Brief Introduction to Number Theoretic Transform

From the Perspective of Ring Theory

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Motivation and Goals

- **Core Problem**: Fast polynomial multiplication in $\mathbb{Z}_p[x]/(x^n - 1)$, where n is a power of 2.
  The problems we deal with include more general quotient rings, but in this slide, we firstly focus on such simple ring for the purpose of illustration.

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- **Background**:
  - In many applications in cryptography, we have to multiply the elements in the ring of the kind $\mathbb{Z}_p[x]/(x^n - 1)$, which is a very time-consuming operation.
  - Suppose we can perform such operation more efficiently, then in the same cost of computational resource (i.e., time), we can perform such multiplication for larger n, and thus improve the security level of the cryptographic scheme.

Residual Number System and Theory of Quotient Rings

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- Note that $105 = 3 \cdot 5 \cdot 7$, and the factors are co-prime (ideals), so we have the following decomposition:

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- Finally, we recombine the result to get the final answer: The process involves the so-called Chinese Remainder Theorem (CRT), that is, to find the solution to the system

$$x \equiv 0 \pmod 3, \quad x \equiv 2 \pmod 5, \quad x \equiv 4 \pmod 7$$

  The solution is $x = 102$, which is the answer to the original multiplication.

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- Such technique is applicable in real-world, for example, to compute a discrete logarithm in the ring $\mathbb{Z}/n\mathbb{Z}$.
- For more discussion on the RNS, see the paper: Modular exponentiation via the explicit Chinese remainder theorem by DJB.

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$$\mathbb{Z}/n\mathbb{Z}, \quad \mathbb{Z}[x]/(x^n - 1), \quad \mathbb{Z}_p[x]/(x^n - 1)$$

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- We can extend the idea to polynomial operations:
  - the set $\mathbb{Z}[x]$ is the set of all polynomials with integer coefficients. Operations are performed as usual.
  - the set $\mathbb{Z}[x]/(x^n - 1)$ thus meanes that all operations are performed modulo $x^n - 1$.
  - For example, in the ring $\mathbb{Z}[x]/(x^2 - 1)$,

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- We also write the quotient integer rings $\mathbb{Z}/n\mathbb{Z}$ as $\mathbb{Z}_n$
- Hence, the meaning of $\mathbb{Z}_p[x]/(x^n - 1)$ is that the coefficients are reduced modulo p, and the polynomial is reduced modulo $x^n - 1$.

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Residual Number System and Theory of Quotient Rings

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- It seems that the recombination is very hard to solve. But, no, see the next slide.

Residual Number System and Theory of Quotient Rings

Decomposition of $\mathbb{Z}[x]/(x^2 - 1)$

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  3. Recombine the result to get the final answer:

     $$\frac{A+B}{2} + \frac{A-B}{2}x.$$

     It takes two add/sub operations and two divided by 2 operations in this step. Note that divided by 2 is a shift operation, which is very fast.

- The algorithm we just invented takes:
  - 6 add/sub operations
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Residual Number System and Theory of Quotient Rings

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- The analysis we just made are based on number of mathematic operations. This is illustrative, but not the whole story. In practice, please benchmark the performance by the actual cycle-count.

Decomposition of $\mathbb{Z}[x]/(x^4 - 1)$

- After experience the fast multiplication brought by the decomposition of $\mathbb{Z}[x]/(x^2 - 1)$, we now try to decompose the ring $\mathbb{Z}[x]/(x^4 - 1)$.

Residual Number System and Theory of Quotient Rings

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- After experience the fast multiplication brought by the decomposition of $\mathbb{Z}[x]/(x^2 - 1)$, we now try to decompose the ring $\mathbb{Z}[x]/(x^4 - 1)$.
- Easy to see that $x^4 - 1 = (x^2 - 1)(x^2 + 1)$, and the factors are co-prime (ideals), so we have the following decomposition:

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- The first ring can be decomposed as we just did. However, for the second one,

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there is no obvious factorization of $x^2 + 1$.

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  The same trick as above can be applied in the cost of introducing complex numbers. Such trick is called the Discrete Fourier Transform (DFT).

