

PHYS 411 (Topics in Many-Body Dynamics) Notes

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Introduction:

This is a set of lecture notes taken from UChicago's PHYS 411 (Topics in Many-Body Dynamics), taught by Vincenzo Vitelli.

Contents

1 Continuum Mechanics	3
1.1 Course Overview	3
1.2 Continuum Mechanics Review	3
1.3 Constitutive Equation and Conservation Laws	4
1.4 Non-Conservative Forces	6
2 Classification/Constraints of Elastic Moduli	7
2.1 Summary of Last Lecture	7
2.2 Representations of Stress/Strain Tensors	8
2.3 Classifying elastic moduli	9
2.4 Quantifying Work	11
3 Microscopic Sources of Exotic Stiffness	11
3.1 Review	11
3.2 Non-reciprocal gadget and constructions	12
3.3 Microscopic source of A	13
3.4 Other contemporary experiments	15
4 Odd Elastodynamics I	15
4.1 Algebraic Representation of Stiffness Tensor	16
4.2 Equation of Motion	16
4.3 Limits and Physical Interpretations	17
5 Odd Elastodynamics II	18
5.1 Review of Last Lecture	18
5.2 Phonons in damped odd solids - heuristic argument	18
5.3 Phonons in damped odd solids - formal argument	19
6 Odd Elastodynamics III	21
6.1 Review	21
6.2 Instability, Phase Diagram	22
6.3 Exceptional Points	23
6.4 Poisson Ratio and Odd Ratio	24
7 Odd Elastodynamics IV	24
7.1 Topological Defects/Dislocations	25
7.2 Relaxing linear momentum conservation	27

8 Self-Propelled Particles/Polar Active Fluids	28
8.1 Overview of Self-propelled particles	28
8.2 Vicsek Model	29
8.3 Toner-Tu Model	31

1 Continuum Mechanics

1.1 Course Overview

This course is a topics course in many-body dynamics. What does that mean? It is inspired by a Polykov used to teach in Princeton, called “Topics”, where he would arrive 10 minutes before and decide what would be an important topic in theoretical physics to teach. I (Vincenzo) don’t have quite this level of improvisation, but this will largely be a course in research topics of non-equilibrium many-body physics/active matter.

We do have a broad theme - we want to treat various aspects of physics (mechanics, solid physics, phase transitions) from the perspective of dynamical systems. The formal pre-reqs are the intro soft matter course (currently being taught in parallel) as well as a course in statical field theory, but these are in some sense excessive - an enterprising student would be able to follow along, with some additional reading.

There is a recommended book (Soft Matter by Vitelli and others). The book will make the lectures look good in that the lectures will have better explanations. But since this course is being taught for the first time, there will be no lecture notes, so some sections of the book may be useful.

Assessment will be from problem sets (3 problem sets of 3-4 problems each). They may be difficult (they will be taken from the book) and long, but they will be stimulating and well written out. They will largely be mini research topics (it will be like working on research with a soft matter theorist like me (Vincenzo), or perhaps someone less crazy) - the grading will be lenient accordingly, as well as policies around deadlines, working together.

1.2 Continuum Mechanics Review

We will start by looking at equilibrium theory of physics, by looking at elasticity. This is normally a problem treated in the context of variational classical field theory. But we will approach this problem from a non-variational perspective (odd elasticity), and we will see interesting phenomena - namely chaos - emerge.

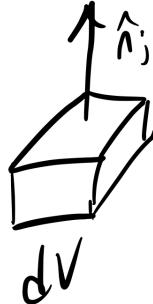
Our starting point is Newton’s second law:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \quad (1.1)$$

For continuous media, we can consider the force as coming from a volume integral of the pressure over space:

$$f_i = \frac{d}{dt} \iiint dV P_i = \iint dA \sigma_{ij} n_j \quad (1.2)$$

where we have used Stoke’s theorem to rewrite the volume integral as an integral over the area.



Therein, σ_{ij} is the linear momentum flux, or a rank-2 stress tensor. Note that throughout, we use Einstein summation convention where repeated indices are summed over (e.g. $u_{ij}x_j := \sum_j u_{ij}x_j$).

We can further write:

$$f_i = \iiint \frac{\partial \sigma_{ij}}{\partial x_j} dV \quad (1.3)$$

Another way to package this; we can write the force as a volume integral of a force density, which is a divergence of the stress tensor:

$$f_i = \iiint dV F_i \quad (1.4)$$

with:

$$F_i = \partial_j \sigma_{ij} \quad (1.5)$$

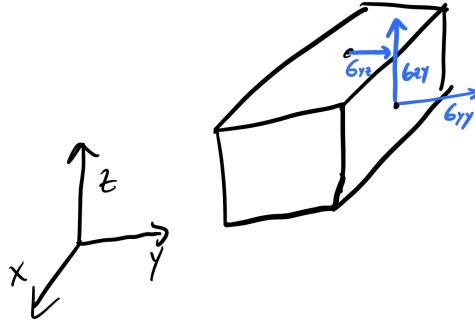
one can also take this as the definition of the stress tensor. If you have never seen the stress-tensor, it is recommended you read the first chapter of Soft Matter.

There is an assumption we made here. When we set the volume integral of $dV P_i$ equal to the area integral, we assumed that there were no sources. We will see what happens soon when this assumption is violated. For example we may have a solid immersed in a fluid, in which case Newton's third law may not hold and linear momentum is generically not conserved. One thing that this class will emphasize is what happens to the conclusions when assumptions are challenged!

Let's look at a differential of the force density:

$$dF_i = \sigma_{ij} n_j dA \quad (1.6)$$

As a concrete example, we can look at the forces acting on different faces of our cube:



1.3 Constitutive Equation and Conservation Laws

We want a relationship between the stress tensor and the order parameter, relating to the displacements. We assume that there is translation invariance for now, so there is no dependence on the displacements themselves, only on gradients of the displacements (this is not true always, e.g. in the case of a solid in fluid, or a solid on a table with glue - these will be additional terms that we consider later). In linear response theory, we connect the stress and strain tensors (rank-2) via a stiffness tensor (rank-4):

$$\sigma_{ij} = K_{ijkl} \partial_k u_l = K_{ijkl} u_l \quad (1.7)$$

The above is a linearized constitutive equation. As ugly as the K_{ijkl} above may be, forms the identity of the material. This is "Hooke's Law on steroids". Note that if we assume that microscopically a solid is composed of masses connected by Hookean springs with spring constant k , it is possible to derive the macroscopic stiffness tensor from the microscopic theory; this may be one of your problems later.

In 2-D, we have:

$$u_{kl} = \begin{pmatrix} \partial_x u_x & \partial_x u_y \\ \partial_y u_x & \partial_y u_y \end{pmatrix} \quad (1.8)$$

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \quad (1.9)$$

Note that we do not symmetrize here. In fact we can explicitly think about what happens in the cases when the off-diagonal entries are not equal. When $\sigma_{ij} \neq \sigma_{ji}$, it means that there is a non-vanishing net-torque (imagine back to the forces on the cube picture) and so the medium can spontaneously rotate. What does this teach us? Buried in the cumbersome algebraic structure of the rank-4 tensor, we see the presence and or violation of conservation laws. In this case, $\sigma_{ij} \neq \sigma_{ji}$ it implies that angular momentum is *not* conserved. Most books assume the fundamental conservation laws of nature hold and write down theories accordingly. But, we can allow for exotic mechanisms of the breaking of such fundamental laws, e.g. from interactions with external sources which give rise to effective theories.

We are rapidly marching through conservation laws; we have imposed conservation of linear momentum from the get-go, but we do not impose $\sigma_{ij} = \sigma_{ji}$ /impose angular momentum conservation (If we did, then the rank-4 tensor would reduce to rank-3). This means that we have external torques present/sources of angular momentum. As we go on, we will study symmetries of our system through the symmetries of the (admittedly complicated) K_{ijkl} object, and understand the conservation laws through it.

Now, we *assume* that there is a potential energy. In Hooke's law, we assume $F = -\partial_x E$ with $E = \frac{1}{2}kx^2$ so we have a conservative force. Here, we don't work on the level of single particles, so we write the total potential energy as the integral over energy density:

$$U = \iiint d^3x \epsilon(\mathbf{x}) \quad (1.10)$$

The question is now what is $\epsilon(\mathbf{x})$? If we want a linear force, we require that $\epsilon(\mathbf{x})$ be a quadratic form in \mathbf{x} :

$$\epsilon(\mathbf{x}) = \frac{1}{2}c_{ijkl}u_{ij}(\mathbf{x})u_{kl}(\mathbf{x}) \quad (1.11)$$

Now, we want the force in the l th direction, which is a functional derivative of the energy density:

$$(\text{Force})_l = -\frac{\delta \epsilon}{\delta u_l} = -\left(\frac{\partial \epsilon}{\partial u_l} - \partial_k \frac{\partial \epsilon}{\partial u_{lk}} \right) \quad (1.12)$$

where again we recall $u_{lk} = \partial_l u_k$. Let's try to unweird this expression via analogy (maybe you want to explain to a friend, or pick up someone at a bar by telling them that you work on functional derivatives) - in classical mechanics, we have an action:

$$S = \int L(q(t)) dt \quad (1.13)$$

we then have the trajectory that is followed is that for which the functional derivative of the action vanishes¹:

$$\frac{\delta S}{\delta q} = 0 \implies \frac{\partial L}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \quad (1.14)$$

And we can now see that the form of this expression is equivalent to the functional derivative of the energy density we wrote above.

Since we that the energy density cannot depend on displacement u_l /it only depend on gradients of displacement; hence the first term of the force expression vanishes:

$$(\text{Force})_l = \partial_k \frac{\partial \epsilon}{\partial u_{lk}} \quad (1.15)$$

¹Of course, this is something your date is expected to know.

Now if we recall Eq. (1.5) that the force density is the divergence of the stress tensor, we can write the above as:

$$\sigma_{lk} = \frac{\partial \epsilon}{\partial u_{lk}} \quad (1.16)$$

Now, if ϵ is quadratic in u , we will find that the relationship of stress and strain will be linear. The question then becomes; what is the relationship between K_{ijkl} and between c_{ijlm} (the coefficients of force and the coefficients of energy)? In the Hooke's law case they coincide and are both the spring constant k . However, this coincidence is only because we have a conservative force - we will see that even in this more general/continuum setting that if we have energy conservation the coefficients coincide, and in non-conservative settings they may differ.

