Turunan

A. PENDAHULUAN

Turunan/differensial adalah laju sesaat perubahan fungsi f(x) pada interval x_2 dan x_1 yang mendekati nol.

Laju rata-rata perubahan fungsi

Jika $x_1 = a$, $x_2 = a + b$, dan a adalah domain dari f(x), maka:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(a+b) - f(a)}{(a+b) - a}$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+b) - f(x)}{b}$$

Laju sesaat perubahan fungsi (turunan)

Adalah nilai limit dari laju rata-rata perubahan fungsi f(x) pada interval x_2 dan x_1 mendekati nol.

Jika $x_1 = a$, $x_2 = a + b$, a adalah domain dari f(x), dan nilai b mendekati nol, maka:

$$\frac{dy}{dx} = \lim_{b\to 0}\frac{\Delta y}{\Delta x} = \lim_{b\to 0}\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{b\to 0}\frac{f(a+b) - f(a)}{(a+b) - a}$$

$$\mathbf{\frac{dy}{dx} = \frac{d[f(x)]}{dx} = y' = f'(x) = \lim_{b\to 0}\frac{f(x+b) - f(x)}{b}}$$

B. RUMUS-RUMUS TURUNAN

Rumus-rumus turunan fungsi pada beberapa bentuk:

Fungsi (f(x))	Turunan fungsi (f'(x))
U ± V	U' ± V'
U.V	U'.V + U.V'
U.V.W	U'.V.W + U.V'.W + U.V.W'
$\frac{U}{V}$	$\frac{U'.V - U.V'}{V^2}$
U^n	$n.U^{n-1}.U'$
U∘V = U(V(x))	U'(V(x)).V'(x)
U∘V∘W = U(V(W(x)))	U'(V(W(x))).(V(W(x)))'
y = f(u) u = g(x)	$\frac{dy}{du}.\frac{du}{dx} = \frac{dy}{dx}$
y = f(u) v = h(x) u = g(v)	$\frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx} = \frac{dy}{dx}$

C. TURUNAN FUNGSI ALJABAR

Aturan-aturan yang digunakan pada turunan fungsi aljabar:

f(x)	f'(x)
k (konstanta)	0
k.x	k
$k.x^n$	$n.k.x^{n-1}$

Contoh pengerjaan bentuk U ± V:

Contoh 1: $y = x^4 - 5x^2 - 7$, tentukan turunannya!

$y' = 4.x^{4-1} - 2.5.x^{2-1} - 0$

$y' = 4x^3 - 10x$

Contoh 2: f(x) = (x – 5)(x + 7), tentukan turunan pertama dan keduanya!

$f(x) = x^2 + 2x - 35$

$f'(x) = 2.x^{2-1} + 2 - 0$

$f'(x) = 2x + 2$

$f''(x) = 2$

Contoh 3: $f(x) = 3x\sqrt{x} - 7\sqrt{x} - 5x$, tentukan f'(x)!

$f(x) = 3x^{3/2} - 7x^{1/2} - 5x$

$f'(x) = 3.\frac{3}{2}.x^{1/2} - 7.\frac{1}{2}.x^{-1/2} - 5$

$f'(x) = \frac{9}{2}\sqrt{x} - \frac{7}{2\sqrt{x}} - 5$

Contoh 4: $y = 2a^2x^2 - 3ax^4 + 5x + a + 7$, tentukan turunan y terhadap x!

$\frac{dy}{dx} = 2.2a^2.x^{2-1} - 4.3a.x^{4-1} + 5 + 0$

$\frac{dy}{dx} = 4a^2x - 12ax^3 + 5$

Contoh pengerjaan bentuk U.V:

Contoh 1: Turunan pertama dari $y = 2x^2\sqrt{2-x}$ adalah?

$U = 2x^2$ $\quad U' = 4x$

$V = \sqrt{2-x} = (2-x)^{1/2}$ $\quad V' = \frac{1}{2}.(2-x)^{-1/2}.(-1) = \frac{-1}{2\sqrt{2-x}}$

$y' = U'V + U.V'$

$y' = 4x\sqrt{2-x} + 2x^2.\frac{-1}{2\sqrt{2-x}}$

$y' = \frac{8x - 4x^2 - x^2}{\sqrt{2-x}}$ $\quad y' = \frac{8x - 5x^2}{\sqrt{2-x}}$

Contoh 2: f(x) = (3x + 4)(8 – x), tentukan f'(x)!

