Arbitrage-free SVI volatility surfaces

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Outline

- History of SVI
- Static arbitrage
- Equivalent SVI formulations
- Simple closed-form arbitrage-free SVI surfaces
- How to eliminate butterfly arbitrage
- How to interpolate and extrapolate
- Fit quality on SPX
- An alternative to SABR?

History of SVI

- SVI was originally devised at Merrill Lynch in 1999 and subsequently publicly disseminated in [4].
- SVI has two key properties that have led to its subsequent popularity with practitioners:
  - For a fixed time to expiry t, the implied Black-Scholes variance $\sigma^2_{\mathrm{BS}}(k,t)$ is linear in the log-strike k as $|k| \to \infty$ consistent with Roger Lee's moment formula [11].
  - It is relatively easy to fit listed option prices whilst ensuring no calendar spread arbitrage.
- The consistency of the SVI parameterization with arbitrage bounds for extreme strikes has also led to its use as an extrapolation formula [9].
- As shown in [6], the SVI parameterization is not arbitrary in the sense that the large-maturity limit of the Heston implied volatility smile is exactly SVI.

Previous work

- Calibration of SVI to given implied volatility data (for example [12]).
- [2] showed how to parameterize the volatility surface so as to preclude dynamic arbitrage.
- Arbitrage-free interpolation of implied volatilities by [1], [3], [8], [10].
- Prior work has not successfully attempted to eliminate static arbitrage.

- Efforts to find simple closed-form arbitrage-free parameterizations of the implied volatility surface are widely considered to be futile.

Notation

- Given a stock price process $(S_t)_{t\geq 0}$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$, the forward price process $(F_t)_{t\geq 0}$ is $F_t := \mathbb{E}\left(S_t|\mathcal{F}_0\right)$.
- For any $k \in \mathbb{R}$ and $t > 0$, $C_{\mathrm{BS}}(k, \sigma^2 t)$ denotes the Black-Scholes price of a European Call option on S with strike $F_t \mathrm{e}^k$, maturity t and volatility $\sigma > 0$.
- $\sigma_{\mathrm{BS}}(k, t)$ denotes Black-Scholes implied volatility.
- Total implied variance is $w(k, t) = \sigma_{\mathrm{BS}}^2(k, t)t$.
- The implied variance $v(k, t) = \sigma_{\mathrm{BS}}^2(k, t) = w(k, t)/t$.
- The map $(k, t) \mapsto w(k, t)$ is the volatility surface.
- For any fixed expiry $t > 0$, the function $k \mapsto w(k, t)$ represents a slice.

Characterisation of static arbitrage

Definition 2.1

A volatility surface is free of static arbitrage if and only if the following conditions are satisfied:

(i) *it is free of calendar spread arbitrage;*

(ii) *each time slice is free of butterfly arbitrage.*

Calendar spread arbitrage

Lemma 2.2

If dividends are proportional to the stock price, the volatility surface w is free of calendar spread arbitrage if and only if

$$\partial_t w(k,t) \geq 0, \quad \text{for all } k \in \mathbb{R} \text{ and } t > 0.$$

- Thus there is no calendar spread arbitrage if there are no crossed lines on a total variance plot.

Butterfly arbitrage

Definition 2.3

A slice is said to be free of butterfly arbitrage if the corresponding density is non-negative.

Now introduce the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(k) := \left(1 - \frac{kw'(k)}{2w(k)}\right)^2 - \frac{w'(k)^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2}.$$

Lemma 2.4

A slice is free of butterfly arbitrage if and only if $g(k) \geq 0$ for all $k \in \mathbb{R}$ and $\lim_{k \to +\infty} d_+(k) = -\infty$.

The raw SVI parameterization

For a given parameter set $\chi_R = \{a, b, \rho, m, \sigma\}$, the *raw SVI parameterization* of total implied variance reads:

Raw SVI parameterization

$$w(k; \chi_R) = a + b\left\{\rho\,(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right\}$$

where $a \in \mathbb{R}$, $b \geq 0$, $|\rho| < 1$, $m \in \mathbb{R}$, $\sigma > 0$, and the obvious condition $a + b\,\sigma\,\sqrt{1-\rho^2} \geq 0$, which ensures that $w(k, \chi_R) \geq 0$ for all $k \in \mathbb{R}$. This condition ensures that the minimum of the function $w(\cdot, \chi_R)$ is non-negative.