Let us work through the algebra. Computing the derivative of ϵ :

$$\begin{aligned} \frac{\partial \epsilon}{\partial u_{pq}} &= \frac{1}{2} c_{ijkl} [\delta_{ip} \delta_{jq} u_{kl} + u_{ij} \delta_{pk} \delta_{ql}] \\ &= \frac{1}{2} [c_{pqkl} u_{kl} + c_{ijpq} u_{ij}] \\ &= \frac{1}{2} [c_{pqkl} u_{kl} + c_{klpq}] u_{kl} \end{aligned} \quad (1.17)$$

Note in the third equality we replace ij with kl ; they are dummy/repeated/summed over indices and thus we can call them whatever we want². Now, identifying:

$$K_{pqkl} = \frac{1}{2} [c_{pqkl} + c_{klpq}] \quad (1.18)$$

we have:

$$\sigma_{pq} = K_{pqkl} u_{kl} \quad (1.19)$$

And thus we manifestly see that the stiffness tensor is symmetric under interchange of pairs of indices, so long as the force is derived from a variation of a quadratic energy density:

$$K_{pqkl} = K_{klpq} \quad (1.20)$$

This relation is known as the Maxwell-Betti reciprocity. Later we will identify $\alpha := pq$ and $\beta := kl$ and better appreciate the symmetry algebraically.

1.4 Non-Conservative Forces

What if we have a world where we deform the material and through interaction with the environment, the system can suck up or expel energy? Then, we will find that work is not a state function, and:

$$\oint \mathbf{F} \cdot d\mathbf{l} \neq 0 \quad (1.21)$$

In this context, the linear stress-strain relation says *nothing* about the conservative nature of the forces, only that the forces are linear. It only says something about the energy with the constraint that the forces are conservative.

How do we think about the tensors in this setting? Recall that we can always write a tensor in terms of a symmetric and antisymmetric part, e.g.:

$$\sigma_{ij} = \sigma_{ij}^S + \sigma_{ij}^A \quad (1.22)$$

²Name-calling is not good, but it's ok with tensors - they don't mind.

with:

$$\sigma_{ij}^S = \frac{\sigma_{ij} + \sigma_{ji}}{2}, \quad \sigma_{ij}^S = \sigma_{ji}^S \quad (1.23)$$

$$\sigma_{ij}^A = \frac{\sigma_{ij} - \sigma_{ji}}{2}, \quad \sigma_{ij}^A = -\sigma_{ji}^A \quad (1.24)$$

here σ_{ij}^S conserves angular momentum, and the antisymmetric part is the culprit of angular momentum conservation violation. Next class, we will do a similar decoupling for our stiffness tensor:

$$K_{ijkl} = K_{ijkl}^e + K_{ijkl}^o \quad (1.25)$$

with:

$$K_{ijkl}^e = K_{klij}^e \quad (1.26)$$

$$K_{ijkl}^o = -K_{klij}^o \quad (1.27)$$

the odd/o part will be new moduli that emerge only when our system is open/not closed. The existence of this term is known as odd elasticity. This subject started in 2020, from a Nature Physics paper first authored by a grad student here! This is not just a mathematical possibility, but indeed appears in nature, e.g. in chiral tissues.

Next time, we will understand these tensors via group theory. We will then see what the operational meanings of the tensors are. We will discuss what predictions we can make given a set of symmetries and conservation laws.

2 Classification/Constraints of Elastic Moduli

For a less detailed discussion of what we discuss here, you can consult chapter 9 (“Active Matter”) of the Soft Matter textbook.

2.1 Summary of Last Lecture

Last lecture, we studied solids, and the relationship between the stress and strain tensor:

$$\sigma_{ij} = K_{ijkl} \underbrace{\frac{\partial u_k}{\partial x_l}}_{\partial_k u_l} \quad (2.1)$$

the two are related via the stiffness tensor K_{ijkl} , which summarizes the identity of the solid.

If the forces are conservative, then the stress is the variation of an energy density:

$$\sigma_{ij} = \frac{\partial \epsilon}{\partial u_{ij}} \quad (2.2)$$

And it follows that K_{ijkl} is symmetric under exchange of pairs of indices:

$$K_{ijkl} = K_{klij} \quad (2.3)$$

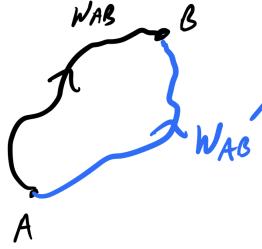
But, say we want to consider a medium where the forces are not conservative. Then neither of Eq. (2.2), (2.3) have to hold. In this more general case, we can split K_{ijkl} into an even and odd part:

$$K_{ijkl} = K_{ijkl}^e + K_{ijkl}^o \quad (2.4)$$

where even/odd means that the tensor acquires a \pm sign under swap of pairs of indices. In a soft matter or elasticity course you have seen the first term - but the second term is novel, and comes up when we study open systems.

Since the forces are non-conservative, work is no longer a state function:

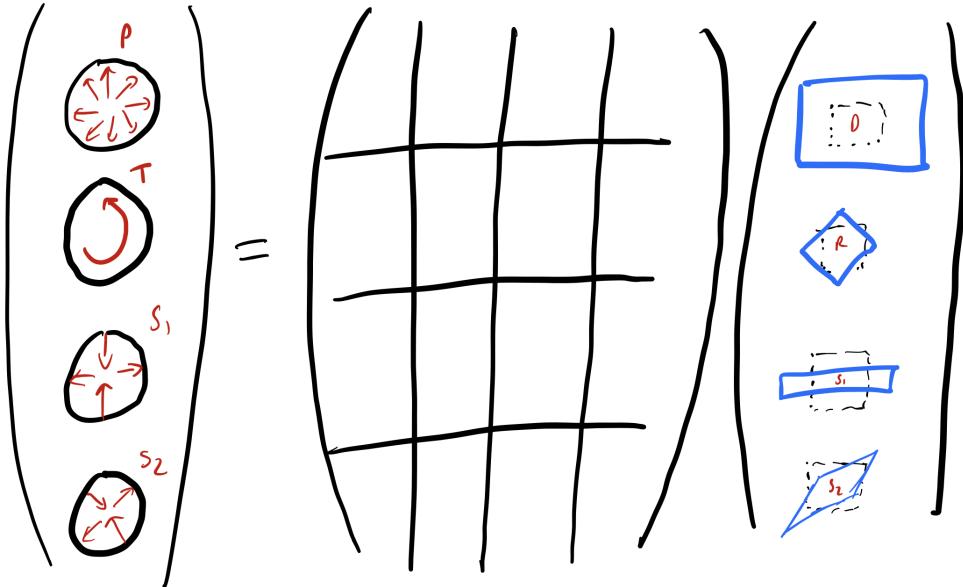
$$W_{AB} \neq U_B - U_A \quad (2.5)$$



In this case, we have net work (positive or negative, depending on the sign) when we go around a cycle. The work done is rate-independent (no time derivatives here!). Further, the medium must be chiral because there is a difference in the sign of work depending on whether we traverse a loop clockwise or counterclockwise.

2.2 Representations of Stress/Strain Tensors

Let's try to classify all possible entries of the stiffness tensor K_{ijkl} using the representations of σ and u ; this will allow us to classify possible elastic moduli. In particular, let us try to classify odd elasticity K_{ijkl}^o in 2-dimensions. In 2-D, stress σ_{ij} and strain u_{kl} each have 4-independent components, which we may package into a 4-entry vector. Therein the stiffness $K_{\alpha\beta}$ is a 4x4 matrix that connects them. We "play God" by analyzing/studying the form of $K_{\alpha\beta}$ without knowing any details about our system - we can deduce constraints purely mathematically.



$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \quad (2.6)$$

The four entries of strain u_β will be the projections onto the basis elements of dilation, rotation, 2 shears. We can do the same for stress, writing down the forces that result from the deformations - pressure, torque, and shear stress 1/2. Then the matrix elements of $K_{\alpha\beta}$ relate the two.

What is the analytical meaning of the drawings above? We write:

$$u_{ij} = \sum_{\alpha=0}^3 u^\alpha \tau_{ij}^\alpha \quad (2.7)$$

with:

$$\tau^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.8)$$

Corresponding to dilation, rotation, shear 1, and shear 2 respectively. Note that these obey the relation (note that we take a trace below, since the indices are repeated and hence summed):

$$\boxed{\tau_{ij}^\alpha \tau_{ij}^\gamma = 2\delta^{\alpha\gamma}} \quad (2.9)$$

This is the algebra for the generators of the rotation group in 2-D.

This was a statement about the geometry of the strain deformation. We can do a similar procedure for the stress:

$$\sigma_{ij} = \sum_{\alpha=0}^3 \sigma^\alpha \tau_{ij}^\alpha \quad (2.10)$$

where now the projections onto $\tau^0, \tau^1, \tau^2, \tau^3$ correspond to pressure, torque, shear stress 1, shear stress 2.

2.3 Classifying elastic moduli

Now, we want to discover what is in the $K_{\alpha\beta}$ matrix. To this end, we can think about symmetries of the medium. One such symmetry/idea is isotropy - wherein there is no preferred direction to the material. With this assumption we can already set a lot of the entries of K to zero.

In a world that is isotropic, I cannot couple a scalar/pseudoscalars to a vector/bivectors - why? Because this would involve a preferred direction. So, look at the matrix element of K that connects dilation with shear stress. We cannot measure the angle of shear in a isotropic medium, so this matrix element must vanish.

More generally, the dilation/rotation subspaces are scalar/pseudoscalars, and the stress subspaces are pseudovectors. Further, pressure/torque are scalar/pseudoscale and shear stress are pseudovector. By isotropy the subspaces cannot be connected, which allows us to conclude that 8 of the entries vanish:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \quad (2.11)$$

Next, consider the symmetry of objectivity/frame invariance. We can consider the solid to be "self-standing" - in this case, applying a rigid body rotation applies no stress. Note that we can break this if the solid was standing on some sticky surface (in which case we could have stresses applied from the contact) - symmetries can be broken by perturbations, e.g. isotropy via a magnetic field. But what does this frame invariance give us for the stiffness tensor? Since rotation generates no stresses, the entire second column is set to zero (rotations cannot generate pressure or torques).

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} \cdot & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \quad (2.12)$$

We have assumed that linear momentum is conserved. We have not used energy conservation here - let's see what this implies for us now.

To learn about the last remaining entries, let us work through some algebra. We write:

$$\sigma_{ij} = \sigma^\gamma \tau_{ij}^\gamma = K_{ijkl} = u_{kl} = K_{ijkl} \tau_{kl}^\beta u^\beta \quad (2.13)$$

Now multiplying the blue by $\frac{1}{2}\tau_{ij}^\alpha$, then on the LHS I get:

$$\frac{1}{2}\tau_{ij}^\alpha \sigma^\gamma \tau_{ij}^\gamma = \frac{1}{2}\sigma^\gamma 2\delta^{\alpha\gamma} = \sigma^\alpha \quad (2.14)$$

and on the RHS we get:

$$\frac{1}{2}\tau_{ij}^\alpha K_{ijkl} \tau_{kl}^\beta u^\beta \quad (2.15)$$

so:

$$\sigma^\alpha = \frac{1}{2}\tau_{ij}^\alpha K_{ijkl} \tau_{kl}^\beta u^\beta = K^{\alpha\beta} u^\beta \quad (2.16)$$

So, if we assume that the theory is conservative/we have reciprocity $K_{ijkl} = K_{klji}$, then $K^{\alpha\beta} = K^{\beta\alpha}$ - the matrix is symmetric, and so we can set the matrix element that connects dilation to torque is zero:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \quad (2.17)$$

We can name some of the remaining entries; the entry connecting dilation to pressure is the bulk modulus B :

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \quad (2.18)$$

Now, it is an exercise that you may have on a problem set that isotropy constrains the lower diagonal entries to be the same, and the lower off-diagonal entries to be equal and opposite. The diagonals we can call the shear modulus G , and the off-diagonals must be zero (as energy conservation constrains $K^{\alpha\beta}$ to be symmetric). Thus:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \quad (2.19)$$

So, if we assume isotropy, frame invariance, and conservative forces, we just have B, G above. Let's repeat the analysis if we do not assume conservative forces. The diagonal components are not affected, but the off-diagonal entries we constrained to be zero are now generically nonzero:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} B & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & G & -K^0 \\ 0 & 0 & +K^0 & G \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \quad (2.20)$$

So, this is the conclusion of our analysis - we have (relaxing energy conservation) 4 moduli describing our material.

