PHYS 411 (Topics in Many-Body Dynamics) Notes

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Introduction:

This is a set of lecture notes taken from UChicago's PHYS 411 (Topics in Many-Body Dynamics), taught by Vincenzo Vitelli.

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1 Continuum Mechanics

1.1 Course Overview

This course is a topics course in many-body dynamics. What does that mean? It is inspired by a Polykov used to teach in Princeton, called "Topics", where he would arrive 10 minutes before and decide what would be an important topic in theoretical physics to teach. I (Vincenzo) don't have quite this level of improvisation, but this will largely be a course in research topics of non-equilibrium many-body physics/active matter.

We do have a broad theme - we want to treat various aspects of physics (mechanics, solid physics, phase transitions) from the perspective of dynamical systems. The formal pre-reqs are the intro soft matter course (currently being taught in parallel) as well as a course in statical field theory, but these are in some sense excessive - an enterprising student would be able to follow along, with some additional reading.

There is a recommended book (Soft Matter by Vitelli and others). The book will make the lectures look good in that the lectures will have better explanations. But since this course is being taught for the first time, there will be no lecture notes, so some sections of the book may be useful.

Assessment will be from problem sets (3 problem sets of 3-4 problems each). They may be difficult (they will be taken from the book) and long, but they will be stimulating and well written out. They will largely be mini research topics (it will be like working on research with a soft matter theorist like me (Vincenzo), or perhaps someone less crazy) - the grading will be lenient accordingly, as well as policies around deadlines, working together.

1.2 Continuum Mechanics Review

We will start by looking at equilibrium theory of physics, by looking at elasticity. This is normally a problem treated in the context of variational classical field theory. But we will approach this problem from a non-variational perspective (odd elasticity), and we will see interesting phenomena - namely chaos - emerge.
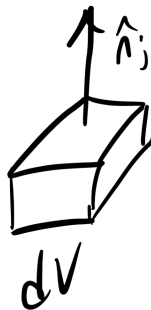
Our starting point is Newton's second law:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{1.1}$$

For continuous media, we can consider the force as coming from a volume integral of the pressure over space:

$$f_i = \frac{d}{dt} \iiint dV P_i = \oiint dA \sigma_{ij} n_j \tag{1.2}$$

where we have used Stoke's theorem to rewrite the volume integral as an integral over the area.

Therein, σ_{ij} is the linear momentum flux, or a rank-2 stress tensor. Note that throughout, we use Einstein summation convention where repeated indices are summed over (e.g. $u_{ij}x_j := \sum_j u_{ij}x_j$).

We can further write:

$$f_i = \iiint \frac{\partial \sigma_{ij}}{\partial x_j} \, dV \tag{1.3}$$

Another way to package this; we can write the force as a volume integral of a force density, which is a divergence of the stress tensor:

$$f_i = \iiint dV F_i \tag{1.4}$$

with:

$$\boxed{F_i = \partial_j \sigma_{ij}} \tag{1.5}$$

one can also take this as the definition of the stress tensor. If you have never seen the stress-tensor, it is recommended you read the first chapter of Soft Matter.

There is an assumption we made here. When we set the volume integral of $dV P_i$ equal to the area integral, we assumed that there were no sources. We will see what happens soon when this assumption is violated. For example we may have a solid immersed in a fluid, in which case Newton's third law may not hold and linear momentum is generically not conserved. One thing that this class will emphasize is what happens to the conclusions when assumptions are challenged!

