

# PHYS 411 (Entanglement in Many-Body Systems) Notes

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## Introduction:

This is a set of lecture notes taken from UChicago's PHYS 411 (Entanglement in Many-Body Systems), taught by Michael Levin. Topics covered include...

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# 1 Toric Code I

## 1.1 Course Overview + Logistics

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Evaluation: Once every  $\sim 2$  weeks, 100%.

Textbook: None; papers/references will be provided.

This class will cover topics at the interface of quantum many-body/condensed matter theory and quantum information. This has been a dynamic interface for a couple decades now, with the two fields inspiring each other. The topics chosen are both interesting from a physics point of view, but also deeply important in QI. A rough schedule is as follows:

- (I) **Anyons and topological quantum computation.** Anyons exist in 2-d quantum systems that have exchange statistics that are not bosonic or fermionic; the exchange phase can be anything (hence the name). Anyons first emerge in discussion of the fractional quantum hall effect, but in the last 20 years, people (lead by Alexei Kitaev) have found interesting connections between anyons and quantum computing.
- (II) **Symmetry-protected topological phases.** This is another important topic in condensed matter, but is deeply connected to concepts in quantum information, e.g. finite depth-circuits (indeed they provide a quantum-information theoretic way to define phases of matter).
- (III) **Entanglement entropy in many-body systems.** EE gives a lot of insight into the physics of MB systems. This also has practical applications, leading to numerical algorithms using...
- (IV) **Matrix product states.**

## 1.2 Defining the Toric Code Model

References: Kitaev's lecture notes arXiv:0904.2771, original paper arXiv:quant-ph/9707021.

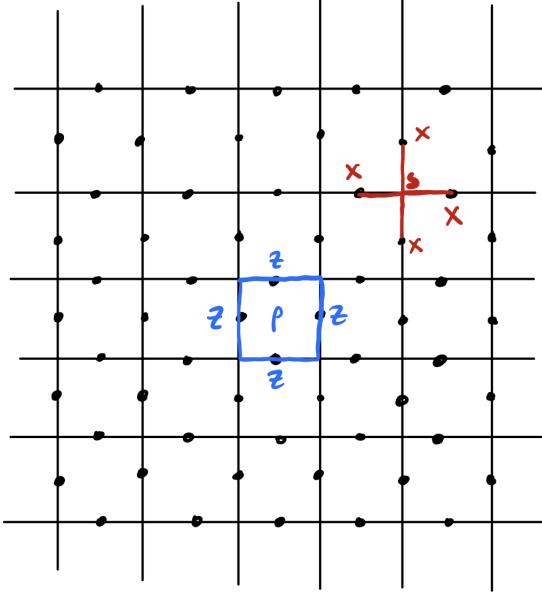
The toric code is an exactly solvable spin model (it can also be thought of a quantum error correcting code, but we introduce it as a spin model to start). It has:

1. Anyon excitations
2. Topological ground state degeneracy

We consider this model on different kinds of lattices and geometries, but for now we consider the square lattice, and place a spin-1/2 degree of freedom on each of the edges of the lattice (we don't specify the boundary conditions yet). The Hilbert space has dimension  $2^N$  with  $N$  the total number of edges/spins. The Hamiltonian takes the following form:

$$H = - \sum_s A_s - \sum_p B_p \quad (1.1)$$

the  $s$  are vertices on the lattice and  $p$  are plaquettes.



The  $A_s$  term is a product of Pauli-X operators on stars about vertices  $s$ :

$$A_s = \prod_{j \in \text{star}(s)} X_j \quad (1.2)$$

And the  $B_p$  term is a product of Pauli-Z operators on the boundaries of plaquettes:

$$B_p = \prod_{j \in \partial p} Z_j \quad (1.3)$$

We adopt the QI notation:

$$X = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1.4)$$

$$Y = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (1.5)$$

$$Z = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.6)$$

### 1.3 Solving the toric code model

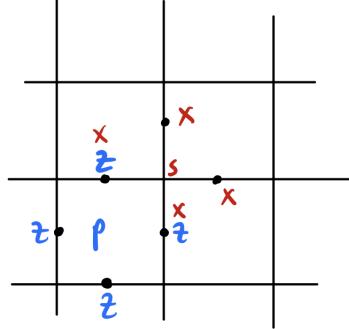
Notice that all of the  $A_s$  and  $B_p$  terms commute with one another. For example, it's trivial to see that:

$$[A_s, A_{s'}] = [B_p, B_{p'}] = 0 \quad (1.7)$$

because Xs are mutually commuting and Zs are mutually commuting. Slightly less obvious is that the star terms commute with the plaquettes:

$$[A_s, B_p] = 0 \quad (1.8)$$

We might be worried if this holds because  $X$  and  $Z$  anticommute. But in fact the above holds; if the star and plaquette are faraway then there are no overlapping  $X, Z$ s so they commute. In the case where the star/plaquette overlap, we have that two  $X, Z$ s overlap (see picture below) so the anticommutation cancels to a commutation.



Thus, we are able to simultaneously diagonalize  $\{A_s, B_p\}$ . Denote the eigenstates by  $|\{a_s, b_p\}\rangle$  where the  $a_s, b_p$  are the eigenvalues. If we have residual degeneracy, we may require additional quantum numbers to specify the state, but let us not worry about this quite yet. Note that because:

$$A_s^2 = B_p^2 = \mathbb{I} \quad (1.9)$$

this implies that the eigenvalues are:

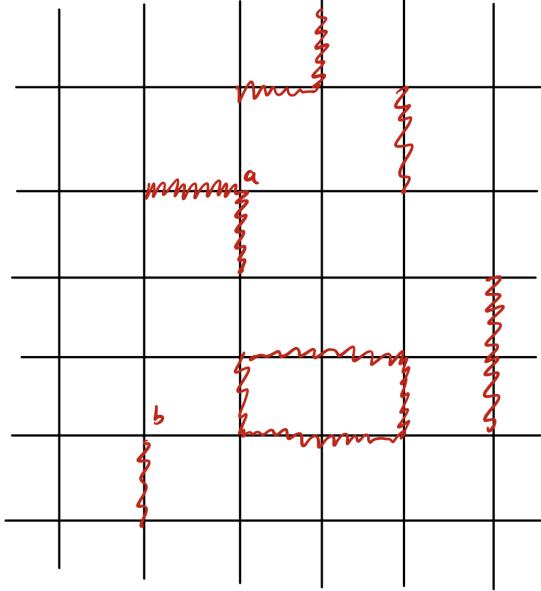
$$a_s, b_p = \pm 1. \quad (1.10)$$

And the  $|seta_s, b_p\rangle$  are also the energy eigenstates, with energy:

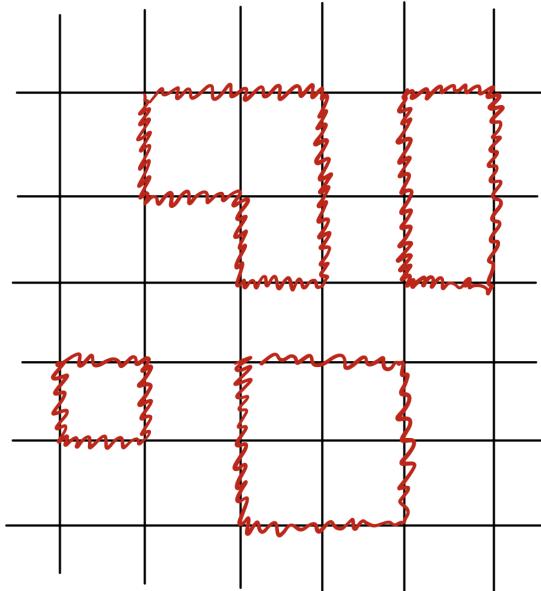
$$E = - \sum_s a_s - \sum_p b_p. \quad (1.11)$$

This is the sense in which the problem is exactly solvable. We have found all of the eigenstates, and can find their degeneracies by figuring out the degeneracies of  $\{a_s, b_p\}$ .

For the ground state, we set  $a_s = b_p = 1$ . We can ask the question; how many states are there that satisfy this? This is equivalent to asking what the ground state degeneracy is. The answer to this question is dependent on the geometry of the model. Let's start with the simplest, and arguably the most important case; the infinite plane. Let us work in the  $X$ -basis. In this basis, the different basis states are  $|\pm\rangle \otimes |\pm\rangle \otimes \dots$  for each link. A useful visualization will be in terms of strings on this lattice. In this string picture, we will say that  $X_j = -1$  corresponds to a string on link  $j$ . On the other hand, if  $X_j = +1$  then we will say that there is no string on the link. Thus, we can view an arbitrary  $X$  basis state as strings occupying the lattice in some configuration.



With this point of view, let's figure out what the ground states look like in terms of the  $X$ -eigenstates. We want  $a_s = 1$  for every star, thus  $\prod_{j \in \text{star}(s)} X_j = 1$ . This implies that the number of -1s in the product must be even. This means that the number of strings that touch a given site must be even. In the above picture,  $\text{star}(a)$  obeys this condition while  $\text{star}(b)$  does not. But, if  $a_s = 1$  for every single star, this implies that the strings must form closed loops (else - at the endpoints of strings we end up with  $a_1 = -1$ ). An allowed configuration is sketched below:



Note that since  $XZ = -ZX$ , a given  $B_p$  flips string occupation around a plaquette  $p$ , e.g.:

$$B_p | \square \square \square \rangle = | \square \square \square \rangle$$

But then all of the  $a_s = b_p = 1$  requires that we have an equal amplitude superposition of all closed loop states (if this was not the case, the application of some  $B_p$  would flip some loops and change the state - not an eigenstate!). Thus! There is a unique ground state:

$$|\Omega\rangle = |a_s = b_p = 1\rangle = \sum_{\text{closed loop config } C} |C\rangle. \quad (1.12)$$

We could draw an example  $|C\rangle$  pictorially as:

$$|C\rangle = | \square \square \square \rangle$$

It can be seen that  $B_p$  only permutes the different  $C$ s to each other without changing their weight, so indeed the above is the +1 eigenstate of the  $B_p$ s. More formally, if  $B_p|\psi\rangle = |\psi\rangle$  for all  $p$ , then:

$$\langle C|\psi\rangle = \langle C|B_p|\psi\rangle = \langle C'|\psi\rangle \quad (1.13)$$

where  $|C'\rangle$  is a different closed loop configuration. This implies that all of the amplitudes of the closed loop configurations have equal weight.

Note that formally there is an infinite number of closed loop configurations on an infinite plane, so there is a bit of a subtlety in normalizing the state (which requires the machinery of operator algebras, etc.) which we sweep under the rug.

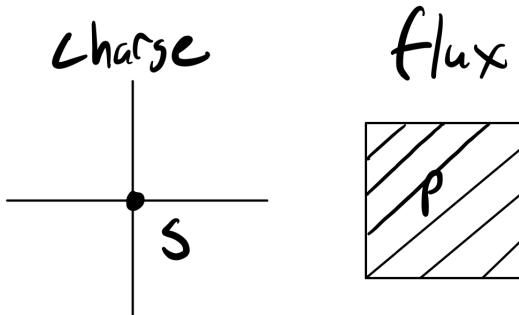
Note that the above argument gave a unique state for  $a_s = b_p = 1$ , but for any choice of  $\{a_s, b_p\}$  we can get a statement of a similar flavour. Thus, this justifies the choice of notation  $|\{a_s, b_p\}\rangle$  as for a given choice of  $a_s/b_p$  the state is unique.

Now, we have the full energy spectrum, as we have determined the degeneracy of all of the eigenspaces. We can thus determine the energy gap, which is the energy difference between the ground state and first excited state. Flipping one of the  $a_s$  or the  $b_p$  results in an energy penalty of 2, and thus the energy gap is  $\Delta = 2$ . This is important because it means that we have a “gapped Hamiltonian”. In comparison, there are “gapless” Hamiltonians for which the energy difference goes to zero in the thermodynamic limit. This distinction/property will be important when we discuss anyons - gapped Hamiltonians are the context in which they are currently understood.

## 1.4 Excitations and String Operators

There are two types of elementary excitations:

1.  $a_s = -1$  for some  $s$ ; this is a “charge”, and lives on a site.
2.  $b_p = -1$  for some  $p$ ; this is a “flux”, and lives on a plaquette.



The terminology comes from  $\mathbb{Z}_2$  gauge theory. We will see that these excitations are not conventional bosons or fermions that we may be familiar with.

If we visualize these excitations, adding a charge is like taking our superposition of all closed loops and then to each of those states adding one defect (with a string that goes off to infinity). Adding a plaquette takes the superposition of loops, and we count the number of loops that go around  $p$  (a kind of winding number) and take that to be the sign of the configuration in the superposition. Pictorially, we have defects in the first case and vortices in the second.

Now, the question becomes how can we create charge or flux excitations? In spin systems you might be familiar with (e.g. creating a magnon in a Heisenberg spin chain) you would create them via a local operator. But here we actually apply a non-local string operator to create these excitations.

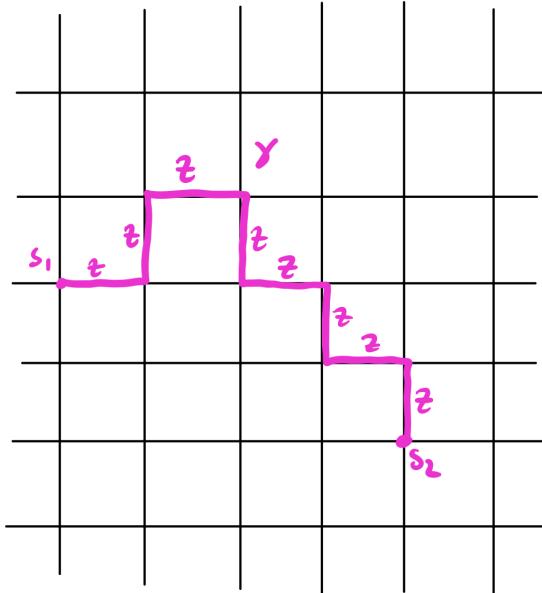
We define:

$$W^Z(\gamma) = \prod_{j \in \gamma} Z_j \quad (1.14)$$

where  $\gamma$  is some open path on the lattice. The  $W^Z(\gamma)$  is a string of  $Z$ s along  $\gamma$ . This string operator creates charge excitations at the two endpoints; notice that:

$$[W^Z(\gamma), B_p] = 0 \quad (1.15)$$

this is obvious as the  $B_p$ s consist of  $Z$ s only. Less obvious is that the  $W^Z(\gamma)$  commutes with almost all of the  $A_s$  operators - all except at the endpoints of  $\gamma$ ,  $s_1$  and  $s_2$ .



Let us argue this. At intermediate points along the path each of the points have an even number of strings so we have commutation. At the endpoints, we only have 1 link and so we have anticommutation:

$$W^Z(\gamma)A_{s_1,2} = -A_{s_1,2}W^z(\gamma) \quad (1.16)$$

What are the implications of this? Looking at the ground state  $|\Omega\rangle = |\{a_s = b_p = 1\}\rangle$ :

$$W^Z(\gamma)|\Omega\rangle = |\{a_{s_1} = a_{s_2} = -1, \text{others} = 1\}\rangle. \quad (1.17)$$

and we can see this because the string operator only flips the stars at the endpoints, and commutes with everything else. Thus, we conclude that the  $W^Z(\gamma)$  creates charges at the two endpoints of  $\gamma$ , as claimed.

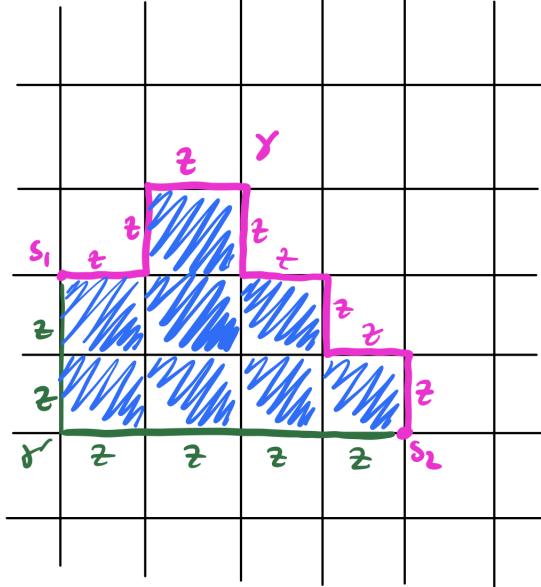
An important observation with regard to string operators; If  $\gamma'$  is a different path with the same endpoints, then the string operator  $W^z(\gamma')$  applied to the ground state yields the *exact* same state (even up to the same phase):

$$W^Z(\gamma)|\Omega\rangle = W^z(\gamma')|\Omega\rangle \quad (1.18)$$

this is the notion in which string operators are “flexible”. To see this, note that  $W^Z(\gamma') = W^Z(\gamma) \prod_{p \in \text{int}(\gamma' \cup \gamma)} B_p$ , i.e. the two operators are related via a product of plaquette operators in the interior of the two paths. Thus:

$$W^Z(\gamma')|\Omega\rangle = W^z(\gamma) \prod_{p \in \text{int}(\gamma' \cup \gamma)} B_p |\Omega\rangle = W^Z(\gamma)|\Omega\rangle \quad (1.19)$$

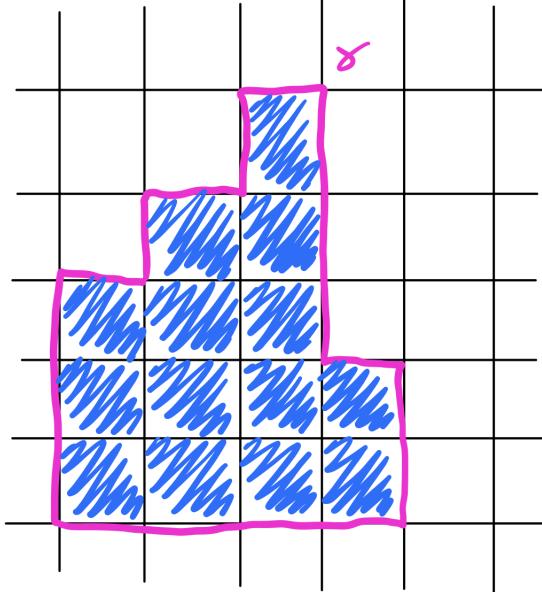
where we have used that  $|\Omega\rangle$  is the +1-eigenstate of all the  $B_p$ s.



