Macroeconometrics PS1

Q1. Start from the assumptions that:

- $\{y_t\}_{t=1}^T$ is generated by VAR(p) ~~$\otimes$~~.
- $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is iid with $E\varepsilon_t = 0$, $E\varepsilon_t\varepsilon_t^T = \Sigma_0$ has full rank, $\|\varepsilon_t\|_{2+\delta} < \infty$ ...
- $\{y_t\}_{t=1}^T$ is stationary.

Moreover, $\hat{\varepsilon}_t$ is defined such that:

$$\hat{\varepsilon}_t := y_t - \sum_{i=1}^{p} \hat{\Phi}_i y_{t-i} = \varepsilon_t + Z_{t-1}^T(\beta_0 - \hat{\beta}_T)$$

Then first, observe that:

$$\hat{\varepsilon}_t\hat{\varepsilon}_t^T = \left[\varepsilon_t + Z_{t-1}^T(\beta_0 - \hat{\beta}_T)\right]\left[\varepsilon_t + Z_{t-1}^T(\beta_0 - \hat{\beta}_T)\right]^T$$

$$= \left[\varepsilon_t \quad Z_{T-1}^T(\beta_0 - \hat{\beta}_T)\right]\begin{bmatrix} 1 \\ 1 \end{bmatrix}\left(\left[\varepsilon_t \quad Z_{t-1}^T(\beta_0 - \hat{\beta}_T)\right]\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^T$$

$$= \left[\varepsilon_t \quad Z_{t-1}^T(\beta_0 - \hat{\beta}_T)\right]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \varepsilon_t^T \\ (\beta_0 - \hat{\beta}_T)^T Z_{t-1} \end{bmatrix}$$

$$= \left[\varepsilon_t + Z_{t-1}^T(\beta_0 - \hat{\beta}_T) \quad \varepsilon_t + Z_{t-1}^T(\beta_0 - \hat{\beta}_T)\right]\begin{bmatrix} \varepsilon_t^T \\ (\beta_0 - \hat{\beta}_T)^T Z_{t-1} \end{bmatrix}$$

$$= \varepsilon_t \varepsilon_t^T + Z_{t-1}^T(\beta_0 - \hat{\beta}_T)\varepsilon_t^T + \varepsilon_t(\beta_0 - \hat{\beta}_T)^T Z_{t-1}$$

$$+ Z_{t-1}^T(\beta_0 - \hat{\beta}_T)(\beta_0 - \hat{\beta}_T)^T Z_{t-1}$$

We now examine what each term converges to in probability. Start with $\varepsilon_t \varepsilon_t^T$. Since $\varepsilon_t \sim iid$ and $E\varepsilon_t\varepsilon_t^T = \Sigma_0$, we have:

$$\frac{1}{T}\sum_{t=1}^{T} \varepsilon_t \varepsilon_t^T \xrightarrow{P} E\varepsilon_t\varepsilon_t^T = \Sigma_0$$

by the law of large number.

~~Next, we examine $Z_{t-1}^T(\beta_0 - \hat{\beta}_T)\varepsilon_t^T$. Since $\{y_t\}_{t=1}^T$ is generated by a VAR(p), ε_t is orthogonal to $Z_{t-1}^T\hat{\beta}_T$. Moreover, recall that $E(\varepsilon_t) = 0$. Hence:~~

$$\cancel{E\left(Z_{t-1}^T(\beta_0 - \hat{\beta}_T)\varepsilon_t^T\right) = E\left(Z_{t-1}^T(\beta_0 - \hat{\beta}_T)\right)E\,\varepsilon_t^T}$$

$$\cancel{= 0.}$$

Next, we examine $Z_{t-1}^T(\beta_0 - \hat{\beta}_1)\varepsilon_t^T$. By all the assumptions made above, Theorem 2.4.1 in the notes show that

$$T^{1/2}(\beta_0 - \hat{\beta}_T) \xrightarrow{d} N[0, \Gamma^{-1} \otimes \Sigma_0].$$

$$\Rightarrow T^{-1/2}\sum_{t=1}^{T}(\beta_0 - \hat{\beta}_T) \xrightarrow{P} 0 \; \dots \; (1)$$

Moreover, defining:

$$\sum_{t=1}^{T} \mu_{T,t} = T^{-1/2}\sum_{t=1}^{T} Z_{t-1}\varepsilon_t$$

Let $\mathcal{F}_{T,t} = \sigma(\{\{y_s\}_{s\le t}\})$. Since $z_{t-1} \in \mathcal{F}_{T,t-1}$ and ~~ε_t is~~ $E\varepsilon_t = 0$, $\varepsilon_t \sim$ iid (and therefore independent of $\mathcal{F}_{T,t-1}$), we have

$$E(u_{T,t} \mid \mathcal{F}_{T,t-1}) = T^{-1/2} z_{t-1} E(\varepsilon_t \mid \mathcal{F}_{T,t-1}) = 0$$

so that $\{u_{T,t}, \mathcal{F}_{T,t}\}$ is a mean-zero martingale difference vector.

