

# PHYS 411 (Entanglement in Many-Body Systems) Notes

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## Introduction:

This is a set of lecture notes taken from UChicago's PHYS 411 (Entanglement in Many-Body Systems), taught by Michael Levin. Topics covered include...

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# 1 Toric Code I

## 1.1 Course Overview + Logistics

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Evaluation: Once every  $\sim 2$  weeks, 100%.

Textbook: None; papers/references will be provided.

This class will cover topics at the interface of quantum many-body/condensed matter theory and quantum information. This has been a dynamic interface for a couple decades now, with the two fields inspiring each other. The topics chosen are both interesting from a physics point of view, but also deeply important in QI. A rough schedule is as follows:

- (I) **Anyons and topological quantum computation.** Anyons exist in 2-d quantum systems that have exchange statistics that are not bosonic or fermionic; the exchange phase can be anything (hence the name). Anyons first emerge in discussion of the fractional quantum hall effect, but in the last 20 years, people (lead by Alexei Kitaev) have found interesting connections between anyons and quantum computing.
- (II) **Symmetry-protected topological phases.** This is another important topic in condensed matter, but is deeply connected to concepts in quantum information, e.g. finite depth-circuits (indeed they provide a quantum-information theoretic way to define phases of matter).
- (III) **Entanglement entropy in many-body systems.** EE gives a lot of insight into the physics of MB systems. This also has practical applications, leading to numerical algorithms using...
- (IV) **Matrix product states.**

## 1.2 Defining the Toric Code Model

References: Kitaev's lecture notes [arXiv:0904.2771](https://arxiv.org/abs/0904.2771), original paper [arXiv:quant-ph/9707021](https://arxiv.org/abs/quant-ph/9707021).

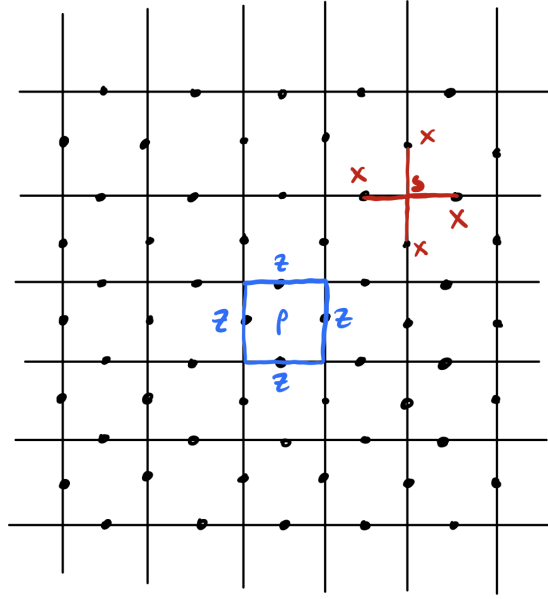
The toric code is an exactly solvable spin model (it can also be thought of a quantum error correcting code, but we introduce it as a spin model to start). It has:

1. Anyon excitations
2. Topological ground state degeneracy

We consider this model on different kinds of lattices and geometries, but for now we consider the square lattice, and place a spin-1/2 degree of freedom on each of the edges of the lattice (we don't specify the boundary conditions yet). The Hilbert space has dimension  $2^N$  with  $N$  the total number of edges/spins. The Hamiltonian takes the following form:

$$H = - \sum_s A_s - \sum_p B_p \tag{1.1}$$

the  $s$  are vertices on the lattice and  $p$  are plaquettes.



The  $A_s$  term is a product of Pauli-X operators on stars about vertices  $s$ :

$$A_s = \prod_{j \in \text{star}(s)} X_j \quad (1.2)$$

And the  $B_p$  term is a product of Pauli-Z operators on the boundaries of plaquettes:

$$B_p = \prod_{j \in \partial p} Z_j \quad (1.3)$$

We adopt the QI notation:

$$X = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1.4)$$

$$Y = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (1.5)$$

$$Z = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.6)$$

### 1.3 Solving the toric code model

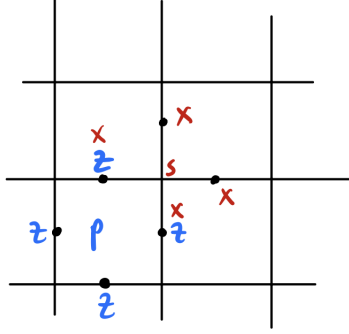
Notice that all of the  $A_s$  and  $B_p$  terms commute with one another. For example, its trivial to see that:

$$[A_s, A_{s'}] = [B_p, B_{p'}] = 0 \quad (1.7)$$

because Xs are mutually commuting and Zs are mutually commuting. Slightly less obvious is that the star terms commute with the plaquettes:

$$[A_s, B_p] = 0 \quad (1.8)$$

We might be worried if this holds because  $X$  and  $Z$  anticommute. But in fact the above holds; if the star and plaquette are faraway then there are no overlapping  $X, Z$ s so they commute. In the case where the star/plaquette overlap, we have that two  $X, Z$ s overlap (see picture below) so the anticommutation cancels to a commutation.



Thus, we are able to simultaneously diagonalize  $\{A_s, B_p\}$ . Denote the eigenstates by  $|\{a_s, b_p\}\rangle$  where the  $a_s, b_p$  are the eigenvalues. If we have residual degeneracy, we may require additional quantum numbers to specify the state, but let us not worry about this quite yet. Note that because:

$$A_s^2 = B_p^2 = \mathbb{I} \quad (1.9)$$

this implies that the eigenvalues are:

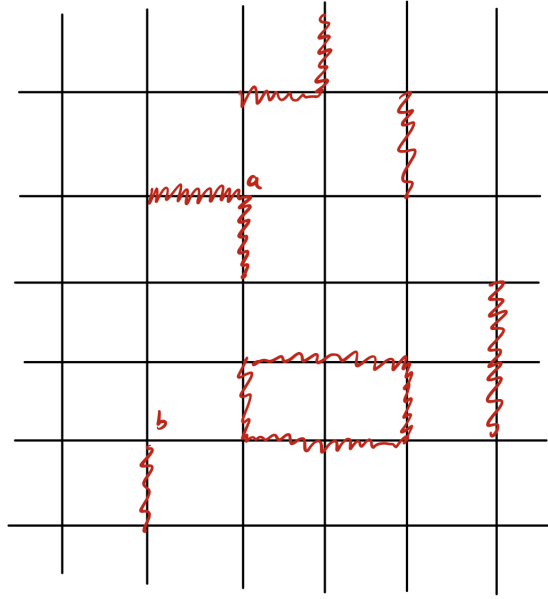
$$a_s, b_p = \pm 1. \quad (1.10)$$

And the  $|\{a_s, b_p\}\rangle$  are also the energy eigenstates, with energy:

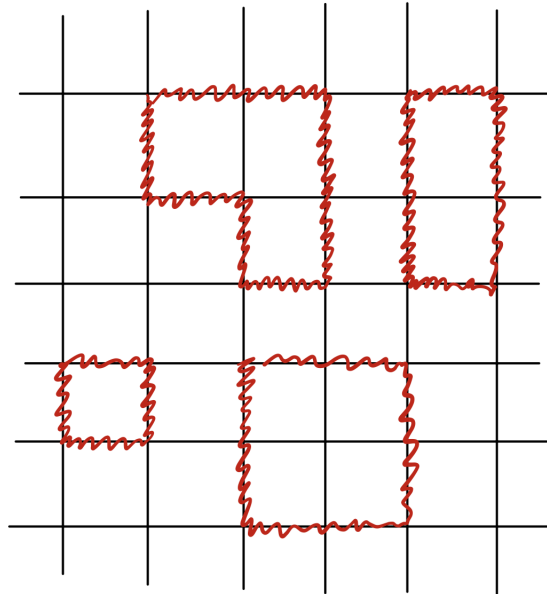
$$E = -\sum_s a_s - \sum_p b_p. \quad (1.11)$$

This is the sense in which the problem is exactly solvable. We have found all of the eigenstates, and can find their degeneracies by figuring out the degeneracies of  $\{a_s, b_p\}$ .

For the ground state, we set  $a_s = b_p = 1$ . We can ask the question; how many states are there that satisfy this? This is equivalent to asking what the ground state degeneracy is. The answer to this question is dependent on the geometry of the model. Let's start with the simplest, and arguably the most important case; the infinite plane. Let us work in the  $X$ -basis. In this basis, the different basis states are  $|\pm\rangle \otimes |\pm\rangle \otimes \dots$  for each link. A useful visualization will be in terms of strings on this lattice. In this string picture, we will say that  $X_j = -1$  corresponds to a string on link  $j$ . On the other hand, if  $X_j = +1$  then we will say that there is no string on the link. Thus, we can view an arbitrary  $X$  basis state as strings occupying the lattice in some configuration.



With this point of view, let's figure out what the ground states look like in terms of the  $X$ -eigenstates. We want  $a_s = 1$  for every star, thus  $\prod_{j \in \text{star}(s)} X_j = 1$ . This implies that the number of  $-1$ s in the product must be even. This means that the number of strings that touch a given site must be even. In the above picture,  $\text{star}(a)$  obeys this condition while  $\text{star}(b)$  does not. But, if  $a_s = 1$  for every single star, this implies that the strings must form closed loops (else - at the endpoints of strings we end up with  $a_1 = -1$ ). An allowed configuration is sketched below:



Note that since  $XZ = -ZX$ , a given  $B_p$  flips string occupation around a plaquette  $p$ , e.g.:

$$B_p | \text{loop with } p \rangle = | \text{loop without } p \rangle$$

But then all of the  $a_s = b_p = 1$  requires that we have an equal amplitude superposition of all closed loop states (if this was not the case, the application of some  $B_p$  would flip some loops and change the state - not an eigenstate!). Thus! There is a unique ground state:

$$|\Omega\rangle = |a_s = b_p = 1\rangle = \sum_{\text{closed loop config } C} |C\rangle. \quad (1.12)$$

We could draw an example  $|C\rangle$  pictorially as:

$$|C\rangle = | \text{two loops} \rangle$$

It can be seen that  $B_p$  only permutes the different  $C$ s to each other without changing their weight, so indeed the above is the  $+1$  eigenstate of the  $B_p$ s. More formally, if  $B_p|\psi\rangle = |\psi\rangle$  for all  $p$ , then:

$$\langle C|\psi\rangle = \langle C|B_p|\psi\rangle = \langle C'|\psi\rangle \quad (1.13)$$

where  $|C'\rangle$  is a different closed loop configuration. This implies that all of the amplitudes of the closed loop configurations have equal weight.

Note that formally there is an infinite number of closed loop configurations on an infinite plane, so there is a bit of a subtlety in normalizing the state (which requires the machinery of operator algebras, etc.) which we sweep under the rug.