Let's look at a differential of the force density:

$$dF_i = \sigma_{ij} n_j dA \tag{1.6}$$

As a concrete example, we can look at the forces acting on different faces of our cube:
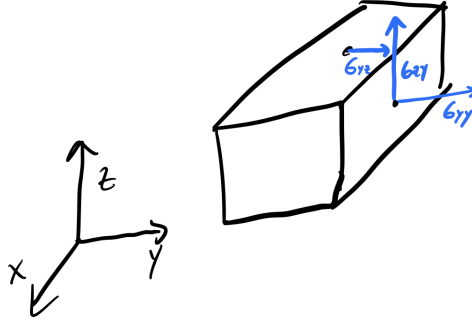
1.3 Constitutive Equation and Conservation Laws

We want a relationship between the stress tensor and the order parameter, relating to the displacements. We assume that there is translation invariance for now, so there is no dependence on the displacements themselves, only on gradients of the displacements (this is not true always, e.g. in the case of a solid in fluid, or a solid on a table with glue - these will be additional terms that we consider later). In linear response theory, we connect the stress and strain tensors (rank-2) via a stiffness tensor (rank-4):

$$\sigma_{ij} = K_{ijkl} \partial_k u_l = K_{ijkl} u_{kl} \tag{1.7}$$

The above is a linearized constitutive equation. As ugly as the K_{ijkl} above may be, forms the identity of the material. This is "Hooke's Law on steroids". Note that if we assume that microscopically a solid is composed of masses connected by Hookian springs with spring constant k, it is possible to derive the macroscopic stiffness tensor from the microscopic theory; this may be one of your problems later.

In 2-D, we have:

$$u_{kl} = \begin{pmatrix} \partial_x u_x & \partial_x u_y \\ \partial_y u_x & \partial_y u_y \end{pmatrix} \tag{1.8}$$

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$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \tag{1.9}$$

Note that we do not symmetrize here. In fact we can explicitly think about what happens in the cases when the off-diagonal entries are not equal. When $\sigma_{ij} \neq \sigma_{ji}$, it means that there is a non-vanishing net-torque (imagine back to the forces on the cube picture) and so the medium can spontaneously rotate. What does this teach us? Buried in the cumbersome algebraic structure of the rank-4 tensor, we see the presence and or violation of conservation laws. In this case, $\sigma_{ij} \neq \sigma_{ji}$ it implies that angular momentum is *not* conserved. Most books assume the fundamental conservation laws of nature hold and write down theories accordingly. But, we can allow for exotic mechanisms of the breaking of such fundamental laws, e.g. from interactions with external sources which give rise to effective theories.

We are rapidly marching through conservation laws; we have imposed conservation of linear momentum from the get-go, but we do not impose $\sigma_{ij} = \sigma_{ji}$/impose angular momentum conservation (If we did, then the rank-4 tensor would reduce to rank-3). This means that we have external torques present/sources of angular momentum. As we go on, we will study symmetries of our system through the symmetries of the (admittedly complicated) K_{ijkl} object, and understand the conservation laws through it.

Now, we *assume* that there is a potential energy. In Hooke's law, we assume $F = -\partial_x E$ with $E = \frac{1}{2}kx^2$ so we have a conservative force. Here, we don't work on the level of single particles, so we write the total potential energy as the integral over energy density:

$$U = \iiint d^3 x \epsilon(\mathbf{x}) \tag{1.10}$$

The question is now what is $\epsilon(\mathbf{x})$? If we want a linear force, we require that $\epsilon(\mathbf{x})$ be a quadratic form in $\mathbf{x}$:

$$\epsilon(\mathbf{x}) = \frac{1}{2} c_{ijkl} u_{ij}(\mathbf{x}) u_{kl}(\mathbf{x}) \tag{1.11}$$

Now, we want the force in the lth direction, which is a functional derivative of the energy density:

$$(\text{Force})_l = -\frac{\delta \epsilon}{\delta u_l} = -\left(\frac{\partial \epsilon}{\partial u_l} - \partial_k \frac{\partial \epsilon}{\partial u_{lk}} \right) \tag{1.12}$$

where again we recall $u_{lk} = \partial_l u_k$. Let's try to unweird this expression via analogy (maybe you want to explain to a friend, or pick up someone at a bar by telling them that you work on functional derivatives) - in classical mechanics, we have an action:

$$S = \int L(q(t)) dt \tag{1.13}$$

we then have the trajectory that is followed is that for which the functional derivative of the action vanishes[1]:

$$\frac{\delta S}{\delta q} = 0 \implies \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \tag{1.14}$$

And we can now see that the form of this expression is equivalent to the functional derivative of the energy density we wrote above.

