# PHYS 444 (Quantum Field Theory II) Notes

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**Introduction:**

This is a set of lecture notes taken from UChicago's PHYS 444 (Quantum Field Theory II), taught by Luca Delecretaz. Topics covered include...

## Contents

# 1 Fermions - Representations of Lorentz

## 1.1 Introduction

In QFTI we explored scalar QFT deeply; both perturbatively and non-perturbatively. We looked at the path integral formalism, renormalization, and scattering. But the elephant in the room is we did this all with scalar fields, and most particles in the universe... are not. They transform with different representations of the Lorentz group; the scalar field is simply the simplest one.

In this course, we will look at fermions; they are qualitatively different due to Pauli exclusion and will lead to new physics. Building on this, we can build Lagrangians in 3+1 dimensions with dimensionless couplings, and moreover different kinds of dimensionless couplings. This will lead us into quantum electrodynamics eventually; but let's start with a discussion of fermions.

## 1.2 Representations of the Lorentz group; scalar and vector representations

Recall the Lorentz group $O(1,3)$; it consists of all transformations that leaves spacetime distance invariant, i.e. all transformations:

$$x^\mu \to \Lambda^\mu_\nu x^\nu \tag{1.1}$$

such that:

$$\Lambda^\mu_\alpha \Lambda^\nu_\beta \eta^{\alpha\beta} = \eta^{\mu\nu} \tag{1.2}$$

with $\eta = \mathrm{diag}(-1,1,1,1)$ the Minkowski metric.

We have already studied its simplest representation on fields, the scalar field:

$$\phi(x) \to \phi'(x') = \phi(x) \tag{1.3}$$

This is not a trivial representation (the field does transform), but in a way that is completely absorbed by the transformation of the coordinates.

This is realized on Hilbert space/quantum fields $\hat{\phi}$ by a unitary operator:

$$\hat{\phi}(x) \to U(\Lambda)^{-1}\hat{\phi}(x)U(\Lambda) = \hat{\phi}(\Lambda^{-1}x). \tag{1.4}$$

Out of scalar fields, we could build composite objects that transform in potentially other representations. The simplest example is taking higher powers of the field, $[\phi(x)]^n$ still transforms as a scalar. But, consider $A_\mu = \partial_{x^\mu}\phi(x)$; this transforms in a 4-vector representation:

$$A_\mu(x) \equiv \partial_{x^\mu}\phi(x) \to U(\Lambda)^{-1}\partial_{x^\mu}\phi(x)U(\Lambda) = \partial_{x^\mu}\phi(\Lambda^{-1}x) = \partial_{x^\mu}\phi(\bar{x}) = \frac{\partial \bar{x}^\nu}{\partial \bar{x}^\mu}\partial_{x^\mu}\phi(\bar{x}) = \Lambda_\mu^{\ \nu}\partial_{\bar{x}^\nu}\phi(\bar{x}) \tag{1.5}$$

Where in the last equality we use that new coordinates are linearly related to the old coordinates, with:

$$\frac{\partial \bar{x}^\nu}{\partial x^\mu} = (\Lambda^{-1})^\mu_{\ \nu} = \Lambda_\mu^{\ \nu} \tag{1.6}$$

And thus:

$$A_\mu(x) \to \Lambda_\mu^{\ \nu} A_\nu(\bar{x}) \tag{1.7}$$

or:

$$U(\Lambda)^{-1}A_\mu(x)U(\Lambda) = \Lambda_\mu^{\ \nu}A_\nu(\Lambda^{-1}x) \tag{1.8}$$

Under the spatial rotation subgroup $O(3) \subset O(1,3)$, this splits into a scalar part for the time component $A_0$ and a 3-vector part for the spatial components $A_i$.

The photon is a particle in this 4-vector representation of Lorentz. In that case, $A_\mu$ is the (fundamental) group field in Maxwell's equations, with the field:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.9}$$

or:

$$E_i = \partial_0 A_i - \partial_i A_0, \quad B_i = \epsilon_{ijk}\partial_j A_k \tag{1.10}$$

## 1.3 General Representation of Lorentz

We will come back to the photon later in the course, but let us turn to the question of building a general representation of the Lorentz group. To this end, we study its algebra. It contains the generator of rotations $J_i$:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad (\mathfrak{su}(2) \text{ algebra}) \tag{1.11}$$

which forms a subalgebra. It also contains the generator of boosts $K_i$:

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \tag{1.12}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k. \tag{1.13}$$

but the $K_i$ do not form a closed subalgebra. Rotations and boosts together form the full Lorentz algebra, which we derived last quarter through the study of infinitesimal transformations.

We consider the following linear combinations of generators:

$$J_i^\pm = \frac{1}{2}(J_i \pm iK_i). \tag{1.14}$$

It can be easily checked that the $J^+$s form a closed subalgebra:

$$[J_i^+, J_j^+] = \frac{1}{4}[J_i + iK_i, J_j + iK_j] = \frac{1}{2}(i\epsilon_{ijk}J_k - \epsilon_{ijk}K_k) = i\epsilon_{ijk}\frac{1}{2}(J_k + iK_k) = i\epsilon_{ijk}J^+ \tag{1.15}$$

Moreover the $J^+$s commute with the $J^-$s:

$$[J_i^+, J_j^-] = \frac{1}{4}[J_i + iK_i, J_j - iK_j] = 0. \tag{1.16}$$

and the $J^-$s have the same commutation relation with each other as the $J^+$s:

$$[J_i^-, J_j^-] = i\epsilon_{ijk}J_k^- \tag{1.17}$$

Thus, we conclude that the Lorentz algebra is just 2 commuting $\mathfrak{su}(2)$ algebras! We thus conclude that:

$$\mathfrak{so}(1,3) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \tag{1.18}$$

The usual representation of $\mathfrak{su}(2)$ is labelled by a half-integer, and thus the irreducible representations of the Lorentz group are thus labelled by two half-integers $(j^-, j^+)$, with:

$$j^\pm = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \tag{1.19}$$

Thus a general field $\Psi_{ab}(x)$ has two indices, with $a = 1, \dots, 2j_- + 1$ and $b = 1, \dots, 2j_+ + 1$.

## 1.4 Examples of Lorentz Representations

For a scalar field, we have the $(0,0)$ representation. In this case, the indices $a, b$ run from 1 to 1 so we just omit the indices and write $\phi(x)$.

For rotations, we have $J_i = J_i^+ + J_i^-$. The $(j^-, j^+)$ irrep (irreducible representation) of Lorentz therefore decomposes into spin:

$$|j^- - j^+|, |j^- - j^+| + 1, \dots, j^- + j^+ \tag{1.20}$$

under rotations.

Let's look at a slightly more interesting example. What is the 4-vector representation? A first guess is that a vector looks like spin-1, so we might put $(1,0)$; but this doesn't work because under rotation it doesn't split into a scalar rotation $A_0$ and a 3-vector $A_i$. Thus, the correct representation turns out to be $(\frac{1}{2}, \frac{1}{2})$. Under spatial rotation, this decomposed into a spin 0 ($A_0$) and spin 1 ($A_i$) piece, which is what we need.

This skipped one representation though! What about $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ (these are the smallest non-scalar irreps)? Well, first note that these reps are related by a parity $\mathbf{x} \to -\mathbf{x}$, as under this reflection angular momentum is invariant and boosts are flipped, i.e.:

$$\mathbf{J} \to \mathbf{J}, \quad \mathbf{K} \to -\mathbf{K} \tag{1.21}$$

and so:

$$J_i^{\pm} \to J_i^{\mp} \tag{1.22}$$

Hence a $(\frac{1}{2}, 0)$ particle cannot be parity invariant on its own. We will later study these in "chiral" theories. This rep is the left or right handed Weyl (or Majorana[1]) spinor.

