

MATH 20C Notes - Week Four

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Introduction

Deep

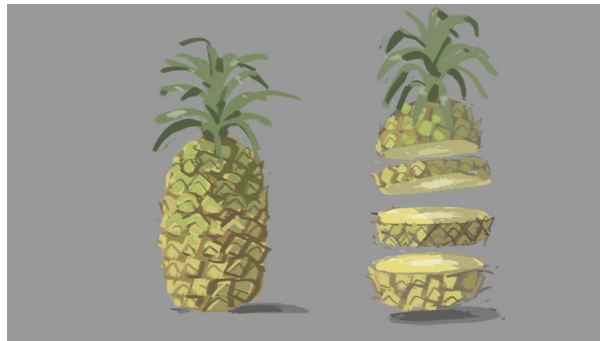


Figure 1: A pineapple and a cross section of a pineapple

1 Paths and Curves

$$\bar{f} : \mathbb{R} \rightarrow \mathbb{R}^n$$

If the domain of f is the real numbers of some interval $[a, b]$, we say f is a path
The range is called a curve.

Example

The parameterization of a line $\bar{f}(t) = (1, 1, 0) + t(2, 1, 1)$

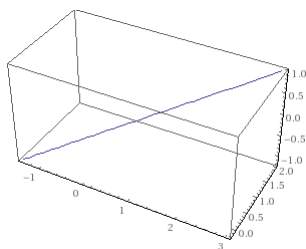


Figure 2: The range is the line

Example

$$\begin{aligned} \bar{f}(t) : [0, 2\pi] &\rightarrow \mathbb{R} \\ t &\rightarrow \bar{f}(t) = (\cos(t), \sin(t)) \end{aligned}$$

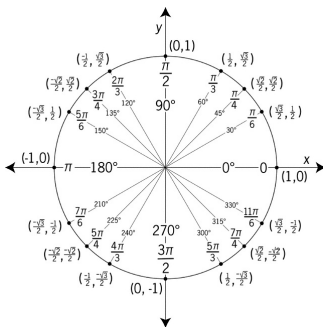


Figure 3: Curve is the Unit Circle

Example: Wheel with Ant

In this example, we have wheel moving at $1m/s$ with a radius of 1 meter, and an ant on the perimeter of the wheel. We want to parameterize the ant's position at time t .

Let $\text{ant}(t)$ denote the location of the ant.

The location of the center is $(0, 1)$ at $t = 0$, and the location of the center is $(t, 1)$

At the time t , we need the angle of the ant $\theta(t)$

$$\text{Starting angle} = \frac{-\pi}{2}$$

At angle θ , the wheel has traveled $\theta = t$

$$\theta_{\text{ant}}(t) = -\frac{\pi}{2} - t$$

$$\text{ant}(t) = (t, 1) + (\cos(\frac{-\pi}{2} - t), \sin(\frac{-\pi}{2} - t))$$

Given a path $\bar{f}(t) : \mathbb{R} \rightarrow \mathbb{R}^3$

The velocity of the path at time t is $D\bar{f}(t)$

The speed of the path at time t is $\|D\bar{f}(t)\|$

The velocity is the direction of the tangent line of the path. If velocity $\neq 0$, then the velocity points in the direction of the curve. The tangent line of time t_o is