In addition, given $\|z_{t-1}\varepsilon_t\| \le \|z_{t-1}\|\|\varepsilon_t\|$ and by independence of ε_t and $\mathcal{F}_{T,t-1}$,

$$E\left(\|z_{t-1}\varepsilon_t\|_{2+\delta} \mid \mathcal{F}_{T,t-1}\right) \le \|z_{t-1}\|_{2+\delta} \, E\|\varepsilon_t\|_{2+\delta}$$

$$= C\|z_{t-1}\|_{2+\delta}$$

where we have used the assumption that $\|\varepsilon_t\|_{2+\delta} < \infty$. Hence,

$$\sum_{t=1}^{T} E\left(\|u_{T,t}\|_{2+\delta}^{\cancel{2+\delta}} \mid \mathcal{F}_{T,t-1}\right) \le \frac{C}{T^{1+\delta/2}} \sum_{t=1}^{T} \|z_{t-1}\|_{2+\delta}^{\cancel{2+\delta}}$$

$\|z_{t-1}\|_{2+\delta}^{\cancel{2+\delta}}$ is a transformation of y_{t-1}, which is ~~itself~~ stationary and ergodic by assumption, such that $\|z_{t-1}\|_{2+\delta}$ is itself stationary and ergodic (Lemma 2.3.1). By the ergodic theorem,

$$T^{-1} \sum_{t=1}^{T} \|z_{t-1}\|_{2+\delta} \xrightarrow{a.s.} E\|z_t\|_{2+\delta} < \infty$$

such that the r.h.s is $O_p(T^{-\delta/2})$:

$$\sum_{t=1}^{T} E\left(\|u_{T,t}\|_{2+\delta} \mid \mathcal{F}_{T,t-1}\right) \le \frac{C}{T^{1+\delta/2}} \sum_{t=1}^{T} \|z_{t-1}\|_{2+\delta} \xrightarrow{p} 0$$

Moreover, we have by the ergodic theorem,

$$\sum_{t=1}^{T} E\left(u_{T,t} u_{T,t}^T \mid \mathcal{F}_{T,t-1}\right) = T^{-1}\sum_{t=1}^{T} Z_{t-1} E\left(\varepsilon_t\varepsilon_t^T \mid \mathcal{F}_{t-1}\right) Z_{t-1}$$

$$= T^{-1}\sum_{t=1}^{T} Z_{t-1}\Sigma_0 Z_{t-1}^T$$

$$\xrightarrow{p} E Z_t \Sigma_0 Z_t^T.$$

* where

$$E Z_t \Sigma_0 Z_t^T = E(\alpha_{t-1} \otimes I_k)(1 \otimes \Sigma_0)(\alpha_{t-1}^T \otimes I_k)$$

$$= E\,\alpha_{t-1}\alpha_{t-1}^T \otimes \Sigma_0 = \Gamma \otimes \Sigma_0$$

with $\Gamma \otimes \Sigma_0$ being a positive semi-definite matrix (see in the notes)

We can now invoke the Martingale central limit theorem to show that

$$T^{-1/2}\sum_{t=1}^{T} Z_{t-1}\varepsilon_t \xrightarrow{d} N(0, \Gamma\otimes\Sigma_0)$$

$$\Longrightarrow T^{-1/2}\sum_{t=1}^{T} Z_{t-1}\varepsilon_t \xrightarrow{p} 0 \quad \ldots ②$$

By Slutsky's Theorem and ① ; ②, we have that

$$T^{-1}\sum_{t=1}^{T} Z_{t-1}^T(\beta_0 - \hat{\beta}_T)\varepsilon_t^T \longrightarrow 0.$$

Finally, we examine $Z_{t-1}^T(\beta_0 - \hat{\beta}_T)(\beta_0 - \hat{\beta}_T)^T Z_{t-1}$. Since $Z_{t-1}Z_{t-1}^T$ is a transformation of y_{t-1}, a ~~non lin~~ stationary and ergodic process, and since $E\|Z_{t-1}Z_{t-1}^T\| < \infty$ (see notes), $Z_t Z_t^T$ is also a stationary and ergodic process. Using the ergodic theorem,

$$T^{-1}\sum_{t=1}^{T} Z_{t-1}Z_{t-1}^T \xrightarrow{p} E Z_{t-1}Z_{t-1}^T \quad \ldots ③$$

