MATH 20C Notes - Week Three

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Introduction

Deep

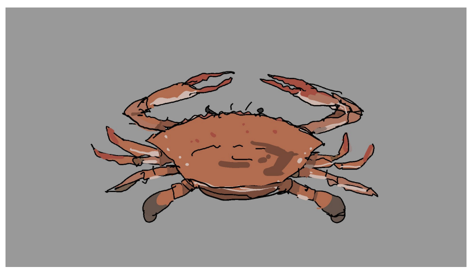

Figure 1: A crab

1 Limits and Continuity

In 1D, $f : \mathbb{R} \to \mathbb{R}$, and

$$\lim_{\vec{v} \to \vec{a}} \vec{f}(\vec{v}) = \vec{b}$$

if $\vec{f}(\vec{v})$ gets arbitrarily close to $\vec{b}$ as $\vec{v}$ approaches $\vec{a}$ (close means the distance $||\vec{f}(\vec{v}) - \vec{b}||$ is small).

1.1 Precise Definition

$$\text{Say } \lim_{\vec{v} \to \vec{a}} \vec{f}(\vec{v}) = \vec{b}$$

if for all $\epsilon \leq 0$ there exists $\delta \geq 0$ such that whenever $||\vec{v} - \vec{a}|| \leq \delta$, then $||\vec{f}(\vec{v}) - \vec{b}|| \leq \epsilon$

1.2 Properties of Limits

Suppose $\lim_{\vec{v}\to\vec{a}} \vec{f}(\vec{v}) = \vec{b}$ and $\lim_{\vec{v}\to\vec{a}} \vec{g}(\vec{v}) = \vec{c}$ and $f : \mathbb{R}^n \to \mathbb{R}^m$

1. $\lim_{\vec{v}\to\vec{a}} \vec{f}(\vec{v}) + \vec{g}(\vec{v}) = \vec{b} + \vec{c}$
2. If λ is a real number, then $\lim_{\vec{v}\to\lambda\vec{a}} \vec{f}(\vec{v}) = \lambda\vec{b}$
3. $f(x_1, \ldots, x_n) = \overbrace{(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))}^{m different real valued functions}$

$\lim_{\vec{v}\to\vec{a}} \vec{f}(\vec{v}) = \vec{b} = (b_1, \ldots, b_n)$ if and only if $\lim_{\vec{v}\to\vec{a}} \vec{f_i}(\vec{v}) = \vec{b_i}$ for all $1 \leq i \leq m$

A function is **continuous** if for any $\vec{a}$, we have $\lim_{\vec{v}\to\vec{a}} \vec{f}(\vec{v}) = \vec{f}(\vec{a})$

Nonexample

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ to be $(x, y) \to \frac{x^2}{x^2+y^2}$
What is the limit of $f(x, y)$?
To understand the question, we take sections from the function.

For $x = 0$

$$f(0, y) = \frac{0}{0 + y^2} = 0 \quad \text{When } y \neq 0$$

$$\text{So } \lim_{y\to 0} f(0, y) = 0$$

For $y = 0$

$$f(x, 0) = \frac{x^2}{x^2 + 0} = 1$$

$$\text{So } \lim_{x\to 0} f(x, 0) = 1$$

For $x = y$

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

$$\text{So } \lim_{x\to 0} f(x) = \frac{1}{2}$$

Example

$$f(x, y) = \frac{2x^2y}{x^2 + y^2}$$

x Section

$$f(0, y) = \frac{0}{y^2} = 0$$

y Section

$$f(x, 0) = \frac{0}{x^2} = 0$$

$x = y$ Section

$$\lim_{(x,y)\to(0,0)} f(x, x) = \frac{2x^2y}{x^2 + y^2} = 0$$

1.3 Properties of Continuous Functions

- Polynomial functions are continuous

$$f(x, y, z) = (x^3 + xyz + z^3)$$

- Compositions of two continuous functions are continuous

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

$$g : \mathbb{R}^m \to \mathbb{R}^p$$

if f and g are continuous, then $g \cdot f$ is continuous.

