University of Lagos, Akoka
Department of Mathematics
Second Semester Examination 2017/2018 Session
Course Code: Mat 204/ 222 Course Title: Linear Algebra I
Instructions: Answer all questions in both section A and B Time Allowed: **2hrs.**

Section A.

1. Find all values of t which makes the vectors $(t, -1, 0, 0)$, $(1, 3, t, 0)$, $(0, -1, -1, 0)$, $(-2, 1, t, 1)$ linearly dependent.
2. Find the value(s) of c if $v = (-3, -5, 1, c, -2)$ *and* $\|v\| = \sqrt{55}$
3. If $f(x) = x^2 - 4x + 6$, find $f(A)$ where $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$.
4. Factorize the determinant of $\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{bmatrix}$.
5. Use Crammers rule to solve the system of linear equation over R
   $2x - y + 3z = 7$
   $x + 2y - z = 1$
   $x + y + z = 3$
6. Verify $(AB)^{-1} = B^{-1}A^{-1}$ where $A = \begin{bmatrix} 4 & 2 \\ 5 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 6 \\ 3 & 2 \end{bmatrix}$.
7. If $u = (6, -1, 2)$, $v = (-3, 2, 1)$, $w = (-2, 3, 4)$ are vectors in R^3 Compute $5u + 4v - 3w$.
8. If $R = \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix}$, find r, s such that $R + rR + sI = 0$ and Determine R^{-1}.
9. Show that $det. \begin{bmatrix} n! & (n-1)! & (n-2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{bmatrix} = n!\,(n+1)!\,(n+2)!\,2.$
10. If $P = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ and $QP = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ determine Q.

Section B

Question One

a. Define the following
   i. Linear Independence.
   ii. Subspace.
   iii. Basis.
b. Show that $w = \{ f : f(2) = 0\}$ is a subspace of V, if V is the set of functions from R into R.
c. Let $T : R^3 \to R^3$ be a linear operator defined by $T(x, y, z) = (2y + z, x - 4y, 3x)$.
   i. Find the matrix $[T]_e$ of T in the basis $e = \{e_1 = (1,1,0), e_2 = (1, 0, 0), e_3 = (0, 0, 1)\}$.
   ii. If $v = (3, 5, -2)$, verify that $[T]_e[v]_e = [T(v)]_e$.

Question Two

a. What are Similar matrices?

b. Let $f: R^2 \to R^2$ be a mapping defined by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ 3x + 10y \end{bmatrix}$ with respect to the basis i and j.

i. Show that f is bijective.

ii. Define the inverse mapping f^{-1}.

iii. Write down the matrix of transformation for the mapping f with respect to the basis $\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$ of R^2.

c. Establish the linear dependence or independence of vectors $v_1 = (1, 0, -1, 2)$, $v_2 = (1, 3, 1, 6)$, $v_3 = (1, 5, -1, 16)$, $v_4 = (4, 1, 0, 2)$ in R^4 and find the dimension of the space spanned by these vectors.

Question Three

a. Give the definition of the following

i. Equal matrices

ii. Upper triangular matrices

iii. Scalar matrices

b. Let V be a vector space of all $n \times n$ matrices over a field K. if A is a given matrix in V, let $f: V \to V$ be a mapping defined by $f(M) = MA + AM$. Show that f is a linear transformation.

c. If $A = \begin{pmatrix} t-4 & 0 & 0 \\ -2 & t+1 & 1 \\ 3 & 4 & t-2 \end{pmatrix}$, find the value of t for which A is singular

d. Find the row rank of the matrix $\begin{pmatrix} 8 & -1 & 6 & 3 & 7 \\ 3 & 0 & -1 & 4 & 3 \\ -1 & -1 & 9 & -9 & -2 \end{pmatrix}$.

Section A Solution to 2017/18 Examination

Bolu-make-nails

2) $v = (-3, -5, 1, c, -2)$

$\|v\| = \sqrt{(-3)^2 + (-5)^2 + (1)^2 + c^2 + (-2)^2}$

$\sqrt{55} = \sqrt{9 + 25 + 1 + c^2 + 4}$

$39 + c^2 = 55$

$c^2 = 16$

$c = \pm 4$

3) $f(x) = x^2 - 4x + 6$ $\quad A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$

$f(A) = A^2 - 4A + 6I$

$A^2 = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1+6 & -2-8 \\ -3-12 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & -10 \\ -15 & 22 \end{bmatrix}$

$-4A = -4\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 8 \\ 12 & -16 \end{bmatrix}$

$6I = 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

$f(A) = \begin{bmatrix} 7 & -10 \\ -15 & 22 \end{bmatrix} + \begin{bmatrix} -4 & 8 \\ 12 & -16 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