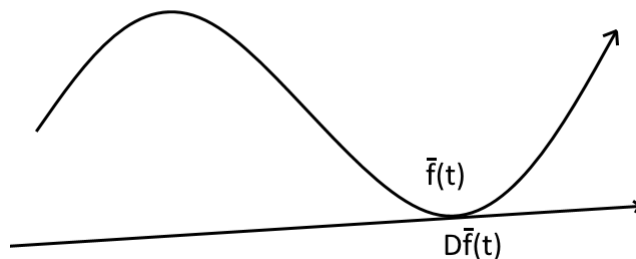


Figure 4: Can parameterize l

parameterized by $l(t) = f(t_o) + t \cdot (D\bar{f}(t_o))$

Example: The Unit Circle

$$\bar{f}(t) = (t, \cos(t), \sin(t))$$

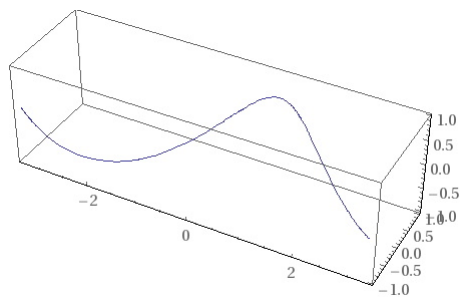


Figure 5: The Spiral

We want to find the equation for the tangent line at $t = 0$. To do this, we need to find the direction (i.e. $D\bar{f}(0)$).

$$D\bar{f}(0) = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -\sin(t) \\ \cos(t) \end{bmatrix} = \overbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}^{\text{Think of this like a row vector}}$$

$$l(t) = (0, 1, 0) + t(1, 0, 1)$$

Exercise

Find the speed of the path (t, t^2, e^t) at time 0.

$$\bar{f}(0) = (0, 0, 1)$$

$$D\bar{f}(0) = \begin{bmatrix} 1 \\ 2t \\ e^t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\|D\bar{f}(0)\| = \|(1, 0, 1)\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$l(t) = (0, 0, 1) + t(1, 0, 1)$$

2 Properties of the Derivative

The derivative of $D\bar{f}$ is an $\overbrace{n}^{\text{row}} \times \overbrace{m}^{\text{columns}}$ matrix
 $\underbrace{\bar{f} : \mathbb{R}^m \rightarrow \mathbb{R}^n}$

Theorem

- If c is a real number (constant)
 - $Dc\bar{f}(\vec{v}) = c \cdot D\bar{f}(\vec{v})$
 - Every entry in the matrix is multiplied by c
- Suppose $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$
 - $D(\bar{f} + \bar{g})(\vec{v}) = (D\bar{f}(\vec{v})) + (D\bar{g}(\vec{v}))$
 - Add each component individually based on their coordinate
- (Product Rule) Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are two scalar functions
 - $D(f \circ g)(\vec{v}) = f(\vec{v}) \cdot Dg(\vec{v}) + g(\vec{v}) \cdot Df(\vec{v})$

4. (Quotient Rule)

$$\bullet D\left(\frac{f}{g}\right)(\vec{v}) = \frac{g(\vec{v}) \cdot Df(\vec{v}) - f(\vec{v}) \cdot Dg(\vec{v})}{(g(\vec{v}))^2}$$

Example

$$\begin{aligned} h : \mathbb{R}^3 &\rightarrow \mathbb{R} \\ h(x, y, z) &= \frac{x^2 + y^2}{z^2 + 1} \\ Dh(x, y, z) &= \frac{(z^2 + 1)D(x^2 + y^2) - (x^2 + y^2)D(z^2 + 1)}{(z^2 + 1)^2} \\ &= \frac{(z^2 + 1) \cdot (2x, 2y, 0) - (x^2 + y^2) \cdot (0, 0, 2z)}{(z^2 + 1)^2} \\ &= \left(\frac{2x}{z^2 + 1}, \frac{2y}{z^2 + 1}, \frac{-(x^2 + y^2)(2z)}{(z^2 + 1)^2} \right) \end{aligned}$$

Exercise

$$f(x, y, z) = \sin(xy) \cos(yz)$$

Compute $\nabla f(x, y, z)$

$$\begin{aligned} f(x, y, z) &= \sin(xy) \cos(yz) \\ Df(x, y, z) &= \sin(xy) \cdot D \cos(yz) + \cos(yz) \cdot D \sin(xy) \\ &= \sin(xy)(0, -z \sin(yz), -y \sin(yz)) + \cos(yz) \cdot (y \cos(xy), x \cos(xy), 0) \\ &= (y \cos(yz) \cos(xy), x \cos(xy) \cos(yz) - z \sin(xy) \sin(yz), -y \sin(xy) \sin(yz)) \end{aligned}$$

3 Chain Rule

The derivative of composed functions is a Product

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^k \quad g : \mathbb{R}^k \rightarrow \mathbb{R}^m$$