Meaning of raw SVI parameters

Changes in the parameters have the following effects:

- Increasing a increases the general level of variance, a vertical translation of the smile;
- Increasing b increases the slopes of both the put and call wings, tightening the smile;
- Increasing ρ decreases (increases) the slope of the left(right) wing, a counter-clockwise rotation of the smile;
- Increasing m translates the smile to the right;
- Increasing σ reduces the at-the-money (ATM) curvature of the smile.

The natural SVI parameterization

For a given parameter set $\chi_N = \{\Delta, \mu, \rho, \omega, \zeta\}$, the *natural SVI parameterization* of total implied variance reads:

Natural SVI parameterization

$$w(k;\chi_N) = \Delta + \frac{\omega}{2}\left\{1 + \zeta\rho\,(k-\mu) + \sqrt{(\zeta(k-\mu)+\rho)^2 + (1-\rho^2)}\right\},$$

where $\omega \geq 0$, $\Delta \in \mathbb{R}$, $\mu \in \mathbb{R}$, $|\rho| < 1$ and $\zeta > 0$.

- This parameterization is a natural generalization of the time ∞ Heston smile explored in [6].

The SVI Jump-Wings (SVI-JW) parameterization

- Neither the raw SVI nor the natural SVI parameterizations are intuitive to traders.
- There is no reason to expect these parameters to be particularly stable.
- The *SVI-Jump-Wings (SVI-JW) parameterization* of the implied variance v (rather than the implied total variance w) was inspired by a similar parameterization attributed to Tim Klassen, then at Goldman Sachs.

SVI-JW

For a given time to expiry $t > 0$ and a parameter set $\chi_J = \{v_t, \psi_t, p_t, c_t, \widetilde{v}_t\}$ the SVI-JW parameters are defined from the raw SVI parameters as follows:

SVI-JW parameterization

$$
\begin{aligned}
v_t &= \frac{a + b\left\{-\rho\, m + \sqrt{m^2 + \sigma^2}\right\}}{t}, \\
\psi_t &= \frac{1}{\sqrt{w_t}}\,\frac{b}{2}\left(-\frac{m}{\sqrt{m^2+\sigma^2}} + \rho\right), \\
p_t &= \frac{1}{\sqrt{w_t}}\, b\,(1-\rho)\,, \\
c_t &= \frac{1}{\sqrt{w_t}}\, b\,(1+\rho)\,, \\
\widetilde{v}_t &= \left(a + b\,\sigma\,\sqrt{1-\rho^2}\right)/t
\end{aligned}
$$

with $w_t := v_t\, t$.

Interpretation of SVI-JW parameters

The SVI-JW parameters have the following interpretations:

- v_t gives the ATM variance;
- ψ_t gives the ATM skew;
- p_t gives the slope of the left (put) wing;
- c_t gives the slope of the right (call) wing;
- $\widetilde{v}_t$ is the minimum implied variance.

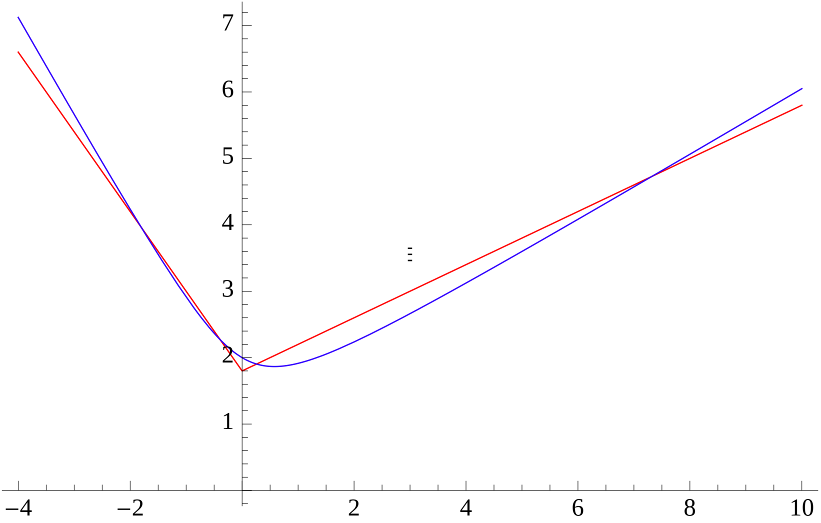
Features of the SVI-JW parameterization

- If smiles scaled perfectly as $1/\sqrt{w_t}$, SVI-JW parameters would be constant, independent of the slice t.
  - This makes it easy to extrapolate the SVI surface to expirations beyond the longest expiration in the data set.
- The choice

