PHYS 411 (Entanglement in Many-Body Systems) Notes

Rio Weil

This document was typeset on January 16, 2025

Introduction:
This is a set of lecture notes taken from UChicago's PHYS 411 (Entanglement in Many-Body Systems), taught by Michael Levin. Topics covered include...

Contents

1 Toric Code I

1.1 Course Overview + Logistics

Instructor email: malevin@uchicago.edu
Office: MCP 447
Evaluation: Once every $\sim$ 2 weeks, 100%.
Textbook: None; papers/references will be provided.

This class will cover topics at the interface of quantum many-body/condensed matter theory and quantum information. This has been a dynamic interface for a couple decades now, with the two fields inspiring each other. The topics chosen are both interesting from a physics point of view, but also deeply important in QI. A rough schedule is as follows:

(I) **Anyons and topological quantum computation.** Anyons exist in 2-d quantum systems that have exchange statistics that are not bosonic or fermionic; the exchange phase can be anything (hence the name). Anyons first emerge in discussion of the fractional quantum hall effect, but in the last 20 years, people (lead by Alexei Kitaev) have found interesting connections between anyons and quantum computing.

(II) **Symmetry-protected topological phases.** This is another important topic in condensed matter, but is deeply connected to concepts in quantum information, e.g. finite depth-circuits (indeed they provide a quantum-information theoretic way to define phases of matter).

(III) **Entanglement entropy in many-body systems.** EE gives a lot of insight into the physics of MB systems. This also has practical applications, leading to numerical algorithms using...

(IV) **Matrix product states.**

1.2 Defining the Toric Code Model

References: Kitaev's lecture notes arXiv:0904.2771, original paper arXiv:quant-ph/9707021.

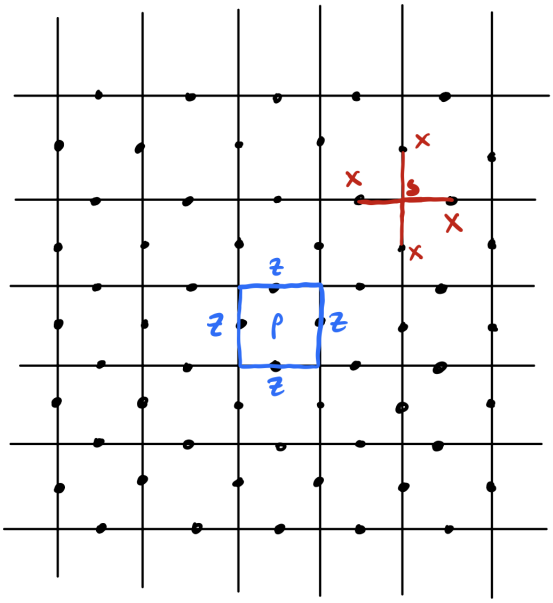
The toric code is an exactly solvable spin model (it can also be thought of a quantum error correcting code, but we introduce it as a spin model to start). It has:

1. Anyon excitations
2. Topological ground state degeneracy

We consider this model on different kinds of lattices and geometries, but for now we consider the square lattice, and place a spin-1/2 degree of freedom on each of the edges of the lattice (we don't specify the boundary conditions yet). The Hilbert space has dimension 2^N with N the total number of edges/spins. The Hamiltonian takes the following form:

$$H = -\sum_s A_s - \sum_p B_p \tag{1.1}$$

the s are vertices on the lattice and p are plaquettes.



The A_s term is a product of Pauli-X operators on stars about vertices s:

$$A_s = \prod_{j \in \text{star}(s)} X_j \tag{1.2}$$

And the B_p term is a product of Pauli-Z operators on the boundaries of plaquettes:

$$B_p = \prod_{j \in \partial p} Z_j \tag{1.3}$$

We adopt the QI notation:

$$X = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.4}$$

$$Y = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.5}$$

$$Z = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.6}$$

,

1.3 Solving the toric code model

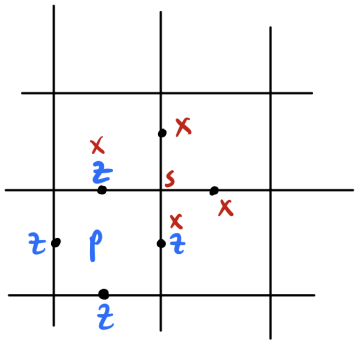
Notice that all of the A_s and B_p terms commute with one another. For example, its trivial to see that:

$$[A_s, A_{s'}] = [B_p, B_{p'}] = 0 \tag{1.7}$$

because Xs are mutually commuting and Zs are mutually commuting. Slightly less obvious is that the star terms commute with the plaquettes:

$$[A_s, B_p] = 0 \tag{1.8}$$

We might be worried if this holds because X and Z anticommute. But in fact the above holds; if the star and plaquette are faraway then there are no overlapping X, Zs so they commute. In the case where the star/plaquette overlap, we have that two X, Zs overlap (see picture below) so the anticommutation cancels to a commutation.



Thus, we are able to simultaneously diagonalize $\left\{A_s, B_p\right\}$. Denote the eigenstates by $|\left\{a_s, b_p\right\}\rangle$ where the a_s, b_p are the eigenvalues. If we have residual degeneracy, we may require additional quantum numbers to specify the state, but let us not worry about this quite yet. Note that because:

$$A_s^2 = B_p^2 = \mathbb{I} \tag{1.9}$$

this implies that the eigenvalues are:

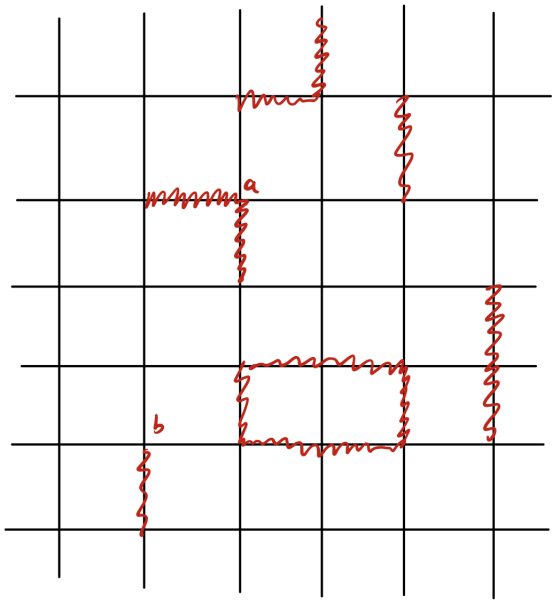
$$a_s, b_p = \pm 1. \tag{1.10}$$

And the $|seta_s, b_p\rangle$ are also the energy eigenstates, with energy:

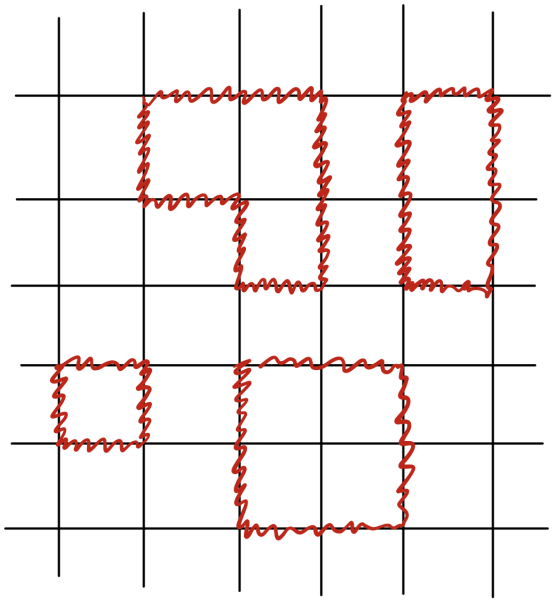
$$E = -\sum_s a_s - \sum_p b_p. \tag{1.11}$$

This is the sense in which the problem is exactly solvable. We have found all of the eigenstates, and can find their degeneracies by figuring out the degeneracies of $\left\{a_s, b_p\right\}$.