Since we that the energy density cannot depend on displacement u_l/it only depend on gradients of displacement; hence the first term of the force expression vanishes:

$$(\text{Force})_l = \partial_k \frac{\partial \epsilon}{\partial u_{lk}} \tag{1.15}$$

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[1]Of course, this is something your date is expected to know.

Now if we recall Eq. (1.5) that the force density is the divergence of the stress tensor, we can write the above as:

$$\sigma_{lk} = \frac{\partial \epsilon}{\partial u_{lk}} \tag{1.16}$$

Now, if ϵ is quadratic in u, we will find that the relationship of stress and strain will be linear. The question then becomes; what is the relationship between K_{ijkl} and between c_{ijnm} (the coefficients of force and the coefficients of energy)? In the Hooke's law case they coincide and are both the spring constant k. However, this coincidence is only because we have a conservative force - we will see that even in this more general/continuum setting that if we have energy conservation the coefficients coincide, and in non-conservative settings they may differ.

Let us work through the algebra. Computing the derivative of ϵ:

$$\begin{aligned}\frac{\partial \epsilon}{\partial u_{pq}} &= \frac{1}{2} c_{ijkl} \left[\delta_{ip} \delta_{jq} u_{kl} + u_{ij} \delta_{pk} \delta_{ql} \right] \\ &= \frac{1}{2} \left[c_{pqkl} u_{kl} + c_{ijpq} u_{ij} \right] \\ &= \frac{1}{2} \left[c_{pqkl} u_{kl} + c_{klpq} \right] u_{kl} \end{aligned} \tag{1.17}$$

Note in the third equality we replace ij with kl; they are dummy/repeated/summed over indices and thus we can call them whatever we want[2]. Now, identifying:

$$K_{pqkl} = \frac{1}{2} [c_{pqkl} + c_{klpq}] \tag{1.18}$$

we have:

$$\sigma_{pq} = K_{pqkl} u_{kl} \tag{1.19}$$

And thus we manifestly see that the stiffness tensor is symmetric under interchange of pairs of indices, so long as the force is derived from a variation of a quadratic energy density:

$$\boxed{K_{pqkl} = K_{klpq}} \tag{1.20}$$

This relation is known as the Maxwell-Betti reciprocity. Later we will identify $\alpha := pq$ and $\beta := kl$ and better appreciate the symmetry algebraically.

1.4 Non-Conservative Forces

What if we have a world where we deform the material and through interaction with the environment, the system can suck up or expel energy? Then, we will find that work is not a state function, and:

$$\oint \mathbf{F} \cdot d\mathbf{l} \neq 0 \tag{1.21}$$

In this context, the linear stress-strain relation says *nothing* about the conservative nature of the forces, only that the forces are linear. It only says something about the energy with the constraint that the forces are conservative.

How do we think about the tensors in this setting? Recall that we can always write a tensor in terms of a symmetric and antisymmetric part, e.g.:

$$\sigma_{ij} = \sigma_{ij}^S + \sigma_{ij}^A \tag{1.22}$$

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[2]Name-calling is not good, but its ok with tensors - they don't mind.

with:

$$\sigma_{ij}^S = \frac{\sigma_{ij} + \sigma_{ji}}{2}, \quad \sigma_{ij}^S = \sigma_{ji}^S \tag{1.23}$$

$$\sigma_{ij}^A = \frac{\sigma_{ij} - \sigma_{ji}}{2}, \quad \sigma_{ij}^A = -\sigma_{ji}^A \tag{1.24}$$

here σ_{ij}^S conserves angular momentum, and the antisymmetric part is the culprit of angular momentum conservation violation. Next class, we will do a similar decoupling for our stiffness tensor:

$$K_{ijkl} = K_{ijkl}^e + K_{ijkl}^o \tag{1.25}$$

with:

$$K_{ijkl}^e = K_{klij}^e \tag{1.26}$$

$$K_{ijkl}^o = -K_{klij}^o \tag{1.27}$$

the odd/o part will be new moduli that emerge only when our system is open/not closed. The existence of this term is known as odd elasticity. This subject started in 2020, from a Nature Physics paper first authored by a grad student here! This is not just a mathematical possibility, but indeed appears in nature, e.g. in chiral tissues.

