Master in Computer Vision Barcelona

Module: 3D Vision

Lecture 3: 3D projective geometry and transformations. Camera models

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Outline

1. Projective geometry and transformations in 3D:
   - A little bit of motivation.
   - The 3D projective space and transformations in 3D.
2. Camera models.

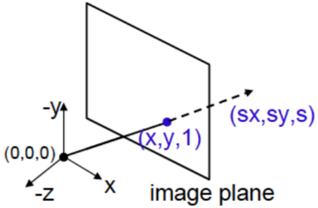
Remember:

Remember: why projective geometry?

- **The <u>image capture of the camera</u> is described with a <u>linear transformation</u>.**
- **The intersection of two lines and the line that passes through two points is a <u>linear operation</u>.**
- **<u>Points at infinity</u> have a natural representation.**
- 2D projective geometry:

The projective plane

A point in the image is a **ray** in the projective space.

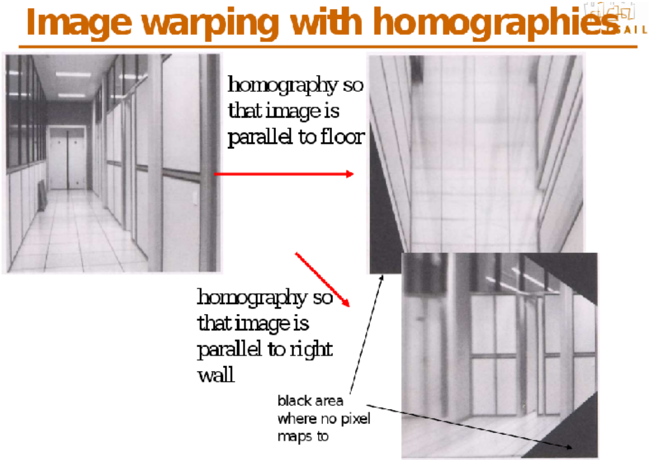


Each point (x, y) on the (image) plane is represented by a ray: (sx, sy, s), the **visual ray**, **visual direction** or **view direction**.

All points on the ray are equivalent: $(x, y, 1) \equiv (sx, sy, s)$

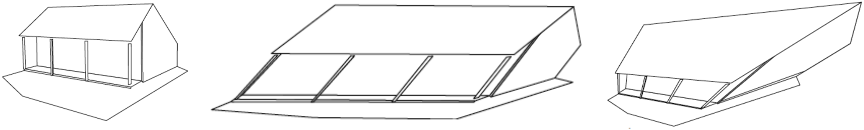
Remember: 2D projective geometry

We have seen that it allows, e.g., to remove the projective distortion of flat objects and build image mosaics.



3D Projective geometry and transformations

Now, **3D projective geometry:** **It models the camera projection and allows the 3D reconstruction, the calibration and auto-calibration.**



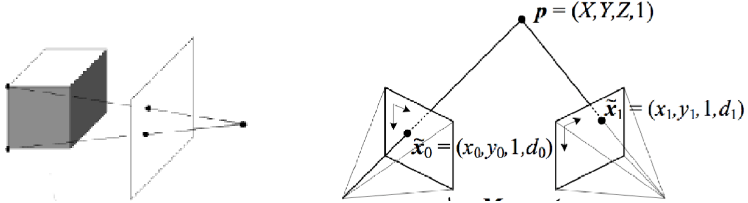
3D projective geometry and transformations

Why is 3D projective geometry useful?

Motivation (and movie trailer): **Projective camera**

$$\mathbf{x}_{\mathrm{im}} = \mathbf{P}\mathbf{X}_{\mathrm{w}}.$$

where $\mathbf{X}_{\mathrm{w}} \in \mathbb{P}^3$ represents the coordinates of a point in the world coordinate frame, $\mathbf{x}_{\mathrm{im}} \in \mathbb{P}^2$ are image coordinates, and $\mathbf{P}$ is the (3×4) **projective matrix** or **camera matrix** (see later in this notes).

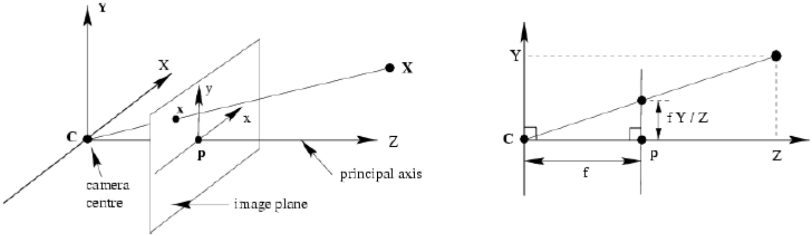


Basic example: Pinhole camera

3D projective geometry and transformations

Why is 3D projective geometry useful? **Example: Pinhole camera**

We will see:



Written in projective coordinates of $\mathbb{P}^3$ and $\mathbb{P}^2$, it is a linear map :

$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}.$$

Central projection in homogeneous coordinates.