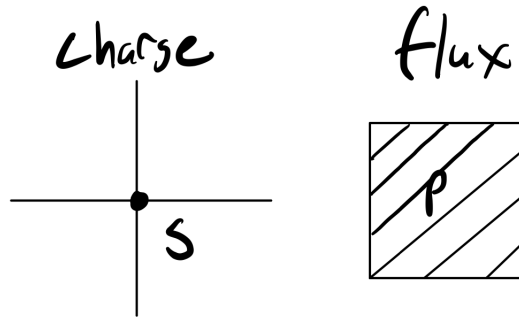
Note that the above argument gave a unique state for  $a_s = b_p = 1$ , but for any choice of  $\{a_s, b_p\}$  we can get a statement of a similar flavour. Thus, this justifies the choice of notation  $|\{a_s, b_p\}\rangle$  as for a given choice of  $a_s/b_p$  the state is unique.

Now, we have the full energy spectrum, as we have determined the degeneracy of all of the eigenspaces. We can thus determine the energy gap, which is the energy difference between the ground state and first excited state. Flipping one of the  $a_s$  or the  $b_p$  results in an energy penalty of 2, and thus the energy gap is  $\Delta = 2$ . This is important because it means that we have a “gapped Hamiltonian”. In comparison, there are “gapless” Hamiltonians for which the energy difference goes to zero in the thermodynamic limit. This distinction/property will be important when we discuss anyons - gapped Hamiltonians are the context in which they are currently understood.

## 1.4 Excitations and String Operators

There are two types of elementary excitations:

1.  $a_s = -1$  for some  $s$ ; this is a “charge”, and lives on a site.
2.  $b_p = -1$  for some  $p$ ; this is a “flux”, and lives on a plaquette.



The terminology comes from  $\mathbb{Z}_2$  gauge theory. We will see that these excitations are not conventional bosons or fermions that we may be familiar with.

If we visualize these excitations, adding a charge is like taking our superposition of all closed loops and then to each of those states adding one defect (with a string that goes off to infinity). Adding a plaquette takes the superposition of loops, and we count the number of loops that go around  $p$  (a kind of winding number) and take that to be the sign of the configuration in the superposition. Pictorially, we have defects in the first case and vortices in the second.

Now, the question becomes how can we create charge or flux excitations? In spin systems you might be familiar with (e.g. creating a magnon in a Heisenberg spin chain) you would create them via a local operator. But here we actually apply a non-local string operator to create these excitations.

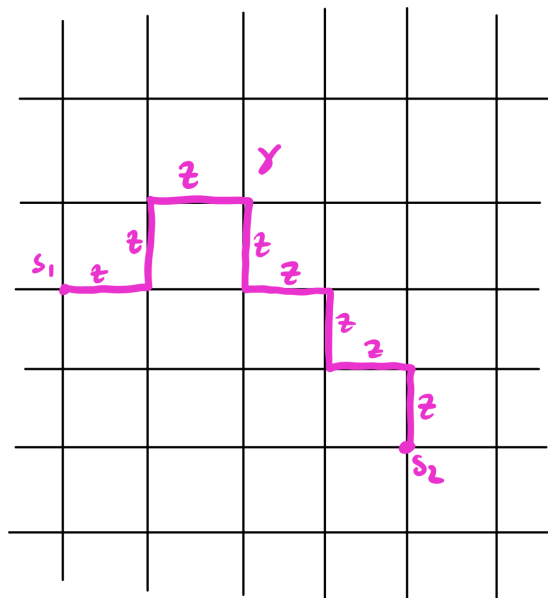
We define:

$$W^Z(\gamma) = \prod_{j \in \gamma} Z_j \quad (1.14)$$

where  $\gamma$  is some open path on the lattice. The  $W^Z(\gamma)$  is a string of Zs along  $\gamma$ . This string operator creates charge excitations at the two endpoints; notice that:

$$[W^Z(\gamma), B_p] = 0 \quad (1.15)$$

this is obvious as the  $B_p$ s consist of Zs only. Less obvious is that the  $W^Z(\gamma)$  commutes with almost all of the  $A_s$  operators - all except at the endpoints of  $\gamma$ ,  $s_1$  and  $s_2$ .



Let us argue this. At intermediate points along the path each of the points have an even number of strings so we have commutation. At the endpoints, we only have 1 link and so we have anticommutation:

$$W^Z(\gamma)A_{s_{1,2}} = -A_{s_{1,2}}W^Z(\gamma) \quad (1.16)$$

What are the implications of this? Looking at the ground state  $|\Omega\rangle = |\{a_s = b_p = 1\}\rangle$ :

$$W^Z(\gamma)|\Omega\rangle = |\{a_{s_1} = a_{s_2} = -1, \text{others} = 1\}\rangle. \quad (1.17)$$

and we can see this because the string operator only flips the stars at the endpoints, and commutes with everything else. Thus, we conclude that the  $W^Z(\gamma)$  creates charges at the two endpoints of  $\gamma$ , as claimed.

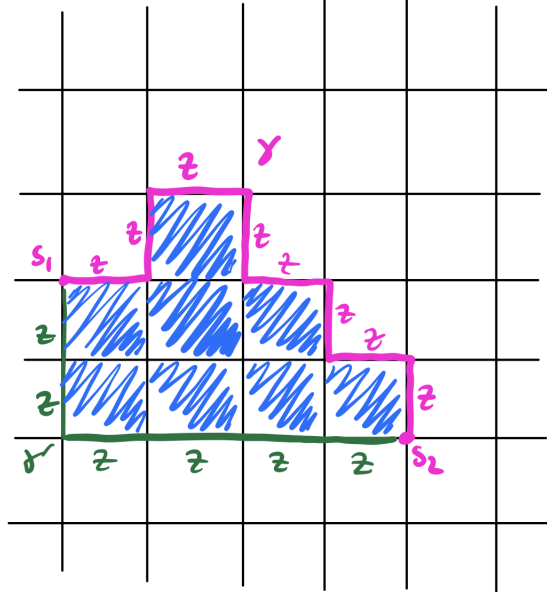
An important observation with regard to string operators; If  $\gamma'$  is a different path with the same endpoints, then the string operator  $W^Z(\gamma')$  applied to the ground state yields the *exact* same state (even up to the same phase):

$$W^Z(\gamma)|\Omega\rangle = W^Z(\gamma')|\Omega\rangle \quad (1.18)$$

this is the notion in which string operators are “flexible”. To see this, note that  $W^Z(\gamma') = W^Z(\gamma) \prod_{p \in \text{int}(\gamma' \cup \gamma)} B_p$ , i.e. the two operators are related via a product of plaquette operators in the interior of the two paths. Thus:

$$W^Z(\gamma')|\Omega\rangle = W^Z(\gamma) \prod_{p \in \text{int}(\gamma' \cup \gamma)} B_p |\Omega\rangle = W^Z(\gamma)|\Omega\rangle \quad (1.19)$$

where we have used that  $|\Omega\rangle$  is the +1-eigenstate of all the  $B_p$ s.

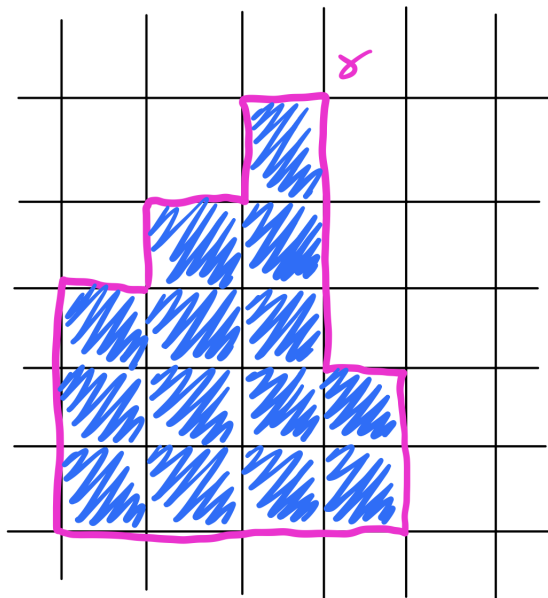


Relatedly, if  $\gamma$  is a closed loop then:

$$W^Z(\gamma)|\Omega\rangle = |\Omega\rangle \quad (1.20)$$

which follows from the fact that a closed loop is just a product of  $B_p$ s.





These features - which currently seem pretty specific to the toric code model - are in fact quite general. Any system with anyons have string operators with these properties!

A last comment to provide some physical intuition for what the string operator is. We can view it as the physical process of first creating two charges (by applying a single  $Z$ ), then moving that charge via the application of further  $Z$ s along the path. I.e. a string operator is just creating two charges and separating them. This makes the notion that the string is flexible intuitive; we should get the same state (up to some phase) if the particles are created and end up in some separated location(s), no matter how we move them there.

## 2 Toric Code II

## 2.1 Review

A quick review of some definitions; the Toric code has Hamiltonian (defined on a 2D lattice with qubits placed on the edges):

$$H = - \sum_s A_s - \sum_p B_p \quad (2.1)$$

with  $A_s$  star operators around each lattice vertex:

$$A_s = \prod_{j \in \text{star}(s)} X_s \quad (2.2)$$

and  $B_p$  plaquette operators around each lattice plaquette

$$B_p = \prod_{j \in \partial p} Z_p \quad (2.3)$$

we constructed the ground state  $|\Omega\rangle = |a_s = b_p = 1\rangle$ . We also discussed a string operator, which creates charge excitations:

$$W^Z(\gamma) = \prod_{j \in \gamma} Z_j \quad (2.4)$$

where  $\gamma$  is a path on the lattice. It creates charges ( $a_s = -1$ ) at the endpoints of  $\gamma$ . Further, these string operators are flexible, in the sense that:

$$W^Z(\gamma)|\Omega\rangle = W^Z(\gamma')|\Omega\rangle \quad (2.5)$$

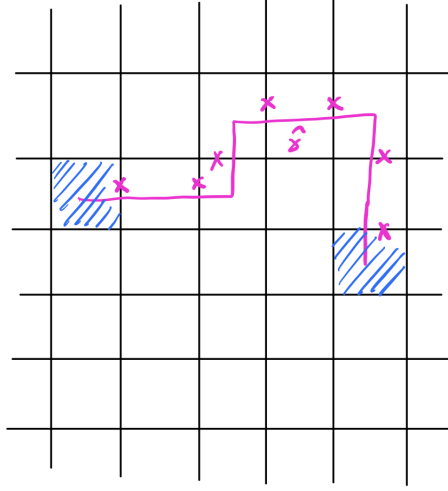
for two paths  $\gamma, \gamma'$  with the same endpoints.

