

Supersymmetric Quantum Mechanics and Index Theorem

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Supersymmetric quantum mechanics is discussed in the spaces of one and three dimensions. The structure of the continuum portion of the spectrum of the hamiltonian is investigated by making use of Callias' and Weinberg's generalized index theorems. The difference of the densities of bosonic and fermionic states at a fixed energy is calculated and it turns out to be topologically invariant and non-vanishing in the presence of continuous spectra.

§ 1. Introduction

Supersymmetric quantum mechanics (SQM) is the simplest supersymmetric model invented by Witten¹⁾ to envisage how the dynamical breakdown of supersymmetry takes place. He considered SQM whose energy spectrum consisted only of discrete levels. He defined a convenient quantity which is now called Witten's index.²⁾

On the other hand, it is known that there exists an interesting generalization of the index for a class of Dirac operators if the space is of odd dimension and is not compact.^{3),4)} The conventional index J is generalized to a one-parameter dependent quantity $J(z)$, z being an arbitrary parameter. Both J and $J(z)$ are invariant under any finite changes of potential functions in the Dirac operator concerned. $J(z)$ equals J when the parameter z vanishes. The non-compactness of the space corresponds to the existence of the continuum in the spectrum of the Dirac operator.

In this paper, in contrast to Witten, we study SQM whose energy spectrum consists of continuum as well as discrete levels. We discuss the SQM-version of the generalized index theorem and investigate the structure of the continuum spectrum of the SQM hamiltonian. The naive discussion that some expectation values summed over degenerate fermionic and bosonic states cancel out turns out to be misleading. The index theorem also supplies us with some sum rules which are independent of the detailed structure of the superpotential.

In §2, we consider SQM in the one-dimensional space. The superpotential here is taken to be of kink-type. In §3, we extend the discussion of §2 to the case of the three-dimensional space. We investigate the case of the background potential of non-Abelian monopole-type. Section 4 is devoted to summary. Finally in the Appendix, we present an elementary proof of the generalized index theorem which is made use of in §2.

§ 2. The case of kink-type superpotential

We consider SQM in the one-dimensional space. The super-charges Q and Q^\dagger are defined by^{1),5)}

$$\begin{aligned} Q &= L\psi, & L &= p - i\phi(x), \\ Q^\dagger &= \psi^\dagger L^\dagger, & L^\dagger &= p + i\phi(x), \end{aligned} \quad (2.1)$$

where $\phi(x)$ is the real superpotential, $p=(1/i)(d/dx)$, and ψ and ψ^\dagger are the fermionic variables which satisfy

$$\begin{aligned}[x, \psi] &= [p, \psi] = 0, \\ \psi^2 &= 0\end{aligned}\tag{2.2}$$

and

$$\{\psi, \psi^\dagger\} = 1.$$

The hamiltonian H is given by

$$\begin{aligned}H &= \{Q, Q^\dagger\} \\ &= L^\dagger L F + L L^\dagger (1 - F),\end{aligned}\tag{2.3}$$

where F is the fermion number operator defined by

$$F = \psi^\dagger \psi.\tag{2.4}$$

It is clear that H commutes with F . H also commutes with Q and Q^\dagger because of the nilpotency

$$Q^2 = (Q^\dagger)^2 = 0,\tag{2.5}$$

so that the theory is supersymmetric. We assume that $\phi(x)$ tends to some finite value at spatial infinity:

$$\phi(+\infty) = \mu$$

and

$$\phi(-\infty) = \lambda.\tag{2.6}$$

Operators $L^\dagger L$ and $L L^\dagger$ are given by

$$L^\dagger L = -\frac{d^2}{dx^2} + U(x), \quad U(x) = (\phi(x))^2 - \frac{d\phi(x)}{dx}$$

and

$$L L^\dagger = -\frac{d^2}{dx^2} + V(x), \quad V(x) = (\phi(x))^2 + \frac{d\phi(x)}{dx}.\tag{2.7}$$

The spectrum of H consists of possible discrete levels and of the continuum. The lowest tip of the continuum is given by $\min(U(+\infty), U(-\infty), V(+\infty), V(-\infty)) = \min(\mu^2, \lambda^2)$.