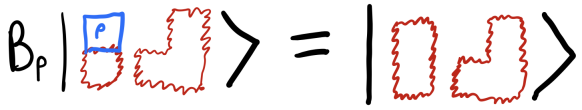
For the ground state, we set $a_s = b_p = 1$. We can ask the question; how many states are there that satisfy this? This is equivalent to asking what the ground state degeneracy is. The answer to this question is dependent on the geometry of the model. Let's start with the simplest, and arguably the most important case; the infinite plane. Let us work in the X-basis. In this basis, the different basis states are $|\pm\rangle \otimes |\pm\rangle \otimes \ldots$ for each link. A useful visualization will be in terms of strings on this lattice. In this string picture, we will say that $X_j = -1$ corresponds to a string on link j. On the other hand, if $X_j = +1$ then we will say that there is no string on the link. Thus, we can view an arbitrary X basis state as strings occupying the lattice in some configuration.



With this point of view, let's figure out what the ground states look like in terms of the X-eigenstates. We want $a_s = 1$ for every star, thus $\prod_{j \in \text{star}(s)} X_j = 1$. This implies that the number of -1s in the product must be even. This means that the number of strings that touch a given site must be even. In the above picture, star(a) obeys this condition while star(b) does not. But, if $a_s = 1$ for every single star, this implies that the strings must form closed loops (else - at the endpoints of strings we end up with $a_1 = -1$). An allowed configuration is sketched below:



Note that since $XZ = -ZX$, a given B_p flips string occupation around a plaquette p, e.g.:



But then all of the $a_s = b_p = 1$ requires that we have an equal amplitude superposition of all closed loop states (if this was not the case, the application of some B_p would flip some loops and change the state - not an eigenstate!). Thus! There is a unique ground state:

$$|\Omega\rangle = |a_s = b_p = 1\rangle = \sum_{\text{closed loop config } C} |C\rangle. \tag{1.12}$$

We could draw an example $|C\rangle$ pictorially as:

It can be seen that B_p only permutes the different Cs to each other without changing their weight, so indeed the above is the $+1$ eigenstate of the B_ps. More formally, if $B_p|\psi\rangle = |\psi\rangle$ for all p, then:

$$\langle C|\psi\rangle = \langle C|B_p|\psi\rangle = \langle C'|\psi\rangle \tag{1.13}$$

where $|C'\rangle$ is a different closed loop configuration. This implies that all of the amplitudes of the closed loop configurations have equal weight.

Note that formally there is an infinite number of closed loop configurations on an infinite plane, so there is a bit of a subtlety in normalizing the state (which requires the machinery of operator algebras, etc.) which we sweep under the rug.

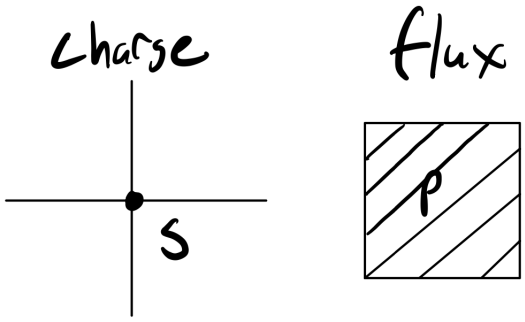
Note that the above argument gave a unique state for $a_s = b_p = 1$, but for any choice of $\left\{a_s, b_p\right\}$ we can get a statement of a similar flavour. Thus, this justifies the choice of notation $|\left\{a_s, b_p\right\}\rangle$ as for a given choice of a_s/b_p the state is unique.

Now, we have the full energy spectrum, as we have determined the degeneracy of all of the eigenspaces. We can thus determine the energy gap, which is the energy difference between the ground state and first excited state. Flipping one of the a_s or the b_p results in an energy penalty of 2, and thus the energy gap is $\Delta = 2$. This is important because it means that we have a "gapped Hamiltonian". In comparison, there are "gapless" Hamiltonians for which the energy difference goes to zero in the thermodynamic limit. This distinction/property will be important when we discuss anyons - gapped Hamiltonians are the context in which they are currently understood.

1.4 Excitations and String Operators

There are two types of elementary excitations:

1. $a_s = -1$ for some s; this is a "charge", and lives on a site.
2. $b_p = -1$ for some p; this is a "flux", and lives on a plaquette.



The terminology comes from $\mathbb{Z}_2$ gauge theory. We will see that these excitations are not conventional bosons or fermions that we may be familiar with.

If we visualize these excitations, adding a charge is like taking our superposition of all closed loops and then to each of those states adding one defect (with a string that goes off to infinity). Adding a plaquette takes the superposition of loops, and we count the number of loops that go around p (a kind of winding number) and take that to be the sign of the configuration in the superposition. Pictorially, we have defects in the first case and vortices in the second.

Now, the question becomes how can we create charge or flux excitations? In spin systems you might be familiar with (e.g. creating a magnon in a Heisenberg spin chain) you would create them via a local operator. But here we actually apply a non-local string operator to create these excitations.

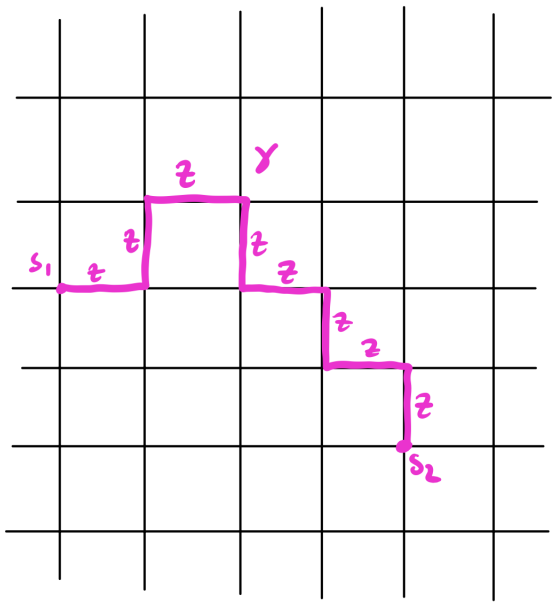
We define:

$$W^Z(\gamma) = \prod_{j \in \gamma} Z_j \tag{1.14}$$

where γ is some open path on the lattice. The $W^Z(\gamma)$ is a string of Zs along γ. This string operator creates charge excitations at the two endpoints; notice that:

$$[W^Z(\gamma), B_p] = 0 \tag{1.15}$$

this is obvious as the B_ps consist of Zs only. Less obvious is that the $W^Z(\gamma)$ commutes with almost all of the A_s operators - all except at the endpoints of γ, s_1 and s_2.



Let us argue this. At intermediate points along the path each of the points have an even number of strings so we have commutation. At the endpoints, we only have 1 link and so we have anticommutation:

$$W^Z(\gamma)A_{s_{1,2}} = -A_{s_{1,2}}W^z(\gamma) \tag{1.16}$$

What are the implications of this? Looking at the ground state $|\Omega\rangle = |\left\{a_s = b_p = 1\right\}\rangle$:

$$W^Z(\gamma)|\Omega\rangle = |\{a_{s_1} = a_{s_2} = -1, \text{others} = 1\}\rangle. \tag{1.17}$$

and we can see this because the string operator only flips the stars at the endpoints, and commutes with everything else. Thus, we conclude that the $W^Z(\gamma)$ creates charges at the two endpoints of γ, as claimed.

