Macroeconometrics

Q1. A piecewise linear model of the form:

$$f_0(y_t) = \Psi_0^+ s_t y_t + \Psi_0^- (1 - s_t) y_t$$

is coherent iff det Ψ_0^+ and det Ψ_0^- have the same sign (see slide 4, Lecture 8). That is, if:

$$\det(\Psi_0^+) = \Psi_{0,11}^+ \Psi_{0,22} - \Psi_{0,12} \Psi_{0,21}$$

and $$\det(\Psi_0^-) = \Psi_{0,11}^- \Psi_{0,22} - \Psi_{0,12} \Psi_{0,21}$$

have the same sign, or $(\Psi_{0,11}^+ \Psi_{0,22} - \Psi_{0,12} \Psi_{0,21})(\Psi_{0,11}^- \Psi_{0,22} - \Psi_{0,12} \Psi_{0,21}) > 0$.

Q2. Assuming $\Psi_{0,11}^+ \neq \Psi_{0,11}^-$, there are five parameters to identify, $\{\Psi_{0,11}^+, \Psi_{0,11}^-, \Psi_{0,12}, \Psi_{0,21}, \Psi_{0,22}\}$.

The variances on the reduced-form model are:

$$\Omega_0^+ = (\Psi_0^+)^{-1} I_2 (\Psi_0^+)^{-1T} = (\Psi_0^+)^{-1} (\Psi_0^+)^{-1T}$$

$$\Omega_0^- = (\Psi_0^-)^{-1} I_2 (\Psi_0^-)^{-1T} = (\Psi_0^-)^{-1} (\Psi_0^-)^{-1T}$$

which should contain $2 \times k(k+1)/2 = 6$ restriction with $k = 2$, if $\Psi_0^- \neq \Psi_0^+$. The model is over-identified and the order condition holds.

Both identification via regime change and identification via heteroskedasticity use switches between regimes and the corresponding ~~variation~~ changes in variances to identify the model's parameters, some of which are constant across the regimes. While identification via heteroskedasticity assumes that the regime switch is exogenous (e.g. exogenous change in monetary policy regime due to a new governor), identification via regime switch assumes regime switch is endogenous (e.g. the ZLB binding or not binding).

Q3. Start with the log-likelihood function:

$$L(\psi) = \sum_{t=p+1}^{T} \log \Pr(y_t \mid y_{1:t-1}, \psi)$$

Now, notice that given $\eta_t \sim N(0, I_2)$:

$$(y_t \mid y_{1:t-1}, \psi) \sim N\left((\Phi_0(s_{1t}))^{-1}\left(c + \sum_{i=1}^{p} \Phi_i y_{t-i}\right), (\Phi_0(s_{1t}))^{-1} (\Phi_0(s_{1t}))^{-1T} \right)$$

Hence, from the bivariate normal distribution, we have:

$_t = \Phi_0(s_{1t})$

$$\Pr(y_t \mid y_{1:t-1}, \psi) = \frac{\exp\left(-\frac{1}{2}\left(y_t - \Phi_{0t}^{-1}\left(c + \sum_{i=1}^{p} \Phi_i y_{t-i}\right)\right)^T \Sigma_{\varepsilon t}^{-1} \left(y_t - \Phi_{0t}^{-1}(c + \dots)\right)\right)}{2\pi \sqrt{\det(\Sigma_{\varepsilon t})}}$$

First, consider the numerator, which simplifies to:

$$-\frac{1}{2}\left[y_t - \Phi_{0t}^{-1}\left(c + \sum \Phi_i y_{t-i}\right)\right]^T \Sigma_{\varepsilon t}^{-1} \left[y_t - \Phi_{0t}^{-1}\left(c + \sum \Phi_i y_{t-i}\right)\right]$$

$$= -\frac{1}{2}\left[\Phi_{0t} y_t - \left(c + \sum \Phi_i y_{t-i}\right)\right]^T \underbrace{(\Phi_{0t}^{-1})^T (\Phi_{0t}^T \Phi_{0t})(\Phi_{0t}^{-1})}_{I_2.}$$

$$\left[\Phi_{0t} y_t - \left(c + \sum \Phi_i y_{t-i}\right)\right]$$

$$= -\frac{1}{2} \| \Phi_{0t} y_t - c - \Sigma \Phi_i y_{t-i} \|^2$$

Next, consider the denominator:

$$\Sigma_{\varepsilon t} = \Phi_{0t}^{-1} \Phi_{0t}^{-1T}$$

$$\Longrightarrow \det(\Sigma_{\varepsilon t}) = \det(\Phi_{0t}^{-1}) \det(\Phi_{0t}^{-1T})$$

$$= \det(\Phi_{0t}^{-1})^2$$

$$\Longrightarrow \det(\Sigma_{\varepsilon t})^{1/2} = \det(\Phi_{0t}^{-1}) = \det(\Phi_{0t})^{-1}$$

Combining everything together:

$$\log \Pr(y_t | y_{1:t-1}, \Phi) = -\frac{1}{2} \| \Phi_{0t} y_t - c - \sum_{i=1}^{p} \Phi_i y_{t-i} \|^2$$

$$- \log 2\pi + \log \det(\Phi_{0t})$$

$$\Longrightarrow L(\Phi) = \sum_{t=p+1}^{T} \Big(\log \det(\Phi_{0t}) - \log 2\pi - \frac{1}{2} \| \Phi_{0t} y_t - c$$

$$- \sum_{i=1}^{p} \Phi_i y_{t-i} \|^2 \Big)$$