We here recall the generalized index theorem for the elliptic operator L . It reads³⁾

$$J(z) \equiv \text{Tr} \left(\frac{z}{L^\dagger L + z} - \frac{z}{L L^\dagger + z} \right) = \frac{1}{2} \left(\frac{\mu}{\sqrt{\mu^2 + z}} - \frac{\lambda}{\sqrt{\lambda^2 + z}} \right),\tag{2.8}$$

where z is a parameter which is not real negative. The simple derivation and the physical contents of (2.8) are discussed in the Appendix. When the values of $|\mu|$ and $|\lambda|$ are infinite, the z -dependence of $J(z)$ disappears and $J(z) = J(0)$ only counts the difference of the number of zero-modes of L and L^\dagger . It is remarkable that, even for non-vanishing z , $J(z)$ remains invariant under any finite changes of $\phi(x)$ in finite spatial

regions. It should also be noted that the theorem (2·8) plays a key role in the discussion of the phenomenon of fermion fractionization in one-dimensional polymer.⁶⁾

The SQM-version of (2·8) can be obtained easily. We see from (2·8) that

$$(-1)^F g(H)|f\rangle = -g(L^\dagger L)|f\rangle$$

and

$$(-1)^F g(H)|b\rangle = +g(LL^\dagger)|b\rangle, \quad (2\cdot9)$$

where $g(O)$ is an arbitrary function of O , and $|f\rangle$ and $|b\rangle$ are fermionic and bosonic states defined by

$$F|f\rangle = |f\rangle$$

and

$$F|b\rangle = 0, \quad (2\cdot10)$$

respectively. From (2·8) and (2·9), we have

$$I(z) \equiv \text{Tr} \left\{ (-1)^F \frac{z}{H+z} \right\} = -\frac{1}{2} \left(\frac{\mu}{\sqrt{\mu^2+z}} - \frac{\lambda}{\sqrt{\lambda^2+z}} \right), \quad (2\cdot11)$$

where Tr denotes the sum over all the fermionic and bosonic eigenstates of H . From the discussion in the Appendix, we find that $I(z)$ can be divided into three parts:

$$I(z) = I(0) + I_D + I_C(z). \quad (2\cdot12)$$

$I(0)$ is given by

$$I(0) = n_b^{E=0} - n_f^{E=0} = -\frac{1}{2}(\text{sign } \mu - \text{sign } \lambda), \quad (2\cdot13)$$

where $n_b^{E=0}$ ($n_f^{E=0}$) is the number of normalizable zero-modes of H which are the eigenstates of F with the eigenvalue $O(1)$. We see that $I(0)$ is nothing but Witten's index. We clearly understand from (2·13) how Witten's index for SQM is determined by the behaviour of the superpotential. In Refs. 1), 2) and 5), the dynamical breakdown of supersymmetry was discussed for infinite values of $|\mu|$ and $|\lambda|$: The supersymmetry is preserved as long as the sign of (infinite) μ is opposite to that of (infinite) λ . We see that the same conclusion holds also for finite values of μ and λ . I_D denotes the contribution from the non-zero discrete levels of H and vanishes due to the supersymmetry,

$$I_D = 0. \quad (2\cdot14)$$

$I_C(z)$ is the contribution from the continuum of spectrum of H and is given by

$$I_C(z) = \frac{1}{2}(\text{sign } \mu - \text{sign } \lambda) - \frac{1}{2} \left(\frac{\mu}{\sqrt{\mu^2+z}} - \frac{\lambda}{\sqrt{\lambda^2+z}} \right). \quad (2\cdot15)$$