Relatedly, if  $\gamma$  is a closed loop then:

$$W^Z(\gamma)|\Omega\rangle = |\Omega\rangle \quad (1.20)$$

which follows from the fact that a closed loop is just a product of  $B_p$ s.



These features - which currently seem pretty specific to the toric code model - are in fact quite general. Any system with anyons have string operators with these properties!

A last comment to provide some physical intuition for what the string operator is. We can view it as the physical process of first creating two charges (by applying a single  $Z$ ), then moving that charge via the application of further  $Z$ s along the path. I.e. a string operator is just creating two charges and separating them. This makes the notion that the string is flexible intuitive; we should get the same state (up to some phase) if the particles are created and end up in some separated location(s), no matter how we move them there.

## 2 Toric Code II

### 2.1 Review

A quick review of some definitions; the Toric code has Hamiltonian (defined on a 2D lattice with qubits placed on the edges):

$$H = - \sum_s A_s - \sum_p B_p \quad (2.1)$$

with  $A_s$  star operators around each lattice vertex:

$$A_s = \prod_{j \in \text{star}(s)} X_j \quad (2.2)$$

and  $B_p$  plaquette operators around each lattice plaquette

$$B_p = \prod_{j \in \partial p} Z_j \quad (2.3)$$

we constructed the ground state  $|\Omega\rangle = |a_s = b_p = 1\rangle$ . We also discussed a string operator, which creates charge excitations:

$$W^Z(\gamma) = \prod_{j \in \gamma} Z_j \quad (2.4)$$

where  $\gamma$  is a path on the lattice. It creates charges ( $a_s = -1$ ) at the endpoints of  $\gamma$ . Further, these string operators are flexible, in the sense that:

$$W^Z(\gamma)|\Omega\rangle = W^Z(\gamma')|\Omega\rangle \quad (2.5)$$

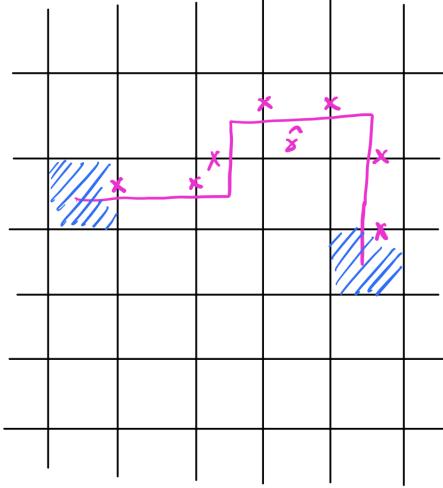
for two paths  $\gamma, \gamma'$  with the same endpoints.

## 2.2 String operator for flux excitations

There is a similar string operator for flux excitations. Just a heads up that the structure of having string operators (one for each anyon type - here for charges and fluxes/ $e$  and  $m$ ) is quite general. We define:

$$W^X(\hat{\gamma}) = \prod_{j \in \hat{\gamma}} X_j \quad (2.6)$$

where  $\hat{\gamma}$  is an open path on the dual lattice, i.e. that go through the center of plaquettes:

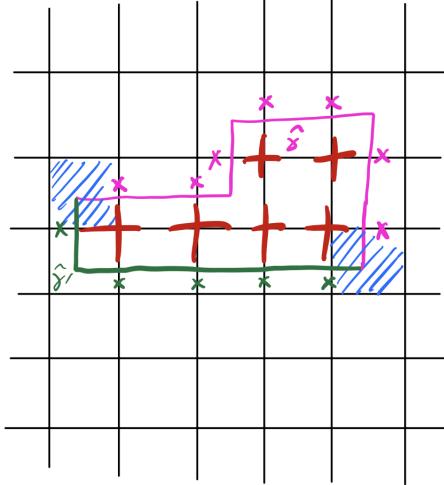


Much like the string operator for the charges:

- It can be checked that  $W^X(\hat{\gamma})$  creates fluxes  $b_p = -1$  at the two endpoints of  $\hat{\gamma}$ .
- The string operators are flexible, with:

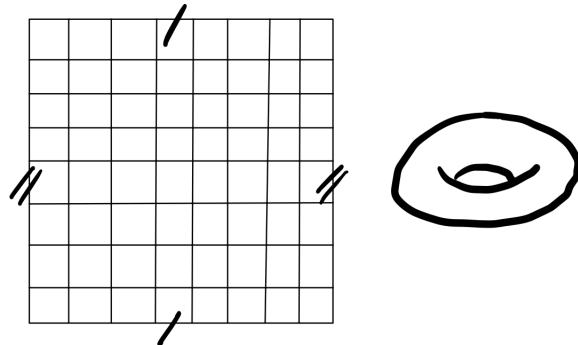
$$W^X(\hat{\gamma})|\Omega\rangle = W^X(\hat{\gamma}')|\Omega\rangle \quad (2.7)$$

for  $\hat{\gamma}, \hat{\gamma}'$  with the same endpoints. They are related by the product of star operators on the interior.



The existence of these flexible, non-commuting string operators is very fundamental to the structure of the toric code, and to anyon systems more generally. The existence of these is independent of geometries. But we will see that it will have implications when we consider the model for specific systems.

### 2.3 Ground state degeneracy on a torus



Consider the toric code, on a finite  $(L \times L)$  torus. We may have different allowable states for a given  $a_s$  and  $b_p$ , and thus may find that there are different degeneracies, compared to the infinite plane case. We can now ask what is the ground state degeneracy  $D$ ? Indeed, this question is equivalent to asking what the dimension of the eigenspace with  $a_s = b_p = 1$  is. To find this, we look at the trace of the projector onto the eigenspace<sup>1</sup>:

$$\begin{aligned} D &= \text{Tr}(\text{proj. onto } a_s = b_p = 1 \text{ subspace}) \\ &= \text{Tr}\left(\prod_s \left(\frac{\mathbb{I} + A_s}{2}\right) \prod_p \left(\frac{\mathbb{I} + B_p}{2}\right)\right) \end{aligned} \quad (2.8)$$

where the second line follows from the fact that the product of the (mutually commuting) projectors gives

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<sup>1</sup>This is a very formal way to find it, we will soon get different perspectives on this question

the projector onto the subspace. Computing this:

$$D = \frac{1}{2^{N_s}} \frac{1}{2^{N_p}} \text{Tr}(\prod_s (\mathbb{I} + A_s) \prod_p (\mathbb{I} + B_p)) \quad (2.9)$$

Now we expand out this product, and can think about the traces of the individual terms. A single  $A_s, B_p$  will be traceless (as the Paulis are traceless), and so will most products of  $A_s, B_p$ ; the only non-traceless terms will be those that simplify to the identity. If we stare at this, there are only a few combinations for which this occurs; there is the term with all identity, the term with all stars (all the Xs cancel), the term with all plaquettes (all the Zs cancel), and the term with all stars and all plaquettes.

$$D = \frac{1}{2^{N_s}} \frac{1}{2^{N_p}} \text{Tr}(\mathbb{I} + \prod_s A_s + \prod_p B_p + \prod_s \prod_p A_s B_p) = \frac{1}{2^{N_s}} \frac{1}{2^{N_p}} \text{Tr}(4\mathbb{I}) \quad (2.10)$$

Now looking at the trace of the identity:

$$\text{Tr}(\mathbb{I}) = 2^{\dim(\mathcal{H})} = 2^{N_{\text{links}}} \quad (2.11)$$

Thus:

$$D = \frac{1}{2^{L^2}} \cdot \frac{1}{2^{L^2}} \cdot 4 \cdot 2^{2L^2} = 4 \quad (2.12)$$

so:

$$\boxed{D = 4} \quad (2.13)$$

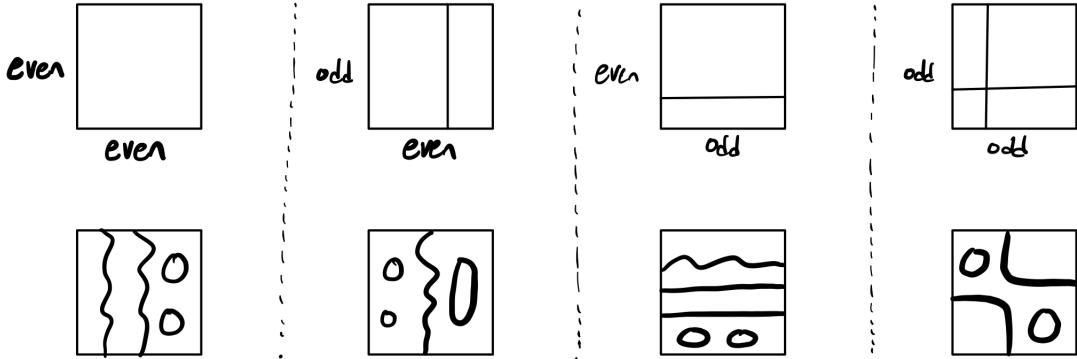
A nice feature of this argument is we can repeat it for any of the eigenspaces. In fact, every  $\{a_s, b_p\}$  eigenspace with  $\prod_s a_s = \prod_p b_p = 1$  (i.e. an even number of charges) is 4-fold degenerate.

## 2.4 Ground states in the string picture

Now, let's see if we can understand the ground state degeneracy in the string picture. To review, we work in the X-basis, and  $X_j = \pm 1$  corresponds to there being a string (plus) or no string on link  $j$ .  $a_s = 1$  requires the product of Xs on the star to be one, implying that the strings form closed loops.  $b_p = 1$  implies that there is an equal amplitude superposition of string states, as different string states are related by  $B_p$  moves. We used these two conditions on the infinite plane geometry (Where  $B_p$  moves are "ergodic") to say that there was a unique ground state, namely that with an equal weight superposition of all closed loop configurations:

$$|\Omega\rangle = \sum_{\text{closed loop config } C} |C\rangle \quad (2.14)$$

On a torus, the closed loop states can be divided into 4 classes; even/even, even/odd, odd/even, and odd/odd.



What does this mean? this means that if we draw a line going across the torus (on the dual lattice), we “cross” an even/odd number of strings. Within each sector/class, the  $B_p$  moves are ergodic. But,  $B_p$  moves cannot change the parity of the crossings. Thus the four degenerate ground states correspond to the equal weight superpositions of closed loop configurations within a given class. We can label the ground states via their winding number:

$$|\Omega_{(e/o,e/o)}\rangle = \sum_{\text{closed loop config } C \text{ with (e/o, e/o) winding}} |C\rangle \quad (2.15)$$

So, so far we have understood the ground state degeneracy from two perspectives. But neither of these tells us the deeper principle underlying the degeneracy. Let us discuss this now - it will allow us to see why the degeneracy is topologically protected.

## 2.5 Origin of the ground state degeneracy

The punchline is that the GSD comes from the existence of non-commuting string operators.

Define the charge string operators:

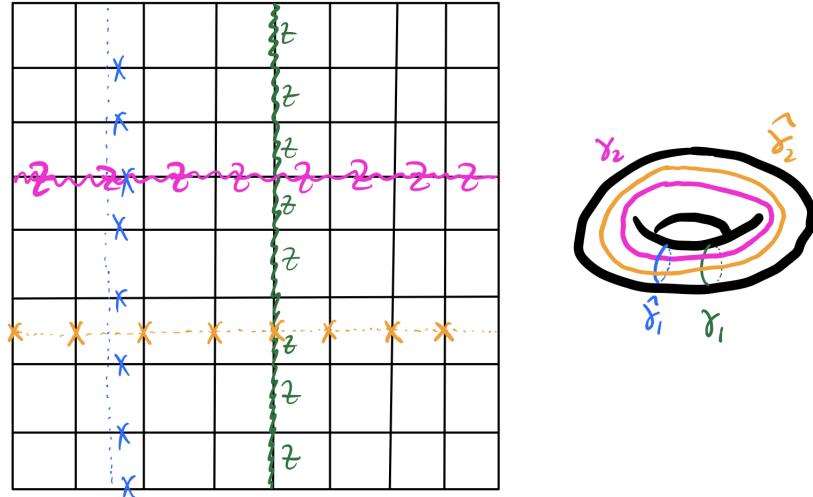
$$W_1^Z = \prod_{j \in \gamma_1} Z_j \quad (2.16)$$

$$W_2^Z = \prod_{j \in \gamma_2} Z_j \quad (2.17)$$

These string operators correspond to the creation and subsequent annihilation of a charge as the string wraps around the torus. We can also define the string operators for the fluxes;

$$W_1^X = \prod_{j \in \hat{\gamma}_1} X_j \quad (2.18)$$

$$W_2^X = \prod_{j \in \hat{\gamma}_2} X_j \quad (2.19)$$



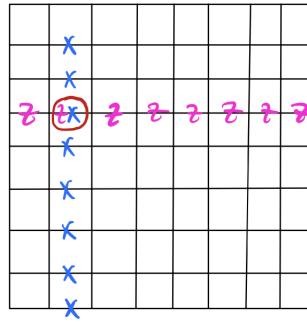
Notice that:

$$\begin{aligned} [W_i^Z, A_s] &= [W_i^Z, B_p] = 0 \\ [W_i^X, A_s] &= [W_i^X, B_p] = 0 \end{aligned} \quad (2.20)$$

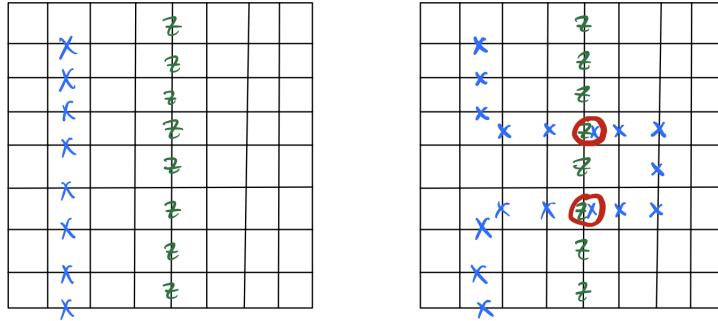
so they map ground states to ground states. They have an interesting commutation algebra.  $\{W_1^Z, W_2^Z, W_1^X, W_2^X\}$  all commute, except for two exceptions:

$$\begin{aligned} W_1^X W_2^Z &= -W_2^Z W_1^X \\ W_2^X W_1^Z &= -W_1^Z W_2^X \end{aligned} \quad (2.21)$$

this is because they anticommute in one place.



The fact that the other commute is clear; the fact that the Xs mutually commute and Zs mutually commute is immediate. For  $W_1^X, W_1^Z$ , they act on disjoint regions. Even if you were to pick representatives that overlap, they will overlap an even amount of times.



This algebra is quite interesting; we have symmetry generators that commute with the Hamiltonian, but not with each other. A simple HW exercise you will do is that the eigenstates/ground states will come in multiplets of 4 (the fact that it is exactly 4 in this case comes from the microscopic calculation).

A final comment; we said how the  $\{a_s, b_p\}$  do not uniquely specify the ground (or excited) states. But in fact the string operators provide the missing quantum numbers necessary to specify the state. Specifically, we can uniquely label the eigenstates by choosing (in addition to the  $\{a_s, b_p\}$ ) the values of the Ws, e.g.  $| \{a_s, b_p\}, w_1^x, w_2^x \rangle$  where  $w_1^x = \pm 1$  and  $w_2^x = \pm 1$ . We can thus denote the four ground states as

$|\Omega, \pm\pm\rangle$ . These are *exactly* the same what we called before the  $|\Omega_{(e/0,e/0)}\rangle$  - in fact the  $W^x$  string operator counts the number of crossing of X strings.

Next time, we will discuss how the GSD is robust to arbitrary local perturbations - it is topologically protected.

### 3 Toric Code III, Berry Phase

#### 3.1 Robustness of toric code GSD

We saw the degeneracy of the toric code from both the explicit/formal calculation as well as from string operators. GSD in itself is not interesting, and is quite fragile; perturbations tend to split it - what is interesting about the TC GSD is that it is extremely robust; local perturbations cannot split it.

More precisely, consider an arbitrary local perturbation of  $H$ :

$$H' = H + \lambda \sum_j V_j \quad (3.1)$$

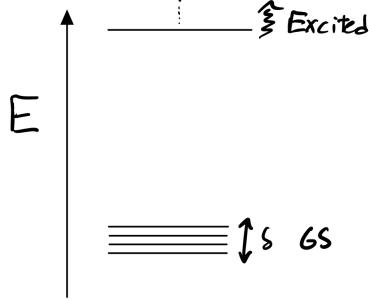
where  $V_j$  is a local operator supported near site  $j$ .



Claim: For sufficiently small  $\lambda$ ,  $H'$  has 4 nearly degenerate ground states with splitting:

$$\delta \leq e^{-C(\lambda)L} \quad (3.2)$$

with  $C(\lambda)$  a  $\lambda$ -dependent constant. The idea is that with an arbitrary local perturbation we have an exponentially small (in the thermodynamic limit) splitting of the ground state manifold, and thus the GSD is robust. From a QI perspective, this is useful because it tells us that we have a robust encoding of a qubit - we can use the ground state as a robust subspace. This degeneracy is often called a “protected”, or topological because it is automatically protected.



#### 3.2 Argument for robustness

Using the fact that  $\{W_1^X, W_2^X, W_1^Z, W_2^Z\}$  (string operators) are flexible and non-commuting, we can establish a key property of the unperturbed ground states  $|\Omega, \pm\rangle$ . Namely, for any operator  $O$  supported on

less than  $L$  sites:

$$\langle \Omega, \pm, \pm | O | \Omega, \pm, \pm \rangle = \text{diag}(c, c, c, c) \quad (3.3)$$

for some constant  $c$ . A shorthand for the above is:

$$\langle \Omega, \alpha | O | \Omega, \beta \rangle = c \delta_{\alpha\beta}. \quad (3.4)$$

You will show this relation on the homework.

Let's unpack this equation. The first thing it tells us is that:

$$\langle \Omega, \alpha | O | \Omega, \alpha \rangle = \langle \Omega, \alpha' | O | \Omega, \alpha' \rangle \quad (3.5)$$

which tells us that local operators cannot distinguish different ground states. The second thing it tells us is that:

$$\langle \Omega, \alpha' | O | \Omega, \alpha \rangle = 0 \quad \text{for } \alpha \neq \alpha' \quad (3.6)$$

in other words, local operators cannot connect ground states.

As a comparison, consider  $|\uparrow\rangle^{\otimes N}, |\downarrow\rangle^{\otimes N}$  the 2-fold degenerate ground states of the Ising model  $H = -\sum_{ij} Z_i Z_j$ . A local operator cannot connect them, but it is possible to distinguish the two states by measuring  $Z_i$ . In a symmetry breaking state, we do not have the structure of flexible strings, and hence do not have the same notion of local indistinguishability to arbitrary operators (only to symmetric ones).