## 2.2 String operator for flux excitations

There is a similar string operator for flux excitations. Just a heads up that the structure of having string operators (one for each anyon type - here for charges and fluxes/ $e$  and  $m$ ) is quite general. We define:

$$W^X(\hat{\gamma}) = \prod_{j \in \hat{\gamma}} X_j \quad (2.6)$$

where  $\hat{\gamma}$  is an open path on the dual lattice, i.e. that go through the center of plaquettes:

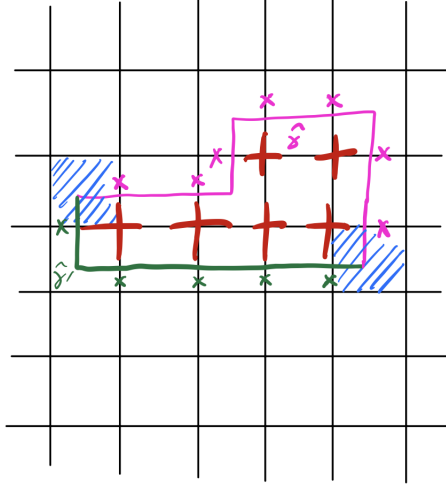


Much like the string operator for the charges:

- It can be checked that  $W^X(\hat{\gamma})$  creates fluxes  $b_p = -1$  at the two endpoints of  $\hat{\gamma}$ .
- The string operators are flexible, with:

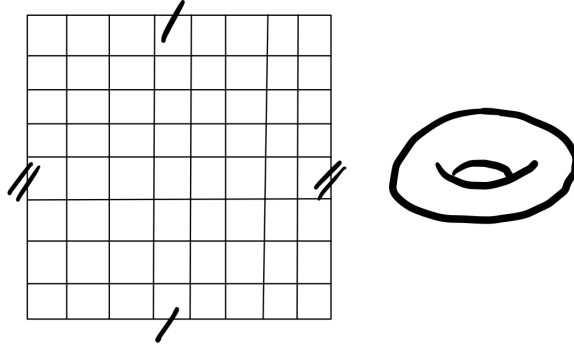
$$W^X(\hat{\gamma})|\Omega\rangle = W^X(\hat{\gamma}')|\Omega\rangle \quad (2.7)$$

for  $\hat{\gamma}, \hat{\gamma}'$  with the same endpoints. They are related by the product of star operators on the interior.



The existence of these flexible, non-commuting string operators is very fundamental to the structure of the toric code, and to anyon systems more generally. The existence of these is independent of geometries. But we will see that it will have implications when we consider the model for specific systems.

### 2.3 Ground state degeneracy on a torus



Consider the toric code, on a finite  $(L \times L)$  torus. We may have different allowable states for a given  $a_s$  and  $b_p$ , and thus may find that there are different degeneracies, compared to the infinite plane case. We can now ask what is the ground state degeneracy  $D$ ? Indeed, this question is equivalent to asking what the dimension of the eigenspace with  $a_s = b_p = 1$  is. To find this, we look at the trace of the projector onto the eigenspace<sup>1</sup>:

$$\begin{aligned} D &= \text{Tr}(\text{proj. onto } a_s = b_p = 1 \text{ subspace}) \\ &= \text{Tr}\left(\prod_s \left(\frac{\mathbb{I} + A_s}{2}\right) \prod_p \left(\frac{\mathbb{I} + B_p}{2}\right)\right) \end{aligned} \quad (2.8)$$

where the second line follows from the fact that the product of the (mutually commuting) projectors gives

<sup>1</sup>This is a very formal way to find it, we will soon get different perspectives on this question

the projector onto the subspace. Computing this:

$$D = \frac{1}{2^{N_s}} \frac{1}{2^{N_p}} \text{Tr} \left( \prod_s (\mathbb{I} + A_s) \prod_p (\mathbb{I} + B_p) \right) \quad (2.9)$$

Now we expand out this product, and can think about the traces of the individual terms. A single  $A_s, B_p$  will be traceless (as the Paulis are traceless), and so will most products of  $A_s, B_p$ ; the only non-traceless terms will be those that simplify to the identity. If we stare at this, there are only a few combinations for which this occurs; there is the term with all identity, the term with all stars (all the Xs cancel), the term with all plaquettes (all the Zs cancel), and the term with all stars and all plaquettes.

$$D = \frac{1}{2^{N_s}} \frac{1}{2^{N_p}} \text{Tr} (\mathbb{I} + \prod_s A_s + \prod_p B_p + \prod_s \prod_p A_s B_p) = \frac{1}{2^{N_s}} \frac{1}{2^{N_p}} \text{Tr} (4\mathbb{I}) \quad (2.10)$$

Now looking at the trace of the identity:

$$\text{Tr}(\mathbb{I}) = 2^{\dim(\mathcal{H})} = 2^{N_{\text{links}}} \quad (2.11)$$

Thus:

$$D = \frac{1}{2^{L^2}} \cdot \frac{1}{2^{L^2}} \cdot 4 \cdot 2^{2L^2} = 4 \quad (2.12)$$

so:

$$\boxed{D = 4} \quad (2.13)$$

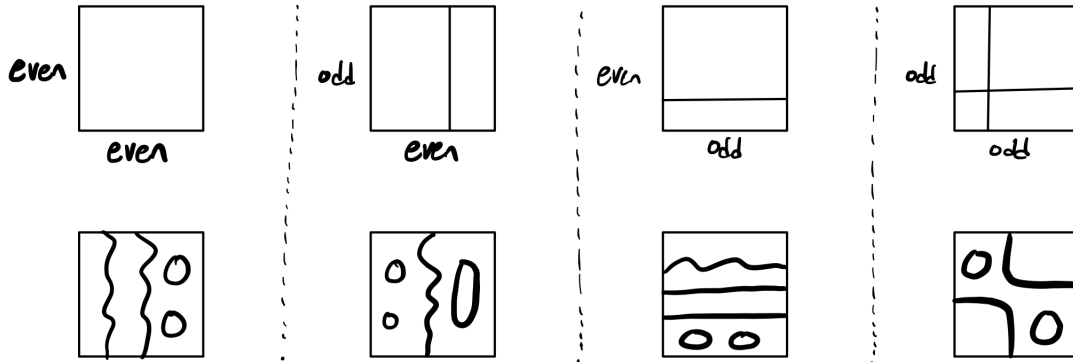
A nice feature of this argument is we can repeat it for any of the eigenspaces. In fact, every  $\{a_s, b_p\}$  eigenspace with  $\prod_s a_s = \prod_p b_p = 1$  (i.e. an even number of charges) is 4-fold degenerate.

## 2.4 Ground states in the string picture

Now, let's see if we can understand the ground state degeneracy in the string picture. To review, we work in the X-basis, and  $X_j = \pm 1$  corresponds to there being a string (plus) or no string on link  $j$ .  $a_s = 1$  requires the product of Xs on the star to be one, implying that the strings form closed loops.  $b_p = 1$  implies that there is an equal amplitude superposition of string states, as different string states are related by  $B_p$  moves. We used these two conditions on the infinite plane geometry (Where  $B_p$  moves are "ergodic") to say that there was a unique ground state, namely that with an equal weight superposition of all closed loop configurations:

$$|\Omega\rangle = \sum_{\text{closed loop config } C} |C\rangle \quad (2.14)$$

On a torus, the closed loop states can be divided into 4 classes; even/even, even/odd, odd/even, and odd/odd.



What does this mean? this means that if we draw a line going across the torus (on the dual lattice), we “cross” an even/odd number of strings. Within each sector/class, the  $B_p$  moves are ergodic. But,  $B_p$  moves cannot change the parity of the crossings. Thus the four degenerate ground states correspond to the equal weight superpositions of closed loop configurations within a given class. We can label the ground states via their winding number:

$$|\Omega_{(e/o, e/o)}\rangle = \sum_{\text{closed loop config } C \text{ with } (e/o, e/o) \text{ winding}} |C\rangle \quad (2.15)$$

So, so far we have understood the ground state degeneracy from two perspectives. But neither of these tells us the deeper principle underlying the degeneracy. Let us discuss this now - it will allow us to see why the degeneracy is topologically protected.

## 2.5 Origin of the ground state degeneracy

The punchline is that the GSD comes from the existence of non-commuting string operators.

Define the charge string operators:

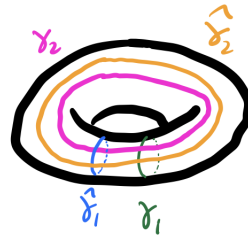
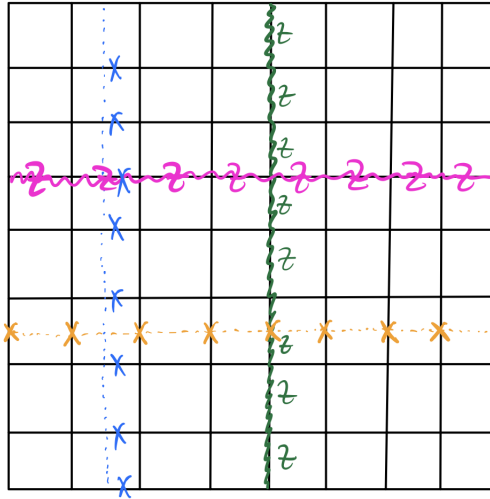
$$W_1^Z = \prod_{j \in \gamma_1} Z_j \quad (2.16)$$

$$W_2^Z = \prod_{j \in \gamma_2} Z_j \quad (2.17)$$

These string operators correspond to the creation and subsequent annihilation of a charge as the string wraps around the torus. We can also define the string operators for the fluxes;

$$W_1^X = \prod_{j \in \hat{\gamma}_1} X_j \quad (2.18)$$

$$W_2^X = \prod_{j \in \hat{\gamma}_2} X_j \quad (2.19)$$



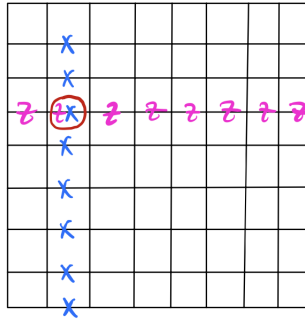
Notice that:

$$\begin{aligned} [W_i^Z, A_s] &= [W_i^Z, B_p] = 0 \\ [W_i^X, A_s] &= [W_i^X, B_p] = 0 \end{aligned} \quad (2.20)$$

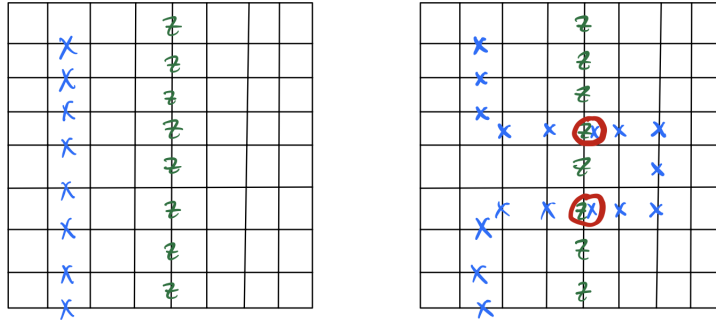
so they map ground states to ground states. They have an interesting commutation algebra.  $\{W_1^Z, W_2^Z, W_1^X, W_2^X\}$  all commute, except for two exceptions:

$$\begin{aligned} W_1^X W_2^Z &= -W_2^Z W_1^X \\ W_2^X W_1^Z &= -W_1^Z W_2^X \end{aligned} \quad (2.21)$$

this is because they anticommute in one place.



The fact that the other commute is clear; the fact that the Xs mutually commute and Zs mutually commute is immediate. For  $W_1^X, W_1^Z$ , they act on disjoint regions. Even if you were to pick representatives that overlap, they will overlap an even amount of times.