We now consider the applications of (2·15). For simplicity, we hereafter put

$$\lambda = -\mu, \quad (2\cdot16)$$

so that (2·15) turns to

$$I_c(z) = \text{sign } \mu - \frac{\mu}{\sqrt{\mu^2 + z}}. \quad (2 \cdot 17)$$

The state $|k, \alpha\rangle$ in the continuum is defined by

$$H|k, \alpha\rangle = (k^2 + \mu^2)|k, \alpha\rangle, \quad \alpha = b, f, \quad (2 \cdot 18)$$

$$F|k, f\rangle = |k, f\rangle$$

and

$$F|k, b\rangle = 0. \quad (2 \cdot 19)$$

The state $|k, f\rangle$ ($|k, b\rangle$) transforms into $|k, b\rangle$ ($|k, f\rangle$) under the operation of $Q/\sqrt{k^2 + \mu^2}$ ($Q^\dagger/\sqrt{k^2 + \mu^2}$) and is assumed to be properly normalized. We further define $\Delta(x, k)$ by

$$\Delta(x, k) = |\langle x|k, b\rangle|^2 - |\langle x|k, f\rangle|^2. \quad (2 \cdot 20)$$

Then we have

$$\begin{aligned} I_c(z) &= \text{Tr}' \left\{ (-1)^F \frac{z}{H + z} \right\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx dk}{2\pi} \frac{z}{k^2 + \mu^2 + z} \Delta(x, k), \end{aligned} \quad (2 \cdot 21)$$

where Tr' is the sum over all the states except for zero-modes. Comparing (2·21) with (2·17) and expanding them in z , we obtain

$$\text{Tr}' \left\{ \frac{(-1)^F}{H^m} \right\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx dk}{2\pi} \frac{\Delta(x, k)}{(k^2 + \mu^2)^m} = \frac{(2m-1)!!}{m! 2^m} \frac{\text{sign } \mu}{\mu^{2m}}, \quad m = 1, 2, 3, \dots \quad (2 \cdot 22)$$

The above result is a counterexample against the discussion that the sum of the expectation values of $(-1)^F g(H)$, g being arbitrary, over degenerate bosonic and fermionic states would vanish. Such kind of naive arguments is not valid in the presence of the continuum spectrum of H because of divergent norms of state vectors.

If we put $z = -E + i\varepsilon$, $E \geq \mu^2$, $\varepsilon = 0+$ and $z = -E - i\varepsilon$ in (2·17) and (2·21), take the difference between two cases, take into account that $\sqrt{\mu^2 - E \pm i\varepsilon} = \pm i\sqrt{E - \mu^2}$ for $E \geq \mu^2$, we obtain a sum rule

$$\begin{aligned} \int_{-\infty}^{\infty} dx [\Delta(x, \sqrt{E - \mu^2}) + \Delta(x, -\sqrt{E - \mu^2})] &= \frac{4\mu}{E}, \\ E &\geq \mu^2. \end{aligned} \quad (2 \cdot 23)$$

We have thus seen that Callias' theorem supplies us with much information about the continuum of the spectrum of H .

§ 3. The case of monopole-type superpotential

We next discuss SQM in the three dimensional space. The index theorem finds an interesting application in the case that the background potential is topologically non-trivial. The simplest potential of such kind is realized by the $SU(2)$ monopole.⁷⁾ We introduce the static isovector Higgs field ϕ and the Yang-Mills field A_i , $i = 1, 2, 3$, ($A_0 = 0$),

and assume that their configurations are characterized by the Bogomol'ny condition^{7),8)}

$$\mathbf{B}_i = D_i \boldsymbol{\phi}, \quad (3.1)$$

where \mathbf{B}_i and D_i are defined by

$$\mathbf{B}_i = \frac{1}{2} \varepsilon_{ijk} (\partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + \mathbf{A}_i \times \mathbf{A}_j), \quad (3.2)$$

$$D_i = \partial_i + \mathbf{A}_i \cdot \mathbf{T}. \quad (3.3)$$

Here the gauge coupling is set equal to unity and \mathbf{T} denotes the isospin matrix of $\boldsymbol{\phi}$:

$$(T^a)^{bc} = \varepsilon^{bac}. \quad (3.4)$$

The asymptotic behaviour of $\boldsymbol{\phi}$ is assumed to be

$$\boldsymbol{\phi} \sim \nu \hat{\boldsymbol{\phi}}, \quad \nu = \text{const}, \quad x_i x_i \sim \infty, \quad (3.5)$$

