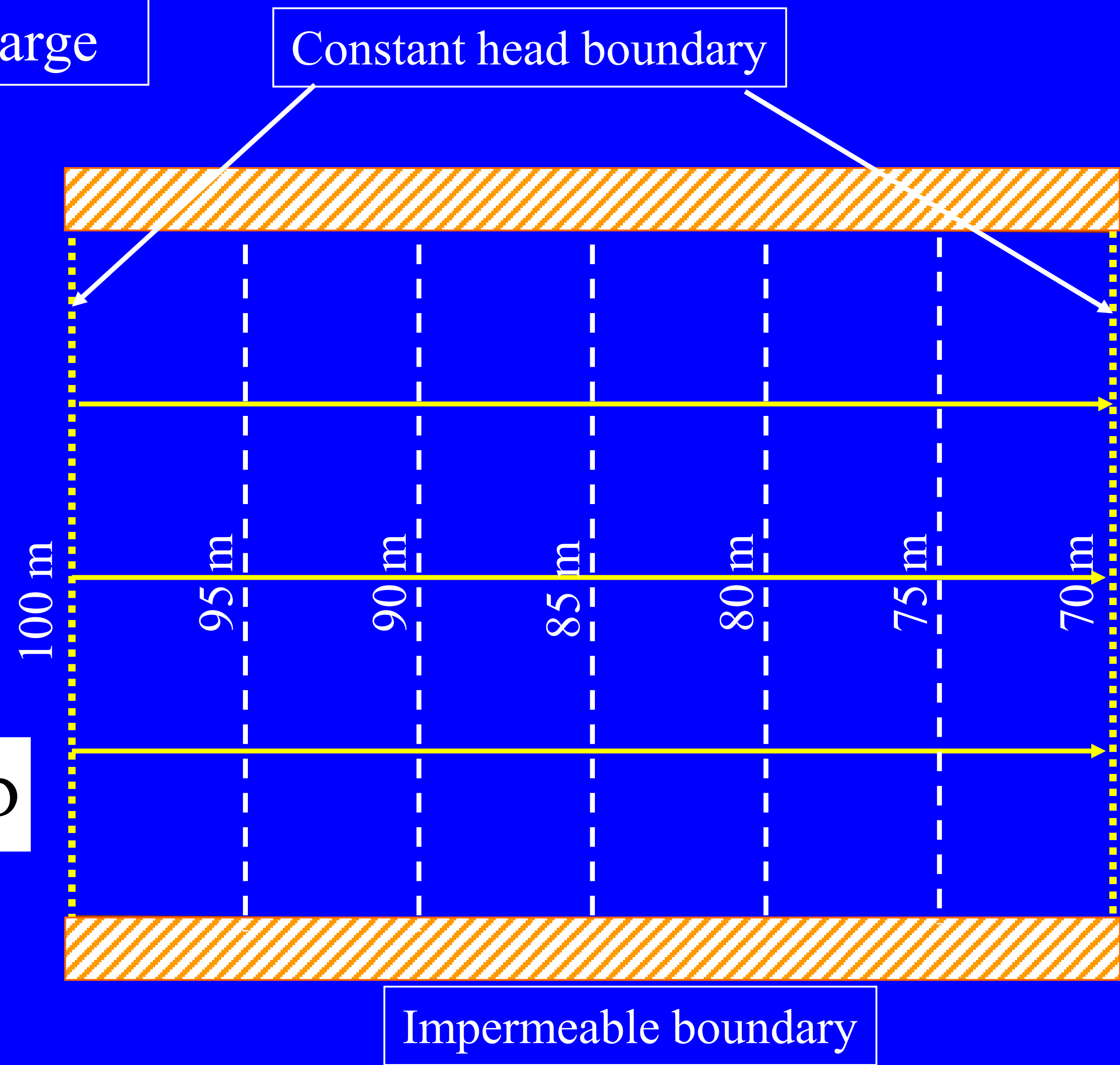
$$q = K\frac{\Delta\varphi}{\Delta s}.\Delta n = K\Delta\varphi$$

Δs: distance between two potential lines;

Δn: distance between two flow lines.

$$Q = mq = mK\Delta\varphi$$

m: number of flow channels.



