MATH 20C Notes - Week Two

C-Rin

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1 Introduction

Why the yellow chalk?

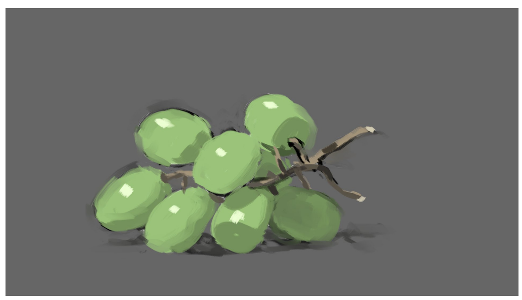
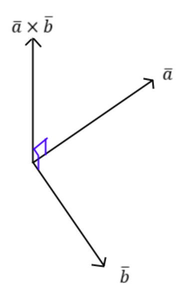


Figure 1: Some grapes

2 The Cross Product

While the dot product is used to take two equal-length sequences of numbers and return a single number, we can use **the cross product** in order to multiply and get vectors in 3D.



$$\begin{bmatrix} i & j & k \\ a & b & c \\ d & e & f \end{bmatrix} \begin{array}{l} \leftarrow unit \\ \leftarrow \vec{v} \\ \leftarrow \vec{w} \end{array}$$

This method does not work in 2D.

Example

$$\vec{v} = (1,2,3) \quad \vec{w} = (4,5,6)$$

$$\vec{v} \times \vec{w} = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = (2 \cdot 6 - 3 \cdot 5)\hat{i} - (1 \cdot 6 - 3 \cdot 4)\hat{j} + (1 \cdot 5 - 2 \cdot 4)\hat{k}$$

$$= -3\hat{i} + 6\hat{j} - 3\hat{k}$$

This gives us the direction of the cross product.

2.1 Triple Product Formula

We can find the product of three 3-dimensional vectors with the **triple product**

$$\vec{a} = (a_1, a_2, a_3) \quad \vec{b} = (b_1, b_2, b_3) \quad \vec{c} = (c_1, c_2, c_3)$$

$$\text{Triple Product} = (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

3x3 Determinants equal zero if and only if the 3 row vectors lie on a single plane.

Because of this, $\vec{a} \times \vec{b}$ is orthogonal to the plane spanned by $\vec{a} \times \vec{b}$

$\vec{a} \times \vec{b}$ is orthogonal/perpendicular to both $\vec{a}$ and $\vec{b}$

Consequences

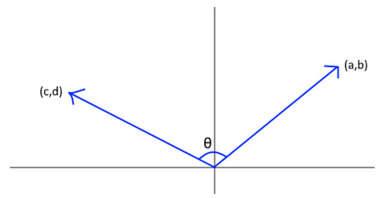
1. If $\vec{a}$ and $\vec{b}$ are co-linear (the same line through the origin), then $\vec{a} \times \vec{b} = (0,0,0)$
2. If $\vec{a}$ and $\vec{b}$ are unit vectors, then $||\vec{a} \times \vec{b} = \sin(\theta)$
3. If $\vec{a}$ and $\vec{b}$ are perpendicular, then $||\vec{a}|| \cdot ||\vec{b}|| = ||\vec{a} \times \vec{b}||$

3 Determinant Signs

3.1 2x2 Determinant Signs

If the angle from (a,b) to (c,d) $0 \leq \theta \leq \pi$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \geq 0$

If the angle from (a,b) to (c,d) $-\pi \leq \theta \leq 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \leq 0$



3.2 3x3 Determinant Signs

$$\vec{a} = (a_1, a_2, a_3) \quad \vec{b} = (b_1, b_2, b_3) \quad \vec{c} = (c_1, c_2, c_3)$$

If the vectors $\vec{a}, \vec{b}, \vec{c}$ satisfy the right hand rule, then $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \leq 0$

If the vectors $\vec{a}, \vec{b}, \vec{c}$ fail the right hand rule (or satisfy the left hand rule), then $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \leq 0$

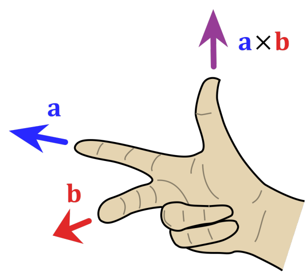


Figure 2: The Right Hand Rule helps us find the direction of the cross product using two vectors

If $\vec{a} \times \vec{b} \neq (0, 0, 0)$, then $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ satisfy the right hand rule.