3D projective geometry

- We will study the geometric properties and objects of the 3D projective space, denoted by $\mathbb{P}^3$. Many of the concepts are generalizations of the 2D projective space, the projective plane $\mathbb{P}^2$.
- For instance, the points of $\mathbb{P}^3$ increase the 3D Euclidean space $\mathbb{R}^3$ by adding a set of ideal points, which are on a plane at infinity $\mathbf{\Pi}_\infty$ (analogous to ℓ_∞ in $\mathbb{P}^2$; $\mathbb{P}^2$ was constructed by adding to $\mathbb{R}^2$ the points on ℓ_∞).
- The parallel lines and now also the parallel planes meet at $\mathbf{\Pi}_\infty$.
- The points are represented in homogeneous coordinates which increase in one component the cartesian coordinates in 3D.
- On the other hand, there appear new properties thanks to the new dimension that is added.

3D projective space

- Euclidean space $\mathbb{R}^3$: $(X, Y, Z)^T$ **cartesian coordinates** (inhomogeneous).
- Projective space $\mathbb{P}^3$: $\mathbf{X} = (x_1, x_2, x_3, x_4)^T$ **homogeneous coordinates** (or projective), except the point $(0, 0, 0, 0)^T$.

  If $x_4 \neq 0$, $\mathbf{X}$ represents the point $(X, Y, Z)^T = (\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4})^T$ in $\mathbb{R}^3$.
- Notation:
  $P^3 := \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_4 \neq 0\} \cup \Pi_0(\mathbb{R}^4) = \mathbb{R}^4 \setminus \{(0, 0, 0, 0),$

  where $\Pi_0(\mathbb{R}^4) = \{(x_1, x_2, x_3, 0)^T : (x_1, x_2, x_3)^T \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}\}$.

  We say that $P^3 = \{\text{clasic points}\} \cup \{\text{ideal points (at infinity)}\}$.
- Given $\mathbf{X}, \mathbf{X}' \in P^3$, we define the equivalence relation

$$\mathbf{X} \equiv \mathbf{X}' \text{ if there exists } \lambda \neq 0 \text{ such that } \mathbf{X} = \lambda \mathbf{X}'.$$

- We define $\mathbb{P}^3 = P^3 / \equiv$ and we call it the **3D projective space**.

  (compare with the projective plane $\mathbb{P}^2$.)

3D projective space

Points at ∞

Homogeneous points $(x_1, x_2, x_3, 0)$ represent points at infinity (ideal points).

They form a plane: $x_4 = 0$ (the plane at infinity $\mathbf{\Pi}_\infty$).

Equation: $< (0,0,0,1)^T, (x_1, x_2, x_3, x_4)^T >= 0$,

or $(0,0,0,1)(x_1, x_2, x_3, x_4)^T = 0$,

or $\mathbf{\Pi}_\infty^T \mathbf{X} = 0$, where $\mathbf{\Pi}_\infty = (0,0,0,1)^T$.

3D projective space

Planes in $\mathbb{P}^3$

A plane in $\mathbb{P}^3$ can be written as

$$\pi_1 x_1 + \pi_2 x_2 + \pi_3 x_3 + \pi_4 x_4 = 0,$$

or in short as

$$\mathbf{\Pi}^T \mathbf{X} = 0,$$

where $\mathbf{\Pi} = (\pi_1, \pi_2, \pi_3, \pi_4)^T$.

The first three coordinates of $\mathbf{\Pi}$ correspond to the normal $\mathbf{n} \in \mathbb{R}^3$ of Euclidean geometry.

Using inhomogeneous coordinates we can write $\mathbf{X} = (\widetilde{\mathbf{X}}, 1)^T$ where $\widetilde{\mathbf{X}} \in \mathbb{R}^3$, and $\mathbf{\Pi} = (\mathbf{n}, d)^T$ with $\|\mathbf{n}\| = 1$ and $d =$ distance to the origin, and we may write the plane equation as

$$\mathbf{n} \cdot \widetilde{\mathbf{X}} + d = 0.$$

3D projective space

Some facts:

- Three different points determine a unique plane.
- Two different planes intersect in a unique line.
- Three different planes determine a unique point.