$f(A) = \begin{bmatrix} 9 & 2 \\ -3 & 12 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{bmatrix} = 1\begin{vmatrix} y & z \\ y^3 & z^3 \end{vmatrix} - \begin{vmatrix} x & z \\ x^3 & z^3 \end{vmatrix}$

$+1\begin{vmatrix} x & y \\ x^3 & y^3 \end{vmatrix} = yz^3 - y^3z - xz^3 + x^3z + xy^3 - x^3y$

$Vol = yz^3 - y^3z - xz^3 + x^3z + xy^3 - x^3y$

5) We were told to use Cramer's rule which we were not taught so Bolu-make-nails will use the method we were taught 😊

$2x - y + 3z = 7$
$x + 2y - 2z = 1$
$x + y + z = 3$
$\Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 3 & 7 \\ 1 & 2 & -2 & 1 \\ 1 & 1 & 1 & 3 \end{array}\right]$

$R_3' = R_2 - R_1$ $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 7 \\ 1 & 2 & -2 & 1 \\ 0 & 1 & -3 & -2 \end{array}\right]$

$R_2' = 2R_2 - R_1$ $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 7 \\ 0 & 5 & -7 & -5 \\ 0 & 1 & -3 & -2 \end{array}\right]$

$R_3'' = 5R_3' - R_2$ $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 7 \\ 0 & 5 & -7 & -5 \\ 0 & 0 & -8 & -5 \end{array}\right]$

$\frac{R_1}{2}, \frac{R_2}{5}, -\frac{R_3}{8}$ $\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 7/2 \\ 0 & 1 & -\frac{7}{5} & -1 \\ 0 & 0 & 1 & 5/8 \end{array}\right]$

$z = \frac{5}{8}$ $\quad y - \frac{7}{5}z = -1$

$y = -1 + \frac{7}{5} \times \frac{5}{8} = -1 + \frac{7}{8}$

$y = -1/8$

$x - \frac{y}{2} + \frac{3z}{2} = \frac{7}{2}$

$x + \frac{1}{16} + \frac{15}{16} = \frac{7}{2}$

$x + 1 = \frac{7}{2}$

$x = \frac{7}{2} - \frac{2}{2} = 5/2$

$\therefore x = 5/2, \; y = -1/8, \; z = 5/8$

6) $A = \begin{bmatrix} 4 & 2 \\ 5 & -4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 6 \\ 3 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 0+6 & 24+4 \\ 0-12 & 30-8 \end{bmatrix} = \begin{bmatrix} 6 & 28 \\ -12 & 22 \end{bmatrix} = C$

Let $C = AB$

$C^{-1} = ?$

Minor $\begin{bmatrix} 22 & -12 \\ 28 & 6 \end{bmatrix}$ Co-factor $\begin{bmatrix} 22 & 12 \\ -28 & 6 \end{bmatrix}$

Adjoint $\frac{1}{468}\begin{bmatrix} 22 & -28 \\ 12 & 6 \end{bmatrix} = (AB)^{-1}$

$A = \begin{bmatrix} 4 & 2 \\ 5 & -4 \end{bmatrix}$ $\quad A^{-1} = ?$

Minor $\begin{bmatrix} -4 & 5 \\ 2 & 4 \end{bmatrix}$ Cofactor $\begin{bmatrix} -4 & -5 \\ -2 & 4 \end{bmatrix}$

Adjoint $\frac{-1}{26}\begin{bmatrix} -4 & -2 \\ -5 & 4 \end{bmatrix} = A^{-1}$

$B = \begin{bmatrix} 0 & 6 \\ 3 & 2 \end{bmatrix}$; Minor $\begin{bmatrix} 2 & 3 \\ 6 & 0 \end{bmatrix}$ Cofactor $\begin{bmatrix} 2 & -3 \\ -6 & 0 \end{bmatrix}$

Adjoint $\frac{-1}{18}\begin{bmatrix} 2 & -6 \\ -3 & 0 \end{bmatrix} = B^{-1}$

#Make Noise...