- Sums of $\overbrace{\text{continuous functions}}^{\text{or scalar multiples}}$ are continuous
- $f : \mathbb{R}^m \to \mathbb{R}^n$

$$(x_1, \ldots, x_m) \to f(x_1, \ldots, x_m) = (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m))$$

f is continuous if and only if all the f_is are continuous.

$$f : \mathbb{R} \to \mathbb{R}^2$$

$$t \to (\overbrace{t}^{f_1(t)}, \overbrace{t^2}^{f_2(t)})$$

2 Differentiation

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

$$\frac{\partial f}{\partial x}\mathbb{R}^3 \to \mathbb{R}$$

What is the meaning of the derivative?

2.1 Partial Derivative

$\frac{\partial f}{\partial x}$ is the partial derivative with respect to x.

$$f : \mathbb{R}^3 \to \mathbb{R}^1 \quad \text{for } (x, y, z)$$

$$\frac{\partial f}{\partial x}(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

$$\frac{\partial f}{\partial y}(x, y, z) = \lim_{h \to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$$

$$\frac{\partial f}{\partial z}(x, y, z) = \lim_{h \to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$$

2.2 Tangent Planes

As the derivative of a single variable function creates a tangent line, the derivative of a surface creates tangent planes.

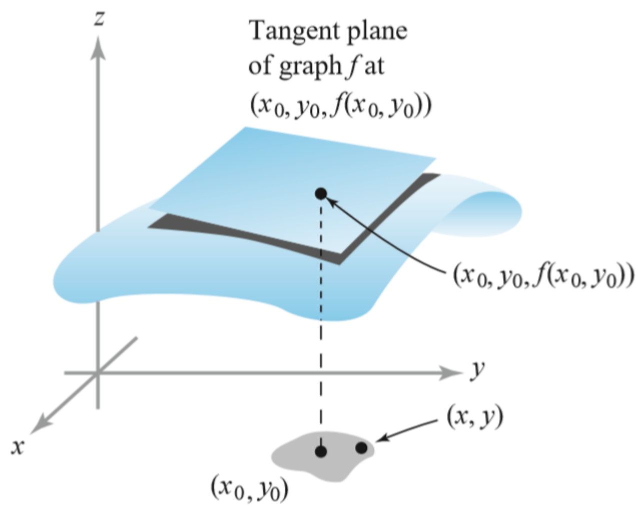

Figure 2:

What is the relation between $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ and the tangent plane?

The graph of f has two tangent vectors.

$$(1, 0, \frac{\partial g}{\partial x}, (x_o, y_o))$$

$$(0, 1, \frac{\partial g}{\partial y}, (x_o, y_o))$$

2.3 Equation for the Tangent Plane

The Normal Vector

$$\bar{n} = v_1 \times v_2 = \overbrace{(-\frac{\partial g}{\partial x}(x_o, y_o), -\frac{\partial g}{\partial y}(x_o, y_o), 1)}^{\text{recovers the "slope" of the tangent plane}}$$

Looking at the section of $z = g$

$$\frac{\partial g}{\partial x}(x_o, y_o) = \lim_{h \to 0} \frac{g(x_o + h, y_o) - g(x_o, y_o)}{h}$$

$$\bar{n} \cdot (x - x_o, y - y_o, z - f(x_o, y_o)) = 0$$

$$z = g(x, y) + \frac{\partial g}{\partial x}(x - x_o) + \frac{\partial g}{\partial y}(y - y_o)$$

Also called the linear approximation to g

Example

Compare the linear approximation to $z = x^2 + y^4$ at $(x_o, y_o) = 1, 0$

$$\frac{\partial z}{\partial x} = 2x + 0 = 2x \qquad \frac{\partial z}{\partial y} = 4y^3$$

$$\frac{\partial z}{\partial x}(1, 0) = 2 \cdot 1 = 2 \qquad \frac{\partial z}{\partial y}(1, 0) = 0$$

$$\bar{n} = (-2, 0, 1)$$

$$z = g(x_o, y_o) + 2 \cdot (x - 1) + 0(y - 0) = 1 + 2(x - 1 = 2x - 1)$$

Exercise

Find the equation for the tangent plane to $z = x^2 - y^3 x$ at $(1, 1)$

$$\frac{\partial z}{\partial x} = 2x - y^3 \qquad \frac{\partial z}{\partial y} = -3y^2 x$$