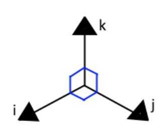


Figure 3:

$$ij = k \quad jk = i \quad ki = j$$
$$ji = k \quad kj = i \quad ik = j$$

$$i \times j = \begin{bmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = i(0)j(0) + k(1100) = k$$

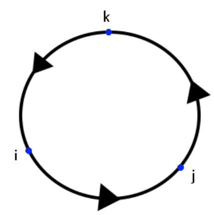


Figure 4:

4 Properties of the Cross Product

- $\vec{a} \times \vec{b} = (0,0,0)$ if and only if $\vec{a}$ and $\vec{b}$ are parallel
- (Skew Symmetry) $\vec{a} \times \vec{b} = -\vec{a} \times \vec{b}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ and $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
- $(\alpha\vec{a}) \times \vec{b} = \alpha(\vec{a} \times \vec{b})$

Warning

The cross product is not associative, so

$$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$$

The cross product also does not interact well with the dot product (except for the triple product formula).

Exercise

Compute $(0,0,1) \times (1,1,0)$ AND $(1,0,0) \times (1,1,0)$

$$(0,0,1) \times (1,1,0) = \begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \hat{i}(0 \cdot 0 - 1 \cdot 1) - \hat{j}(0 \cdot 0 - 1 \cdot 1) + \hat{k}(0 \cdot 1 - 0 \cdot 1) = -\hat{i} + \hat{j}$$

OR

$$(0,0,1) = \hat{k} \quad (1,1,0) = \hat{i} + \hat{j}$$

$$\hat{k} \times (\hat{i} + \hat{j})$$

$$(1,0,0)\times(1,1,0) = \begin{bmatrix} i & j & k \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \hat{i}(0\cdot 0 - 0\cdot 1) - \hat{j}(1\cdot 0 - 1\cdot 0) + \hat{k}(1\cdot 1 - 1\cdot 0) = \hat{k}$$

OR

$$\hat{i} \times (\hat{i} + \hat{j})$$

5 Equation of Planes

$$\overbrace{Ax + By + Cz = 0}^{(A,B,C)\cdot(x,y,z)}$$

$$P = \text{solution set} = \text{all the vectors perpendicular to } (A,B,C)$$

Solutions

Call (A,B,C) the normal vector denoted by $\bar{n}$
If P is a plane with normal vector $\bar{n}$ containing $P_o = (x_o, y_o, z_o)$
Alternate Equations for Plane P

$$= \bar{n} \cdot (x - x_o, y - y_o, z - z_o) = 0$$

$$= A(x - x_o) + B(y - y_o) + C(z - z_o) = 0$$

$$= Ax + By + Cz + D = 0$$

$$(D = -Ax_o - By_o - Cz_o)$$

Example

Find the equation for the plane through $(1,1,0), (1,0,1), (0,1,1)$

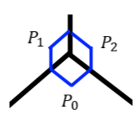


Figure 5: Needs a normal vector, given by $\vec{a} \times \vec{b}$

Consider

$$\vec{a} = P_1 - P_0 = (0, -1, 1)$$
$$\vec{b} = P_2 - P_0 = (-1, 0, 1)$$
$$(0, -1, 1) \times (-1, 0, 1) = (-1, -1, -1) = (A, B, C)$$
$$P_o = (1, 1, 0) \text{ is a point in } P$$

Plane P can be found with

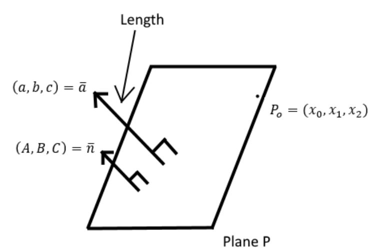
$$(A, B, C) \cdot (x - 1, y - 1, z - 0) = 0$$

OR

$$(-x + 1, -y + 1 - z) = 0$$
$$-x - y - z + 2 = 0 \quad x + y + z = 2$$

6 Distance from a Point to a Plane

Find the distance between the point $\vec{a}$ and the plane P



Equations

$$\bar{n} \cdot ((x, y, z) - P_o) = 0$$

OR

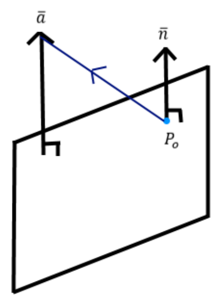
$$\bar{n} \cdot ((x - x_o, y - y_o, z - z_o) = 0$$

OR

$$A(x - x_o) + B(y - y_o) + C(z - z_o) = 0 = Ax + By + Cz + D$$

We can find the line segment from $\vec{a}$ and the plane P using the scalar multiple of the normal vector, $\bar{n}$

The answer should be the length of the projection of $(\vec{a} - P_o)$ onto $\bar{n}$



$$\overbrace{||\frac{(\vec{a} - P_o) \cdot \bar{n}}{\vec{n} \cdot \vec{n}} \cdot \vec{n}||}^{\text{The projection of } (\vec{a} - P_o) \text{ onto } \bar{n}}$$

$$= ||\frac{((a,b,c) - (x_0, x_1, x_2)) \cdot (A,B,C)}{(A,B,C) \cdot (A,B,C)} \cdot (A,B,C)||$$

$$= \frac{|Aa + Bb + Cc + D}{A^2 + B^2 + C^2} ||\bar{n}||$$

$$= \frac{|Aa + Bb + Cc + D}{\sqrt{A^2 + B^2 + C^2}}$$

$$\overbrace{D = (x_0, x_1, x_2) \cdot (A,B,C)}^{\text{D is the collection of constants}}$$