This algebra is quite interesting; we have symmetry generators that commute with the Hamiltonian, but not with each other. A simple HW exercise you will do is that the eigenstates/ground states will come in multiplets of 4 (the fact that it is exactly 4 in this case comes from the microscopic calculation).

A final comment; we said how the  $\{a_s, b_p\}$  do not uniquely specify the ground (or excited) states. But in fact the string operators provide the missing quantum numbers necessary to specify the state. Specifically, we can uniquely label the eigenstates by choosing (in addition to the  $\{a_s, b_p\}$ ) the values of the  $W$ s, e.g.  $|\{a_s, b_p\}, w_1^x, w_2^x\rangle$  where  $w_1^x = \pm 1$  and  $w_2^x = \pm 1$ . We can thus denote the four ground states as

$|\Omega, \pm \pm\rangle$ . These are *exactly* the same what we called before the  $|\Omega_{(e/o,e/o)}\rangle$  - in fact the  $W^x$  string operator counts the number of crossing of X strings.

Next time, we will discuss how the GSD is robust to arbitrary local perturbations - it is topologically protected.

### 3 Toric Code III, Berry Phase

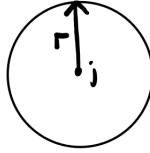
#### 3.1 Robustness of toric code GSD

We saw the degeneracy of the toric code from both the explicit/formal calculation as well as from string operators. GSD in itself is not interesting, and is quite fragile; perturbations tend to split it - what is interesting about the TC GSD is that it is extremely robust; local perturbations cannot split it.

More precisely, consider an arbitrary local perturbation of  $H$ :

$$H' = H + \lambda \sum_j V_j \quad (3.1)$$

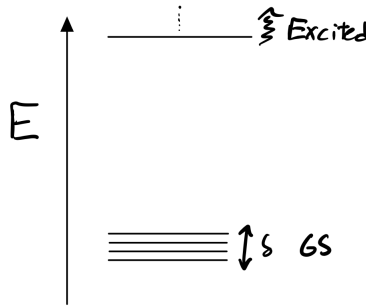
where  $V_j$  is a local operator supported near site  $j$ .



Claim: For sufficiently small  $\lambda$ ,  $H'$  has 4 nearly degenerate ground states with splitting:

$$\delta \leq e^{-C(\lambda)L} \quad (3.2)$$

with  $C(\lambda)$  a  $\lambda$ -dependent constant. The idea is that with an arbitrary local perturbation we have an exponentially small (in the thermodynamic limit) splitting of the ground state manifold, and thus the GSD is robust. From a QI perspective, this is useful because it tells us that we have a robust encoding of a qubit - we can use the ground state as a robust subspace. This degeneracy is often called a “protected”, or topological because it is automatically protected.



#### 3.2 Argument for robustness

Using the fact that  $\{W_1^X, W_2^X, W_1^Z, W_2^Z\}$  (string operators) are flexible and non-commuting, we can establish a key property of the unperturbed ground states  $|\Omega, \pm\rangle$ . Namely, for any operator  $O$  supported on

less than  $L$  sites:

$$\langle \Omega, \pm, \pm | O | \Omega, \pm, \pm \rangle = \text{diag}(c, c, c, c) \quad (3.3)$$

for some constant  $c$ . A shorthand for the above is:

$$\langle \Omega, \alpha | O | \Omega, \beta \rangle = c \delta_{\alpha\beta}. \quad (3.4)$$

You will show this relation on the homework.

Let's unpack this equation. The first thing it tells us is that:

$$\langle \Omega, \alpha | O | \Omega, \alpha \rangle = \langle \Omega, \alpha' | O | \Omega, \alpha' \rangle \quad (3.5)$$

which tells us that local operators cannot distinguish different ground states. The second thing it tells us is that:

$$\langle \Omega, \alpha' | O | \Omega, \alpha \rangle = 0 \quad \text{for } \alpha \neq \alpha' \quad (3.6)$$

in other words, local operators cannot connect ground states.

As a comparison, consider  $|\uparrow\rangle^{\otimes N}, |\downarrow\rangle^{\otimes N}$  the 2-fold degenerate ground states of the Ising model  $H = -\sum_{ij} Z_i Z_j$ . A local operator cannot connect them, but it is possible to distinguish the two states by measuring  $Z_i$ . In a symmetry breaking state, we do not have the structure of flexible strings, and hence do not have the same notion of local indistinguishability to arbitrary operators (only to symmetric ones).

Using the local distinguishability/unconnectability, let us sketch an argument for the toric code GSD. For concreteness, consider a perturbation:

$$H' = H + \lambda \sum_j X_j. \quad (3.7)$$

To obtain the first-order splitting (in degenerate perturbation theory), we need to find the matrix elements:

$$\langle \Omega, \pm, \pm | \sum_j X_j | \Omega, \pm, \pm \rangle \quad (3.8)$$

and then diagonalize. By Eq. (3.3), we know that:

$$\langle \Omega, \pm, \pm | X_j | \Omega, \pm, \pm \rangle = c_j \mathbb{I} \quad (3.9)$$

Therefore the ground state degeneracy is not split to first order. Looking at second order, we need to find matrix elements:

$$\langle \Omega, \pm, \pm | (\sum_j X_j) (H - E_{\text{gs}})^{-1} \Pi_{\text{ex}} (\sum_j X_j) | \Omega, \pm, \pm \rangle \quad (3.10)$$

and then diagonalize (third, fourth order (and so on) we have the same procedure, just add a factor of  $(H - E_{\text{gs}})^{-1} \Pi_{\text{ex}} (\sum_j X_j)$  each order). A given  $X_j$  takes us to an excited state, the  $\Pi_{\text{ex}}$  projector is then irrelevant<sup>2</sup>, and then  $(H - E_{\text{gs}})^{-1}$  is just a number (difference between the ground and excited state energy), so we end up evaluating matrix elements of the form:

$$\langle \Omega, \pm, \pm | X_j X_k | \Omega, \pm, \pm \rangle = c \mathbb{I} \quad (3.11)$$

where we again use Eq. (3.3). So, again at second order we have no splitting, and the argument follows the same way for third, fourth order etc. using the same property. The argument only breaks when the property no longer holds, which occurs at  $L$ th order of perturbation theory when we end up looking at the matrix element of an operator supported on  $L$  sites. In particular, we get terms of the form:

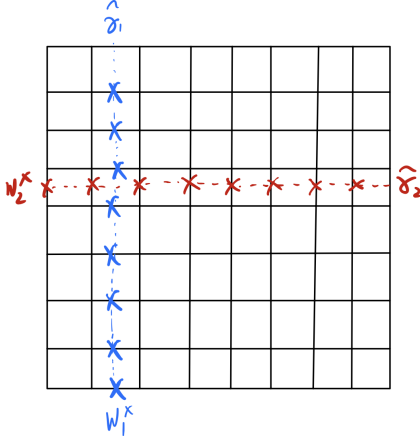
$$\prod_{j \in \hat{\gamma}_1} X_j = W_1^X, \quad \prod_{j \in \hat{\gamma}_2} X_j = W_2^X \quad (3.12)$$

i.e. the string operators.

---

<sup>2</sup>For a general operator, we instead can use that the projector can be restricted to a local operator as we only need to look at some local patch to tell that we are in an excited state





Thus we end up the matrix elements:

$$\langle \Omega, \pm, \pm | W | \Omega, \pm, \pm \rangle \sim \lambda^L \text{diag}(c_1, c_2, c_3, c_4) \quad (3.13)$$

and so then the splitting between the ground states is:

$$\delta \sim \lambda^L = e^{-L \log(\frac{1}{\lambda})} \quad (3.14)$$

Note that this is quite heuristic, and to make it rigorous you require more precise arguments, namely that the perturbation theory converges, with a finite radius of convergence  $\lambda_0$  (which holds for arbitrarily large  $L$ ). Without a formal argument, we expect such a finite radius of convergence for “typical” gapped local Hamiltonians, with  $\lambda_0 \sim \Delta$  (so actually, in addition to the local indistinguishability, we are also using that the toric code is gapped).

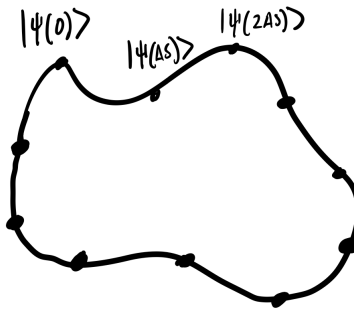
### 3.3 A Review of Berry Phase

Before we move to a general discussion of anyons, we first review the notion of a Berry phase, which is a very related idea.

Let:

$$\left\{ |\psi(s)\rangle, 0 \leq s \leq T, |\psi(T)\rangle = e^{i\phi} |\psi(0)\rangle \right\} \quad (3.15)$$

be a closed path in the set of normalized quantum states (rays in Hilbert space). Let us split up the path into  $N$  parts of length  $\Delta s$ , with  $N\Delta s = T$ . Graphically:



and for brevity we denote  $|\psi(\cdot)\rangle = |\psi\rangle$ . Now, we define the Berry phase as:

$$e^{i\theta_B} = \lim_{N \rightarrow \infty} \langle \psi_0 | \psi_{N-1} \rangle \dots \langle \psi_2 | \psi_1 \rangle \langle \psi_1 | \psi_0 \rangle \quad (3.16)$$

The Berry phase has properties:

1.  $|e^{i\theta_B}| = 1$ , i.e.  $e^{i\theta_B}$  is a  $U(1)$  phase. This can be seen from the fact that  $\langle \psi_1 | \psi_0 \rangle \sim \frac{1}{N}$  and so the  $N$ -fold product is of order  $\sim 1$ .
2.  $e^{i\theta_B}$  only depends on the path and not its parameterization. That is, it is invariant under  $s \rightarrow s' = f(s)$  with  $f(0) = 0$  and  $f(T) = T'$ .
3.  $e^{i\theta_B}$  does not depend on the phase of  $|\psi(s)\rangle$ . That is, it is invariant under  $|\psi(s)\rangle \rightarrow e^{i\varphi(s)}|\psi(s)\rangle$ . This is easily seen from the definition - the phase of the ket is cancelled out by that of the bra in the  $N$ -fold product.

Taking the limit  $N \rightarrow \infty$ , we have the formula:

$$\theta_B = \int_0^T ds \, i \langle \psi(s) | \frac{d}{ds} | \psi(s) \rangle \quad (3.17)$$

there is however the caveat when we evaluate the Berry phase in this way. We have to add the assumption that  $|\psi(0)\rangle = |\psi(T)\rangle$  without the phase factor.

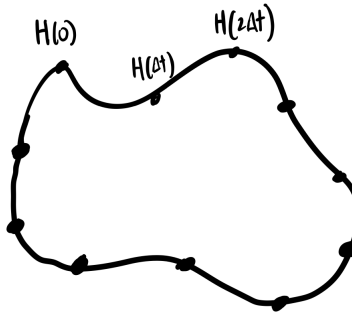
### 3.4 Berry phase and adiabatic evolution

The Berry phase shows up in two places; in adiabatic processes/cycles, and in path integrals. Today, we talk about the former.