where $\hat{\boldsymbol{\phi}}$ satisfies

$$(\hat{\boldsymbol{\phi}})^2 = 1 \quad (3.6)$$

and

$$N = \frac{1}{8\pi} \int_{\Sigma} d^2 S_i \varepsilon_{ijk} \hat{\boldsymbol{\phi}} \cdot (\partial_j \hat{\boldsymbol{\phi}} \times \partial_k \hat{\boldsymbol{\phi}}), \quad \Sigma; \text{ sphere of an infinite radius}, \quad (3.7)$$

with N being an integer. N is known as the Kronecker index of the map $S^2(x_i x_i = \infty) \rightarrow S^2((\hat{\boldsymbol{\phi}})^2 = 1)$. It can be shown from (3.2), (3.5) and the Bianchi identity $D_i \mathbf{B}_i = 0$ that

$$\nu N = \frac{1}{4\pi} \int (\mathbf{B}_i)^2 d^3 x \geq 0. \quad (3.8)$$

In other words, \mathbf{B}_i can be put as in (3.1) only when νN is not negative. The magnetic charge of the monopole is identified with $\int_{\Sigma} \hat{\boldsymbol{\phi}} \cdot \mathbf{B}_i d^2 S_i$ and calculated to be $4\pi N$.^{7),9)}

Now we consider SQM in the isovector sector. We define the supercharges Q and Q^\dagger by

$$\begin{aligned} Q &= L\psi, \\ Q^\dagger &= \psi^\dagger L^\dagger, \end{aligned} \quad (3.9)$$

where L and L^\dagger are given by

$$\begin{aligned} L &= -i\sigma_i D_i + \boldsymbol{\phi} \cdot \mathbf{T}, \\ L^\dagger &= -i\sigma_i D_i - \boldsymbol{\phi} \cdot \mathbf{T}, \end{aligned} \quad (3.10)$$

σ_i 's being Pauli matrices. ψ and ψ^\dagger are fermionic variables satisfying the three dimensional version of (2.2). The hamiltonian H and the fermion number operator F are defined in the same way as in § 2:

$$H = \{Q, Q^\dagger\} = L^\dagger L F + L L^\dagger (1 - F) \quad (3.11)$$

and

$$F = \psi^\dagger \psi. \quad (3.12)$$

H commutes with Q , Q^\dagger and F . From (3.1) and (3.10), we have⁹⁾

$$L^\dagger L = -(D_i)^2 - (\boldsymbol{\phi} \cdot \mathbf{T})^2 - 2i\sigma_i(\mathbf{B}_i \cdot \mathbf{T}), \quad (3.13)$$

$$LL^\dagger = -(D_i)^2 - (\boldsymbol{\phi} \cdot \mathbf{T})^2. \quad (3.14)$$

We find that σ_i 's disappear in LL^\dagger . The eigenfunction Ψ of the Schrödinger equation

$$H\Psi = E\Psi \quad (3.15)$$

can now be regarded as describing an assembly of a two-component Pauli spinor $\boldsymbol{\chi}$ and two scalars S_1 and S_2 . All of $\boldsymbol{\chi}$, S_1 and S_2 are isovectors and are defined by the following equations:

$$L^\dagger L\boldsymbol{\chi} = E\boldsymbol{\chi}, \quad F\boldsymbol{\chi} = \boldsymbol{\chi}, \quad \boldsymbol{\chi} = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \quad (3.16)$$

and

$$LL^\dagger S = ES, \quad FS = 0, \quad S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}. \quad (3.17)$$

For non-vanishing values of E , $\boldsymbol{\chi}$ and S transit to each other when operated by Q/\sqrt{E} or Q^\dagger/\sqrt{E} .

The index theorem for the Dirac operator L of (3.10) was derived by Weinberg.⁹⁾ It is given by

$$\text{Tr} \left(\frac{z}{L^\dagger L + z} - \frac{z}{LL^\dagger + z} \right) = \frac{2N\nu}{\sqrt{z + \nu^2}}, \quad (3.18)$$

where N is defined by (3.7), ν by (3.5) and z is an arbitrary parameter. Just as in the previous section, formula (3.18) can be cast into its SQM-form:

$$\begin{aligned} I(z) &\equiv \text{Tr} \left\{ \frac{(-1)^F z}{H + z} \right\} \\ &= I(0) + I_c(z) \\ &= -\frac{2N\nu}{\sqrt{z + \nu^2}}, \end{aligned} \quad (3.19)$$

