MATH 20C Notes - Week Five

C-Rin

October 2019

Introduction

Deep

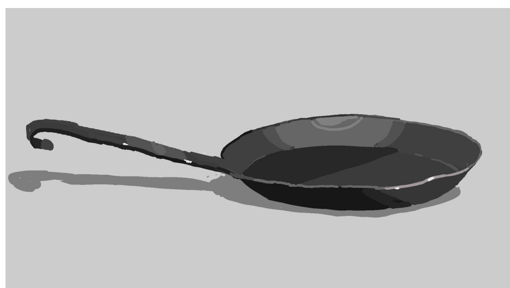

Figure 1: A pan

1 Gradients and Directional Derivatives

With $f : \mathbb{R}^3 \to \mathbb{R}$, the gradient of f is the derivative, thought of as a vector.

$$\nabla f(x,y,z) = (\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(z,y,z))$$

Example

$$f(x,y,z) = xy - z^2$$
$$\nabla f(x,y,z) = (y, z, -2z)$$
$$\nabla f(1,1,0) = (0,1,0)$$

∇f gives us a lot of information we can use.
Definition: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar valued differentiable funciton, and let $\vec{v}, \vec{w}$ be in $\mathbb{R}^3$

The directional derivative of f at $\vec{v}$ in the direciton of $\vec{w}$ be

$$i < \frac{\partial f}{\partial t}(\vec{v} + t\vec{w})(0)$$

$\vec{v} + t\vec{w}$ is the line through $\vec{v}$ in the $\vec{w}$ direction.

Theorem

$$\frac{\partial f}{\partial t}(\vec{v} + t\vec{w})(0) = \overbrace{\nabla f(\vec{v})) \cdot \vec{w}}^{\text{the dot product}}$$

This is the application of the chain rule.

Example

$$f(x,y,z) = x^2 e^{-yz}$$

Compute the directional derivative @ $(1,0,0)$ in the direction of $(1,1,1)$

$$\nabla f(x,y,z) = (2xe^{-yz}, -zx^2e^{-yz}, -yx^2e^{-yz})$$
$$\nabla f(1,0,0) = (2 \cdot 1e^0, 0, 0)$$
$$\nabla f(1,0,0) \cdot (1,1,1) = (2,0,0) \cdot (1,1,1) = 2$$

Remark

What does this mean?

The directional derivative approximates ∇f if you move from $(1,0,0) + (1,1,1) = (2,1,1)$

Exercise

$$g(x, y, z) = x^2 y + z\cos(2\pi x)$$

Compute the directional derivative of g @ $(1, 2, 3)$ in the direction of $(-1, 0, 1)$

$$\nabla g(x, y, z) = (2xy + (-2\pi z\sin(2\pi x)), x^2 + 0, \cos(2\pi x))$$
$$\nabla g(1, 2, 3) = (4 + (-2\pi(3)\sin(2\pi)), 1, \cos(2\pi))$$
$$= (4 - 6\pi(0), 1, 0) = (-4 + 0 + 0 + 1) = 3$$

2 Geometric Interpretation of the Gradient

$$f : \mathbb{R}^3 \to \mathbb{R} \text{ is differentiable and scalar valued}$$
$$\nabla f(a, b, c) = \text{ a 3D vector}$$

$\nabla f(a, b, c)$ is a normal vector to the tangent plane at a level set.

Theorem

If $\nabla f(a, b, c) \neq (0, 0, 0)$, then the gradient is normal to the tangent plane of the level set of f @ (a, b, c)
Consequently, the equation for the tangent plane of the level set at (a, b, c) is

$$\nabla f(a, b, c) \cdot ((x, y, z) - (a, b, c)) = 0$$

Example

Consider the ellipsoid $9x^2+2y^2+2z^2 = 25$. What is the equation for the tangent plane at $(1, 2, 2)$?

$f(x, y, z) = 9x^2 + 2y^2 + 2z^2 = 25$ at level 25.

$$\nabla f(x, y, z) = (18x, 4y, 4z)$$

$$\nabla f(1, 2, 2) = (18, 8, 8)$$

$$(18, 8, 8) \cdot ((x, y, z) - (1, 2, 2) = 0)$$
$$18(x - 1) + 8(y - 2) + 8(z - 2) = 0$$

Theorem

If f is differentiable and $\nabla f(a, b, c) \neq 0$, then $\nabla f(a, b, c)$ is a normal vector to the tangent plane to the level set.

