MATH 20C Notes - Week Six

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Introduction

Deep

Figure 1: A stylized tree (unspecified species)

1 Acceleration/Newton's 2nd Law

Let $\bar{c} : \mathbb{R} \to \mathbb{R}^3$ be a differentiable path (a particle's position at time (t)).

The velocity of $\bar{c}$ is

$$\bar{c}\prime(t) = (\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\frac{\partial z}{\partial t})$$

The speed is the magnitude $||\bar{c}\prime(t)||$

The acceleration is $\bar{c}\prime\prime(t) = (x\prime\prime(t), y\prime\prime(t), z\prime\prime(t))$

1.1 Differentiation Rules for Paths

Let $\bar{b} : \mathbb{R} \to \mathbb{R}^3$ and $\bar{c} : \mathbb{R} \to \mathbb{R}^3$ be differentiable paths.

Let $p(t)$ be a scalar function

Product Rules

1. Sum Rule
$$\frac{\partial}{\partial t}(\bar{b}(t) + \bar{c}(t)) = \bar{b}\prime(t) + \bar{c}\prime(t)$$

2. Dot Product Rule
$$\frac{\partial}{\partial t}(\bar{b}(t) \cdot \bar{c}(t)) = \bar{b}\prime(t) \cdot \bar{c}(t) + \bar{c}\prime(t) \cdot \bar{b}(t)$$

3. Scalar Product Rule
$$\frac{\partial}{\partial t}(p(t) \cdot \bar{c}(t)) = p\prime(t)\bar{c}(t) + p(t)\bar{c}\prime(t)$$

4. Cross Product Rule
$$\frac{\partial}{\partial t}(\bar{b}(t) \times \bar{c}(t)) = \bar{b}\prime(t) \times \bar{c}(t) + \bar{c}\prime(t) \times \bar{b}(t)$$

Example

Suppose
$$\bar{b}(t) = (\sin(t), \cos(t), 0)$$
$$\bar{c}(t) = (0, t, t^2)$$

$$\bar{b}(t) \times \bar{c}(t) = (t^2 \cos(t), -t^2 \sin(t), t \sin(t))$$

$$\frac{\partial}{\partial t}(\bar{b} \times \bar{c}) = (2t \cos(t) - t^2 \sin(t), -2t \sin(t)) - t^2 \cos(t), sin(t) + t \cos(t)$$

$$\bar{b}\prime(t) = (\cos(t), -\sin(t), 0) \qquad \bar{c}\prime(t) = (0, 1, 2t)$$

$$\bar{b}\prime \times \bar{c} + \bar{b} \times \bar{c}\prime = (-t^2 \sin(t), -t^2 \cos(t), t \cos(t)) + (2t \cos(t), -2t \sin(t), t \sin(t))$$

Newton's Second Law

$$\bar{F} = m\bar{a}$$

Exercise

Suppose there is a particle with initial position

$$\bar{r}(0) = (6, -2, 1) \qquad \bar{v}(0) = (-5, 1, 3) \qquad \bar{a}(t) = (2, -6, -4)$$

Find the trajectory/path of the particle.

$$\int a(t) = v(t) = (2t, -6t, -4t) + c \qquad v(0) = (-5, 1, 3) = (0, 0, 0)$$

$$v(t) = (2t - 5, -6t + 1, -4t + 3)$$

$$\int v(t) = r(t) = (t^2 - 5t, -3t^2 + t, -2t^2 + 3t) + c$$

$$r(0) = (6, -2, 1) = (0, 0, 0)$$

$$r(t) = (t^2 - 5t + 6, -3t^2 + t - 2, -2t^2 + 3t + 1)$$

2 Arc Length

Consider a path

$$\bar{c} : [a, b] \to \mathbb{R}^3$$

This path traces out a curve, and the length of that curve is the arc length.

$$L = \int_a^b ||c\prime(t)||dt \qquad \bar{c}(t) = (x(t), y(t), z(t))$$

$$\int \sqrt{x\prime(t)^2 + y\prime(t)^2 + z\prime(t)^2}dt$$

Example - Deriving the Circumference of a circle

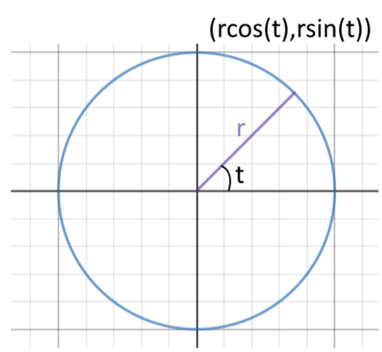
t is in $[0, 2\pi]$ and the arc length is $2\pi r$