Flow net

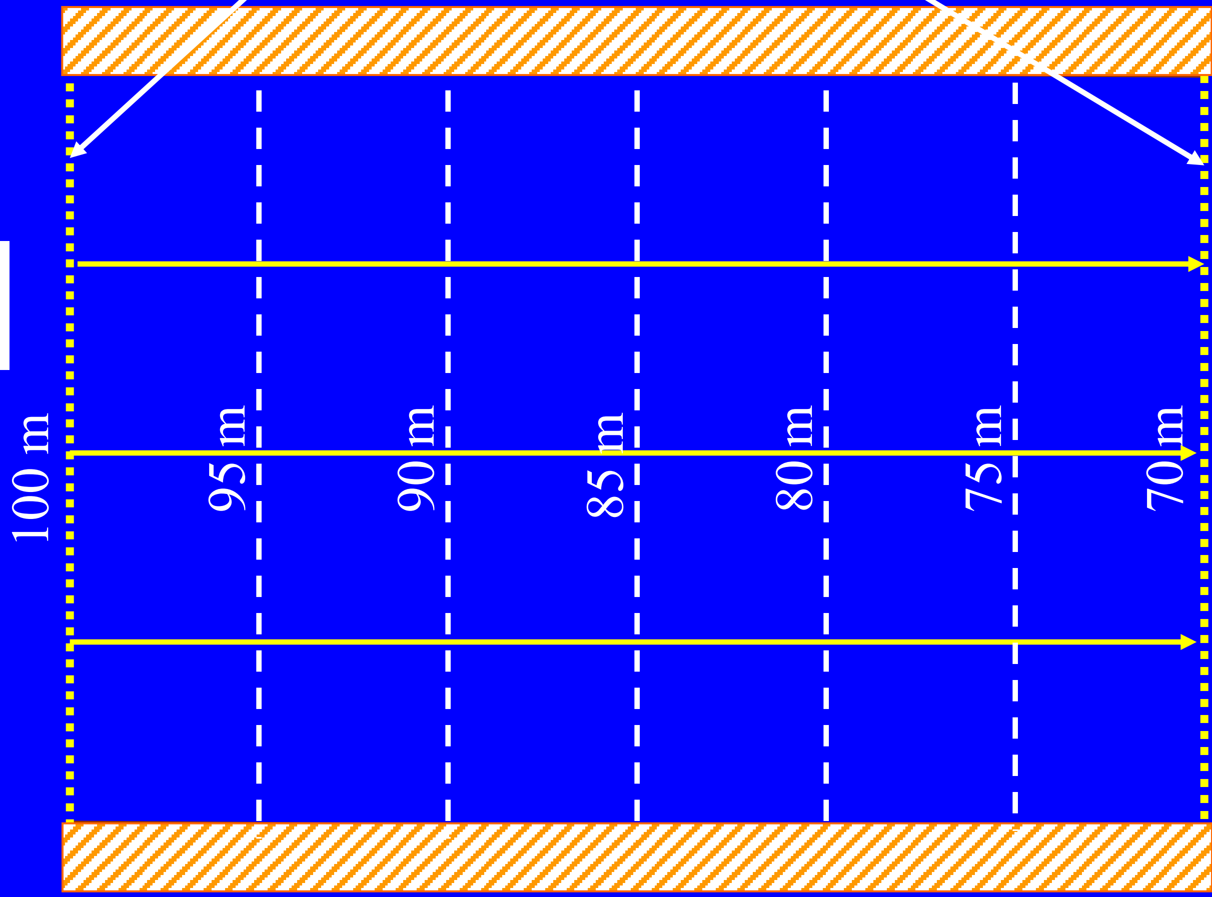
- Calculation of discharge

$$Q = 4q$$

$$Q = 4*K*5 = 20K$$

$$Q_{total} = 20T$$

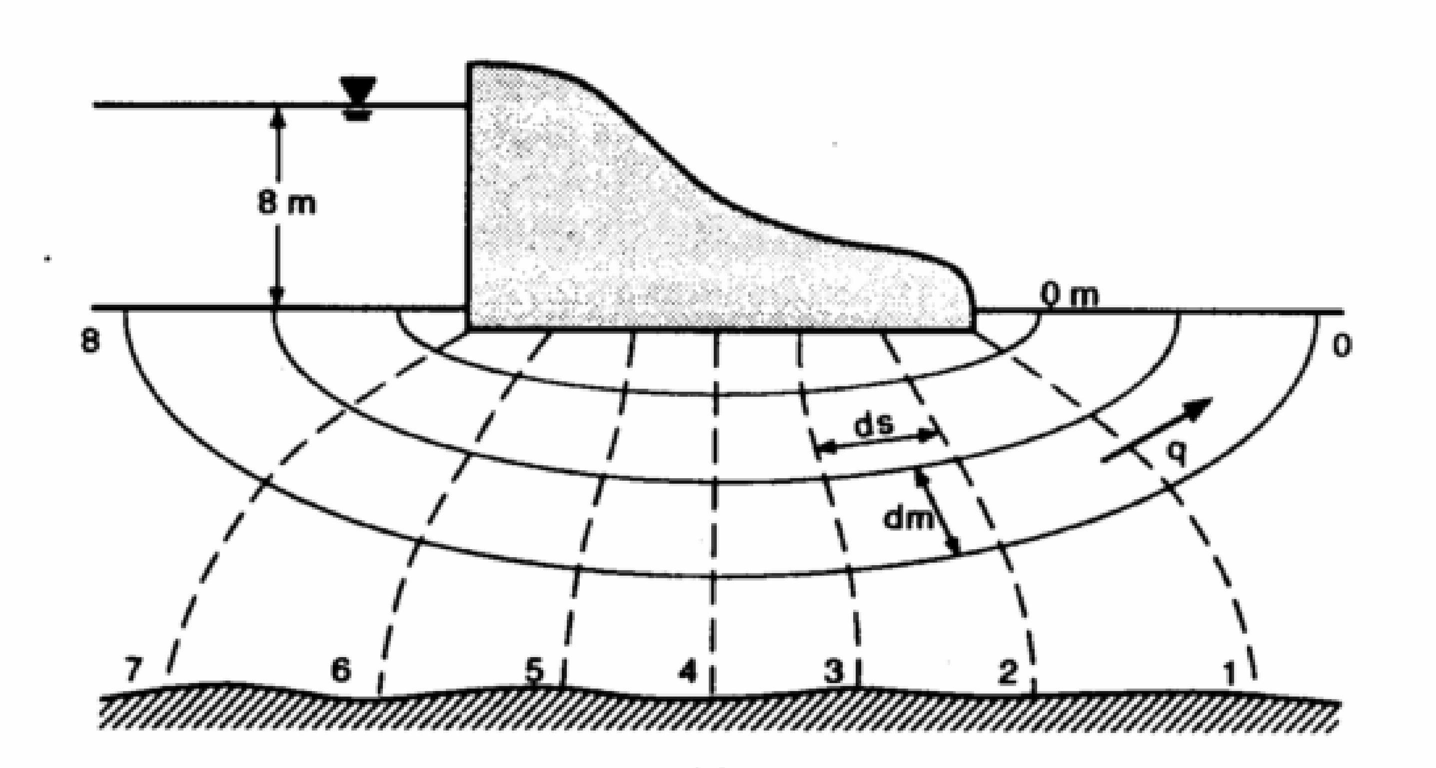
Constant head boundary



Impermeable boundary

Flow net

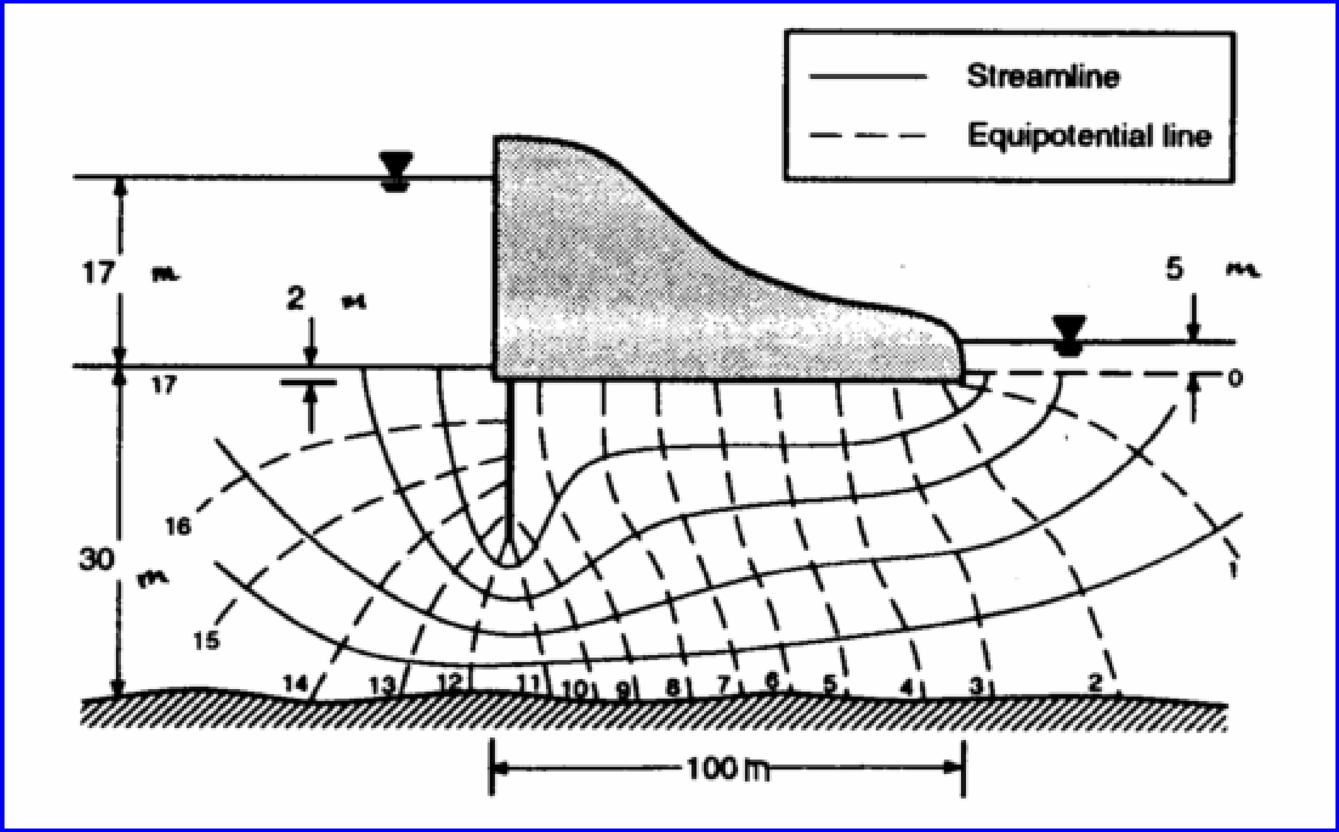