$$\frac{\partial z}{\partial x}(1, 1) = 2 - 1 = 1 \qquad \frac{\partial z}{\partial y}(1, 1) = -3$$

$$\bar{n} = (-1, 3, 0)$$

$$z = g(x_o, y_o) + 1 \cdot (x - 1) + (-3)(y - 1) = 0 + x - 1 - 3 + 3 = x - 3y + 2$$

$$z = x - 3y + 2$$

3 The Derivative of Multi-Variable Functions

The derivative of $\vec{f}$ denoted by $D\vec{f}$ is the matrix

$$D\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{v}) & \frac{\partial f_1}{\partial x_2}(\vec{v}) & \dots & \frac{\partial f_1}{\partial x_m}(\vec{v}) \\ \frac{\partial f_2}{\partial x_1}(\vec{v}) & \frac{\partial f_2}{\partial x_2}(\vec{v}) & \dots & \frac{\partial f_2}{\partial x_m}(\vec{v}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{v}) & \frac{\partial f_n}{\partial x_2}(\vec{v}) & \dots & \frac{\partial f_n}{\partial x_m}(\vec{v}) \end{bmatrix}$$

Example

$$\vec{f}(x,y) = (\cos(xy), e^{xy})$$

$$\text{Domain: } \mathbb{R}^2 \text{ Co-Domain: } \mathbb{R}^2$$

$$\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$$

Compute $D\vec{f}(1,1)$

$$D\vec{f}(1,1) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(1,1) & \frac{\partial f_1}{\partial y}(1,1) \\ \frac{\partial f_2}{\partial x}(1,1) & \frac{\partial f_2}{\partial y}(1,1) \end{bmatrix}$$

$$\begin{aligned} f_1(x,y) &= \cos(xy) & \frac{\partial \vec{f_1}}{\partial x} &= -y\sin(xy) & \frac{\partial \vec{f_2}}{\partial y} &= ye^{x} \\ f_2(x,y) &= e^{xy} & \frac{\partial f_1}{\partial y} &= -x\sin(xy) & \frac{\partial f_2}{\partial y} &= xe^{xy} \end{aligned}$$

$$D\vec{f}(1,1) = \begin{bmatrix} -\sin(1\cdot 1) & -\sin(1\cdot 1) \\ e^{1\cdot 1} & e^{1\cdot 1} \end{bmatrix} = \begin{bmatrix} -\sin(1) & -\sin(1) \\ e & e \end{bmatrix}$$

Exercise

$\vec{f}(t) = (1,2,3) + t(1,1,1)$

Compute $D\vec{f}(0)$
Domain: $\mathbb{R}$ and Co-domain: $\mathbb{R}^3$

$$\vec{f}(t) = (1,2,3) + t(1,1,1)$$

$$\vec{f}(t) = (\overbrace{1+t}^{f_1(t)}\overbrace{2+t,}^{f_2(t)}\overbrace{3+t}^{f_3(t)})$$

$$\frac{\partial f_1}{\partial t} = 1 \quad \frac{\partial f_2}{\partial t} = 1 \quad \frac{\partial f_3}{\partial t} = 1$$

$$D\vec{f} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3.1 Definition

$$f : \mathbb{R}^n \to \mathbb{R}$$

Df is also denoted ∇f and called the gradient of f.

If $f : \mathbb{R}^2 \to \mathbb{R}$

$$\overbrace{z = f(x_o, y_o) + \frac{\partial f}{\partial x}(x_o, y_o)(x - x_o) + \frac{\partial f}{\partial y}(x_o, y_o)(y - y_o)}^{\text{Linear Approximation}}$$

$$\nabla f(x_o, y_o) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_o, y_o) & \frac{\partial f}{\partial y}(x_o, y_o) \end{bmatrix}$$

$$\overbrace{z = f(x_o, y_o) + \nabla f(x_o, y_o) \cdot (x - x_o, y - y_o)}^{\text{The same as linear approximation}}$$