MATH 20C Notes - Week One

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Introduction

This will be the beginning document for my collection of MATH 20C notes this quarter.

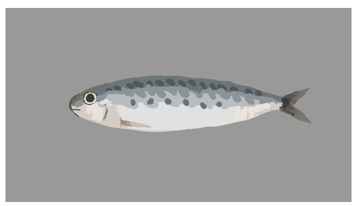


Figure 1: A Fish

1 9/27/29 Lecture

1.1 Vectors in 2D and 3D

Vectors are a quantity with both direction and magnitude that is shown in a tuple of numbers.

$$(1, 2)$$
$$(4, 5, 6)$$

1.2 Adding Vectors

We are able to add vectors coordinate-wise, adding x-values together, y-values, and so on.

$$\vec{v} = (a, b, c)$$
$$\vec{w} = (d, e, f)$$
$$\vec{v} + \vec{w} = (a + d, b + e, c + f)$$

1.3 Real Number Notation

For real numbers, $\mathbb{R}$ is used to notate real numbers and vectors, where

$$\mathbb{R} \text{ and } \mathbb{R}^1 \text{ notate the set of real numbers}$$

$$\mathbb{R}^2 \text{ notates the set of pairs of real numbers (2D Vectors)}$$

$$\mathbb{R}^3 \text{ notates the set of 3D Vectors}$$

1.4 Scaling Vectors

We can also scale vectors using a real number λ, where

$$\lambda \text{ is a real number, } \mathbb{R}$$

$$\vec{w} = (a, b)$$

$$\lambda\vec{w} = (\lambda a, \lambda b)$$

Example

Solve the equation $(-21, 23) - (x, 6) = (-25, y)$ for x and y

$$\text{For } x,$$

$$-21 - x = -25;\ x = 4$$

$$\text{For } y,$$

$$23 - 6 = y;\ y = 17$$

$$x = 4, y = 17$$

2 The Dot Product

2.1 Unit Vectors

Also known as "Standard Basis Vectors," unit vectors describe the linear combination of vectors.

$$i = (1, 0, 0)$$

$$j = (0, 1, 0)$$

$$k = (0, 0, 1)$$

$$\vec{v} = (a, b, c) = a\hat{i} + b\hat{j} + c\hat{k}$$

$\vec{v}$ is a linear combination of $\hat{i}, \hat{j}, \hat{k}$

2.2 The Inner/Dot Product

There are a few ways to multiply vectors, of which we have already discussed scalar multiplication, where we multiply a vector with a real number. Another method we can use is the **Dot Product**, also known as the inner product.

Example

The Dot Product of $\vec{v} = (a, b, c)$ and $\vec{w} = (d, e, f)$ is the real number

$$\vec{v} \cdot \vec{w} = a \cdot d + b \cdot e + c \cdot f$$

We can multiply each corresponding coordinate and add them together.

2.3 The Length of a Vector

The length of a vector can be found with

$$\vec{v} = \sqrt{a^2 + b^2 + c^2} = \sqrt{\vec{v} \cdot \vec{v}}$$

A unit vector is a vector of length, and any non-zero vector $\vec{v}$ is a scalar multiple of a unit vector.
A 2D vector with a length of 1 creates a unit circle with a radius of 1, while a 3D vector with a length of 1 creates a sphere with radius 1.

$$\vec{v} = ||\vec{v}|| \quad \overbrace{(\frac{\vec{v}}{||\vec{v}||})}^{\text{unit vector}}$$

2.4 The Geometric Interpretation of the Dot Product

Let $\vec{v}$ be a non-zero vector with a normalized vector $\vec{u}$

$$\vec{u} \cdot \vec{v} = \quad \overbrace{||\vec{v}||}^{\text{length of } \vec{v}}$$

$$\vec{v} = ||\vec{v}|| \cdot \vec{u}$$
$$\vec{u} \cdot \vec{v} = \vec{u} \cdot$$

If $\vec{v}$ and $\vec{w}$ point in the same direction, then $\vec{v} \cdot \vec{w} = ||\vec{v}|| \cdot ||\vec{w}||$