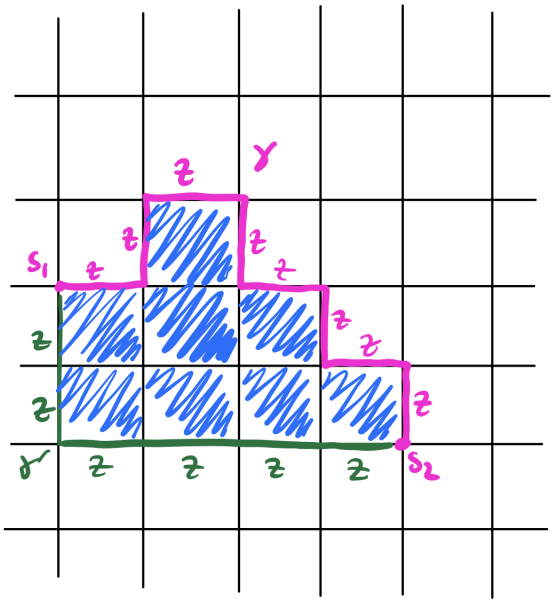
An important observation with regard to string operators; If γ' is a different path with the same endpoints, then the string operator $W^z(\gamma')$ applied to the ground state yields the *exact* same state (even up to the same phase):

$$W^Z(\gamma)|\Omega\rangle = W^z(\gamma')|\Omega\rangle \tag{1.18}$$

this is the notion in which string operators are "flexible". To see this, note that $W^Z(\gamma') = W^Z(\gamma)\prod_{p\in\text{int}(\gamma'\cup\gamma)} B_p$, i.e. the two operators are related via a product of plaquette operators in the interior of the two paths. Thus:

$$W^Z(\gamma')|\Omega\rangle = W^z(\gamma)\prod_{p\in\text{int}(\gamma'\cup\gamma)} B_p|\Omega\rangle = W^Z(\gamma)|\Omega\rangle \tag{1.19}$$

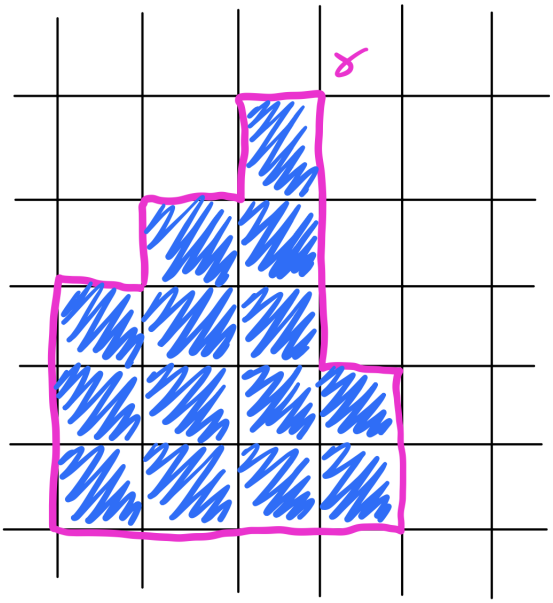
where we have used that $|\Omega\rangle$ is the $+1$-eigenstate of all the B_ps.



Relatedly, if γ is a closed loop then:

$$W^Z(\gamma)|\Omega\rangle = |\Omega\rangle \tag{1.20}$$

which follows from the fact that a closed loop is just a product of B_ps.



These features - which currently seem pretty specific to the toric code model - are in fact quite general. Any system with anyons have string operators with these properties!

A last comment to provide some physical intuition for what the string operator is. We can view it as the physical process of first creating two charges (by applying a single Z), then moving that charge via the application of further Zs along the path. I.e. a string operator is just creating two charges and separating them. This makes the notion that the string is flexible intuitive; we should get the same state (up to some phase) if the particles are created and end up in some separated location(s), no matter how we move them there.

2 Toric Code II

2.1 Review

A quick review of some definitions; the Toric code has Hamiltonian (defined on a 2D lattice with qubits placed on the edges):

$$H = -\sum_s A_s - \sum_p B_p \tag{2.1}$$

with A_s star operators around each lattice vertex:

$$A_s = \prod_{j \in \text{star}(s)} X_s \tag{2.2}$$

and B_p plaquette operators around each lattice plaquette

$$B_p = \prod_{j \in \partial p} Z_p \tag{2.3}$$

we constructed the ground state $|\Omega\rangle = |a_s = b_p = 1\rangle$. We also discussed a string operator, which creates charge excitations:

$$W^Z(\gamma) = \prod_{j \in \gamma} Z_j \tag{2.4}$$

where γ is a path on the lattice. It creates charges ($a_s = -1$) at the endpoints of γ. Further, these string operators are flexible, in the sense that:

$$W^Z(\gamma)|\Omega\rangle = W^Z(\gamma')|\Omega\rangle \tag{2.5}$$

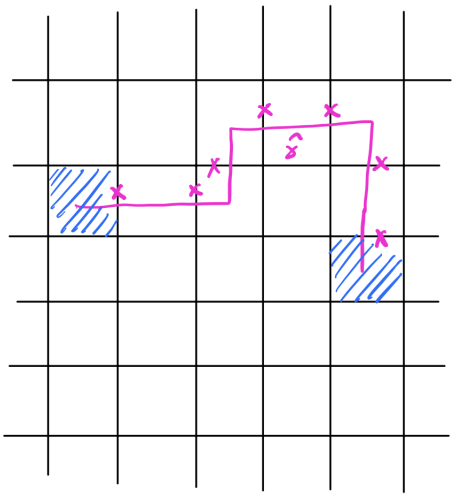
for two paths γ, γ' with the same endpoints.

2.2 String operator for flux excitations

There is a similar string operator for flux excitations. Just a heads up that the structure of having string operators (one for each anyon type - here for charges and fluxes/e and m) is quite general. We define:

$$W^X(\hat{\gamma}) = \prod_{j \in \hat{\gamma}} X_j \tag{2.6}$$

where $\hat{\gamma}$ is an open path on the dual lattice, i.e. that go through the center of plaquettes:

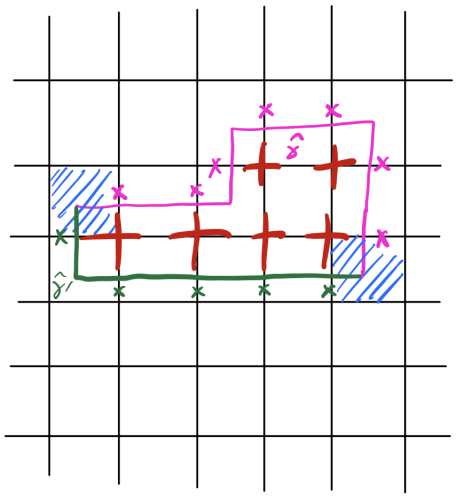


Much like the string operator for the charges:

- It can be checked that $W^X(\hat{\gamma})$ creates fluxes $b_p = -1$ at the two endpoints of $\hat{\gamma}$.
- The string operators are flexible, with:

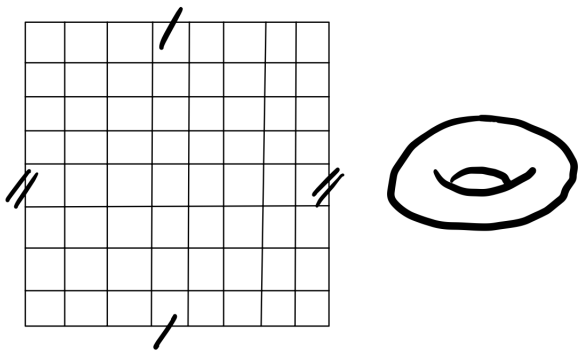
$$W^X(\hat{\gamma})|\Omega\rangle = W^X(\hat{\gamma}')|\Omega\rangle \tag{2.7}$$

  for $\hat{\gamma}, \hat{\gamma}'$ with the same endpoints. They are related by the product of star operators on the interior.