where $I(0)$ is equal to Witten's index,

$$I(0) = n_b^{E=0} - n_f^{E=0} = -2N \text{sign } \nu, \quad (3.20)$$

and $I_c(z)$ is given by

$$I_c(z) = \text{Tr}' \left\{ \frac{(-1)^F z}{H + z} \right\} = 2N \left(\text{sign } \nu - \frac{\nu}{\sqrt{z + \nu^2}} \right) \quad (3.21)$$

with $n_b^{E=0}$, $n_f^{E=0}$ and Tr' being of the same meaning as in § 2.

The continuum portion of the spectrum of H extends down to zero since one of the eigenvalues of $(\boldsymbol{\phi} \cdot \mathbf{T})^2$ in (3.13) and (3.14) is vanishing. We define $\Delta(\mathbf{x}, \mathbf{k})$ by

$$\Delta(\mathbf{x}, \mathbf{k}) = \sum_{\beta} |\langle \mathbf{x} | \mathbf{k}, b, \beta \rangle|^2 - \sum_{\beta} |\langle \mathbf{x} | \mathbf{k}, f, \beta \rangle|^2, \quad (3.22)$$

where $|\mathbf{k}, \alpha, \beta\rangle$ are characterized by

$$\begin{aligned} H|\mathbf{k}, \alpha, \beta\rangle &= k^2|\mathbf{k}, \alpha, \beta\rangle, & k &= |\mathbf{k}|, & \alpha &= b, f, \\ F|\mathbf{k}, f, \beta\rangle &= |\mathbf{k}, f, \beta\rangle, & F|\mathbf{k}, b, \beta\rangle &= 0 \end{aligned} \quad (3.23)$$

and β collectively represents indices other than \mathbf{k} and α . \mathbf{x} in (3.16) corresponds to $\langle \mathbf{x} | \mathbf{k}, f, \beta \rangle$, while S_1 and S_2 in (3.17) to $\langle \mathbf{x} | \mathbf{k}, b, \beta \rangle$. Then $I_c(z)$ is written as

$$I_c(z) = \int \frac{d^3k d^3x}{(2\pi)^3} \frac{z}{k^2 + z} \Delta(\mathbf{x}, \mathbf{k}). \quad (3.24)$$

If we define $\Delta\rho(E)$ by

$$\Delta\rho(E) = \int \frac{d^3k d^3x}{(2\pi)^3} \delta(k^2 - E) \Delta(\mathbf{x}, \mathbf{k}), \quad (3.25)$$

$\Delta\rho(E)$ measures the difference of the densities of bosonic and fermionic states at an energy E . Combining (3.21), (3.24) and (3.25) and noticing that $\sqrt{\nu^2 - E + i\varepsilon} - \sqrt{\nu^2 - E - i\varepsilon} = -2i\theta(E - \nu^2)/\sqrt{E - \nu^2}$, $\varepsilon = 0+$, we obtain

$$\Delta\rho(E) = \frac{2\nu N}{\pi E \sqrt{E - \nu^2}} \theta(E - \nu^2). \quad (3.26)$$

We see that $\Delta\rho(E)$ vanishes only for $0 \leq E < \nu^2$ and blows up as E tends to ν^2 from the above.

If the product νN is negative, \mathbf{B}_i cannot be identified with $D_i\phi$. It is easy, however, to understand that the identification $\mathbf{B}_i = -D_i\phi$ leads us to a consistent SQM for $\nu N < 0$.

§ 4. Summary

We discussed SQM in the cases that the spectrum of H consists of discrete and continuous levels. The SQM-versions of Callias' and Weinberg's index theorems turned out to be helpful to study rather subtle structures of continuum portions of spectra. We obtained formulae such as (2.22), (2.23) and (3.26). We hope that some aspects of Witten's general and abstract discussion²⁾ have been clarified by our concrete and simple examples.