Using the local distinguishability/unconnectability, let us sketch an argument for the toric code GSD. For concreteness, consider a perturbation:

$$H' = H + \lambda \sum_j X_j. \quad (3.7)$$

To obtain the first-order splitting (in degenerate perturbation theory), we need to find the matrix elements:

$$\langle \Omega, \pm, \pm | \sum_j X_j | \Omega, \pm, \pm \rangle \quad (3.8)$$

and then diagonalize. By Eq. (3.3), we know that:

$$\langle \Omega, \pm, \pm | X_j | \Omega, \pm, \pm \rangle = c_j \mathbb{I} \quad (3.9)$$

Therefore the ground state degeneracy is not split to first order. Looking at second order, we need to find matrix elements:

$$\langle \Omega, \pm, \pm | (\sum_j X_j) (H - E_{\text{gs}})^{-1} \Pi_{\text{ex}} (\sum_j X_j) | \Omega, \pm, \pm \rangle \quad (3.10)$$

and then diagonalize (third, fourth order (and so on) we have the same procedure, just add a factor of  $(H - E_{\text{gs}})^{-1} \Pi_{\text{ex}} (\sum_j X_j)$  each order). A given  $X_j$  takes us to an excited state, the  $\Pi_{\text{ex}}$  projector is then irrelevant<sup>2</sup>, and then  $(H - E_{\text{gs}})^{-1}$  is just a number (difference between the ground and excited state energy), so we end up evaluating matrix elements of the form:

$$\langle \Omega, \pm, \pm | X_j X_k | \Omega, \pm, \pm \rangle = c \mathbb{I} \quad (3.11)$$

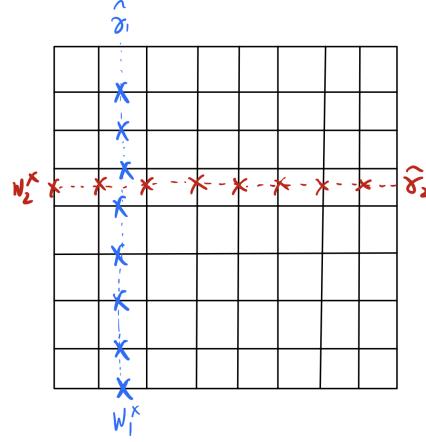
where we again use Eq. (3.3). So, again at second order we have no splitting, and the argument follows the same way for third, fourth order etc. using the same property. The argument only breaks when the property no longer holds, which occurs at  $L$ th order of perturbation theory when we end up looking at the matrix element of an operator supported on  $L$  sites. In particular, we get terms of the form:

$$\prod_{j \in \hat{\gamma}_1} X_j = W_1^X, \quad \prod_{j \in \hat{\gamma}_2} X_j = W_2^X \quad (3.12)$$

i.e. the string operators.

---

<sup>2</sup>For a general operator, we instead can use that the projector can be restricted to a local operator as we only need to look at some local patch to tell that we are in an excited state



Thus we end up the matrix elements:

$$\langle \Omega, \pm, \pm | W | \Omega, \pm, \pm \rangle \sim \lambda^L \text{diag}(c_1, c_2, c_3, c_4) \quad (3.13)$$

and so then the splitting between the ground states is:

$$\delta \sim \lambda^L = e^{-L \log(\frac{1}{\lambda})} \quad (3.14)$$

Note that this is quite heuristic, and to make it rigorous you require more precise arguments, namely that the perturbation theory converges, with a finite radius of convergence  $\lambda_0$  (which holds for arbitrarily large  $L$ ). Without a formal argument, we expect such a finite radius of convergence for “typical” gapped local Hamiltonians, with  $\lambda_0 \sim \Delta$  (so actually, in addition to the local indistinguishability, we are also using that the toric code is gapped).

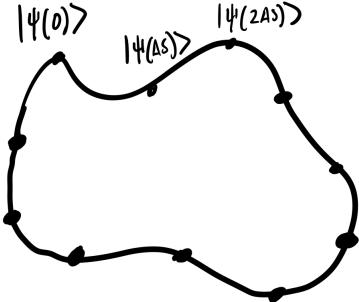
### 3.3 A Review of Berry Phase

Before we move to a general discussion of anyons, we first review the notion of a Berry phase, which is a very related idea.

Let:

$$\left\{ |\psi(s)\rangle, 0 \leq s \leq T, |\psi(T)\rangle = e^{i\phi} |\psi(0)\rangle \right\} \quad (3.15)$$

be a closed path in the set of normalized quantum states (rays in Hilbert space). Let us split up the path into  $N$  parts of length  $\Delta s$ , with  $N\Delta s = T$ . Graphically:



and for brevity we denote  $|\psi(\cdot)\rangle = |\psi\rangle$ . Now, we define the Berry phase as:

$$e^{i\theta_B} = \lim_{N \rightarrow \infty} \langle \psi_0 | \psi_{N-1} \rangle \dots \langle \psi_2 | \psi_1 \rangle \langle \psi_1 | \psi_0 \rangle \quad (3.16)$$

The Berry phase has properties:

1.  $|e^{i\theta_B}| = 1$ , i.e.  $e^{i\theta_B}$  is a  $U(1)$  phase. This can be seen from the fact that  $\langle \psi_1 | \psi_0 \rangle \sim \frac{1}{N}$  and so the  $N$ -fold product is of order  $\sim 1$ .
2.  $e^{i\theta_B}$  only depends on the path and not its parameterization. That is, it is invariant under  $s \rightarrow s' = f(s)$  with  $f(0) = 0$  and  $f(T) = T'$ .
3.  $e^{i\theta_B}$  does not depend on the phase of  $|\psi(s)\rangle$ . That is, it is invariant under  $|\psi(s)\rangle \rightarrow e^{i\varphi(s)}|\psi(s)\rangle$ . This is easily seen from the definition - the phase of the ket is cancelled out by that of the bra in the  $N$ -fold product.

Taking the limit  $N \rightarrow \infty$ , we have the formula:

$$\theta_B = \int_0^T ds i \langle \psi(s) | \frac{d}{ds} |\psi(s)\rangle \quad (3.17)$$

there is however the caveat when we evaluate the Berry phase in this way. We have to add the assumption that  $|\psi(0)\rangle = |\psi(T)\rangle$  without the phase factor.

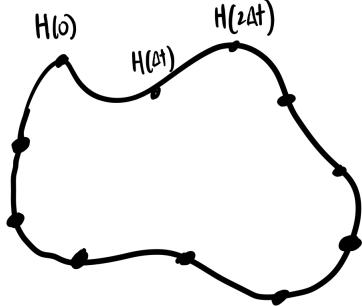
### 3.4 Berry phase and adiabatic evolution

The Berry phase shows up in two places; in adiabatic processes/cycles, and in path integrals. Today, we talk about the former.

First, a reminder of the adiabatic theorem. Let  $H(t)$  be a time-dependent Hamiltonian with  $0 \leq t \leq T$ . Suppose  $H(t)$  has a unique ground state  $|\psi(t)\rangle$  with energy  $E(t)$  and gap  $\Delta(t)$ . Suppose  $H(t)$  varies on a timescale  $\tau \gg \frac{1}{\min_t \Delta(t)}$ . Then:

$$|\psi(0)\rangle \xrightarrow{\text{evolve under } H(t)} (\text{phase}) |\psi(T)\rangle. \quad (3.18)$$

Now, consider a closed path  $H(T) = H(0)$ , which we may consider an “adiabatic cycle”.



Then:

$$|\psi(0)\rangle \xrightarrow{\text{evolve}} (\text{phase}) |\psi(0)\rangle \quad (3.19)$$

Let's compute this phase factor! It is given by:

$$(\text{phase}) = \langle \psi(0) | \mathcal{T} \exp(-i \int_0^T dt H(t)) | \psi(0) \rangle \quad (3.20)$$

with  $\mathcal{T}$  denoting time ordering. We compute the phase factor by discretizing the time-dependent Hamiltonian to  $H(0) = H_0, H(\Delta t) = H_1, H(2\Delta t) = H_2, \dots$  with  $N\Delta t = T$  and associated instantaneous ground states  $|\psi(0)\rangle = |\psi_0\rangle, |\psi(\Delta t)\rangle = \psi_1, \dots$ . Then the expression for the phase factor becomes:

$$(\text{phase}) = \lim_{N \rightarrow \infty} \langle \psi(0) | e^{-i\Delta t H_{N-1}} e^{-i\Delta t H_{N-2}} \dots e^{-i\Delta t H_1} e^{-i\Delta t H_0} | \psi(0) \rangle \quad (3.21)$$

According to the adiabatic theorem, we know that:

$$e^{-i\Delta t H_0} |\psi_0\rangle = |\psi_1\rangle \langle \psi_1| e^{-i\Delta t H_0} |\psi_0\rangle \quad (3.22)$$

as the adiabatic evolution takes  $|\psi_0\rangle \rightarrow |\psi_1\rangle$  in the first time interval. We can thus insert projectors about each time step:

$$(\text{phase}) = \lim_{N \rightarrow \infty} \langle \psi(0) | e^{-i\Delta t H_{N-1}} |\psi_{N-1}\rangle \langle \psi_{N-1}| e^{-i\Delta t H_{N-2}} |\psi_{N-2}\rangle \langle \psi_{N-2}| \dots |\psi_2\rangle \langle \psi_2| e^{-i\Delta t H_1} |\psi_1\rangle \langle \psi_1| e^{-i\Delta t H_0} |\psi(0)\rangle \quad (3.23)$$

Each of the expectation values only gives a phase factor of the energy, and so:

$$(\text{phase}) = \lim_{N \rightarrow \infty} \exp(-i\Delta t \sum_{k=0}^{N-1} E_k) \langle \psi_0 | \psi_{N-1} \rangle \dots \langle \psi_2 | \psi_1 \rangle \langle \psi_1 | \psi_0 \rangle \quad (3.24)$$

and so:

$$(\text{phase}) = \exp(-i \int_0^T E(t) dt) e^{i\theta_B}$$

(3.25)

i.e. we have a path-dependent dynamical phase part and a Berry phase part.

We'll stop here for now, and next time we will discuss the Berry phase associated with anyons.

## 4 Abelian Anyons I

Today we discuss anyons and derive them from first principles. We focus on the 2-D case in our discussion.

### 4.1 Single-particle Berry Phase

Let  $H$  be a 2-D gapped Hamiltonian with short-ranged interactions (sum of local terms). Suppose  $H$  has a particle-like excitation (the rough idea is that there is a state in  $\mathcal{H}$  that looks like the ground state everywhere, except for a localized region in space). In general, these excitations/particles can be in different locations in space<sup>3</sup>  $\mathbf{r}$ . For each different position, we will have a distinct many-body state, which we can label as  $|\mathbf{r}\rangle$ . We can then consider the Berry phase  $\theta_B(\gamma)$  associated with a closed path  $\gamma$ :

$$\theta_B(\gamma) = \int_0^T \langle \mathbf{r}(t) | i \frac{d}{dt} |\mathbf{r}(t)\rangle dt \quad (4.1)$$




---

<sup>3</sup>Physically, we could imagine these  $\mathbf{r}$  as minima of some trapping potential

We can define the Berry connection, which is a vector:

$$\mathcal{A}(\mathbf{r}) = \langle \mathbf{r} | i \nabla_{\mathbf{r}} | \mathbf{r} \rangle = \begin{pmatrix} \langle \mathbf{r} | i \frac{\partial}{\partial x} | \mathbf{r} \rangle \\ \langle \mathbf{r} | i \frac{\partial}{\partial y} | \mathbf{r} \rangle \end{pmatrix} \quad (4.2)$$

Using the chain rule, we are able to write the berry phase as an integral over the Berry connection:

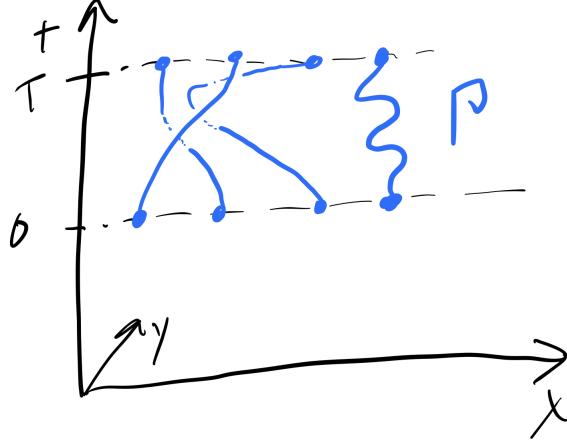
$$\theta_B(\gamma) = \int_{\gamma} \mathcal{A}(\mathbf{r}) \cdot d\mathbf{r} \quad (4.3)$$

a comment; this looks a lot like a vector potential (and indeed this choice of notation is not a coincidence); the effect that  $\mathcal{A}$  has on the physics is the same as if the particle was coupled to a background vector potential/magnetic field  $\mathbf{A}$ . This has very little to do with anyons - but when we go to multiple particles, we see the physics of anyons start to emerge.

## 4.2 Multi-particle Berry Phase and the Locality constraint

The extra part that appears when we look at the multi-particle Berry phase is exchange statistics (and this is how we will “see” anyons emerge)! Consider a state with  $n$  identical excitations/particles, which we can parameterize by  $|\{\mathbf{r}_1, \dots, \mathbf{r}_n\}\rangle$ . Since the particles are identical, we need not specify the order, only the positions.

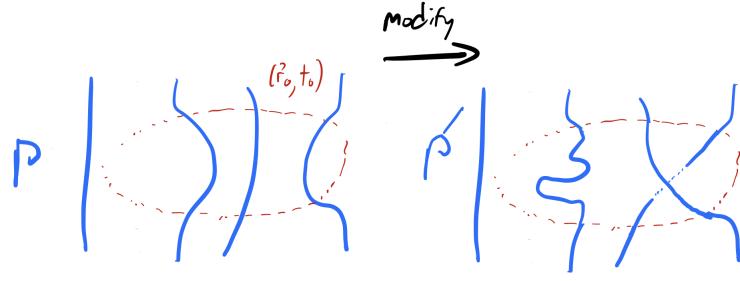
Now, let us consider the Berry phase associated with an  $n$ -particle closed path  $\Gamma$ . A picture of this path directly is a bit tricky (e.g. for 2 particles we have a 4-dimensional configuration space, which is not even Euclidean due to the lack of ordering). But we can draw it as particle worldlines through time, e.g. for four particles:



The Berry phase is then:

$$\theta_B(\Gamma) = \int_0^T \langle \{\mathbf{r}_1(t), \dots, \mathbf{r}_n(t)\} | i \frac{d}{dt} | \{\mathbf{r}_1(t), \dots, \mathbf{r}_n(t)\} \rangle dt \quad (4.4)$$

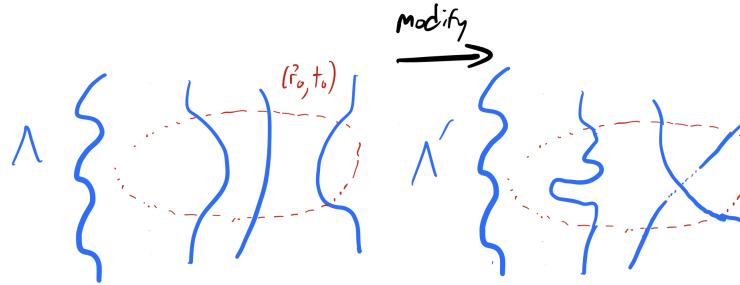
The question to understand is then; what does this look like? What are general constraints on  $\theta_B$ ? The answer is that  $\theta_B$  has to be “local”. More precisely, imagine modifying a multi-particle path  $\Gamma$  near  $(\mathbf{r}_0, t_0)$ :



Then,  $\theta_B(\Gamma') - \theta_B(\Gamma)$  depends only on what  $\Gamma, \Gamma'$  look like near  $(\mathbf{r}_0, t_0)$ . In other words:

$$\theta_B(\Gamma') - \theta_B(\Gamma) = \theta_B(\Lambda') - \theta_B(\Lambda) \quad (4.5)$$

if  $\Lambda, \Lambda'$  looks like  $\Gamma, \Gamma'$  near  $(\mathbf{r}_0, t_0)$  and differ by the same local move:



The claim is local change near  $(\mathbf{r}_0, t_0)$  is insensitive to faraway modifications. Why does Eq. (4.5) hold? It is because the difference in Berry phase  $\theta_B(\Gamma') - \theta_B(\Gamma)$  can be measured by a local operator acting near  $(\mathbf{r}_0, t_0)$  (Physically, we can imagine an interference or adiabatic experiment there). The equation then follows, assuming:

1.  $|\Psi\rangle = |\{\mathbf{r}_1, \dots, \mathbf{r}_n\}\rangle$  has short-ranged correlations, i.e.:

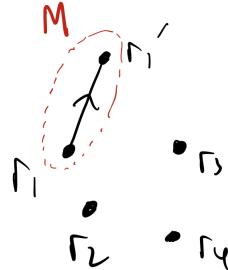
$$\langle A_{\mathbf{r}} A'_{\mathbf{r}'} \rangle_{\Psi} = \langle A_{\mathbf{r}} \rangle_{\Psi} \langle A'_{\mathbf{r}'} \rangle_{\Psi} + \mathcal{O}(e^{-\frac{|\mathbf{r}-\mathbf{r}'|}{\xi}}) \quad (4.6)$$

for  $A_{\mathbf{r}}, A'_{\mathbf{r}'}$  local operators supported near  $\mathbf{r}, \mathbf{r}'$ . This is where the gapped assumption comes in; the ground state of a gapped Hamiltonian has short-ranged correlations.

2. Particles can be moved by local operators. In other words:

$$|\{\mathbf{r}'_1, \mathbf{r}_2, \dots, \mathbf{r}_n\}\rangle = M |\{\mathbf{r}_1, \dots, \mathbf{r}_n\}\rangle \quad (4.7)$$

where  $M$  is an operator supported near  $\mathbf{r}_1, \mathbf{r}'_1$ .



These two conditions together imply the locality constraint on the Berry phase.