First, a reminder of the adiabatic theorem. Let  $H(t)$  be a time-dependent Hamiltonian with  $0 \leq t \leq T$ . Suppose  $H(t)$  has a unique ground state  $|\psi(t)\rangle$  with energy  $E(t)$  and gap  $\Delta(t)$ . Suppose  $H(t)$  varies on a timescale  $\tau \gg \frac{1}{\min_t \Delta(t)}$ . Then:

$$|\psi(0)\rangle \xrightarrow{\text{evolve under } H(t)} (\text{phase}) |\psi(t)\rangle. \quad (3.18)$$

Now, consider a closed path  $H(T) = H(0)$ , which we may consider an “adiabatic cycle”.



Then:

$$|\psi(0)\rangle \xrightarrow{\text{evolve}} (\text{phase}) |\psi(0)\rangle \quad (3.19)$$

Let's compute this phase factor! It is given by:

$$(\text{phase}) = \langle \psi(0) | \mathcal{T} \exp(-i \int_0^T dt H(t)) | \psi(0) \rangle \quad (3.20)$$

with  $\mathcal{T}$  denoting time ordering. We compute the phase factor by discretizing the time-dependent Hamiltonian to  $H(0) = H_0, H(\Delta t) = H_1, H(2\Delta t) = H_2, \dots$  with  $N\Delta t = T$  and associated instantaneous ground states  $|\psi(0)\rangle = |\psi_0\rangle, |\psi(\Delta t)\rangle = |\psi_1\rangle, \dots$ . Then the expression for the phase factor becomes:

$$(\text{phase}) = \lim_{N \rightarrow \infty} \langle \psi(0) | e^{-i\Delta t H_{N-1}} e^{-i\Delta t H_{N-2}} \dots e^{-i\Delta t H_1} e^{-i\Delta t H_0} | \psi(0) \rangle \quad (3.21)$$

According to the adiabatic theorem, we know that:

$$e^{-i\Delta t H_0} |\psi_0\rangle = |\psi_1\rangle \langle \psi_1 | e^{-i\Delta t H_0} | \psi_0 \rangle \quad (3.22)$$

as the adiabatic evolution takes  $|\psi_0\rangle \rightarrow |\psi_1\rangle$  in the first time interval. We can thus insert projectors about each time step:

$$(\text{phase}) = \lim_{N \rightarrow \infty} \langle \psi(0) | e^{-i\Delta t H_{N-1}} | \psi_{N-1} \rangle \langle \psi_{N-1} | e^{-i\Delta t H_{N-2}} | \psi_{N-2} \rangle \langle \psi_{N-2} | \dots | \psi_2 \rangle \langle \psi_2 | e^{-i\Delta t H_1} | \psi_1 \rangle \langle \psi_1 | e^{-i\Delta t H_0} | \psi(0) \rangle \quad (3.23)$$

Each of the expectation values only gives a phase factor of the energy, and so:

$$(\text{phase}) = \lim_{N \rightarrow \infty} \exp(-i\Delta t \sum_{k=0}^{N-1} E_k) \langle \psi_0 | \psi_{N-1} \rangle \dots \langle \psi_2 | \psi_1 \rangle \langle \psi_1 | \psi_0 \rangle \quad (3.24)$$

and so:

$$(\text{phase}) = \exp(-i \int_0^T E(t) dt) e^{i\theta_B} \quad (3.25)$$

i.e. we have a path-dependent dynamical phase part and a Berry phase part.

We'll stop here for now, and next time we will discuss the Berry phase associated with anyons.

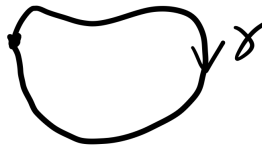
## 4 Abelian anyons I

Today we discuss anyons and derive them from first principles. We focus on the 2-D case in our discussion.

### 4.1 Single-particle Berry Phase

Let  $H$  be a 2-D gapped Hamiltonian with short-ranged interactions (sum of local terms). Suppose  $H$  has a particle-like excitation (the rough idea is that there is a state in  $\mathcal{H}$  that looks like the ground state everywhere, except for a localized region in space). In general, these excitations/particles can be in different locations in space<sup>3</sup>  $\mathbf{r}$ . For each different position, we will have a distinct many-body state, which we can label as  $|\mathbf{r}\rangle$ . We can then consider the Berry phase  $\theta_B(\gamma)$  associated with a closed path  $\gamma$ :

$$\theta_B(\gamma) = \int_0^T \langle \mathbf{r}(t) | i \frac{d}{dt} | \mathbf{r}(t) \rangle dt \quad (4.1)$$



<sup>3</sup>Physically, we could imagine these  $\mathbf{r}$  as minima of some trapping potential

We can define the Berry connection, which is a vector:

$$\mathcal{A}(\mathbf{r}) = \langle \mathbf{r} | i \nabla_{\mathbf{r}} | \mathbf{r} \rangle = \begin{pmatrix} \langle \mathbf{r} | i \frac{\partial}{\partial x} | \mathbf{r} \rangle \\ \langle \mathbf{r} | i \frac{\partial}{\partial y} | \mathbf{r} \rangle \end{pmatrix} \quad (4.2)$$

Using the chain rule, we are able to write the berry phase as an integral over the Berry connection:

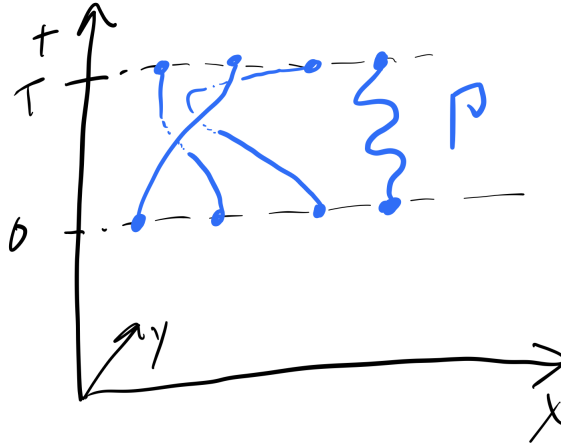
$$\theta_B(\gamma) = \int_{\gamma} \mathcal{A}(\mathbf{r}) \cdot d\mathbf{r} \quad (4.3)$$

a comment; this looks a lot like a vector potential (and indeed this choice of notation is not a coincidence); the effect that  $\mathcal{A}$  has on the physics is the same as if the particle was coupled to a background vector potential/magnetic field  $\mathbf{A}$ . This has very little to do with anyons - but when we go to multiple particles, we see the physics of anyons start to emerge.

## 4.2 Multi-particle Berry Phase and the Locality constraint

The extra part that appears when we look at the multi-particle Berry phase is exchange statistics (and this is how we will “see” anyons emerge)! Consider a state with  $n$  identical excitations/particles, which we can parameterize by  $|\{\mathbf{r}_1, \dots, \mathbf{r}_n\}\rangle$ . Since the particles are identical, we need not specify the order, only the positions.

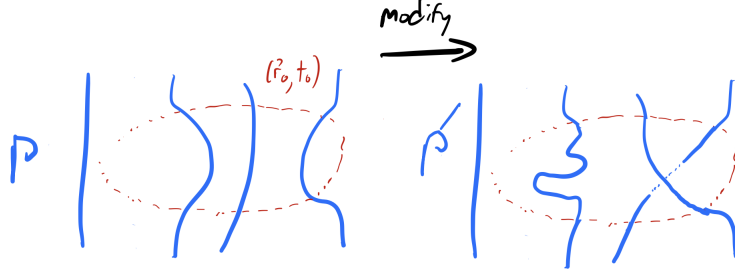
Now, let us consider the Berry phase associated with an  $n$ -particle closed path  $\Gamma$ . A picture of this path directly is a bit tricky (e.g. for 2 particles we have a 4-dimensional configuration space, which is not even Euclidean due to the lack of ordering). But we can draw it as particle worldlines through time, e.g. for four particles:



The Berry phase is then:

$$\theta_B(\Gamma) = \int_0^T \langle \{\mathbf{r}_1(t), \dots, \mathbf{r}_n(t)\} | i \frac{d}{dt} | \{\mathbf{r}_1(t), \dots, \mathbf{r}_n(t)\} \rangle dt \quad (4.4)$$

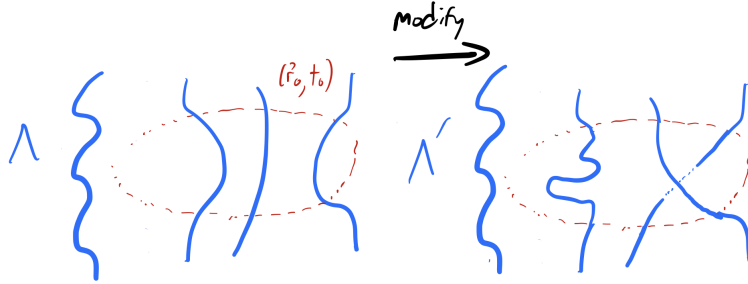
The question to understand is then; what does this look like? What are general constraints on  $\theta_B$ ? The answer is that  $\theta_B$  has to be “local”. More precisely, imagine modifying a multi-particle path  $\Gamma$  near  $(\mathbf{r}_0, t_0)$ :



Then,  $\theta_B(\Gamma') - \theta_B(\Gamma)$  depends only on what  $\Gamma, \Gamma'$  look like near  $(\mathbf{r}_0, t_0)$ . In other words:

$$\theta_B(\Gamma') - \theta_B(\Gamma) = \theta_B(\Lambda') - \theta_B(\Lambda) \quad (4.5)$$

if  $\Lambda, \Lambda'$  looks like  $\Gamma, \Gamma'$  near  $(\mathbf{r}_0, t_0)$  and differ by the same local move:



The claim is local change near  $(\mathbf{r}_0, t_0)$  is insensitive to faraway modifications. Why does Eq. (4.5) hold? It is because the difference in Berry phase  $\theta_B(\Gamma') - \theta_B(\Gamma)$  can be measured by a local operator acting near  $(\mathbf{r}_0, t_0)$  (Physically, we can imagine an interference or adiabatic experiment there). The equation then follows, assuming:

1.  $|\Psi\rangle = |\{\mathbf{r}_1, \dots, \mathbf{r}_n\}\rangle$  has short-ranged correlations, i.e.:

$$\langle A_{\mathbf{r}} A_{\mathbf{r}'} \rangle_{\Psi} = \langle A_{\mathbf{r}} \rangle_{\Psi} \langle A_{\mathbf{r}'} \rangle_{\Psi} + \mathcal{O}(e^{-\frac{|\mathbf{r}-\mathbf{r}'|}{\xi}}) \quad (4.6)$$

for  $A_{\mathbf{r}}, A_{\mathbf{r}'}$  local operators supported near  $\mathbf{r}, \mathbf{r}'$ . This is where the gapped assumption comes in; the ground state of a gapped Hamiltonian has short-ranged correlations.

2. Particles can be moved by local operators. In other words:

$$|\{\mathbf{r}'_1, \mathbf{r}_2, \dots, \mathbf{r}_n\}\rangle = M |\{\mathbf{r}_1, \dots, \mathbf{r}_n\}\rangle \quad (4.7)$$

where  $M$  is an operator supported near  $\mathbf{r}_1, \mathbf{r}'_1$ .