- Seepage underneath the dam



$$Q = 4q = 4K$$

Flow net

- Seepage underneath the dam

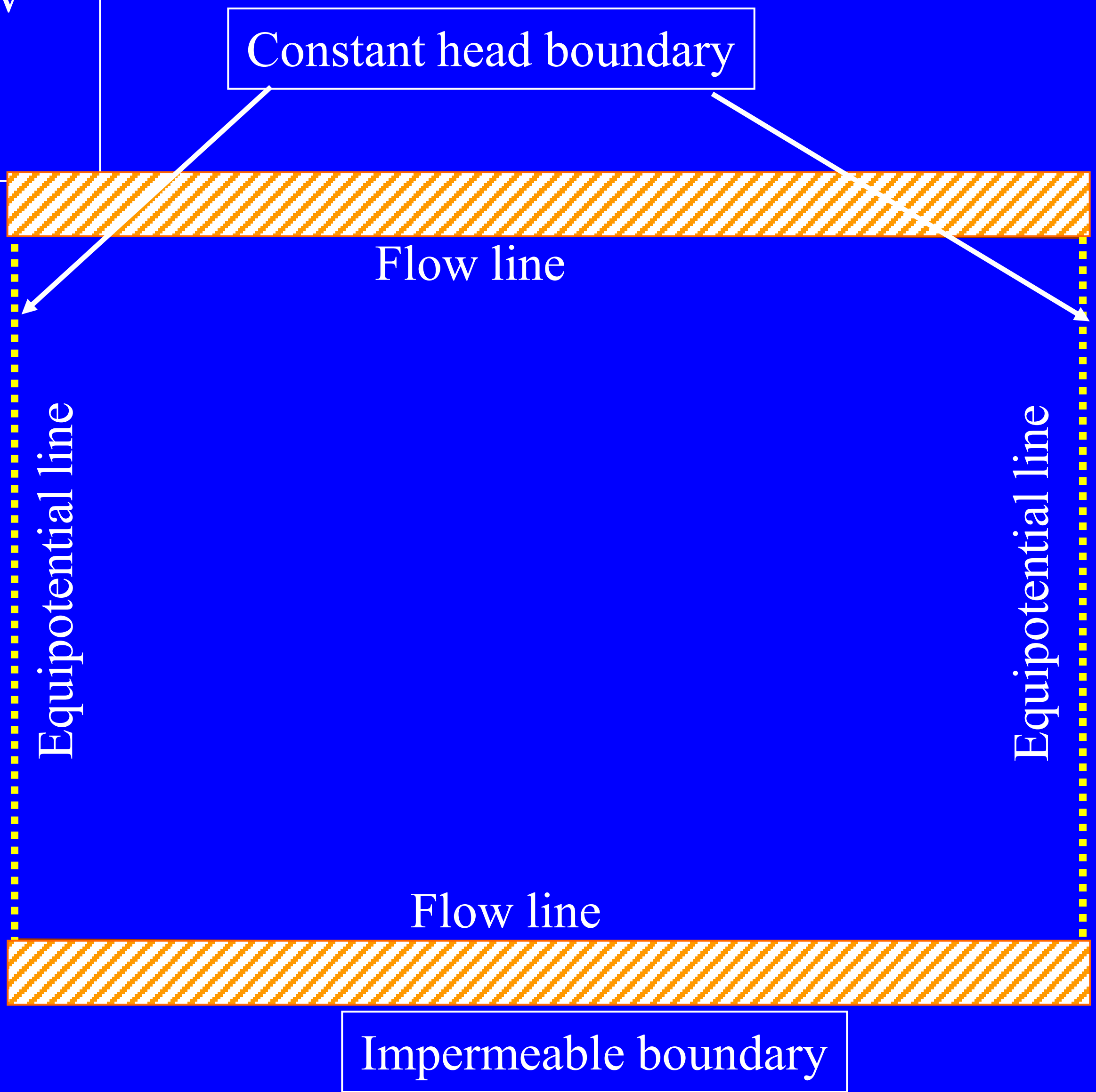


$$Q = 5q = 5K$$

Flow net

- Construction of the flow net in the isotropic and homogeneous aquifer

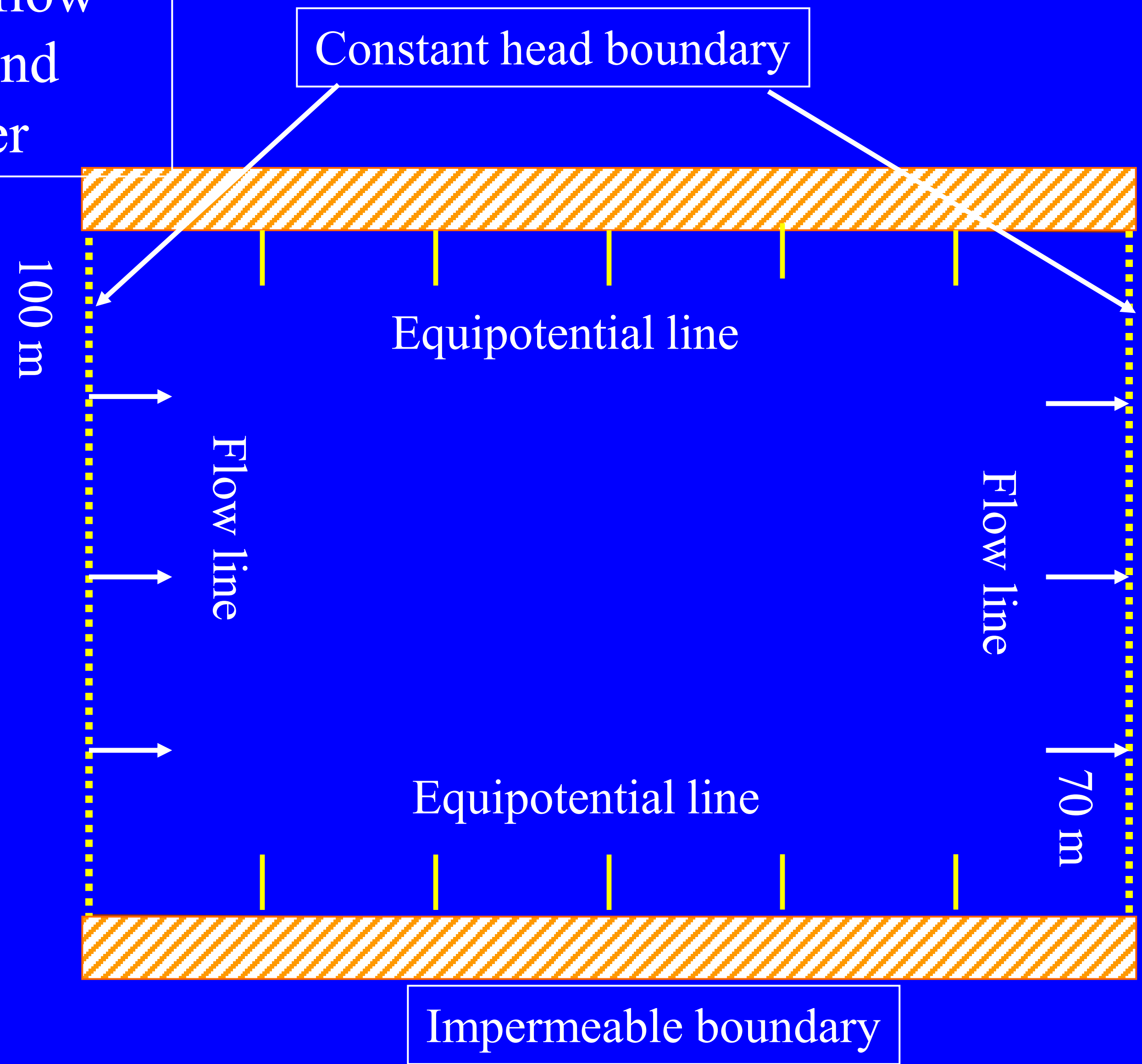