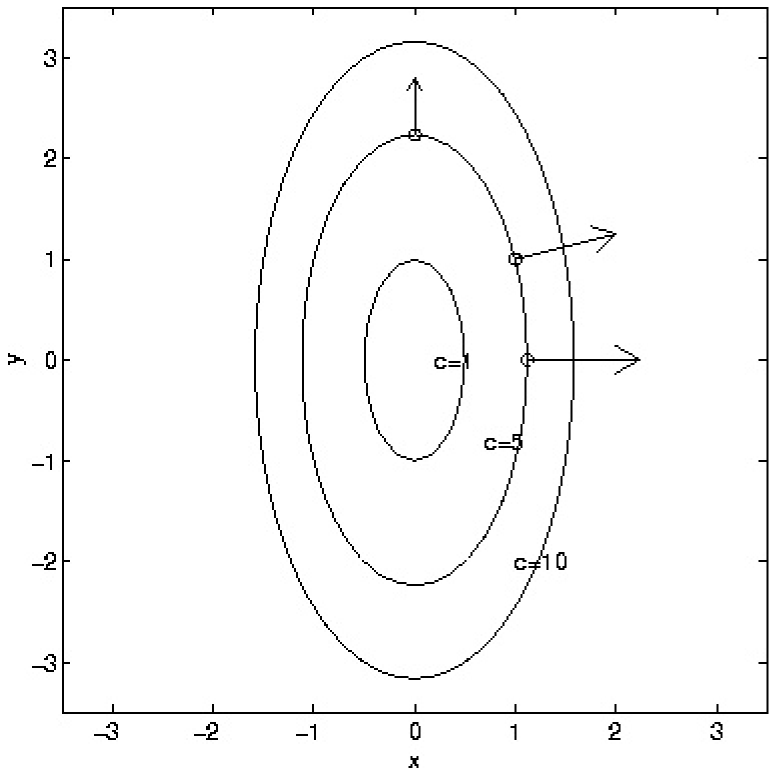

Figure 2: A vector field

The Equation for Plane P

$$\nabla f(a, b, c) \cdot ((x, y, z) - (a, b, c)) = 0$$

We can "visualize" the gradient as a vector at each point, creating a "vector field" on the graph.

Theorem

The gradient points in the direction where the function is increasing the greatest. $-\nabla f(a, b)$ is the direction where the function is decreasing the greatest.

3 Iterated Partial Derivatives

Definition

Let $f : \mathbb{R}^3 \to \mathbb{R}$
We say f is twice continuously differentiable, or c^2

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \text{ are all differentiable}$$

and

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial \frac{\partial f}{\partial x}}{\partial x}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial z \partial x}$$

are all differentiable.

Theorem

If $f : \mathbb{R}^3 \to \mathbb{R}$ is c^2, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ (Also true for } \frac{\partial^2 f}{\partial x \partial z} \text{ and } \frac{\partial^2 f}{\partial y \partial z}$$

Example

$$f(x, y) = \cos(xy^2)$$

$$\frac{\partial f}{\partial x} = -y^2 \sin(xy^2) \qquad \frac{\partial f}{\partial y} = -2xy \sin(xy^2)$$

$$\frac{\partial^2 f}{\partial x^2} = -y^4 \cos(xy^2) \qquad \frac{\partial^2 f}{\partial y^2} = -4x^2y^2 \cos(xy^2)$$

$$\frac{\partial^2 f}{\partial y \partial x} = -2y \sin(xy^2) - 2xy^3 \cos(xy^2) = \frac{\partial^2 f}{\partial x \partial y} = -2y \sin(xy^2) - 2xy^3 \cos(xy^2)$$

Exercise

Compute all the 2nd derivatives

1) $f(x, y) = \log(x + 2y)$

$$\frac{\partial f}{\partial x} = \frac{1}{x + 2y} \qquad \frac{\partial f}{\partial y} = \frac{2}{x + 2y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{x + 2y(0) - (x + 2y)}{(x + 2y)^2} = \frac{-1}{x + 2y}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2}{(x + 2y)^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(x + 2y)^{-1} = \frac{-2}{(x + 2y)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \frac{2}{x + 2y} = \frac{-2}{(x + 2y)^2}$$

2) $g(x,y) = x^3y + 2y^2$

$$\frac{\partial f}{\partial x} = 3x^2y \qquad \frac{\partial f}{\partial y} = x^3 + 4y$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy \qquad \frac{\partial^2 f}{\partial y^2} = 4$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2y) = 3x^2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 3x^2$$

4 Subscript Notation

$$\frac{\partial f}{\partial x} = f_x \qquad \frac{\partial^2 f}{\partial x^2} = f_{xx} = f_{x^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy} \qquad \frac{\partial^2 f}{\partial y^2} = f_y^2$$