$\ldots f(1) = \{\} \quad g(2) = 0$

$v_3 = (6,5,-1,10) \quad v_4 = \ldots$

$B^{-1}A^{-1} = \begin{bmatrix} 2 & -6 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} -4 & -2 \\ -5 & 4 \end{bmatrix}$

$= \begin{bmatrix} -8+30 & -4-24 \\ 12+0 & 6+0 \end{bmatrix} = \frac{1}{44}\begin{bmatrix} 22 & -28 \\ 12 & 6 \end{bmatrix} = (AB)^{-1}$

Proved

7) $u = (6,-1,2)$ $v = (-3,2,1)$ $w = (-2,3,4)$

$5u + 4v - 3w = 5(6,-1,2) + 4(-3,2,1) - 3(-2,3,4)$

$= (30,-5,10) + (-12,8,4) + (6,-9,-12)$

$= (24,-6,2)$

~~8) $B = \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix}$ let $r = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $s = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$~~

~~$rB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2a+3b & -5a+b \\ 2c+3d & -5c+d \end{bmatrix}$~~

~~$sI = \begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$~~

~~$B + rB + sI = \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2a+3b & -5a+b \\ 2c+3d & -5c+d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix}$~~

~~$\begin{bmatrix} 2a+3b+w+2 & -5a+b+x-5 \end{bmatrix}$~~

8) $B = \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix}$ $rB = \begin{bmatrix} 2r & -5r \\ 3r & r \end{bmatrix}$ $sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$

$B + rB + sI = \begin{bmatrix} 2+2r-s & -5-5r \\ 3+3r & 1+r+s \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$2+2r-s = 0 \qquad -5-5r = 0$

$1+r+s = 0 \qquad r = -1$

$2+(-1)(2) - s = 0$

$2-2-s = 0$

$s = 0$

$r = -1, \; s = 0$

$B^{-1}: \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix}$ Minor $\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$

Cofactor $\begin{bmatrix} 1 & -3 \\ 5 & 2 \end{bmatrix}$ Trans; Adjoint $\begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix}$

9) $\begin{vmatrix} n(n-1)! & (n-1)(n+1)(n-1)! \\ (n+1)! & (n+2)(n+1)! & (n+1)(n+2)! \\ (n+2)! & (n+3)(n+2)! & (n+4)(n+3)! \end{vmatrix}$

$n!(n+1)!(n+2)! \begin{vmatrix} 1 \end{vmatrix}$

10) $P = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $QP = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

let $Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$QP = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3a & b \\ 3c & d \end{bmatrix}$

$\begin{bmatrix} 3a & b \\ 3c & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$a = 2/3 \quad c = 0 \quad b = 1 \quad d = 2$

$Q = \begin{bmatrix} \frac{2}{3} & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1 \\ 0 & 2 \end{bmatrix}$

Section B

i) Linear independence: A set of vectors $S = \{u, v, \ldots\}$ is said to be linearly independent if $\forall a, b, \ldots \in F$

$au + bv + \cdots = 0$

ii) Subspace: A vector W is said to be the subspace of a vector V over a field F if W is not empty and scalar multiplication vector addition holds on W.

iii) Basis: A set $S = \{v_1, \ldots, v_n\}$ of vectors is a basis of V if S is linearly independent and S spans V.

1b) $W = \{f : f(2) = 0\}$

To show that W is a subspace, W is non empty set and vector addition and multiplication holds on W.

$\therefore$ since $0 \in W$

W is non-empty set.

ii) $W_1, W_2 \in W \ni W_1 = \{g : g(x) = 0\}$

$W_2 = \{h : h(x) = 0\}$

$W_1 + W_2 = g(x) + h(x)$

$= 0 + 0 = 0 \in W.$

iii) Let $k \in$ a field F and $W_1 \in W$

$kW_1 = k\,g(x) = k \times 0$

$= 0 \in W.$

$\therefore W$ is a subspace of V.

(c) $T(x,y,z) = (2y+z,\ x-4y,\ 3x)$

$e_1 = (1,1,0)\ Te_1 = (2,-3,3)$

$e_2 = (1,0,0)\ Te_2 = (0,1,3)$

$e_3 = (0,0,1)\ Te_3 = (1,0,0)$

$$[T]_e = \begin{bmatrix} 2 & -3 & 3 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix}$$

2) Symmetric Matrices

A matrix is said to be symmetric if $A = A^T$.

b) $f\left(\begin{vmatrix} x \\ y \end{vmatrix}\right) = \begin{bmatrix} 2x+3y \\ 3x+10y \end{bmatrix}$

$f(x,y) = (2x+3y,\ 3x+10y)$

i) To show that f is bijective

$f(a,b) = f(x,y)$ implies that $a = x,\ b = y$

$(2a+3b,\ 3a+10b) = (2x+3y)\ 3x+10y)$

$2a+3b = 2x+3y$
$3a+10b = 3x+10y$ $\Rightarrow a = x\ \ b = y$

$f(x,y) = (a,b) \Rightarrow f^{-1}(a,b) = (x,y)$

$f(x,y) = (a,b)$

$(2x+3y;\ 3x+10y) = (a,b)$

1c) $V_1 = (1,0,-1,2)\ V_2 = (1,3,1,6)$

$V_3 = (1,5,-1,10)\ V_4 = (4,1,0,2)$

$aV_1 + bV_2 + cV_3 + dV_4 = 0$

$(a+b+c+4d,\ 3b+5c+d,\ -a+b-c,\ 2a+6b+10c+2d)$

$a+b+c+4d = 0$

$3b+5c+d = 0$

$-a+b-c = 0$

$2a+6b+10c+2d = 0$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 0 \\ 0 & 3 & 5 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 2 & 6 & 10 & 2 & 0 \end{array}\right]$$