$$\psi_t = \left.\frac{\partial \sigma_{\mathrm{BS}}(k,t)}{\partial k}\right|_{k=0}$$

  of volatility skew as the skew measure rather than variance skew for example, reflects the empirical observation that volatility is roughly lognormally distributed.
- Since both features are roughly consistent with empirical observation, we expect (and see) greater parameter stability over time.
  - Traders can keep parameters in their heads.

SVI slices may cross at no more than four points



Condition for no calendar spread arbitrage

Lemma 3.1

Two raw SVI slices admit no calendar spread arbitrage if a certain quartic polynomial has no real root.

Ferrari Cardano

The idea is as follows:

- Two total variance slices cross if

$$\begin{aligned} & a_1 + b_1 \left\{ \rho_1 (k - m_1) + \sqrt{(k - m_1)^2 + \sigma_1^2} \right\} \\ = \quad & a_2 + b_2 \left\{ \rho_2 (k - m_2) + \sqrt{(k - m_2)^2 + \sigma_2^2} \right\} \end{aligned}$$

- Rearranging and squaring gives a quartic polynomial equation of the form

$$\alpha_4 \, k^4 + \alpha_3 \, k^3 + \alpha_2 \, k^2 + \alpha_1 \, k + \alpha_0 = 0,$$

  where each of the coefficients are lengthy yet explicit expressions in terms of the raw SVI parameters.

    - If this quartic polynomial has no real root, then the slices do not intersect.

SVI butterfly arbitrage

Recall the definition:

$$g(k) := \left(1 - \frac{kw'(k)}{2w(k)}\right)^2 - \frac{w'(k)^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2}.$$

- The highly nonlinear behavior of g makes it seemingly impossible to find general conditions on the parameters that would eliminate butterfly arbitrage.
- We now provide an example where butterfly arbitrage is violated.

Axel Vogt post on Wilmott.com

AVt
Senior Member

Posts: 971
Joined: Dec 2001

Thu Apr 06, 06 08:37 PM

It works for observables and far beyond for extrapolation.

But for a (theoretical) experiment try the following data

a = -.40998372001772e-1,
b = .13308181151379,
m = .35858898335748,
rho = .30602086142471,
sigma = .41531878803777

The Vogt smile

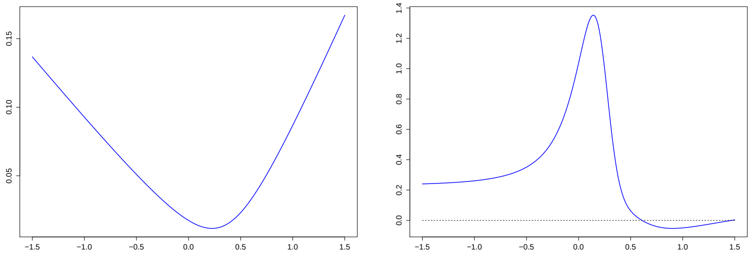


Figure 1: Plots of the total variance smile w (left) and the function g (right), using Axel Vogt's parameters

Surface SVI

Consider now the following extension of the natural SVI parameterization:

Surface SVI (SSVI) parameterization

$$w(k,\theta_t) = \frac{\theta_t}{2}\left\{1 + \rho\varphi(\theta_t)k + \sqrt{(\varphi(\theta_t)k+\rho)^2 + (1-\rho^2)}\right\} \quad (1)$$

with $\theta_t > 0$ for $t > 0$, and where φ is a smooth function from $(0,\infty)$ to $(0,\infty)$ such that the limit $\lim_{t\to 0}\theta_t\varphi(\theta_t)$ exists in $\mathbb{R}$.