The existence of these flexible, non-commuting string operators is very fundamental to the structure of the toric code, and to anyon systems more generally. The existence of these is independent of geometries. But we will see that it will have implications when we consider the model for specific systems.

2.3 Ground state degeneracy on a torus



Consider the toric code, on a finite $(L \times L)$ torus. We may have different allowable states for a given a_s and b_p, and thus may find that there are different degeneracies, compared to the infinite plane case. We can now ask what is the ground state degeneracy D? Indeed, this question is equivalent to asking what the dimension of the eigenspace with $a_s = b_p = 1$ is. To find this, we look at the trace of the projector onto

the eigenspace[1]:

$$\begin{aligned} D &= \mathrm{Tr}(\text{proj. onto } a_s = b_p = 1 \text{ subspace}) \\ &= \mathrm{Tr}(\prod_s \left(\frac{\mathbb{I} + A_s}{2}\right) \prod_p \left(\frac{\mathbb{I} + B_p}{2}\right)) \end{aligned} \tag{2.8}$$

where the second line follows from the fact that the product of the (mutually commuting) projectors gives the projector onto the subspace. Computing this:

$$D = \frac{1}{2^{N_s}} \frac{1}{2^{N_p}} \mathrm{Tr}(\prod_s (\mathbb{I} + A_s) \prod_p (\mathbb{I} + B_p)) \tag{2.9}$$

Now we expand out this product, and can think about the traces of the individual terms. A single A_s, B_p will be traceless (as the Paulis are traceless), and so will most products of A_s, B_p; the only non-traceless terms will be those that simplify to the identity. If we stare at this, there are only a few combinations for which this occurs; there is the term with all identity, the term with all stars (all the Xs cancel), the term with all plaquettes (all the Zs cancel), and the term with all stars and all plaquettes.

$$D = \frac{1}{2^{N_s}} \frac{1}{2^{N_p}} \mathrm{Tr}(\mathbb{I} + \prod_s A_s + \prod_p B_p + \prod_s \prod_p A_s B_p) = \frac{1}{2^{N_s}} \frac{1}{2^{N_p}} \mathrm{Tr}(4\mathbb{I}) \tag{2.10}$$

Now looking at the trace of the identity:

$$\mathrm{Tr}(\mathbb{I}) = 2^{\dim(\mathcal{H})} = 2^{N_{\text{links}}} \tag{2.11}$$

Thus:

$$D = \frac{1}{2^{L^2}} \cdot \frac{1}{2^{L^2}} \cdot 4 \cdot 2^{2L^2} = 4 \tag{2.12}$$

so:

$$\boxed{D = 4} \tag{2.13}$$

A nice feature of this argument is we can repeat it for any of the eigenspaces. In fact, every $\left\{a_s, b_p\right\}$ eigenspace with $\prod_s a_s = \prod_p b_p = 1$ (i.e. an even number of charges) is 4-fold degenerate.

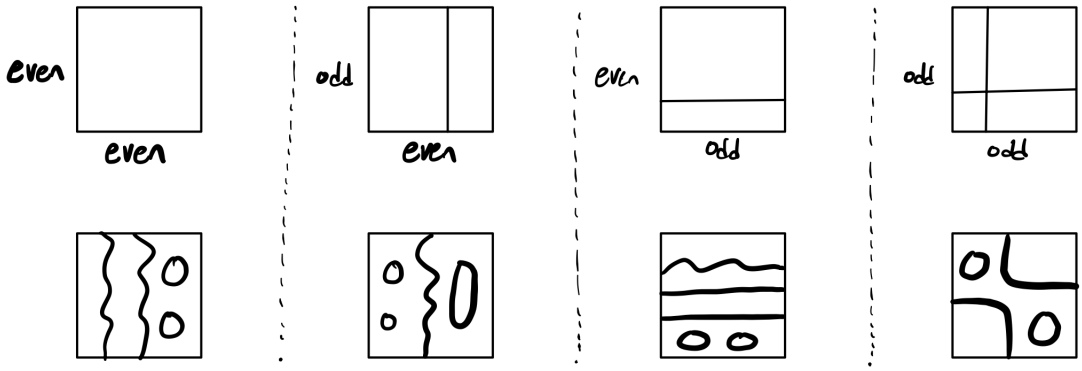
2.4 Ground states in the string picture

Now, let's see if we can understand the ground state degeneracy in the string picture. To review, we work in the X-basis, and $X_j = \pm 1$ corresponds to there being a string (plus) or no string on link j. $a_s = 1$ requires the product of Xs on the star to be one, implying that the strings form closed loops. $b_p = 1$ implies that there is an equal amplitude superposition of string states, as different string states are related by B_p moves. We used these two conditions on the infinite plane geometry (Where B_p moves are "ergodic") to say that there was a unique ground state, namely that with an equal weight superposition of all closed loop configurations:

$$|\Omega\rangle = \sum_{\text{closed loop config } C} |C\rangle \tag{2.14}$$

On a torus, the closed loop states can be divided into 4 classes; even/even, even/odd, odd/even, and odd/odd.

[1] This is a very formal way to find it, we will soon get different perspectives on this question



What does this mean? this means that if we draw a line going across the torus (on the dual lattice), we "cross" an even/odd number of strings. Within each sector/class, the B_p moves are ergodic. But, B_p moves cannot change the parity of the crossings. Thus the four degenerate ground states correspond to the equal weight superpositions of closed loop configurations within a given class. We can label the ground states via their winding number:

$$|\Omega_{(e/o,e/o)}\rangle = \sum_{\text{closed loop config } C \text{ with (e/o, e/o) winding}} |C\rangle \tag{2.15}$$

So, so far we have understood the ground state degeneracy from two perspectives. But neither of these tells us the deeper principle underlying the degeneracy. Let us discuss this now - it will allow us to see why the degeneracy is topologically protected.

2.5 Origin of the ground state degeneracy

The punchline is that the GSD comes from the existence of non-commuting string operators.

Define the charge string operators:

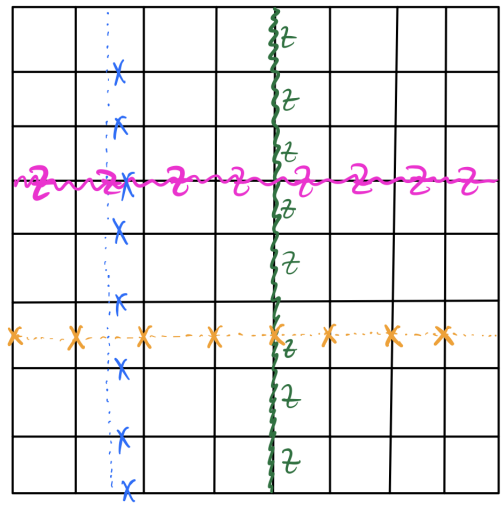
$$W_1^Z = \prod_{j\in\gamma_1} Z_j \tag{2.16}$$

$$W_2^Z = \prod_{j\in\gamma_2} Z_j \tag{2.17}$$

These string operators correspond to the creation and subsequent annihilation of a charge as the string wraps around the torus. We can also define the string operators for the fluxes;

$$W_1^X = \prod_{j\in\hat{\gamma}_1} X_j \tag{2.18}$$

$$W_2^X = \prod_{j\in\hat{\gamma}_2} X_j \tag{2.19}$$



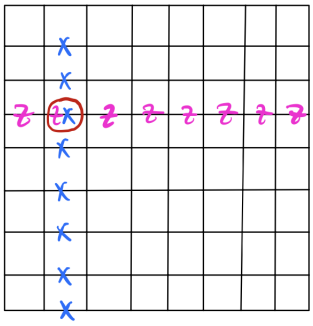
Notice that:

$$\begin{aligned}[W_i^Z, A_s] = [W_i^Z, B_p] = 0 \\ [W_i^X, A_s] = [W_i^X, B_p] = 0\end{aligned} \tag{2.20}$$