Step 1: Sketch the flow system and identify the specified equipotential lines and flow lines.



Flow net

- Construction of the flow net in the isotropic and homogeneous aquifer

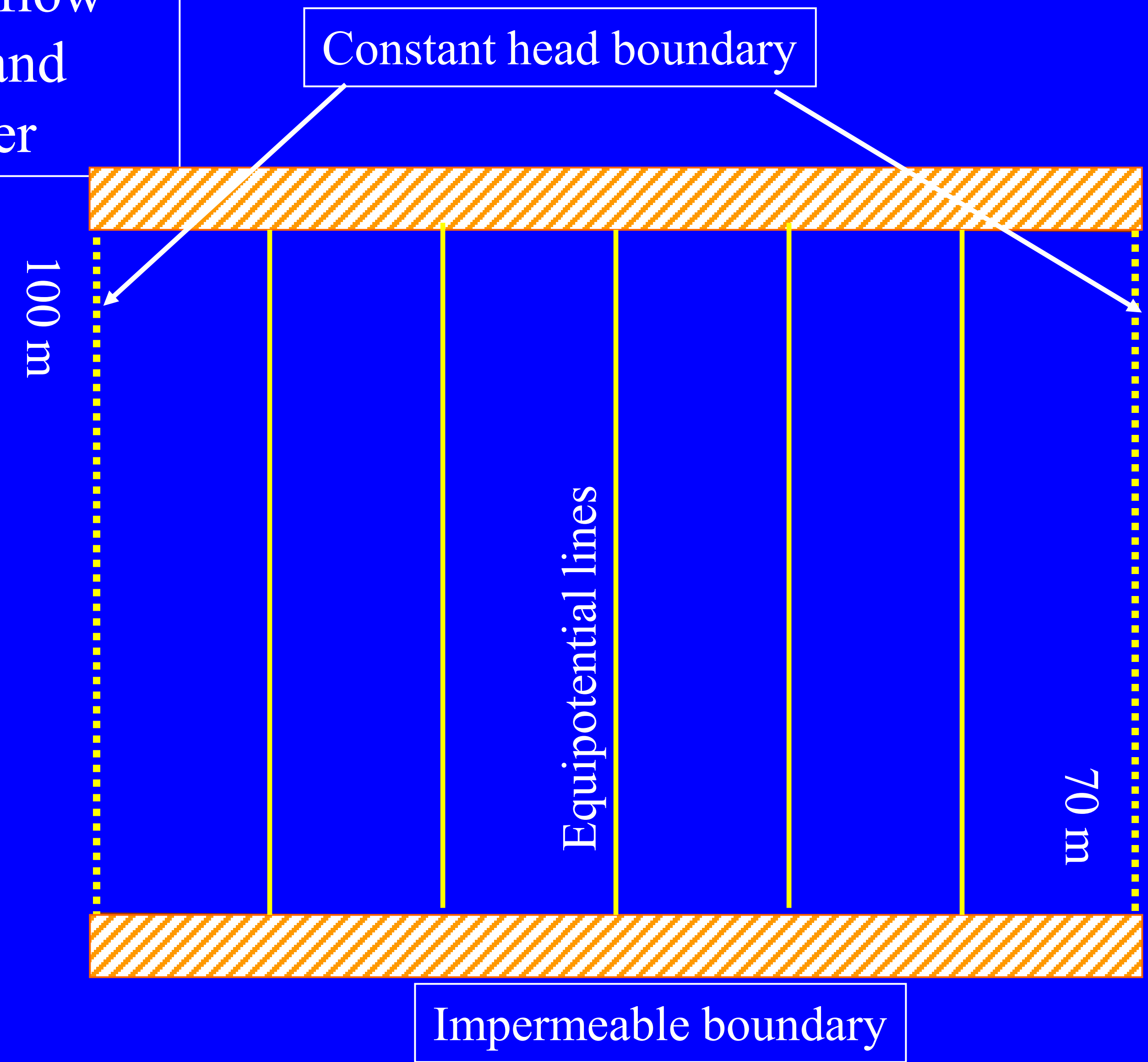
Step 2: Mark the positions of equipotential lines and flow lines.