## 1.5 Properties of the Weyl spinor

We're in the USA, so let's pick the right-handed representation $(0, \frac{1}{2})$. In this rep:

$$J_+^i = \frac{1}{2}\sigma^i \tag{1.23}$$

which satisfy:

$$[J_+^i, J_+^j] = i\epsilon_{ijk} J_k \tag{1.24}$$

which is a 2-d rep of $\mathfrak{su}(2)_+$. The minus matrices are trivial, with:

$$J_-^i = 0. \tag{1.25}$$

giving the 1-d rep of $\mathfrak{su}(2)_-$. These two objects act on different spaces (one on the right, one on the left index). If we want to see them as the representation of the entire group $SU(2) \times SU(2)$, we can write:

$$J_i^+ = \mathbb{I} \otimes \frac{1}{2}\sigma_i, \quad J_i^- = 0 \otimes \mathbb{I} \tag{1.26}$$

From this we can also get the generators of rotations and boosts:

$$J_i = J_i^+ + J_i^- = \frac{1}{2}\sigma_i \tag{1.27}$$

$$K_i = \frac{1}{i}(J_i^+ - J_i^-) = -\frac{i}{2}\sigma_i \tag{1.28}$$

We therefore have a 2-d representation of Lorentz $(0, \frac{1}{2})$ on fields $\psi_R = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. For a Lorentz transformation $\Lambda$ of angles $\theta_i$ and rapidities $\eta_i$, i.e. $\Lambda = e^{i(\theta^i J_i + \eta^i K_i)}$, we have the fields transform as:

$$U(\Lambda)^{-1}\psi_R(x)U(\Lambda) = e^{\frac{i}{2}(\theta_i - i\eta^i)\sigma_i}\psi_R(\Lambda^{-1}x) \tag{1.29}$$

note that the rotation part is unitary but the boost part is not. Because the boost generators are related to the $J^{\pm}$s via an $i$, in any representation it will be anti-unitary:

$$K_i^{\dagger} = -K_i \tag{1.30}$$

[1] in 3+1 dimensions they coincide, but in other dimensions they differ, hence the two different names.

but do not confuse this with the full transformation on the Hilbert space, which is always unitary. Another comment; this implies that $\psi_R$ must be complex, as the transformation does not take real to real. In fact, the anti-unitarity of the $K_i$s implies that $\psi_R^*$ transforms in the parity flipped $(\frac{1}{2},0)$ representation.

Another interesting feature of this representation is that under a $2\pi$ rotation, say, around the $\hat{\mathbf{z}}$ axis (so $\theta^z = 2\pi$), the field transforms to:

$$\psi_R'(x) = e^{\frac{i}{2}(2\pi\sigma_z)}\psi_R(\Lambda^{-1}x) = e^{i\pi\sigma_z}\psi_R(x) = -\psi_R(x) \tag{1.31}$$

note that $\Lambda^{-1}x = x$ as the coordinate comes back to itself. So, in scalar field theory a $2\pi$ rotation leaves the field invariant. But here, the field does *not* transform back into itself due to the negative sign out front. This is a first hint of the spin-statistics theorem, which we will soon discuss; but this theorem will tell us that the particles must be fermionic. The statistics are tied to the representation of the Lorentz group.

## 1.6 Building a Lagrangian for the Weyl spinor

Let us start by building a quadratic/Gaussian Lagrangian for $\psi_R$. We could try tensoring $\psi_{Ra}^*\psi_{Rb}$. Since $\psi_R \in (0,\frac{1}{2})$ and $\psi_R^* \in (\frac{1}{2},0)$, from group theory:

$$\psi_{Ra}^*\psi_{Rb} \in (0,\frac{1}{2}) \otimes (\frac{1}{2},0) = (0 \otimes \frac{1}{2}, \frac{1}{2} \otimes 0) = (\frac{1}{2},\frac{1}{2}) \tag{1.32}$$

which as we discussed previously, transforms like a 4-vector. But it doesn't look like one... it looks like a $2 \times 2$ matrix. But there should be a way to map it to a 4-vector. What are the possible bilinears we can build? Well, we can consider:

$$A_0 = \psi_R^\dagger\psi_R \tag{1.33}$$

$$A_i = \psi_R^\dagger\sigma_i\psi_R \tag{1.34}$$

How do these transform? Under infinitesimal transformations, we have that:

$$\psi_R \to \psi_R' \approx (1 + iJ_i\theta_i)\psi_R = (1 + \frac{i}{2}\sigma_i\theta_i)\psi_R \tag{1.35}$$

from these we can deduce how the other objects transform. Namely:

$$A_0 \to \psi_R^\dagger(1 - \frac{i}{2}\sigma_i\theta_i)(1 + \frac{i}{2}\sigma_i\theta_i)\Psi_R \approx \psi_R^\dagger\psi_R \tag{1.36}$$

which we see is invariant under infinitesimal transformations $\theta \ll 1$, making it a good candidate for $A_0$/the scalar part. Let's next consider $A_i$:

$$A_i \to \psi_R^\dagger(1 - \frac{i}{2}\sigma_j\theta_j)\sigma_i(1 + \frac{i}{2}\sigma_j\theta_j)\psi_R \approx \psi_R^\dagger(\sigma_i + \frac{i}{2}[\sigma_i,\sigma_j]\theta_j)\Psi_R = A_i - \epsilon_{ijk}\theta_j A_k \tag{1.37}$$

This is how a 3-vector transfrms under spatial rotations, again justifying our guess for what the vector part $A_i$ should be as a good one. Repeating this exercise with boosts, one finds that the entire object is indeed a 4-vector under rotations:

$$A_\mu \equiv \Psi_R^\dagger\sigma_\mu\Psi_R \tag{1.38}$$

with:

$$\sigma_\mu = (\sigma_0 = \mathbb{I}, \sigma_i = \text{Paulis}) \tag{1.39}$$

is a 4-vector field under rotations. To recap, we had group theory technology that told us that the product $\psi_{Ra}^*\psi_{Rb}$ should transform like a 4-vector. We then constructed a map which actually gave us what the 4 components should be, inspired by the group theoretic trick.

Now that we know this, do we have ideas for what Lagrangians we can build out of the Weyl spinor? Note that the Lagrangian must be a scalar field. Consider as a first go[2]:

$$\mathcal{L} = \partial_\mu(\psi_R^\dagger \sigma^\mu \psi_R) \tag{1.40}$$

but then this is a total derivative, so there is no interesting (bulk) physics. Refining our guess, let's let the derivative act on one of the fields:

$$\mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R \tag{1.41}$$

which is indeed the Lagrangian for a (right-handed) Weyl spinor. It is qualitatively different from the scalar in that it only has one derivative (c.f. the scalar field Lagrangian had two derivatives) - the counting of degrees of degrees of freedom are different.

On Thursday, we will try to construct a mass term. We will also remedy the fact that $\mathcal{L}$ is not a parity-invariant theory, which is something that we will need for electrons.

# 2 Fermions - Dirac Fermions

## 2.1 Review of Representations

A general lorentz-invariant field is labelled by 2 $\mathfrak{su}(2)$ quantum numbers $(j^-, j^+)$. For example, $\phi \in (0,0)$, $A_\mu \in (\frac{1}{2}, \frac{1}{2})$, $\psi_R \in (0, \frac{1}{2})$, $\psi_L \in (\frac{1}{2}, 0)$. The matrices $\sigma^\mu = (\mathbb{I}, \sigma^i)$ provide an explicit map from $\psi_{Ra}^\dagger \psi_{Rb} \in (0, \frac{1}{2}) \otimes (\frac{1}{2}, 0) = (\frac{1}{2}, \frac{1}{2})$ to $A_\mu \equiv \psi_R^\dagger \sigma_\mu \psi_R$ (you will look at this in more detail in PS1). This gave us the free-field/Gaussian Lagrangian for $\psi_R$s of the form:

$$\mathcal{L}_{\text{Weyl, R}} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R. \tag{2.1}$$

This is a fine Lagrangian, and you will study it in PS1, and construct its mass term. However, it's not good for describing particles such as the electron, as it is not parity invariant (we saw that parity exchanges $J_i^+ \leftrightarrow J_i^-$ (with $J_i^\pm = \frac{1}{2}(J_i + iK_i)$)). Our goal today is to construct the simplest parity invariant theory of spinors. To this end, we will need both $\psi_R$ and $\psi_L$.