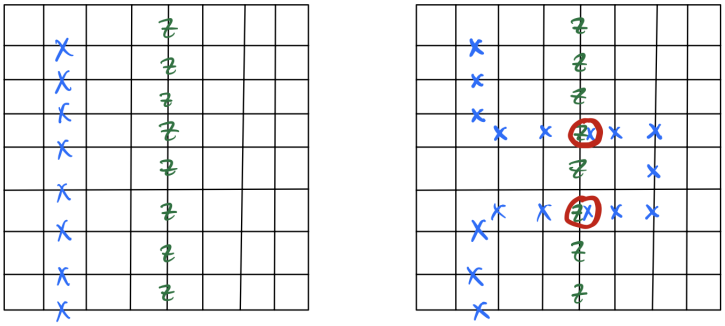
so they map ground states to ground states. They have an interesting commutation algebra. $\left\{W_1^Z, W_2^Z, W_1^X, W_2^X\right\}$ all commute, except for two exceptions:

$$\begin{aligned}W_1^X W_2^Z = -W_2^Z W_1^X \\ W_2^X W_1^Z = -W_1^Z W_2^X\end{aligned} \tag{2.21}$$

this is because they anticommute in one place.



The fact that the other commute is clear; the fact that the Xs mutually commute and Zs mutually commute is immediate. For W_1^X, W_1^Z, they act on disjoint regions. Even if you were to pick representatives that overlap, they will overlap an even amount of times.



This algebra is quite interesting; we have symmetry generators that commute with the Hamiltonian, but not with each other. A simple HW exercise you will do is that the eigenstates/ground states will come in multiplets of 4 (the fact that it is exactly 4 in this case comes from the microscopic calculation).

A final comment; we said how the $\left\{a_s, b_p\right\}$ do not uniquely specify the ground (or excited) states. But in fact the string operators provide the missing quantum numbers necessary to specify the state. Specifically, we can uniquely label the eigenstates by choosing (in addition to the $\left\{a_s, b_p\right\}$) the values of the Ws, e.g $|\left\{a_s, b_p\right\}, w_1^x, w_2^x\rangle$ where $w_1^x = \pm 1$ and $w_2^x = \pm 1$. We can thus denote the four ground states as $|\Omega, \pm\pm\rangle$. These are *exactly* the same what we called before the $|\Omega_{(e/o,e/o)}\rangle$ - in fact the W^x string operator counts the number of crossing of X strings.

Next time, we will discuss how the GSD is robust to arbitrary local perturbations - it is topologically protected.

3 Toric Code III, Berry Phase

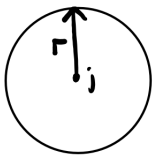
3.1 Robustness of toric code GSD

We saw the degeneracy of the toric code from both the explicit/formal calculation as well as from string operators. GSD in itself is not interesting, and is quite fragile; perturbations tend to split it - what is interesting about the TC GSD is that it is extremely robust; local perturbations cannot split it.

More precisely, consider an arbitrary local perturbation of H:

$$H' = H + \lambda \sum_j V_j \tag{3.1}$$

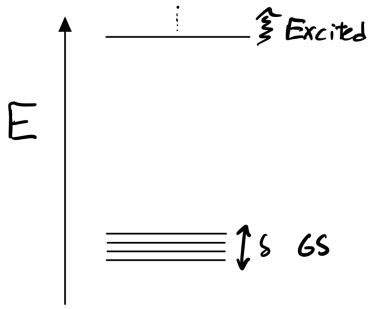
where V_j is a local operator supported near site j.



Claim: For sufficiently small λ, H' has 4 nearly degenerate ground states with splitting:

$$\delta \leq e^{-C(\lambda)L} \tag{3.2}$$

with $C(\lambda)$ a λ-dependent constant. The idea is that with a arbitrary local perturbation we have an exponentially small (in the thermodynamic limit) splitting of the ground state manifold, and thus the GSD is robust. From a QI perspective, this is useful because it tells us that we have a robust encoding of a qubit - we can use the ground state as a robust subspace. This degeneracy is often called a "protected", or topological because it is automatically protected.



3.2 Argument for robustness

Using the fact that $\left\{W_1^X, W_2^X, W_1^Z, W_2^Z\right\}$ (string operators) are flexible and non-commuting, we can establish a key property of the unperturbed ground states $|\Omega, \pm\rangle$. Namely, for any operator O supported on less than L sites:

$$\langle \Omega, \pm, \pm | O | \Omega, \pm, \pm \rangle = \text{diag}(c, c, c, c) \tag{3.3}$$

for some constant c. A shorthand for the above is:

$$\langle \Omega, \alpha | O | \Omega, \beta \rangle = c\delta_{\alpha\beta}. \tag{3.4}$$

You will show this relation on the homework.

Let's unpack this equation. The first thing it tells us is that:

$$\langle \Omega, \alpha | O | \Omega, \alpha \rangle = \langle \Omega, \alpha' | O | \Omega, \alpha' \rangle \tag{3.5}$$

which tells us that local operators cannot distinguish different ground states. The second thing it tells us is that:

$$\langle \Omega, \alpha' | O | \Omega, \alpha \rangle = 0 \quad \text{for } \alpha \neq \alpha' \tag{3.6}$$

in other words, local operators cannot connect ground states.

As a comparison, consider $|\uparrow\rangle^{\otimes N}, |\downarrow\rangle^{\otimes N}$ the 2-fold degenerate ground states of the Ising model $H = -\sum_{ij} Z_i Z_j$. A local operator cannot connect them, but it is possible to distinguish the two states by measuring Z_i. In a symmetry breaking state, we do not have the structure of flexible strings, and hence do not have the same notion of local indistinguishability to arbitrary operators (only to symmetric ones).

Using the local distinguishability/unconnectability, let us sketch an argument for the toric code GSD. For concreteness, consider a perturbation:

$$H' = H + \lambda \sum_j X_j. \tag{3.7}$$

To obtain the first-order splitting (in degenerate perturbation theory), we need to find the matrix elements:

$$\langle \Omega, \pm, \pm | \sum_j X_j | \Omega, \pm, \pm \rangle \tag{3.8}$$

and then diagonalize. By Eq. (3.3), we know that:

$$\langle \Omega, \pm, \pm | X_j | \Omega, \pm, \pm \rangle = c_j \mathbb{I} \tag{3.9}$$

Therefore the ground state degeneracy is not split to first order. Looking at second order, we need to find matrix elements:

$$\langle \Omega, \pm, \pm | (\sum_j X_j)(H - E_{\text{gs}})^{-1} \Pi_{\text{ex}} (\sum_j X_j) | \Omega, \pm, \pm \rangle \tag{3.10}$$

and then diagonalize (third, fourth order (and so on) we have the same procedure, just add a factor of $(H - E_{\text{gs}})^{-1}\Pi_{\text{ex}}(\sum_j X_j)$ each order). A given X_j takes us to an excited state, the Π_{ex} projector is then irrelevant[2], and then $(H - E_{\text{gs}})^{-1}$ is just a number (difference between the ground and excited state energy), so we end up evaluating matrix elements of the form:

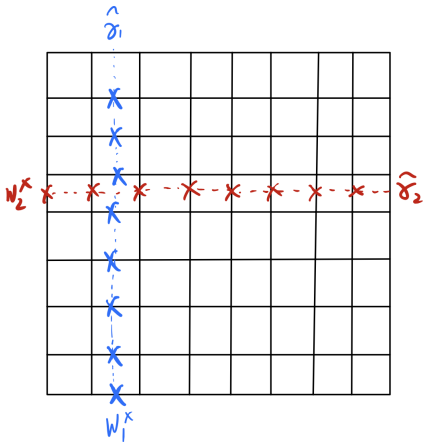
$$\langle \Omega, \pm, \pm | X_j X_k | \Omega, \pm, \pm \rangle = c \mathbb{I} \tag{3.11}$$

where we again use Eq. (3.3). So, again at second order we have no splitting, and the argument follows the same way for third, fourth order etc. using the same property. The argument only breaks when the property no longer holds, which occurs at Lth order of perturbation theory when we end up looking at the matrix element of an operator supported on L sites. In particular, we get terms of the form:

$$\prod_{j \in \hat{\gamma}_1} X_j = W_1^X, \quad \prod_{j \in \hat{\gamma}_2} X_j = W_2^X \tag{3.12}$$

i.e. the string operators.