Note that they are analogous results to the ones for points and lines in the projective plane $\mathbb{P}^2$. For instance, for the last one, the point $\mathbf{X} \in \mathbb{P}^3$ intersection of three planes $\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{\Pi}_3$ is given by the null space of

$$\begin{pmatrix} \mathbf{\Pi}_1^T \\ \mathbf{\Pi}_2^T \\ \mathbf{\Pi}_3^T \end{pmatrix} \mathbf{X} = \mathbf{0},$$

which is the analogous of the equation for the point $\mathbf{p} \in \mathbb{P}^2$ intersection of two lines ℓ_1, ℓ_2.

Quadrics and dual quadrics

- A **quadric** is a surface of $\mathbb{P}^3$ given by the equation

$$\mathbf{X}^T Q \mathbf{X} = 0, \quad \text{where } Q \text{ is a symmetric } 4 \times 4 \text{ matrix.}$$

  It is the analogous object to a conic in $\mathbb{P}^2$. Many of the properties of quadrics follow directly from those of conics.

- Degrees of freedom $= 9 = 10 - 1 =$ the 10 parameters of a symmetric 4×4 matrix minus one for the scale factor.

- The plane $\mathbf{\Pi} = Q\mathbf{X}$ is called the **polar plane** of $\mathbf{X}$ with respect to Q.

  If $\mathbf{X}$ is in the quadric Q, then the plane $\mathbf{\Pi} = Q\mathbf{X}$ is tangent to the quadric at $\mathbf{X}$. If $\mathbf{X}$ is not on Q and Q is non-singular, the polar plane $\mathbf{\Pi} = Q\mathbf{X}$ is defined by the points of contact with Q of the cone of rays through $\mathbf{X}$ tangent to Q.

- **Dual quadric:** The dual of a quadric is also a quadric.

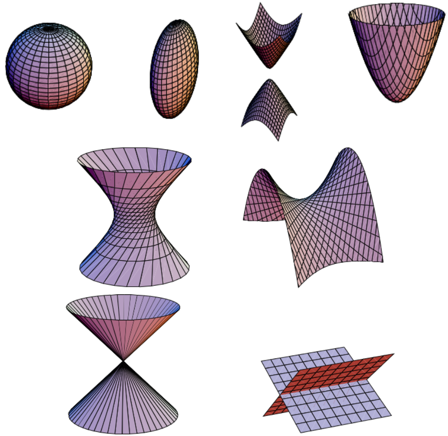
  A dual quadric Q^* is an equation on planes: the tangent planes $\mathbf{\Pi}$ to the point quadric Q satisfy

$$\mathbf{\Pi}^T Q^* \mathbf{\Pi} = 0,$$

  where $Q^* = Q^{-1}$ if Q is invertible, and $=$ adjoint of Q, if Q is singular.

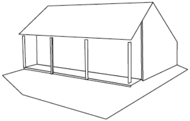
Quadrics

Some examples:

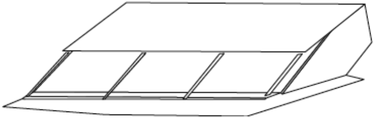


(we refer to [Hartley Zisserman 2004], 3.2.4, for details)

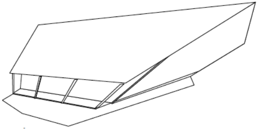
3D Projective Transformations



Similarity



Affine



Projective

3D Projective Transformations

A projective transformation (also called homography) of $\mathbb{R}^3$ is given by a non-singular 4×4 matrix H.

If $\mathbf{X}$ and $\mathbf{X}'$ are homogeneous coordinates in $\mathbb{P}^3$, it can be written as

$$\mathbf{X}' = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix} \mathbf{X}.$$

In short,

$$\mathbf{X}' = H\mathbf{X}.$$

Degrees of freedom: 15 ($\rightarrow 4 \times 4$ elements - 1 multiplicative factor).

As in the case of planar projective transformations, lines are mapped to lines, it preserves incidence relations such as the intersection point of a line with a plane, and order of contact.

Isometry

Let $\mathbf{X}$ and $\mathbf{X}'$ be homogeneous coordinates in $\mathbb{P}^3$.

$$\mathbf{X}' = H_e \mathbf{X} = \begin{pmatrix} R & \vec{t} \\ \vec{0}^T & 1 \end{pmatrix} \mathbf{X}.$$

where H_e is a 4×4 matrix,
R is a rotation 3×3 matrix (orthogonal matrix) and
$\vec{t}$ is a 3×1 translation vector.