7 Multi-variable Functions

We consider functions with domains/co-domains either $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$

Case 1

A function with domains $\mathbb{R}$ and co-domains $\mathbb{R}$

$$f : \mathbb{R} \to \mathbb{R}^2$$

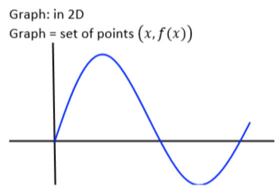


Figure 6: A Regular Real Function

Case 2

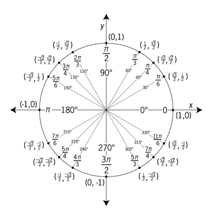
$$\left.\begin{array}{l} f:\mathbb{R}\to\overbrace{\mathbb{R}^2}^{\text{2D Vector}} \\ \textbf{OR} \\ g:\mathbb{R}^2\to\overbrace{\mathbb{R}}^{\text{Real Number}} \end{array}\right\} \left\{\text{For these functions, we can still graph them}\right\}$$

The graph of f is the set of points $(x, f(x))$ in $\overbrace{\mathbb{R}^3}^{\text{3D Vector}}$

Example 1

$$f(x) = (\cos(x), \sin(x))$$

Question: What is the range of the function?
Answer: The Unit Circle



The graph of $f(x) = (\cos(x), \sin(y))$ creates a perfect spiral.

$$g : \mathbb{R}^2 \to \mathbb{R}$$

The graph is a set of points in 3D of the form $(x, y, g(x, y))$

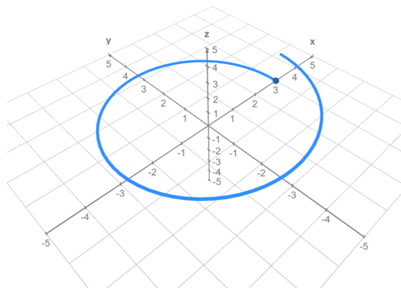


Figure 7:

Example 2

$$g((x,y)) = (x,y) \cdot (x,y) = x^2 + y^2$$

The range of the function = All non-negative real numbers $\mathbb{R}$

$$(x,y) = (0,0)$$

This creates a surface

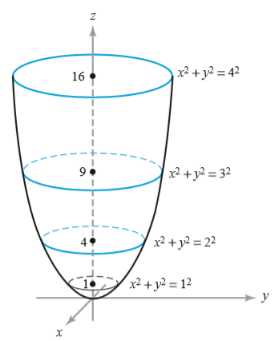


Figure 8:

7.1 Hard to Graph Functions

Graphs are useful when the domain and co=domain have dimensions less than three. If $f : \mathbb{R}^3 \to \mathbb{R}$ is a function, the level set with level c, where c is a real number, is the subject of points in $\mathbb{R}$ that map to c

Example

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$(x, y, z) \to x^2 + y^2 + z^2 = f(x, y, z)$$

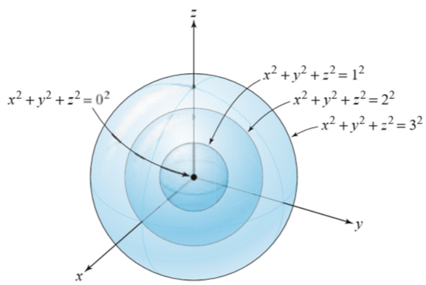


Figure 9: Also known as level surfaces

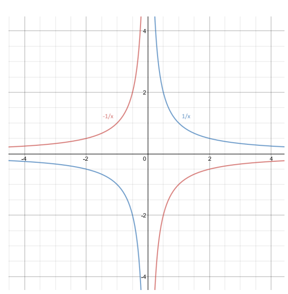
- Level 1
  - A set with length = 1
  - Sphere of radius $\sqrt{1} = 1$
  - Set of a unit vector
- Level 2
  - A set with length 2
  - Sphere of radius $\sqrt{2}$

Exercise

$$g : \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \to x \cdot y$$

Find the level sets with level -1, 0, 1



- Level Set 0

- $xy = 0$
- $x = 0$ or $y = 0$

- Level Set 1
  - $xy = 1$
  - $y = \frac{1}{x}$
- Level Set -1
  - $xy = -1$
  - $y = -\frac{1}{x}$

7.2 Taking Sections

A section of a graph in 3D is the intersection of the graph with a plane.

Example

$$h : \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \to x^2 - y^2 = h(x, y)$$

@ $x = 0$

$$z = h(0, y) = -y^2$$

@$y = 0$

$$z = h(x, 0) = x^2$$