Interpretation of SSVI

- This representation amounts to considering the volatility surface in terms of ATM variance time, instead of standard calendar time.
- The ATM total variance is $\theta_t = \sigma_{\mathrm{BS}}^2(0,t)\,t$ and the ATM volatility skew is given by

$$\partial_k \sigma_{\mathrm{BS}}(k,t)|_{k=0} = \frac{1}{2\sqrt{\theta_t t}} \partial_k w(k,\theta_t)\Big|_{k=0} = \frac{\rho\sqrt{\theta_t}}{2\sqrt{t}}\varphi(\theta_t).$$

- The smile is symmetric around at-the-money if and only if $\rho = 0$, a well-known property of stochastic volatility models.

Conditions on SSVI for no calendar spread arbitrage

Theorem 4.1

The SSVI surface (1) is free of calendar spread arbitrage if and only if

1. $\partial_t \theta_t \geq 0$, *for all* $t \geq 0$;
2. $0 \leq \partial_\theta(\theta\varphi(\theta)) \leq \frac{1}{\rho^2}\left(1+\sqrt{1-\rho^2}\right)\varphi(\theta)$, *for all* $\theta > 0$,

where the upper bound is infinite when $\rho = 0$.

- In particular, SSVI is free of calendar spread arbitrage if:
  - the skew in total variance terms is monotonically increasing in trading time and
  - the skew in implied variance terms is monotonically decreasing in trading time.
- In practice, any reasonable skew term structure that a trader defines will have these properties.

Conditions on SSVI for no butterfly arbitrage

Theorem 4.2

The volatility surface (1) is free of butterfly arbitrage if the following conditions are satisfied for all $\theta > 0$:

1. $\theta\varphi(\theta)\left(1+|\rho|\right) < 4$;
2. $\theta\varphi(\theta)^2\left(1+|\rho|\right) \leq 4$.

Remark

Condition 1 needs to be a strict inequality so that $\lim_{k\to+\infty} d_+(k) = -\infty$ *and the SVI density integrates to one.*

Are these conditions necessary?

Lemma 4.2

The volatility surface (1) is free of butterfly arbitrage only if

$$\theta\varphi(\theta)\left(1+|\rho|\right) \leq 4, \quad \textit{for all } \theta > 0.$$

Moreover, if $\theta\varphi(\theta)\left(1+|\rho|\right) = 4$, *the surface (1) is free of butterfly arbitrage only if*

$$\theta\varphi(\theta)^2\left(1+|\rho|\right) \leq 4.$$

So the theorem is almost if-and-only-if.

No butterfly arbitrage in terms of SVI-JW parameters

A volatility smile of the form (1) is free of butterfly arbitrage if

$$\sqrt{v_t\, t}\max\left(p_t, c_t\right) < 4, \quad \text{and} \quad \left(p_t + c_t\right)\max\left(p_t, c_t\right) \leq 8,$$

hold for all $t > 0$.

The Roger Lee arbitrage bounds

- The asymptotic behavior of the surface (1) as $|k|$ tends to infinity is

$$w(k, \theta_t) = \frac{(1 \pm \rho)\,\theta_t}{2} \varphi(\theta_t)\,|k| + \mathcal{O}(1), \quad \text{for any } t > 0.$$

- Thus the condition $\theta\varphi(\theta)\,(1 + |\rho|) \leq 4$ of Theorem 4.2 corresponds to the upper bound of 2 on the asymptotic slope established by Lee [11].
  - Again, Condition 1 of the theorem is necessary.

No static arbitrage with SSVI

Corollary 4.1

The SSVI surface (1) is free of static arbitrage if the following conditions are satisfied:

1. $\partial_t \theta_t \geq 0$, *for all* $t > 0$
2. $0 \leq \partial_\theta(\theta\varphi(\theta)) \leq \frac{1}{\rho^2}\left(1 + \sqrt{1-\rho^2}\right)\varphi(\theta)$, *for all* $\theta > 0$;
3. $\theta\varphi(\theta)\left(1+|\rho|\right) < 4$, *for all* $\theta > 0$;
4. $\theta\varphi(\theta)^2\left(1+|\rho|\right) \leq 4$, *for all* $\theta > 0$.