[2]For a general operator, we instead can use that the projector can be restricted to a local operator as we only need to look at some local patch to tell that we are in an excited state



Thus we end up the matrix elements:

$$\langle \Omega, \pm, \pm | W | \Omega, \pm, \pm \rangle \sim \lambda^L \text{diag}(c_1, c_2, c_3, c_4) \tag{3.13}$$

and so then the splitting between the ground states is:

$$\delta \sim \lambda^L = e^{-L \log(\frac{1}{\lambda})} \tag{3.14}$$

Note that this is quite heuristic, and to make it rigorous you require more precise arguments, namely that the perturbation theory converges, with a finite radius of convergence λ_0 (which holds for arbitrarily large L). Without a formal argument, we expect such a finite radius of convergence for "typical" gapped local Hamiltonians, with $\lambda_0 \sim \Delta$ (so actually, in addition to the local indistinguishability, we are also using that the toric code is gapped).

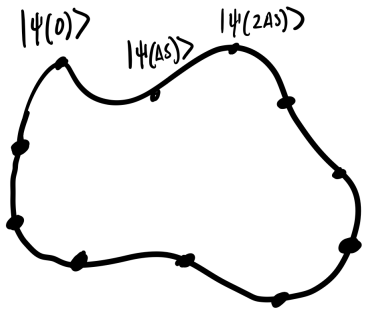
3.3 A Review of Berry Phase

Before we move to a general discussion of anyons, we first review the notion of a Berry phase, which is a very related idea.

Let:

$$\left\{ |\psi(s)\rangle, 0 \leq s \leq T, |\psi(T)\rangle = e^{i\phi} |\psi(0)\rangle \right\} \tag{3.15}$$

be a closed path in the set of normalized quantum states (rays in Hilbert space). Let us split up the path into N parts of length Δs, with $N\Delta s = T$. Graphically:



and for brevity we denote $|\psi(\cdot)\rangle = |\psi_\cdot\rangle$. Now, we define the Berry phase as:

$$e^{i\theta_B} = \lim_{N\to\infty} \langle\psi_0|\psi_{N-1}\rangle \dots \langle\psi_2|\psi_1\rangle\langle\psi_1|\psi_0\rangle \tag{3.16}$$

The Berry phase has properties:

1. $|e^{i\theta_B}| = 1$, i.e. $e^{i\theta_B}$ is a $U(1)$ phase. This can be seen from the fact that $\langle\psi_1|\psi_0\rangle \sim \frac{1}{N}$ and so the N-fold product is of order ~ 1.
2. $e^{i\theta_B}$ only depends on the path and not its parameterization. That is, it is invariant under $s \to s' = f(s)$ with $f(0) = 0$ and $f(T) = T'$.
3. $e^{i\theta_B}$ does not depend on the phase of $|\psi(s)\rangle$. That is, it is invariant under $|\psi(s)\rangle \to e^{i\varphi(s)}|\psi(s)\rangle$. This is easily seen from the definition - the phase of the ket is cancelled out by that of the bra in the N-fold product.

Taking the limit $N \to \infty$, we have the formula:

$$\theta_B = \int_0^T ds\; i\langle\psi(s)| \frac{\mathrm{d}}{\mathrm{d}s} |\psi(s)\rangle \tag{3.17}$$

there is however the caveat when we evaluate the Berry phase in this way. We have to add the assumption that $|\psi(0)\rangle = |\psi(T)\rangle$ without the phase factor.

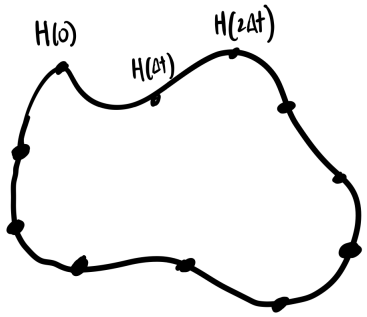
3.4 Berry phase and adiabatic evolution

The Berry phase shows up in two places; in adiabatic processes/cycles, and in path integrals. Today, we talk about the former.

First, a reminder of the adiabatic theorem. Let $H(t)$ be a time-dependent Hamiltonian with $0 \le t \le T$. Suppose $H(t)$ has a unique ground state $|\psi(t)\rangle$ with energy $E(t)$ and gap $\Delta(t)$. Suppose $H(t)$ varies on a timescale $\tau \gg \frac{1}{\min_t \Delta(t)}$. Then:

$$|\psi(0)\rangle \overset{\text{evolve under } H(t)}{\longrightarrow} (\text{phase})|\psi(t)\rangle. \tag{3.18}$$

Now, consider a closed path $H(T) = H(0)$, which we may consider an "adiabatic cycle".



Then:

$$|\psi(0)\rangle \overset{\text{evolve}}{\longrightarrow} (\text{phase})|\psi(0)\rangle \tag{3.19}$$

Let's compute this phase factor! It is given by:

$$(\text{phase}) = \langle\psi(0)|\mathcal{T}\exp(-i\int_0^T dt H(t))|\psi(0)\rangle \tag{3.20}$$

with $\mathcal{T}$ denoting time ordering. We compute the phase factor by discritizing the time-dependent Hamiltonian to $H(0) = H_0, H(\Delta t) = H_1, H(2\Delta t) = H_2, \ldots$ with $N\Delta t = T$ and associated instantaneous ground states $|\psi(0)\rangle = |\psi_0\rangle, |\psi(\Delta t)\rangle = \psi_1, \ldots$. Then the expression for the phase factor becomes:

$$(\text{phase}) = \lim_{N\to\infty} \langle\psi(0)|e^{-i\Delta t H_{N-1}} e^{-i\Delta t H_{N-2}} \ldots e^{-i\Delta t H_1} e^{-i\Delta t H_0}|\psi(0)\rangle \tag{3.21}$$

According to the adiabatic theorem, we know that:

$$e^{-i\Delta t H_0}|\psi_0\rangle = |\psi_1\rangle\langle\psi_1|e^{-i\Delta t H_0}|\psi_0\rangle \tag{3.22}$$

as the adiabatic evolution takes $|\psi_0\rangle \to |\psi_1\rangle$ in the first time interval. We can thus insert projectors about each time step:

$$(\text{phase}) = \lim_{N\to\infty} \langle\psi(0)|e^{-i\Delta t H_{N-1}}|\psi_{N-1}\rangle\langle\psi_{N-1}|e^{-i\Delta t H_{N-2}}|\psi_{N-2}\rangle\langle\psi_{N-2}|\ldots|\psi_2\rangle\langle\psi_2|e^{-i\Delta t H_1}|\psi_1\rangle\langle\psi_1|e^{-i\Delta t H_0}|\psi(0)\rangle \tag{3.23}$$

Each of the expectation values only gives a phase factor of the energy, and so:

$$(\text{phase}) = \lim_{N\to\infty} \exp(-i\Delta t \sum_{k=0}^{N-1} E_k)\langle\psi_0|\psi_{N-1}\rangle \ldots \langle\psi_2|\psi_1\rangle\langle\psi_1|\psi_0\rangle \tag{3.24}$$

and so:

$$\boxed{(\text{phase}) = \exp(-i\int_0^T E(t)dt)e^{i\theta_B}} \tag{3.25}$$

i.e. we have a path-dependent dynamical phase part and a Berry phase part.