Flow net

- Construction of the flow net in the isotropic and homogeneous aquifer

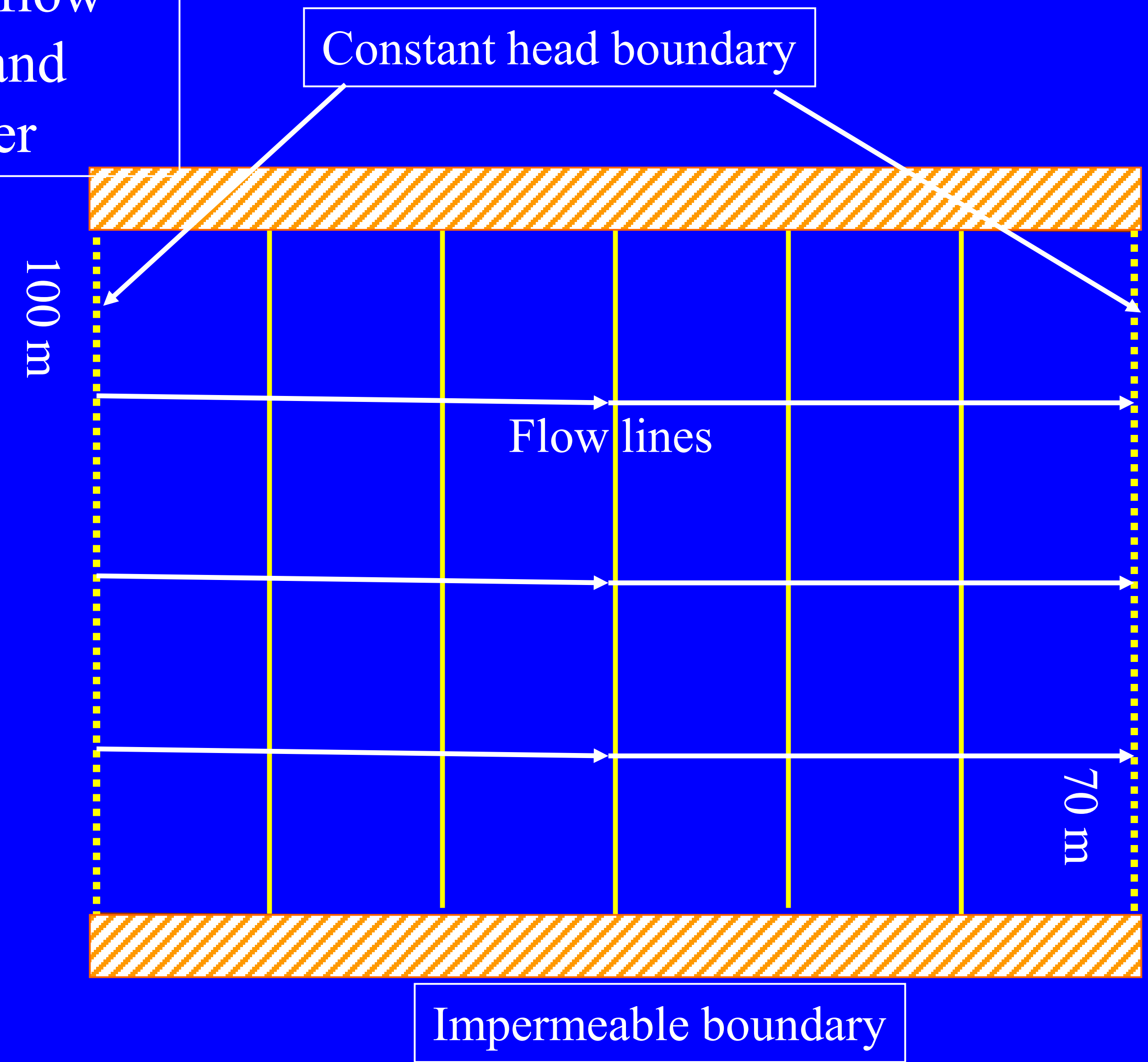
Step 3: Draw trial set of equipotential lines.



Flow net

- Construction of the flow net in the isotropic and homogeneous aquifer

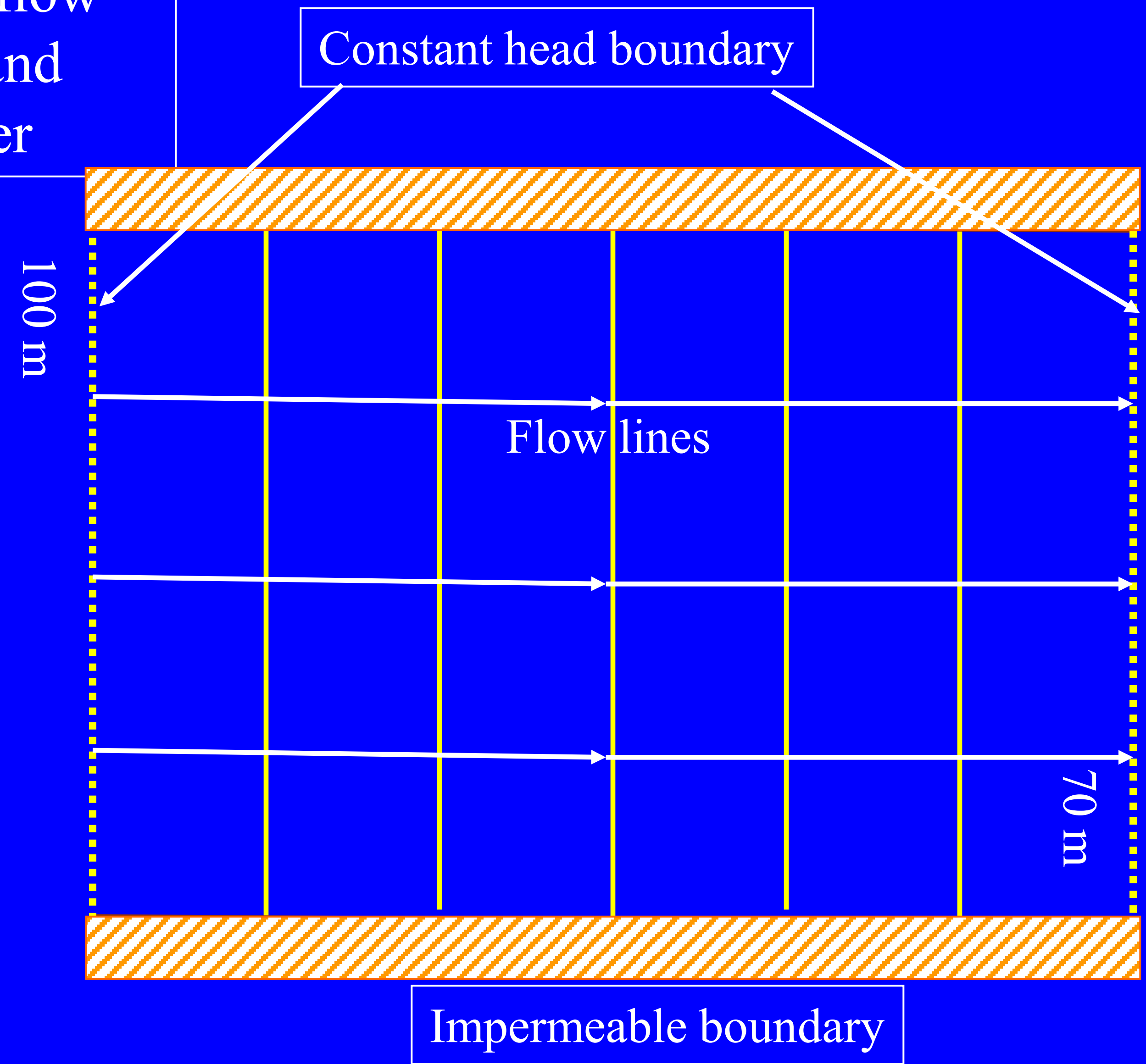
Step 4: Draw trial set of flow lines perpendicular to the equipotential lines.



