# MATH 20C Notes - Week Six

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October 2019

## Introduction

Deep

Figure 1: A plant (unspecified species)

# 1 Extrema of Real Valued Functions

$$f : U \to \mathbb{R} \qquad U \overbrace{\subseteq}^{\text{subset}} \mathbb{R}^2 \qquad U \text{ is an "open subset"}$$

An open subset is a subset value for any point $x \in U$
In an open subset, you can move an arbitrarily small amount in any direction while staying in the subset.

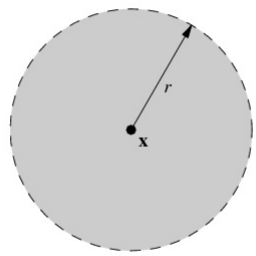



Figure 2: An open disk with $U = \{\vec{v} \in \mathbb{R}^2 \Big| ||\vec{v}|| < 1\}$

## Maximum and Minimum Extrema

Q: What is the definition of a maximum (or minimum) of $f$?

A: A point $\vec{v} \in U$ is a maximum if $f(\vec{v})$ is greater or equal to $f(\vec{w})$ for any $\vec{w} \in U$ (Similarly defined for a minimum)

## Definitions

1. An open neighborhood of a point $x \in U$ is an open set $v \in U$ which contains $x$.
2. A point $x \in U$ is a local max (respectively a local minimum) if there is an open neighborhood of $x$ where $x$ is a maximum (respectively a minimum)
3. A $x \in U$ is a <u>critical point</u> if either
   (a) $f$ is not differentiable at $x$
   **OR**
   (b) $Df(x) = 0$
4. A critical point which is not a local maximum or minimum is a saddle point

## Theorem

Let $f : U \to \mathbb{R}$ and $U \subseteq \mathbb{R}^n$ be a differentiable function.

If $\vec{v} \in U$ and $\vec{v}$ is a local extremum, then $Df(\vec{v}) = 0$ (i.e. $\vec{v}$ is a critical point).

## Example

$f : \mathbb{R}^2 \to \mathbb{R} \qquad f(x, y) = \log(x^2 + y^2 + 1)$

Find all the critical points.

$$\overbrace{\mathbb{R}^2 \;\to\; \mathbb{R}^3}^{g(x)=x^2+y^2+1}\!\!\!\!\underbrace{\phantom{\mathbb{R}^3}\;\to\; \mathbb{R}}_{h(z)=\log(z)}$$

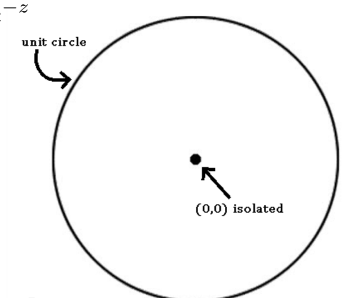
$$\begin{aligned} Df(x, y) &= Dh(x^2 + y^2 + 1) \cdot Dg(x, y) \\ &= \left[\tfrac{1}{x^2+y^2+1}\right] \cdot \begin{bmatrix} 2x & 2y \end{bmatrix} \\ &= \begin{bmatrix} \frac{2x}{x^2+y^2+1} & \frac{2y}{x^2+y^2+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \end{aligned}$$

$$(x, y) = (0, 0)$$

## Example

Find all the critical points in $\mathbb{R}^2$ for $f(x, y) = 2(x^2 + y^2) \cdot e^{-x^2-y^2}$

$f(x, y) = 2(x^2{+}y^2){\cdot}e^{-(x^2+y^2)} \qquad h(z) = x^2{+}y^2 \qquad g(z) = 2z{\cdot}e^{-z}$




$$\begin{aligned} Dg(z) &= 2ze^{[} - z] + 2z(-e^{-z}) = 2e^{-z} - 2ze^{-z} = [2e^{-z}(1 - z)] \\ &= [2e^{-x^2-y^2}(1 - x^2 - y^2)] \end{aligned}$$

$$\begin{aligned} Dg \circ h &= \begin{bmatrix} 2e^{-x^2-y^2}(1 - x^2 - y^2) \end{bmatrix} \begin{bmatrix} 2x & 2y \end{bmatrix} \\ &= \begin{bmatrix} 4xe^{-x^2-y^2}(1 - x^2 - y^2) & 4ye^{-x^2-y^2}(1 - x^2 - y^2) \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 4xe^{-x^2-y^2}(1 - x^2 - y^2) & 4ye^{-x^2-y^2}(1 - x^2 - y^2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$(x, y) = (0, 0), (1, 0), (0, 1), \text{ and all the values on the unit circle}$$

# 2 2nd Derivative Test for Extrema

## Theorem

Let $f(x, y)$ be a $c^2$ function and suppose

1. The derivative $Df(x_o, y_o) = 0$
2. $\frac{\partial^2 f}{\partial x^2}(x_o, y_o) > 0$
3. $D = (\frac{\partial^2 f}{\partial x^2})(\frac{\partial^2 f}{\partial y^2}) - (\frac{\partial^2 f}{\partial x \partial y})^2 > 0$

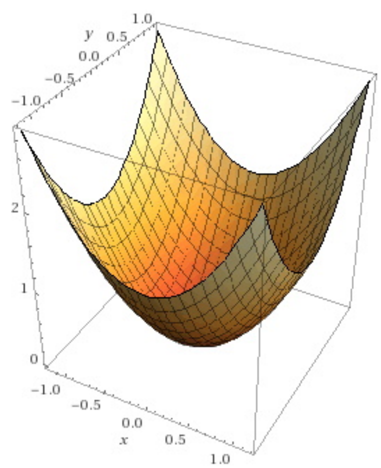
Then $(x_o, y_o)$ is an isolated local minimum. If $\frac{\partial^2 f}{\partial x^2}(\vec{v} < 0)$, and 1. and 3. hold, then $\vec{v}$ is a local maximum.