Comment: In an anisotropic medium, we could have the moduli coupling the two shears, but they will *not* be equal and opposite. One could then imagine matrix elements of the form $\epsilon + K^0, \epsilon - K^0$; one must

look at the *difference* to conclude nonconservative forces (just looking at one element, you cannot be sure if the source is anisotropy). Additionally, a nonzero A does not necessarily mean nonconservative forces, it means nonconservative forces *and* frame invariance, so to conclude nonconservative forces concretely we would need to compare with the other matrix element across the diagonal.

2.4 Quantifying Work

We derive the expression for the work coming from the stresses. For a force we have the contour integral:

$$W = - \oint \mathbf{F} \cdot d\mathbf{l} \quad (2.21)$$

For stresses, we consider a loop integral in 4-D space:

$$W = - \oint \sigma_{ij} du_{ij} = - \oint \sigma^\beta du^\beta \quad (2.22)$$

Now using Stokes' theorem:

$$W = \iint \epsilon^{\alpha\beta} \frac{\partial \sigma^\beta}{\partial u^\alpha} dA \quad (2.23)$$

With $\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Now, we can use that $\sigma^\beta = K^{\beta\alpha} u^\alpha$, and take the derivative. Note that we can focus solely on the entries of the tensor coming from the non-conservative forces (the parts coming from conservative forces give trivially zero contribution)! Writing $K^{\alpha\beta} = K^0 e^{\alpha\beta}$ (we focus on the subspace of shears), we get:

$$W = - \iint \epsilon^{\alpha\beta} e^{\beta\alpha} K^0 dA \quad (2.24)$$

Then:

$$\epsilon^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \implies -\epsilon^{\alpha\beta} e^{\beta\alpha} = 2 \quad (2.25)$$

and so:

$$W = 2K^0 \cdot \text{Area} \quad (2.26)$$

So we see that we do indeed have a nonzero amount of work, proportional to the odd elastic modulus and enclosed area.

3 Microscopic Sources of Exotic Stiffness

3.1 Review

Last time, we tried to constrain the stiffness tensor:

$$\sigma^\alpha = K^{\alpha\beta} u^\beta \quad (3.1)$$

Assuming isotropy, the subspaces of scalars and vectors could not be connected (so the off diagonal blocks vanish), and assuming frame invariance, rotations cannot generate any stresses, so:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} \cdot & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \quad (3.2)$$

Isotropy further contains that the lower diagonals are the same and lower off diagonals are the same, so the most general form of the tensor is:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} B & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & G & -K^0 \\ 0 & 0 & +K^0 & G \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \quad (3.3)$$

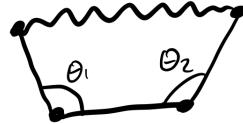
Important point - the existence of the odd elastic moduli K^0 implies two things. First, the structure of the theory is non-variational; we can go around quasi-statically in the space of strains, and we find the work is nonzero:

$$W = \oint \sigma_{ij} du_{ij} \neq 0 \quad (3.4)$$

This tells us that there is a chirality in the material (this was also manifest from the fact that - if $G = 0$, then the part of the stiffness matrix acting on the shear subspace looks like $K^0\epsilon$). Another clear manifestation of chirality is if $A \neq 0$, in which case we can have torques (note - no angular momentum conservation!) as a result of dilations.

3.2 Non-reciprocal gadget and constructions

This is an example of a $K^0 \neq 0, A = 0$ system from a recent Nature paper Veenstra, Scheibner, Brandenbourger, Binysh, Souslov, Vitelli, Coulais. *Nature* (2025). As warm-up, consider the following spring system:



If we have a bond-bending interaction, we have a bond-bending stiffness:

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} -\kappa & \kappa^s \\ \kappa^s & -\kappa \end{pmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix} \quad (3.5)$$

where the diagonals come from the bond-bending, and the off-diagonals (symmetric!) come from the coupling spring. For this system:

$$\nabla \times \tau = 0 \quad (3.6)$$

but we can introduce batteries/motors to break this symmetry. If Maxwell-Betti reciprocity is obeyed, then pushing one side of the object should repel the other symmetrically. But we can introduce motors to make this building block non-reciprocal; pressing on the left side the right side repels, pressing on the right side the left side *attracts*.



This means that we have a stiffness tensor of the form:

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} -\kappa & -\kappa^a \\ \kappa^a & -\kappa \end{pmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix} \quad (3.7)$$

A technical comment - notice that the interaction we've written above is a striking example of a non-pairwise interaction. The torque at point 1 doesn't just depend on the pair of rods emanating out of it, but also from the other components of the system.

What is the physical consequence of this non-reciprocal interaction? Now, we can go through a cycle and generate non-zero work. If we make a ring of these objects - with the motors off, the system jiggles a bit before being slowed to rest with friction. With the motors on, we perturb the medium and in trying to undo the deformation/get back to the starting point the motors can induce energies into the system and the system can keep jiggling forever (using the energy provided by the external motors).

If we think about the dynamics, we have:

$$\begin{pmatrix} I\delta\ddot{\theta}_1 \\ I\delta\ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} -\kappa & -\kappa^a \\ \kappa^a & \kappa \end{pmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix} \quad (3.8)$$

If we diagonalize to find the normal frequencies, we have:

$$\omega^2 = \frac{\kappa}{I} \pm i \frac{\kappa^a}{I} \quad (3.9)$$

The first term makes total sense - this is what we have for just a regular spring system that you've seen many times before in classical mechanics. But the antisymmetric term produces an imaginary component to ω^2 ! This means that (in the absence of nonlinearities - we have assumed linear response for this whole discussion) we have unbounded growth/amplification of the motion! Of course building this gadget in real life we do in fact have nonlinearities - when the angles become large the components of the gadget can touch - which prevents unbounded growth.

Now, we can take our object - where the antisymmetric κ^a breaks the chiral symmetry - and map out the dynamics in strain space (with the two shears being the two axes). The system can exhibit chaotic behaviour!

Let us "rediscover" the wheel (again, still composed of the the gadget). Original conception of a wheel has a circular, fixed shape which moves via force supplied to it from axle (via animal, motor etc.) But our new wheel is adaptive, and moves via deformations. It can come across a rough uphill landscape and use the deformations caused by the rough landscape to travel through it! In fact we can make the problem even harder - we can make a landscape of granular uphill matter, and our sophisticated wheel can still go up it (we can also flip the sign of κ^a - we are allowed to pick/tune whatever sign for these - and it can travel the other way).

We can also build up a crystal out of this gadget by tiling. We can throw a projectile onto the crystal, which is then ejected back upwards at an angle (the angle/asymmetric response is expected! The medium is chiral. Again, we can flip the angle by flipping the sign of κ^a).

Motivating a homework problem you will do - there is a microscopic law Eq. (3.7) and the macroscopic/continuum description Eq. (4.3). Without calculation, we can deduce $K^0 \sim \kappa^a$ and $G \sim \kappa$. We can do a more detailed derivation and derive the macroscopic dependence on the microscopic parameters (and then check that this dependence works out in experiment)! In your homework, you will just take a normal (not odd!) hexagonal microscopic lattice, and see how the macroscopic properties/parameters and their dependencies emerge.

3.3 Microscopic source of A

If we have a central force, we can generically write things as a gradient of a potential. We introduce non-conservative forces in terms of a non-central part. Consider the spring:



$$\mathbf{F} = -(\kappa \hat{\mathbf{r}} + \kappa^a \hat{\phi}) \delta r \quad (3.10)$$

Or as a matrix equation:

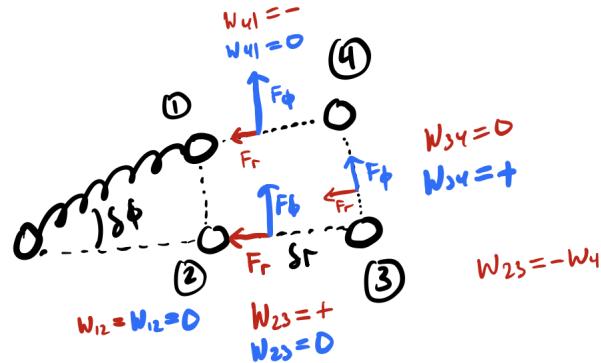
$$\begin{pmatrix} F_r \\ F_\phi \end{pmatrix} = \begin{pmatrix} -\kappa & 0 \\ -\kappa^a & 0 \end{pmatrix} \quad (3.11)$$

Notice again the crucial point - the off-diagonals are *not* equal. This implies that:

$$\oint \mathbf{F} \cdot d\mathbf{l} \propto \kappa^a \quad (3.12)$$

(and does not vanish). The central part does not contribute to the net work as to the central part we can always prescribe a potential. Generally here, $\mathbf{F} \neq -\nabla U$ so long as $\kappa^a \neq 0$. If we built up a lattice/continuum out of this microscopic piece, we could derive how the A modulus emerges from the nonzero κ^a .

Let us consider a simple exercise - let's derive the work done in a cycle.



- From 1 to 2, there is no force and hence no work done.
- From 2 to 3, the direction of motion is radial, so only the radial component of the force contributes (positive) work.
- From 3 to 4, the direction of motion is tangential, so the tangential component of the force picks up (positive) work.
- From 4 to 1, the direction of motion is radial, so the radial component of the force contributes (negative) work. This work is equal and opposite to the work in step 2→3.