- A large class of simple closed-form arbitrage-free volatility surfaces!

A Heston-like surface

Example 4.1

The function φ defined as

$$\varphi(\theta) = \frac{1}{\lambda\theta}\left\{1 - \frac{1 - \mathrm{e}^{-\lambda\theta}}{\lambda\theta}\right\},$$

with $\lambda \geq (1 + |\rho|)/4$ satisfies the conditions of Corollary 4.1.

- This function is consistent with the implied variance skew in the Heston model as shown in [5] (equation 3.19).

A power-law surface

Example 4.2

The choice

$$\varphi(\theta) = \frac{\eta}{\theta^\gamma (1+\theta)^{1-\gamma}}$$

gives a surface that is completely free of static arbitrage provided that $\gamma \in (0, 1/2]$ *and* $\eta (1 + |\rho|) \leq 2$.

- This function is more consistent with the empirically-observed term structure of the volatility skew.

Even more flexibility...

Theorem 4.3

Let $(k,t) \mapsto w(k,t)$ be a volatility surface free of static arbitrage, and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ a non-negative and increasing function of time. Then the volatility surface $w_\alpha(k,\theta_t) := w(k,\theta_t) + \alpha_t$ is also free of static arbitrage.

- Corollary 4.1 gives us the freedom to match three features of one smile (level, skew, and curvature say) but only two features of all the other smiles (level and skew say), subject of course to the given smiles being themselves arbitrage-free.
- Theorem 4.3 may allow us to match an additional feature of each smile through α_t.

How to eliminate butterfly arbitrage

- We have shown how to define a volatility smile (SSVI) that is free of butterfly arbitrage.
- This smile is completely defined given three observables.
  - The ATM volatility and ATM skew are obvious choices for two of them.
  - The most obvious choice for the third observable in equity markets would be the asymptotic slope for k negative and in FX markets and interest rate markets, perhaps the ATM curvature of the smile might be more appropriate.

How to fix butterfly arbitrage

- Supposing we choose to fix the SVI-JW parameters v_t, ψ_t and p_t of a given SVI smile, we may guarantee a smile with no butterfly arbitrage by choosing the remaining parameters c'_t and $\widetilde{v}'_t$ according to SSVI as

$$c'_t = p_t + 2\,\psi_t, \quad \text{and} \quad \widetilde{v}'_t = v_t \, \frac{4 p_t c'_t}{(p_t + c'_t)^2}.$$

- That is, given a smile defined in terms of its SVI-JW parameters, we are guaranteed to be able to eliminate butterfly arbitrage by changing the call wing c_t and the minimum variance $\widetilde{v}_t$, both parameters that are hard to calibrate with available quotes in equity options markets.

Example: Fixing the Vogt smile

- The SVI-JW parameters corresponding to the Vogt smile are:

$$\begin{aligned} & (v_t, \psi_t, p_t, c_t, \widetilde{v}_t) \\ = \; & (0.01742625, -0.1752111, 0.6997381, 1.316798, 0.0116249)\,. \end{aligned}$$

- We know then that choosing $(c_t, \widetilde{v}_t) = (0.3493158, 0.01548182)$ must give a smile free of butterfly arbitrage.
- There must exist some pair of parameters $\{c_t, \widetilde{v}_t\}$ with $c_t \in (0.349, 1.317)$ and $\widetilde{v}_t \in (0.0116, 0.0155)$ such that the new smile is free of butterfly arbitrage and is as close as possible to the original one in some sense.

Numerical optimization

- In this particular case, choosing the objective function as the sum of squared option price differences plus a large penalty for butterfly arbitrage, we arrive at the following "optimal" choices of the call wing and minimum variance parameters that still ensure no butterfly arbitrage:

$$(c_t, \widetilde{v}_t) = (0.8564763, 0.0116249)\,.$$

- Note that the optimizer has left $\widetilde{v}_t$ unchanged but has decreased the call wing.
- The resulting smiles and plots of the function g are shown in Figure 2.