These two conditions together imply the locality constraint on the Berry phase.

### 4.3 Possible Forms of the Berry Phase & Topological Classes

The next question is then - what is the most general Berry phase  $\theta_B(\Gamma)$  that satisfies the locality constraint? One solution, and the one you probably would have guessed, is:

$$\theta_B(\Gamma) = \sum_i \int_{\Gamma} \mathcal{A}(\mathbf{r}_i) \cdot d\mathbf{r}_i. \quad (4.8)$$

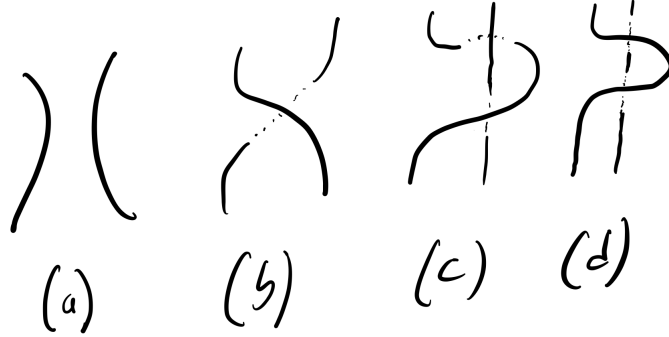
This is manifestly local. We could get a little more general:

$$\theta_B(\Gamma) = \sum_i \int_{\Gamma} \left( \mathcal{A}(\mathbf{r}_i) + \sum_j \mathcal{B}(\mathbf{r}_i, \mathbf{r}_j) + \sum_{jk} \mathcal{C}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) + \dots \right) \cdot d\mathbf{r}_i \quad (4.9)$$

where  $\mathcal{A}$  is the single-particle Berry connection and  $\mathcal{B}, \mathcal{C}$  are the two, three (and so on) particle terms so long as the multi-particle terms are short-ranged, i.e. only they are nonzero where  $\mathbf{r}_j$  is close to  $\mathbf{r}_i$  and so on.

Is this the only possible solution consistent with Eq. (4.5)? No! Indeed the first proposed solution is the form of the Berry phase consistent with bosons, but there are other solutions corresponding to fermions and anyons. What does the first solution miss? Indeed it is possible to have topological terms that look highly non-local, but such that the Berry phase still has the locality constraint.

In this line, we say that two (non-intersecting)  $n$ -particle paths with the same endpoints are topologically equivalent if they can be continuously deformed into one another without bringing particles near each other (worldlines cannot pass through one another). For example, consider:



$(a) \sim (d)$ , but  $(a) \not\sim (b) \not\sim (c)$ . This defines an equivalence relation on (non-intersecting)  $n$ -particle paths, which splits the set of  $n$ -particle paths into equivalence (topological) classes.

Coming back to the Berry phase, the claim is that the most general  $\theta_B$  that satisfies the locality constraint Eq. (4.5) can be written as:

$$\boxed{\theta_B(\Gamma) = \theta_{\text{short-range}}(\Gamma) + \theta_{\text{top}}(\Gamma)} \quad (4.10)$$

where  $\theta_{\text{top}}(\Gamma)$  only depends on the topological class of  $\Gamma$  and the short-range piece is given by Eq. (4.9)

$$\theta_{\text{short-range}}(\Gamma) = \sum_i \int_{\Gamma} \left( \mathcal{A}(\mathbf{r}_i) + \sum_j \mathcal{B}(\mathbf{r}_i, \mathbf{r}_j) + \sum_{jk} \mathcal{C}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) + \dots \right) \cdot d\mathbf{r}_i. \quad (4.11)$$

Eq. (4.10) is the key result. From here, we will classify the different possible topological terms we can have. We will find in 3D that we only get two possible classes and in 2D that we get many more.

Let us argue for Eq. (4.10) in the special case of  $\Gamma$  being a topologically trivial path, i.e. where we only have the first term. For the sake of drawing, let's look at a 2 particle path.

$$\theta_B(\Gamma) = \theta_B(\gamma_1 \& \gamma_2) = ? \quad (4.12)$$

$$\theta_B(\Gamma) = \theta_B(\text{diagram}) = ?$$

where  $\gamma_1, \gamma_2$  are the single particle paths. We can then define:

$$\Delta(\Gamma) = \theta_B(\gamma_1 \& \gamma_2) - \theta_B(\gamma_1) - \theta_B(\gamma_2) \quad (4.13)$$

Now, using Eq. (4.5) we can use that  $\Delta(\Gamma)$  is topologically invariant, i.e. it is invariant to local deformations of  $\gamma_1$  far from  $\gamma_2$  and vice versa.

$$\begin{aligned} & \theta_B(\text{diagram}) - \theta_B(\text{diagram}) - \theta_B(\text{diagram}) \\ &= \\ & \theta_B(\text{diagram}) - \theta_B(\text{diagram}) - \theta_B(\text{diagram}) \end{aligned}$$

Therefore we can deform  $\Gamma$  to the trivial path, for which  $\Delta$  is easily seen to vanish.

$$\Delta(\Gamma) = \Delta(\Gamma_{\text{trivial}}) = 0 \quad (4.14)$$

$$\Delta(\Gamma) = \theta_B(\text{diagram}) - \theta_B(\text{diagram}) - \theta_B(\text{diagram})$$

And thus since  $\Delta(\Gamma) = 0$ , we find:

$$\theta_B(\Gamma) = \theta_B(\gamma_1) + \theta_B(\gamma_2) = \int_{\gamma_1} \mathcal{A}(\mathbf{r}_1) \cdot d\mathbf{r}_1 + \int_{\gamma_2} \mathcal{A}(\mathbf{r}_2) \cdot d\mathbf{r}_2 = \theta_{\text{short-range}}(\Gamma). \quad (4.15)$$

## 5 Abelian anyons II

Consider a gapped  $d$ -dimensional Hamiltonian  $H$  with short-ranged interaction. Supports  $H$  supports a particle-like excitation. Last class, we looked at the single and multi-particle Berry phase for closed paths of such particle-like excitations. We argued that the most general form for the  $n$ -particle Berry phase took the form:

$$\theta_B(\Gamma) = \theta_{s-r}(\Gamma) + \theta_{\text{top}}(\Gamma) \quad (5.1)$$

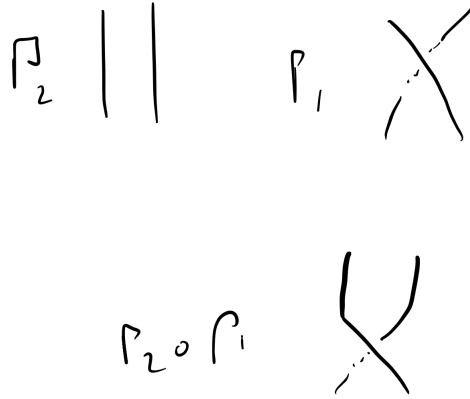
where the short-ranged piece looks like:

$$\theta_{s-r}(\Gamma) = \sum_{i=1}^n \int_{\Gamma} \left( \mathcal{A}(\mathbf{r}_i) + \sum_j \mathcal{B}(\mathbf{r}_i, \mathbf{r}_j) + \sum_{jk} \mathcal{C}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) \right) d\mathbf{r}_i \quad (5.2)$$

where the multi-particle terms are short-ranged. The topological term we discussed only depends on the topological class of  $\Gamma$  (c.f. the short-range part is non-universal and depends on the microscopic details of the path). This topological term will be what informs the exchange statistics. We now ask - what are the possible  $\theta_{\text{top}}$  terms that can appear in Eq. (5.1)? An important constraint is that it must be multiplicative under composition:

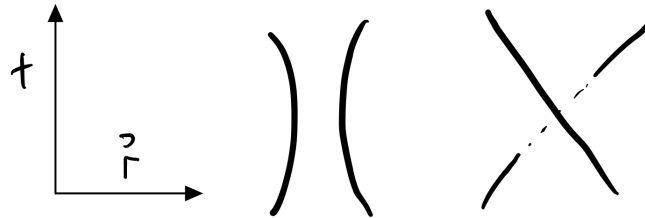
$$e^{i\theta_{\text{top}}(\Gamma_2 \circ \Gamma_1)} = e^{i\theta_{\text{top}}(\Gamma_2)} e^{i\theta_{\text{top}}(\Gamma_1)}. \quad (5.3)$$

This is because  $e^{i\theta_B}$  and  $e^{i\theta_{s-r}}$  both obey this property. Thus, we look for topological terms that obey this property. Formally, this amounts to finding 1-D representations of the braid group.



## 5.1 Topological Berry Phase in 3D

For simplicity, we focus on 2-particle paths. There are 2 such paths in 3D; one where the particles do not exchange, and then one without (picture below slightly misleading because it's hard to draw in 3+1D). We might ask; what if we do 2 exchanges? In 2D we cannot unwrap this, but in 3D a double exchange can be continuously deformed to no exchanges (by using the third spatial dimension).



There are then two possible values for the topological Berry phase:

$$e^{i\theta_{\text{top}}(\text{no exchange})} = a, \quad e^{i\theta_{\text{top}}(\text{exchange})} = b \quad (5.4)$$

By Eq. (5.3):

$$e^{i\theta_{\text{top}}(\text{double exchange})} = b^2 \quad (5.5)$$

but also, since a double exchange can be topologically deformed to the trivial path:

$$e^{i\theta_{\text{top}}(\text{double exchange})} = e^{i\theta_{\text{top}}(\text{no exchange})} = a \quad (5.6)$$



So then we obtain the constraint:

$$b^2 = a \quad (5.7)$$

We can write a very similar equation if we do a trivial path twice; again by Eq. (5.3)

$$e^{i\theta_{\text{top}}(\text{double no exchange})} = a^2 \quad (5.8)$$

but this is also topologically equivalent to the trivial path/a single no exchange path:

$$e^{i\theta_{\text{top}}(\text{double no exchange})} = e^{i\theta_{\text{top}}(\text{no exchange})} = a \quad (5.9)$$

and thus:

$$a^2 = a \quad (5.10)$$

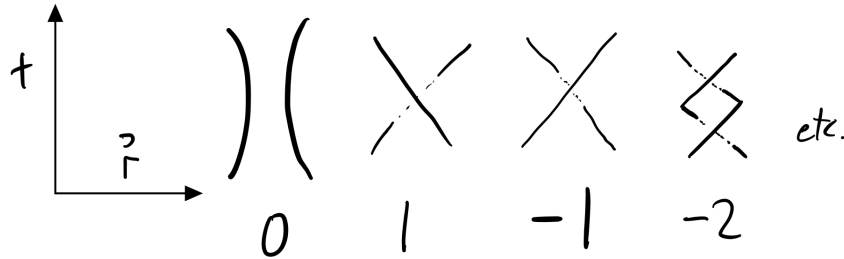
So, we have two solutions:

1.  $a = 1, b = 1$  - this is the case where there is no Berry phase for an exchange. This is the physical definition of a boson.
2.  $a = 1, b = -1$  - this is the case where an exchange gives a Berry phase of -1. This is the physical definition of a fermion.