## 2.2 Parity invariant theory of spinors - kinetic term

The Weyl Lagrangian for $\psi_L$ has the form:

$$\mathcal{L}_{\text{Weyl, L}} = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L \tag{2.2}$$

but the $\bar{\sigma}^\mu$s here are not the same as before - what are they? We could construct them like last time, but we can instead play a trick. Last class we discussed that $\psi_L^* \in (0, \frac{1}{2})$. In PS1, we will show that more precisely, $\sigma_2 \psi_L^*$ transforms like $\psi_R$. This tells us that we can again build a 4-vector:

$$(\sigma_2 \psi_L^*)^\dagger \sigma_\mu \sigma_2 \psi_L^* = \psi_L^T \sigma_2 \sigma_\mu \sigma_2 \psi_L^* = (\psi_L^T \sigma_2 \sigma_\mu \sigma_2 \psi_L^*)^T = -\psi_L^\dagger \sigma_2^T \sigma_\mu^T \sigma_2^T \psi_L = -\psi_L^\dagger \sigma_2 \sigma_\mu^T \sigma_2 \psi_L \tag{2.3}$$

from which we find that:

$$\bar{\sigma}_\mu = \sigma_2 \sigma_\mu^T \sigma_2 = \begin{cases} \mathbb{I} & \text{if } \mu = 0 \\ -\sigma_i & \text{if } \mu = 1,2,3 \end{cases} \tag{2.4}$$

which follows from the fact that $\left\{\sigma_i, \sigma_j\right\} = 2\delta_{ij}$ (the Paulis anticommute unless they are the same). Thus:

$$\bar{\sigma}_\mu = (\mathbb{I}, -\sigma_i) \tag{2.5}$$

[2] $A_\mu A^\mu$ is also a valid choice for Lorentz invariance, but is quartic in $\psi_R$, so we will consider it when we go to interacting theories

It is now easy to make the Lagrangian parity invariant! We simply add both the left and right moving kinetic terms with the same coefficent:

$$\mathcal{L}_{\text{kin}} = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R \sigma^\mu \partial_\mu \psi_R \tag{2.6}$$

Does this Lagrangian have additional symmetries? Our guiding principle was Lorentz invariance (and we also have parity and translation invariance). But we have the additional symmetry of $U(1)_L \times U(1)_R$, where $\psi_L \to e^{i\alpha_L}\psi_L$ and $\psi_R \to e^{i\alpha_R}\psi_R$. It's good that we have this if we want this theory to model electrons, because electric charge is conserved (though for electrons we do not need two conserved charges, of course).

## 2.3 Parity invariant theory of spinors - mass term

The next question we can ask is whether we can construct a mass term for this theory. We can use our group theory logic - if we look at:

$$\psi_R\psi_R \in (0, \frac{1}{2}) \otimes (0, \frac{1}{2}) = (0, \frac{1}{2} \otimes \frac{1}{2}) = (0, 0) + (0, 1) \tag{2.7}$$

we see that we should be able to get a scalar part $(0, 0)$ - so we should get a mass term! In PS1, you will find the unique way to build this Weyl/Majorana mass term looks like $\psi_R^T \sigma_2 \psi_R$. Now, the issue is this mass term no longer has the $U(1)_R$ symmetry acting in the way we described above. But is there away to add the mass term to the total Lagrangian (for both spinors) such that the theory still has some $U(1)$ symmetry? Indeed, we recall that $\psi_L^* \in (0, \frac{1}{2})$ and so if we replace $\psi_R \to \sigma_2\psi_L^*$ we will be able to recover a $U(1)$ symmetry. Our candidate mass term is:

$$(\sigma_2\psi_L^*)^T \sigma_2 \psi_R = \psi_L^\dagger \sigma_2^T \sigma_2 \psi_R = -\psi_L^\dagger \psi_R \tag{2.8}$$

To make it real we add this to its complex conjugate, so the mass term is:

$$\mathcal{L}_m = -m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \tag{2.9}$$

If you doubt the Lorentz invariance of this Lagrangian, you can check that:

$$\begin{aligned} \delta\psi_L &= i(\theta^i + i\eta^i)\frac{1}{2}\sigma_i \psi_i \\ \delta\psi_L &= i(\theta^i - i\eta^i)\frac{1}{2}\sigma_i \psi_i \end{aligned} \tag{2.10}$$

so at linear order the Lagrangian is invariant under Lorentz transformations/the contributions cancel.

## 2.4 The Dirac Fermion

We can package left and right moving spinors into a single 4-component spinor:

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_{L,1} \\ \psi_{L,2} \\ \psi_{R,1} \\ \psi_{R,2} \end{pmatrix} \tag{2.11}$$

which allows us to write the mass term as:

$$\mathcal{L}_m = -m(\psi_L^\dagger \psi_R^\dagger) \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = -m\Psi^\dagger \gamma^0 \Psi = -m\bar{\Psi}\Psi \tag{2.12}$$

where we have defined $\Psi^\dagger\gamma^0 = \bar{\Psi}$. We use this simplifying notation so that we can quickly identify Lorentz invariant quantities. Note that $(\gamma^0)^2 = 1$.

Doing the same with the kinetic term, we have:

$$\begin{aligned}
\mathcal{L}_{\text{kin}} &= i(\psi_L^\dagger \psi_R^\dagger) \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \partial_\mu \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\
&= i\bar{\Psi}\gamma^0 \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \partial_\mu \Psi \\
&= i\bar{\Psi} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \partial_\mu \Psi \\
&= i\bar{\Psi} \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \partial_\mu \Psi \\
&= i\bar{\Psi}\gamma^\mu \partial_\mu \Psi
\end{aligned} \tag{2.13}$$

where in the second equality we use that $(\psi_L^\dagger \psi_R^\dagger) = (\psi_L^\dagger \psi_R^\dagger)(\gamma^0)^2 = \bar{\Psi}\gamma^0$. And so the total Lagrangian is that of a Dirac fermion:

$$\boxed{\mathcal{L}_{\text{Dirac}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_m = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi} \tag{2.14}$$

## 2.5 $\gamma$-matrix anticommutation relations

We defined the Dirac $\gamma$-matrices above, which have simple anti-commutation relations:

$$\gamma^\mu \gamma^\nu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{\sigma}^\nu \\ \sigma^\nu & 0 \end{pmatrix} = \begin{pmatrix} \bar{\sigma}^\mu \sigma^\nu & 0 \\ 0 & \sigma^\mu \bar{\sigma}^\nu \end{pmatrix} \tag{2.15}$$

So then:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + (\mu \leftrightarrow \nu) = \begin{pmatrix} \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu & 0 \\ 0 & \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu \end{pmatrix} \tag{2.16}$$

If $\mu = \nu = 0$ then we get $2\mathbb{I}$, if $\mu = 0, \nu = i$ then we get 0, and if $\mu = i, \nu = i$ we then have $-\left\{\sigma^i, \sigma^j\right\} = -2\delta^{ij}\mathbb{I}$. So at the end of the day, we get:

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} = -2\eta^{\mu\nu} \tag{2.17}$$

Here we really derived this, but there is an alternative approach that Dirac used. In his approach, you instead start with $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$ as the defining property of $\gamma$ matrices; this is what you will explore in PS2. Nevertheless, this anticommutation property will be a very useful one for manipulating/simplifying expressions (without using the explicit form of the $\gamma$ matrices).