The Vogt smile fixed

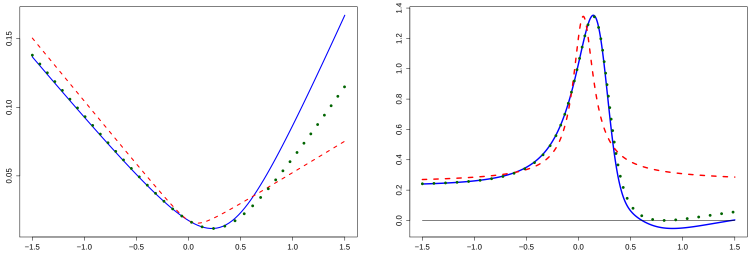


Figure 2: Plots of the total variance smile (left) and the function g (right). The graphs corresponding to the original Axel Vogt parameters is solid, to the guaranteed butterfly-arbitrage-free parameters dashed, and to the "optimal" choice of parameters dotted.

Why extra flexibility may not help

- The additional flexibility potentially afforded to us through the parameter α_t of Theorem 4.3 sadly does not help us with the Vogt smile.
- For α_t to help, we must have $\alpha_t > 0$; it is straightforward to verify that this translates to the condition $v_t\left(1-\rho^2\right) < \tilde{v}_t$ which is violated in the Vogt case.

Quantifying lines crossing

- Consider two SVI slices with parameters χ_1 and χ_2 where $t_2 > t_1$.
- We first compute the points k_i $(i = 1, \ldots, n)$ with $n \leq 4$ at which the slices cross, sorting them in increasing order. If $n > 0$, we define the points $\widetilde{k}_i$ as

$$\begin{aligned}
\widetilde{k}_1 &:= k_1 - 1, \\
\widetilde{k}_i &:= \frac{1}{2}\left(k_{i-1} + k_i\right), \quad \text{if } 2 \leq i \leq n, \\
\widetilde{k}_{n+1} &:= k_n + 1.
\end{aligned}$$

- For each of the $n+1$ points $\widetilde{k}_i$, we compute the amounts c_i by which the slices cross:

$$c_i = \max\left[0, w(\widetilde{k}_i, \chi_1) - w(\widetilde{k}_i, \chi_2)\right].$$

Crossedness

Definition 5.1

The crossedness *of two SVI slices is defined as the maximum of the* c_i $(i = 1, \ldots, n)$. *If* $n = 0$, *the crossedness is null.*

A sample calibration recipe

Calibration recipe

- Given mid implied volatilities $\sigma_{ij} = \sigma_{\mathrm{BS}}(k_i, t_j)$, compute mid option prices using the Black-Scholes formula.
- Fit the square-root SVI surface by minimizing sum of squared distances between the fitted prices and the mid option prices. This is now the initial guess.
- Starting with the square-root SVI initial guess, change SVI parameters slice-by slice so as to minimize the sum of squared distances between the fitted prices and the mid option prices with a big penalty for crossing either the previous slice or the next slice (as quantified by the crossedness from Definition 5.1).

Interpolation

Lemma 5.1

Given two volatility smiles $w(k, t_1)$ and $w(k, t_2)$ with $t_1 < t_2$ where the two smiles are free of butterfly arbitrage and such that $w(k, \tau_2) \geq w(k, \tau_1)$ for all k, there exists an interpolation such that the interpolated volatility surface is free of static arbitrage for $t_1 < t < t_2$.

- For example;

$$\frac{C_t}{K_t} = \alpha_t \frac{C_1}{K_1} + (1 - \alpha_t) \frac{C_2}{K_2},$$

where for any $t \in (t_1, t_2)$, we define

$$\alpha_t := \frac{\sqrt{\theta_{t_2}} - \sqrt{\theta_t}}{\sqrt{\theta_{t_2}} - \sqrt{\theta_{t_1}}} \in [0, 1].$$

works.

A possible choice of extrapolation

- At time $t_0 = 0$, the value of a call option is just the intrinsic value.
- Then we can interpolate between t_0 and t_1 using the above algorithm, guaranteeing no static arbitrage.
- For extrapolation beyond the final slice, first recalibrate the final slice using the simple SVI form (1).
- Then fix a monotonic increasing extrapolation of θ_t and extrapolate the smile for $t > t_n$ according to

$$w(k, \theta_t) = w(k, \theta_{t_n}) + \theta_t - \theta_{t_n},$$

which is free of static arbitrage if $w(k, \theta_{t_n})$ is free of butterfly arbitrage by Theorem 4.3.

