MATH 20C Notes - Week Six

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Introduction

Deep

Figure 1: A glass with red liquid (unspecified)

1 Double Integrals via Riemann Sums

$$f : \mathbb{R}^2 \to \mathbb{R} \text{ is a continuous function}$$

$[a, b] \times [c, d] = \{(x, y) \text{ in } \mathbb{R}^2 \text{ so that } a \leq x \leq b \text{ and } c \leq y \leq d\}$

Goal: Define integral over $[a, b] \times [c, d]$ using riemann sums.

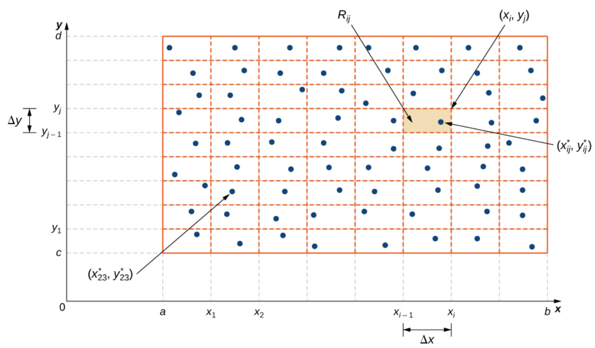

Figure 2: The region cut into n-even pieces of area $\Delta x \cdot \Delta y$

$$\int_n = \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_i) \Delta y \Delta x$$

Defition

The double integral of $f(x, y)$ on $[a, b] \times [c, d]$ is $\lim_n \to \infty \int_n = \int_a^b \int_c^d f(x, y) dy dx$.

Remarks

1. If f takes non negative values, then the double integral computes the appropriate values
2. We do not need f to be continuous to define $\int\int(f) dy dx$

Important Note: f is integratable if you get the same limit no matter which way you divide $[a, b] \times [c, d]$

Properties of the Integral

$f(x, y), g(x, y)$ are two continous functions

Additively

$$\int_a^b \int_c^d f(x, y) + g(x, y) dy dx = \int_a^b \int_c^d f(x, y) dy dx + \int_a^b \int_c^d g(x, y) dy dx$$

Scaling

$$\int_a^b \int_c^d \lambda f(x,y)dydx = \lambda \int_a^b \int_c^d f(x,y)dydx$$

While λ is a real number

Iteration

Let $\int_c^d f(x,y)dy = A(x)$

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b A(x)dx$$

Switching the Order

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Excercise

$$\int_0^1 \int_0^1 x^3 + y^2 dydx = \int_0^1 (x^3+y^2)dy = \left[x^3y + \frac{y^3}{3}\right]_0^1$$

$$= x^3 + \frac{1}{3}$$

$$\int_0^1 (x^3 + \frac{1}{3})dx = \left[\frac{x^4}{4} + \frac{1}{3}x\right]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

Exercise

$$\int_0^1 \int_0^1 ye^{xy}dydx = \int_0^1 \int_0^1 ye^x ydxdy$$

$$\int_0^1 ye^{xy}dx = [e^x y]_{x=0}^1 = e^y - 1$$

$$\int_0^1 e^y - 1dy = [e^y - y]_0^1 = e - 1 - 1 + 0 = e - 2$$

Exercise

$$\int_0^1 \int_0^1 \ln((x+1)(y+1))dydx = \int_0^1 \int_0^1 \ln(x+1) + \ln y + 1dydx$$

$$= \int_0^1 \int_0^1 \ln(x+1)dydx + \int_0^1 \int_0^1 \ln(y+1)dydx$$

$$A(x) = \int_0^1 \ln(y+1)dy = [(y+1)\ln(y+1) - (y+1)]_0^1 = 2\ln(2) - 2 - ((-1)(ln(1)) - 1)$$

$$2\ln(2) - 2 + 1 = 2\ln(2) - 1$$

$$\int_0^1 \int_0^1 \ln(y+1)dydx = \int_0^1 (2\ln(2) - 1)dx = 2\ln(2) - 1$$

$$2\ln 2 - 1 + 2\ln 2 - 1 = 4\ln 2 - 2$$