## 2.6 The Dirac equation

By varying the action w.r.t. the spinor, we obtain the EoM:

$$0 = \frac{\delta S}{\delta \bar{\Psi}} \implies \boxed{0 = (i\gamma^\mu \partial_\mu - m)\Psi} \tag{2.18}$$

this equation *implies* the Klein-Gordon equation. How to see this? Apply $(i\gamma^\mu\partial_\mu + m)$ to both sides:

$$0 = (i\gamma^\mu \partial_\mu + m)(i\gamma^\mu \partial_\mu - m)\Psi = -(m^2 + \gamma^\mu\gamma^\nu \partial_\mu \partial_\nu)\Psi \tag{2.19}$$

Since derivatives commute $\partial_\mu \partial_\nu$ is symmetric, WLOG we can replace $\gamma^\mu \gamma^\nu$ with the symmetric part:

$$0 = -(m^2 + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu)\Psi = (\Box - m^2)\Psi \tag{2.20}$$

where we have used $\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu = -\eta^{\mu\nu}\partial_\mu \partial_\nu = -\Box$. This tells us that there will be many similar structures as we saw in our analysis of the scalar field. Historically, Dirac wanted to find the relativistic version of the Schrodinger equation; to him $\Psi$ was a wavefunction. To us, it is a field.

## 2.7 Parity of the Dirac Lagrangian

Let us say one more thing about parity; we wanted L.I. when constructing our Lagrangian, but we also wanted it to be parity invariant. Parity flips $\psi_L$ and $\psi_R$, so we expect it to flip:

$$\Psi(t, \mathbf{x}) \leftrightarrow \eta \gamma^0 \Psi(t, -\mathbf{x}) \tag{2.21}$$

with $\eta \in \mathbb{C}$. We then observe:

$$(P^{-1})^2 \Psi P^2 = \eta^2 \gamma^0 \gamma^0 \Psi = \eta^2 \Psi \tag{2.22}$$

We may want to say that this should be equal to $\Psi$ because twicefold parity should leave $\Psi$ invariant. But $\Psi$ in itself is not an observable, but $\Psi\Psi$ will be. Thus this gives us the option of $\eta^2 = \pm 1$. Srednicki chooses $\eta = i$. For now, we take $\eta = 1$. Let's check the invariance of the Dirac lagrangian explicitly. We start with the mass term:

$$\mathcal{L}_m(x) = -m\bar{\Psi}(x)\Psi(x) = -m\Psi^\dagger(x)\gamma^0\Psi(x) \to -m\Psi^\dagger(Px)(\gamma^0)^\dagger\gamma^0\gamma^0\Psi(Px) = -m\bar{\Psi}(Px)\Psi(Px) = \mathcal{L}_m(Px) \tag{2.23}$$

with $Px = (t, -\mathbf{x})$. Let's also check the kinetic term:

$$\mathcal{L}_{\text{kin}}(x) = i\Psi^\dagger(x)\gamma^0\gamma^\mu\partial_\mu\Psi(x) \to i\Psi^\dagger(Px)\gamma^0\gamma^0\gamma^\mu\gamma^0(\partial_0, -\partial_i)\Psi(Px) \tag{2.24}$$

But now $\gamma^\mu\gamma^0 = \gamma^0\gamma^\mu$ if $\mu = 0$ and $= -\gamma^0\gamma^\mu$ if $\mu = 1, 2, 3$, so the signs from the derivative and the gamma matrices cancel, and so:

$$\mathcal{L}_{\text{kin}}(x) = i\Psi^\dagger(Px)\gamma^0\gamma^0\gamma^0\gamma^\mu\partial_\mu\Psi(Px) = i\bar{\Psi}(Px)\gamma^\mu\partial_\mu\Psi(Px). \tag{2.25}$$

Thus the Dirac Lagrangian is even under parity. It is also easy to construct Lagrangians that are parity odd. We could have instead built terms of the form $-\mathcal{L}_L + \mathcal{L}_R$, or by using a final gamma matrix:

$$\gamma^5 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} = \text{diag}(-1, -1, 1, 1). \tag{2.26}$$

For example, a parity odd kinetic term would simply be:

$$i\bar{\Psi}\gamma^\mu\gamma^5\partial_\mu\Psi \tag{2.27}$$

without the $\gamma^5$, this evaluates to $i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L + i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R$. With it, it flips the sign of the left part, so:

$$i\bar{\Psi}\gamma^\mu\gamma^5\partial_\mu\Psi = -i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L + i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R. \tag{2.28}$$

We could also have a parity odd mass term:

$$\tilde{m}i\bar{\Psi}\gamma^5\Psi = i\tilde{m}(\psi_L^\dagger\psi_R^\dagger)\gamma^0\gamma^5\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = i\tilde{m}(\psi_L^\dagger\psi_R^\dagger)\begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = i\tilde{m}(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L) \tag{2.29}$$

It is pretty clear that this is parity odd by looking at the Weyl fermions, but in fact we can check this directly from the expressions for the Dirac fermions. Using that:

$$\left\{\gamma^5, \gamma^0\right\} = \gamma^5\gamma^0 + \gamma^0\gamma^5 = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} = 0 \tag{2.30}$$

(and in fact $\left\{\gamma^5, \gamma^\mu\right\} = 0$) we find:

$$\bar{\Psi}\gamma^5\Psi \to \Psi^\dagger\gamma^0\gamma^0\gamma^5\gamma^0\Psi = -\Psi^\dagger\gamma^0\gamma^5\Psi = -\bar{\Psi}\gamma^5\Psi \tag{2.31}$$

Now that we have the $\gamma$s, it is easy to build bi-spinor objects in a given Lorentz representation. For parity-even, we have scalars:

$$\bar{\Psi}\Psi, \bar{\Psi}\Box\Psi, \ldots \tag{2.32}$$

we have vectors:

$$\bar{\Psi}\gamma_\mu\Psi, \ldots \tag{2.33}$$

we have tensors:

$$\bar{\Psi}\gamma_\mu\gamma_\nu\Psi, \ldots \tag{2.34}$$

and if we want axial/parity-odd objects we can insert $\gamma^5$s. For example axial scalars:

$$\bar{\Psi}\gamma^5\Psi, \ldots \tag{2.35}$$

axial vectors:

$$\bar{\Psi}\gamma^5\gamma_\mu\Psi, \ldots \tag{2.36}$$

and axial tensors:

$$\bar{\Psi}\gamma^5\gamma_\mu\gamma_\nu\Psi, \ldots \tag{2.37}$$

# 3 Spin-Statistics Theorem, Quantizing the Dirac Lagrangian

## 3.1 Exchange statistics

We have seen that a general representation of the Lorentz group on a field takes the form:

$$\psi(x) \to \psi'(x) = U(\Lambda)^{-1}\psi(x)U(\Lambda) = D(\Lambda)\psi(\Lambda^{-1}x) \tag{3.1}$$

where $D(\Lambda)$ is the finite-dimensional representation of the Lorentz group acting on the indices $\psi_a$. In other words, it acts on the coordinate and then also acts on the fields themselves via a matrix.

We have also seen that for half-integer spin representations, $D(2\pi) = -1$ (i.e. the matrix that realizes a $2\pi$ rotation of spinors gives negative of the identity). We saw this in the case of Weyl and Dirac spinors/fermions, but this is much more general; any half-integer representation (any where $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$ sums to a half-integer) of $SU(2)$ has this property. This is a hint towards the spin-statistics theorem and will be an ingredient in its proof. We will show that this requires that the corresponding field has fermionic statistics, i.e. obey the anticommutation relations:

$$\left\{\hat{\psi}(\mathbf{x}), \hat{\psi}(\mathbf{y})\right\} = 0 \quad (\mathbf{x} \neq \mathbf{y}) \tag{3.2}$$

These operators create states/wavefunctions that are antisymmetric under exchange of identical particles, i.e.:

$$|\ldots, \mathbf{x}, \mathbf{y}, \ldots\rangle = -|\ldots, \mathbf{y}, \mathbf{x}, \ldots\rangle. \tag{3.3}$$

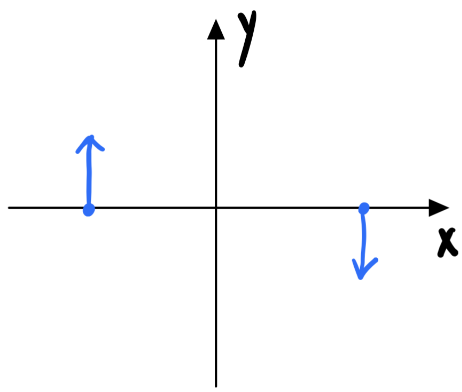
Note that $\pm$ are the only options (at least, in 3+1d). Because the particles are indistinguishable, the states have to be equivalent; they can at most differ by a phase:

$$|\ldots, \mathbf{x}, \mathbf{y}, \ldots\rangle = e^{i\phi}|\ldots, \mathbf{y}, \mathbf{x}, \ldots\rangle \tag{3.4}$$

But exchanging twice is continuously deformable to doing nothing (we can use the "third dimension" to continuously deform this double exchange to the identity identity), so $e^{2i\phi} = 1$ which implies $e^{i\phi} = \pm 1$. But note that being in 3-d is crucial; in 2D we are confined to the plane, so if we try to take our particles and deform the double exchange to the identity, we can only accomplish this by making the particles hit each other.