Next time, we will understand these tensors via group theory. We will then see what the operational meanings of the tensors are. We will discuss what predictions we can make given a set of symmetries and conservation laws.

2 Classification/Constraints of Elastic Moduli

For a less detailed discussion of what we discuss here, you can consult chapter 9 ("Active Matter") of the Soft Matter textbook.

2.1 Summary of Last Lecture

Last lecture, we studied solids, and the relationship between the stress and strain tensor:

$$\sigma_{ij} = K_{ijkl} \underbrace{u_{kl}}_{\partial_k u_l} \tag{2.1}$$

the two are related via the stiffness tensor K_{ijkl}, which summarizes the identity of the solid.

If the forces are conservative, then the stress is the variation of an energy density:

$$\sigma_{ij} = \frac{\partial \epsilon}{\partial u_{ij}} \tag{2.2}$$

And it follows that K_{ijkl} is symmetric under exchange of pairs of indices:

$$K_{ijkl} = K_{klij} \tag{2.3}$$

But, say we want to consider a medium where the forces are not conservative. Then neither of Eq. (2.2), (2.3) have to hold. In this more general case, we can split K_{ijkl} into an even and odd part:

$$K_{ijkl} = K_{ijkl}^e + K_{ijkl}^o \tag{2.4}$$

where even/odd means that the tensor acquires a ± sign under swap of pairs of indices. In a soft matter or elasticity course you have seen the first term - but the second term is novel, and comes up when we study open systems.

Since the forces are non-conservative, work is no longer a state function:

$$W_{AB} \neq U_B - U_A \tag{2.5}$$
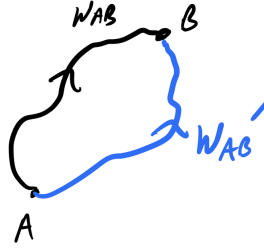
In this case, we have net work (positive or negative, depending on the sign) when we go around a cycle. The work done is rate-independent (no time derivatives here!). Further, the medium must be chiral because there is a difference in the sign of work depending on whether we traverse a loop clockwise or counterclockwise.

2.2 Representations of Stress/Strain Tensors

Let's try to classify all possibel entries of the stiffness tensor K_{ijkl} using the representations of σ and u; this will allow us to classify possible elastic moduli. In particular, let us try to classify odd elasticity K^o_{ijkl} in 2-dimensions. In 2-D, stress σ_{ij} and strain u_{kl} each have 4-independent components, which we may package into a 4-entry vector. Therein the stiffness $K_{\alpha\beta}$ is a 4x4 matrix that connects them. We "play God" by analyzing/studying the form of $K_{\alpha\beta}$ without knowing any details about our system - we can deduce constraints purely mathematically.
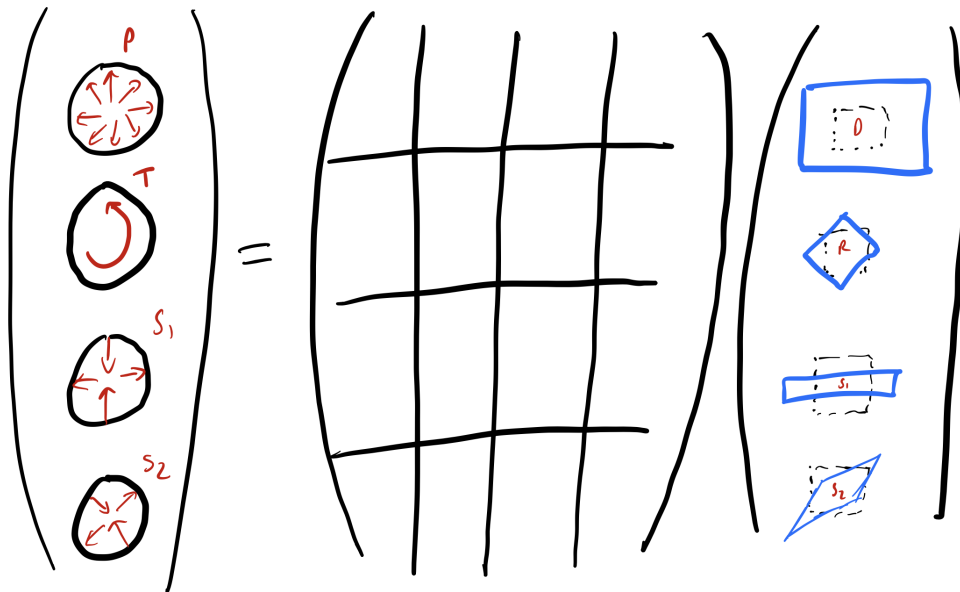
$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \tag{2.6}$$