Degrees of freedom: 6
$\rightarrow$ 3 for the rotation angles + 3 for the translation coefficients

Invariants: lengths, angles.

Remark: Isometries are transformations of $\mathbb{R}^3$ that preserve the Euclidean distance; i.e., $\|\mathcal{T}\vec{X} - \mathcal{T}\vec{Y}\|_2 = \|\vec{X} - \vec{Y}\|_2$ for all $\vec{X}, \vec{Y} \in \mathbb{R}^3$. An isometry $\mathcal{T} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is $\mathcal{T}\vec{X} = R\vec{X} + \vec{t}$. In homogeneous coordinates, $\mathbf{X}' = H_e\mathbf{X}$, where $\mathbf{X} = (\vec{X}, 1)$.

Rotation matrix in 3D

The **rotation matrix** R is represented by three rotation matrices that represent rotations about the x, y, and z axis:

$$R = R_z(\alpha)R_y(\theta)R_x(\phi)$$

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarity

Let $\mathbf{X}$ and $\mathbf{X}'$ be homogeneous coordinates in $\mathbb{P}^3$.

$$\mathbf{X}' = H_s \mathbf{X} = \begin{pmatrix} sR & \vec{t} \\ \vec{0}^T & 1 \end{pmatrix} \mathbf{X}.$$

where H_e is a 4×4 matrix,
s is an isotropic scaling factor,
R is a rotation 3×3 matrix (orthogonal matrix) and
$\vec{t}$ is a translation vector.

Degrees of freedom: 7
$\rightarrow$ 3 for the rotation angles $+$ 3 for the translation coefficients
$+$ 1 for scaling factor

Invariants: angles, ratios, the absolute conic Ω_∞.

Remark: Similarities are matrices that represent changes of coordinates from Euclidean to Euclidean frames (modulo scalings).

Similarity

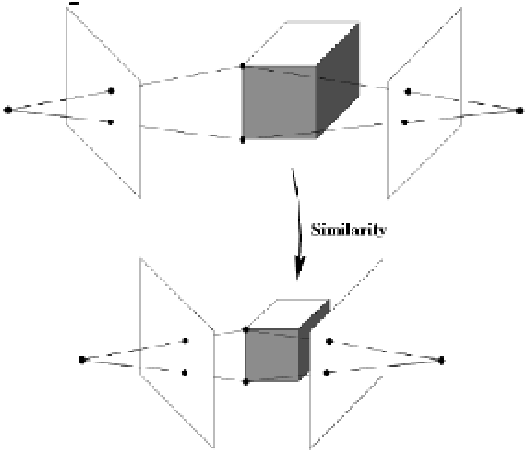


Image source: [Hartley Zisserman 2004]

Affine transformation

Let $\mathbf{X}$ and $\mathbf{X}'$ be homogeneous coordinates in $\mathbb{P}^3$.

$$\mathbf{X}' = H_a \mathbf{X} = \begin{pmatrix} A & \vec{t} \\ \vec{0}^T & 1 \end{pmatrix} \mathbf{X}.$$

where A is a non-singular 3×3 matrix ,
$\vec{t}$ is a translation vector.

Degrees of freedom: 12
$\rightarrow$ 9 for A + 3 for the translation coefficients

Invariants: parallelism, ratio of two volumes, the plane at infinity $\mathbf{\Pi}_\infty$.

Projective transformation

Let $\mathbf{X}$ and $\mathbf{X}'$ be homogeneous coordinates in $\mathbb{P}^3$.

$$\mathbf{X}' = H_p \mathbf{X} = \begin{pmatrix} A & \vec{t} \\ \vec{v}^T & v \end{pmatrix} \mathbf{X}.$$

where H_p is a homography

$$H_p = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix}$$

Degrees of freedom: 15
$\rightarrow$ 4×4 elements - 1 multiplicative factor

Invariants: concurrency, collinearity, order of contact, cross ratio.