So in summary, in 3+1d we can only have bosonic $(+1)$ or fermionic $(-1)$ statistics. But in 2+d1 we can in general have $e^{2i\phi} \neq 1$, and have "any" statistics $e^{i\phi}$, i.e. we have "anyons". These are not fundamental particles (because we don't live in 2d) but emerge in condensed matter systems such as in the fractional quantum Hall effect and toric code.

## 3.2 Proof of spin-statistics in 3+1d (almost)




We consider a two-point function involving two fermionic fields in the vacuum:

$$G(x) = \langle 0|D(\pi)\hat{\psi}(\mathbf{x})\hat{\psi}(-\mathbf{x})|0\rangle \tag{3.5}$$

Let us introduce $\hat{U}(\Lambda)\hat{U}(\Lambda)^{-1}$ for $\hat{U}(\Lambda) = \hat{R}$ a rotation by $\pi$. We use the Lorentz invariance of the vaccum, i.e. $\hat{U}(\Lambda)|0\rangle = |0\rangle$.

$$\begin{aligned} G(x) &= \langle 0|\hat{U}(\Lambda)\hat{U}(\Lambda)^{-1}D(\pi)\hat{\psi}(\mathbf{x})\hat{U}(\Lambda)\hat{U}(\Lambda)^{-1}\hat{\psi}(-\mathbf{x})\hat{U}(\Lambda)\hat{U}(\Lambda)^{-1}|0\rangle \\ &= \langle 0|\hat{U}(\Lambda)^{-1}D(\pi)\hat{\psi}(\mathbf{x})\hat{U}(\Lambda)\hat{U}(\Lambda)^{-1}\hat{\psi}(-\mathbf{x})\hat{U}(\Lambda)|0\rangle \\ &= \langle 0|D(\pi)D(\pi)\hat{\psi}(-\mathbf{x})D(\pi)\hat{\psi}(\mathbf{x})|0\rangle \\ &= \langle 0|D(2\pi)\hat{\psi}(-\mathbf{x})D(\pi)\hat{\psi}(\mathbf{x})|0\rangle \\ &= -\langle 0|\hat{\psi}(-\mathbf{x})D(\pi)\hat{\psi}(\mathbf{x})|0\rangle \end{aligned} \tag{3.6}$$

where in the third equality we note that $\hat{U}(\Lambda)^{-1}\hat{\psi}(\mathbf{x})\hat{U}(\Lambda) = D(\pi)\hat{\psi}(-\mathbf{x})$ and in the final equality we use that $D(2\pi) = -1$ for fermions (note that $D(\pi)$ is just a matrix acting on the indices of $\hat{\psi}$ and so we can move it around freely in the expression). This establishes that the $\psi$ operators have fermionic statistics.

One thing we swept under the rug is the possibility that $G(x) = 0$. This comes from the CPT theorem; invariance under charge/parity/time-reversal symmetry follows that $G(x) > 0$. □

Rio note: Asked whether it matters that we only showed it for $\pm\mathbf{x}$ - answer was that it suffices to show it for one correlator. I think maybe you could argue this (from spatial translation/rotation invariance) that all possible coordinate pairs would reduce to the $\pm\mathbf{x}$ case?

Also, note that there are different available proofs of spin-statistics, and each offer a different perspective; see Schwartz for more details and insight.

## 3.3 Review of quantizing the free scalar

We follow Srednicki 37-39 for this discussion.

We recall the Dirac Lagrangian we had constructed:

$$\mathcal{L} = i\bar{\Psi}\partial\!\!\!/\Psi - m\bar{\Psi}\Psi \tag{3.7}$$

where $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ and $\partial\!\!\!/ = \partial_\mu \gamma^\mu$ with $\gamma$ matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{3.8}$$

so:

$$\partial\!\!\!/ = \begin{pmatrix} 0 & 0 & \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ 0 & 0 & \partial_1 + i\partial_2 & \partial_0 - \partial_3 \\ \dots & \dots & 0 & 0 \\ \dots & \dots & 0 & 0 \end{pmatrix}. \tag{3.9}$$

We want to quantize this Lagrangian, but let us first review how we quantized the free scalar. For the free scalar that we had the action:

$$S = -\frac{1}{2}\int (\partial\phi)^2 + m^2\phi^2 \tag{3.10}$$

the conjugate momenta:

$$\pi(x) = \frac{\delta S}{\delta\dot{\phi}} = \dot{\phi} \tag{3.11}$$

which obeyed the canonical Poisson bracket relation:

$$\{\phi(\mathbf{x}), \pi(\mathbf{y})\}_{\text{PB}} = \delta^3(\mathbf{x} - \mathbf{y}) \tag{3.12}$$

which we promoted to a commutator relation of operators via canonical quantization:

$$[\hat{\phi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}). \tag{3.13}$$

We had the Hamiltonian, which we quantized and diagonalized:

$$\begin{aligned} H &= \int \Pi\dot{\phi} - \mathcal{L} \\ &= \int d^3x \frac{1}{2}\Pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}|\Pi_{\mathbf{k}}|^2 + \frac{1}{2}(\mathbf{k}^2 + m^2)|\phi_{\mathbf{k}}|^2 \\ &= \int \frac{d^3k}{(2\pi)^3 2\epsilon_{\mathbf{k}}}\epsilon_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \end{aligned} \tag{3.14}$$

with:

$$\begin{aligned} a_{\mathbf{k}} &= \epsilon_{\mathbf{k}}\phi_{\mathbf{k}} + i\Pi_{\mathbf{k}} \\ a_{\mathbf{k}}^\dagger &= \epsilon_{\mathbf{k}}\phi_{-\mathbf{k}} - i\Pi_{-\mathbf{k}} \end{aligned} \tag{3.15}$$

and so:

$$\phi_{\mathbf{k}} = \frac{1}{2\epsilon_{\mathbf{k}}}(a_{\mathbf{k}} + a^\dagger_{\mathbf{k}}) \tag{3.16}$$

so we found the full time evolution of the fields:

$$\phi(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3 2\epsilon_{\mathbf{k}}}(a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}) \tag{3.17}$$

with $k^0 = \epsilon_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. The Hilbert space of our scalar QFT was Fock space, with:

$$a_{\mathbf{k}}|0\rangle = 0 \quad \forall \mathbf{k} \tag{3.18}$$

$$a^\dagger_{\mathbf{k}}|0\rangle = |\mathbf{k}\rangle, a^\dagger_{\mathbf{k}_1} a^\dagger_{\mathbf{k}_2} = |\mathbf{k}_1, \mathbf{k}_2\rangle, \dots \tag{3.19}$$

## 3.4 Quantizing the Dirac Lagrangian

With that review under our belt, we want to quantize:

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu p_\mu \Psi - m\bar{\Psi}\Psi \tag{3.20}$$

so we have the canonical momenta:

$$\Pi = \frac{\delta S}{\delta \dot{\Psi}} = i\bar{\Psi}\gamma^0 = i\Psi^\dagger\gamma^0\gamma^0 = i\Psi^\dagger \implies \Pi_a = i\Psi^*_a \tag{3.21}$$

Thus canonical quantization promotes:

$$\left\{\Psi_a(\mathbf{x}), \Pi_b(\mathbf{y})\right\}_{\text{PB}} = \delta^3(\mathbf{x} - \mathbf{y})\delta_{ab} \tag{3.22}$$

to the commutator:

$$[\hat{\Psi}_a(\mathbf{x}), \hat{\Pi}_b(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})\delta_{ab}. \tag{3.23}$$