We'll stop here for now, and next time we will discuss the Berry phase associated with anyons.

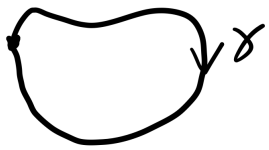
4 Abelian anyons

Today we discuss anyons and derive them from first principles. We focus on the 2-D case in our discussion.

4.1 Single-particle Berry Phase

Let H be a 2-D gapped Hamiltonian with short-ranged interactions (sum of local terms). Suppose H has a particle-like excitation (the rough idea is that there is a state in $\mathcal{H}$ that looks like the ground state everywhere, except for a localized region in space). In general, these excitations/particles can be in different locations in space[3] $\mathbf{r}$. For each different position, we will have a distinct many-body state, which we can label as $|\mathbf{r}\rangle$. We can then consider the Berry phase $\theta_B(\gamma)$ associated with a closed path γ:

$$\theta_B(\gamma) = \int_0^T \langle \mathbf{r}(t)| i \frac{\mathrm{d}}{\mathrm{d}t} |\mathbf{r}(t)\rangle dt \tag{4.1}$$



We can define the Berry connection, which is a vector:

$$\boldsymbol{\mathcal{A}}(\mathbf{r}) = \langle \mathbf{r}| i \boldsymbol{\nabla}_{\mathbf{r}} |\mathbf{r}\rangle = \begin{pmatrix} \langle \mathbf{r}| i \frac{\partial}{\partial x} |\mathbf{r}\rangle \\ \langle \mathbf{r}| i \frac{\partial}{\partial y} |\mathbf{r}\rangle \end{pmatrix} \tag{4.2}$$

Using the chain rule, we are able to write the berry phase as an integral over the Berry connection:

$$\theta_B(\gamma) = \int_\gamma \boldsymbol{\mathcal{A}}(\mathbf{r}) \cdot d\mathbf{r} \tag{4.3}$$

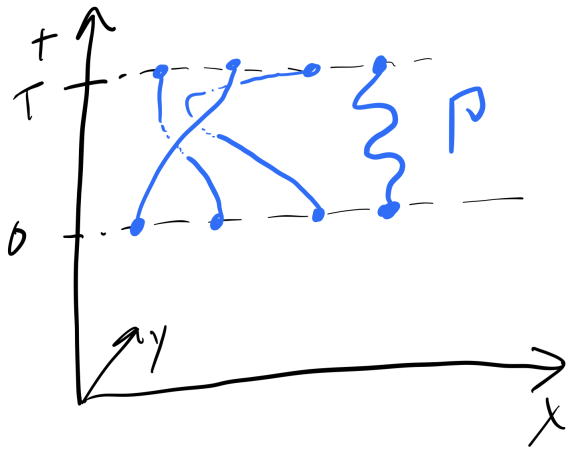
a comment; this looks a lot like a vector potential (and indeed this choice of notation is not a coincidence); the effect that $\boldsymbol{\mathcal{A}}$ has on the physics is the same as if the particle was coupled to a background vector potential/magnetic field $\mathbf{A}$. This has very little to do with anyons - but when we go to multiple particles, we see the physics of anyons start to emerge.

4.2 Multi-particle Berry Phase and the Locality constraint

The extra part that appears when we look at the multi-particle Berry phase is exchange statistics (and this is how we will "see" anyons emerge)! Consider a state with n identical excitations/particles, which we can parameterize by $|\{\mathbf{r}_1, \ldots \mathbf{r}_n\}\rangle$. Since the particles are identical, we need not specify the order, only the positions.

Now, let us consider the Berry phase associated with an n-particle closed path Γ. A picture of this path directly is a bit tricky (e.g. for 2 particles we have a 4-dimensional configuration space, which is not even Euclidean due to the lack of ordering). But we can draw it as particle worldlines through time, e.g. for four particles:

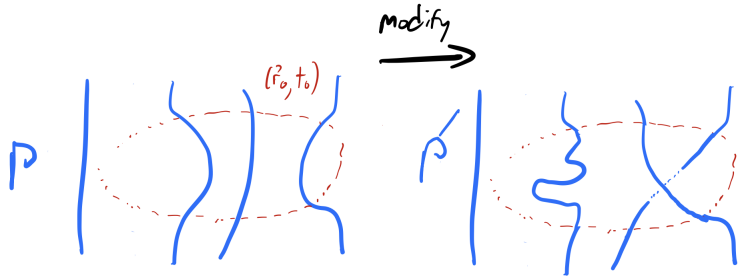
[3] Physically, we could imagine these $\mathbf{r}$ as minima of some trapping potential



The Berry phase is then:

$$\theta_B(\Gamma) = \int_0^T \langle \{\mathbf{r}_1(t), \dots \mathbf{r}_n(t)\} | i \frac{\mathrm{d}}{\mathrm{d}t} | \{\mathbf{r}_1(t), \dots \mathbf{r}_n(t)\} \rangle dt \tag{4.4}$$

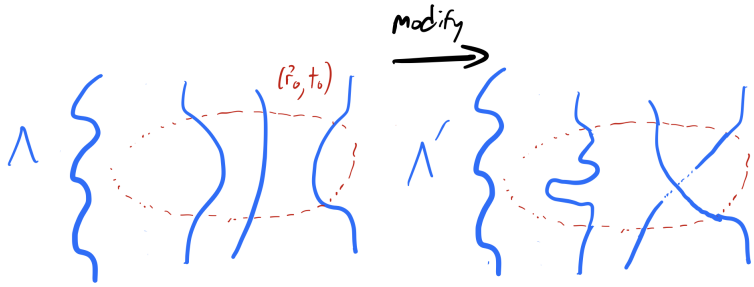
The question to understand is then; what does this look like? What are general constraints on θ_B? The answer is that θ_B has to be "local". More precisely, imagine modifying a multi-particle path Γ near $(\mathbf{r}_0, t_0)$:



Then, $\theta_B(\Gamma') - \theta_B(\Gamma)$ depends only on what Γ, Γ' look like near $(\mathbf{r}_0, t_0)$. In other words:

$$\theta_B(\Gamma') - \theta_B(\Gamma) = \theta_B(\Lambda') - \theta_B(\Lambda) \tag{4.5}$$

if Λ, Λ' looks like Γ, Γ' near $(\mathbf{r}_0, t_0)$ and differ by the same local move:



The claim is local change near $(\mathbf{r}_0, t_0)$ is insensitive to faraway modifications. Why does Eq. (4.5) hold? It is because the difference in Berry phase $\theta_B(\Gamma') - \theta_B(\Gamma)$ can be measured by a local operator acting near $(\mathbf{r}_0, t_0)$ (Physically, we can imagine an interference or adiabatic experiment there). The equation then follows, assuming:

1. $|\Psi\rangle = |\{\mathbf{r}_1, \ldots \mathbf{r}_n\}\rangle$ has short-ranged correlations, i.e.:

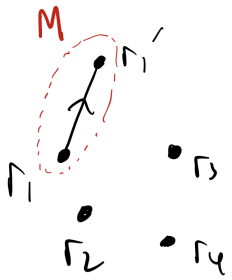
$$\left\langle A_{\mathbf{r}} A'_{\mathbf{r}'} \right\rangle_\Psi = \langle A_{\mathbf{r}} \rangle_\Psi \langle A_{\mathbf{r}'}; \rangle_\Psi + \mathcal{O}(e^{-\frac{|\mathbf{r}-\mathbf{r}'|}{\xi}}) \tag{4.6}$$

   for $A_{\mathbf{r}}, A_{\mathbf{r}'}$ local operators supported near $\mathbf{r}, \mathbf{r}'$. This is where the gapped assumption comes in; the ground state of a gapped Hamiltonian has short-ranged correlations.