Flow net

- Construction of the flow net in the isotropic and homogeneous aquifer

Step 5: Erase and redraw the trial equipotential and flow lines until the orthogonal flow net is obtained.



Flow net

- Refraction of flow lines in the heterogeneous aquifer

$$Q_1 = aK_1 \frac{dh_1}{dl_1} \qquad Q_2 = cK_2 \frac{dh_2}{dl_2}$$

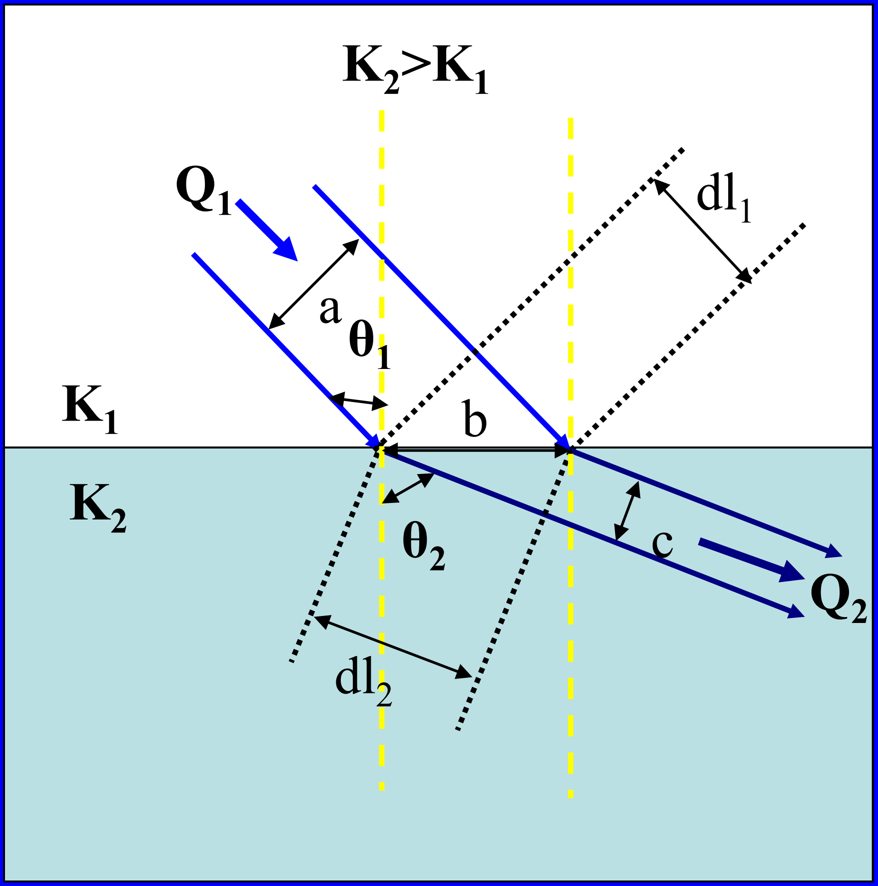
$$Q_1 = Q_2 \qquad dh_1 = dh_2$$

$$a = b\cos(\theta_1) \qquad c = b\cos(\theta_2)$$