The Hamiltonian reads:

$$H = \Pi\dot{\Psi} - \mathcal{L} = -i\bar{\Psi}\gamma^i\partial_i\Psi + m\bar{\Psi}\Psi \tag{3.24}$$

## 3.5 General Solution for Dirac Equation

Let's try to find the most general solution $\Psi(t, \mathbf{x})$ of the Dirac equation:

$$(i\partial\!\!\!/ - m)\Psi = 0. \tag{3.25}$$

Because $\Psi(x)$ satisfies the Klein-Gordon equation (which follows from the Dirac equation):

$$(\Box + m^2)\Psi = 0 \tag{3.26}$$

a general solution will take the form:

$$\Psi(t, \mathbf{x}) = u(\mathbf{p})e^{ipx} + v(\mathbf{p})e^{-ipx} \tag{3.27}$$

where $u, v$ are 4-component spinors:

$$u(\mathbf{p}) = \begin{pmatrix} u_1(\mathbf{p}) \\ u_2(\mathbf{p}) \\ u_3(\mathbf{p}) \\ u_4(\mathbf{p}) \end{pmatrix} \tag{3.28}$$

and we have the on-shell condition for $p = (p^0, \mathbf{p})$:

$$p^0 = \epsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}. \tag{3.29}$$

But, the solution also has to solve the Dirac equation, which gives us the further condition on the spinors:

$$0 = (i\partial\!\!\!/ - m)\Psi = (-p\!\!\!/ - m)u(\mathbf{p})e^{ipx} + (p\!\!\!/ - m)v(\mathbf{p})e^{-ipx} \implies \begin{cases} (p\!\!\!/ + m)u(\mathbf{p}) = 0 \\ (p\!\!\!/ - m)v(\mathbf{p}) = 0 \end{cases} \tag{3.30}$$

The trick will be to solve these equations in a preferred frame - namely the rest frame, where $p^\mu_{\text{rest}} = (m, \mathbf{0})$ - and then boost back to an arbitrary frame $p^\mu = \Lambda^\mu_{\ \nu} p^\nu_{\text{rest}}$:

$$u(\mathbf{p}) = D(\Lambda)u(\mathbf{0}) \tag{3.31}$$

This works because the Dirac equation is Lorentz invariant, but let's check:

$$\begin{aligned} 0 \stackrel{?}{=} (p\!\!\!/ + m)D(\Lambda)u(\mathbf{0}) &= D(\Lambda)D(\Lambda)^{-1}(p\!\!\!/ + m)D(\Lambda)u(\mathbf{0}) \\ &= D(\Lambda)(p_\mu D(\Lambda)^{-1}\Gamma^\mu D(\Lambda) - m)u(\mathbf{0}) \\ &= D(\Lambda)(p_\mu \Lambda^\mu_{\ \nu}\gamma^\nu - m)u(\mathbf{0}) \\ &= D(\Lambda)(p\!\!\!/_{\text{rest}} - m)u(\mathbf{0}) \\ &= 0. \end{aligned} \tag{3.32}$$

So indeed this strategy works!

In the rest frame we have $p^0_{\text{rest}} = m$ and $p_{0,\text{rest}} = -m$ and so:

$$p\!\!\!/_{\text{rest}} = p_0\gamma^0 = -m \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{3.33}$$

Thus our equation for $u$ is:

$$m \begin{pmatrix} \mathbb{I} & -\mathbb{I} \\ -\mathbb{I} & \mathbb{I} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0 \tag{3.34}$$

This implies that $u_1 - u_3 = 0$ and $u_2 - u_4 = 0$. One possible basis for the rest frame solution is therefore:

$$u_+(\mathbf{0}) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_-(\mathbf{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \tag{3.35}$$

This basis is convenient because these solutions have a well-defined spin in the $\hat{\mathbf{z}}$-direction (namely $\pm\frac{1}{2}$). Let us do the same for $v$:

$$m \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & \mathbb{I} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0 \tag{3.36}$$

This implies that $v_1 + v_3 = 0$ and $v_2 + v_4 = 0$. We then have a basis of solutions:

$$v_+(\mathbf{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_-(\mathbf{0}) = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{3.37}$$

Now, we boost to get the general solution. What is the Lorentz transformation that takes $p^\mu_{\text{rest}} = (m, \mathbf{0})$ into $p^\mu = (\sqrt{m^2 + \mathbf{p}^2}, \mathbf{p})$? It is clear that we must boost in the $\hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}$ direction, so we want the boost $e^{i\eta \hat{\mathbf{p}} \cdot \mathbf{k}}$ by an amount (rapidity) $\eta = \sinh(\frac{|\mathbf{p}|}{m})$. What is then $D(\mathbf{k})$ (representation of boosts on spinors)? This we can construct by recalling that $\mathbf{k} = \frac{\mathbf{J}_+ - \mathbf{J}_-}{i}$ so then:

$$K_i = \begin{pmatrix} \frac{i}{2}\sigma_i & 0 \\ 0 & -\frac{i}{2}\sigma_i \end{pmatrix}. \tag{3.38}$$

So, in summary:

$$u(\mathbf{p}) = D(\Lambda) u(\mathbf{0}) = e^{i\eta \hat{p}^i K_i} \tag{3.39}$$

with the $\eta, K_i$ as described the above. There are solutions for $u_\pm, v_\pm$. At the end of the day, a general solution to the Dirac equation takes the form of a most general linear combination of these solutions:

$$\Psi(t, \mathbf{x}) = \sum_{s=\pm} \int \frac{d^3 p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} [b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx}] \tag{3.40}$$

Classically, $b, d^\dagger$ are coefficients, but after we finish quantizing the theory on Thursday we will see that they play similar roles to the creation/annihilation operators as in the scalar case, and they will create/annihilate spin-1/2 particles. Moreover, we will see that they have to come with a particle-pair; the Dirac equation solutions predict the existence of antiparticles! Historically, this was how the positron (the antiparticle of the electron) was predicted.

# 4 The Free Dirac QFT

## 4.1 Reviewing the General Solution

We found that the general solution to the Dirac equation took the form:

$$\psi(x) = \sum_{s=\pm} \int \frac{d^3 p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \left[ b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right] \tag{4.1}$$

With:

$$u_+(\mathbf{0}) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_-(\mathbf{0}) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_+(\mathbf{0}) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_-(\mathbf{0}) = \sqrt{m} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{4.2}$$

the spinors in the rest frame, and:

$$u_s(\mathbf{p}) = \exp(i \sinh^{-1}(\frac{|\mathbf{p}|}{m}) \hat{\mathbf{p}} \cdot \mathbf{k}) u_s(\mathbf{0}) \tag{4.3}$$

(same for the $v_s$) which are the spinors in an arbitrary frame (boosted from the rest frame via $D(\Lambda) = \exp(i \sinh^{-1}(\frac{|\mathbf{p}|}{m}) \hat{\mathbf{p}} \cdot \mathbf{k})$). The generators are:

$$K^i = \begin{pmatrix} i\sigma_i/2 & 0 \\ 0 & -i\sigma_i/2 \end{pmatrix}, \quad J^i = \begin{pmatrix} \sigma_i/2 & 0 \\ 0 & \sigma_i/2 \end{pmatrix} \tag{4.4}$$