By Slutsky's Theorem and ①, ③, we have:

$$T^{-1}\sum_{t=1}^{T} Z_{t-1}^T(\beta_0-\hat{\beta}_T)(\beta_0-\hat{\beta}_T)^T Z_{t-1} \xrightarrow{P} 0$$

Putting all previous results together, we have shown that

$$\cancel{T^{-1}\sum_{t=1}^{T}\hat{\varepsilon}_t\hat{\varepsilon}_t^T = \underbrace{\varepsilon_t\varepsilon_t^T}_{\to\Sigma_0} + \underbrace{Z_{t-1}^T(\beta_0-\hat{\beta}_T)\varepsilon_t^T}_{\to 0}}$$

$$\cancel{+\underbrace{\varepsilon_t(\beta_0-\hat{\beta}_T)^T Z_{t-1}}_{\to 0} + \underbrace{Z_{t-1}^T(\beta_0-\hat{\beta}_T)(\beta_0-\hat{\beta}_T)^T Z_{t-1}}_{\to 0}}$$

$$\cancel{\to E\,\varepsilon_t\varepsilon_t^T = \Sigma_0}$$

$$T^{-1}\sum_{t=1}^{T}\hat{\varepsilon}_t\hat{\varepsilon}_t^T = \overset{\to\Sigma_0}{T^{-1}\sum_{t=1}^{T}\varepsilon_t\varepsilon_t^T} + \overset{\to 0}{T^{-1}\sum_{t=1}^{T} Z_{t-1}^T}\; \overset{\to 0}{T^{-1}\sum_{t=1}^{T}(\beta_0-\hat{\beta}_T)\varepsilon_t^T}$$

$$\underset{\to 0}{}\qquad\underset{\to 0}{}$$

$$+\ T^{-1}\sum_{t=1}^{T}\varepsilon_t \wedge_{T^{-1}\sum_{t=1}^{T}} (\beta_0-\hat{\beta}_T)^T Z_{t-1}$$

$$+\ \underset{\to E Z_{t-1}Z_{t-1}^T}{T^{-1}\sum_{t=1}^{T} Z_{t-1}^T}\; \overset{\to 0}{T^{-1}\sum_{t=1}^{T}(\beta_0-\hat{\beta}_T)}\; \overset{\to 0}{T^{-1}\sum_{t=1}^{T}(\beta_0-\hat{\beta}_T)^T} Z_{t-1}$$

$$\xrightarrow{P} \Sigma_0$$

using the fact that $(\beta_0-\hat{\beta}_T) = T^{-1}\sum_{t=1}^{T}(\beta_0-\hat{\beta}_T)$

Macroeconometrics PS1

Q2. Since $\{y_t\}$ is given its stationary initialisation and $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a stationary and ergodic mds sequence with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{Z}}$, $\{y_t\}$, defined as a stable VAR(1) process:

$$y_t = \Phi y_{t-1} + \varepsilon_t$$

is ~~also~~ stationary. Therefore, the proof in section 2.4.1 applies up to ~~equation (2.4.5)~~ before equation (2.4.5) in the notes.

However, $\{\varepsilon_t\}$ is no longer iid so that the application of the MGCLT to the numerator of

$$T^{1/2}(\hat{\beta}_T - \beta_0) = \left(T^{-1}\sum_{t=1}^{T} z_{t-1} z_{t-1}^{T}\right)^{-1} T^{-1/2}\sum_{t=1}^{T} z_{t-1}\varepsilon_t$$

needs to be revised.

First, define $u_{T,t}$ such that:

$$\sum_{t=1}^{T} u_{T,t} := T^{-1/2}\sum_{t=1}^{T} z_{t-1}\varepsilon_t .$$

Let $\mathcal{F}_t = \sigma(\{y_s\}_{s\le t})$. Since $z_{t-1}\in\mathcal{F}_{t-1}$ and $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is given as an mds, we have:

$$E(u_{T,t}\,|\,\mathcal{F}_{t-1}) = T^{-1/2} z_{t-1} E(\varepsilon_t\,|\,\mathcal{F}_{t-1}) = 0$$

so that $\{u_{T,t}, \mathcal{F}_t\}$ is an mds sequence.