3 The Law of Cosines

$$||\vec{u} - \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 + 2||\vec{u}|| \cdot ||\vec{v}|| \cos(\theta)$$

$$2\cos(\theta) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - (\vec{u} - \vec{v})(\vec{u} - \vec{v}) = 2\vec{u} \cdot \vec{v}$$

3.1 Master Equation

For any two vectors $\vec{v}, \vec{w}$

$$\vec{v} \cdot \vec{w} = ||\vec{v}|| \cdot ||\vec{w}|| \cos(\theta)$$

The two vectors $\vec{v}, \vec{w}$ are perpendicular if and only if $\vec{v} \cdot \vec{w} = 0$

Example

What is the angle between $2\hat{j} - \hat{i}$ and $\hat{i} + \hat{j}$

$$(\hat{i} + \hat{j}) \cdot (2\hat{j} - \hat{i}) = ||\hat{i} + \hat{j}|| \cdot ||2\hat{j} - \hat{i}|| \cos(\theta)$$
$$(\hat{i} + \hat{j}) \cdot (2\hat{j} - \hat{i}) = -1 + 2 = 1$$
$$||\hat{i} + \hat{j}|| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

3.1.1 Compute the angle between $\hat{i} = (1, 0)$ and $\hat{j} = (0, 1)$

$$\theta = 90°$$

3.1.2 Compute the angle between $(2, 4)$ and $(3, 7)$

$$(2, 4) \cdot (3, 7) = \sqrt{20} \cdot \sqrt{58} \cos(\theta)$$
$$(6 + 28) = \sqrt{20} \cdot \sqrt{58} \cos(\theta)$$
$$\cos(\theta) = \frac{34}{\sqrt{20 \cdot 58}}$$
$$\theta = \arccos(\frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \cdot ||\vec{w}||})$$

3.2 Cauchy-Schwarz Inequality

$$|\vec{v} \cdot \vec{w}| \leq ||\vec{v}|| \cdot ||\vec{w}||$$
$$\vec{a} \cdot \vec{v} = ||\vec{a}|| \cdot ||\vec{b}|| \cos(\theta)$$

The equality hold exactly when $\vec{v}$ and $\vec{w}$ are scalar multiples of each other

4 Orthogonal Projection

Vectors are orthogonal when they are perpendicular, so the dot product of two vectors that are perpendicular equal zero.

$\vec{p}$ is a vector on the line through $\vec{o}$ (the origin) in the direction of $\vec{w}$

$\vec{q}$ is perpendicular (orthogonal) to $\vec{w}$

So $\vec{q} \cdot \vec{w} = 0$

4.1 Definition

$$\vec{p} \text{ is the projection of } \vec{v} \text{ onto } \vec{w}$$

How do we find $\vec{p}$

$$\vec{p} = \lambda\vec{w} \text{ for some real number } \lambda$$
$$\vec{q} = \vec{v} - \vec{p} = \vec{v} - \lambda\vec{w}$$
$$0 = \vec{w} \cdot \vec{q} = \vec{w} \cdot \vec{v} - \lambda\vec{w} \cdot \vec{w}$$

Solve for λ

$$\vec{w} \cdot \vec{v} = \lambda(\vec{w} \cdot \vec{w})$$
$$\text{For } \lambda = \frac{\vec{w} \cdot \vec{v}}{\vec{w} \cdot \vec{w}}$$
$$\vec{p} = \frac{\vec{w} \cdot \vec{v}}{\vec{w} \cdot \vec{w}} \cdot \vec{w}$$

Example

$$\vec{v} = (2,2); \vec{w} = (1,2)$$

We want to find the orthogonal projection of $\vec{v}$ onto $\vec{w}$

$$\vec{p} = \frac{\vec{w} \cdot \vec{v}}{\vec{w} \cdot \vec{w}} \cdot \vec{w} = \frac{(1,2) \cdot (2,2)}{(1,2) \cdot (1,2)} \cdot (1,2)$$
$$= (\frac{6}{5})(1,2) = (\frac{6}{5}, \frac{12}{5})$$