Residual Number System and Theory of Quotient Rings

Decomposition of $\mathbb{Z}[x]/(x^4 - 1)$

- After experience the fast multiplication brought by the decomposition of $\mathbb{Z}[x]/(x^2 - 1)$, we now try to decompose the ring $\mathbb{Z}[x]/(x^4 - 1)$.
- Easy to see that $x^4 - 1 = (x^2 - 1)(x^2 + 1)$, and the factors are co-prime (ideals), so we have the following decomposition:

$$\mathbb{Z}[x]/(x^4 - 1) \cong \mathbb{Z}[x]/(x^2 - 1) \times \mathbb{Z}[x]/(x^2 + 1)$$

- Until here, we can already develop a fast multiplication algorithm by such decomposition. But it the first coordinate-ring can be further decomposed, and it seems that decomposing then one more time can bring more speedup.
- The first ring can be decomposed as we just did. However, for the second one,

$$\mathbb{Z}[x]/(x^2 + 1)$$

  there is no obvious factorization of $x^2 + 1$.
- One way is to employ the complex number field, and we have the following decomposition:

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- The concept of the Number Theoretic Transform (NTT) is similar to DFT, but required more well-strucutred ring.

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RESIDUAL NUMBER SYSTEM AND THEORY OF QUOTIENT RINGS

DECOMPOSITION OF $\mathbb{Z}[x]/(x^4 - 1)$

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Residual Number System and Theory of Quotient Rings

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- Recall that our core problem is to deal with the multiplication of the ring $\mathbb{Z}_p[x]/(x^n - 1)$. Here, we in particular consider the ring $\mathbb{Z}_p[x]/(x^4 - 1)$.

Residual Number System and Theory of Quotient Rings

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Residual Number System and Theory of Quotient Rings

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- The important observation is that, there may be some element in $\mathbb{Z}_p$, denoted as ω, such that $\omega^2 = -1$, which, serves as the imaginary unit as in the DFT.
- Take, for example, the ring $\mathbb{Z}_{17}[x]/(x^4 - 1)$ for example. In this ring,

$$4^2 = 16 \equiv -1 \pmod{17}.$$

We thus have the following decomposition:

$$\begin{aligned}\mathbb{Z}_{17}[x]/(x^2 + 1) &= \mathbb{Z}_{17}[x]/(x^2 - (-1)) = \mathbb{Z}_{17}[x]/(x^2 - 4^2) \\ &\cong \mathbb{Z}_{17}[x]/(x - 4) \times \mathbb{Z}_{17}[x]/(x + 4).\end{aligned}$$

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- Hence we have the full decomposition:

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- We can analogously develop a fast multiplication algorithm for the ring $\mathbb{Z}_{17}[x]/(x^4 - 1)$: projection to coordinate-ring, coordinate-wise multiplication, recombination.

- The projection goes like:

$$
\begin{aligned}
\mathbb{Z}_{17}[x]/(x^4-1) &\cong \mathbb{Z}_{17}[x]/(x^2-1) \times \mathbb{Z}_{17}[x]/(x^2+1) \\
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3+1x+4x^2+2x^3 &\mapsto (7+3x, -1-1x) \cong (7+3x, 16+16x), \\
&\mapsto (10, -4, -5, 3) \cong (10, 13, 12, 3) \\
2+7x+1x^2+2x^3 &\mapsto (3+9x, 1+5x) \\
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Residual Number System and Theory of Quotient Rings

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Residual Number System and Theory of Quotient Rings

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- Coordinate-wise multiplication is straightforward:

$$(10, 13, 12, 3) \cdot (12, 11, 4, 15) = (120, 143, 48, 45) \cong (1, 7, 14, 11).$$

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- Recombination is not obvious, see the next slide.

Residual Number System and Theory of Quotient Rings

Decomposition of $\mathbb{Z}[x]/(x^4 - 1)$

- We now try to recombine the result by the information of coordinates

$$(1, 7, 14, 11) \in \mathbb{Z}_{17}[x]/(x-1) \times \mathbb{Z}_{17}[x]/(x+1) \times \mathbb{Z}_{17}[x]/(x-4) \times \mathbb{Z}_{17}[x]/(x+4).$$

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- For the first two coordinate, we can partially recombine them into the ring $\mathbb{Z}_{17}[x]/(x^2-1)$, since they came from the colored projection as shown:

$$\begin{aligned}\mathbb{Z}_{17}[x]/(x^4-1) &\cong {\color{blue}\mathbb{Z}_{17}[x]/(x^2-1)} \times \mathbb{Z}_{17}[x]/(x^2+1)\\ &\cong {\color{blue}\mathbb{Z}_{17}[x]/(x-1)} \times {\color{blue}\mathbb{Z}_{17}[x]/(x+1)} \times \mathbb{Z}_{17}[x]/(x-4) \times \mathbb{Z}_{17}[x]/(x+4).\end{aligned}$$

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- Such partial recombination is easy, as we done before:

$$4-3x = \frac{1+7}{2} + \frac{1-7}{2}x.$$

- We now try to recombine the result, denoted $f(x)$, by the information of coordinates

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- Check that $4+11x$ projects to $(14,11)$.

