Macroeconometrics

Q7. (a) We are looking for the bivariate distribution of $(y_{1T+1} | \hat{y}_{1T+1|T}, y_{2T+1} | \hat{y}_{2T+1|T})$. First, from slide 13 of Lecture 1, recall that:

$$y_{T+h} = \hat{y}_{T+h|T} + \sum_{i=0}^{h-1} \phi_i \varepsilon_{T+h-i}$$

In particular, with $h = 1, 2$:

$$y_{T+1} = \hat{y}_{T+1|T} + \phi_0 \varepsilon_{T+1}$$

$$y_{T+2} = \hat{y}_{T+2|T} + \phi_0 \varepsilon_{T+2} + \phi_1 \varepsilon_{T+1}$$

By assumption, we know that $\varepsilon_t \sim N(0, \Sigma)$. Moreover,

$$E(y_{T+1} | \hat{y}_{T+1|T}) = \hat{y}_{T+1|T}$$

$$E(y_{T+2} | \hat{y}_{T+2|T}) = \hat{y}_{T+2|T}.$$

and we have:

$$Var(y_{T+1} | \hat{y}_{T+1|T}) = Var(\phi_0 \varepsilon_{T+1}) := \Sigma_1$$

$$Var(y_{T+2} | \hat{y}_{T+2|T}) = Var(\phi_0 \varepsilon_{T+2} + \phi_1 \varepsilon_{T+1}) := \Sigma_2.$$

where Σ_1 and Σ_2 can be estimated following slide 16 as the h-step ahead forecast error.

Moreover, we have:

$$\mathrm{Cov}(y_{T+1} | \hat{y}_{T+1|T} \ , \ y_{T+2} | \hat{y}_{T+2|T})$$

$$= \mathrm{Cov}(\phi_0 \varepsilon_{T+1} \ , \ \phi_1 \varepsilon_{T+2} + \phi_1 \varepsilon_{T+1})$$

$$= E\left[\phi_0 \varepsilon_{T+1} (\phi_0 \varepsilon_{T+2} + \phi_1 \varepsilon_{T+1})^T\right]$$

$$= E\left[\phi_0 \varepsilon_{T+1} (\varepsilon_{T+1}^T \phi_1^T)\right] \quad \text{since } \varepsilon_t \sim iid$$

$$= E\left[\phi_0 \varepsilon_{T+1} \varepsilon_{T+1}^T \phi_0\right] \phi_1^T \quad \text{by normalizing } \phi_0 = I_3$$

$$= \Sigma_1 \phi_1^T$$

~~Hence, the bivariate distribution is~~:

In particular, the bivariate distribution of $(y_{1,T+1} | \hat{y}_{1,T+1|T} \ , \ y_{1,T+2} | \hat{y}_{1,T+2|T})$ is:

$$\begin{pmatrix} y_{1,T+1} | \hat{y}_{1,T+1|T} \\ y_{1,T+2} | \hat{y}_{1,T+2|T} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \hat{y}_{1,T+1|T} \\ \hat{y}_{1,T+2|T} \end{pmatrix} , \begin{pmatrix} \hat{\Sigma}_{1,11} & (\hat{\Sigma}_1 \hat{\phi}_1^T)_{11} \\ (\hat{\Sigma}_1 \hat{\phi}_1^T)_{11} & \hat{\Sigma}_{2,11} \end{pmatrix} \right)$$