The four entries of strain u_β will be the projections onto the basis elements of dilation, rotation, 2 shears. We can do the same for stress, writing down the forces that result from the deformations - pressure, torque, and shear stress 1/2. Then the matrix elements of $K_{\alpha\beta}$ relate the two.

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What is the analytical meaning of the drawings above? We write:

$$u_{ij} = \sum_{\alpha=0}^{3} u^\alpha \tau_{ij}^\alpha \tag{2.7}$$

with:

$$\tau^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.8}$$

Corresponding to dilation, rotation, shear 1, and shear 2 respectively. Note that these obey the relation (note that we take a trace below, since the indices are repeated and hence summed):

$$\boxed{\tau_{ij}^\alpha \tau_{ij}^\gamma = 2\delta^{\alpha\gamma}} \tag{2.9}$$

This is the algebra for the generators of the rotation group in 2-D.

This was a statement about the geometry of the strain deformation. We can do a similar procedure for the stress:

$$\sigma_{ij} = \sum_{\alpha=0}^{3} \sigma^\alpha \tau_{ij}^\alpha \tag{2.10}$$

where now the projections onto $\tau^0, \tau^1, \tau^2, \tau^3$ correspond to pressure, torque, shear stress 1, shear stress 2.

2.3 Classifying elastic moduli

Now, we want to discover what is in the $K_{\alpha\beta}$ matrix. To this end, we can think about symmetries of the medium. One such symmetry/idea is isotropy - wherein there is no preferred direction to the material. With this assumption we can already set a lot of the entries of K to zero.

In a world that is isotropic, I cannot couple a scalar/pseudoscalars to a vector/bivectrs - why? Because this would involve a preferred direction. So, look at the matrix element of K that connects dilation with shear stress. We cannot measure the angle of shear in a isotropic medium, so this matrix element must vanish.

More generally, the dilation/rotation subspaces are scalar/pseudoscalars, and the stress subspaces are pseudovectors. Further, pressure/torque are scalar/pseudoscale and shear stress are pseudovector. By isotropy the subspaces cannot be connected, which allows us to conclude that 8 of the entries vanish:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \tag{2.11}$$

Next, consider the symmetry of objectivity/frame invariance. We can consider the solid to be "self-standing" - in this case, applying a rigid body rotation applies no stress. Note that we can break this if the solid was standing on some sticky surface (in which case we could have stresses applied from the contact) - symmetries can be broken by perturbations, e.g. isotropy via a magnetic field. But what does this frame invariance give us for the stiffness tensor? Since rotation generates no stresses, the entire second column is set to zero (rotations cannot generate pressure or torques).

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} \cdot & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \tag{2.12}$$

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We have assumed that linear momentum is conserved. We have not used energy conservation here - let's see what this implies for us now.