Thus only surviving term is W_{34} arising from F_ϕ , which we can calculate to be:

$$\oint \mathbf{F} \cdot d\mathbf{l} = W_{34} = F_\phi dl = \kappa^a \delta r (r + \delta r) \delta \phi \approx \kappa^a (r \delta \phi) \delta r = \kappa^a \cdot \text{Area} \quad (3.13)$$

So in the harmonic approximation, with:

$$F''(r) = F''(a) - \kappa(r - a), \quad F_\perp \approx F^\perp(a) - \kappa^a(r - a) \quad (3.14)$$

we can derive the bulk and shear moduli:

$$B = \frac{\sqrt{3}}{2} \left(\kappa + \frac{F''(a)}{a} \right), \quad G = \frac{\sqrt{3}}{4} \left(\kappa - 3 \frac{F''(a)}{a} \right) \quad (3.15)$$

as well as the K^0/A moduli:

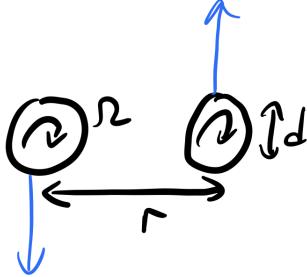
$$A = -\frac{\sqrt{3}}{2} \left(\kappa^a - \frac{F^\perp(a)}{a} \right), \quad K^0 = \frac{\sqrt{3}}{4} \left(\kappa^a - 3 \frac{F^\perp(a)}{a} \right) \quad (3.16)$$

note that the precise dependence depends on not just the force mechanism but also the layout of the lattice (e.g. will change with hexagonal, square etc.). If we did a similar derivation with the non-reciprocal robot gadget, we could also derive a macro-from-micro result from above (see the paper!), with $A = 0$ and $K^0 \neq 0$.

3.4 Other contemporary experiments

There have been a number of recent experiments that measure interesting non-central crystals. For example colloidal spinners in a magnetic field that form odd elastic crystals Bililign, Usabiaga, Ganan, Poncet, Soni, Magkiriadou, Shelley, Bartolo, Irvine. *Nature* (2022) or crystal formation in biological platforms Tan, Mietke, Higinbotha, Li, Chen, Foster, Gokhale, Dunkel, Fakhri. *Nature* (2022).

In such non-central crystals, there is the potential to measure lubrication forces:



where:

$$F^\perp \propto \Omega \log \left(\frac{r-d}{d} \right) \quad (3.17)$$

Next time - we will take the macroscopic theory, and study the stability of the lattice, as well as excitations (sound modes) in it. These have been concretely measured in robotic platform, and allegedly in the biological platform above (and perhaps possible to measure it in the colloidal spinner crystal?) however in the latter two cases the micro-to-macro correspondence is much less understood.

4 Odd Elastodynamics I

Last time, we looked at microscopic models where odd elasticity could arise. We also looked in detail at how energy could be generated in cycles of such system. Today we look odd elastodynamics, and study the dynamical equation for the displacement field. The presentation will be more algebraic.

In the last few lectures, we had pictures of different stresses/strains - these were pictorial representations of tensors, and in this lecture we make a dictionary to map between the pictures and algebra. For example we'll see how the tensors:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \delta_{ij}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow \epsilon_{ij} \quad (4.1)$$

connect to stress/strain. For simplicity, we stay in 2D.

We recall the stress/strain relation:

$$\sigma_{ij} = K_{ijmn} \partial_m u_n = K_{ijmn} u_{mn} \quad (4.2)$$

Where we have the stiffness tensor K_{ijmn} . We wrote this relation (using symmetry constraints) as:

$$\begin{pmatrix} P \\ T \\ SS1 \\ SS2 \end{pmatrix} = \begin{pmatrix} B & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & G & -K^0 \\ 0 & 0 & +K^0 & G \end{pmatrix} \begin{pmatrix} D \\ R \\ S1 \\ S2 \end{pmatrix} \quad (4.3)$$

Note that if $\sigma_{ij} = \frac{\delta\epsilon}{\delta u_{ij}}$, then $K_{ijmn} = K_{mni,j}$, but if σ_{ij} is not of this form then K is not necessary symmetric, so we can have the elastic moduli K^0, A emerge in the stiffness tensor.

4.1 Algebraic Representation of Stiffness Tensor

Let's now switch to an algebraic representation:

$$K_{ijmn} = B\delta_{ij}\delta_{mn} - A\epsilon_{ij}\delta_{mn} + G(\delta_{in}\delta_{jm} + \delta_{im}\delta_{jn} - \delta_{ij}\delta_{mn}) + K^0 \frac{1}{2}(\epsilon_{im}\delta_{jn} + \epsilon_{in}\delta_{jm} + \epsilon_{jm}\delta_{in} + \epsilon_{jn}\delta_{im}) \quad (4.4)$$

The first term corresponds to the bulk modulus, which connects the dilation (projection of u_{mn} onto identity/ δ_{mn}) and the pressure (projection of σ_{ij} onto identity/ δ_{ij}). Thus it is not surprising that it has the form above; explicitly, they are derived via:

$$P\delta_{ij} = B(\nabla \cdot \mathbf{u})\delta_{ij} \implies \sigma_{ij} = [B\delta_{ij}\delta_{mn}] \partial_m u_n \quad (4.5)$$

For torque, we project σ_{ij} onto the antisymmetric matrix ϵ_{ij} (the sign of A in K_{ijmn} is up to convention of dilation - we choose it to be negative so we get a positive sign in the equation of motion):

$$T\epsilon_{ij} = A(\nabla \cdot \mathbf{u})\epsilon_{ij} \implies \sigma_{ij} = [A\epsilon_{ij}\delta_{mn}] \partial_m u_n \quad (4.6)$$

Note that the G term is invariant/same sign (even) under $ij \leftrightarrow mn$, while the K^0 term flips sign (odd) under $ij \leftrightarrow mn$.

4.2 Equation of Motion

I can now calculate the stress in terms of the strain, and taking one more derivative of the stress I can get the force. Then we obtain the equation of motion for odd elasticity!

$$\rho\partial_t^2 u_j + \Gamma\partial_t u_j = F_j = \partial_i \sigma_{ji} = \partial_i (K_{jimm} \partial_m u_n) = K_{jimm} \partial_i \partial_m u_n \quad (4.7)$$

where we note that the ∂_i commutes across the K_{jimm} as the stiffness tensor is position independent (just a tensor of numbers). Thus, we have an EoM that relates a first + second time derivative to a second spatial derivative of displacement/strain.

If we looked at the robots of last time (centimeter scale), the first/inertial term $\rho\partial_t^2 u_j$ is important. But for very small systems we can consider the overdamped limit (equivalently - the low Reynolds number³ limit):

$$\Gamma\dot{u}_j \gg \rho\ddot{u}_j. \quad (4.8)$$

Thus, in the analysis of our equations of motion we neglect the inertial term.

³ $R = \frac{uL}{v}$ with u the velocity, L the characteristic size, v the viscosity

A good exercise is to massage the derivatives until the terms with two spatial derivatives have recognizable form. For the B term, we get:

$$\Gamma \partial_t u_j = \partial_i K_{jimn} \stackrel{B-term}{=} \partial_i B \delta_{ji} \delta_{mn} u_{mn} = B \partial_j \partial_i u_i \quad (4.9)$$

or as a vector:

$$\Gamma \partial_t \mathbf{u} = B \nabla (\nabla \cdot \mathbf{u}) \quad (4.10)$$

For the A term, we get:

$$\Gamma \partial_t u_j = \partial_i K_{jimn} \stackrel{A-term}{=} A \epsilon_{ji} \delta_{mn} \partial_i u_{mn} = A \epsilon_{jk} \partial_k \partial_i u_i \quad (4.11)$$

or as a vector:

$$\Gamma \partial_t \mathbf{u} = A \hat{\mathbf{z}} \times \nabla (\nabla \cdot \mathbf{u}) \quad (4.12)$$

Writing down the other terms:

$$\Gamma \partial_t u_j = B \nabla (\nabla \cdot \mathbf{u}) + G \nabla^2 u_j + A \epsilon_{jk} \partial_k \partial_i u_i + K^0 \epsilon_{jk} \nabla^2 u_k \quad (4.13)$$

or vectorially:

$$\Gamma \partial_t \mathbf{u} = B \nabla (\nabla \cdot \mathbf{u}) + G \nabla^2 \mathbf{u} + A \hat{\mathbf{z}} \times \nabla (\nabla \cdot \mathbf{u}) + K^0 \hat{\mathbf{z}} \times (\nabla^2 \mathbf{u}) \quad (4.14)$$

4.3 Limits and Physical Interpretations

Note that if $B = A = K^0 = 0$, then we just have a diffusion equation. So one interpretation of what we have derived is a non-reciprocal diffusion equation.

Note that if instead we worked in the underdamped limit, we would have:

$$\rho \partial_t^2 \mathbf{u} = G \nabla^2 \mathbf{u} \quad (4.15)$$

which is a wave equation, with wave speed $c = \sqrt{\frac{G}{\rho}}$. This wave equation is predicated on two things - that we have a nonzero shear modulus and that we have a second derivative in time. If we kill the second derivative, we overdamp the oscillations. If we work in a medium with $B \approx G \approx 0$, so the medium is not elastic/we've killed the ability of the medium to rattle kinetic energy to potential energy (you can interpret B/G as the spring constants of your medium).

Now the question is - can we be in the overdamped regime and in the regime where $K^0 \gg G, B, A$ and have the medium still support waves? The answer is, strikingly, yes! We'll watch a movie next time, but for now let's write down the equation in this limit:

$$\Gamma \begin{pmatrix} \dot{u}_x \\ \dot{u}_y \end{pmatrix} = \begin{pmatrix} 0 & -K^0 \\ K^0 & 0 \end{pmatrix} \nabla^2 \begin{pmatrix} u_x \\ u_y \end{pmatrix} \quad (4.16)$$

If we did not throw away the shear modulus G , then we would have diagonal components and diffusion transport. But in this limit we still have the interesting cross-diffusion terms. This might remind us of Hamilton's equation for the harmonic oscillator, where for $U = \frac{1}{2}x^2$ ($m = k = 1$):

$$\dot{x} = p, \dot{p} = -\partial_x U \implies \partial_t \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \quad (4.17)$$

We note the symplectic matrix:

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (4.18)$$

which is what is responsible for the oscillatory behavior (the fact that they are conjugate coordinates is what further gives us energy conservation). Here u_x, u_y are not canonical conjugate coordinates, but seeing that

it has the same symplectic structure, we will have oscillatory behaviour (you might object that we have the Laplacian ∇^2 - but we can get rid of this via Fourier transform, so the Laplacian just gives a k^2). Interestingly, the shear modulus G actually hinders the elasticity, while the odd elasticity K^0 is responsible for the wave propagation.

Formally, we can write solutions:

$$u_i(\mathbf{x}) = \tilde{u}_i(\mathbf{q}) e^{i(\mathbf{q} \cdot \mathbf{x} - \omega t)} \quad (4.19)$$

where:

$$\omega = \frac{K^0}{\Gamma} q^2 \quad (4.20)$$

Ok, so we get oscillatory behaviour. But why? The intuition is the following - work is no longer a state function, so if we have cycles, we can take in or put out energy in a rate-independent manner. So, if we have an odd elastic medium, we perturb it, then the medium can go through a cyclic deformation, wherein it can take in energy to keep the motion going (rather than dissipating it).