## 4.2 Spinor relations

We note that we have the boosted spinor:

$$u_s(\mathbf{p}) = D(\Lambda)u_s(\mathbf{p}) \tag{4.5}$$

and also the barred spinor:

$$\bar{u}_s(\mathbf{p}) = u_s^\dagger(\mathbf{p})\gamma^0 = u_s^\dagger(\mathbf{0})D(\Lambda)^\dagger\gamma^0 = \bar{u}_s(\mathbf{0})\gamma^0 D(\Lambda)^\dagger\gamma^0 = \bar{u}_s(\mathbf{0})D(\Lambda)^{-1} \tag{4.6}$$

where we note that $\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$ and so $\gamma^0 K_i \gamma^0 = -K_i$ so with the $\theta \leftrightarrow \theta$ flip from the dagger operation we indeed get the inverse transformation. Note that we get simple orthogonality relations between the $u, v$s from studying them in the rest frame:

$$\bar{u}_s(\mathbf{p})u_{s'}(\mathbf{p}) = \bar{u}_s(\mathbf{0})D(\Lambda)^{-1}D(\Lambda)u_{s'}(\mathbf{0}) = \bar{u}_s(\mathbf{0})u_{s'}(\mathbf{0}) = 2m\delta_{ss'} \tag{4.7}$$

$$\bar{v}_s(\mathbf{p})\bar{v}_{s'}(\mathbf{p}) = \bar{v}_s(\mathbf{0})v_{s'}(\mathbf{0}) = v_{s'}^\dagger(\mathbf{0})\gamma^0 v_s(\mathbf{0}) = -2m\delta_{ss'} \tag{4.8}$$

Note that for the $v$s we have minus signs inside of the spinor, so explicitly the $\gamma^0$ becomes relevant.

$$\bar{v}_s(\mathbf{p})u_{s'}(\mathbf{p}) = 0. \tag{4.9}$$

Let's try computing a slightly more complicated object:

$$\bar{u}_s(\mathbf{p})\gamma^\mu u_{s'}(\mathbf{p}) = \bar{u}_s(\mathbf{0})D(\Lambda)^{-1}\gamma^\mu D(\Lambda)u_{s'}(\mathbf{0}) = \Lambda^\mu{}_\nu \bar{u}_s(\mathbf{0})\gamma^\nu u_{s'}(\mathbf{0}) = \Lambda^\mu{}_\nu 2m\delta^\nu_0\delta_{ss'} = 2p^\mu\delta_{ss'} \tag{4.10}$$

which we could have partly guessed. There is a very similar equation for the $v$s:

$$\bar{v}_s(\mathbf{p})\gamma^\mu v_{s'}(\mathbf{p}) = 2p^\mu\delta_{ss'} \tag{4.11}$$

Finally:

$$\bar{u}_s(\mathbf{p})\gamma^0 v_{s'}(-\mathbf{p}) = u_s^\dagger(\mathbf{p})v_{s'}(-\mathbf{p}) = u_s^\dagger(\mathbf{0})D(\Lambda)^\dagger D(\Lambda_{-\mathbf{p}})v_{s'}(\mathbf{0}) = u_s^\dagger(\mathbf{0})\exp(i\eta\hat{\mathbf{p}}\cdot\mathbf{K})\exp(-i\eta\hat{\mathbf{p}}\cdot\mathbf{K})v_{s'}(\mathbf{0}) = u_s^\dagger(\mathbf{0})v_{s'}(\mathbf{0}) = 0 \tag{4.12}$$

## 4.3 Dirac reation and Annihilation Operators

We will promote $\psi$s to fermionic anticommuting operators. This is fine and dandy, but to extract what the behaviour of $b/d^\dagger$ are we will have to use the above spinor technology.

First, let's fourier transform Eq. (4.1):

$$\int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}}\psi(x) = \sum_{s=\pm}\frac{1}{2\epsilon_\mathbf{p}}[b_s(\mathbf{p})u_s(\mathbf{p}) + d_s^\dagger(-\mathbf{p})v_s(-\mathbf{p})] \tag{4.13}$$

Now, we act on the above with $\bar{u}_s(\mathbf{p})\gamma^0$ and use our spinor relations we derived in the previous section:

$$\begin{aligned}\int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}}\bar{u}_s(\mathbf{p})\gamma^0\psi(x) &= \sum_{s=\pm}\frac{1}{2\epsilon_\mathbf{p}}(b_s(\mathbf{p})\bar{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) + d_s^\dagger(-\mathbf{p})\bar{u}_s(\mathbf{p})\gamma^0 v_s(-\mathbf{p})) \\ &= \sum_{s=\pm}\frac{1}{2\epsilon_\mathbf{p}}(b_s(\mathbf{p})2p^0\delta_{ss'} + 0) \\ &= b_s(\mathbf{p})\end{aligned} \tag{4.14}$$

The analogous barred relation is:

$$\int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} \bar{\psi}(x)\gamma^0 u_s(\mathbf{p}) = b_s^\dagger(\mathbf{p}). \tag{4.15}$$

Via canonical quantization, we promote the $\psi$s to operators with the equal time commutation relations:

$$\left\{\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{0})\right\} = \delta_{ab}\delta^3(\mathbf{x}) \tag{4.16}$$

$$\{\psi(\mathbf{x}), \psi(\mathbf{0})\} = 0 \tag{4.17}$$

From this we find that:

$$\{b, b\} = \left\{b^\dagger, b^\dagger\right\} = 0 \tag{4.18}$$

as out expressions for $b, b^\dagger$ only involve $\psi, \psi^\dagger$s respectively. The only nontrivial relation is the anticommutator between $b$ and $b^\dagger$:

$$\begin{aligned}
\left\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\right\} &= \int d^3x d^3x' e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{p}'\cdot\mathbf{x}')} \left\{\bar{u}_s(\mathbf{p})\gamma^0\psi(\mathbf{x}), \bar{\psi}(\mathbf{x}')\gamma^0 u_s(\mathbf{p})\right\} \\
&= \int d^3x d^3x' e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{p}'\cdot\mathbf{x}')} (\bar{u}_s(\mathbf{p})\gamma^0)^a \left\{\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{0})\right\} (\gamma^0\gamma^0 u_s(\mathbf{p}))_b \\
&= \int d^3x d^3x' e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{p}'\cdot\mathbf{x}')} (\bar{u}_s(\mathbf{p})\gamma^0)^a \delta_{ab}\delta^3(\mathbf{x} - \mathbf{x}')(\gamma^0\gamma^0 u_s(\mathbf{p}))_b \\
&= \int d^3x d^3x' e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{p}'\cdot\mathbf{x}')} \delta^3(\mathbf{x} - \mathbf{x}')\bar{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}') \\
&= \int d^3x e^{-i(\mathbf{p} - \mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}') \\
&= \int d^3x e^{-i(\mathbf{p} - \mathbf{p}')\cdot\mathbf{x}} \delta_{ss'} 2p^0 \\
&= (2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}')\delta_{ss'} 2\epsilon_{\mathbf{p}}
\end{aligned} \tag{4.19}$$

Now we can do the same with the $d$s. Act on Eq. (4.1) with $\bar{v}_s(-\mathbf{p})\gamma^0$. We then obtain:

$$d_s^\dagger(-\mathbf{p}) = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} v_s(-\mathbf{p})\gamma^0\psi(x) \tag{4.20}$$

and flipping $\mathbf{p} \to -\mathbf{p}$:

$$d_s^\dagger(\mathbf{p}) = \int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} v_s(\mathbf{p})\gamma^0\psi(x) \tag{4.21}$$

Taking the dagger of this relation:

$$d_s(\mathbf{p}) = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \bar{\psi}(x)\gamma^0 v_s(\mathbf{p}) \tag{4.22}$$

So then:

$$\{d, d\} = \left\{d^\dagger, d^\dagger\right\} = 0 \tag{4.23}$$

and again the only nontrivial relation is between $d, d^\dagger$:

$$\begin{aligned}
\left\{d_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}')\right\} &= \int_{\mathbf{x}\mathbf{x}'} e^{i(\mathbf{p}\cdot\mathbf{x} - \mathbf{p}'\cdot\mathbf{x}')} \left\{\bar{v}_s(\mathbf{p})\gamma^0\psi(\mathbf{x}), \psi^\dagger(\mathbf{x}')\gamma^0 v_s(\mathbf{p}')\right\} \\
&= \int_{\mathbf{x}\mathbf{x}'} e^{i(\mathbf{p}\cdot\mathbf{x} - \mathbf{p}'\cdot\mathbf{x}')} \delta^3(\mathbf{x} - \mathbf{x}')\bar{v}_s(\mathbf{p})\gamma^0 v_{s'}(\mathbf{p}') \\
&= \int d^3x e^{i(\mathbf{p} - \mathbf{p}')\cdot\mathbf{x}} 2p^0\delta_{ss'} \\
&= (2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}') 2\epsilon_{\mathbf{p}}\delta_{ss'}
\end{aligned} \tag{4.24}$$