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Appendix

— Elementary Derivation of Callias' Theorem —

To clarify the physical meaning of Callias' theorem (2.8), we here give an elementary proof of it. We follow and generalize Jackiw's method,^{10), 11)} by which he discussed the property of solitons in one-dimensional polymer. We define $u_E(x)$ and $v_E(x)$ as the properly normalized eigenfunctions of $L^\dagger L$ and LL^\dagger , respectively:

$$\begin{aligned} L^\dagger L u_E(x) &= E u_E(x), \\ LL^\dagger v_E(x) &= E v_E(x), \end{aligned} \quad (\text{A}\cdot 1)$$

where $\phi(x)$ is assumed to satisfy condition (2.6). For nonvanishing E , disregarding the phase ambiguity of $u_E(x)$ and $v_E(x)$, we can put⁽⁶⁾

$$v_E(x) = \frac{1}{\sqrt{E}} L u_E(x),$$

and conversely

$$u_E(x) = \frac{1}{\sqrt{E}} L^\dagger v_E(x). \quad (\text{A}\cdot 2)$$

We see that operators $L^\dagger L$ and LL^\dagger have common eigenvalues except for possible zero-modes. With the help of (A.1) and (A.2), we obtain^{(10), (11)}

$$|u_E(x)|^2 - |v_E(x)|^2 = -\frac{1}{E} \left[\frac{1}{2} \frac{d^2}{dx^2} |u_E(x)|^2 + \frac{d}{dx} \{ \phi(x) |u_E(x)|^2 \} \right], \quad E \neq 0. \quad (\text{A}\cdot 3)$$

The generalized index $J(z)$ of Eq. (2.8) consists of three parts:

$$\begin{aligned} J(z) &= \int_{-\infty}^{\infty} dx \mathcal{S}_E \frac{z}{E+z} \{ |u_E(x)|^2 - |v_E(x)|^2 \} \\ &= J_0 + J_D + J_C(z), \end{aligned} \quad (\text{A}\cdot 4)$$

where \mathcal{S}_E denotes the sum and integral over all the possible values of E . J_0 , J_D and $J_C(z)$ are contributions from zero-modes, non-zero discrete levels and the continuum, respectively. Because of the afore-mentioned degeneracy of the spectra of $L^\dagger L$ and LL^\dagger , J_D vanishes:

$$J_D = 0. \quad (\text{A}\cdot 5)$$

The zero-modes of $L^\dagger L$ is analyzed easily since $L^\dagger L u_0(x) = 0$ implies $L u_0(x) = 0$. The solution of the latter equation is given by $u_0(x) = \text{const} \cdot \exp[-\int_0^x \phi(x') dx']$ and can be normalized only when $\mu > 0 > \lambda$. Similarly, the operator LL^\dagger has a normalized zero-modes only in the case that $\lambda > 0 > \mu$. We conclude that

$$J_0 = \frac{1}{2} (\text{sign } \mu - \text{sign } \lambda). \quad (\text{A}\cdot 6)$$

As for $J_C(z)$, we must be careful since both $\int_{-\infty}^{\infty} |u_E(x)|^2 dx$ and $\int_{-\infty}^{\infty} |v_E(x)|^2 dx$ diverge. The naive discussion that the difference of these two pieces should vanish because of the relation (A.2) does not work. We now calculate $J_C(z)$ by temporarily assuming that $\mu^2 \geq \lambda^2$. For $E \geq \mu^2$, we have two solutions $u_E^{(1)}(x)$ and $u_E^{(2)}(x)$ of (A.1). The solution $u_E^{(1)}(x)$ ($u_E^{(2)}(x)$) expresses the wave incident from the left (right). The asymptotic behaviours of $u_E^{(1)}(x)$ and $u_E^{(2)}(x)$ are given by

$$\begin{aligned} u_E^{(1)}(x) &\sim e^{ik_1 x} + R^{(1)}(E) e^{-ik_1 x}, & x &\sim -\infty \\ u_E^{(1)}(x) &\sim T^{(1)}(E) e^{ik_2 x}, & x &\sim +\infty \\ u_E^{(2)}(x) &\sim e^{-ik_2 x} + R^{(2)}(E) e^{ik_2 x}, & x &\sim +\infty \end{aligned} \quad (\text{A}\cdot 7)$$