$$\int_0^{2\pi} \sqrt{(-r\sin(t))^2 + (rcos(t))^2}dt$$

$$= \int_0^{2\pi} \sqrt{r^2\sin^2(t) + r^2\cos^2(t)}dt$$

$$= \int_0^{2\pi} \sqrt{r^2(\sin^2 t + \cos^2(t))}dt$$

$$= \int_0^{2\pi} \sqrt{r^2(1)}dt = \int_0^{2\pi} rdt$$



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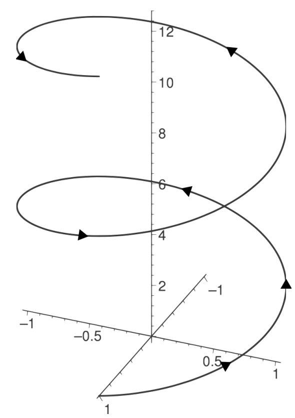
$$= 2\pi r$$

Example



$$\bar{c}(t) = (t, \cos(t), \sin(t)) \qquad 0 \leq t \leq 2\pi$$

$$x'(t) = 1$$

$$y'(t) = -\sin(t)$$

$$z'(t) = \cos(t)$$

$$= \int_0^{2\pi} \sqrt{2} dt$$

$$= 2\pi\sqrt{2}$$

Exercise

Find the length of the curve

$$\bar{c}(t) = (2t, \frac{4}{3}t^{\frac{3}{2}}, \frac{1}{2}t^2) \text{ for } 0 \leq t \leq 3$$

$$x'(t) = 2 \qquad y'(t) = 2\sqrt{t} \qquad z'(t) = t$$

$$\int_0^3 \sqrt{4 + 4t + t^2} dt = \int_0^3 \sqrt{(t+2)^2} dt = \int_{0^3} (t+2) dt$$

$$\left[\frac{t^2}{2} + 2t\right]_0^3 = \frac{9}{2} + 6 = \frac{21}{2}$$

Remark

Arc length can be computed in any dimension

Let $\bar{c} : [a, b] \to \mathbb{R}^n$

Arc length $= \int_a^b ||\bar{b}'(t)|| dt$

$$\int_a^b x'(t) dt = x(b) - x(a)$$

3 Double Integral

The Intuitive Meaning

$$f : \mathbb{R}^2 \to \mathbb{R}$$

(Which takes values greater than 0

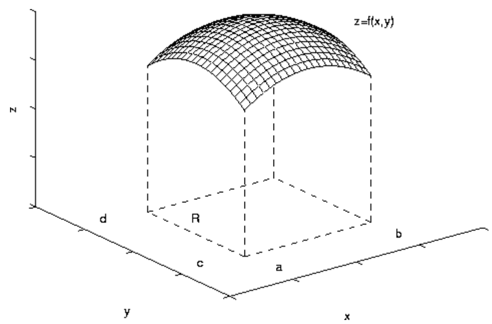


Figure 2:

The volume contained above $a \le x \le b$ and $c \le y \le d$ and below $z = f(x, y)$

$$\underbrace{= \int_a^b \int_c^d f(x,y)dydx}_{\text{Double Integral}}$$

Example

What is the volume of $z = z - x$ on the rectangular region of $0 \le x \le 2$ and $0 \le y \le 3$

$$= \int_0^2 \int_0^3 (2-x)dydx = \frac{2 \cdot 2 \cdot 3}{2} = 6$$

Remark

If we are integrating over a more interesting region, we can represent it as

$$\iint f(x,y)dydx$$

How in practice do we compute these volumes?

We can use Cavalieri's Principle to calculate the volume using cross sectioned areas where

$$\text{Volume } = \int_b^a A(x)dx$$

Example

$$\int_0^1 \int_0^1 (x^2 + y^2) dx dy$$

Suppose x is fixed (i.e. a constant) on $0 \leq x \leq 1$

$$A(x) = \int_0^1 x^2 + y^2 dy = \left[x^2 y + \frac{1}{3} y^3 \right]_0^1 = x^2 + \frac{1}{3}$$

$$\int_0^1 x^2 + \frac{1}{3} dx = \frac{x^3}{3} + \frac{1}{3} x = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Intuitive Definition of Integrals in 2D

$f : \mathbb{R}^2 \to \mathbb{R}$ with nonnegotiable values

$$int_a^b \int_c^d f(x,y) dy dx = \text{the volume of the solid}$$

1. bounded below by $\mathbb{R}$
2. bounded by the graph $z = f(x,y)$

   We use iterative integrals to find the volume.

$$\underbrace{\int_a^b A(x) dx = int_a^b \int_c^d f(x,y) dy dx}_{IterativeIntegral}$$