Moreover, $\|Z_{t-1}\varepsilon_t\| \leq \|Z_{t-1}\|\|\varepsilon_t\|$ and given $Z_{t-1} \in \mathcal{F}_{t-1}$:

$$E\left(\|Z_{t-1}\varepsilon_t\|^{2+\delta} \,\middle|\, \mathcal{F}_{t-1}\right) \leq \|Z_{t-1}\|^{2+\delta} E\left(\|\varepsilon_t\|^{2+\delta} \,\middle|\, \mathcal{F}_{t-1}\right)$$

$$= C\|Z_{t-1}\|^{2+\delta}$$

given the assumption that there exists $\delta > 0$ such that:

$$E\left(\|\varepsilon\|^{2+\delta} \,\middle|\, \mathcal{F}_{t-1}\right) \leq C, \quad \forall t \in \mathbb{Z}.$$

Hence,

$$\sum_{t=1}^{T} E\left(\|u_{T,t}\|^{2+\delta} \,\middle|\, \mathcal{F}_{t-1}\right) \leq \frac{C}{T^{1+\delta/2}} \sum_{t=1}^{T} \|Z_{t-1}\|^{2+\delta}.$$

$\|Z_{t-1}\|^{2+\delta}$ is a measurable transformation of z_{t-1}, which is itself stationary and ergodic. By applying the ergodic theorem, we immediately see that the RHS is $O_p(T^{-\delta/2}) = o_p(1)$, as required to use the MGCLT.

Next, observe that:

$$\left\langle \sum_{t=1}^{T} u_{T,t} \right\rangle = \sum_{t=1}^{T} E\left(u_{T,t}u_{T,t}^{T} \,\middle|\, \mathcal{F}_{t-1}\right)$$

$$= T^{-1}\sum_{t=1}^{T} Z_{t-1} E\left(\varepsilon_t\varepsilon_t^{T} \,\middle|\, \mathcal{F}_{t-1}\right) Z_{t-1}^{T}$$

$$\xrightarrow{P} E\left(Z_{t-1} E\left(\varepsilon_t\varepsilon_t^{T} \,\middle|\, \mathcal{F}_{t-1}\right) Z_{t-1}^{T}\right)$$

by applying the ergodic theorem.

Now, we are given the assumption that:

$$T^{-1}\sum_{t=1}^{T} E\left(\varepsilon_t\varepsilon_t^T \,\middle|\, \mathcal{F}_{t-1}\right) \xrightarrow{P} \Sigma.$$

$$\Longrightarrow T^{-1}\sum_{t=1}^{T} E\left(\varepsilon_t\varepsilon_t^T \,\middle|\, \mathcal{F}_{t-1}\right) \xrightarrow{P} E\left(E\left(\varepsilon_t\varepsilon_t^T \,\middle|\, \mathcal{F}_{t-1}\right)\right)$$

$$= E\left(\varepsilon_t\varepsilon_t^T\right) = \Sigma.$$

Using this fact, notice that:

$$E\left(Z_{t-1}\, E\left(\varepsilon_t\varepsilon_t^T \,\middle|\, \mathcal{F}_{t-1}\right) Z_{t-1}^T\right)$$

$$= E\left(y_{t-1} \otimes I_k\right)\left(1 \otimes E\left(\varepsilon_t\varepsilon_t^T \,\middle|\, \mathcal{F}_{t-1}\right)\right)\left(y_{t-1}^T \otimes I_k\right)$$

$$= E\left(y_{t-1}\, y_{t-1}^T\right) \otimes E\left(E\left(\varepsilon\varepsilon_t^T \,\middle|\, \mathcal{F}_{t-1}\right)\right)$$

$$= E\left(y_{t-1}\, y_{t-1}^T\right) \otimes \Sigma.$$

$$=_{(1)} E\left(y_0\, y_0^T\right) \otimes \Sigma$$

$$= \Gamma \otimes \Sigma.$$

where $=_{(1)}$ follows by stationarity. The stability condition is therefore satisfied.

By applying the MGCLT, we have:

$$T^{-1/2}\sum_{t=1}^{T} Z_{t-1}\,\varepsilon_t \xrightarrow{d} N(0, \Gamma \otimes \Sigma).$$

. The rest of the proof follows the notes. From the results shown above, equations (2.2.4) and (2.4.3), using Slutsky's Theorem, we have:

$$T^{1/2}(\hat{\beta}_T - \beta_0) = \left(T^{-1}\sum_{t=1}^{T} Z_{t-1} Z_{t-1}^{T}\right)^{-1} T^{-1/2}\sum_{t=1}^{T} Z_{t-1}\varepsilon_t$$

$$\xrightarrow{d} (\Gamma \otimes I_k)^{-1} N(0, \Gamma \otimes \Sigma_0)$$

$$=_d N(0, \Gamma^{-1} \otimes \Sigma).$$