$U = 3x + 4$ $\quad U' = 3$

$V = 8 - x$ $\quad V' = -1$

$f'(x) = U'V + U.V'$

$f'(x) = (3)(8 - x) + (3x + 4)(-1)$

$f'(x) = 24 - 3x - 3x - 4$

$f'(x) = 20 - 6x$

Contoh 3: $f(x) = (x - 2)^2(3 - x)$, tentukan turunan kedua dari f(x) dan nilai f''(1).

$U = (x - 2)^2$ $\quad U' = 2(x - 2)(1) = 2x - 4$

$V = 3 - x$ $\quad V' = -1$

$f'(x) = U'V + U.V'$

$f'(x) = (2x - 4)(3 - x) + (x - 2)^2(-1)$

$f'(x) = 6x - 2x^2 - 12 + 4x - x^2 + 4x - 4$

$f'(x) = -3x^2 + 14x - 16$

$f''(x) = (2)(-3x^{2-1}) + 14 - 0$

$\underline{f''(x) = -6x + 14}$

$f''(1) = -6(1) + 14 \qquad \underline{f''(1) = 8}$

Contoh 4: $a = (2b - 4)(b - 1)(3 - b)$, tentukan $\frac{da}{db}$!

$U = 2b - 4 \qquad U' = 2$

$V = b - 1 \qquad V' = 1$

$W = 3 - b \qquad W' = -1$

$\frac{da}{db} = U'.V.W + U.V'.W + U.V.W'$

$= 2(b-1)(3-b) + (2b-4)(1)(3-b) + (2b-4)(b-1)(-1)$

$= 2(3b - b^2 - 3 + b) + (6b - 2b^2 - 12 + 4b) -$

$(2b^2 - 2b - 4b + 4)$

$= 8b - 2b^2 - 6 + 10b - 2b^2 - 12 - 2b^2 + b - 4$

$\underline{\frac{da}{db} = 19b - 6b^2 - 22}$

Contoh pengerjaan bentuk $\frac{U}{V}$:

Contoh 1: Tentukan y' dari $y = \frac{3x+2}{2x+3}$!

$U = 3x + 2 \qquad U' = 3$

$V = 2x + 3 \qquad V' = 2$

$y' = \frac{U'.V - U.V'}{V^2}$

$y' = \frac{(3)(2x+3) - (3x+2)(2)}{(2x+3)^2}$

$y' = \frac{6x + 9 - 6x - 4}{4x^2+12x+9} \qquad y' = \frac{5}{4x^2+12x+9}$

Contoh 2: Tentukan nilai f'(x) dari $f(x) = \frac{1}{1+\frac{1}{x}}$!

$U = 1 \qquad U' = 0$

$V = 1 + x^{-1} \qquad V' = -x^{-2}$

$f'(x) = \frac{U'.V - U.V'}{V^2}$

$f'(x) = \frac{(0)(1+x^{-1}) - (1)(-x^{-2})}{(1+x^{-1})^2}$

$f'(x) = \frac{x^{-2}}{1+2x^{-1}+x^{-2}} = \frac{\frac{1}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}}$

$\underline{f'(x) = \frac{1}{x^2+2x+1}}$

Contoh pengerjaan bentuk U^n:

Contoh 1: $y = (1 - 5x)^6$, maka nilai y'?

$y' = n.U^{n-1}.U'$

$y' = 6.(1 - 5x)^{6-1}.(-5)$

$\underline{y' = -30(1 - 5x)^5}$

Contoh 2: $y = (x - 2)^3$, tentukan turunan pertama dan kedua y.

$y' = n.U^{n-1}.U'$

$y' = 3.(x - 2)^{3-1}.(1)$

$y' = 3(x - 2)^2 = 3(x^2 - 4x + 4)$

$\underline{y' = 3x^2 - 12x + 12}$

$y'' = 2.3.x^{2-1} - 12$

$\underline{y'' = 6x - 12}$

Contoh 3: $g(x) = (\sqrt{x} - 5)^2 + 2\sqrt{x} + 2$, nilai g'(x)?