The modern view is that fermions have nonlocal behaviour, and the anticommutation relations are a way to package this nonlocality in a local way.

## 5.2 Topological Berry Phase in 2D

Again, we focus on 2-particle paths (this fully determines the  $n$ -particle behaviour). Unlike in 3D, we get many topological classes, with different classes corresponding to different braids.



We can label braids by the number of times particles exchange in the clockwise direction. We let:

$$e^{i\theta_{\text{top}}(\text{n clockwise exchanges})} = a_n \quad (5.11)$$

Then, we know by Eq. (5.3) that:

$$a_n = a_1^n \quad (5.12)$$

Since we have no other constraints, we can let:

$$a_1 = e^{i\theta} \quad (5.13)$$

Then:

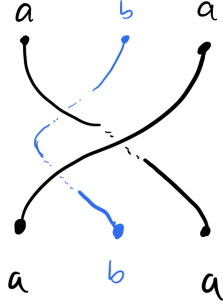
$$e^{i\theta_{\text{top}}(\text{n clockwise exchanges})} = e^{in\theta} \quad (5.14)$$

The  $\theta$  is thus known as the “statistical angle of excitations”. There are three cases:

1.  $\theta = 0$  - no phase from exchanges - bosons
2.  $\theta = \pi$  -  $-1$  phase from exchanges - fermions
3.  $\theta \neq 0, \pi$  - some other  $e^{i\theta}$  phase from excitations - anyons

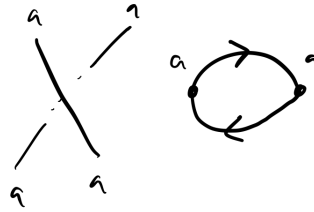
### 5.3 Generalization to multiple anyon types

Suppose  $H$  supports several types of particle excitations  $A = \{a, b, \dots\}$ . We can then consider multi-particle closed paths  $\Gamma$ , for example:

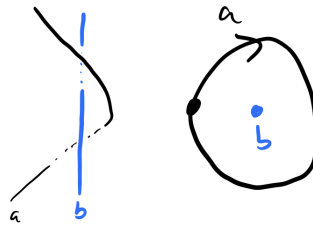


In this case, what are the possible topological terms? Without going through the entire argument,  $\theta_{\text{top}}(\Gamma)$  is characterized by multiple statistical angles (we consider the 2-particle case, which again characterizes all multi-particle paths):

1.  $\{e^{i\theta_a} : a \in A\}$ ; “exchange statistics of  $a$ ”

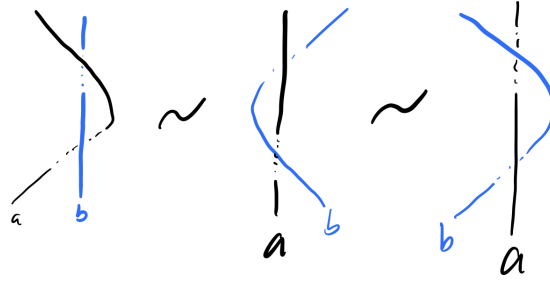


2.  $\{e^{i\theta_{ab}} : a, b \in A\}$  “mutual statistics”

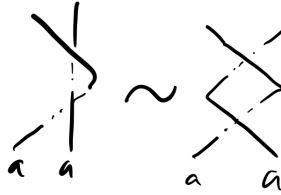


There are some important constraints on the statistical angles:

1.  $e^{i\theta_{ab}} = e^{i\theta_{ba}}$  - this is because wrapping  $a$  around  $b$  is topologically equivalent to wrapping  $b$  around  $a$ .



2.  $e^{i\theta_{aa}} = e^{2i\theta_a}$  - wrapping  $a$  around an  $a$  is topologically equivalent to a double exchange.

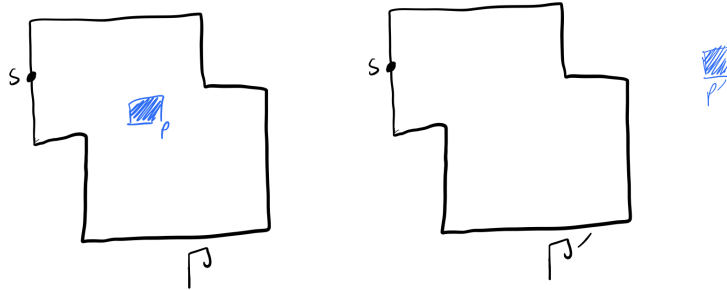


Terminology: A “non-trivial” anyon  $a$  is a particle such that  $e^{i\theta_{ab}} \neq 1$  for some  $b$  (allowing for  $b = a$ ). In other words, an anyon is a particle which braids non-trivially with another particle. Note that this means that some particles may have trivial exchange statistics, but still be anyons by virtue of non-trivial mutual statistics. This will be the case for the Toric code, where  $e, m$  individually have trivial exchange statistics but braid non-trivially.

So, let us summarize. For a general gapped system with particle-like excitation, the topological Berry phase is completely characterized by the exchange statistics and the mutual statistics. Now, to have a concrete example, let’s compute these both for the toric code.

## 5.4 Anyons in the Toric Code

Let’s compute the mutual statistics of the charge and flux excitations. We denote the mutual statistics by  $e^{i\theta_{em}}$  (we denote the charge by  $e$  and the flux by  $m$ , as in Gauge theory). To compute this, we have to somehow extract the topological part of the Berry phase. To this end, we compare 2 different paths such that the uninteresting/short ranged parts of the Berry phase cancel out.



We can write this as:

$$\theta_B(\Gamma) - \theta_B(\Gamma') = [\theta_{s-r}(\Gamma) - \theta_{s-r}(\Gamma')] + [\theta_{\text{top}}(\Gamma) - \theta_{\text{top}}(\Gamma')] \quad (5.15)$$

By construction, the  $s - r$  parts cancel because locally the two paths look exactly the same (locally, you don't see the flux you are surrounding or avoiding). Further, we notice that  $\Gamma'$  is topologically trivial because  $\Gamma'$  can be shrunk to a point (there is no plaquette inside it). Thus:

$$\theta_B(\Gamma) - \theta_B(\Gamma') = \theta_{\text{top}}(\Gamma) = \theta_{\text{em}} \quad (5.16)$$

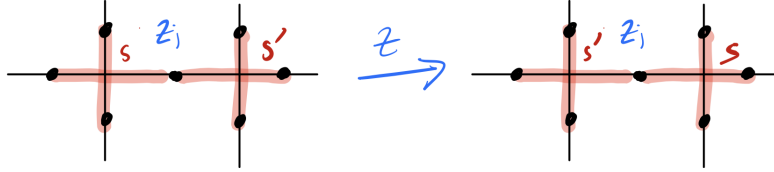
Likewise, if  $\Gamma, \Gamma'$  are adiabatic cycles, then:

$$\theta(\Gamma) - \theta(\Gamma') = [\theta_B(\Gamma) - \theta_B(\Gamma')] + [\theta_{\text{dyn}}(\Gamma) - \theta_{\text{dyn}}(\Gamma')] = \theta_{\text{em}} \quad (5.17)$$

as the dynamical phase between the two cycles cancel, as well. In fact,  $\Gamma, \Gamma'$  do not have to be adiabatic; they can be composed out of any sequence of *local* "movement operators", and we will still get:

$$\theta(\Gamma) - \theta(\Gamma') = \theta_{\text{em}} \quad (5.18)$$

and this will be how we compute  $\theta_{\text{em}}$  for the toric code. Let's try this. Notice that  $Z_j$  moves a charge:



and this is because  $Z_j$  anticommutes with  $A_s, A_{s'}$ , resulting in:

$$\begin{aligned} Z_j |a_s = 1, a_{s'} = 1\rangle &= |a_s = -1, a_{s'} = -1\rangle \\ Z_j |a_s = -1, a_{s'} = -1\rangle &= |a_s = 1, a_{s'} = 1\rangle \\ Z_j |a_s = 1, a_{s'} = -1\rangle &= |a_s = -1, a_{s'} = 1\rangle \\ Z_j |a_s = -1, a_{s'} = 1\rangle &= |a_s = 1, a_{s'} = -1\rangle \end{aligned} \quad (5.19)$$

## 6 Abelian Anyons III, The Quantum Double Model I

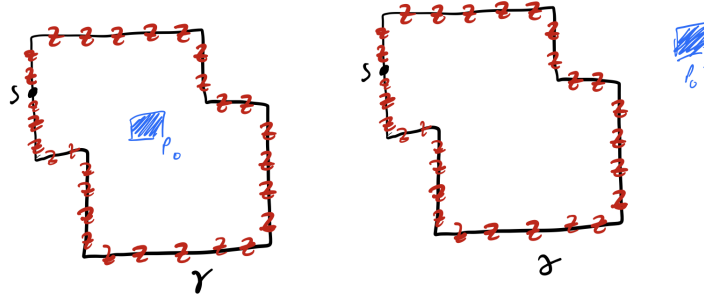
### 6.1 Mutual Statistics of Toric Code Anyons

We want to compute the mutual statistics of charge and flux statistics  $e^{i\theta_{\text{em}}}$  in the Toric code. The strategy was to compare two paths,  $\Gamma$  and  $\Gamma'$  (see figure from last lecture) such that the paths look the same, but  $\Gamma$  has a flux in the center of the path and  $\Gamma'$  has a flux outside of it. By looking at the difference of the two processes, since the two paths "look the same", the uninteresting short-range contributions to the phase will cancel, leaving us with just  $\theta_{\text{em}}$ :

$$\theta(\Gamma) - \theta(\Gamma') = \theta_{\text{em}} \quad (6.1)$$

where  $\theta(\Gamma), \theta(\Gamma')$  are the total phases accumulated by some sequence of local movement operators. Let's compute these and now see what we get. The simplest operator we can write down to move charges is  $Z_j$  - the pauli-Z operator on link  $j$ . This moves a charge from site  $s$  to site  $s'$  (and vice versa), as depicted in the figure from last lecture (the figure is slightly poorly notated; it should really depict movement of charges, as opposed to swapping the site labels. The precise action is captured in the anticommutation, or in the relations of Eq. (5.19)). We can see this from the anticommutation of  $Z_j$  with  $A_s, A_{s'}$ .

Denote the initial state by  $|p_0, s\rangle$ . Let  $\Gamma$  denote the process composed out of a sequence of  $Z$  operators, which moves the charge  $s$  along the path  $\gamma$ .