$R_4' = -2R_1 + R_4$ $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 0 \\ 0 & 3 & 5 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & 4 & 8 & 6 & 0 \end{array}\right]$

$R_3' = R_1 + R_3$ $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 0 \\ 0 & 3 & 5 & 1 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 4 & 8 & 6 & 0 \end{array}\right]$

~~$2R_3 - R_4 = R_4'$~~ $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 0 \\ 0 & 3 & 5 & 1 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & -8 & 2 & 0 \end{array}\right]$

$R_4' = 2R_3 - R_4$ $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 0 \\ 0 & 3 & 5 & 1 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & -8 & 2 & 0 \end{array}\right]$

$-8c + 2d = 0$

$-8c = -2d \quad c = \frac{d}{4}$

$2b + 4d = 0$

$b = -2d$

$3b + 5c + d = 0$

$-2d + \frac{d}{4} + d = 0$

$\frac{-3}{4}d = 0 \quad d = 0$

$c = 0$

$b = 0$

$a = 0$

They are dependent

Bolu. Make. Marks. 3

Dimension:

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 3 & 1 & 6 \\ 1 & 5 & -1 & 16 \\ 4 & 1 & 0 & 2 \end{bmatrix}$$

$R_3 = R_2 - R_3 \begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 3 & 1 & 6 \\ 0 & -2 & 2 & -10 \\ 4 & 1 & 0 & 2 \end{bmatrix}$

$R_2 = R_2 - R_1 \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 2 & 4 \\ 0 & -2 & 2 & -10 \\ 4 & 1 & 0 & 2 \end{bmatrix}$

$R_3 = R_2 - R_3 \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 2 & 4 \\ 0 & 5 & 0 & 14 \\ 4 & 1 & 0 & 2 \end{bmatrix}$

$R_2 = 5R_2 - 3R_3$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 10 & -22 \\ 0 & 5 & 0 & 14 \\ 4 & 1 & 0 & 2 \end{bmatrix}$$

$R_4 = 5R_4 + R_3 \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 10 & -22 \\ 0 & 5 & 0 & 14 \\ -20 & 0 & 0 & 4 \end{bmatrix}$

$20R_1 + R_4 = R_4 \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 10 & -22 \\ 0 & 5 & 0 & 14 \\ 0 & 0 & -20 & 44 \end{bmatrix}$

$R_4 = 2R_2 + R_4 \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 10 & -22 \\ 0 & 5 & 0 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\frac{R_3}{5} \leftrightarrow \frac{R_2}{10} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 14/5 \\ 0 & 0 & 1 & -22/10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Dimension $= \underline{\underline{3}}$

3)i) Equal matrices: Matrices A, B are said to be equal if their entries are equal.

ii) Upper triangular matrices are matrices with zero on the lower triangle.

E.g. $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix}$

iii) Scalar matrices: This is basically square matrix whose all-off diagonal elements are zero and all on-diagonal elements are equal.

3c) $A = \begin{pmatrix} t-4 & 0 & 0 \\ -2 & t+1 & 1 \\ 3 & 4 & t-2 \end{pmatrix}$

Singular matrix means $|A| = 0$

$t-4 \begin{vmatrix} t+1 & 1 \\ 4 & t-2 \end{vmatrix} - 0 + 0 = 0$

$t-4((t+1)(t-2)-4) = 0$

$(t-4)(t^2-t-6) = 0$

$(t-4)(t-3)(t+2) = 0$

$t = 4$ or $\underline{\underline{3}}$ or -2

d) $\begin{pmatrix} 8 & -1 & 6 & 3 & 7 \\ 3 & 0 & -1 & 4 & 3 \\ -1 & -1 & 9 & -9 & -2 \end{pmatrix}$

$R_3 = R_1 + R_3 \begin{pmatrix} 8 & -1 & 6 & 3 & 7 \\ 3 & 0 & -1 & 4 & 3 \\ 9 & 0 & -3 & 12 & 9 \end{pmatrix}$

$R_3 = 3R_2 - R_3 \begin{pmatrix} 8 & -1 & 6 & 3 & 7 \\ 3 & 0 & -1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Rank $= \underline{\underline{2}}$

Solved By Mamsn's A.G.S elect

#MakeNoise

@Bolu_make_noise