SVI square-root calibration

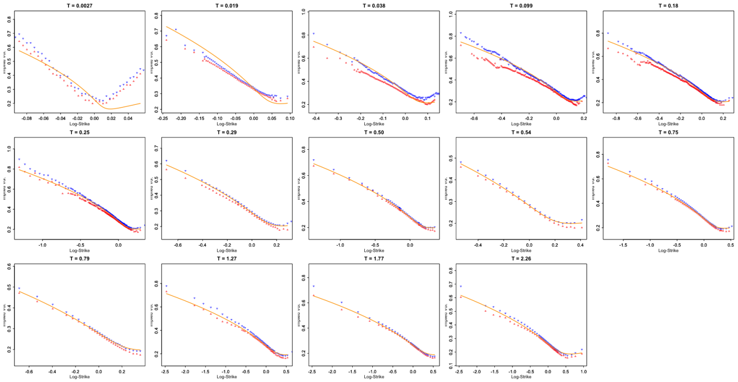


Figure 3: SPX option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the square-root SVI fit

SVI square-root calibration: December 2011 detail

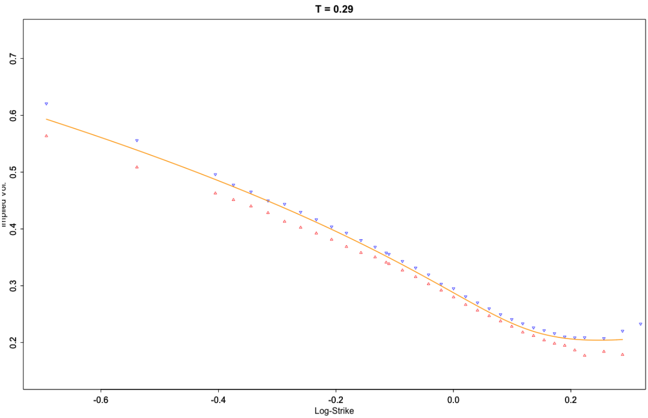


Figure 4: SPX Dec-2011 option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the square-root SVI fit

Full SVI calibration

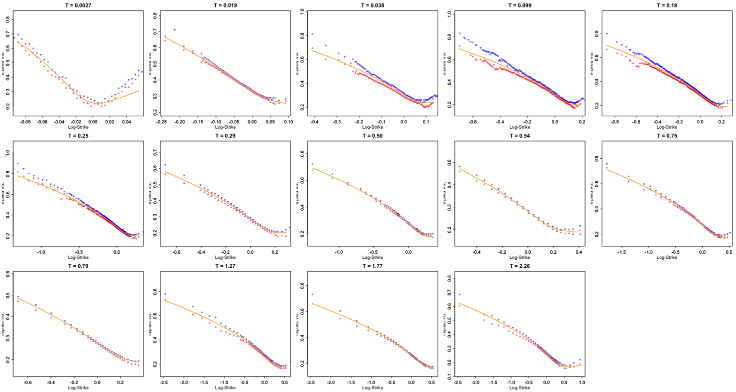


Figure 5: SPX option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the SVI fit

Full SVI calibration: March 2012 detail

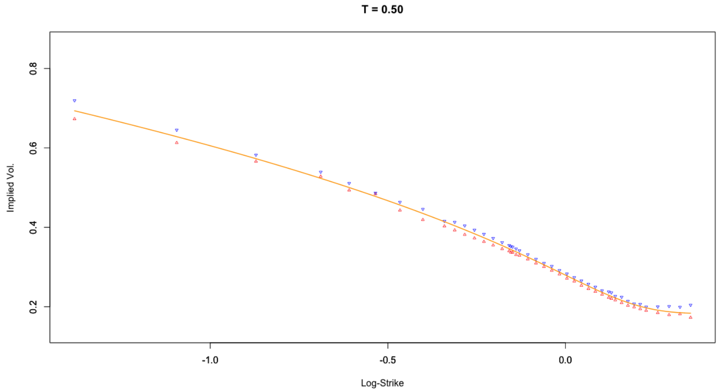


Figure 6: SPX Mar-2012 option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the SVI fit