$$dl_1/b = \sin(\theta_1) \qquad dl_2/b = \sin(\theta_2)$$

$$\frac{K_1}{K_2} = \frac{\tan(\theta_1)}{\tan(\theta_2)}$$

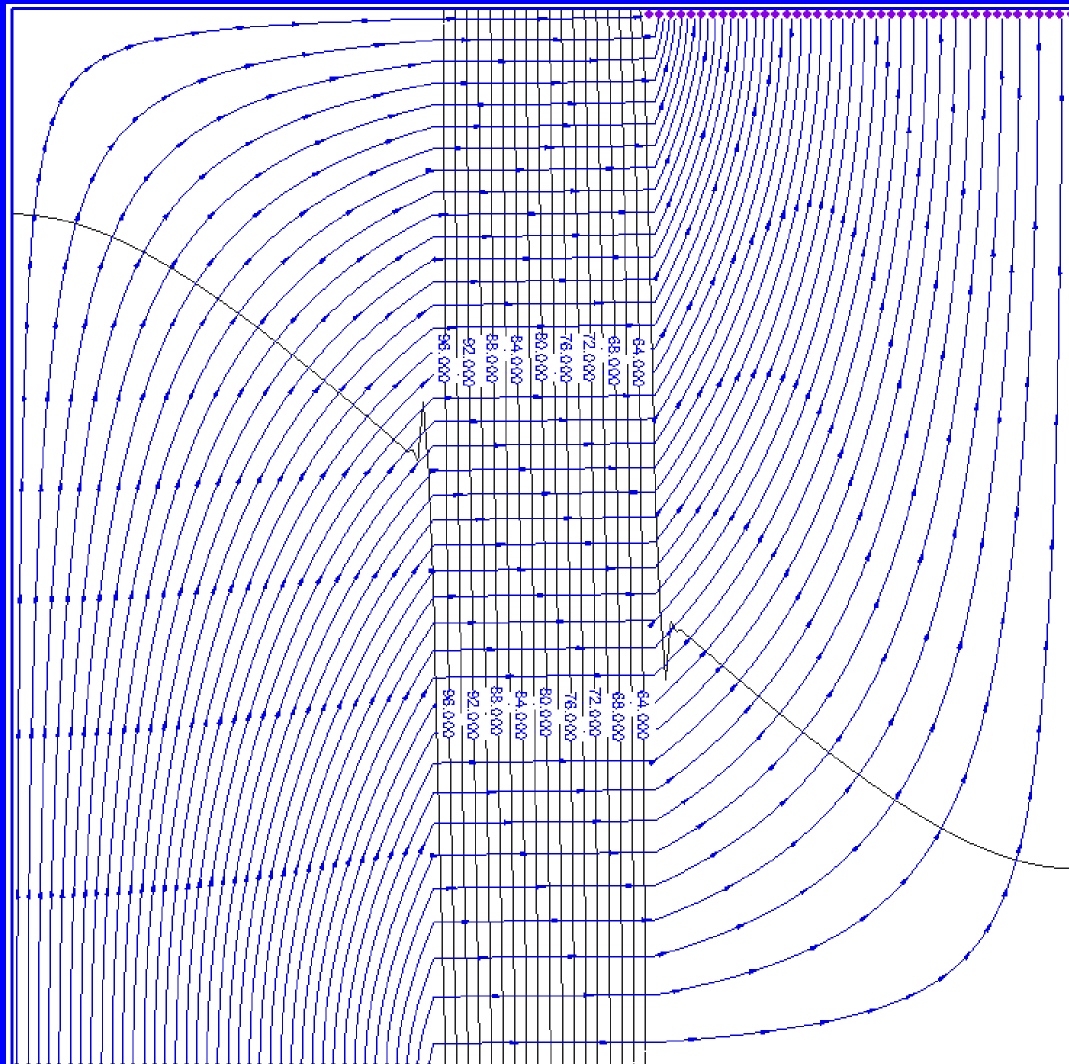
$\theta_2 > \theta_1$



Flow net

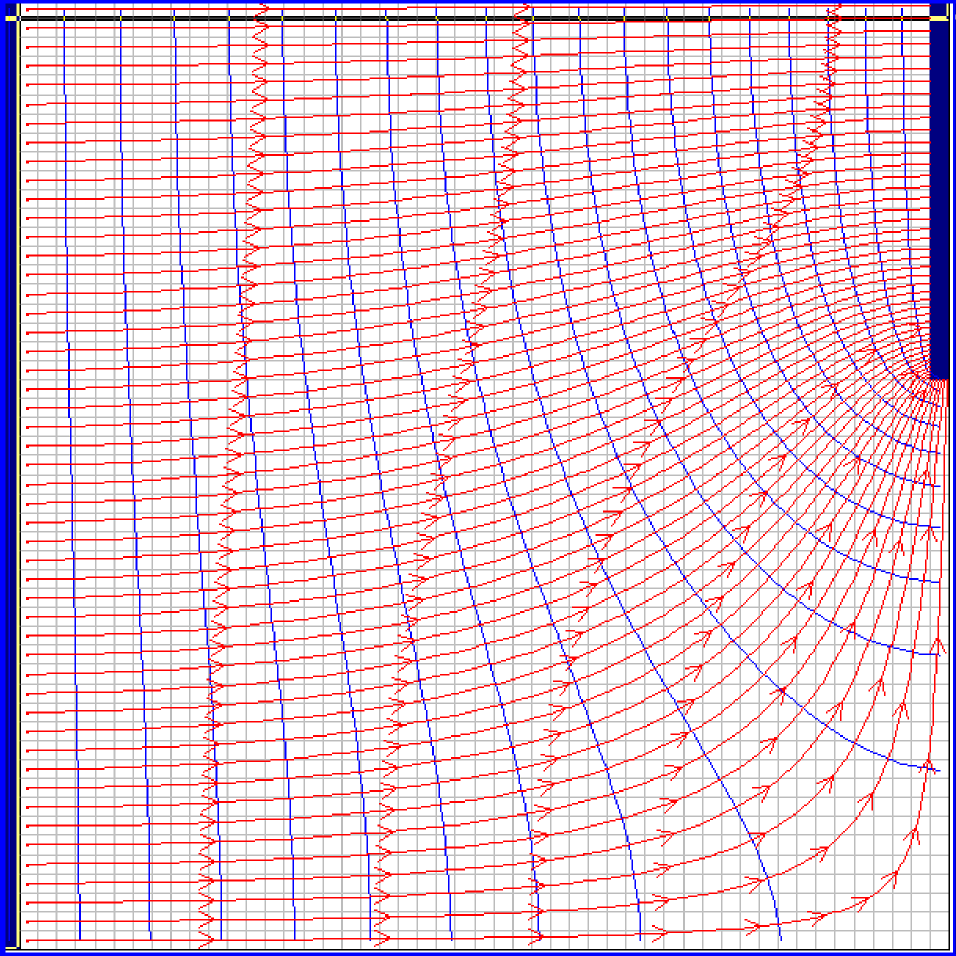
- Refraction of flow lines in the heterogeneous aquifer

Example:
Confined aquifer
Thickness = 50m
5050x5050m
H_1=100m
H_2=60m
K_1=100m/d
K_2=1m/d
Travel time between two marks is 10 years