5 Odd Elastodynamics II

5.1 Review of Last Lecture

We wrote down an equation of motion for the displacement field:

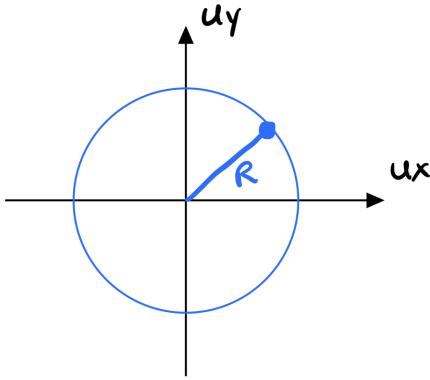
$$\underbrace{\rho \partial_t^2 u_j + \Gamma \partial_t u_j}_{\text{inertia}} + \underbrace{B \partial_j \partial_i u_i + G \nabla^2 u_j + A \epsilon_{jk} \partial_k \partial_i u_i + K^0 \epsilon_{jk} \nabla^2 u_k}_{\text{drag}} = 0 \quad (5.1)$$

what we keep on the LHS depend on whether we work in the overdamped/underdamped limit. What is present on the RHS depends on the terms that appear in the constitutive equation $\sigma_{ij} = K_{ijmn} u_{mn}$. B is the bulk modulus, G is the shear modulus, A, K^0 are the odd moduli - A couples to the antisymmetric part of the stress tensor and gives rise to torque. K^0 relates to the shears, and can arise even if $\sigma_{ij} = \sigma_{ji}$.

We consider the overdamped limit where the $\rho \partial_t^2 u_j$ inertia term vanishes. We also take $K^0 \gg B, A, G$ - this is a peculiar thing to do if we want to study waves, because we are overdamped and have no rigidity ($B, G \sim 0$). But we shall see such a medium can host phonons - perturbing such a system could inject enough motion to make the system go through a cycle (which could then repeat), and overcome the energy dissipated by the drag. There are some conditions that must be met for this to be observed. We start with an intuitive/heuristic argument, then proceed to write down the phase diagram/stability of such phonons more concretely.

5.2 Phonons in damped odd solids - heuristic argument

We consider motion of radius R in displacement space (u_x, u_y) :



Then in strain space (S_1, S_2) we have a circle of radius qR will be traced out over one cycle. The work done is (as we have previously calculated):

$$W_{\text{odd}} = 2K^0 \cdot \text{Area} = 2K^0\pi(qR)^2 \quad (5.2)$$

Where does this energy go? We can imagine that this energy is dissipated in each cycle, so that the wave neither dies down, nor does it get amplified without bound. So, let us write down the energy being dissipated, and then balance this with the work done. The dissipated energy is:

$$W_{\text{dis}} = P\Delta t = \Gamma|\dot{u}|^2 T = \Gamma\left(\frac{2\pi R}{T}\right)^2 T = \Gamma(2\pi R)^2 \frac{1}{T} = \Gamma R^2 2\pi\omega \quad (5.3)$$

where T is the period of the cycle. Note in the third equality that we rewrite \dot{u} to be the perimeter divided by the period, and in the last equality we replace $\omega = \frac{2\pi}{T}$. Now, doing an energy balance argument, we say that the energy gained is lost by the dissipation:

$$W_{\text{odd}} = W_{\text{dis}} \implies 2K^0\pi(qR)^2 = \Gamma R^2 2\pi\omega \implies \boxed{\omega = \frac{K^0}{\Gamma} q^2} \quad (5.4)$$

Note that since the conclusion only tells us that $\omega \sim \frac{K^0}{\Gamma} q^2$ since the heuristic derivation should not necessarily give us the correct prefactor (though it does, here - a more careful derivation gives the same result).

Note that the wave speed is given by:

$$c = \frac{K^0 q}{\Gamma} \quad (5.5)$$

so the speed of the wave scales with the wavevector/moment q . Different parts of the spectrum travel at different speeds! Why no square root? We do *not* have an inertial wave, where we would have two time derivatives and hence a ω^2 . Instead we have a dispersive wave. Note that if we take $K^0 \rightarrow 0$ (the limit of the medium becoming conservative) or take $\Gamma \rightarrow \infty$, then the wave speed goes to zero/the waves vanish.

An important note; if we have EoM $\ddot{x} = -kx$, then $k = \frac{\partial^2 U}{\partial x^2}$, we can write it as a variation of the energy. But, given that we have an odd component, $K_{ijmn}^0 \neq \frac{\delta e}{\delta u_{ij} \delta u_{mn}}$, we *cannot* write it as a variation of energy. This is an example of us taking a system of solids - where traditionally everything is written as the variation of an energy - and generalizing it to a dynamical system. The dispersions we will generate as a result will have different character than what we would normally encounter in physics, either classical or quantum.

5.3 Phonons in damped odd solids - formal argument

Next, we want to calculate the dispersion relationship formally, and derive the phase diagram to see for what regions/phases in parameter space are phonons stable. We have equation of motion:

$$\Gamma \partial_t u_j = K_{ijmn} \partial_i \partial_m u_n \quad (5.6)$$

we guess the ansatz:

$$u_i(\mathbf{x}) = u_i(\mathbf{q})e^{-i(\mathbf{q} \cdot \mathbf{x} + \omega t)} \quad (5.7)$$

Leaving the algebra as an exercise, we define the parallel and transverse components:

$$u_{\parallel} \equiv \hat{q}_i \tilde{u}_i \quad (5.8)$$

$$u_{\perp} \equiv \epsilon_{ij} \hat{q}_i \tilde{u}_j \quad (5.9)$$

then the outcome of the exercise would be:

$$q_i q_m K_{ijmn} \tilde{u}_n = q^2 \begin{pmatrix} B + G & K^0 \\ -K^0 - A & G \end{pmatrix} \begin{pmatrix} u_{\parallel} \\ u_{\perp} \end{pmatrix} \quad (5.10)$$

If we also write the LHS in terms of u_{\parallel}, u_{\perp} then:

$$-i\omega \Gamma \begin{pmatrix} u_{\parallel} \\ u_{\perp} \end{pmatrix} = -q^2 \begin{pmatrix} B + G & K^0 \\ -K^0 - A & G \end{pmatrix} \begin{pmatrix} u_{\parallel} \\ u_{\perp} \end{pmatrix} \quad (5.11)$$

We can then diagonalize to find the normal frequencies/modes, wherein we shall find:

$$\omega = -i \left[\frac{B}{2} + G \pm \sqrt{\left(\frac{B}{2} \right)^2 - K^0 A - (K^0)^2} \right] \frac{q^2}{\Gamma} \quad (5.12)$$

Now, the $\frac{B}{2} + G$ term corresponds to an exponential decay. It is proportional to B, G . There is something very subtle happening here. We perturb our solid, starting a cycle that generates energy due to K^0, A . The bulk and shear moduli B, G do not like the cycles in strain space - from their perspective these cycles cost energy. The presence of the elastic moduli causes the damping of the waves! This is in stark contrast to oscillations that come from a potential, wherein the elastic moduli support the waves.

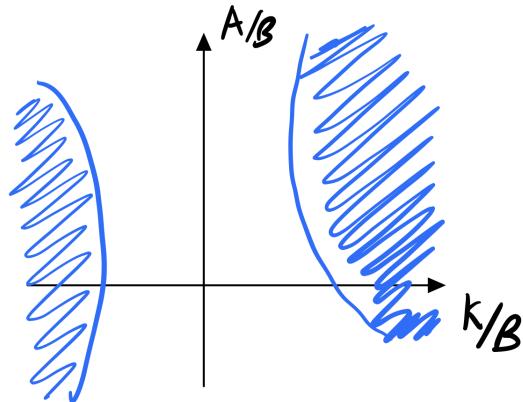
Going back to our expression for the frequency, the onset of active waves is whenever we have oscillatory motion of the form $e^{\pm i \text{Re}(\omega)t}$. The imaginary component can only give us decay or amplification. To get a real part, we need the square root term to be imaginary. For this, we require that the argument of the square root is negative. Thus the condition for active waves is (defining $\tilde{K}^0 = \frac{2K^0}{B}$, $\tilde{A} = \frac{2A}{B}$):

$$\frac{B}{2} \left(1 - \tilde{K}^0 \tilde{A} - (\tilde{K}^0)^2 \right) < 0 \quad (5.13)$$

the phase transition then occurs on the hyperbola:

$$\tilde{A} = \frac{1}{\tilde{K}^0} - \tilde{K}^0 \quad (5.14)$$

so we have the rough phase diagram:



where in the blue regions/phases we get active waves, and in the central condition we have no propagation.

Next class we look at this phase diagram more carefully. We also look at what happens along the $\tilde{A} = \frac{1}{K^0} - \tilde{K}^0$ lines and look for exceptional points. We also look at limits of large odd elastic moduli and see cases where we have amplification.

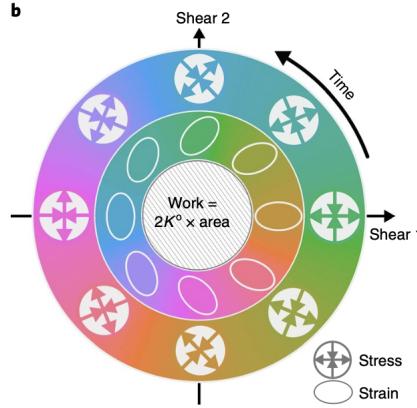
6 Odd Elastodynamics III

6.1 Review

We continue our discussion of odd elastodynamics. Last time, we considered the limit where $G, B, A \ll K^0$ and where we had overdamped dynamics. We then showed that we have oscillations with frequency:

$$\omega = \frac{K^0}{\Gamma} q^2 \quad (6.1)$$

Remarkably, we get standing waves, where stress and strain are 90 degrees out of phase (Figure below taken from <https://www.nature.com/articles/s41567-020-0795-y>).



In strain space, we trace out an area, corresponding to work:

$$W = 2K^0 \text{Area} \quad (6.2)$$

with wave speed:

$$c = \frac{2K^0 q}{\Gamma}. \quad (6.3)$$

Decomposing the displacements into parallel and perpendicular components:

$$u_{\parallel} \equiv \hat{q}_i \tilde{u}_i, \quad u_{\perp} \equiv \epsilon_{ij} \hat{q}_j \tilde{u}_i \quad (6.4)$$

we obtain the relation:

$$-i\omega\Gamma \begin{pmatrix} u_{\parallel} \\ u_{\perp} \end{pmatrix} = -q^2 \begin{pmatrix} B + G & K^0 \\ -K^0 - A & G \end{pmatrix} \begin{pmatrix} u_{\parallel} \\ u_{\perp} \end{pmatrix} \quad (6.5)$$

which inform us about the normal modes of the elastic spectrum. We will momentarily see how the dynamical matrix for an odd solid differs from a regular one - namely the matrix is not Hermitian. We want

to use the above matrix equation to solve for the eigenfrequencies and eigenvectors. Taking the determinant and setting it to zero, we find:

$$\omega = -i \left[\underbrace{\frac{B}{2} + G}_{\text{decay}} \pm \sqrt{\left(\frac{B}{2}\right)^2 - K^0 A - (K^0)^2} \right] \frac{q^2}{\Gamma} \quad (6.6)$$

The decay term comes from the fact that the odd elasticity tries to deform the object, but this costs energy in terms of the bulk/shear modulus (because it strains the material). The more we pay that energy cost (i.e. larger B, G) the more the wave gets attenuated. Fortunately, we look at the $B, G \ll K^0$ limit; this is somewhat equivalent to studying fluid density waves in the low-viscosity limit. Microscopically this is quite odd (though there are some physical systems, e.g. Abrikosov lattices in superconductors), because this corresponds to the $\kappa \rightarrow 0$ limit of the microscopic force law $F = -(\kappa^0 \hat{\phi} + \kappa \hat{r})\delta r$.