SVI-SABR

- Consider the (lognormal) SABR formula with $\beta = 1$:

$$\sigma_{BS}(k) = \alpha\, f\left(\frac{k}{\alpha}\right)$$

with

$$f(y) = -\frac{\nu\, y}{\log\left(\frac{\sqrt{\nu^2\, y^2 + 2\,\rho\,\nu\, y + 1} - \nu\, y - \rho}{1-\rho}\right)}. \tag{2}$$

- Compare this with the simpler SVI-SABR formula:

$$\sigma^2_{\mathrm{BS}}(k) = \frac{\alpha^2}{2}\left\{1 + \rho\,\frac{\nu}{\alpha}\,k + \sqrt{\left(\frac{\nu}{\alpha}\,k + \rho\right)^2 + (1-\rho^2)}\right\} \tag{3}$$

which is guaranteed free of butterfly arbitrage if $\alpha\,\nu\,(1+|\rho|) < 4$ and $\nu^2\,(1+|\rho|) < 4$.

Butterfly arbitrage

- It is well known that the SABR volatility smile is susceptible to butterfly arbitrage.
    - The corresponding density is often negative for extreme strikes.
- On the other hand, the SVI-SABR density is guaranteed positive so long as $\alpha\,\nu\,t\,(1+|\rho|) < 4$ and $\nu^2\,t\,(1+|\rho|) < 4$.
    - Typical values of these parameters for SPX are $\nu^2\,t = 0.6$, $\alpha = 0.2$, $\rho = -0.7$ so for SPX there is empirically no butterfly arbitrage.
    - SABR and SVI-SABR fit parameters are not identical but they are similar.

An example: March 2012 again

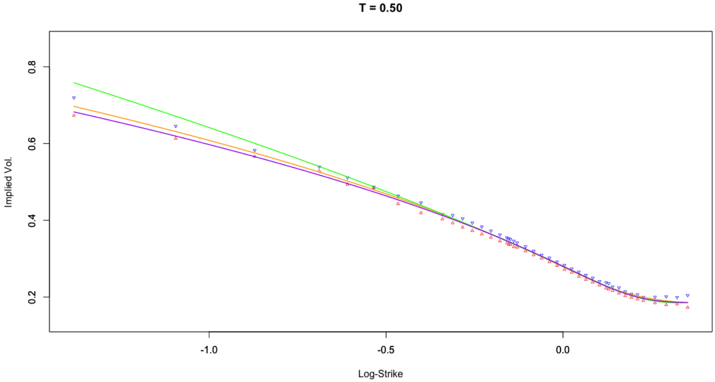


Figure 7: SPX Mar-2012 option quotes as of 3pm on 15-Sep-2011. Red and blue triangles are bid and ask implied volatilities; the orange solid line is the SVI fit, the green line the SABR fit, the purple line the SVI-SABR fit

Plots of $g(k)$

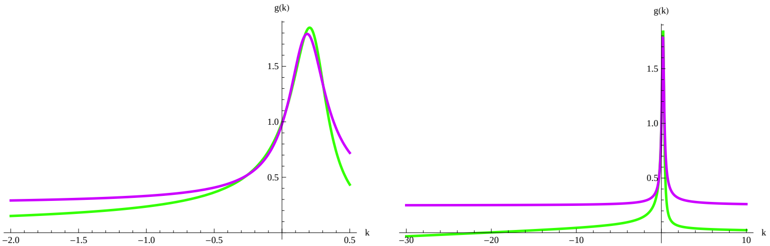


Figure 8: $g(k)$ for the SABR fit is in green, $g(k)$ for the SVI-SABR fit in purple. The negative SABR density is clearly visible in the extreme left wing.

- We note that around at-the-money, the two densities are very similar. However, as the strike moves away from ATM, the densities diverge and the SABR density goes negative.

Summary

- We have found and described a large class of arbitrage-free SVI volatility surfaces with a simple closed-form representation.
- Taking advantage of the existence of such surfaces, we showed how to eliminate both calendar spread and butterfly arbitrages when calibrating SVI to implied volatility data.
- We further demonstrated the high quality of typical SVI fits with a numerical example using recent SPX options data.
- Finally, we showed how a guaranteed arbitrage-free simple SVI smile could potentially replace SABR in applications.

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