## Remarks

1. D is the determinant of the following 2x2 matrix, known as a Hessian Matrix.
2. If $\vec{v}$ is a critial point, but $D < 0$ at $\vec{v}$, then $\vec{v}$ is a saddle point.

$$\underbrace{\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}}_{\text{A Hessian Matrix}}$$

A saddle point is a point on the surface of the graph of a function where the slopes (derivatives) in orthogonal directions are all zero (a critical point), but which is not a local extremum of the function.




## Example

Find the critical points of $x^2 + y^2 = f(x, y)$

### Critical Points

We need to find $\frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}$

$$\nabla f(x, y) = \begin{bmatrix} 2x & 2y \end{bmatrix} \qquad \frac{\partial^2 f}{\partial x^2} = 2 > 0$$

$$\frac{\partial^2 f}{\partial y^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

This is a local minimum due to the second derivative test.

### Exercise

Find the critical points of $f(x, y) = x^5y + xy^3 + xy$ and determine if they are local extrema or saddle points.

$$Df(x, y) = \begin{bmatrix} 5x^4y + y^3 + y & x^5 + 3xy^2 + x \end{bmatrix}$$

$$\text{Critical points } (x, y) = (0, 0)$$

$$D = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2$$

## 3 Closed Sets

A closed set is a set which contains all of its boundary points.

A closed unit disc would be an example with the bounds $\{\vec{v} \in \mathbb{R}^2 \Big| ||\vec{v}|| \leq 1\}$

A nonexample would be an open unit disc, which does not contain any of its boundary points.

A subset of $\mathbb{R}^2$ is bounded if it is contained in some arbitrarily "large enough" circle.

### Example

Every Square is bounded.

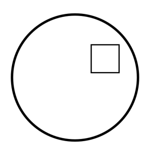


Figure 3: Every square has a circle that can make it bounded

### Nonexample

The x-axis is always unbounded, where at some point it is not in the circle.

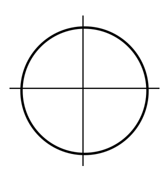


Figure 4: At some point, the line passes through the circle, no matter how big the circle is.

### Theorem

If you have any two sets $f : z \to \mathbb{R}$ is a continuous function and $z \subseteq \mathbb{R}^2$ is closed and bounded, then the function has a maximum and a minimum.

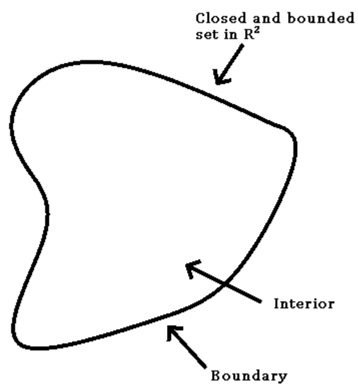



Figure 5: A closed and bounded set in $\mathbb{R}^2$

The set of interior points is an open set.

# 4 The Optimization Method

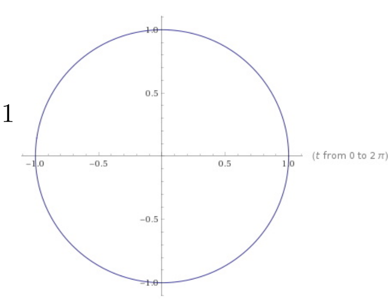
1. Find all the critical points on the interior
2. Use the path and try to find all the critical points of $f(\vec{g}(t))$
3. Compare the values of the critical points (i.e. plug them in)

### Example

Find the maximum/minimum $f(x, y) = x^2 + y^2 - x - y + 1$ on the closed unit disk.

To do this, we need to parameterize the boundary.

$$Df(x, y) = \begin{bmatrix} 2x - 1 & 2y - 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad (x, y) = (1/2, 1/2)$$




$$\begin{aligned} f(\vec{g}(t)) &= \overbrace{\cos^2(t) + \sin^2(t)}^{1} - \cos(t) - \sin(t) + 1 \\ &= 2 - \cos(t) - sin(t) \end{aligned}$$

What is the domain of $t$?
In this situation, $[0, 2\pi]$ works.

$$f'(\vec{g}(t)) = \sin(t) - \cos(t) = 0 \text{ when } \sin(t) = \cos(t)$$

$$t = \frac{\pi}{4}, \frac{5\pi}{4} \text{ (and possibly 0 and } 2\pi)$$

$$f(1/2, 1/2) = \frac{1}{2} \qquad f(\vec{g}(0)) = 1$$

$$f(\vec{g}(\frac{\pi}{4})) = \frac{1}{2} + \frac{1}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 1 - \sqrt{2} \qquad f(\vec{g}(\frac{5\pi}{4})) = 2 + \sqrt{2}$$

# 5 Constrained Extrema and Lagrange Multipliers

## Optimizing functions and level sets (A.K.A) The Lagrange Error

## Theorem

$U \subseteq \mathbb{R}^n$ is an open set, and $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are two differentiable functions.

Let S be a level set of $g$ on level $c$.

If $f$ has a local maximum or minimum on S at a point $\vec{v}$ in S and $\nabla g(\vec{v}) \neq 0$, then there exists a real number $\lambda$ such that

$$\nabla f(\vec{v}) = \lambda \nabla g(\vec{v})$$

## Example

$$\underbrace{S \subseteq \mathbb{R}^2}_{S \leftarrow y = x+1} \qquad f(x, y) = x^2 + y^2$$

Find the local extrema of $f$ on $S$.