### 4.3 Possible Forms of the Berry Phase & Topological Classes

The next question is then - what is the most general Berry phase  $\theta_B(\Gamma)$  that satisfies the locality constraint? One solution, and the one you probably would have guessed, is:

$$\theta_B(\Gamma) = \sum_i \int_{\Gamma} \mathcal{A}(\mathbf{r}_i) \cdot d\mathbf{r}_i. \quad (4.8)$$

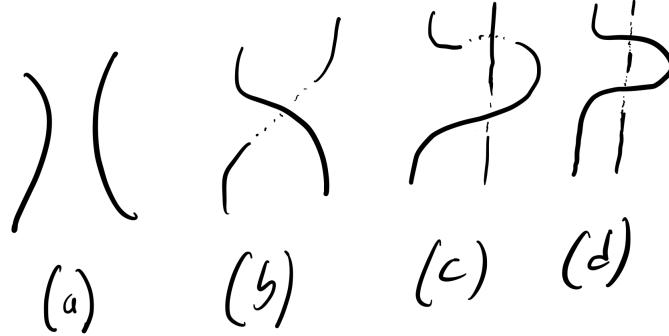
This is manifestly local. We could get a little more general:

$$\theta_B(\Gamma) = \sum_i \int_{\Gamma} \left( \mathcal{A}(\mathbf{r}_i) + \sum_j \mathcal{B}(\mathbf{r}_i, \mathbf{r}_j) + \sum_{jk} \mathcal{C}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) + \dots \right) \cdot d\mathbf{r}_i \quad (4.9)$$

where  $\mathcal{A}$  is the single-particle Berry connection and  $\mathcal{B}, \mathcal{C}$  are the two, three (and so on) particle terms so long as the multi-particle terms are short-ranged, i.e. only they are nonzero where  $\mathbf{r}_j$  is close to  $\mathbf{r}_i$  and so on.

Is this the only possible solution consistent with Eq. (4.5)? No! Indeed the first proposed solution is the form of the Berry phase consistent with bosons, but there are other solutions corresponding to fermions and anyons. What does the first solution miss? Indeed it is possible to have topological terms that look highly non-local, but such that the Berry phase still has the locality constraint.

In this line, we say that two (non-intersecting)  $n$ -particle paths with the same endpoints are topologically equivalent if they can be continuously deformed into one another without bringing particles near each other (worldlines cannot pass through one another). For example, consider:



(a)  $\sim$  (d), but (a)  $\not\sim$  (b)  $\not\sim$  (c). This defines an equivalence relation on (non-intersecting)  $n$ -particle paths, which splits the set of  $n$ -particle paths into equivalence (topological) classes.

Coming back to the Berry phase, the claim is that the most general  $\theta_B$  that satisfies the locality constraint Eq. (4.5) can be written as:

$$\boxed{\theta_B(\Gamma) = \theta_{\text{short-range}}(\Gamma) + \theta_{\text{top}}(\Gamma)} \quad (4.10)$$

where  $\theta_{\text{top}}(\Gamma)$  only depends on the topological class of  $\Gamma$  and the short-range piece is given by Eq. (4.9)

$$\theta_{\text{short-range}}(\Gamma) = \sum_i \int_{\Gamma} \left( \mathcal{A}(\mathbf{r}_i) + \sum_j \mathcal{B}(\mathbf{r}_i, \mathbf{r}_j) + \sum_{jk} \mathcal{C}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) + \dots \right) \cdot d\mathbf{r}_i. \quad (4.11)$$

Eq. (4.10) is the key result. From here, we will classify the different possible topological terms we can have. We will find in 3D that we only get two possible classes and in 2D that we get many more.

Let us argue for Eq. (4.10) in the special case of  $\Gamma$  being a topologically trivial path, i.e. where we only have the first term. For the sake of drawing, let's look at a 2 particle path.

$$\theta_B(\Gamma) = \theta_B(\gamma_1 \& \gamma_2) = ? \quad (4.12)$$

$$\Theta_B(\Gamma) = \Theta_B\left(\begin{array}{c} \gamma_1 \\ \gamma_2 \end{array}\right) = ?$$

where  $\gamma_1, \gamma_2$  are the single particle paths. We can then define:

$$\Delta(\Gamma) = \theta_B(\gamma_1 \& \gamma_2) - \theta_B(\gamma_1) - \theta_B(\gamma_2) \quad (4.13)$$

Now, using Eq. (4.5) we can use that  $\Delta(\Gamma)$  is topologically invariant, i.e. it is invariant to local deformations of  $\gamma_1$  far from  $\gamma_2$  and vice versa.

$$\begin{aligned} \Theta_B\left(\begin{array}{c} \gamma_1 \\ \gamma_2 \end{array}\right) - \Theta_B\left(\begin{array}{c} \gamma_1 \\ \gamma_2 \end{array}\right) - \Theta_B\left(\begin{array}{c} \gamma_1 \\ \gamma_2 \end{array}\right) \\ = \\ \Theta_B\left(\begin{array}{c} \gamma_1 \\ \gamma_2 \end{array}\right) - \Theta_B\left(\begin{array}{c} \gamma_1 \\ \gamma_2 \end{array}\right) - \Theta_B\left(\begin{array}{c} \gamma_1 \\ \gamma_2 \end{array}\right) \end{aligned}$$

Therefore we can deform  $\Gamma$  to the trivial path, for which  $\Delta$  is easily seen to vanish.

$$\Delta(\Gamma) = \Delta(\Gamma_{\text{trivial}}) = 0 \quad (4.14)$$

$$\Delta(\Gamma) = \Theta_B\left(\begin{array}{c} | \\ \gamma_1 \\ \gamma_2 \end{array}\right) - \Theta_B\left(\begin{array}{c} | \\ \gamma_1 \end{array}\right) - \Theta_B\left(\begin{array}{c} | \\ \gamma_2 \end{array}\right)$$

And thus since  $\Delta(\Gamma) = 0$ , we find:

$$\theta_B(\Gamma) = \theta_B(\gamma_1) + \theta_B(\gamma_2) = \int_{\gamma_1} \mathcal{A}(\mathbf{r}_1) \cdot d\mathbf{r}_1 + \int_{\gamma_2} \mathcal{A}(\mathbf{r}_2) \cdot d\mathbf{r}_2 = \theta_{\text{short-range}}(\Gamma). \quad (4.15)$$

## 5 Abelian Anyons II

Consider a gapped  $d$ -dimensional Hamiltonian  $H$  with short-ranged interaction. Supports  $H$  supports a particle-like excitation. Last class, we looked at the single and multi-particle Berry phase for closed paths of such particle-like excitations. We argued that the most general form for the  $n$ -particle Berry phase took the form:

$$\theta_B(\Gamma) = \theta_{s-r}(\Gamma) + \theta_{\text{top}}(\Gamma) \quad (5.1)$$

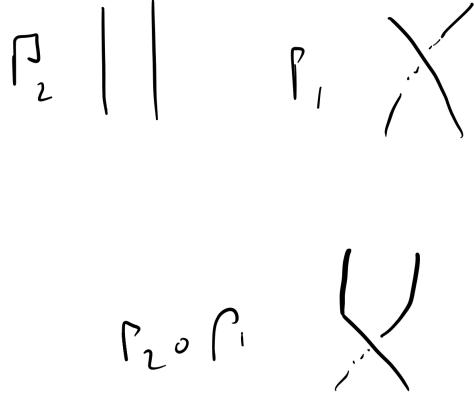
where the short-ranged piece looks like:

$$\theta_{s-r}(\Gamma) = \sum_{i=1}^n \int_{\Gamma} \left( \mathcal{A}(\mathbf{r}_i) + \sum_j \mathcal{B}(\mathbf{r}_i, \mathbf{r}_j) + \sum_{jk} \mathcal{C}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) \right) d\mathbf{r}_i \quad (5.2)$$

where the multi-particle terms are short-ranged. The topological term we discussed only depends on the topological class of  $\Gamma$  (c.f. the short-range part is non-universal and depends on the microscopic details of the path). This topological term will be what informs the exchange statistics. We now ask - what are the possible  $\theta_{\text{top}}$  terms that can appear in Eq. (5.1)? An important constraint is that it must be multiplicative under composition:

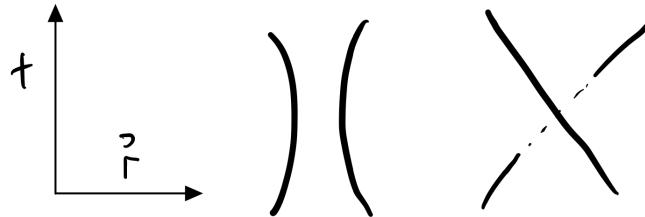
$$e^{i\theta_{\text{top}}(\Gamma_2 \circ \Gamma_1)} = e^{i\theta_{\text{top}}(\Gamma_2)} e^{i\theta_{\text{top}}(\Gamma_1)}. \quad (5.3)$$

This is because  $e^{i\theta_B}$  and  $e^{i\theta_{s-r}}$  both obey this property. Thus, we look for topological terms that obey this property. Formally, this amounts to finding 1-D representations of the braid group.



## 5.1 Topological Berry Phase in 3D

For simplicity, we focus on 2-particle paths. There are 2 such paths in 3D; one where the particles do not exchange, and then one without (picture below slightly misleading because it's hard to draw in 3+1D). We might ask; what if we do 2 exchanges? In 2D we cannot unwrap this, but in 3D a double exchange can be continuously deformed to no exchanges (by using the third spatial dimension).



There are then two possible values for the topological Berry phase:

$$e^{i\theta_{\text{top}}(\text{no exchange})} = a, \quad e^{i\theta_{\text{top}}(\text{exchange})} = b \quad (5.4)$$

By Eq. (5.3):

$$e^{i\theta_{\text{top}}(\text{double exchange})} = b^2 \quad (5.5)$$

but also, since a double exchange can be topologically deformed to the trivial path:

$$e^{i\theta_{\text{top}}(\text{double exchange})} = e^{i\theta_{\text{top}}(\text{no exchange})} = a \quad (5.6)$$

So then we obtain the constraint:

$$b^2 = a \quad (5.7)$$

We can write a very similar equation if we do a trivial path twice; again by Eq. (5.3)

$$e^{i\theta_{\text{top}}(\text{double no exchange})} = a^2 \quad (5.8)$$

but this is also topologically equivalent to the trivial path/a single no exchange path:

$$e^{i\theta_{\text{top}}(\text{double no exchange})} = e^{i\theta_{\text{top}}(\text{no exchange})} = a \quad (5.9)$$

and thus:

$$a^2 = a \quad (5.10)$$

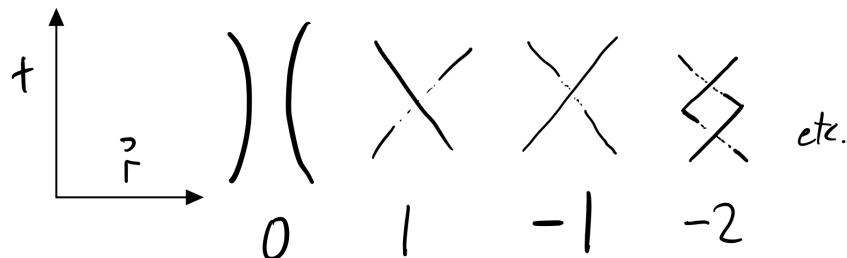
So, we have two solutions:

1.  $a = 1, b = 1$  - this is the case where there is no Berry phase for an exchange. This is the physical definition of a boson.
2.  $a = 1, b = -1$  - this is the case where an exchange gives a Berry phase of -1. This is the physical definition of a fermion.

The modern view is that fermions have nonlocal behaviour, and the anticommutation relations are a way to package this nonlocality in a local way.

## 5.2 Topological Berry Phase in 2D

Again, we focus on 2-particle paths (this fully determines the  $n$ -particle behaviour). Unlike in 3D, we get many topological classes, with different classes corresponding to different braids.



We can label braids by the number of times particles exchange in the clockwise direction. We let:

$$e^{i\theta_{\text{top}}(n \text{ clockwise exchanges})} = a_n \quad (5.11)$$

Then, we know by Eq. (5.3) that:

$$a_n = a_1^n \quad (5.12)$$

Since we have no other constraints, we can let:

$$a_1 = e^{i\theta} \quad (5.13)$$

Then:

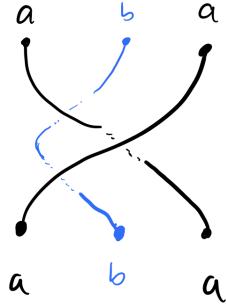
$$e^{i\theta_{\text{top}}(n \text{ clockwise exchanges})} = e^{in\theta} \quad (5.14)$$

The  $\theta$  is thus known as the “statistical angle of excitations”. There are three cases:

1.  $\theta = 0$  - no phase from exchanges - bosons
2.  $\theta = \pi$  - -1 phase from exchanges - fermions
3.  $\theta \neq 0, \pi$  - some other  $e^{i\theta}$  phase from excitations - anyons

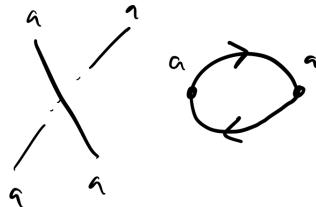
### 5.3 Generalization to multiple anyon types

Suppose  $H$  supports several types of particle excitations  $A = \{a, b, \dots\}$ . We can then consider multi-particle closed paths  $\Gamma$ , for example:

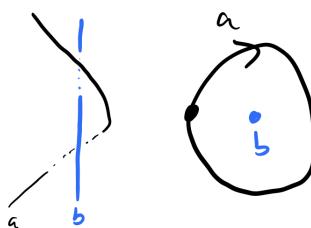


In this case, what are the possible topological terms? Without going through the entire argument,  $\theta_{\text{top}}(\Gamma)$  is characterized by multiple statistical angles (we consider the 2-particle case, which again characterizes all multi-particle paths):

1.  $\{e^{i\theta_a} : a \in A\}$ ; “exchange statistics of  $a$ ”

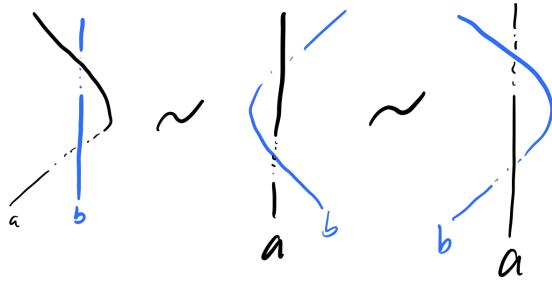


2.  $\{e^{i\theta_{ab}} : a, b \in A\}$  “mutual statistics”

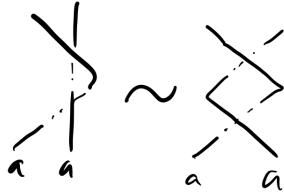


There are some important constraints on the statistical angles:

1.  $e^{i\theta_{ab}} = e^{i\theta_{ba}}$  - this is because wrapping  $a$  around  $b$  is topologically equivalent to wrapping  $b$  around  $a$ .



2.  $e^{i\theta_{aa}} = e^{2i\theta_a}$  - wrapping  $a$  around an  $a$  is topologically equivalent to a double exchange.

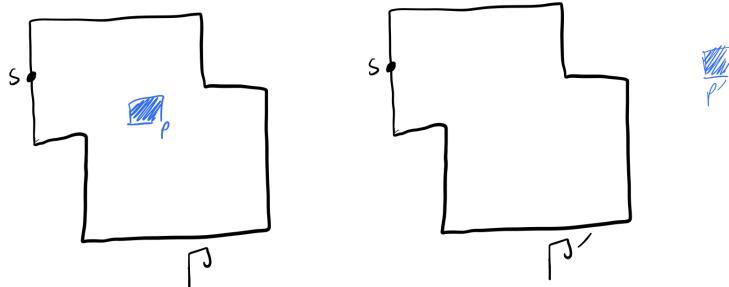


Terminology: A “non-trivial” anyon  $a$  is a particle such that  $e^{i\theta_{ab}} \neq 1$  for some  $b$  (allowing for  $b = a$ ). In other words, an anyon is a particle which braids non-trivially with another particle. Note that this means that some particles may have trivial exchange statistics, but still be anyons by virtue of non-trivial mutual statistics. This will be the case for the Toric code, where  $e, m$  individually have trivial exchange statistics but braid non-trivially.

So, let us summarize. For a general gapped system with particle-like excitation, the topological Berry phase is completely characterized by the exchange statistics and the mutual statistics. Now, to have a concrete example, let’s compute these both for the toric code.

## 5.4 Anyons in the Toric Code

Let’s compute the mutual statistics of the charge and flux excitations. We denote the mutual statistics by  $e^{i\theta_{em}}$  (we denote the charge by  $e$  and the flux by  $m$ , as in Gauge theory). To compute this, we have to somehow extract the topological part of the Berry phase. To this end, we compare 2 different paths such that the uninteresting/short ranged parts of the Berry phase cancel out.