Projective transformation

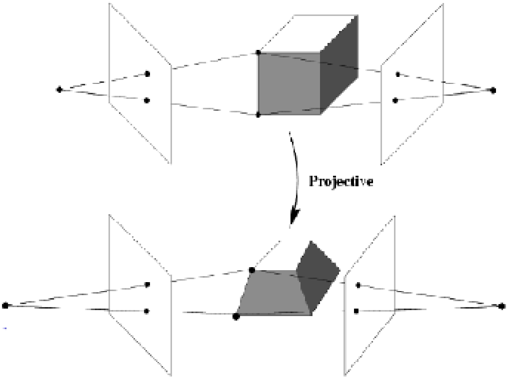
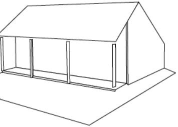
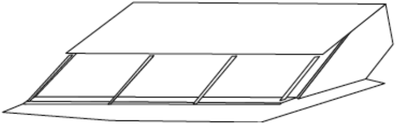


Image source: [Hartley Zisserman 2004]

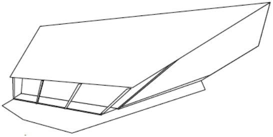
3D projective transformations



Similarity



Affine



Projective

Hierarchy of 3D transformations

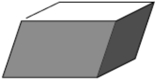
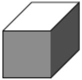
Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^{\mathsf{T}} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, π_∞, (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Volume.

Image source: [Hartley Zisserman 2004]

3D Projective Transformations

Law of transformations of geometric onjects by a 3D homography:

Projective transformation of points: $\mathbf{X}' = H\mathbf{X}$

Projective transformation of planes: $\mathbf{\Pi}' = H^{-T}\mathbf{\Pi}$

Projective transformation of quadrics: $Q' = H^{-T}QH^{-1}$

Projective transformation of dual quadrics: $Q^{*\prime} = HQ^{*}H^{T}$

Some more comments on the plane at infinity in $\mathbb{P}^3$

- The plane at infinity has the canonical position $\mathbf{\Pi}_\infty = (0,0,0,1)^T$ in affine 3-space. (analogous to $\ell_\infty = (0,0,1)^T$ in $\mathbb{P}^2$)
- It contains the points (and directions) at infinity $(x_1, x_2, x_3, 0)^T$.

- **$\mathbf{\Pi}_\infty$ enables the identification of affine properties such as parallelism**:
  - Two planes are parallel if and only if they intersect at a line in $\mathbf{\Pi}_\infty$.
  - Two lines are parallel (also, a line is parallel to a plane), if they intersect at a point in $\mathbf{\Pi}_\infty$.
- In $\mathbb{P}^3$, any pair of planes intersect in a line. When the planes are parallel, they intersect in a line at $\mathbf{\Pi}_\infty$.

- **Result:** The plane at infinity, $\mathbf{\Pi}_\infty$, is a fixed plane under the projective transformation H if, and only if H is an affinity.

  That is: The plane $\mathbf{\Pi}_\infty$ is a fixed plane (doesn't vary) when we do an affine transformation.

  On the other hand, a projective transformation do move $\mathbf{\Pi}_\infty$.

Some more comments on the plane at infinity in $\mathbb{P}^3$

This is why

Π_∞ will play an important role in the autocalibration of a camera (in the same waw that ℓ_∞ allowed to remove projective distorsion).

Π_∞ allows to remove the projective distortion in a 3D reconstruction from N cameras and therefore allows to identify parallel lines.

When we transform a 3D reconstruction with a transformation fixing the plane at infinity to $(0,0,0,1)^T$, **the 3D reconstruction is related to the 3D scene by an affine transformation**.

Camera models

1.- The pinhole camera model

  1.1- From world in the camera system to infinite resolution image

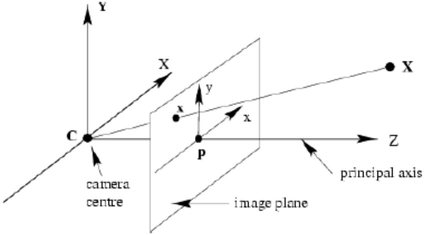
  1.2- From world to world in camera system

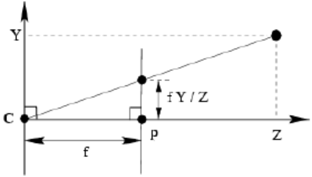
2.- Camera CCD

3.- Finite projective camera

4.- Cameras at infinity

1. The pinhole camera model

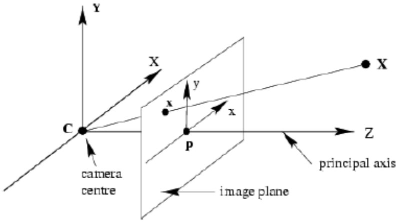


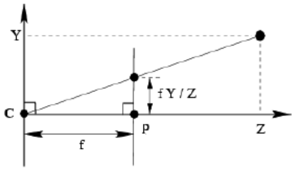


Assuming that the optic is perfect and the camera realizes a central projection from the world to the image plane, consider a reference frame where its origin is the center of projection and the image plane is given by $Z = f$ where f is the focal length of the camera. A point $(X, Y, Z)^T$ in the world is mapped to the point $(fX/Z, fY/Z, f)^T$ in the image plane. This gives a map from $\mathbb{R}^3$ to $\mathbb{R}^2$ written as:

$$(X, Y, Z)^T \rightarrow (fX/Z, fY/Z)^T.$$

Pinhole camera

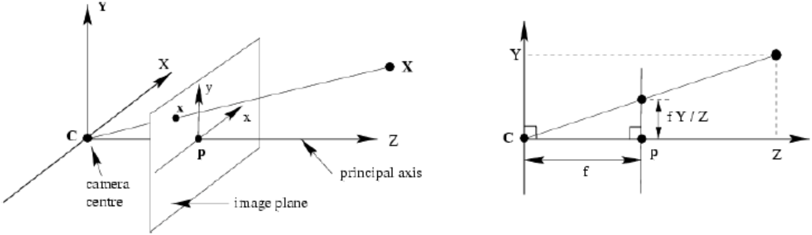




Camera anatomy:
Camera centre (or optical centre): the centre of projection.
Principal axis (or principal ray): the lne from the camera centre perpendicular to the **image plane**.
Principal point P: where the principal axis meets the image plane.
Principal plane: the plane through the camera centre parallel to the image plane.

Image source: [Hartley Zisserman 2004]

Pinhole camera



$$(X, Y, Z)^T \rightarrow (fX/Z, fY/Z)^T.$$

Notice that this map is non-linear in (X, Y, Z). It will be linear if we write it in projective coordinates of $\mathbb{P}^3$ and $\mathbb{P}^2$:

$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}.$$

Image and camera coordinate systems

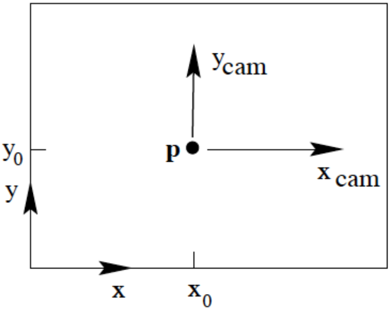


Fig. 6.2. *Image* (x, y) *and camera* $(x_{\mathrm{cam}}, y_{\mathrm{cam}})$ *coordinate systems.*

Image source: [Hartley Zisserman 2004]

Transformation between the world and camera reference systems

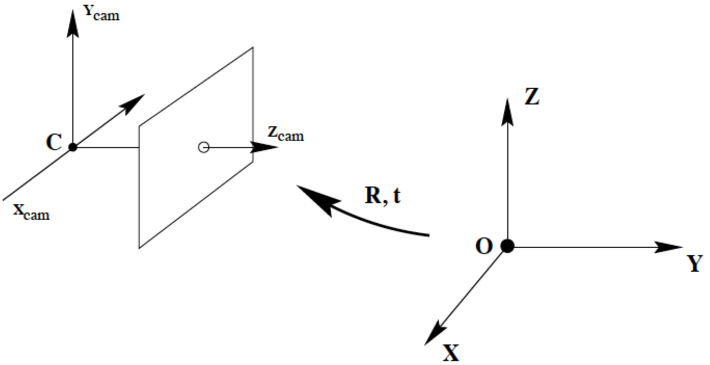


Fig. 6.3. *The Euclidean transformation between the world and camera coordinate frames.*

Image source: [Hartley Zisserman 2004]

Affine camera

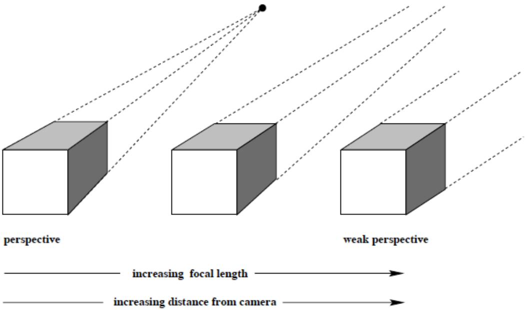


Image source: [Hartley Zisserman 2004]

References

[Hartley and Zisserman 2004] R.I. Hartley and A. Zisserman, Multiple View Geometry in Computer Vision, Cambridge University Press, 2004.