# MATH 20C Notes - Week Four

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## Introduction

Deep

Figure 1: A pineapple and a cross section of a pineapple

# 1 Paths and Curves

$$\bar{f} : \mathbb{R} \to \mathbb{R}^n$$

If the domain of $f$ is the real numbers of some interval $[a, b]$, we say $f$ is a path
The range is called a curve.

## Example

The parameterization of a line $\bar{f}(t) = (1, 1, 0) + t(2, 1, 1)$

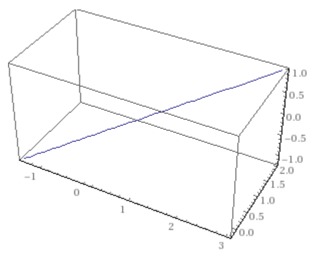


Figure 2: The range is the line

## Example

$$\bar{f}(t) : [0, 2\pi] \to \mathbb{R}$$
$$t \to \bar{f}(t) = (\cos(t), \sin(t))$$

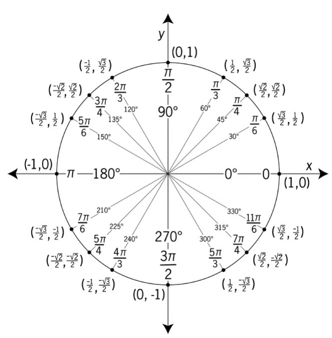


Figure 3: Curve is the Unit Circle

## Example: Wheel with Ant

In this example, we have wheel moving at $1m/s$ with a radius of 1 meter, and an ant on the perimeter of the wheel. We want to parameterize the ant's position at time $t$.

Let ant(t) denote the location of the ant.
The location of the center is $(0,1)$ at $t = 0$, and the location of the center is $(t,1)$
At the time $t$, we need the angle of the ant $\theta(t)$

$$\text{Starting angle} = \frac{-\pi}{2}$$

$$\text{At angle } \theta\text{, the wheel has traveled } \theta = t$$

$$\theta_{ant}(t) = -\frac{\pi}{2} - t$$

$$ant(t) = (t,1) + (\cos(\frac{-\pi}{2} - t), \sin(\frac{-\pi}{2} - t))$$

Given a path $\bar{f}(t) : \mathbb{R} \to \mathbb{R}^3$

The velocity of the path at time($t$) is $D\bar{f}(t)$

The speed of the path at time $t$ is $||D\bar{f}(t)||$

The velocity is the direction of the tangent line of the path If velocity $\neq 0$, then the velocity points in the direction of the curve. The tangent line of time $t_o$ is

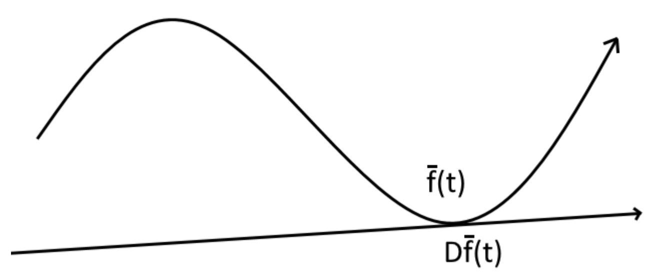


Figure 4: Can parameterize $l$

parameterized by $l(t) == f(t_o) + t \cdot (D\bar{f}(t_o))$

## Example: The Unit Circle

$$\bar{f}(t) = (t, \cos(t), \sin(t))$$

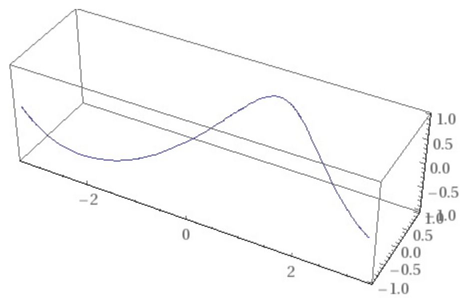


Figure 5: The Spiral

We want to find the equation for the tangent line at $t = 0$.To do this, we need to find the direction (i.e. $D\bar{f}(0)$).

$$D\bar{f}(0) = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -\sin(t) \\ cos(t) \end{bmatrix} = \overbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}^{\substack{Thinkofthis \\ \text{like a row vector}}}$$

$$l(t) = (0, 1, 0) + t(1, 0, 1)$$

### Exercise

Find the speed of the path $(t, t^2, e^t)$ at time 0.

$$\bar{f}(0) = (0, 0, 1)$$

$$D\bar{f}(0) = \begin{bmatrix} 1 \\ 2t \\ e^t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$||D\bar{f}(0) = ||(1, 0, 1)|| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$l(t) = (0, 0, 1) + t(1, 0, 1)$$

## 2 Properties of the Derivative

$$\underbrace{\text{The derivative of } D\bar{f} \text{ is an}}_{\bar{f} : \mathbb{R}^m \to \mathbb{R}^n} \overbrace{n}^{\overset{n}{\text{row}}} \times \overbrace{m}^{\overset{m}{\text{columns}}} \text{ matrix}$$

### Theorem

1. If c is a real number (constant)
   - $Dc\bar{f}(\vec{v}) = c \cdot D\bar{f}(\vec{v})$
   - Every entry in the matrix is multiplied by c
2. Suppose $g : \mathbb{R}^m \to \mathbb{R}^n$
   - $D(\bar{f} + \bar{g})(\vec{v}) = (D\bar{f}(\vec{v})) + (D\bar{f}(\vec{v}))$
   - Add each component indivudually based on their coordinate
3. (Product Rule) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are two scalar functions
   - $D(f \circ g)(\vec{v}) = f(\vec{v}) \cdot Dg(\vec{v}) + g(\vec{v} \cdot Df(\vec{v}))$