Residual Number System and Theory of Quotient Rings

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- But such partial recombination is a little bit tricky,

$$\mathbb{Z}_{17}[x]/(x^2+1) \cong \mathbb{Z}_{17}[x]/(x-4) \times \mathbb{Z}_{17}[x]/(x+4)$$
$$a+bx \mapsto (a+4b, a-4b) = (A,B).$$
$$\frac{1}{2}(A+B) + \frac{1}{2}\frac{A-B}{4}x \mapsto (A,B).$$

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$$\begin{gathered}\mathbb{Z}_{17}[x]/(x^2+1) \cong \mathbb{Z}_{17}[x]/(x-4) \times \mathbb{Z}_{17}[x]/(x+4) \\ a+bx \mapsto (a+4b, a-4b) = (A,B).\end{gathered}$$

$$\frac{1}{2}(A+B) + \frac{1}{2}\frac{A-B}{4}x \mapsto (A,B).$$

- In this case, the recombination goes:

$$\frac{1}{2}(14+11) + \frac{1}{2}\frac{14-11}{4}x = 4+11x.$$

  Note that the division are performed in the ring $\mathbb{Z}_{17}$.
- Check that $4+11x$ projects to $(14,11)$.

Residual Number System and Theory of Quotient Rings

Decomposition of $\mathbb{Z}[x]/(x^4 - 1)$

- So far, we know that the answer of the multiplication, denoted $f(x)$, represented in the first layer of decomposition is:

$$\mathbb{Z}_{17}[x]/(x^4 - 1) \cong \mathbb{Z}_{17}[x]/(x^2 - 1) \times \mathbb{Z}_{17}[x]/(x^2 + 1)$$
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  We have to do one more layer of recombination to get the final answer.

- Check that $f(x) = 4 + 4x + 0x^2 - 7x^3$ projects to $(1, 7, 14, 11)$.

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$$\mathbb{Z}_{17}[x]/(x^4-1) \cong \mathbb{Z}_{17}[x]/(x^2-1) \times \mathbb{Z}_{17}[x]/(x^2+1)$$
$$(a_0+a_1x+a_2x^2+a_3x^3) \mapsto (a_0+a_2+(a_1+a_3)x, a_0-a_2+(a_1-a_3)x) := (A_0+A_1x, A_2+A_3x).$$

  Hence

$$\frac{1}{2}(A_0+A_2)+\frac{1}{2}(A_1+A_3)x+\frac{1}{2}(A_0-A_2)x^2+\frac{1}{2}(A_1-A_3)x^3 \mapsto (A_0+A_1x, A_2+A_3x).$$

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- Apply to our case, the final answer is:

$$\frac{1}{2}(4+4)+\frac{1}{2}(-3+11)x+\frac{1}{2}(4-4)x^2+\frac{1}{2}(-3-11)x^3 = 4+4x+0x^2-7x^3.$$

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  1. Project the operands $a(x), b(x)$ into the coordinate-rings

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  2. Perform the Coordinate-wise multiplication

Residual Number System and Theory of Quotient Rings

Decomposition of $\mathbb{Z}[x]/(x^4 - 1)$

- Let's now give a summary of the fast multiplication algorithm just invented:
  1. Project the operands $a(x), b(x)$ into the coordinate-rings

$$\begin{aligned}\mathbb{Z}_{17}[x]/(x^4 - 1) &\cong \mathbb{Z}_{17}[x]/(x^2 - 1) \times \mathbb{Z}_{17}[x]/(x^2 + 1) \\ &\cong \mathbb{Z}_{17}[x]/(x - 1) \times \mathbb{Z}_{17}[x]/(x + 1) \times \mathbb{Z}_{17}[x]/(x - 4) \times \mathbb{Z}_{17}[x]/(x + 4)\end{aligned}$$

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- Let's now give a summary of the fast multiplication algorithm just invented:
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- We left as an exercise that: The schoolbook of the same calculation requies 16 multiplications and some additions, make an estimation of the number of operations in the fast multiplication algorithm we just invented.
- Another issue is, here we picked a particular modular number 17, how to generalize the algorithm to arbitrary p? What conditions should the number p satisfy in order to have such decomposition (i.e., existence of the element ω such that $\omega^2 = -1$)? We will discuss this in the final part.