We can write this as:

$$\theta_B(\Gamma) - \theta_B(\Gamma') = [\theta_{s-r}(\Gamma) - \theta_{s-r}(\Gamma')] + [\theta_{\text{top}}(\Gamma) - \theta_{\text{top}}(\Gamma')] \quad (5.15)$$

By construction, the  $s - r$  parts cancel because locally the two paths look exactly the same (locally, you don't see the flux you are surrounding or avoiding). Further, we notice that  $\Gamma'$  is topologically trivial because  $\Gamma'$  can be shrunk to a point (there is no plaquette inside it). Thus:

$$\theta_B(\Gamma) - \theta_B(\Gamma') = \theta_{\text{top}}(\Gamma) = \theta_{\text{em}} \quad (5.16)$$

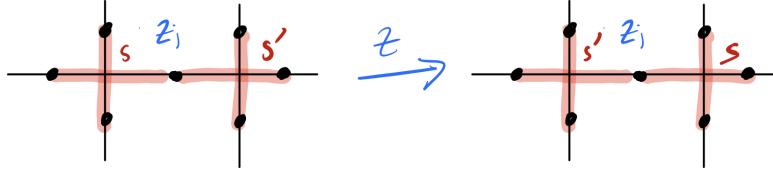
Likewise, if  $\Gamma, \Gamma'$  are adiabatic cycles, then:

$$\theta(\Gamma) - \theta(\Gamma') = [\theta_B(\Gamma) - \theta_B(\Gamma')] + [\theta_{\text{dyn}}(\Gamma) - \theta_{\text{dyn}}(\Gamma')] = \theta_{\text{em}} \quad (5.17)$$

as the dynamical phase between the two cycles cancel, as well. In fact,  $\Gamma, \Gamma'$  do not have to be adiabatic; they can be composed out of any sequence of *local* "movement operators", and we will still get:

$$\theta(\Gamma) - \theta(\Gamma') = \theta_{\text{em}} \quad (5.18)$$

and this will be how we compute  $\theta_{\text{em}}$  for the toric code. Let's try this. Notice that  $Z_j$  moves a charge:



and this is because  $Z_j$  anticommutes with  $A_s, A_{s'}$ , resulting in:

$$\begin{aligned} Z_j |a_s = 1, a_{s'} = 1\rangle &= |a_s = -1, a_{s'} = -1\rangle \\ Z_j |a_s = -1, a_{s'} = -1\rangle &= |a_s = 1, a_{s'} = 1\rangle \\ Z_j |a_s = 1, a_{s'} = -1\rangle &= |a_s = -1, a_{s'} = 1\rangle \\ Z_j |a_s = -1, a_{s'} = 1\rangle &= |a_s = 1, a_{s'} = -1\rangle \end{aligned} \quad (5.19)$$

## 6 Abelian Anyons III, Quantum Double Model I

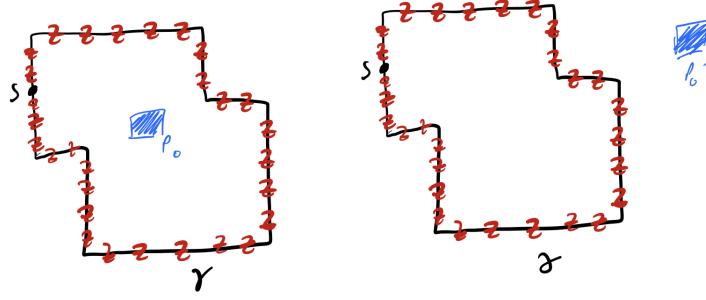
### 6.1 Mutual Statistics of Toric Code Anyons

We want to compute the mutual statistics of charge and flux statistics  $e^{i\theta_{\text{em}}}$  in the Toric code. The strategy was to compare two paths,  $\Gamma$  and  $\Gamma'$  (see figure from last lecture) such that the paths look the same, but  $\Gamma$  has a flux in the center of the path and  $\Gamma'$  has a flux outside of it. By looking at the difference of the two processes, since the two paths "look the same", the uninteresting short-range contributions to the phase will cancel, leaving us with just  $\theta_{\text{em}}$ :

$$\theta(\Gamma) - \theta(\Gamma') = \theta_{\text{em}} \quad (6.1)$$

where  $\theta(\Gamma), \theta(\Gamma')$  are the total phases accumulated by some sequence of local movement operators. Let's compute these and now see what we get. The simplest operator we can write down to move charges is  $Z_j$  - the pauli-Z operator on link  $j$ . This moves a charge from site  $s$  to site  $s'$  (and vise versa), as depicted in the figure from last lecture (the figure is slightly poorly notated; it should really depict movement of charges, as opposed to swapping the site labels. The precise action is captured in the anticommutation, or in the relations of Eq. (5.19)). We can see this from the anticommutation of  $Z_j$  with  $A_s, A_{s'}$ .

Denote the initial state by  $|p_0, s\rangle$ . Let  $\Gamma$  denote the process composed out of a sequence of  $Z$  operators, which moves the charge  $s$  along the path  $\gamma$ .



By definition:

$$\prod_{j \in \gamma} Z_j |p_0, s\rangle = e^{i\theta(\Gamma)} |p_0, s\rangle \quad (6.2)$$

To figure out what  $e^{i\theta(\Gamma)}$ , we observe the operator identity:

$$\prod_{j \in \gamma} Z_j = \prod_{p \in \text{int}(\gamma)} B_p \quad (6.3)$$

Note that this is a kind of Stokes' theorem. We can use this identity to evaluate  $e^{i\theta(\Gamma)}$  - we're in business because there is exactly one  $b_p = -1$  inside of  $\gamma$ :

$$e^{i\theta(\Gamma)} |p_0, s\rangle = \prod_{p \in \text{int}(\gamma)} B_p |p_0, s\rangle = -|p_0, s\rangle \quad (6.4)$$

Thus:

$$e^{i\theta(\Gamma)} = -1 \quad (6.5)$$

We aren't quite done yet. We should compare this to the case where the flux is not in the center. Now, consider  $\Gamma'$  which is an identical process save for the flux is on a plaquette  $p'_0$  outside of the path  $\gamma$ . Then:

$$e^{i\theta(\Gamma')} |p'_0, s\rangle = \prod_{j \in \gamma} Z_j |p'_0, s\rangle = \prod_{p \in \text{int}(\gamma)} B_p |p'_0, s\rangle = +|p'_0, s\rangle \quad (6.6)$$

where the last equality follows since all  $b_p = +1$  in the interior of the path (the only flux is outside). Thus:

$$e^{i\theta(\Gamma')} = 1 \quad (6.7)$$

Therefore looking at the difference:

$$e^{i\theta_m} = e^{i\theta(\Gamma)} e^{-i\theta(\Gamma')} = -1 \cdot 1 = -1 \quad (6.8)$$

Thus:

$$e^{i\theta_m} = -1 \quad (6.9)$$

We compared two phases, one which was nontrivial and one which was trivial, and then the ratio gives us the interesting exchange statistic.

Why was it important that we took the difference between these two paths/compared the two processes? We chose  $Z$  as our movement operator, but someone else could have very well chosen  $e^{i\phi}Z$  for any phase; then that random phase would contribute to  $e^{i\theta(\Gamma)}$  and  $e^{i\theta(\Gamma')}$ . But crucially, it contributes in the exact same way to both of these, and hence does not contribute to the difference.

A couple remarks:

1. The result does not depend on the microscopic details of the path  $\gamma$ . We expected this from what we know about the Berry phase, but also saw this clearly arise from the calculation itself. Moreover,  $\theta_{em}$  does not depend on the choice of movement operator; we won't prove this in full generality, but we did remark how movement operators differing by a phase give the same result. In fact even if the movement operators look more starkly different, we find the same result.
2. Mutual statistics are symmetric, and indeed we could swap the roles of the charges/fluxes and  $Z/X$  and we would get the same results.
3. According to the definition from last class,  $e, m$  are indeed non-trivial anyons since  $e^{i\theta_{em}} = -1$  (non-trivial mutual statistics, with each other).
4. We could also ask what the mutual statistics of taking an  $e$  particle around itself (or  $m$  around itself). Then, its trivial to show that:

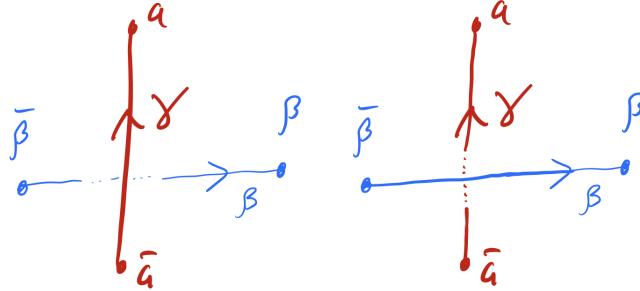
$$e^{i\theta_{ee}} = e^{i\theta_{mm}} = 1 \quad (6.10)$$

5. Note that an even number of charges/fluxes look like having no charges/fluxes at all. Actually, we have four types of "sectors" of excitations  $\{1, e, m, \epsilon\}$  with 1 being even charges/fluxes,  $e$  being odd charges even fluxes,  $m$  being even charges odd fluxes, and  $\epsilon$  being odd charges and fluxes (you will study this one on the homework).

## 6.2 Connection between Abelian Anyons and String Operators

We sketch the connection between mutual statistics and the algebra of string operators. In the previous section we gave a physically transparent way of computing mutual statistics, now we give the shortcut. Let  $a, b$  be Abelian anyons, and let  $W_a, W_b$  be associated string operators.  $W_a$  creates an excitation  $a$  at one end and its antiparticle  $\bar{a}$  at its other end (in the toric code, the particles and antiparticles coincide). Consider paths  $\gamma, \beta$ , and then compare:

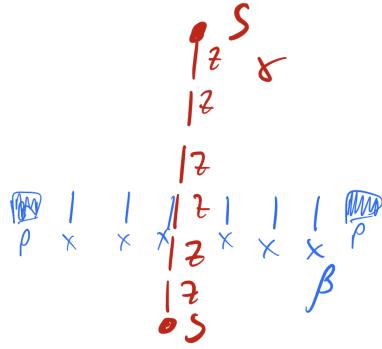
$$W_a(\gamma)W_b(\beta)|\Omega\rangle, \quad W_b(\beta)W_a(\gamma)|\Omega\rangle \quad (6.11)$$



The claim is then:

$$W_a(\gamma)W_b(\beta)|\Omega\rangle = e^{i\theta_{ab}} W_b(\beta)W_a(\gamma)|\Omega\rangle \quad (6.12)$$

For example, lets use  $a = e, b = m$  on the toric code. The  $b$ -string is a string of Xs that creates two fluxes on the end, and the  $a$ -string is a string of Zs that creates two charges on the end:



These two operators anticommute on the operator level:

$$W_e(\gamma)W_m(\beta) = -W_m(\beta)W_e(\gamma) \quad (6.13)$$

and hence also anticommute when acted upon the ground state. Thus:

$$e^{i\theta_{em}} = -1 \quad (6.14)$$

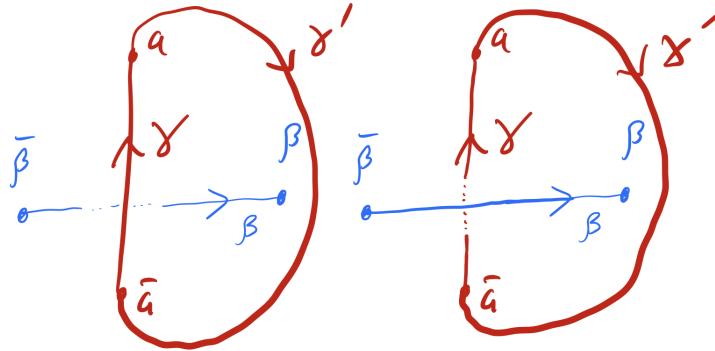
Let us derive the general relation of Eq. (6.12). On general grounds, we know that:

$$W_a(\gamma)W_b(\beta)|\Omega\rangle = e^{i\theta}W_b(\beta)W_a(\gamma)|\Omega\rangle \quad (6.15)$$

for *some* phase  $\theta$ . This is because both of the states correspond to the creation of anyons on both sides - hence both states have the same 4 anyons ( $a, \bar{a}, b, \bar{b}$ ) at the same locations. Hence they must describe the same physical state. To compute the phase, we multiply both the right and left hand side with a further string operator,  $W_a(\gamma')$ :

$$W_a(\gamma')W_a(\gamma)W_b(\beta)|\Omega\rangle = e^{i\theta}W_a(\gamma')W_b(\beta)W_a(\gamma)|\Omega\rangle \quad (6.16)$$

Graphically, the LHS/RHS look like:



The first thing we see at an algebraic level is that  $W_a(\gamma')$  commutes with  $W_b(\beta)$ , as they act on non-overlapping places:

$$W_a(\gamma')W_a(\gamma)W_b(\beta)|\Omega\rangle = e^{i\theta}W_b(\beta)W_a(\gamma')W_a(\gamma)|\Omega\rangle \quad (6.17)$$

Now let's think about the two processes we have here. On the LHS, we have that  $a$  braids around  $b$  (we first create  $b$  and then do the big loop). On the RHS, we first create  $a, \bar{a}$  and do the big loop, and then create  $b$  -  $a$  follows the exact same path, but does *not* braid around  $b$ . Thus, we would conclude that the phase difference  $\theta$  must be the topological Berry phase  $\theta_{ab}$ , as everything else cancels.

To summarize:

(a) Abelian anyon excitations  $\iff$  non-commuting flexible string operators (with the algebra of the SOs encoding the mutual statistics. On the HW, you will also see that the algebra of the SOs imply exchange statistics, but this will require looking at three string operators)

(b) Non commuting flexible string operators  $\implies$  Robust GSD (on torus)

(c) Robust GSD  $\implies$  robust quantum memory (can encode the qubit in the GSD).

Thus any system with Abelian anyons has all these properties... it is harder to formulate the reverse statement, but easy counterexamples don't come to mind.

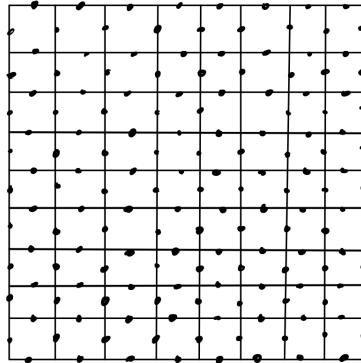
### 6.3 Introduction to the Quantum Double Model

The above concludes our discussion of abelian anyons and the toric code. We thus move onto the discussion of non-abelian anyons and the quantum double model. The reference is quant-ph/9707021.

We can think of the quantum double model as the generalization of the toric code. It's not actually a single model, but a class of models, of which we specify a specific model by choice of a finite group  $G$ . The toric code corresponds to choosing  $G = \mathbb{Z}_2$  (the simplest nontrivial choice). Looking a bit ahead, when we choose  $G$  to be Abelian, the model will host Abelian anyons, but when we choose  $G$  to be a non-Abelian group, the quantum double model will host non-Abelian anyon excitations.

What are non-Abelian anyons? In short, Abelian if we trap  $n$  particles at  $n$  sites have a unique physical state, vs. for non-Abelian anyons the same process results in a ground state with degeneracy. We will see that the braiding processes for these anyons will not be Abelian.

We have a local Hilbert space on each edge site of a lattice, with each edge Hilbert space having dimension  $|G|$ .



It is convenient to define two orthonormal basis for each edge,  $\{\uparrow |g\rangle\}$  and  $\{\downarrow |g\rangle\}$  where  $g \in G$ . The  $\uparrow / \downarrow$  correspond to the orientation of the link. The relationship between these two bases are very simple:

$$\uparrow |g\rangle = \downarrow |g^{-1}\rangle \quad (6.18)$$

For example:

$$\left| g \begin{array}{c} h \\ \nearrow \searrow \\ \square \\ \downarrow \end{array} \begin{array}{c} K \\ \nearrow \searrow \\ \square \\ \downarrow \end{array} e \right\rangle = \left| g \begin{array}{c} h \\ \nearrow \searrow \\ \square \\ \downarrow \end{array} \begin{array}{c} K^{-1} \\ \nearrow \searrow \\ \square \\ \downarrow \end{array} e \right\rangle$$

= ...

We're out of time for today, but it's worth noting that quantum double model with group  $G$  is related to  $G$ -gauge theory; just written in a more concrete way. In such a gauge theory, to transport a charge in the gauge theory along a link you use unitary  $g/g^{-1}$  depending on which direction you want to transport it.

## 7 Quantum Double Model II

Like the toric code, we consider a lattice of local Hilbert spaces; except unlike the toric code where each on-site hilbert space was  $\mathbb{C}^2$  (a qubit), now the model is defined by a group  $G$  with the Hilbert space at each site having dimension  $|G|$ . There is a set of two orthonormal bases for each edge,  $\{\uparrow |g\rangle\}, \{\downarrow |g\rangle\}$  where  $g \in G$ , with the property that flipping the direction of the arrow flips  $g \rightarrow g^{-1}$ . This is captured in the last figure we drew last time.

We might ask why we might not fix an orientation convention. We could do this, but it does break some symmetries in the model. So it's better to keep a "free global" orientation.

### 7.1 Hamiltonian for the Quantum Double Model

We introduce two types of operators  $A_g(s), B(p)$ . These are related, but do not exactly coincide with the operators in the toric code.

We start with the star operator  $A_g(s)$ , which graphically has the action:

$$A_g(s) \left| \begin{array}{c} h_2 \downarrow \\ \nearrow s \nearrow \\ h_1 \quad h_4 \\ \downarrow \end{array} \right\rangle = \left| \begin{array}{c} gh_2 \downarrow \\ \nearrow s \nearrow \\ gh_1 \quad gh_4 \\ \downarrow \end{array} \right\rangle$$

in other words it permutes between bases elements via left-multiplication of  $g$  on edges that are part of the star. It is sometimes called a "gauge transformation". Note that incoming arrows get left-multiplied by  $g$  and outgoing arrows get right-multiplied by  $g^{-1}$  (but we will derive this explicitly, we only need specify the action on incoming arrows).

The plaquette operator has the action:

$$B(p) \left| \begin{array}{c} h_2 \\ \nearrow p \nearrow \\ h_1 \quad h_4 \\ \downarrow \end{array} \right\rangle = \delta_{h_1 h_2 h_3 h_4, 1} \left| \begin{array}{c} h_2 \\ \nearrow p \nearrow \\ h_1 \quad h_4 \\ \downarrow \end{array} \right\rangle$$

where:

$$\delta_{h_1 h_2 h_3 h_4, 1} = \begin{cases} 1 & h_1 h_2 h_3 h_4 = 1 \\ 0 & \text{otherwise} \end{cases} \quad (7.1)$$

The terminology we will use is that  $h_1 h_2 h_3 h_4$  is the “flux through the plaquette  $p$ ”.

There is a bit of convention with the multiplication of the group elements, in that we start with the basepoint in the bottom right and go around counterclockwise. We might ask that does this basepoint choice matter? If we started in the top right, we would instead get:

$$h_2 h_3 h_4 h_1 = h_1^{-1} (h_1 h_2 h_3 h_4) h_1 \quad (7.2)$$

so concretely, different base points change flux by conjugation. But since the identity element is unchanged by conjugation, and here we only care about fluxes as identity, it will turn out to not matter here (in fact we could go in the other direction, and this changes the sign of the flux, but because the identity is self-inverse it again does not matter here. But it will matter when we look at some related operators).

Another comment;  $B(p)$  are diagonal in the group element basis, while  $A_g(s)$  are not.

Now, defining the Hamiltonian in terms of these operators:

$$H = - \sum_s A(s) - \sum_p B(p) \quad (7.3)$$

where:

$$A(s) = \frac{1}{|G|} \sum_g A_g(s) \quad (7.4)$$

A couple comments about this Hamiltonian. Let’s check to make sure that it is indeed Hermitian.  $B(p)$  is clearly Hermitian, because in our chosen orthonormal basis it is real and diagonal in the group element basis. To see that  $A(s)$  is Hermitian requires a tiny bit of work. First, we notice that  $A_g(s)$  is a unitary operator (because it is a permutation matrix):

$$A_g(s)^\dagger = A_g(s)^{-1} \quad (7.5)$$

But since  $A_g(s)$  acts via left multiplication of  $g$ ,  $A_g(s)^{-1}$  should be a left multiplication by  $g^{-1}$ :

$$A_g(s)^\dagger = A_g(s)^{-1} = A_{g^{-1}}(s). \quad (7.6)$$

This tells us that in general  $A_g(s)$  will not be Hermitian. But the sum will be! This is because:

$$A(s)^\dagger = \left( \frac{1}{|G|} \sum_g A_g(s) \right)^\dagger = \frac{1}{|G|} \sum_g A_{g^{-1}}(s) = \frac{1}{|G|} \sum_g A_g(s) = A(s) \quad (7.7)$$

as the sum over all inverses of group elements is the same as the sum over all group elements.