To learn about the last remaining entries, let us work through some algebra. We write:

$$\sigma_{ij} = \sigma^\gamma \tau_{ij}^\gamma = K_{ijkl} = u_{kl} = K_{ijkl}\tau_{kl}^\beta u^\beta \tag{2.13}$$

Now multiplying the blue by $\frac{1}{2}\tau_{ij}^\alpha$, then on the LHS I get:

$$\frac{1}{2}\tau_{ij}^\alpha \sigma^\gamma \tau_{ij}^\gamma = \frac{1}{2}\sigma^\gamma 2\delta^{\alpha\gamma} = \sigma^\alpha \tag{2.14}$$

and on the RHS we get:

$$\frac{1}{2}\tau_{ij}^\alpha K_{ijkl}\tau_{kl}^\beta u^\beta \tag{2.15}$$

so:

$$\sigma^\alpha = \frac{1}{2}\tau_{ij}^\alpha K_{ijkl}\tau_{kl}^\beta u^\beta = K^{\alpha\beta}u^\beta \tag{2.16}$$

So, if we assume that the theory is conservative/we have reciprocity $K_{ijkl} = K_{klij}$, then $K^{\alpha\beta} = K^{\beta\alpha}$ - the matrix is symmetric, and so we can set the matrix element that connects dilation to torque is zero:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{pmatrix}\begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \tag{2.17}$$

We can name some of the remaining entries; the entry connecting dilation to pressure is the bulk modulus B:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{pmatrix}\begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \tag{2.18}$$

Now, it is an exercise that you may have on a problem set that isotropy constrains the lower diagonal entries to be the same, and the lower off-diagonal entries to be equal and opposite. The diagonals we can call the shear modulus G, and the off-diagonals must be zero (as energy conservation constrains $K^{\alpha\beta}$ to be symmetric). Thus:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G \end{pmatrix}\begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \tag{2.19}$$

So, if we assume isotropy, frame invariance, and conservative forces, we just have B, G above. Let's repeat the analysis if we do not assume conservative forces. The diagonal components are not affected, but the off-diagonal entries we constrained to be zero are now generically nonzero:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} B & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & G & -K^0 \\ 0 & 0 & +K^0 & G \end{pmatrix}\begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \tag{2.20}$$

So, this is the conclusion of our analysis - we have (relaxing energy conservation) 4 moduli describing our material.

Comment: In an anisotropic medium, we could have the moduli coupling the two shears, but they will *not* be equal and opposite. One could then imagine matrix elements of the form $\epsilon + K^0, \epsilon - K^0$; one must

look at the *difference* to conclude nonconservative forces (just looking at one element, you cannot be sure if the source is anisotropy). Additionally, a nonzero A does not necessarily mean nonconservative forces, it means nonconservative forces *and* frame invariance, so to conclude nonconservative forces concretely we would need to compare with the other matrix element across the diagonal.

2.4 Quantifying Work

We derive the expression for the work coming from the stresses. For a force we have the contour integral:

$$W = - \oint \mathbf{F} \cdot d\mathbf{l} \tag{2.21}$$

For stresses, we consider a loop integral in 4-D space:

$$W = - \oint \sigma_{ij} du_{ij} = - \oint \sigma^\beta du^\beta \tag{2.22}$$

Now using Stokes' theorem:

$$W = \iint \epsilon^{\alpha\beta} \frac{\partial \sigma^\beta}{\partial u^\alpha} dA \tag{2.23}$$

With $\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Now, we can use that $\sigma^\beta = K^{\beta\alpha} u^\alpha$, and take the derivative. Note that we can focus solely on the entries of the tensor coming from the non-conservative forces (the parts coming from conservative forces give trivially zero contribution)! Writing $K^{\alpha\beta} = K^0 e^{\alpha\beta}$ (we focus on the subspace of shears), we get:

$$W = - \iint \epsilon^{\alpha\beta} e^{\beta\alpha} K^0 dA \tag{2.24}$$

Then:

$$\epsilon^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \implies -\epsilon^{\alpha\beta} e^{\beta\alpha} = 2 \tag{2.25}$$

and so:

$$W = 2K^0 \cdot \text{Area} \tag{2.26}$$

So we see that we do indeed have a nonzero amount of work, proportional to the odd elastic modulus and enclosed area.