DECOMPOSITION OF $\mathbb{Z}_p[x]/(x^n - 1)$

DECOMPOSITION OF $\mathbb{Z}_p[x]/(x^8 - 1)$

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- We have

$$
\begin{aligned}
(x^8 - 1) &= (x^4 - 1)(x^4 + 1) = (x^4 - 1)(x^4 - \omega_4^2) \\
&= (x^2 - 1)(x^2 + 1)(x^2 - \omega_4)(x^2 + \omega_4) = (x^2 - 1)(x^2 - \omega_4^2)(x^2 - \omega_4)(x^2 + \omega_4) \\
&= (x^2 - 1)(x^2 - \omega_4^2)(x^2 - \omega_8^2)(x^2 + \omega_8^2) = (x^2 - 1)(x^2 - \omega_4^2)(x^2 - \omega_8^2)(x^2 - \omega_8^6) \\
&= (x - 1)(x + 1)(x - \omega_4)(x + \omega_4)(x - \omega_8)(x + \omega_8)(x - \omega_8^3)(x + \omega_8^3).
\end{aligned}
$$

- Proceeding from the previous example, we now try to decompose the ring $\mathbb{Z}_p[x]/(x^8-1)$.
- We assume that the modulus number p will make the existence of 8-th primitive root ω_8, that is, $\omega_8^8 = 1$ and $\omega_8^4 = -1$. Existence of ω_4 follows from $\omega_4 = \omega_8^2$.
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$$\begin{aligned}(x^8-1) &= (x^4-1)(x^4+1) = (x^4-1)(x^4-\omega_4^2) \\ &= (x^2-1)(x^2+1)(x^2-\omega_4)(x^2+\omega_4) = (x^2-1)(x^2-\omega_4^2)(x^2-\omega_4)(x^2+\omega_4) \\ &= (x^2-1)(x^2-\omega_4^2)(x^2-\omega_8^2)(x^2+\omega_8^2) = (x^2-1)(x^2-\omega_4^2)(x^2-\omega_8^2)(x^2-\omega_8^6) \\ &= (x-1)(x+1)(x-\omega_4)(x+\omega_4)(x-\omega_8)(x+\omega_8)(x-\omega_8^3)(x+\omega_8^3).\end{aligned}$$

- Hence, we have the decomposition:

$$\begin{aligned}\mathbb{Z}_p[x]/(x^8-1) &\cong (\mathbb{Z}_p[x]/(x^4-1)) \times (\mathbb{Z}_p[x]/(x^4+1)) \\ &\cong (\mathbb{Z}_p[x]/(x^2-1)) \times (\mathbb{Z}_p[x]/(x^2+1)) \times (\mathbb{Z}_p[x]/(x^2-\omega_4)) \times (\mathbb{Z}_p[x]/(x^2+\omega_4)) \\ &\cong (\mathbb{Z}_p[x]/(x-1)) \times (\mathbb{Z}_p[x]/(x+1)) \times (\mathbb{Z}_p[x]/(x-\omega_4)) \times (\mathbb{Z}_p[x]/(x+\omega_4)) \\ &\quad \times (\mathbb{Z}_p[x]/(x-\omega_8)) \times (\mathbb{Z}_p[x]/(x+\omega_8)) \times (\mathbb{Z}_p[x]/(x-\omega_8^3)) \times (\mathbb{Z}_p[x]/(x+\omega_8^3)).\end{aligned}$$

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Decomposition of $\mathbb{Z}_p[x]/(x^n - 1)$

Decomposition of $\mathbb{Z}_p[x]/(x^8 - 1)$

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\end{aligned}
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- It can then be sofisticatedly written as:

$$\mathbb{Z}_p[x]/(x^8-1)$$

The notation $\mathrm{brv}_1(k)$, $\mathrm{brv}_2(k)$, and $\mathrm{brv}_3(k)$ will be discussed in the next slide.