$U = \sqrt{x} - 5 = x^{1/2} - 5 \qquad U' = \frac{1}{2}.x^{-1/2} = \frac{1}{2\sqrt{x}}$

$V = 2\sqrt{x} = 2x^{1/2} \qquad V' = 2.\frac{1}{2}.x^{-1/2} = \frac{1}{\sqrt{x}}$

$W = 2 \qquad W' = 0$

$g'(x) = n.U^{n-1}.U' + V' + W'$

$g'(x) = 2(\sqrt{x} - 5).\frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{x}} + 0$

$g'(x) = \frac{\sqrt{x} - 5}{\sqrt{x}} + \frac{1}{\sqrt{x}} = \frac{\sqrt{x} - 4}{\sqrt{x}}.\frac{\sqrt{x}}{\sqrt{x}} = \frac{x - 4\sqrt{x}}{x}$

$\underline{g'(x) = 1 - \frac{4\sqrt{x}}{x}}$

Contoh pengerjaan bentuk komposisi fungsi dan turunan berantai:

Contoh 1: Jika $f(x) = x^2 + 4$, $g(x) = 3x + 6$, dan $h(x) = f \circ g(x)$, tentukan h'(x)!

$f'(x) = 2x \qquad g'(x) = 3$

$h'(x) = f'(g(x)).g'(x)$

$h'(x) = 2(3x + 6)(3) \qquad \underline{h'(x) = 18x + 36}$

Contoh 2: $y = \sqrt{x+\sqrt{5x-1}}$, tentukan y'.

Kita anggap bahwa:

$y = \sqrt{u} \qquad u = x + \sqrt{5x-1}$

maka,

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$= \frac{1}{2\sqrt{u}}.(1+\frac{5}{2\sqrt{5x+1}}) = \frac{1}{2\sqrt{x+\sqrt{5x-1}}}.(1+\frac{5}{2\sqrt{5x+1}})$

$= \frac{1}{2\sqrt{x+\sqrt{5x-1}}} + \frac{1}{2\sqrt{x+\sqrt{5x-1}}}.(\frac{5}{2\sqrt{5x+1}})$

$\underline{\frac{dy}{dx} = \frac{2\sqrt{5x+1}+5}{4\sqrt{(x+\sqrt{5x-1})(\sqrt{5x+1})}}}$

D. TURUNAN FUNGSI TRIGONOMETRI

Aturan-aturan yang digunakan pada turunan fungsi trigonometri:

f(x)	f'(x)
sin U	cos U. U'
cos U	-sin U. U'
tan U	sec^2 U. U'
sec U	sec U. tan U. U'
cot U	$-cosec^2$ U. U'
cosec U	cosec U. cot U. U'

Contoh pengerjaan bentuk U ± V:

Contoh 1: f(x) = 2.cosx – sin4x + tanx, maka f'($\frac{\pi}{4}$)?

f'(x) = –2.sinx – 4.cos4x + sec^2x

f'($\frac{\pi}{4}$) = –2.sin($\frac{\pi}{4}$) – 4.cos4($\frac{\pi}{4}$) + $sec^2(\frac{\pi}{4})$

f'($\frac{\pi}{4}$) = –2. $^1/_2\sqrt{2}$ – 4.(–sin($\frac{\pi}{2}$)) + $(\sqrt{2})^2$

f'($\frac{\pi}{4}$) = $-\sqrt{2}$ + 4(1) + 2 $\quad$ <u>f'($\frac{\pi}{4}$) = 6 – $\sqrt{2}$</u>

Contoh 2: h(x) = cosx + x.sinx – x^3 + 5, maka h'(x)?

h'(x) = –sinx + (1)(sinx) + (x)(cosx) – $3x^2$ + 0

h'(x) = –sinx + sinx + x.cosx – $3x^2$

<u>h'(x) = x.cosx – $3x^2$</u>

Contoh pengerjaan bentuk U.V:

Contoh 1: y = (sinx – cosx)(sinx + cosx), tentukan turunan pertama dan kedua dari y.