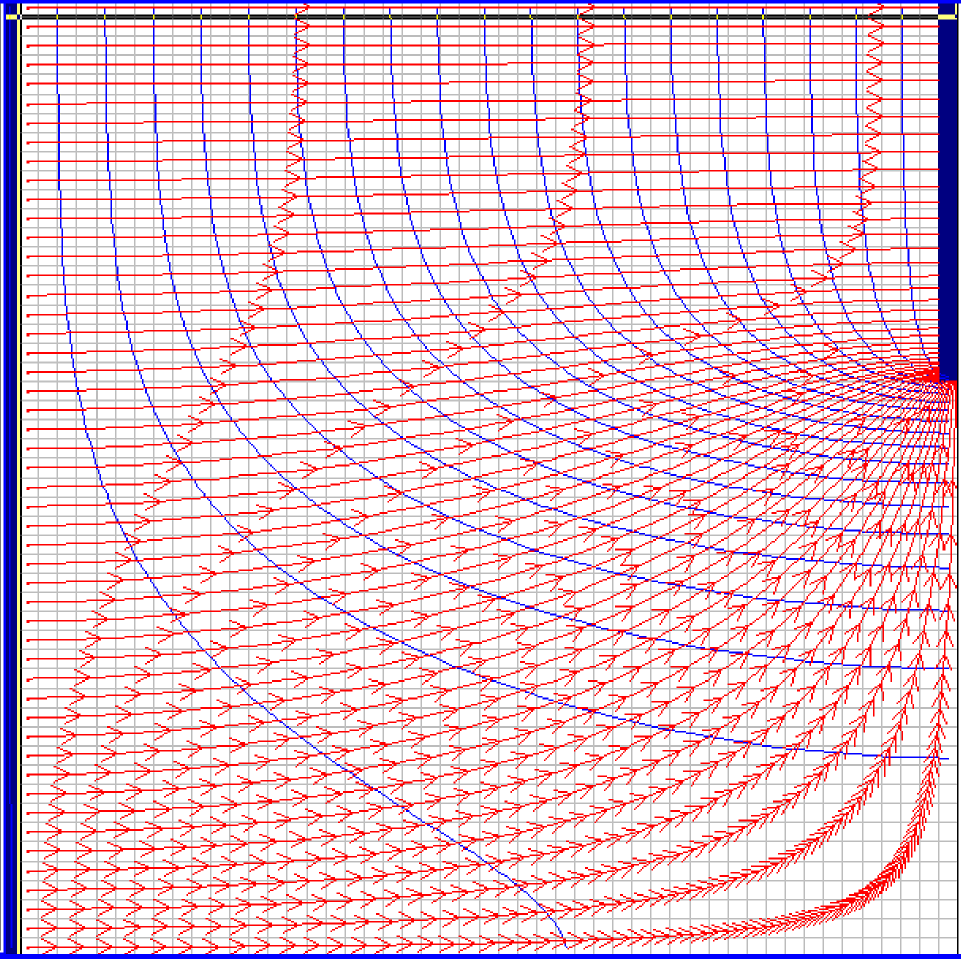


Flow net

- Flow net in the homogeneous and anisotropic aquifer



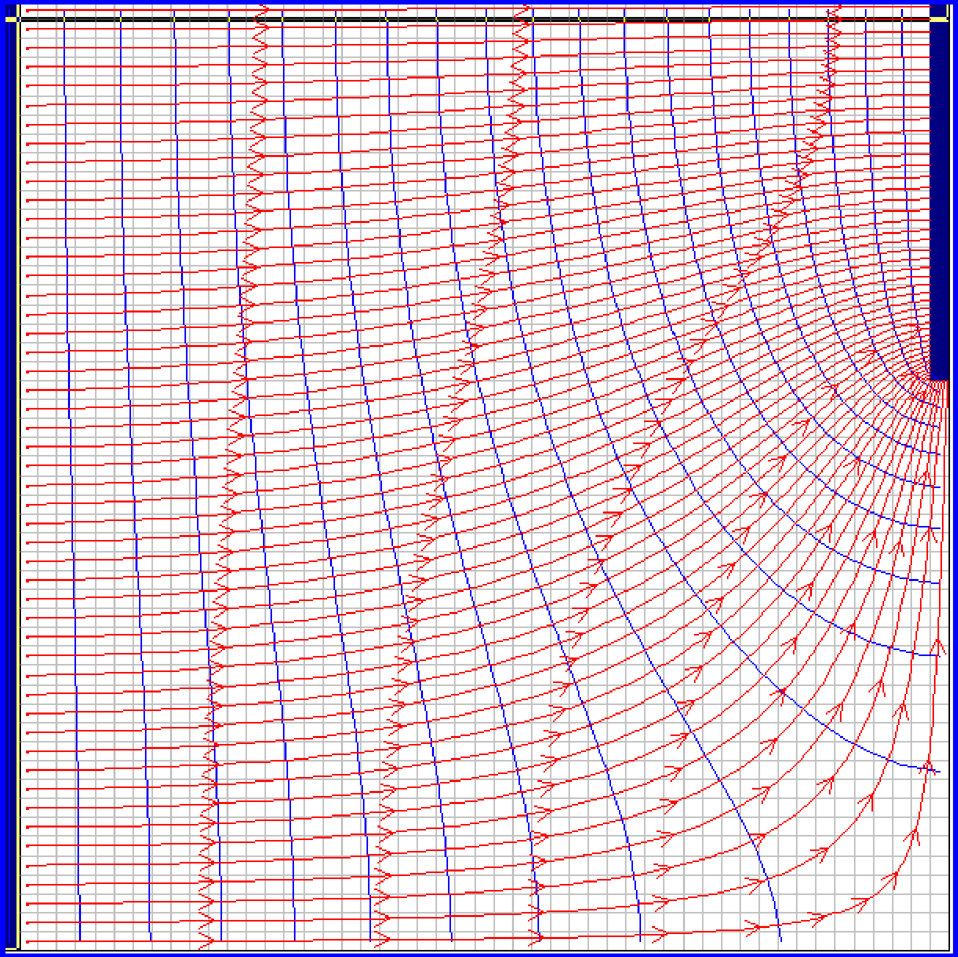
$K_x=K_y=10m/d$



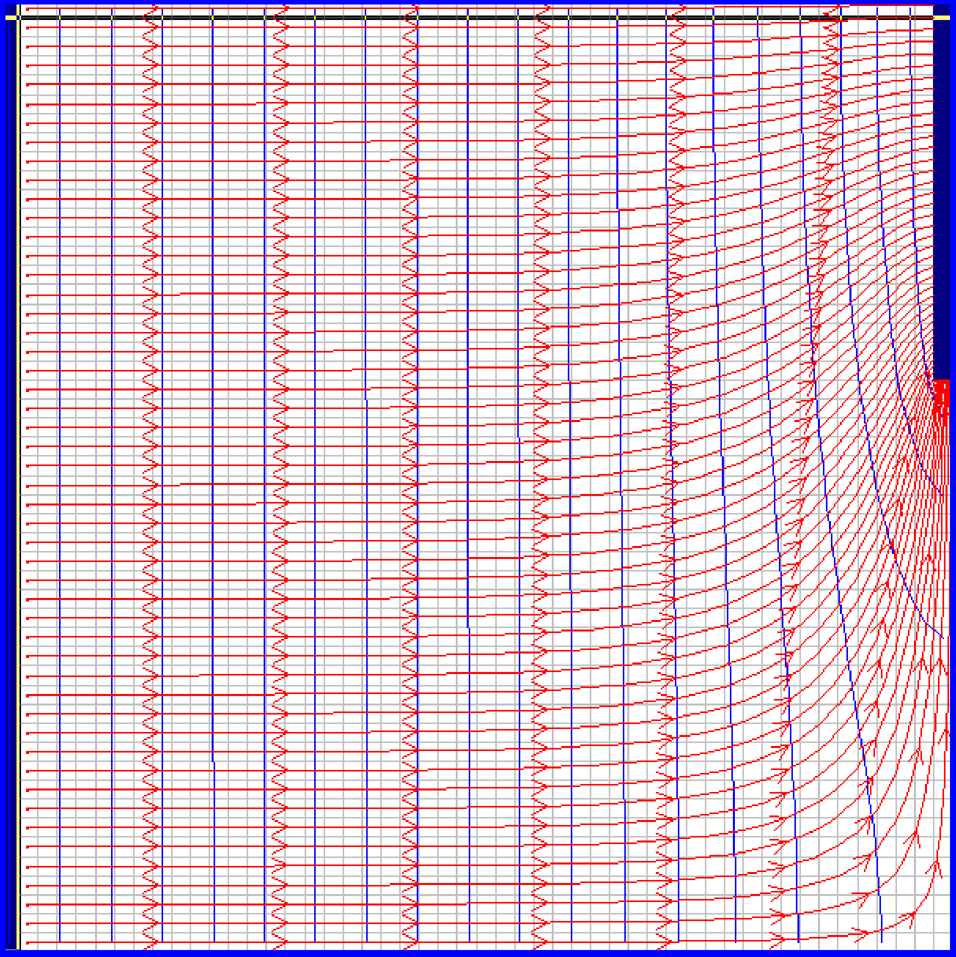
$K_y=0.1\ K_x$

Flow net

- Flow net in the homogeneous and anisotropic aquifer



$K_x = K_y = 10$ m/d



$K_y = 10K_x$