Exercise

Find the orthogonal projection of (1,2,3) onto (1,1,1)

$$\vec{p} = \frac{(1,1,1) \cdot (1,2,3)}{(1,1,1) \cdot (1,1,1)}(1,1,1)$$
$$= \frac{1+2+3}{1+1+1}(1,1,1) = \frac{6}{3}(1,1,1) = 2(1,1,1) = (2,2,2)$$

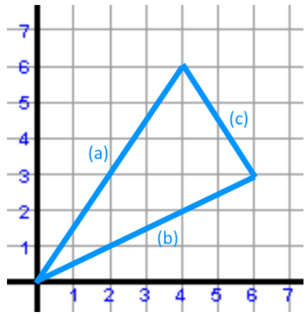
5 The Triangle Inequality

For any two vectors $\vec{a}$ and $\vec{b}$

$$\overbrace{||\vec{a} - \vec{b}||}^{\text{Distance between } \vec{a} \text{ and } \vec{b}} \leq \overbrace{||\vec{a}||}^{length of \vec{a}} + ||\vec{b}||$$

In 2D, the length of the third edge of a triangle is at most the sum of the lengths of the other sides.

$$||b - a|| \leq ||a|| + ||b||$$
$$||a + b|| \leq ||a|| + ||b||$$



6 Matrices, Determinants, and the Cross Products

A 2x2 matrix is an array of vectors

$$\overbrace{(a, b)}^{vector}$$

$$\overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{matrix}$$

Each vector inside the matrix is a row vector

6.1 Determinants

The **determinant** of a 2x2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Example

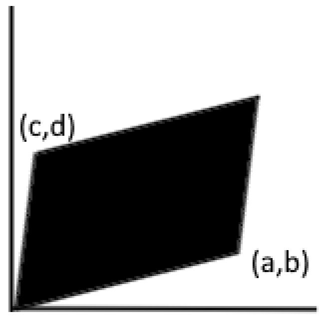
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 1 \cdot 1 - 1 \cdot 1 = 0$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$$

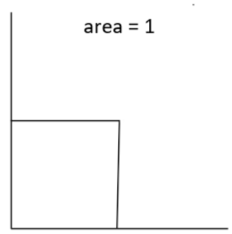
6.2 Geometric Meaning

The absolute value of the determinant equals to the area of the parallelogram of the matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \text{ exactly when} (a, b) \text{ and } (c, d) \text{ lie on the same line through } (0, 0)$$



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1$$



7 3x3 Determinants

When we move from 2D determinants to 3D, some changes must be made

- In 2D, we make a parallelogram
  - The absolute value of the 2D determinant equals to the area
- In 3D, we make a parallelepiped
  - The absolute value of the 3D determinant equals to the volume
- Row vectors still make sense in 3D

$$\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

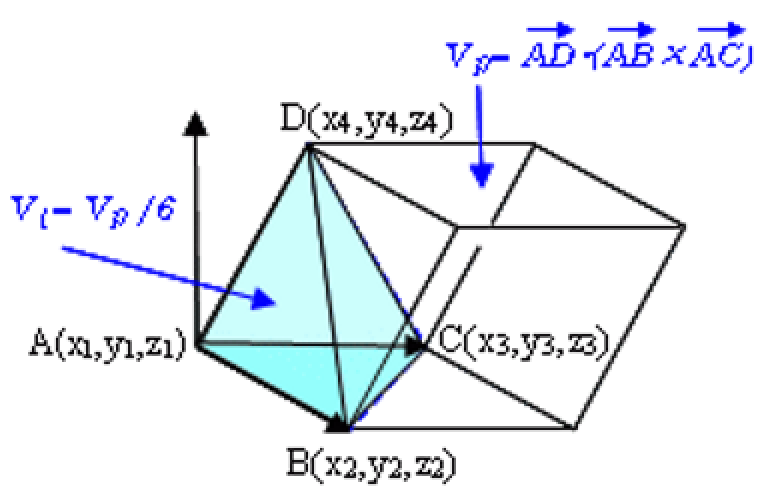


Figure 2: The shape created by the 3D determinant is a parallelepiped