U = sinx – cosx $\quad$ U' = cosx + sinx

V = sinx + cosx $\quad$ V' = cosx – sinx

y' = U'V + UV'

y' = (cosx + sinx)(sinx + cosx) + (sinx – cosx)(cosx – sinx)

y' = $\sin^2x$ + 2.sinx.cosx + $\cos^2x$ – ($\sin^2x$ – 2.sinx.cosx + $\cos^2x$)

<u>y' = 4.sinx.cosx</u>

y' = 2.sin2x $\quad$ <u>y'' = 4.cos2x</u>

Contoh 2: Tentukan y' dari y = 4.$\sin^2x$.cos2x !

U = 4.$\sin^2x$ $\quad$ U' = 2.4.sinx.cosx

U' = 8.sinx.cosx = 4.sin2x

V = cos2x $\quad$ V' = –2.sin2x

y' = U'V + UV'

y' = (4.sin2x)(cos2x) + (4.$\sin^2x$)(–2.sin2x)

<u>y' = 2.sin4x – 8.$\sin^2x$.sin2x</u>

Contoh pengerjaan bentuk $\frac{U}{V}$:

Contoh 1: Jika y = $\frac{\sin x}{1 - \cos x}$, tentukan nilai y'!

U = sinx $\quad$ U' = cosx

V = 1 – cosx $\quad$ V' = sinx

$$y' = \frac{U'.V - U.V'}{V^2}$$

$$y' = \frac{(\cos x)(1 - \cos x) - (\sin x)(\sin x)}{(1 - \cos x)^2}$$

$$y' = \frac{\cos x - \cos^2 x - \sin^2 x}{(1 - \cos x)(1 - \cos x)}$$

$$y' = \frac{-(-\cos x + (\cos^2 x + \sin^2 x))}{(1 - \cos x)(1 - \cos x)} = \frac{-\cancel{(-\cos x + 1)}}{\cancel{(1 - \cos x)}(1 - \cos x)}$$

$$\underline{y' = \frac{1}{\cos x - 1}}$$

Contoh 2: f(x) = $\frac{x + \sin x}{1 + \cos x}$, maka f'(x)?

U = x + sinx $\quad$ U' = 1 + cosx

V = 1 + cosx $\quad$ V' = -sinx

$$f'(x) = \frac{U'.V - U.V'}{V^2}$$

$$f'(x) = \frac{(1 + \cos x)(1 + \cos x) - (x + \sin x)(-\sin x)}{(1 + \cos x)^2}$$

$$f'(x) = \frac{1 + 2.\cos x + \cos^2 x + x.\sin x + \sin^2 x}{(1 + \cos x)^2}$$

$$\underline{f'(x) = \frac{2 + x.\sin x + 2.\cos x}{(1 + \cos x)^2}}$$

Contoh pengerjaan bentuk U^n:

Contoh 1: Tentukan turunan dari y = $\sin^7(5x^2 - \frac{\pi}{2})$!

$y' = n.U^{n-1}.U'$

$y' = 7.\sin^{7-1}(5x^2 - \frac{\pi}{2}).\cos(5x^2 - \frac{\pi}{2}).(2.5x^{2-1} - 0)$

<u>$y' = 70x.\sin^6(5x^2 - \frac{\pi}{2}).\cos(5x^2 - \frac{\pi}{2})$</u>

Contoh 2: f'(x) dari f(x) = $sec^{10}(3 - 5x)$ adalah?

$f'(x) = 10.sec^{10-1}(3 - 5x).sec(3 - 5x).tan(3 - 5x).(-5)$

<u>f'(x) = –50.sec10(3 – 5x).tan(3 – 5x)</u>

Contoh 3: y = $\frac{1}{5}.\cot^5x - \frac{1}{3}.\cot^3x$ + cotx + x, maka turunan pertama dan kedua y adalah?

