Macroeconometrics Assignment 3

Empirical question: asymmetric multipliers?

Consider the following piecewise linear bivariate SVAR model

$$\Psi_0(y_{1t})\, y_t = c + \sum_{i=1}^{p} \Psi_i y_{t-i} + \eta_t, \qquad \eta_t \sim iid\mathrm{N}[0, I_2] \tag{1}$$

for $t \geq p+1$, where

$$\Psi_0(y_{1t}) := \begin{cases} \Psi_0^+, & y_{1t} > 0 \\ \Psi_0^-, & y_{1t} \leq 0 \end{cases}, \quad \Psi_0^+ := \begin{bmatrix} \Psi_{0,11}^+ & \Psi_{0,12} \\ \Psi_{0,21} & \Psi_{0,22} \end{bmatrix}, \; \Psi_0^- := \begin{bmatrix} \Psi_{0,11}^- & \Psi_{0,12} \\ \Psi_{0,21} & \Psi_{0,22} \end{bmatrix}$$

and the initial values $y_{1:p}$ are fixed. For convenience, define the "sign of y_{1t}" indicator

$$s_t = \mathbf{1}\{y_{1t} > 0\}$$

so that, for example, we can write $\Psi_0(y_{1t}) = \Psi_0^+ s_t + \Psi_0^-(1 - s_t)$.

The structural IRF from model (1) is nonlinear and will generally be state-dependent. For example, the most commonly used definition of the response of y_{t+h} to an impulse ς in shock $\eta_{i,t}$ is

$$\mathrm{SIRF}_h := \mathbb{E}[y_{t+h} \mid \{y_{s<t}\}, \eta_{i,t} = \varsigma] - \mathbb{E}[y_{t+h} \mid \{y_{s<t}\}, \eta_{i,t} = 0]. \tag{2}$$

However, when y_{1t} is sufficiently far from zero, (2) can be approximated by

$$\mathrm{SIRF}_h = \begin{cases} C_h^+ \Theta_0^+ & y_{1t} \gg 0 \\ C_h^- \Theta_0^- & y_{1t} \ll 0 \end{cases} \tag{3}$$

where $C_h^{\pm}$ are the moving average coefficients corresponding to the VAR polynomial $\left(\Psi_0^{\pm}\right)^{-1} \sum_{i=1}^{p} \Psi_i L^i$ and $\Theta_0^{\pm} = \left(\Psi_0^{\pm}\right)^{-1}$.

You are asked to use the above model investigate possible asymmetries in the fiscal multiplier using data on real GDP growth (y_1) and fiscal deficit as a share of GDP (y_2).

Data: You can obtain data for this exercise from FRED using the following codes: A191RL1A225NBEA (real GDP growth) and FYFSGDA188S (surplus as a share of GDP) [this the negative of deficit!]. This data is at an annual frequency, so I recommend setting $p = 1$ in your empirical analysis.

The rest of the exercise assumes you have a sample of size T of data y_t.

1. State a necessary and sufficient condition for this model to be coherent, i.e., under which the system of equations (1) can be solved to express y_t as a function of exogenous and predetermined variables, for all possible realizations of the shock $\eta_t \in \mathbb{R}^2$.

2. Explain why the parameters of the structural model (1) appear to be identified on the order condition. (In fact, it can be shown that the rank condition is satisfied if and only if $\Psi_0^+ \neq \Psi_0^-$; you don't need to show this!) Discuss the similarities and differences with identification via heteroskedasticity.

3. Collect all the structural parameters in a vector ψ, and show that the log-likelihood function can be written as ($\|x\|^2 := x^\mathsf{T} x$ for any vector x)

$$L(\psi) = \sum_{t=p+1}^{T} \left(\log \det \Psi_0 (y_{1t}) - \log 2\pi - \frac{1}{2} \left\| \Psi_0 (y_{1t}) y_t - c - \sum_{i=1}^{p} \Psi_i y_{t-i} \right\|^2 \right)$$

4. In this step, you are asked to test the null hypothesis that the model (1) is linear. You can do that using the following steps:

   (a) Compute the restricted maximum likelihood, $\hat{L}_0$ say, under the null hypothesis. [hint: ψ is under-identified under H_0, so you should maximize the likelihood of the (observationally equivalent) reduced-form VAR. This can be computed analytically!]

   (b) Compute $\hat{L} := L(\hat{\psi}) = \max_\psi L(\psi)$ using an optimization method of your choice. [I recommend BFGS using the inverse of the Cholesky factor of $\hat{\Sigma}_\varepsilon$ as initial value for $\Psi_0^\pm$, i.e., initializing at the linear case.] Is the coherency condition satisfied at the MLE $\hat{\psi}$?

   (c) Compute the LR statistic $LR = 2(\hat{L} - \hat{L}_0)$ and calculate the p-value of the test using a χ^2 distribution (you should work out the correct degrees of freedom). [Alternatively, you can compute the critical value of the LR test using a parametric bootstrap, but this can take a long time!]