So, the  $H$  is Hermitian, and hence a valid Hamiltonian.

## 7.2 Relationship to the Toric Code

Suppose we set  $G = \mathbb{Z}_2 = \{1, g\}$  with  $g^2 = 1$ . Then on each link we have two states; we can identify  $|1\rangle \leftrightarrow |Z=1\rangle$  and  $|g\rangle \leftrightarrow |Z=-1\rangle$  with  $Z$  the standard Pauli operator. We can then see what the star and plaquette operators reduce to:

$$A(s) = \frac{1}{2} (A_1(s) + A_g(s)) = \frac{\mathbb{I} + \prod_{j \in \text{star}(s)} X_s}{2} = \frac{\mathbb{I} + A_s}{2} \quad (7.8)$$

So we get the projector onto the star operator of the toric code. The same holds for  $B(p)$ ; to have a product of group elements equal to 1 in  $\mathbb{Z}_2$ , this means we need an even number of gs. This is equivalent to projecting onto states with an even number of  $Z = -1$ :

$$B(p) = \frac{1}{2}(\mathbb{I} + \prod_{j \in \partial p} Z_p) = \frac{\mathbb{I} + B_p}{2} \quad (7.9)$$

which is a projection onto the toric code plaquette operator.

We thus in the  $\mathbb{Z}_2$  case recover the toric code Hamiltonian (save for a bunch of identities, which doesn't affect any of the physics save for a shift in the spectrum).

### 7.3 The Quantum Double Model is a Commuting Projector Hamiltonian

We make a few claims about these operators:

1.  $A(s)^2 = A(s)$
2.  $B(p)^2 = B(p)$
3.  $[A(s), A(s')] = 0$
4.  $[B(p), B(p')] = 0$
5.  $[A(s), B(p)] = 0$

Combining all of these, this tells us that  $\{A(s), B(p)\}$  form a set of commuting projectors. Just like in the toric code where we had a sum of commuting operators with eigenvalues  $\pm 1$ , here we have a sum of commuting operators with eigenvalues 0, 1. Let's go ahead and prove the above 5:

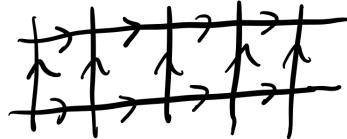
1. We note that  $A_g(s)A_h(s) = A_{gh}(s)$  which follows immediately by the definition. Then:

$$A(s)^2 = \left( \frac{1}{|G|} \sum_g A_g(s) \right) \left( \frac{1}{|G|} \sum_h A_h(s) \right) = \frac{1}{|G|^2} \sum_{gh} A_{gh}(s) \quad (7.10)$$

but now the double sum gives me every element in the group (but  $|G|$  different ways), thus:

$$A(s)^2 = \frac{1}{|G|^2} |G| \sum_g (s) = \frac{1}{|G|} \sum_g A_g(s) = A(s) \quad (7.11)$$

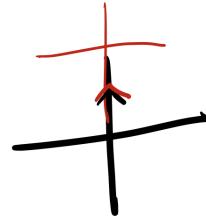
2.  $B(s)^2 = B(s)$  is obvious because it has eigenvalues 1 and 0.
3. Fix an orientation convention on the lattice where everything goes up/right.



Now every star has 2 outgoing and 2 incoming arrows. So, let's work out what  $A_g(s)$  in the outgoing case:

$$\begin{aligned}
A_g(s) \left| \begin{array}{c} h_1 \\ \nearrow h_2 \searrow h_3 \\ s \\ \downarrow h_4 \end{array} \right\rangle &= A_g(s) \left| \begin{array}{c} h_1 \\ \nearrow h_2^{-1} \searrow h_3^{-1} \\ s \\ \downarrow h_4 \end{array} \right\rangle \\
&= \left| \begin{array}{c} gh_1 \\ \nearrow gh_2^{-1} \searrow gh_3^{-1} \\ s \\ \downarrow gh_4 \end{array} \right\rangle \\
&= \left| \begin{array}{c} gh_1 \\ \nearrow h_2^{-1} \searrow h_3^{-1} \\ s \\ \downarrow gh_4 \end{array} \right\rangle
\end{aligned}$$

wherein in the second equality we flip the outgoing arrows so that we may apply our known action for  $A_g(s)$  on incoming arrows, and in the last equality we flip again. Thus, in summary,  $A_g(s)$  acts on left multiplication of  $g$  on incoming arrows, and right multiplication of  $g^{-1}$  on outgoing arrows. Now, it is easy to see that the star operators commute. We only need worry if the star operators overlap (if they don't overlap, they trivially commute):



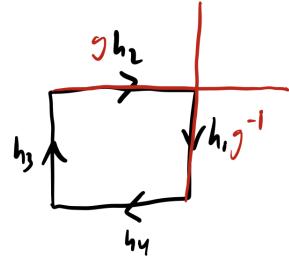
but then one will have an incoming arrow, one has an outgoing arrow (note this is true independent of any choice of orientation!). Since left and right multiplication commute,  $[A_g(s), A_h(s')] = 0$  if  $s \neq s'$  (if on the same site  $gh \neq hg$  in general). Thus,  $[A(s), A(s')] = 0$  for all  $s, s'$  (the case where  $s = s'$  is trivial in this case - its the same operator!)

4.  $[B(p), B(p')] = 0$  is obvious; they are all diagonal in the group element basis.
5. First, we determine how to calculate the action of plaquette operator for this orientation:

$$\begin{aligned}
B(p) \left| \begin{array}{c} h_2 \\ \nearrow p \searrow h_1 \\ h_3 \\ \downarrow h_4 \end{array} \right\rangle &= B(p) \left| \begin{array}{c} h_2 \\ \nearrow h_3^{-1} \searrow h_4^{-1} \\ h_3 \\ \downarrow h_4 \end{array} \right\rangle \\
&= \delta_{h_1^{-1}h_2h_3h_4, 1} \left| \begin{array}{c} h_2 \\ \nearrow h_3^{-1} \searrow h_4^{-1} \\ h_3 \\ \downarrow h_4 \end{array} \right\rangle \\
&= \delta_{h_1^{-1}h_2h_3h_4, 1} \left| \begin{array}{c} h_2 \\ \nearrow h_3^{-1} \searrow h_4^{-1} \\ h_3 \\ \downarrow h_4 \end{array} \right\rangle
\end{aligned}$$

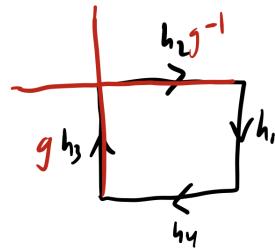
where we can see if that we go against the arrow, we multiply by the inverse of the group element in the kronecker delta instead.

Now for the commutation argument. If the star and plaquette have no overlap the commutation is trivial. What about when they do overlap? We have 4 cases to check. For when they overlap at the top right corner, the  $A_g$  left and right multiplies in a way such that things cancel:



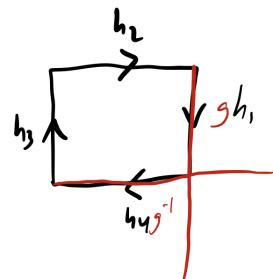
$$h_1 h_2 h_3 h_4 \xrightarrow{A_g} (h_1 g^{-1})(g h_2) h_3 h_4 = h_1 h_2 h_3 h_4 \quad (7.12)$$

For the case when they overlap at the top left corner:



$$h_1 h_2 h_3 h_4 \xrightarrow{A_g} h_1 (h_2 g^{-1})(g h_3) h_4 = h_1 h_2 h_3 h_4 \quad (7.13)$$

the bottom left corner is exactly the same. The most interesting case is the bottom right corner; this is interesting case because this is our chosen basepoint from which we are measuring the flux:



$$h_1 h_2 h_3 h_4 \xrightarrow{A_g} (gh_1) h_2 h_3 (h_4 g^{-1}) = g(h_1 h_2 h_3 h_4)g^{-1} \quad (7.14)$$

So  $A_g$  conjugates the flux via a group element  $g$  in this last interesting case. So, in all cases  $A_g$  always preserves the conjugacy class of the flux  $h_1 h_2 h_3 h_4$ . Thus:

$$[A_g(s), B(p)] = 0 \quad (7.15)$$

as 1 (and 0) are invariant under conjugation. Thus  $[A(s), B(p)] = 0$ .

## 7.4 Solution to the Quantum Double Model

Since  $\{A(s), B(p)\}$  are commuting projectors, i.e. have eigenvalues 0, 1, the ground states correspond to states  $|\Omega\rangle$  with:

$$A(s)|\Omega\rangle = B(p)|\Omega\rangle = |\Omega\rangle \quad (7.16)$$

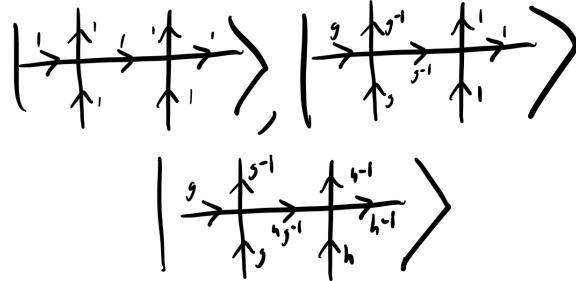
Our central question to answer is how many states  $|\Omega\rangle$  that satisfy the above?

We consider the infinite plane geometry. From the Euler characteristic we intuit that there is a unique ground state in this case. We work in the  $|g\rangle$  basis. Then:

$$B(p)|\Omega\rangle = |\Omega\rangle \quad (7.17)$$

implies that  $|\Omega\rangle$  is a sum of states with “vanishing flux”, i.e.  $h_1 h_2 h_3 h_4 = 1$ . for example, on a very small lattice:

So there are many states with vanishing flux, and we can of course take linear combinations of them, so we have a linear combination with undetermined coefficients. But, we will then find that the constraint that  $A(s) = 1$  everywhere enforces that all of the coefficients to be equal weight.



## 8 Quantum Double Model III

### 8.1 Ground State of QD Model

We recall the QD model with group  $G$ , with local Hilbert spaces on edges of a lattice of dimension  $|G|$ , and Hamiltonian given by:

$$H = - \sum_s A(s) - \sum_p B(p) \quad (8.1)$$

with the star term:

$$A(s) = \frac{1}{|G|} \sum_g A_g(s) \quad (8.2)$$

and  $A_g(s)$  corresponding to a left-multiplication of  $g$  on each of the inward pointing legs, and  $B(p)$  the plaquette term the projector onto the subspace with  $h_1 h_2 h_3 h_4 = 1$  (product of group elements around the plaquette) - whether the flux through the plaquette is vanishing/1 (we can also consider  $B_h(p)$  which checks that the flux is equal to a group element  $h$ ). We showed last time that the  $\{A(s), B(p)\}$  are all projectors and mutually commute. It is thus easy to find the ground states:

$$A(s)|\Omega\rangle = B(p)|\Omega\rangle = |\Omega\rangle \quad (8.3)$$

i.e. they should be the +1 eigenstates of all of the commuting projectors in order to be the lowest energy eigenstate. Now, we can ask; how many ground states are there in the infinite plane geometry?

$$B(p)|\Omega\rangle = |\Omega\rangle \implies |\Omega\rangle \text{ is a sum of states } |\{g_j\}\rangle \text{ with vanishing (= 1) flux} \quad (8.4)$$

At the end of last lecture we wrote down a couple examples of such vanishing flux states. The statement we make above is that we can form any arbitrary linear combination of vanishing flux states, and they will have  $B(p)|\Omega\rangle = (+1)|\Omega\rangle$  for all  $p$ . For now, the coefficients in this linear combinations are arbitrary. This will fix the coefficients (very similar to what we did in the toric code case).

We now want to impose the constraint  $A(s)|\Omega\rangle = |\Omega\rangle$ . We now make a claim that this is equivalent to saying that  $A_g(s)|\Omega\rangle = |\Omega\rangle$  for all  $g$ . The  $A_g(s) = +1 \implies A(s) = +1$  direction is clear. The other direction is not quite as obvious, but it is a straightforward to show, and you will do it on the homework.

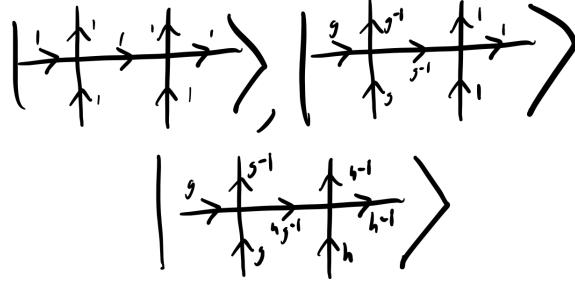
Because  $A_g(s)$  imposes a gauge transformation, it being +1 implies that the amplitude of the pre-gauge transformed and post-transformed states must be the same in the states. More concretely,  $A_g(s)|\Omega\rangle = |\Omega\rangle$  implies that:

$$\langle \{g_j\} | A_g(s) | \Omega \rangle = \langle \{g_j\} | \Omega \rangle \quad (8.5)$$

acting on the left with  $A_g(s)$ :

$$\langle \{g'_j\} | A_g(s) | \Omega \rangle = \langle \{g_j\} | \Omega \rangle \quad (8.6)$$

and hence any two basis states related by a gauge transformation/ $A_g(s)$  must have the same amplitude if we are to have  $A(s)|\Omega\rangle = |\Omega\rangle$ . Now, the key observation is that all of the zero flux states are related by a gauge transformation (the gauge transformations are ergodic); if we again look at our examples:



this becomes clear. The second state is related by a  $g$  transformation to the first, and the third state is related to the second by a  $h$  transformation. Note that this observation that all zero-flux states are connectable is specific to the infinite plane geometry.

The conclusion of the arguments;  $|\Omega\rangle$  is an equal weight superposition of all vanishing flux states<sup>4</sup>

$$|\Omega\rangle = \sum_{\{g_j\} \text{ vanishing flux}} |\{g_j\}\rangle \quad (8.7)$$

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<sup>4</sup>analogous to the Toric code result where we had an equal weight superposition of all closed loop states

A notable comment - out of this analysis, we have found that the ground state is unique. Furthermore, the lowest excited states have  $A(s) = 0$  or  $B(p) = 0$  for some  $s, p$ . The energy gap (though we do not show it here) is therefore  $\Delta = 1$ . So, this is another example of a gapped Hamiltonian, and this is the setting where the notion of anyons are well-defined.

## 8.2 General Definition of Anyons

This is one of the most interesting aspects of the quantum double model. In particular, it is a nice toy model for introducing non-abelian anyons. We have associated anyons with excitations of a Hamiltonian, but this is a bit of a misnomer. Let's more precisely define it.

*Definition (Anyons).* An “anyon excitation” of a gapped (local) Hamiltonian  $H_0$  is any state which is the unique ground state of the of a Hamiltonian of the following form:

$$H = H_0 + V \quad (8.8)$$

with  $V$  a local (Hermitian) operator. We can call this a trapping potential<sup>5</sup>.

The physical idea - if  $H_0$  has ground state  $|\Omega\rangle$ , then the ground state of  $H$  looks like  $|\Omega\rangle$  plus a perturbation/localized defect near  $V$ . We can trap these excitations locally, but we *cannot* create them locally.

*Definition (Equivalent anyons).* Two anyon excitations  $|\psi\rangle, |\psi'\rangle$  (corresponding to potentials  $V, V'$ ) are equivalent/the same (topological) “type” if  $|\psi\rangle = U|\psi'\rangle$  for some local unitary  $U$  (supported near  $V$ ).

We can see that this is a reasonable definition by looking at the  $e, m$  anyons of the toric code. There, we have different choices of  $V$  we can take:

- ( $V = 0$ ):  $H = -\sum_s A_s - \sum_p B_p$  (“1” anyon)
- ( $V = 2A_{s_0}$ ):  $H = -\sum_{s \neq s_0} A_s - \sum_p B_p + A_{s_0}$  (“ $e$ ” anyon)
- ( $V = 2B_{p_0}$ ):  $H = -\sum_s A_s - \sum_{p \neq p_0} B_p + B_{p_0}$  (“ $m$ ” anyon)
- ( $V = 2A_{s_0} + 2B_{p_0}$ ):  $H = -\sum_{s \neq s_0} A_s - \sum_{p \neq p_0} B_p + A_{s_0} + B_{p_0}$  (“ $\epsilon$ ” anyon)

Every other kind of anyon we could create are equivalent up to a local unitary  $U$ . For example two “ $e$ ”s are equivalent to the trivial anyon via a local string operator connecting them.

## 8.3 Flux anyons of Quantum Double Model

The QD model will turn out to have analogs of all of the toric code excitations. We will start by thinking about the flux excitations (generalization of toric code “ $m$ ”).

First, we can ask how many types of flux excitations there are. Claim: For a general group  $G$ , there is one type of flux excitation for every non-trivial conjugacy class  $C \subset G$  (the equivalence classes of a group under the relation of conjugation). For each one of these classes, we can construct a distinct flux excitation.

We will proceed by defining a different trapping potential  $V$  for each conjugacy class and showing that the  $H_0 + V$  has a unique ground state, which cannot be related to each other via local operation.

We consider the modified quantum double Hamiltonian:

$$H = -\sum_s A(s) - \sum_{p \neq p_0} B(p) - B_C(p_0) \quad (8.9)$$

So the trapping potential is:

$$V(p_0) = B(p_0) - B_C(p_0) \quad (8.10)$$

with  $B_C(p_0)$  defined for each conjugacy class  $C \subseteq G$ :

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<sup>5</sup>We think about the infinite plane here, where we have one anyon somewhere and another anyon at infinity. We could also think about this with a pair of anyons created by two trapping potentials.

$$B_C(p) \left| h_3 \begin{array}{c} \nearrow h_2 \\ \downarrow p \\ \swarrow h_4 \end{array} h_1 \right\rangle = \sum_{h_1, h_2, h_3, h_4 \in C} \left| h_3 \begin{array}{c} \nearrow h_2 \\ \downarrow p \\ \swarrow h_4 \end{array} h_1 \right\rangle$$

In other words,  $B_C(p)$  measures the flux through  $p_0$  and checks that the flux lives in the conjugacy class  $C$ . Note that  $B_C(p_0)$  also commutes with  $A(s)$  (how to see this?  $A(s)$  preserves and/or preserves the flux, and thus commutes with  $B_C(p_0)$  which only cares about the conjugacy class of the flux).