2. Particles can be moved by local operators. In other words:

$$|\left\{\mathbf{r}'_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\right\}\rangle = M|\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}\rangle \tag{4.7}$$

   where M is an operator supported near $\mathbf{r}_1, \mathbf{r}'_1$.



These two conditions together imply the locality constraint on the Berry phase.

4.3 Possible Forms of the Berry Phase & Topological Classes

The next question is then - what is the most general Berry phase $\theta_B(\Gamma)$ that satisfies the locality constraint? One solution, and the one you probably would have guessed, is:

$$\theta_B(\Gamma) = \sum_i \int_\Gamma \boldsymbol{\mathcal{A}}(\mathbf{r}_i) \cdot d\mathbf{r}_i. \tag{4.8}$$

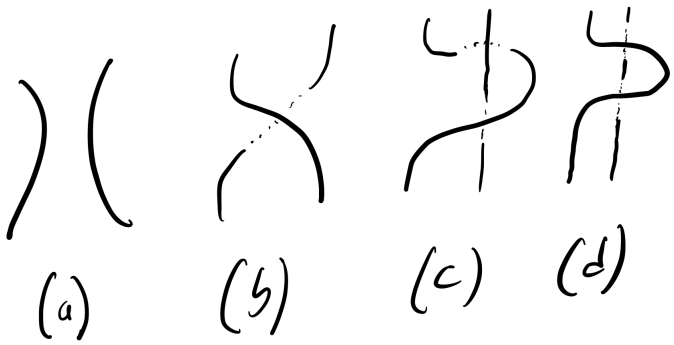
This is manifestly local. We could get a little more general:

$$\theta_B(\Gamma) = \sum_i \int_\Gamma \left(\boldsymbol{\mathcal{A}}(\mathbf{r}_i) + \sum_j \boldsymbol{\mathcal{B}}(\mathbf{r}_i, \mathbf{r}_j) + \sum_{jk} \boldsymbol{\mathcal{C}}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) + \ldots \right) \cdot d\mathbf{r}_i \tag{4.9}$$

where $\boldsymbol{\mathcal{A}}$ is the single-particle Berry connection and $\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}$ are the two, three (and so on) particle terms so long as the multi-particle terms are short-ranged, i.e. only they are nonzero where $\mathbf{r}_j$ is close to $\mathbf{r}_i$ and so on.

Is this the only possible solution consistent with Eq. (4.5)? No! Indeed the first proposed solution is the form of the Berry phase consistent with bosons, but there are other solutions corresponding to fermions and anyons. What does the first solution miss? Indeed it is possible to have topological terms that look highly non-local, but such that the Berry phase still has the locality constraint.

In this line, we say that two (non-intersecting) n-particle paths with the same endpoints are topologically equivalent if they can be continuously deformed into one another without bringing particles near each other (worldlines cannot pass through one another). For example, consider:



$(a) \sim (d)$, but $(a) \not\sim (b) \not\sim (c)$. This defines an equivalence relation on (non-intersecting) n-particle paths, which splits the set of n-particle paths into equivalence (topological) classes.

Coming back to the Berry phase, the claim is that the most general θ_B that satisfies the locality constraint Eq. (4.5) can be written as:

$$\boxed{\theta_B(\Gamma) = \theta_{\text{short-range}}(\Gamma) + \theta_{\text{top}}(\Gamma)} \tag{4.10}$$

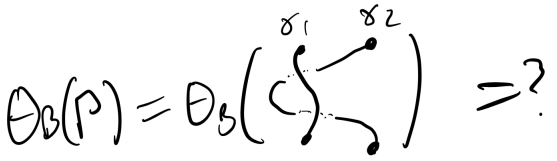
where $\theta_{\text{top}}(\Gamma)$ only depends on the topological class of Γ and the short-range piece is given by Eq. (4.9)

$$\theta_{\text{short-range}}(\Gamma) = \sum_i \int_\Gamma \left(\boldsymbol{\mathcal{A}}(\mathbf{r}_i) + \sum_j \boldsymbol{\mathcal{B}}(\mathbf{r}_i, \mathbf{r}_j) + \sum_{jk} \boldsymbol{\mathcal{C}}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) + \dots \right) \cdot d\mathbf{r}_i. \tag{4.11}$$

Eq. (4.10) is the key result. From here, we will classify the different possible topological terms we can have. We will find in 3D that we only get two possible classes and in 2D that we get many more.

Let us argue for Eq. (4.10) in the special case of Γ being a topologically trivial path, i.e. where we only have the first term. For the sake of drawing, let's look at a 2 particle path.

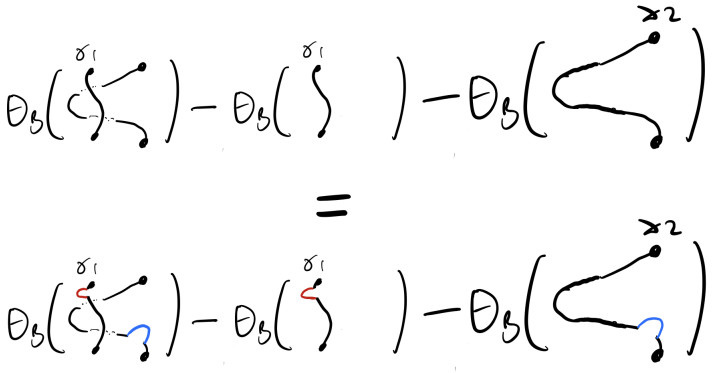
$$\theta_B(\Gamma) = \theta_B(\gamma_1 \& \gamma_2) = ? \tag{4.12}$$



where γ_1, γ_2 are the single particle paths. We can then define:

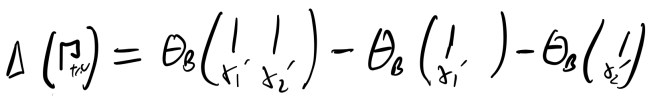
$$\Delta(\Gamma) = \theta_B(\gamma_1 \& \gamma_2) - \theta_B(\gamma_1) - \theta_B(\gamma_2) \tag{4.13}$$

Now, using Eq. (4.5) we can use that $\Delta(\Gamma)$ is topologically invariant, i.e. it is invariant to local deformations of γ_1 far from γ_2 and vise versa.



Therefore we can deform Γ to the trivial path, for which Δ is easily seen to vanish.

$$\Delta(\Gamma) = \Delta(\Gamma_{\text{trivial}}) = 0 \tag{4.14}$$



And thus since $\Delta(\Gamma) = 0$, we find:

$$\theta_B(\Gamma) = \theta_B(\gamma_1) + \theta_B(\gamma_2) = \int_{\gamma_1} \mathcal{A}(\mathbf{r}_1) \cdot d\mathbf{r}_1 + \int_{\gamma_2} \mathcal{A}(\mathbf{r}_2) \cdot d\mathbf{r}_2 = \theta_{\text{short-range}}(\Gamma). \tag{4.15}$$