4. (Quotient Rule)
   - $D(\frac{f}{g})(\vec{v}) = \frac{g(\vec{v})\cdot Df(\vec{v}) - f(\vec{v}\cdot Dg(\vec{v}))}{(g\vec{v})^2}$

### Example

$$h : \mathbb{R}^3 \to \mathbb{R}$$

$$h(x,y,z) = \frac{x^2+y^2}{z^2+1}$$

$$\begin{aligned} Dh(x,y,z) &= \frac{(z^2+1)D(x^2+y^2) - (x^2+y^2)D(z^2+1)}{(z^2+1)^2} \\ &= \frac{(z^2+1)\cdot(2x,2y,0) - (x^2+y^2)\cdot(0,0,2z)}{(z^2+1)^2} \\ &= (\frac{2x}{z^2+1}, \frac{2y}{z^2+1}, \frac{-(x^2+y^2)(2z)}{(z^2+1)^2}) \end{aligned}$$

### Exercise

$f(x,y,z) = \sin(xy)\cos(yz)$
Compute $\nabla f(x,y,z)$

$$f(x,y,z) = \sin(xy)\cos(yz)$$

$$\begin{aligned} Df(x,y,z) &= \sin(xy)\cdot D\cos(yz) + \cos(yz)\cdot D\sin(xy) \\ &= \sin(xy)(0, -z\sin(yz), -y\sin(yz) + \cos(yz)\cdot(y\cos(xy), x\cos(xy), 0) \\ &= (y\cos(yz)\cos(xy), x\cos(xy)\cos(yz) - z\sin(xy)\sin(yz), -y\sin(xy)\sin(yz)) \end{aligned}$$

# 3 Chain Rule

The derivative of composed functions is a Product

$$f : \mathbb{R}^n \to \mathbb{R}^k \;\; g : \mathbb{R}^k \to \mathbb{R}^m$$

## 3.1 Matrix Multiplication

$$A = \overbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}^{\text{2x2}} = \overbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 0 \end{bmatrix}}^{\text{2x3}}$$

$$2 \times [2] \quad \overbrace{\longleftrightarrow}^{\substack{Needs \\ \text{to be} \\ \text{the same}}} \quad [2] \times 3$$

$$A \cdot B = \begin{array}{ccc} \hline (1,2)\cdot(0,3) & (1,2)\cdot(1,4) & (1,2)\cdot(2,0) \\ \hline (3,4)\cdot(0,3) & (3,4)\cdot(1,4) & (3,4)\cdot(2,0) \\ \hline \end{array} = \begin{bmatrix} 6 & 9 & 2 \\ 12 & 19 & 6 \end{bmatrix}$$

$$A = m \times k \qquad B = k \times n$$

$$\overbrace{A \cdot B}^{\text{m} \times \text{n matrix}} = \begin{bmatrix} row(1,A) \cdot column(1,B) & \dots & row(1,A) \cdot column(j,B) \\ \vdots & \ddots & \vdots \\ row(i,A) & \dots & row(i,A) \cdot column(j,B) \end{bmatrix}$$

**Exercise**

**1)**

$$A \cdot B$$

$$A = \begin{bmatrix} 1 & 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\overbrace{A \cdot B}^{1 \times 1} = [13]$$

**2)**

$$C \cdot D$$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$C \cdot D = \begin{bmatrix} (1,1) \cdot (0,1) & (1,1) \cdot (1,0) \\ (1,2) \cdot (0,1) & (1,2) \cdot (1,0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

## 3.2 The Chain Rule

$$\vec{f} : \mathbb{R}^n \to \mathbb{R}$$

$$\vec{g} : \mathbb{R}^k \to \mathbb{R}^m$$

$$D(\vec{g} \circ \vec{f})(\vec{v}) = (D(\vec{g}(f(\vec{v})))) \cdot D\vec{f}(\vec{v})$$

**Example**

$$\underbrace{\vec{f} : \mathbb{R} \to \mathbb{R}}_{t \to (t\cos((2\pi t), t\sin(2\pi t)))} \qquad \underbrace{\vec{g} : \mathbb{R}^2 \to \mathbb{R}}_{(x,y) \to x^2 + y^2}$$

We want to compute $D(g \circ g)(1)$ and $D\vec{g}(\vec{f}(1)) \cdot D\vec{f}(1)$

$$D\vec{f} = \begin{bmatrix} \cos(2\pi t) - 2\pi t \sin(2\pi t) \\ \sin(2\pi t) + 2\pi t \cos(2\pi t) \end{bmatrix}$$

$$D\vec{f}(1) = \begin{bmatrix} \cos(2\pi) - 2\pi \sin(2\pi) \\ \sin(2\pi) - 2\pi \cos(2\pi) \end{bmatrix} = \begin{bmatrix} 1 - 0 \\ 0 + 2\pi \end{bmatrix} = \begin{bmatrix} 1 \\ 2\pi \end{bmatrix}$$

$$Dg(t) = \begin{bmatrix} 2x & 2y \end{bmatrix}$$

$$D\vec{g}(f(1)) = D\vec{g}(1,0) = [2,0]$$

$$D\vec{g} \circ \vec{f}(1) = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2\pi \end{bmatrix} = [2]$$

## Exercise

$$\vec{g}(x,y) = (x^2+1, y^2) \qquad f(u,v) = (\vec{u} = v, u, v^2)$$

$$D(\vec{f} \circ \vec{g})(1,1) = ?$$

$$D\vec{g} = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix} \vec{g}(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad D\vec{f} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{bmatrix} \qquad D\vec{f}(1,1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$D\vec{f}(g(1,1)) = D\vec{f}(2,1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}$$