We can study the onset of waves; this occurs when we have $e^{i\text{Re}(\omega)t}$, but this only occurs when the argument of the square root is negative. The condition we found was:

$$1 - \tilde{K}^0 \tilde{A} - (\tilde{K}^0)^2 < 0 \quad (6.7)$$

where $\tilde{K}^0 = \frac{2K^0}{B}$, $\tilde{A} = \frac{2\tilde{A}}{B}$. The locus/phase transition was given by the hyperbola:

$$\tilde{A} = \frac{1}{\tilde{K}^0} - \tilde{K}^0 \quad (6.8)$$

6.2 Instability, Phase Diagram

Notice the heuristic argument we made based on energy balance relied on the argument that the energy produced in one cycle could overcome the dissipation lost. In particular, we set the energy gained to be equal to that lost. If the energy gained was less, then the wave would die out. If the energy gained was more, then instead we would have an instability/the wave amplitude would continue to grow.

Let's look at the region of this instability more carefully. This occurs when the spectrum acquires a positive imaginary branch. In particular, we look for a situation where $e^{\Im(\omega)t}$ with $\Im(\omega) > 0$, then we have exponential amplification in time. We require:

$$\frac{B}{2} + G - \sqrt{\left(\frac{B}{T}\right)^2 - K^0 A - (K^0)^2} < 0 \quad (6.9)$$

then the crossover happens when:

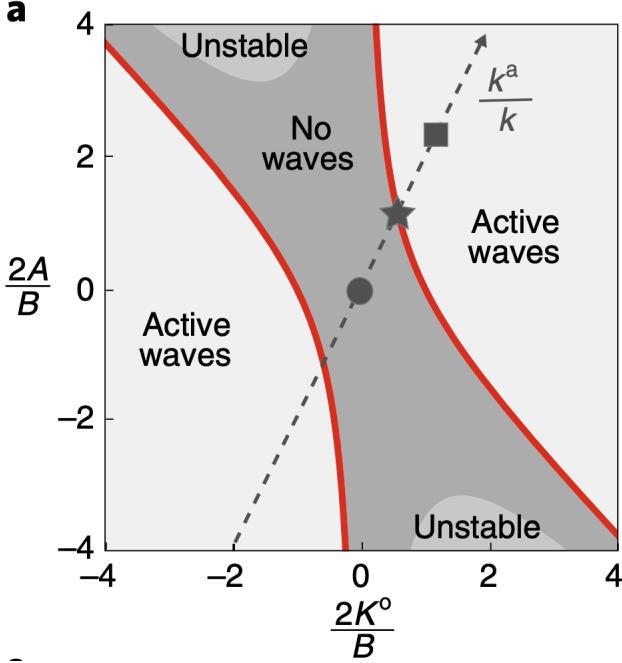
$$(1 + \tilde{G})^2 = 1 - \tilde{K}^0 \tilde{A} - (\tilde{K}^0)^2 \quad (6.10)$$

i.e.:

$$\tilde{A} = -\frac{2\tilde{G} + \tilde{G}^2 + (\tilde{K}^0)^2}{\tilde{K}^0} \quad (6.11)$$

Note that we know that we have instability after this point - we don't know what happens after this (whether we get nonlinear excitations, or whether the lattice begins to melt/crumble etc...)

With this region discussed, we can now understand the entire phase diagram (figure taken from <https://www.nature.com/articles/s41567-020-0795-y>)



A comment; if we had assumed a microscopic force model of $F = -(\kappa^0 \hat{\phi} + \kappa \hat{r})\delta r$, we could coarse grain to find the dependencies of A, K^0, B on the microscopic parameters, and then the knowledge of the microscopic parameter ratio $\frac{\kappa^0}{\kappa}$ would be able to tell us about where in the phase diagram we were located.

Another comment about instability: even if we have a linear microscopic constitutive equation, we could potentially get macroscopic non-linearities coming out of rotations (only 2-d and up - coming from the last term):

$$u_{ij} = \partial_i u_j + \partial_j u_i + \partial_k u_i \partial_k u_j. \quad (6.12)$$

6.3 Exceptional Points

Now, let's talk about what happens on the dashed line of the phase diagram when we cross from the active to the no wave region (star point). We have the ω from Eq. (6.6), and can also compute the eigenvectors. Before, this, a comment about the dynamical matrix - It is not Hermitian if $A, K^0 \neq 0$. This means that the convenient properties - real eigenvalues, orthogonal eigenvectors - are lost. In QM this is necessary for real measurement outcomes and conservation of probability. But for open quantum systems, we require non-Hermitian matrices.

Let's compute the eigenvectors and see that they are non-orthogonal:

$$u_i = \frac{(1 \pm \sqrt{1 - \tilde{K}^0 \tilde{A} - (\tilde{K}^0)^2}) \hat{q}_i - (\tilde{A} + \tilde{K}^0) \epsilon_{ij} \hat{q}_j}{\sqrt{2(1 \pm \sqrt{1 - \tilde{K}^0 \tilde{A} - (\tilde{K}^0)^2}) + \tilde{K}^0 \tilde{A} + (\tilde{K}^0)^2}} \quad (6.13)$$

Unwieldy, but you might be able to convince yourself that we only get orthogonality if $\tilde{K}^0 = A = 0$.

With this obtained we can discuss what happens at the onset of odd waves. There is a point called the *exceptional point*⁴. It is a statement about the fact that a singularity occurs.

⁴In this age of exceptionalism, Vincenzo must apologize and say that this is not that exceptional.

- (a) We can see from Eq. (6.6) that at the exceptional point when $\tilde{A} = \frac{1}{\tilde{K}^0} - \tilde{K}^0$ that the square root term is zero and that the spectrum acquires a degeneracy, where the eigenvalues become the same:

$$\omega = -i \left[\frac{B}{2} + G \pm 0 \right] \quad (6.14)$$

- (b) What happens to the u_+, u_- eigenvectors at this point? They become collinear!

We will encounter these exceptional points again when we study non-reciprocal phase transitions.

6.4 Poisson Ratio and Odd Ratio

Imagine we could measure the spectrum of the odd waves; we would be able to measure $\tilde{K}^0, A, \frac{B}{2}, G$ etc. This would be a dynamical measurement. You might also ask - why could we not do a static measurement? Indeed we could just compress/apply forces to the solid in different ways, and just see how it deforms.

We can consider the Poisson ratio:

$$\nu = -\frac{u_{xx}}{u_{yy}} \quad (6.15)$$

in normal materials, $\nu > 0$ and so if we compress one way, it expands in the other direction. In auxetic materials (e.g. metamaterials like rotating triangular auxetic perforated plates), we have $\nu < 0$ so if we compress it compresses in the other direction too! There is also the case of $\nu \approx 0$, e.g. cork where we compress in one direction and the other direction basically does not change.

What about odd materials? we compress in one direction and then there is a response to an angle. This violates parity - and hence requires parity violating moduli. Note that in this case it is useful to define the odd ratio:

$$\nu^0 = \frac{K^0 B}{(K^0)^2 + G^2 + BG} \quad (6.16)$$

which characterizes the angle at which the material deforms when compressed.

Next time, we look at the case where linear momentum conservation is violated. Then we turn to fluids without viscosity, in particular turbulence in non-viscous fluids.

7 Odd Elastodynamics IV

We study the overdamped limit:

$$\Gamma \partial_t u_i = F_i = \partial_j \sigma_{ij} = \partial_j (K_{ijkl} u_{kl}) = K_{ijkl} \partial_j \partial_l u_k \quad (7.1)$$

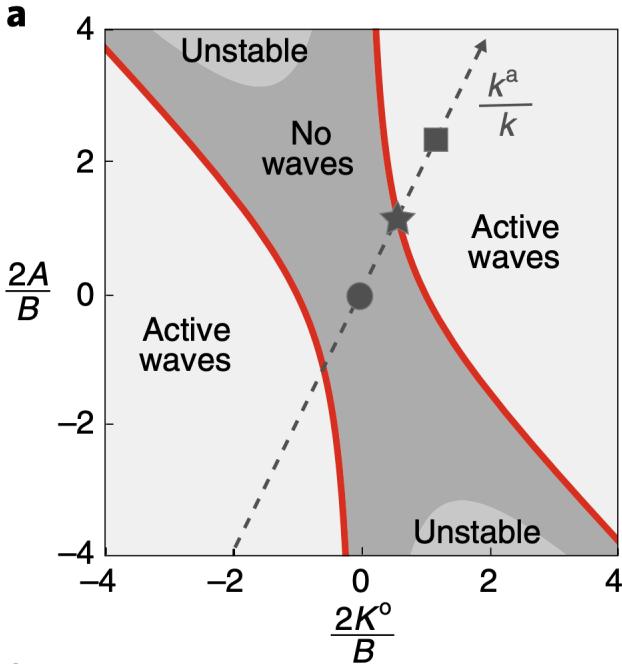
In the space of shears:

$$\Gamma \partial_i \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} G & -K^0 \\ K^0 & G \end{pmatrix} \begin{pmatrix} \nabla^2 u_x \\ \nabla^2 u_y \end{pmatrix} \quad (7.2)$$

or in more full generality, after Fourier transforming and looking at the parallel/perpendicular (to the wavevector) components:

$$-i\Gamma \omega \begin{pmatrix} u_{||} \\ u_{\perp} \end{pmatrix} = q^2 \begin{pmatrix} B + G & K^0 \\ -K^0 - A & G \end{pmatrix} \begin{pmatrix} u_{||} \\ u_{\perp} \end{pmatrix} \quad (7.3)$$

By studying this eigenvalue problem, we get the plot:



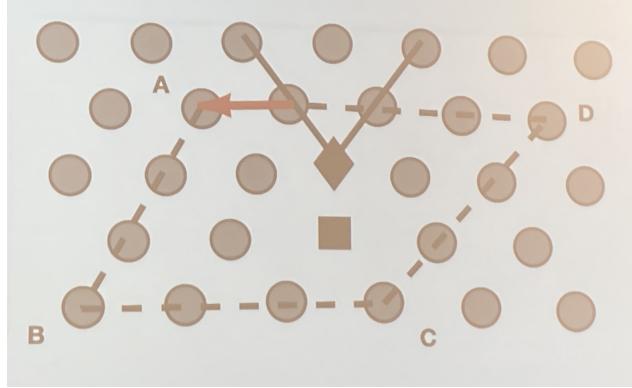
We call the propagating waves “active waves” as they are propagated by the cycles arising from the odd elastic moduli.