5. What does a rejection of your null hypothesis above imply about a possible asymmetry in fiscal multipliers in light of the definition of the SIRF in (3)?

6. Using the reduced-form VAR estimates you obtained in part 4(a), compute the SIRF_h of each shock $\eta_{i,t}$ under the assumption of symmetry ($\Psi_0^+ = \Psi_0^-$) using recursive (Cholesky) identification that places real GDP growth first. Carefully interpret the results, paying particular attention to the signs of the (dynamic) fiscal multipliers.

7. Using the MLE of the structural parameters you obtained in part 4(b), compute and plot the SIRFs in (3) under the unit effect normalization for each shock. [Hint: under the unit-effect normalization that η_{2t} has a unit effect on y_{2t}, we have $\Theta_{0,22} = 1$ and $\Theta_{0,12}^\pm = -\Psi_{12}/\Psi_{11}^\pm$.] Comment on your results and contrast them with the symmetric case in part 6.

8. Briefly discuss the limitations of the above analysis, describing threats to its internal and external validity, and suggesting how the model might be extended in order to address them?

Theoretical question

When answering these questions, please consult the most recent version of the Lecture 5 notes (as posted on 5 March). These questions are drawn the from exercises given in those notes.

1. Provide an example of a bivariate, mean zero random vector $\xi = (\xi_1, \xi_2)^\mathsf{T}$ such that $\mathbb{E}\xi\xi^\mathsf{T}$ is invertible, but $\mathbb{E}vv^\mathsf{T}$ is not, where $v := \operatorname{vech}(\xi\xi^\mathsf{T} - \mathbb{E}\xi\xi^\mathsf{T})$. [Hint: try to construct ξ_1 and ξ_2 such that they are *not* perfectly correlated, but $\xi_1\xi_2 = 0$ always.]

2. Let $\{X_n\}_{n\in\mathbb{N}}$ be a random sequence of m-vectors, and X^* a random m-vector. Suppose that for every subsequence $\{n(k)\}_{k\in\mathbb{N}}$, there exists a further subsequence $\{k(\ell)\}_{\ell\in\mathbb{N}}$ such that

$$X_{n[k(\ell)]} \xrightarrow{p} X^*$$

as $\ell \to \infty$. Show that $X_n \xrightarrow{p} X^*$ as $n \to \infty$. [Hint. Start your argument by supposing that X_n does *not* converge in probability to X^*.]

3. Suppose that for each $T \in \mathbb{N}$, $\{y_{T,t}\}_{t\in\mathbb{Z}}$ is a stationary process given by

$$y_{T,t} = \mu(\lambda_T) + \sum_{i=1}^{p} \Phi_i(\lambda_T) y_{T,t-i} + \varepsilon_{T,t} \qquad\qquad \varepsilon_{T,t} = \Sigma^{1/2}(\lambda_T)\xi_{T,t},$$

where $\xi_{T,t} \sim_{\text{i.i.d.}} \mathrm{N}[0, I_k]$, and $\Sigma^{1/2}$ denotes the positive definite square root of Σ. Here the notation $\Phi_i(\lambda)$, etc., denotes the value of Φ_i under the parameters

$$\lambda = (\mu^\mathsf{T}, (\operatorname{vec}\boldsymbol{\Phi})^\mathsf{T}, (\operatorname{vech}\Sigma)^\mathsf{T})^\mathsf{T}.$$

Suppose that $\lambda_T \to \lambda_0$ as $T \to \infty$; we suppose that the VAR satisfies the stability condition for all T, and in the limit (otherwise $\{y_{T,t}\}$ could not be stationary). To complete the proof of bootstrap consistency given in Section 5.3.3 of the notes, show that:

(a) the martingale difference array $\{u_{T,t}\}$ defined by

$$u_{T,t} := \frac{1}{T^{1/2}} x_{T,t-1} \otimes \varepsilon_{T,t}$$

where $x_{T,t} := (1, \boldsymbol{y}_{t-1}^\mathsf{T})^\mathsf{T}$, satisfies the conditional Lindeberg condition of the MGCLT; and

(b) for A any continuous function of $\beta = (\mu^\mathsf{T}, (\operatorname{vec}\boldsymbol{\Phi})^\mathsf{T})^\mathsf{T}$, show that

$$A(\hat{\beta}_T)(\hat{\Gamma}_T^{-1} \otimes \hat{\Sigma}_T)A(\hat{\beta}_T)^\mathsf{T} \xrightarrow{p}_{\lambda_T} A(\beta_0)(\Gamma_0^{-1} \otimes \Sigma_0)A(\beta_0)^\mathsf{T}$$

as per (5.3.10) in the notes, where $\hat{\Gamma}_T = T^{-1}\sum_{t=1}^{T} x_{T,t-1}x_{T,t-1}^\mathsf{T}$.