DECOMPOSITION OF $\mathbb{Z}_p[x]/(x^n-1)$

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\mathbb{Z}_p[x]/(x^8-1) \cong \prod_{k=0}^{1} \mathbb{Z}_p[x]/(x-\omega_2^{\mathrm{brv}_1(k)})
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\end{aligned}$$

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$$\mathbb{Z}_p[x]/(x^8-1) \cong \prod_{k=0}^{1} \mathbb{Z}_p[x]/(x-\omega_2^{\mathrm{brv}_1(k)}) \cong \prod_{k=0}^{3} \mathbb{Z}_p[x]/(x-\omega_4^{\mathrm{brv}_2(k)})$$

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The notation $\mathrm{brv}_1(k)$, $\mathrm{brv}_2(k)$, and $\mathrm{brv}_3(k)$ will be discussed in the next slide.

Decomposition of $\mathbb{Z}_p[x]/(x^n - 1)$

Decomposition of $\mathbb{Z}_p[x]/(x^8 - 1)$

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- If you think that the above formula is too complicated, it is equivalent to:
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- You can check this

$$\prod_{k=0}^{7} \mathbb{Z}_p[x]/(x - \omega_8^{\mathrm{brv}_3(k)})$$

  equals to

$$(\mathbb{Z}_p[x]/(x - \omega_8^0)) \times (\mathbb{Z}_p[x]/(x - \omega_8^4)) \times (\mathbb{Z}_p[x]/(x - \omega_8^2)) \times (\mathbb{Z}_p[x]/(x - \omega_8^6))$$
$$\times (\mathbb{Z}_p[x]/(x - \omega_8)) \times (\mathbb{Z}_p[x]/(x - \omega_8^5)) \times (\mathbb{Z}_p[x]/(x - \omega_8^3)) \times (\mathbb{Z}_p[x]/(x - \omega_8^7))$$

- Now we can finally state the decomposition formula of $\mathbb{Z}_p[x]/(x^n - 1)$:

$$\begin{aligned}\mathbb{Z}_p[x]/(x^n - 1) &\cong \prod_{k=0}^{1} \mathbb{Z}_p[x]/(x^{\frac{n}{2}} - \omega_2^{\mathrm{brv}_1(k)}) \cong \prod_{k=0}^{3} \mathbb{Z}_p[x]/(x^{\frac{n}{4}} - \omega_4^{\mathrm{brv}_2(k)}) \\ &\cong \prod_{k=0}^{7} \mathbb{Z}_p[x]/(x^{\frac{n}{8}} - \omega_8^{\mathrm{brv}_3(k)}) \\ &\vdots \\ &\cong \prod_{k=0}^{n-1} \mathbb{Z}_p[x]/(x - \omega_n^{\mathrm{brv}_{\log_2 n}(k)}).\end{aligned}$$

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&\cong \prod_{k=0}^{7} \mathbb{Z}_p[x]/(x^{\frac{n}{8}} - \omega_8^{\mathrm{brv}_3(k)}) \\
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  We will say that this is a $\log_2 n$-level decomposition.
- An important observation is that: In order to make the decomposition until the $\log_2 n$-level, we need to have the existence of n-th primitive root. If, say, the current coefficient ring only has 4-th primitive root, then we can only decompose until the 2-level:

$$\mathbb{Z}_p[x]/(x^n - 1) \cong \prod_{k=0}^{1} \mathbb{Z}_p[x]/(x^{\frac{n}{2}} - \omega_2^{\mathrm{brv}_1(k)}).$$

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This is the so-called incomplete NTT. Though not fully decoposed, it is still beneficial (sometimes better) to our purpose. Again, the performance should be measured by the cycle-count.

- By using such decomposition, we can develop a fast algorithm for polynomial multiplication in the ring $\mathbb{Z}_p[x]/(x^8 - 1)$.
- But the projection (and recombination) are more complicated than the previous example. Hence we introduce the concept of butterfly algorithm.
- It is natural to represent a polynomial by an array in the programming language. So here we denote the polynomial $a(x) = a_0 + a_1 x + \cdots + a_7 x^7$ by an array

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}.$$

Decomposition of $\mathbb{Z}_p[x]/(x^n - 1)$

Decomposition of $\mathbb{Z}_p[x]/(x^8 - 1)$

asd

References I