The red lines that separate the regions where the waves do not propagate are called “exceptional lines”, meaning they consist of exceptional points - the two eigenvalues of the dynamical matrix become degenerate, and the two corresponding eigenvectors become collinear. It is worth noting that the dynamical matrix we have written down in Eq. (7.3) is Non-Hermitian (as you may have seen in open quantum systems), and thus the eigenvectors are not orthogonal (and in particular the angle between them goes to zero at the exceptional points).

7.1 Topological Defects/Dislocations

We saw some examples last class of auxetic and odd materials. Another system of interest are crystals with topological defects, which exhibit novel behaviour in the presence of stresses. In your homework you do this for the case without odd elastic moduli, but the calculation can also be done in the $K^0 \neq 0$ case, and such behaviour has also been observed by a group at MIT.

In lecture, we watched a video first of rays propagating through a compressed triangular crystal. We then zoomed into it, and saw that it was perfect save for a dislocation where we have made the coordination number 7/5 as opposed to 6. This defect gives rise to an extra row of atoms; for any loop surrounding the defect we are able to detect this extra step. The rays we saw moving through the crystal arose from the ray-like motion of the defects arising from applying a (shear) stress/perturbation to the lattice.



Mathematically, the dislocation (which are analogous to magnetic monopoles) is characterized by the Burges vector, where we do a loop integral around the dislocation:

$$\mathbf{b} = \oint_{\gamma} d\mathbf{r} \cdot \nabla \mathbf{u} \quad (7.4)$$

Then using the force balance condition:

$$0 = \mathbf{f} = \nabla \cdot \boldsymbol{\sigma} \quad (7.5)$$

We have the displacement field:

$$\mathbf{u}(\mathbf{r}) = \frac{1}{2\pi} \left[\phi \mathbf{b} + \frac{1-\nu}{2} \log(r) \boldsymbol{\epsilon} \cdot \mathbf{b} - \frac{1+\nu}{2} \hat{\mathbf{r}} \cdot \mathbf{b} \hat{\phi} - \nu^0 [\log(r) \mathbf{b} + \hat{\phi} \cdot \mathbf{b} \hat{\phi}] \right] \quad (7.6)$$

where the last term is the odd term, with odd ratio:

$$\nu^0 = \frac{BK^0 - AG}{G(B+G) + K^0(A+K^0)} \quad (7.7)$$

this is zero for a regular crystal (the case you study in your homework), but the odd case was studied theoretically by Lara Braverman and Colin Scheibner, as you can read about in <https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.127.268001>. A couple of years later, a team at MIT used the proposal in this Letter to do an experiment with spinning colloids <https://www.nature.com/articles/s41586-022-04889-6>, where they tried to extract the odd elastic parameters from the shear angle (though the experimental data is very noisy...).

An equation that we will not derive, but use, is that of the Peach-Koehler (PK) force. It is the analog of the Lorentz force in EM theory:

$$F_i = q\epsilon_{ijk}v_jB_k \quad (7.8)$$

this is an example of manifest parity violation that you are very familiar with - cyclotron orbits are in a particular direction. There is a mathematically analogous force for dislocations:

$$F_i^{PK} = -\epsilon_{ij}b_k\sigma_{jk} \quad (7.9)$$

which tells you about the force on the dislocation (characterized by the Burges vector \mathbf{b}) as a result of an applied stress. This force is highly orientation dependent.

The crucial feature of the experimental results using spinning colloids is that there are pre-torques/stresses (which violate angular/linear momentum conservation) in σ_{ij} :

$$\sigma_{ij} = \underbrace{\tau_0 \epsilon_{ij}}_{\text{pre-torque}} + \underbrace{p \delta_{ij}}_{\text{pre-stress}} + K_{ijkl} \partial_k u_l \quad (7.10)$$

If we put the pre-torque into the PK force equation:

$$F_i^{PK} = -\epsilon_{ij}\epsilon_{jk} = \delta_{ik}b_k\tau^0 \quad (7.11)$$

where we have used $-\epsilon_{ij}\epsilon_{jk} = \delta_{ik}$. So, we have a force along the direction of the Burges vector, proportional to the pre-torque density τ^0 .

One last comment that we won't have time for - in the Braverman/Scheibner PRL, we have formulas are derived from continuum mechanics - but the Burges vector is a dislocation arising from meso/microscopic phenomena. These details can actually lead to the breakdown of the field-theoretic description, and lead to a reversal in the direction of the force as predicted by the field theory formula. Sometimes we can learn as much from the cases where field theory holds, as well as where it breaks down! In particular, here the field theoretic description holds if defects interact at far field, but breaks down if defects become close.

7.2 Relaxing linear momentum conservation

All of our discussion thus far has arose because we relaxed the assumption of energy conservation. However, throughout we have been assuming:

$$F_i = \partial_j \sigma_{ij} \quad (7.12)$$

which assumes that the medium conserved linear momentum (the above is the divergence of a flux of linear momentum). This is what gave rise to the force being the second spatial derivative of the displacement/strain.

However, it is possible to write a more general force law, if we relax the conservation law. Let us expand F_i in a general way, expanding in gradients of the deformation:

$$F_i = \alpha u_i + \beta_{ijk} \partial_j u_k + K_{ijkl} \partial_j \partial_l u_k \quad (7.13)$$

Let's consider a simple solid that violates Newton's third law⁵. We want no adhesion forces, so we drop the first term:

$$F_i = \beta_{ijk} \partial_j u_k + K_{ijkl} \partial_j \partial_l u_k \quad (7.14)$$

This could be, for example, a fluid that moves through a collection of colloids/droplets. This clearly violates Newton's third law because there is a preferred direction/the colloids feel different forces against/along the fluid flow. A toy model that can describe this would be a chain of 1D masses connected by non-reciprocal (asymmetric) springs:



In this 1-D case, the equation of motion reduces to (still working in the overdamped limit):

$$\gamma \partial_t u = \beta \partial_x u + \kappa \partial_x^2 u \quad (7.15)$$

simple as this looks, this is still quite different from the standard wave equation that is studied in the context of solids:

$$\rho \partial_t^2 u = \kappa \partial_x^2 u \quad (7.16)$$

wherein the speed of sound is $c = \sqrt{\frac{\kappa}{\rho}}$. The fourier transform of this yields the dispersion relation:

$$\omega^2 = c^2 q^2 = \kappa q^2 \quad (7.17)$$

⁵There is a 2025 PNAS paper by Bartolo, Vitelli, Irvine and others that discusses this - and the classification of β_{ijk} , if you wanted to read more about it.

(Where we work in units where $\rho = 1$), or:

$$\omega(q) = \sqrt{\kappa}|q| \quad (7.18)$$

Note then that:

$$\omega(q) = \omega(-q) \quad (7.19)$$

Let's also try studying the spectrum for Eq. (7.15); fourier transforming (equivalently, using the wave ansatz $u(x, t) = Ae^{i(qx+\omega t)}$). We find:

$$\gamma i\omega = iq\beta - \kappa q^2 \quad (7.20)$$

Or multiplying by $-\frac{i}{\gamma}$:

$$\omega = \frac{\beta}{\gamma}q + i\frac{\kappa}{\gamma}q^2 \quad (7.21)$$

There is a propagating wave term coming from $\frac{\beta}{\gamma}q$, and there is a piece that is attenuating $i\frac{\kappa}{\gamma}q^2$. The part that should draw your attention is that $\omega(q)$ - is *not* invariant under $q \leftrightarrow -q$. Indeed in the $\kappa \rightarrow 0$ limit $\omega(q)$ is actually odd under this transformation:

$$\omega(q) = -\omega(-q) \quad (7.22)$$

The non-reciprocity/breaking of mirror symmetry is what enables this strange feature of the spectrum.

8 Self-Propelled Particles/Polar Active Fluids

8.1 Overview of Self-propelled particles

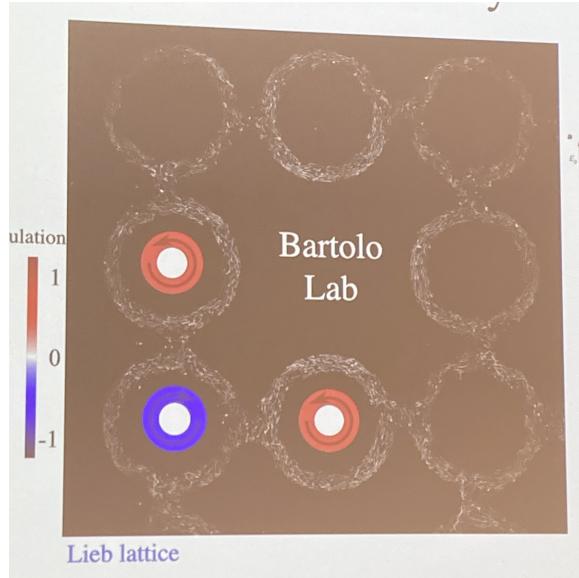
Today we discuss self-propelled particles, also known as polar active fluids. Your next homework will be almost entirely devoted to this topic. This is a new type of active matter, compared to what you have seen thus far in the course. The first experiments on this topic can be found in Pallaci et al. Science (2014), Bricard et al. Nature (2013).

Those in the soft matter class will know that there is one equation that governs the dynamics of simple fluids, namely the Navier-Stokes equation. Those not in the course will learn a bit of simple and complex fluids on the fly.

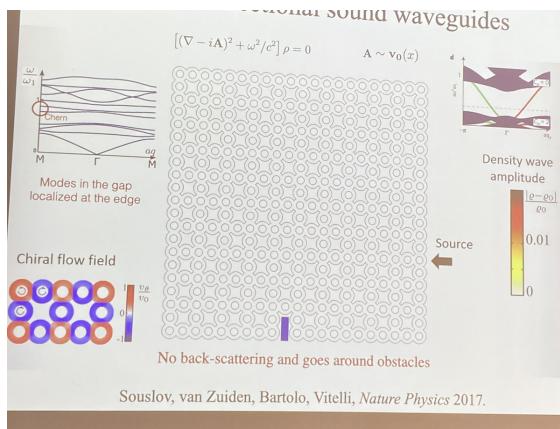
To give a bit of a physical picture, we are imagining that we have a fluid channel, containing many colloids. In the absence of pressure gradient, we see Brownian motion, and in the presence of a pressure gradient, we will see colloid flow. Here we consider the possibility of "active colloids". One example - colloids can have assymmetry ("Janus Particles"), e.g. the two sides can be made up of different material. If we shine light on the particles, the different sides will heat up in different amounts, creating a gradient within the particle itself. It then acts like a rocket that launches fuel more in one direction than the other, triggering the motion.