and

$$u_E^{(2)}(x) \sim T^{(2)}(E)e^{-ik_1x}, \quad x \sim -\infty$$

where k_1 and k_2 are defined by

$$k_1 = \sqrt{E - \lambda^2}$$

and

$$k_2 = \sqrt{E - \mu^2}. \quad (\text{A} \cdot 8)$$

It is well known¹²⁾ that $T(E)$'s and $R(E)$'s are related by

$$k_2 T^{(1)}(E) = k_1 T^{(2)}(E) \quad (\text{A} \cdot 9)$$

and

$$|R^{(1)}(E)|^2 + \frac{k_2}{k_1} |T^{(1)}(E)|^2 = |R^{(2)}(E)|^2 + \frac{k_2}{k_1} |T^{(2)}(E)|^2 = 1. \quad (\text{A} \cdot 10)$$

For $\lambda^2 \leq E < \mu^2$, we should put

$$|u_E^{(i)}(-\infty)|^2 = 2, \quad |u_E^{(i)}(+\infty)|^2 = 0, \quad i = 1, 2. \quad (\text{A} \cdot 11)$$

From (A·3) and (A·4), we have

$$\begin{aligned} J_C(z) &= \int_{-\infty}^{\infty} dx \mathcal{S}_{E \geq \lambda^2} \frac{z}{E+z} \{|u_E(x)|^2 - |v_E(x)|^2\} \\ &= \mathcal{S}_{E \geq \lambda^2} \frac{-z}{E(E+z)} \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2} \frac{d^2}{dx^2} |u_E(x)|^2 + \frac{d}{dx} (\phi(x) |u_E(x)|^2) \right\} \\ &= \mathcal{S}_{E \geq \lambda^2} \frac{-z}{E(E+z)} [\phi(x) |u_E(x)|^2]_{x=-\infty}^{x=+\infty}, \end{aligned}$$

where infinitely-oscillating terms have been dropped. With the aid of (2·6), (A·7), (A·8) and (A·11), the pieces at $x = +\infty$ and $x = -\infty$ are easily given by

$$-\mu z \left[\int_0^{\infty} \frac{dk_2}{2\pi} \frac{1}{E(E+z)} (1 + |R^{(2)}(E)|^2) + \int_0^{\infty} \frac{dk_1}{2\pi} \frac{\theta(E - \mu^2)}{E(E+z)} |T^{(1)}(E)|^2 \right]$$

and

$$\begin{aligned} &-\lambda z \left[\int_0^{\infty} \frac{dk_1}{2\pi} \frac{\theta(E - \mu^2)}{E(E+z)} (1 + |R^{(1)}(E)|^2) \right. \\ &\quad \left. + \int_0^{\infty} \frac{dk_2}{2\pi} \frac{1}{E(E+z)} |T^{(2)}(E)|^2 + \int_0^{\infty} \frac{dk_1}{2\pi} \frac{\theta(\mu^2 - E)}{E(E+z)} \cdot 2 \right], \end{aligned}$$

respectively. Recalling that $dk_1/dk_2 = k_2/k_1$ for $E \geq \mu^2$ and making use of (A·9) and (A·10), we obtain

$$\begin{aligned} J_C(z) &= -\frac{z}{\pi} \int_0^{\infty} dk \left[\frac{\mu}{(k^2 + \mu^2 + z)(k^2 + \mu^2)} - \frac{\lambda}{(k^2 + \lambda^2 + z)(k^2 + \lambda^2)} \right] \\ &= \frac{1}{2} \left(\frac{\mu}{\sqrt{\mu^2 + z}} - \frac{\lambda}{\sqrt{\lambda^2 + z}} \right) - \frac{1}{2} (\text{sign } \mu - \text{sign } \lambda). \quad (\text{A} \cdot 12) \end{aligned}$$

From (A·4)~(A·6) and (A·12), we have (2·8). A similar discussion leads us to (2·8) also for the case $\lambda^2 \geq \mu^2$. After completing the proof of (2·8), we understand that the remarkable theorem of Callias is essentially based on the elementary facts (A·2), (A·9) and (A·10). Relation (A·9) is derived by the fact that $u_E^{(1)}$ and $u_E^{(2)}$ satisfy the same Schrödinger equation, while (A·10) by the unitarity.¹²⁾

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