By definition:

$$\prod_{j \in \gamma} Z_j |p_0, s\rangle = e^{i\theta(\Gamma)} |p_0, s\rangle \quad (6.2)$$

To figure out what  $e^{i\theta(\Gamma)}$ , we observe the operator identity:

$$\prod_{j \in \gamma} Z_j = \prod_{p \in \text{int}(\gamma)} B_p \quad (6.3)$$

Note that this is a kind of Stokes' theorem. We can use this identity to evaluate  $e^{i\theta(\Gamma)}$  - we're in business because there is exactly one  $b_p = -1$  inside of  $\gamma$ :

$$e^{i\theta(\Gamma)} |p_0, s\rangle = \prod_{p \in \text{int}(\gamma)} B_p |p_0, s\rangle = -|p_0, s\rangle \quad (6.4)$$

Thus:

$$e^{i\theta(\Gamma)} = -1 \quad (6.5)$$

We aren't quite done yet. We should compare this to the case where the flux is not in the center. Now, consider  $\Gamma'$  which is an identical process save for the flux is on a plaquette  $p'_0$  outside of the path  $\gamma$ . Then:

$$e^{i\theta(\Gamma')} |p'_0, s\rangle = \prod_{j \in \gamma} Z_j |p'_0, s\rangle = \prod_{p \in \text{int}(\gamma)} B_p |p'_0, s\rangle = +|p'_0, s\rangle \quad (6.6)$$

where the last equality follows since all  $b_p = +1$  in the interior of the path (the only flux is outside). Thus:

$$e^{i\theta(\Gamma')} = 1 \quad (6.7)$$

Therefore looking at the difference:

$$e^{i\theta_m} = e^{i\theta(\Gamma)} e^{-i\theta(\Gamma')} = -1 \cdot 1 = -1 \quad (6.8)$$

Thus:

$$\boxed{e^{i\theta_m} = -1} \quad (6.9)$$

We compared two phases, one which was nontrivial and one which was trivial, and then the ratio gives us the interesting exchange statistic.

Why was it important that we took the difference between these two paths/compared the two processes? We chose  $Z$  as our movement operator, but someone else could have very well chosen  $e^{i\phi}Z$  for any phase; then that random phase would contribute to  $e^{i\theta(\Gamma)}$  and  $e^{i\theta(\Gamma')}$ . But crucially, it contributes in the exact same way to both of these, and hence does not contribute to the difference.

A couple remarks:

1. The result does not depend on the microscopic details of the path  $\gamma$ . We expected this from what we know about the Berry phase, but also saw this clearly arise from the calculation itself. Moreover,  $\theta_{em}$  does not depend on the choice of movement operator; we won't prove this in full generality, but we did remark how movement operators differing by a phase give the same result. In fact even if the movement operators look more starkly different, we find the same result.
2. Mutual statistics are symmetric, and indeed we could swap the roles of the charges/fluxes and  $Z/X$  and we would get the same results.
3. According to the definition from last class,  $e, m$  are indeed non-trivial anyons since  $e^{i\theta_{em}} = -1$  (non-trivial mutual statistics, with each other).
4. We could also ask what the mutual statistics of taking an  $e$  particle around itself (or  $m$  around itself). Then, it's trivial to show that:

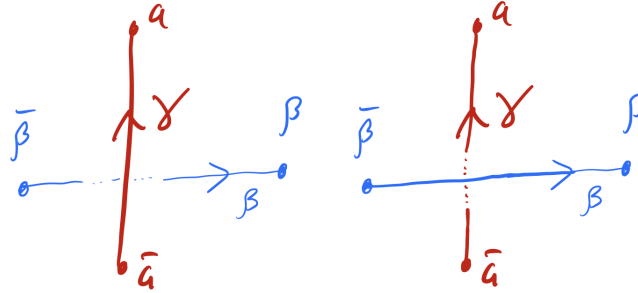
$$e^{i\theta_{ee}} = e^{i\theta_{mm}} = 1 \quad (6.10)$$

5. Note that an even number of charges/fluxes look like having no charges/fluxes at all. Actually, we have four types of "sectors" of excitations  $\{1, e, m, \epsilon\}$  with 1 being even charges/fluxes,  $e$  being odd charges even fluxes,  $m$  being even charges odd fluxes, and  $\epsilon$  being odd charges and fluxes (you will study this one on the homework).

## 6.2 Connection between Abelian Anyons and String Operators

We sketch the connection between mutual statistics and the algebra of string operators. In the previous section we gave a physically transparent way of computing mutual statistics, now we give the shortcut. Let  $a, b$  be Abelian anyons, and let  $W_a, W_b$  be associated string operators.  $W_a$  creates an excitation  $a$  at one end and its antiparticle  $\bar{a}$  at its other end (in the toric code, the particles and antiparticles coincide). Consider paths  $\gamma, \beta$ , and then compare:

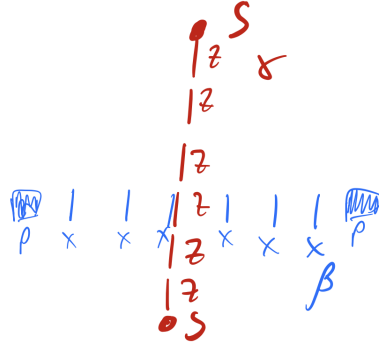
$$W_a(\gamma)W_b(\beta)|\Omega\rangle, \quad W_b(\beta)W_a(\gamma)|\Omega\rangle \quad (6.11)$$



The claim is then:

$$W_a(\gamma)W_b(\beta)|\Omega\rangle = e^{i\theta_{ab}}W_b(\beta)W_a(\gamma)|\Omega\rangle \quad (6.12)$$

For example, let's use  $a = e, b = m$  on the toric code. The  $b$ -string is a string of Xs that creates two fluxes on the end, and the  $a$ -string is a string of Zs that creates two charges on the end:



These two operators anticommute on the operator level:

$$W_e(\gamma)W_m(\beta) = -W_m(\beta)W_e(\gamma) \quad (6.13)$$

and hence also anticommute when acted upon the ground state. Thus:

$$e^{i\theta_{em}} = -1 \quad (6.14)$$

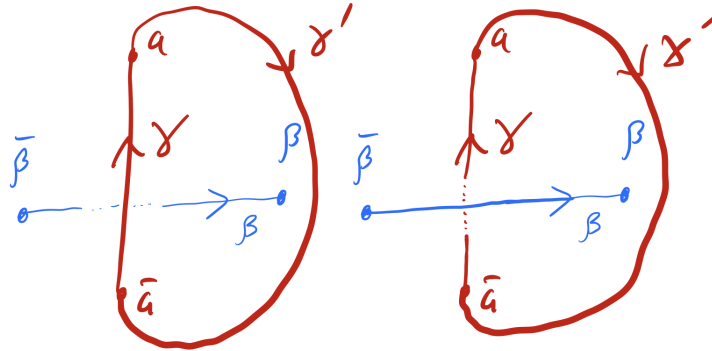
Let us derive the general relation of Eq. (6.12). On general grounds, we know that:

$$W_a(\gamma)W_b(\beta)|\Omega\rangle = e^{i\theta}W_b(\beta)W_a(\gamma)|\Omega\rangle \quad (6.15)$$

for *some* phase  $\theta$ . This is because both of the states correspond to the creation of anyons on both sides - hence both states have the same 4 anyons ( $a, \bar{a}, b, \bar{b}$ ) at the same locations. Hence they must describe the same physical state. To compute the phase, we multiply both the right and left hand side with a further string operator,  $W_a(\gamma')$ :

$$W_a(\gamma')W_a(\gamma)W_b(\beta)|\Omega\rangle = e^{i\theta}W_a(\gamma')W_b(\beta)W_a(\gamma)|\Omega\rangle \quad (6.16)$$

Graphically, the LHS/RHS look like:



The first thing we see at an algebraic level is that  $W_a(\gamma')$  commutes with  $W_b(\beta)$ , as they act on non-overlapping places:

$$W_a(\gamma')W_a(\gamma)W_b(\beta)|\Omega\rangle = e^{i\theta}W_b(\beta)W_a(\gamma')W_a(\gamma)|\Omega\rangle \quad (6.17)$$

Now let's think about the two processes we have here. On the LHS, we have that  $a$  braids around  $b$  (we first create  $b$  and then do the big loop). On the RHS, we first create  $a, \bar{a}$  and do the big loop, and then create  $b$  -  $a$  follows the exact same path, but does *not* braid around  $b$ . Thus, we would conclude that the phase difference  $\theta$  must be the topological Berry phase  $\theta_{ab}$ , as everything else cancels.

To summarize:

- (a) Abelian anyon excitations  $\iff$  non-commuting flexible string operators (with the algebra of the SOs encoding the mutual statistics. On the HW, you will also see that the algebra of the SOs imply exchange statistics, but this will require looking at three string operators)
- (b) Non commuting flexible string operators  $\implies$  Robust GSD (on torus)
- (c) Robust GSD  $\implies$  robust quantum memory (can encode the qubit in the GSD).

Thus any system with Abelian anyons has all these properties... it is harder to formulate the reverse statement, but easy counterexamples don't come to mind.

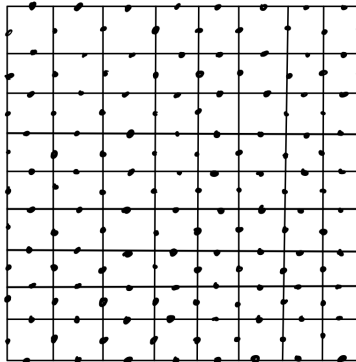
### 6.3 Introduction to the Quantum Double Model

The above concludes our discussion of abelian anyons and the toric code. We thus move onto the discussion of non-abelian anyons and the quantum double model. The reference is quant-ph/9707021.

We can think of the quantum double model as the generalization of the toric code. It's not actually a single model, but a class of models, of which we specify a specific model by choice of a finite group  $G$ . The toric code corresponds to choosing  $G = \mathbb{Z}_2$  (the simplest nontrivial choice). Looking a bit ahead, when we choose  $G$  to be Abelian, the model will host Abelian anyons, but when we choose  $G$  to be a non-Abelian group, the quantum double model will host non-Abelian anyon excitations.

What are non-Abelian anyons? In short, Abelian if we trap  $n$  particles at  $n$  sites have a unique physical state, vs. for non-Abelian anyons the same process results in a ground state with degeneracy. We will see that the braiding processes for these anyons will not be Abelian.

We have a local Hilbert space on each edge site of a lattice, with each edge Hilbert space having dimension  $|G|$ .



It is convenient to define two orthonormal basis for each edge,  $\{\uparrow |g\rangle\}$  and  $\{\downarrow |g\rangle\}$  where  $g \in G$ . The  $\uparrow / \downarrow$  correspond to the orientation of the link. The relationship between these two bases are very simple:

$$\uparrow |g\rangle = \downarrow |g^{-1}\rangle \quad (6.18)$$

For example:



$$\begin{aligned}
 & \left| \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle = \left| \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle \\
 & = \dots
 \end{aligned}$$

The diagrams are rectangular cells with internal lines and labels. Diagram 1 has labels  $h$  (top left),  $k$  (top right),  $l$  (right),  $p$  (bottom right),  $n$  (bottom left), and  $m$  (center). Diagram 2 has labels  $h$  (top left),  $k^{-1}$  (top right),  $l$  (right),  $p$  (bottom right),  $n^{-1}$  (bottom left), and  $m^{-1}$  (center). Arrows indicate the direction of flow for each labeled line.

We're out of time for today, but it's worth noting that quantum double model with group  $G$  is related to  $G$ -gauge theory; just written in a more concrete way.