Another example (which you will become intimately familiar with on the homework) - we can have colloidal rollers that can be polarized via an electric field. Then, by rotating the colloids, we can have the colloids individually flow. If we have a fluid of such colloids, they will want to roll and align with each other - thus without applying any pressure gradient (they are self-driven!), the colloids will spontaneously pick a direction and flow. We will want to derive and analyze the Navier-Stokes equations to describe such self-propelled fluids. On your homework, you will first see how you can get a single particle to self-propel. Then, you will study the linear spectrum/excitations of such a medium. Finally, you will look at non-linear excitations, wherein the colloidal rollers move with a band structure.

You might imagine that there are interesting microfluid experiments that could be done with such exotic fluids. You can engineer lattices to contain such active fluids, and study what the properties of such a lattice are. In a Lieb lattice, such rollers rotate with alternating directions (chirality)!



You can then also study what the sound modes of a channel made of such a fluid would look like - this your task in the homework, to look at density modulations on top of a nonzero velocity field. The sound waves will have a strange structure - density waves that go along with the flow and against the flow will have very different behaviours. You can even put this problem on steroids (Souslov, van Zuiden, Bartolo, Vitelli. *Nature* (2017)), and study such waves in not just one channel but in a 2D lattice - it turns out you can make a topological insulator out of such a lattice! Looking at the band structure, the bulk is gapped. But at the edge, you can have a density wave/excitations, and it is a directed excitation; it travels in a specific direction. In fact such excitations are topologically robust, in that they can flow around obstructions.

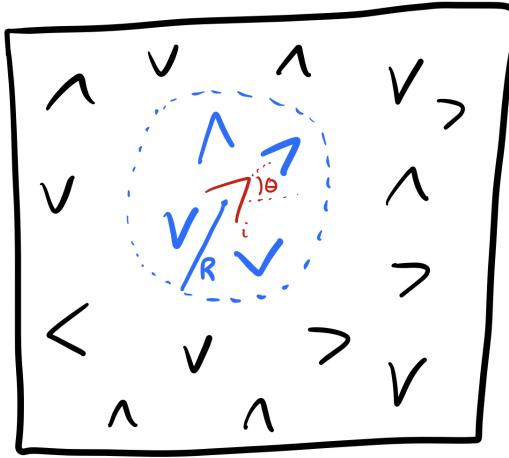


8.2 Vicsek Model

We will study the microscopic model that started this field - the Vicsek model of flying spins. The hydrodynamics of this model shows flocking⁶ behaviour and phase transitions, and will be a useful model for understanding the more contemporary models of active matter that have come after it.

We consider a collection interacting bodies, and consider a mechanism in which the bodies move and align align. Consider the picture below:

⁶Everyone calls this flocking, but no one asked how the birds feel about it...



The update rule for the position of the i th body is:

$$\mathbf{r}_i(t + \Delta t) = \mathbf{r}_i(t) + v_0 \Delta t \begin{pmatrix} \cos(\theta_i(t)) \\ \sin(\theta_i(t)) \end{pmatrix} \quad (8.1)$$

i.e. it takes one step in the direction it is facing. The update rule for the angle/direction of the i th body is:

$$\theta_i(t + \Delta t) = \langle \theta(t) \rangle_{R,i} + \eta_i(t) \quad (8.2)$$

where the first term corresponds to taking the average over the interaction range R of other bodies. $\eta_i(t)$ is a noise term. In the Vicsek model, we can assume that the noise has zero mean:

$$\langle \eta_i(t) \rangle = 0 \quad (8.3)$$

and has correlation (time average):

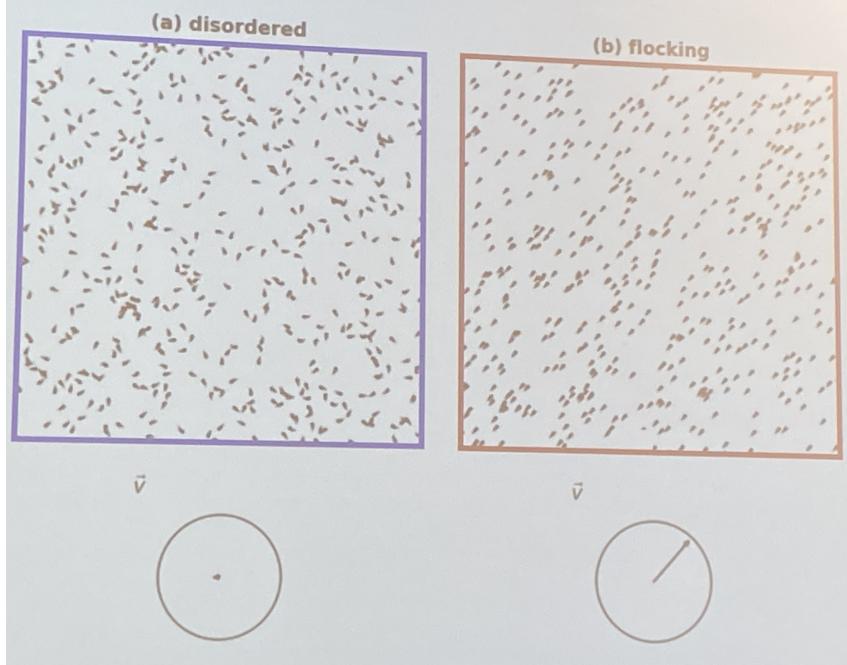
$$\overline{\eta_i(t)\eta_j(t')} = 2\gamma\delta_{ij}\delta_{tt'} \quad (8.4)$$

i.e. the noise is uncorrelated across space and time. The noise is local and Markovian (no memory), sometimes this is also called white noise.

Why is it called flying spins? The alignment of the direction of the bodies is like magnetic/spins which want to align with their neighbours, plus thermal fluctuations. The non-equilibrium dynamics comes from the fact that the spins can move! If the spins were fixed, you can just take the second equation and write down an equilibrium field theory (namely the XY model). But by coupling it into a self-propulsion rule, the model is non-equilibrium! 2D is interesting because the Mermin-Wagner theorem prevents the long-range ordering of the entire system in equilibrium settings.

Vicsek then studied this model just by numerically simulating it (this would be a good thing for you to also simulate as an exercise!) - see Vicsek et al. PRL (1995)

For high noise, we are in the disordered phase where the various spins fly in seemingly random directions. In the low noise, we are in the ordered/flocking phase where all spins move in the same direction. In this phase, we have a nonzero net velocity vector.



If we wanted to promote this to a continuum description, we could write down the energy:

$$E = J \int d^2\mathbf{x} (\nabla \theta)^2 \quad (8.5)$$

If $\frac{J}{k_B T} \ll 1$, thermal fluctuations are strong and we have disorder. For $\frac{J}{k_B T} \gg 1$, it is less clear. In equilibrium settings, in 2D Mermin-Wagner forbids long-range order. But here we have the coupling of the XY spin model to the self-propulsion model, which allows for the restoration of long-range order!

8.3 Toner-Tu Model

See Toner, Tu. PRL (1995) for the original paper.

Now, we will do some phenomenology; we will use the known symmetries of the theory and write down a macroscopic theory. As long as we are smart about it, we will get sensible results. However, we will not get the correct parameter dependencies of the macroscopic model on the microscopic parameters. Let us proceed. What we notice is that depending on the ratio $\frac{J}{\eta}$, we have either disordered or ordered behaviour. If you have taken a course on statistical field theory, you would have seen Landau theories of phase transitions, where we can write down an energy landscale that depends on a continuum spin parameter:

$$E = J \int d^2\mathbf{x} [(\nabla S)^2 + \alpha S^2 + \beta S^4] \quad (8.6)$$

In the limit where S is uniform, the gradient drops out. Then, depending on the α we either have a single minimum at $S = 0$ or degenerate minima at $S = \pm S_0$ (in 2D, we promote the two degenerate minima to a ring of minima, imagine a Sombrero). We can write $\alpha = c(T - T_c) = -c\epsilon$, wherein for $T > T_c$ we have a single mimima and $T < T_c$ we have the degenerate minima. Usually in Landau theory we think about the system being in the energy minimizing configuration. We can also think about the dynamics at which the system comes to an energy minimizing configuration:

$$\partial_t S = -\frac{\delta V(S)}{\delta S} \quad (8.7)$$

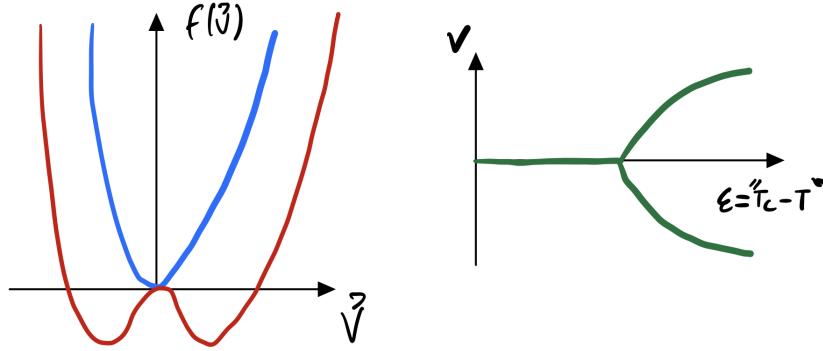
In our case, instead of a continuum spin/magnetization we instead have a velocity field. We have the potential:

$$V(\mathbf{v}) = -\alpha \mathbf{v}^2 + \beta \mathbf{v}^4 \quad (8.8)$$

Which taking the variation we get:

$$\rho \partial_t \mathbf{v} = \alpha \mathbf{v} - \beta |\mathbf{v}|^2 \mathbf{v} \quad (8.9)$$

This describes the bifurcation, but not the fluid aspects of the system. The pictures look like:



Where for $\epsilon > 0$ we have pitchfork bifurcation. Much like in the magnet case, as a function of noise (temperature) we have a phase transition as detectable in the velocity (magnetization).

To account for the fluid aspects of the system, we think back to Navier-Stokes. One equation is the continuity/density conservation equation:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (8.10)$$

and from the other equation, we get insight on how to modify Eq. (8.9) - we add pressure and viscosity terms on the RHS, as well as the convective term on the LHS:

$$\boxed{\rho [\partial_t \mathbf{v} + \lambda_1 (\mathbf{v} \cdot \nabla) \mathbf{v}] = -\nabla P + \eta \nabla^2 \mathbf{v} + \alpha \mathbf{v} - \beta |\mathbf{v}|^2 \mathbf{v}} \quad (8.11)$$

Note the one deviation from Navier-Stokes; there the invariance under boosts enforced $\lambda_1 = 1$, but we lose that here now that there is a preferred velocity. The above is the so-called Toner-Tu model. Minimally, it is as if we “glue together” pitchfork bifurcation with Navier-Stokes.