3.1 Matrix Multiplication

$$\begin{aligned} A &= \overbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}^{2 \times 2} = \overbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 0 \end{bmatrix}}^{2 \times 3} \\ &\quad \begin{array}{c} \text{Needs} \\ \text{to be} \\ \text{the same} \end{array} \\ &\quad 2 \times [2] \quad \longleftrightarrow \quad [2] \times 3 \\ A \cdot B &= \frac{\begin{array}{ccc} (1, 2) \cdot (0, 3) & (1, 2) \cdot (1, 4) & (1, 2) \cdot (2, 0) \\ (3, 4) \cdot (0, 3) & (3, 4) \cdot (1, 4) & (3, 4) \cdot (2, 0) \end{array}}{\quad} = \begin{bmatrix} 6 & 9 & 2 \\ 12 & 19 & 6 \end{bmatrix} \end{aligned}$$

$$A = m \times k \quad B = k \times n$$

$$\underbrace{m \times n \text{ matrix}}_{A \cdot B} = \begin{bmatrix} \text{row}(1, A) \cdot \text{column}(1, B) & \dots & \text{row}(1, A) \cdot \text{column}(j, B) \\ \vdots & \ddots & \vdots \\ \text{row}(i, A) & \dots & \text{row}(i, A) \cdot \text{column}(j, B) \end{bmatrix}$$

Exercise

1)

$$A \cdot B$$

$$A = \begin{bmatrix} 1 & 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underbrace{1 \times 1}_{A \cdot B} = [13]$$

2)

$$C \cdot D$$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$C \cdot D = \begin{bmatrix} (1,1) \cdot (0,1) & (1,1) \cdot (1,0) \\ (1,2) \cdot (0,1) & (1,2) \cdot (1,0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

3.2 The Chain Rule

$$\vec{f}: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\vec{g}: \mathbb{R}^k \rightarrow \mathbb{R}^m$$

$$D(\vec{g} \circ \vec{f})(\vec{v}) = (D(\vec{g}(\vec{f}(\vec{v})))) \cdot D\vec{f}(\vec{v})$$

Example

$$\underbrace{\vec{f}: \mathbb{R} \rightarrow \mathbb{R}}_{t \rightarrow (t \cos(2\pi t), t \sin(2\pi t))} \quad \underbrace{\vec{g}: \mathbb{R}^2 \rightarrow \mathbb{R}}_{(x,y) \rightarrow x^2 + y^2}$$

We want to compute $D(g \circ f)(1)$ and $D\vec{g}(\vec{f}(1)) \cdot D\vec{f}(1)$

$$D\vec{f} = \begin{bmatrix} \cos(2\pi t) - 2\pi t \sin(2\pi t) \\ \sin(2\pi t) + 2\pi t \cos(2\pi t) \end{bmatrix}$$

$$D\vec{f}(1) = \begin{bmatrix} \cos(2\pi) - 2\pi \sin(2\pi) \\ \sin(2\pi) + 2\pi \cos(2\pi) \end{bmatrix} = \begin{bmatrix} 1 - 0 \\ 0 + 2\pi \end{bmatrix} = \begin{bmatrix} 1 \\ 2\pi \end{bmatrix}$$

$$\begin{aligned}
Dg(t) &= [2x \quad 2y] \\
D\vec{g}(f(1)) &= D\vec{g}(1,0) = [2, 0] \\
D\vec{g} \circ \vec{f}(1) &= [2 \quad 0] \begin{bmatrix} 1 \\ 2\pi \end{bmatrix} = [2]
\end{aligned}$$

Exercise

$$\vec{g}(x, y) = (x^2 + 1, y^2) \quad f(u, v) = (\vec{u} = v, u, v^2)$$

$$D(\vec{f} \circ \vec{g})(1, 1) = ?$$

$$D\vec{g} = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix} \quad \vec{g}(1, 1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad D\vec{f} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{bmatrix} \quad D\vec{f}(1, 1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$D\vec{f}(\vec{g}(1, 1)) = D\vec{f}(2, 1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}$$