So, we can think about the ground state of the Hamiltonian of Eq. (8.9) as we did for the quantum double model. In particular the analysis of the plaquette operators goes through in the same way, save for the fact that we now require that our states in our superposition all have nonzero flux through  $p_0$ . Then the  $A(s)$  constraints tell us that the states in this superposition have equal weight. We thus again have a unique ground state:

$$|C\rangle = \sum_{\{g_j\} \text{ flux } C \text{ through } p_0, 1 \text{ elsewhere}} |\{g_j\}\rangle \quad (8.11)$$

We could ask why didn't we just look for an excited state in the original Hamiltonian. The idea is that all conjugacy class eigenstates have the same excited state energy - there is degeneracy in the excited states. But via the local trapping potential definition we split this degeneracy and can identify the distinct anyon excitations.

We need to now argue that different choices of  $C$  give rise to distinct excitations, in the sense that they cannot be connected via a local unitary:

$$|C'\rangle \neq U|C\rangle \quad (\text{if } C' \neq C) \quad (8.12)$$

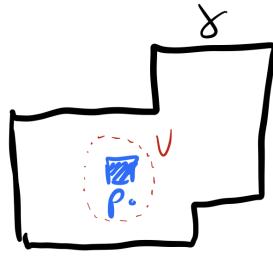
To see this, we define the operator  $B_C(\gamma)$ , which looks at the conjugacy class of a flux around a large loop:

$$B_C(\gamma) \left| \begin{array}{c} \nearrow \gamma \\ \text{large loop} \\ \swarrow h_1 \\ \downarrow h_2 \\ \uparrow h_3 \end{array} \right\rangle = \sum_{h_1, \dots, h_n \in C} \left| \begin{array}{c} \nearrow \gamma \\ \text{large loop} \\ \swarrow h_1 \\ \downarrow h_2 \\ \uparrow h_3 \end{array} \right\rangle$$

Then, it follows that:

$$B_C(\gamma)|C\rangle = |C\rangle \quad (8.13)$$

Which follows from a similar kind of Stokes' theorem that we saw in the toric code case. It can measure the flux inside of the large curve:



But if we have a different conjugacy class:

$$B_C(\gamma)|C'\rangle = 0 \quad (8.14)$$

because it measures the flux inside and sees that it is not equal to  $C$ . This is sufficient to show that  $|C\rangle, |C'\rangle$  cannot be locally connected by a local  $U$ :

$$|C'\rangle \neq U|C\rangle. \quad (8.15)$$

Because  $\gamma$  can be arbitrarily large, and thus have no overlap with a local  $U$  (and hence commute); if the above were true, then:

$$B_C(\gamma)|C'\rangle = B_C(\gamma)U|C\rangle = UB_C(\gamma)|C\rangle = U|C\rangle \neq 0 \quad (8.16)$$

contradiction!

Next time - we will think about trapping  $n$  fluxes (rather than trapping a single flux). And then we will see that this has multiple degenerate ground states, which will be the foundation for non-abelian anyons.

## 9 Quantum Double Model IV

### 9.1 Review - Flux Excitations in QD Model

Last time, we introduced the idea that there is one type of flux excitation for every non-trivial conjugacy class  $C \subseteq G$ . We constructed the explicit states:

$$|C\rangle = \sum_{\{g_j\}} |\{g_j\}\rangle \quad (9.1)$$

with  $\{g_j\}$  such that we have flux  $C$  through plaquette  $p_0$  and 1/no flux elsewhere. This is analogous to the ground state of the original Hamiltonian, except we enforce the condition where there is a nontrivial flux through one plaquette.

In arguing for this, we explained how anyons should be the unique ground state of a trapping potential. In particular, the  $|C\rangle$  above is the unique ground state of:

$$H = - \sum_s A(s) - \sum_{p \neq p_0} B(p) - B_C(p_0) \quad (9.2)$$

with  $B_C(p)$  defined as:

$$B_C(p) |h_3 \xrightarrow{h_2} \xleftarrow{h_4} h_1\rangle = \delta_{h_1, h_2, h_3, h_4 \in C} |h_3 \xrightarrow{h_2} \xleftarrow{h_4} h_1\rangle$$

Note that the  $H$  above is a commuting projector Hamiltonian, as  $B_c(p_0)$  commutes with all the other terms. We showed that this is indeed the unique eigenstate by arguing that  $B_C(\gamma)$  for a sufficiently large loop  $\gamma$  around the flux can detect it, but must commute with any local operator around such a flux.

## 9.2 Multiple Fluxes in QD Model

We can easily generalize the above trapping Hamiltonian to that which traps several fluxes:

$$H = - \sum_s A(s) - \sum_{p \neq p_1, \dots, p_n} B(p) - \sum_{i=1}^n B_{C_i}(p_i) \quad (9.3)$$



interestingly, we will find that in general  $H$  has multiple degenerate ground states. To construct them, we choose different group elements in the different conjugacy classes,  $g_i \in C_i$ , with the constraint:

$$g_1 g_2 \dots g_n = 1 \quad (9.4)$$

This constraint corresponds to ground states having trivial topological charge (and corresponds to the ability to create the ground state via a local term) - we will come back to this. We now construct the ground states in two steps. First, define the basis states:

which produces the fluxes at the desired locations, and then we have branch cuts of non-trivial group elements on links such that all other fluxes are trivial ( $g_i$  on the vertical columns, and then  $g_1 \dots g_{i-1}$  on the horizontal sections).

By construction, the flux through each  $p_i$  is  $g_i \in C_i$ :

$$B_c(p_i)|g_1, \dots, g_n\rangle_0 = |g_1, \dots, g_n\rangle_0 \quad (9.5)$$

and the flux through all other plaquettes is also 1:

$$B(p)|g_1, \dots, g_n\rangle_0 = |g_1, \dots, g_n\rangle_0 \quad p \neq p_1, \dots, p_n \quad (9.6)$$

So, so far the plaquette operators like this state. But the star operators may not. We define the actual ground state to be:

$$|g_1, \dots, g_n\rangle = \prod_s A(s) |g_1, \dots, g_n\rangle_0 \quad (9.7)$$

which is the projection to the  $(s) = 1$  subspace. To get a flavour for what the projection does, we remember the definition of  $A(s)$  as a sum of  $A_g(s)$ s; so:

$$|g_1, \dots, g_n\rangle = \prod_s \left( \frac{1}{|G|} \sum_g A_g(s) \right) |g_1, \dots, g_n\rangle_0 \quad (9.8)$$

So it sums over all possible configurations that can be obtained by  $|g_1, \dots, g_n\rangle_0$  via  $A_g(s)$ s. Because we have now projected into the  $A(s) = 1$  subspace (note; there is no fear that the  $A(s)$  could destroy the state, because they only have positive matrix elements), by construction these states will be eigenstates of  $A(s)$ :

$$A(s) |g_1, \dots, g_n\rangle = |g_1, \dots, g_n\rangle \quad (9.9)$$

and because all of the terms in the Hamiltonian commute, the fact that these are still eigenstates of the  $B$  operators does not change:

$$B(p) |g_1, \dots, g_n\rangle = |g_1, \dots, g_n\rangle \quad p \neq p_1, \dots, p_n \quad (9.10)$$

$$B_{C_i}(p_i) |g_1, \dots, g_n\rangle = |g_1, \dots, g_n\rangle \quad (9.11)$$

so indeed,  $|g_1, \dots, g_n\rangle$  are eigenstates of every operator in  $H$  with eigenvalue +1, so these are indeed ground states of  $H$ .

A side note; we could write the initial state  $|C\rangle$  in this language, just imagining that the cut of non-trivial edges goes off to  $\infty$ .

### 9.3 Ground states - redundancy and completeness

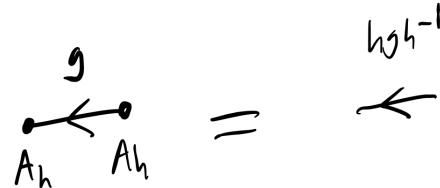
So, we've found some ground states. But these states are actually not all distinct, there is some redundancy. This is because:

$$|g_1, g_2, \dots, g_n\rangle = |hg_1h^{-1}, hg_2h^{-1}, \dots, hg_nh^{-1}\rangle. \quad (9.12)$$

To see this, we note that the  $|g_1, g_2, \dots, g_n\rangle_0$  states are related by a uniform gauge transformation.

$$\prod_s A_h(s) |g_1, g_2, \dots, g_n\rangle_0 = |hg_1h^{-1}, hg_2h^{-1}, \dots, hg_nh^{-1}\rangle_0 \quad (9.13)$$

This is evident from looking at the pictorial definition of the  $|g_1, g_2, \dots, g_n\rangle_0$  states. Indeed, every link gets conjugated by  $h$ :



Because the unprojected states are related by a gauge transformation , when we project onto all the  $A(s)$ s (and sum over all gauge equivalent combinations) we get the same answer. To see this formally:

$$\begin{aligned}
|hg_1h^{-1}, hg_2h^{-1}, \dots, hg_nh^{-1}\rangle &= \prod_s \left( \frac{1}{|G|} \sum_g A_g(s) \right) |hg_1h^{-1}, hg_2h^{-1}, \dots, hg_nh^{-1}\rangle_0 \\
&= \prod_s \left( \frac{1}{|G|} \sum_g A_g(s) \right) \prod_s A_h(s) |g_1, g_2, \dots, g_n\rangle_0 \\
&= \prod_s \left( \frac{1}{|G|} \sum_g A_g(s) A_h(s) \right) |g_1, g_2, \dots, g_n\rangle_0 \\
&= \prod_s \left( \frac{1}{|G|} \sum_g A_{gh}(s) \right) |g_1, g_2, \dots, g_n\rangle_0 \\
&= \prod_s \left( \frac{1}{|G|} \sum_g A_g(s) \right) |g_1, \dots, g_n\rangle_0 \\
&= |g_1, \dots, g_n\rangle
\end{aligned} \tag{9.14}$$

where we have used that the group maps to itself under multiplication in the second to last step.

It is not hard to see that this is the only redundancy. Thus:

$$\langle g_1, \dots, g_n | g'_1, \dots, g'_n \rangle = \delta_{g'_i = hg_i h^{-1} \forall i} \quad \text{for some } h \in G \tag{9.15}$$

The intuition is that only a uniform gauge transformation can map between the different  $|g_1, g_2, \dots, g_n\rangle_0$  states (else, we get a mismatch on the trivial links, no longer making them trivial). The ground states are sums over gauge configurations, and as such the gauge orbits of non-gauge equivalent  $|g_1, g_2, \dots, g_n\rangle_0$  must be non-overlapping and hence orthogonal.

The punchline: There is a distinct ground state for every  $(g_1, \dots, g_n)$  (ordered list) with  $g_1 g_2 \dots g_n = 1$  (allowing this state to be created locally), modulo uniform conjugation.

The next question we can ask is - are these all of the ground states? The answer is yes, at least in a sense. These are all the ground states that can be created from  $|\Omega\rangle$  (the GS of the original QD model) via an operator acting in a finite region, i.e. around  $p_1, \dots, p_n$ . It should be clear that we can create these states locally, the converse (that these states consist of all such states) has not been shown explicitly, but is true. In other words, these are the full set of all ground states with trivial “total topological charge”.

## 9.4 Ground state properties

We label distinct  $|g_1, \dots, g_n\rangle$  states by:

$$\{|\alpha\rangle, \alpha = 1, \dots, D\} \tag{9.16}$$

1. There is an exponentially large ground state degeneracy if conjugacy classes have more than one element. For example take  $G = S_3 = \{\mathbb{I}, (12), (13), (23), (123), (132)\}$ . We have three conjugacy classes:

$$C_1 = \{\mathbb{I}\}, \quad C_2 = \{(12), (13), (23)\}, \quad C_3 = \{(123), (132)\} \tag{9.17}$$

For  $C_2$ , we have:

$$D_n = \frac{3^{n-1} + 3}{6} \tag{9.18}$$

with  $n$  - the number of fluxes - here even (you can verify that for  $C_2$ , no odd number of fluxes can multiply to the identity, as we require).

More generally, if all  $C_i = C$ , then:

$$D_n \sim |C|^n \cdot \text{const. as } n \rightarrow \infty \quad (9.19)$$

2. The ground states are locally indistinguishable. For any  $O$  supported on less than  $L$  sites where  $L = \min_{i,j} \text{dist}(p_i, p_j)$ , then:

$$\langle \alpha | O | \beta \rangle = c \delta_{\alpha\beta} \quad (9.20)$$

Suppose we have  $|g_1, \dots, g_n\rangle, |g'_1, \dots, g'_n\rangle$ . So we might say that there is the ability to distinguish at  $p_1$ . But for any given flux, I can via conjugation make the flux at  $g_1$  look the same  $g'_1 \rightarrow hg'_1h^{-1} = g_1$ . So, we need to be able to see more than one flux.

3. The ground state degeneracy is robust to small local perturbations of  $H$ . If we take  $H \rightarrow H + \lambda V$ , the splitting is  $\delta \sim e^{-\text{const.} L}$ . Roughly (as in the toric code case) it follows from property 2, where we need order  $L$  perturbation theory to connect states in the ground space.

One last comment - we see the above features, and these are all generic/defining features of non-abelian anyons. Next time, we will look more closely at the non-Abelian nature of these objects, and braid the non-Abelian anyons and look at their braid matrices.

## 10 Quantum Double Model V

### 10.1 Review - multiple flux excitations and GSD

Last time, we supposed we had trapped  $n$  flux excitations  $c_1, \dots, c_n$  at plaquettes  $p_1, \dots, p_n$ . What we showed last time was - perhaps unexpectedly - the Hamiltonian had multiple degenerate ground states, labelled by ordered  $n$ -tuples:

$$\{(g_1, \dots, g_n) : g_i \in C_i, g_1 \dots g_n = 1\} \quad (10.1)$$

we found a ground state for each of these  $n$ -tuples, modulo uniform conjugation:

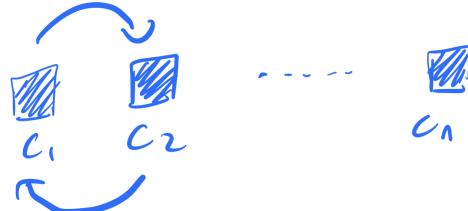
$$|g_1, \dots, g_n\rangle = |hg_1h^{-1}, \dots, hg_nh^{-1}\rangle. \quad (10.2)$$

These are all ground states with trivial total topological charge (which means that we are thinking about ground states that can be created by some local operation).

One point that we want to emphasize from last time - the GSD is slightly special, in the sense that all of the degenerate states are locally indistinguishable. The simplest measurement we could do to distinguish the states involves a string operator that would go around more than one plaquette. Additionally, the size of the degeneracy does not go as a product of the internal degrees of freedom; there is a collective degeneracy coming from defects, here a power of the size of the conjugacy class. This peculiar GSD is the defining characteristic of non-Abelian anyons.

### 10.2 Braiding of fluxes

Let us have  $n$  fluxes  $c_1, \dots, c_n$  and then exchange  $c_1, c_2$  (by some adiabatic evolution).



This braiding operation (since we do it adiabatically) leads to another ground state. If the fluxes are of the same type, then we get the same ground state as we started with, if the fluxes are of different types, we will generically have a different ground state. Either way, we end up with the ground state of a new Hamiltonian:

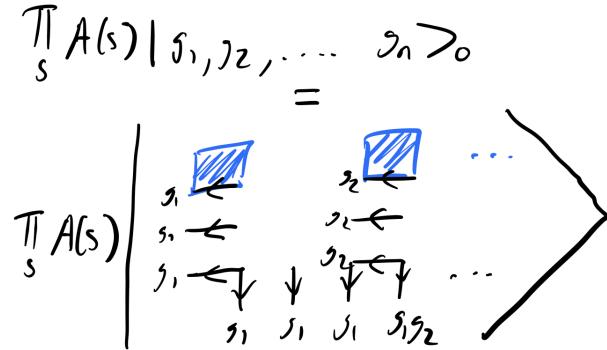
$$|g_1, g_2, \dots, g_n\rangle \xrightarrow{\text{braid}} \sum_{g'_i} (\text{coeff.}) |g'_1, g'_2, \dots, g'_n\rangle \quad (10.3)$$

Let us compute the RHS, up to a phase. With abelian anyons, we get dynamical phases (separate from the topological Berry phase). Here, we will not worry so much about it because we are dealing with Abelian anyons. So, all the details about, e.g., subtraction schemes we need not concern ourselves with.

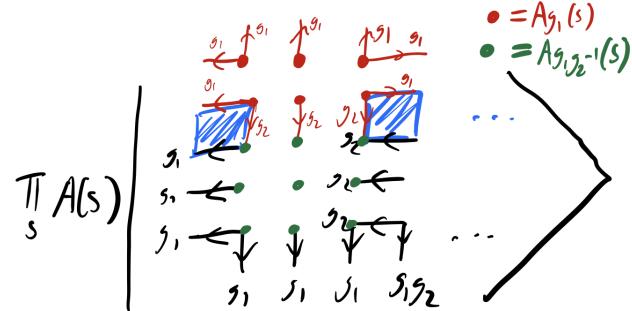
Let's recall the definition:

$$|g_1, \dots, g_n\rangle = \prod_s A(s) |g_1, \dots, g_n\rangle_0 \quad (10.4)$$

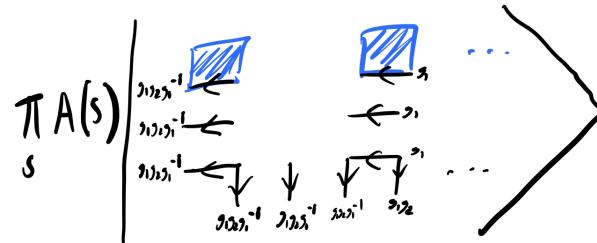
Where (focusing on the first two fluxes):



The final state we can just intuitively write down/guess on the level of the unprojected state, by imagining the operations it would require to move the fluxes in the unprojected setup:



The second picture is gauge equivalent to the first, up to an  $A_{g_1}(s)$  on the red dots and  $A_{g_1 g_2^{-1}}(s)$  on the green. This gives us:



Thus, using our notation for labelling these states, the final result is:

$$\prod_s A(s) |g_1 g_2 g_1^{-1}, g_1, \dots, g_n\rangle_0 = |g_1 g_2 g_1^{-1}, g_1, \dots, g_n\rangle. \quad (10.5)$$

Note that in the Abelian case,  $g_1 g_2 g_1^{-1} = g_2$  so we just get the swap of  $g_1, g_2$ .