$y' = 5.\frac{1}{5}.\cot^{5-1}x.(-cosec^2x) - 3.\frac{1}{3}.\cot^{3-1}x.(-cosec^2x)$ + (–$cosec^2x$) + 1

$y' = -\cot^4x.cosec^2x - \cot^2x.cosec^2x - cosec^2x + 1$

$y' = -\cot^4x.cosec^2x - \cot^2x.cosec^2x + \cot^2x$

$y' = \cot^2x(-\cot^2x.cosec^2x - cosec^2x + 1)$

$y' = \cot^2x(-\cot^2x.cosec^2x + \cot^2x)$

$y' = \cot^4x(-cosec^2x + 1)$

<u>$y' = \cot^6x$</u>

$y'' = 6.\cot^{6-1}x.(-cosec^2x)$ <u>$y = -6.\cot^5x.cosec^2x$</u>

Contoh pengerjaan bentuk komposisi fungsi dan turunan berantai:

Contoh 1: Jika g(x) = x^2, dan h(x) = sin4x, maka turunan dari g∘h(x) adalah?

g'(x) = 2x $\quad$ h'(x) = 4.cos4x

(g∘h(x))' = g'(h(x)).h'(x)

= 2(sin4x).4.cos4x = 8.sin4x.cos4x

(g∘h(x))' = <u>4.sin8x</u>

Contoh 2: y = $\sqrt{\sin\sqrt{\cos 2x}}$, maka y'?

$y = \sqrt{u}$ $\quad$ u = sinv $\quad$ $v = \sqrt{w}$ $\quad$ w = cos2x

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dw}.\frac{dw}{dx}$$

$$= \frac{1}{2\sqrt{u}}.\cos v.\frac{1}{2\sqrt{w}}.(-2.\sin 2x)$$

$$= \frac{1}{2\sqrt{\sin\sqrt{\cos 2x}}}.\cos\sqrt{\cos 2x}.\frac{1}{2\sqrt{\cos 2x}}.(-2.\sin 2x)$$

$$\underline{\frac{dy}{dx} = \frac{-\sin 2x}{(2\sqrt{\sin\sqrt{\cos 2x}})(\sqrt{\cos 2x})}}$$

Contoh pengerjaan dengan menyederhanakan menggunakan dalil-dalil trigonometri:

Contoh 1: $y = \sqrt[3]{\frac{(\sin 2x+\cos 2x)^2}{\sec 4x+\tan 4x}}$, tentukan y'!

$$y = \sqrt[3]{\frac{2.\sin 2x.\cos 2x+\sin^2 x+\cos^2 x}{\frac{1}{\cos 4x}+\frac{\sin 4x}{\cos 4x}}}$$

$$y = \sqrt[3]{\frac{(2.\sin 2x.\cos 2x+1)(\cos 4x)}{1+\sin 4x}}$$

$$y = \sqrt[3]{\frac{\cancel{(\sin 4x+1)}(\cos 4x)}{\cancel{1+\sin 4x}}} = \sqrt[3]{\cos 4x} = \cos^{\frac{1}{3}} 4x$$

$y' = \frac{1}{3}.\cos^{-\frac{2}{3}} 4x.(-\sin 4x)(4)$ $\quad y' = -\frac{4.\sin 4x}{3\sqrt[3]{\cos^2 4x}}$

Contoh 2: $f(x) = (\sin 5x - \cos 5x)^2$, maka nilai f''(x) adalah?

$f(x) = \sin^2 5x - 2.\sin 5x.\cos 5x + \cos^2 5x$

$f(x) = 1 - \sin 10x$ $\quad f'(x) = -10.\cos 10x$

$f''(x) = 100.\sin 10x$

Contoh 3: Tentukan turunan pertama dari persamaan $y = \frac{\sin 3x - \sin 2x + \sin x}{\cos 3x - \cos 2x + \cos x}$!

$$y = \frac{(\sin 3x + \sin x) - \sin 2x}{(\cos 3x + \cos x) - \cos 2x} = \frac{2.\sin 2x.\cos x - \sin 2x}{2.\cos x.\cos x - \cos 2x}$$

$$y = \frac{\cancel{(2\cos x - 1)}.\sin 2x}{\cancel{(2\cos x - 1)}.\cos 2x} = \tan 2x$$

$y' = 2.\sec^2 2x$