More generally, if we exchange fluxes  $i, i+1$ :

$$|\dots, g_i, g_{i+1}, \dots\rangle \rightarrow |\dots, g_i g_{i+1} g_i^{-1}, g_i, \dots\rangle \quad (10.6)$$

which completely defines the braiding of fluxes.

A consistency check; the  $g_1 \dots g_n = 1$  condition must be preserved. Indeed, the exchange (which swaps  $g_i, g_{i+1}$  and then conjugates  $g_{i+1}$ ) preserves this product.

For simplicity, let us specialize to the case where all the fluxes are the same,  $C_1 = C_2 = \dots = C_n = C$ . Now, the Hamiltonian returns to itself, so we can view each braid as defining a  $D \times D$  unitary transformation on the set of degenerate of  $D$  degenerate ground states (of a fixed Hamiltonian). These define what are known as the *braid matrices*. Eq. (10.6) gives us an explicit formula for the braid matrices corresponding to clockwise exchanges. Together with their inverses, clockwise exchanges are the generators of the braid group, so this fully defines braiding.

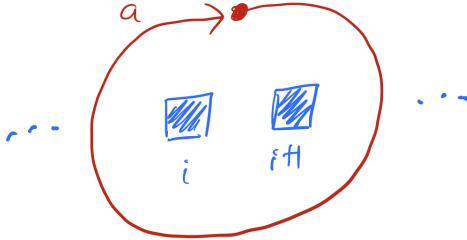
Two properties of (flux) braid matrices:

1. Braid matrices are all permutation matrices, in the  $|g_1, \dots, g_n\rangle$  basis. Acting on a particular basis element, we get a single basis element back (as opposed to a linear combination of such basis elements). This is specific to fluxes.
2. Braid matrices generally do not commute with each other. The outcome is dependent on ordering.

### 10.3 Measurement of total topological charge

We've discussed the unitary operations we can perform on the non-Abelian anyons (braiding), but we should also think about the possible kinds of measurement we can do to distinguish such states. The simplest such measurement is of total topological charge.

We say that two fluxes have trivial total topological charge if braiding with any anyon  $a$  is trivial, i.e.  $U_{\text{braid}} = \mathbb{I}$  for the braiding operation shown below:



Here,  $a$  could be a flux or a charge. The other way to define trivial topological charge is whether the two fluxes can be created locally from the vacuum/ground state. But the former is a bit more operational, because it gives us a physical way to measure the topological charge (by measuring the Non-Abelian Berry phase). This is the way in which we can distinguish degenerate ground states with each other.

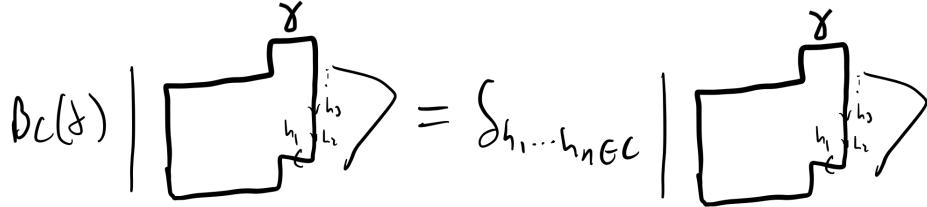
We can now ask - what is the projection  $P_{i,i+1}$  that projects onto the  $i, i+1$  fluxes having trivial topological charge?

The answer (which we will then motivate the correctness of):

$$P_{i,i+1}| \dots, g_i, g_{i+1}, \dots \rangle = \delta_{g_i g_{i+1}, 1} \cdot \frac{1}{|G|} \sum_h | \dots, h g_i h^{-1}, h g_{i+1} h^{-1}, \dots \rangle \quad (10.7)$$

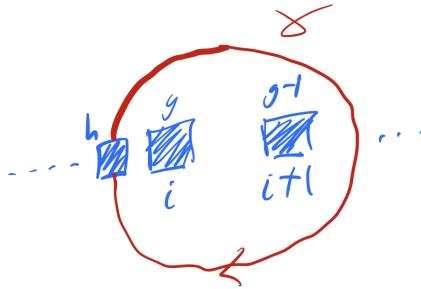
Where we note that only  $g_i, g_{i+1}$  get conjugated. Let us explain why we get the two factors.

1. Why the  $\delta_{g_i g_{i+1}, 1}$ ? This enforces  $g_i g_i + 1 = 1$ , which implies that  $B_1(\gamma)| \dots, g_i, g_{i+1}, \dots \rangle = | \dots, g_i, g_{i+1}, \dots \rangle$  where  $B_1(\gamma)$  is an operator that checks that the total flux is trivial inside of the loop (definition below, taking  $C = 1$ )



This is a necessary condition to have trivial topological charge. It is 1 in the ground state, so if I am able to create the flux excitations locally, it also better be 1. It also has to do with the braiding of anyons, specifically charges;  $B_1(\gamma) = 1$  implies we have trivial braiding with all charge excitations.

2. Why a sum over conjugates of  $g_i, g_{i+1}$ ? It comes from wanting to have trivial braiding with fluxes - we imagine braiding a flux  $h$  around  $g_i, g_{i+1}$ . Assume that we have already fulfilled the  $g_i g_{i+1} = 1$ , so then  $g_i = g_{i+1}^{-1} = g$ .



It is then clear that we need the braiding process  $\sigma_1 \sigma_2 \sigma_2 \sigma_1$  to get the full braid of  $h$  around the two fluxes:



Then it is only a matter of using Eq. (10.6) repeatedly:

$$\begin{aligned}
 |\dots, h, g, g^{-1}, \dots\rangle &\xrightarrow{\sigma_1} |\dots, hgh^{-1}, h, g^{-1}, \dots\rangle \\
 &\xrightarrow{\sigma_2} |\dots, hgh^{-1}, hg^{-1}h^{-1}, h, \dots\rangle \\
 &\xrightarrow{\sigma_2} |\dots, hgh^{-1}, hg^{-1}h^{-1}h(hg^{-1}h^{-1})^{-1}, hg^{-1}h^{-1}, \dots\rangle = |\dots, hgh^{-1}, hg^{-1}hgh^{-1}, hg^{-1}h^{-1}, \dots\rangle \\
 &\xrightarrow{\sigma_1} |\dots, h, hgh^{-1}, hg^{-1}h^{-1}, \dots\rangle
 \end{aligned} \tag{10.8}$$

So; if we braid  $h$  around the two fluxes, nothing happens to the  $h$  (makes sense as the total flux of the two is trivial) and the  $g, g^{-1}$  fluxes get conjugated by  $h$ . The fact that we sum over all possible conjugates  $\sum_h |\dots, hg_i h^{-1}, hg_{i+1} h^{-1}, \dots\rangle$  guarantees that braiding with fluxes will be a trivial operation.

To summarize, there were two terms in the definition of our projection operation. The first term was there to have the braiding with charges to be trivial, the second term was there to have the braiding with fluxes to be trivial.

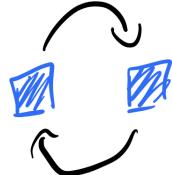
## 11 Topological Quantum Computation, Gapped Phases of Matter

Over the last few lectures, we have discussed Non-abelian anyons in the quantum double model and their properties. We looked at their braiding properties as well as the projective measurement of their total topological charge. These two ideas, when put together, allow us to carry out a quantum computation using Non-abelian anyons. This is a very elegant and new approach. The first proposal by Kitaev was in the context of the quantum double model, so this is where we too shall discuss it.

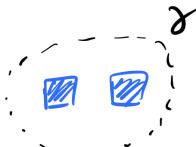
### 11.1 The ingredients

Consider the QD model for some non-abelian group  $G$ . Assume that we can perform the following operations:

1. We can create pairs of fluxes of each “type”, i.e. for each nontrivial conjugacy class.
2. We can braid pairs of (nearby) fluxes.



3. We can measure  $B_1(\gamma)$  for any pair of fluxes:

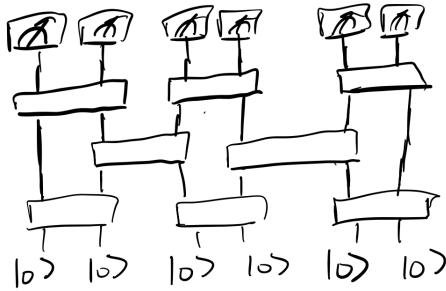


Physically, we can measure this by taking charge excitations and braiding them around the loop. So, we could replace this stipulation with the ability to braid charges around fluxes.

4. We can measure whether a pair of fluxes has trivial total topological charge. For this, we need to braid both charges and fluxes (and both have to be trivial). Physically we could imagine this as some adiabatic evolution.

3 is a subset/coarser measurement than 4. It only measures whether the braid with charges is trivial. 4 measures whether the braid with both charges and fluxes are trivial. Note that we do not need to know *what* the flux is, only whether it is trivial or not.

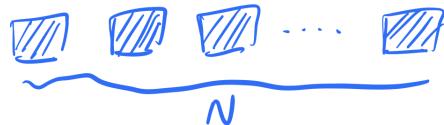
The claim: If  $G$  is a simple (that is,  $G$  does not have any nontrivial normal subgroup, where normal means to be preserved under conjugation) non-abelian group, then we can use efficiently simulate (there is a constant overhead to simulate each gate in the universal gateset via braiding) any quantum circuit with these operations.



As an example, the smallest simple non-abelian group is  $A_5$  (even permutations of 5 elements), with 60 elements. Note that this result has been extended, to smaller groups, e.g.  $S_3$ .

## 11.2 Idea

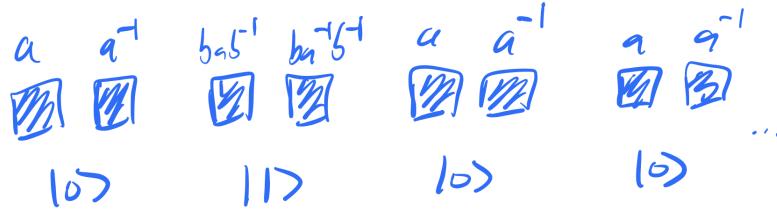
Consider  $N$  fluxes.



There is an exponentially large degeneracy  $\sim |C|^N$ . It turns out that we can define a convenient subspace (sometimes called the computational subspace) of dimension  $2^{N/2}$ . This allows us to represent  $N/2$  qubits. Specifically, choose two elements  $a, b \in G$ , with  $b^2 = 1$  and  $ab \neq ba$  (for a non-abelian simple group, we can always find such  $a, b$ ). Define:

$$|0\rangle = \begin{array}{c} a \\ \diagup \quad \diagdown \\ \blacksquare \quad \blacksquare^{-1} \end{array} \quad |1\rangle = \begin{array}{c} b \quad b^{-1} \\ \diagup \quad \diagdown \\ \blacksquare \quad \blacksquare^{-1} \end{array}$$

A typical computational state (say  $|0100\dots\rangle$ ) then looks like:



Note that there are a few ancillary fluxes<sup>6</sup> which fixes the issue of uniform conjugation which maps  $|0\rangle \leftrightarrow |1\rangle$  - these ancillary fluxes are the “reference fluxes” which allow us to tell apart  $|0\rangle$  and  $|1\rangle$ .

If the group is simple, we can perform the Toffoli gate - a particular 3 qubit gate - using braiding. Note that this is a classical gate universal for classical computation. Further, we can apply and measure single qubit Paulis  $X, Y, Z$ .

Measuring  $Z$  is simple; if we have a reference flux (say,  $a$ ), we can measure  $B_1(\gamma)$  around the reference flux and one half of the pairs of fluxes that make up  $|0\rangle$  or  $|1\rangle$ . For  $|0\rangle$  we find the flux is trivial and for  $|1\rangle$  we find the flux is nontrivial.

Measuring  $X$  has to do with measuring the total topological charge. If we are in the state  $|0\rangle - |1\rangle$ , we have a state which is orthogonal to the state which has trivial total topological charge, because  $|0\rangle, |1\rangle$  have trivial total topological charge, and only a symmetric combination of these will have trivial total topological charge.

Then  $\{\text{Toffoli}, X, Y, Z\}$  is a universal gate set for quantum computation, so we are done! For details, see John Preskill’s lecture notes on quantum computation (Chapter 9.11) <http://theory.caltech.edu/~preskill/ph219/topological.pdf>.

We don’t go through the gory details of this particular protocol, but we do comment that one can prove a similar result for many other types of non-Abelian anyons.

What is the advantage of this kind of model of QC? It is naturally protected against decoherence and errors, so long as anyons are far apart during braiding and measurement (and you work at sufficiently low temperature). In some sense, the quantum error correction is already baked in at the hardware level. This comes from the fact that the system is robust to local perturbations.

### 11.3 Defining gapped phases of matter

Thus far, we have been focusing on very specific models. But we may wonder to the extent to which the properties we have seen are general/persistent, e.g. under perturbations. To discuss this, we want to introduce the notion of a gapped phase of matter.

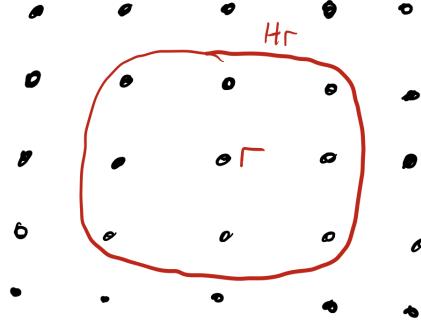
The setting will be qubits on a lattice with local interactions and an energy gap. What do we mean by “local” or “short-range”? The rough definition is that:

$$H = \sum_r H_r \quad (11.1)$$

where  $H_r$  is supported within a finite distance of  $r$ .

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<sup>6</sup>John Preskill calls this the “flux bureau of standards”.



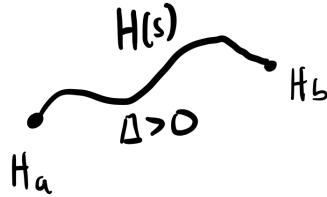
Strictly speaking, we will allow for something slightly more general. Instead of only allowing for strictly short range interactions, we will allow for interactions that decay super-polynomially (faster than any power law):

$$\|[H_r, O_{r'}]\| \leq \mathcal{O}(|r - r'|^{-\infty}) \quad (11.2)$$

where  $\leq \mathcal{O}(|r - r'|^{-\infty})$  means  $\leq \frac{C}{|r - r'|^n}$  for any  $n$ ,  $O_{r'}$  is some single-site operator at  $r'$ , and  $\|\cdot\|$  is the operator norm. The classic example is exponentially decaying tails.

By energy gap, we mean that in the thermodynamic limit (system size  $\rightarrow \infty$ ) we have a finite energy between the ground state(s) and the excited states of the system.

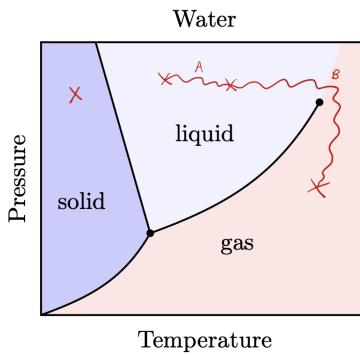
*Definition.* Two local, gapped Hamiltonians  $H_a, H_b$  belong to the same phase if there exists an interpolating family of Hamiltonians  $\{H(s) : 0 \leq s \leq 1\}$  with  $H(0) = H_a, H(1) = H(b)$  such that  $H(s)$  is local and gapped for all  $s$ .



Two comments about the definition:

- Analogy with the finite  $T$  definition of phases. Recall the phase diagram of water:

Here, two points are in the same phase so long as we can find a path connecting them such that the free energy is smooth/analytic along the path (at the phase transition, we have a non-analyticity). This idea of two points in a phase being connected by a “nice” path is analogous.



In the above diagram, two water points are in the same phase (path A). Water and steam are also in the same phase, as (going to high enough temperature) we can find a path that smoothly connects water and steam (path B). However, there is no path that connects ice with water without going through a phase transition, so these points must correspond to different phases. This points out one condition about the definition - it is easier to conclude that two things are in the same phase (because we need only find a single path that works) vs. things are in different phases (in which case we need to show that *all* paths cannot work).

- The existence of an interpolation implies  $H_a, H_b$  are adiabatically connected. We thus intuitively expect that  $H_a, H_b$  have the same physical properties, since we are able to continuously deform one system into the other.

This was the status of phases of matter > 20 years ago. But around 2 decades ago, people developed a way to make this intuition more precise, using tools from quantum information.

## 11.4 Local Unitary Transformations

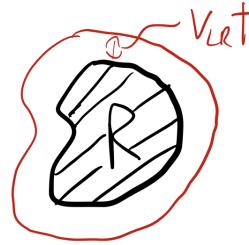
*Definition.* A local unitary transformation  $U$  is any unitary that can be generated by the time evolution of a local Hamiltonian (with local as we defined previous - either finite range or superpolynomially decaying interactions) over a finite time  $t$ . In other words, we can write it as the time ordered exponential:

$$U(T) = \mathcal{T} \exp(-i \int_0^T H(t) dt) \quad (11.3)$$

where  $H$  is local.

In what sense is this  $U$  local? Generated by something local - certainly, but a priori we cannot measure this. The more physical thing we can measure is the fact that it maps local operators to local operators. This is a result due to Lieb and Robinson.

*Theorem (Lieb-Robinson bound).* Let  $U = \mathcal{T} \exp(-i \int_0^T H(t) dt)$  with  $H(t)$  local. Let  $O$  be an operator supported in some region  $R$ . Then,  $U^\dagger O U$  is supported within distance  $v_{LR} T$  of  $R$ , with superpolynomially decaying tails.  $v_{LR}$  is the Lieb-Robinson velocity, and is determined by the range of interactions in  $H(t)$ .



Some intuition; if we imagine calculating  $U^\dagger O U$  for time-independent  $H$ , we want to calculate something like  $e^{iHT} O e^{-iHT}$ . We could then expand this out in a power series:

$$e^{iHT} O e^{-iHT} \approx O + iT[H, 0] - \frac{T^2}{2}[H, [H, O]] + \dots \quad (11.4)$$

wherein if  $H$  has short range interactions, each commutator in the power series spreads  $O$  slightly.

Next time, we will use these concepts to show that two  $H$  in the same phase have ground states that are related by a local unitary transformation.