We are almost done. The last thing to check is that the $b, d$ sets of creation/annihilation operators are independent, i.e. that they anticommute. It is clear that:

$$\left\{d^\dagger, b\right\} = \left\{d, b^\dagger\right\} = 0 \tag{4.25}$$

and the only potentially nontrivial one we should work through is between $b, d$:

$$\begin{aligned}
\left\{b_s(\mathbf{p}), d_{s'}(\mathbf{p}')\right\} &= \int_{\mathbf{x}\mathbf{x}'} e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{p}'\cdot\mathbf{x}')}\left\{\bar{u}_s(\mathbf{p})\gamma^0\psi(x), \bar{\psi}(x)\gamma^0 v_{s'}(\mathbf{p}')\right\} \\
&= \int_{\mathbf{x}\mathbf{x}'} e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{p}'\cdot\mathbf{x}')}\delta^3(\mathbf{x}-\mathbf{x}')\bar{u}_s(\mathbf{p})\gamma^0 v_{s'}(\mathbf{p}') \\
&= (2\pi)^3\delta^3(\mathbf{p}+\mathbf{p}')\bar{u}_s(\mathbf{p})\gamma^0 v_{s'}(-\mathbf{p}) \\
&= 0
\end{aligned} \tag{4.26}$$

so we indeed find that the $b, d$ are independent. The bottom line is we have 4 independent raising and lowering operators $b_\pm, d_\pm$. Notice that $b^\dagger \sim \psi^\dagger$ while $d^\dagger \sim \psi$, so they create particles of opposite "charge". What charge?

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{4.27}$$

has a $U(1)$ symmetry $\psi \to e^{i\alpha}\psi$. Symmetries imply conservation laws via Noether's theorem - we have the conserved current $\partial_\mu j^\mu = 0$. We can identify the Noether charge:

$$Q = \int d^3x j^0 \tag{4.28}$$

which is conserved with $\dot{Q} = 0$. Indeed, we find that:

$$[Q, \psi^\dagger] = i\psi^\dagger, \quad [Q, \psi] = -i\psi \tag{4.29}$$

so the two have opposite charge. When we talk about the electron, this Noether charge is just the familiar electric charge, and this is indeed how the Noether charge got its name.

## 4.4 Hilbert space of the Dirac Fermion

We start with the vacuum state $|0\rangle$ annihilated by all annihilation operators:

$$b_s(\mathbf{p})|0\rangle = d_s(\mathbf{p})|0\rangle = 0 \quad \forall s, \mathbf{p} \tag{4.30}$$

We then have the single particle states:

$$b_s^\dagger(\mathbf{p})|0\rangle = |\mathbf{p}, s, +1\rangle \tag{4.31}$$

which in contrast to the scalar single particle states (which were only labelled by their momentum), are labelled by momentum, spin, and charge. Analogously:

$$d_s^\dagger(\mathbf{p})|0\rangle = |\mathbf{p}, s, -1\rangle \tag{4.32}$$

So for example $|\mathbf{p}, \uparrow, +1\rangle = b_\uparrow^\dagger(\mathbf{p})|0\rangle$ corresponds to an electron[3] with momentum $\mathbf{p}$ and $s_z = +\frac{1}{2}$ and $|\mathbf{p}', \downarrow, -1\rangle = d_\downarrow^\dagger(\mathbf{p})|0\rangle$ corresponds to a positron with momentum $\mathbf{p}$ and spin $s_z = -\frac{1}{2}$. Multiparticle states are then obtained as:

$$\begin{aligned}
|\ldots, (\mathbf{p}_1, s_1, q_1), \ldots, (\mathbf{p}_2, s_2, q_1), \ldots\rangle &= \ldots b_{s_1}^\dagger(\mathbf{p}_1)d_{s_2}^\dagger(\mathbf{p}_2)\ldots|0\rangle \\
&= |\ldots, n_{\mathbf{p}_1,s_1,q_1} = 1, \ldots, n_{\mathbf{p}_2,s_2,q_2} = 1, \ldots\rangle
\end{aligned} \tag{4.33}$$

[3] Let's take the electron to have positive charge, for now...

so we have a bunch of QHO modes labelled by three quantum numbers, but with the distinction to the bosonic case that each mode can only have occupation $n = 0/1$. This is because of the Pauli exclusion principle:

$$b_s^\dagger(\mathbf{p}) b_s^\dagger(\mathbf{p}) = \frac{1}{2}\left\{ b_s^\dagger(\mathbf{p}), b_s^\dagger(\mathbf{p}) \right\} = 0 \tag{4.34}$$

What is the energy of these particles? We should be able to get this from looking at the commutator of the $b$s with the Hamiltonian:

$$\begin{aligned}
[H, b_s(\mathbf{p})] &= -[b_s(\mathbf{p}), H] \\
&= -\int_{\mathbf{x},\mathbf{x}'} e^{-ipx} [\bar{u}_s(\mathbf{p})\gamma^0 \psi(x), \bar{\psi}(x')(-i\gamma^i \partial_i + m)\psi(x')] \\
&= -\int_{\mathbf{x},\mathbf{x}'} e^{-ipx} \bar{u}_s(\mathbf{p})\gamma^0 \left\{ \psi(x), \psi^\dagger(x') \right\} i\gamma^0\gamma^0 \partial_0 \psi(x') \\
&= -\int_{\mathbf{x},\mathbf{x}'} e^{-ipx} \bar{u}_s(\mathbf{p})\gamma^0 \delta^3(\mathbf{x}-\mathbf{x}') i\gamma^0\gamma^0 \partial_0 \psi(x') \\
&= -\int d^3x e^{-ipx} \bar{u}_s(\mathbf{p})\gamma^0 i\partial_0 \psi(x) \\
&\overset{IBP}{=} -p^0 \int d^3x e^{-ipx} \bar{u}_s(\mathbf{p})\gamma^0 \psi(x) \\
&= -\epsilon_{\mathbf{p}} b_s(\mathbf{p})
\end{aligned} \tag{4.35}$$

So $b_s(\mathbf{p})$ lowers the energy by $\epsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, as expected. Note that the final Hamiltonian has a very simple form in terms of the raising and lowering operators (very similar to the Hamiltonian of a scalar QFT):

$$H = \sum_s \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \epsilon_{\mathbf{p}} (b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) + d_s^\dagger(\mathbf{p}) d_s(\mathbf{p})) \tag{4.36}$$

So this is $H$! What we will then do next week is to study our first fermionic observables in the form of propagators:

$$\langle 0 | \psi_a(\mathbf{x}) \bar{\psi}_b(\mathbf{y}) | 0 \rangle \tag{4.37}$$

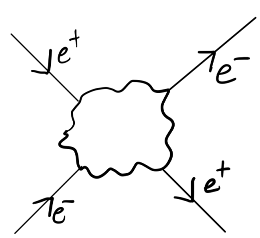
$$\langle 0 | \mathcal{T} \psi_a(\mathbf{x}) \bar{\psi}_b(\mathbf{y}) | 0 \rangle \tag{4.38}$$

We will do a lot of work to get a simple answer, then wonder if there was a simpler way, and in fact we will find that the simple method is using the path integral - the twist there will be that we integrate over a Grassman variable.

# 5 The Fermion Propagator

The relevant section is Srednicki 42.

We want to compute correlation functions fermions, e.g. $\langle 0 | \bar{\psi}_a \psi_b \psi_c \bar{\psi}_d | 0 \rangle$. Through the LSZ formula, they will be related to $e^+/e^-$ scattering. These correlators are also observables in their own right - for example the current (observable in CM experiments) goes as $\langle j^\mu j^\nu \rangle \sim \langle \bar{\psi}\gamma^\mu \psi \bar{\psi}\gamma^\nu \psi \rangle$.



Today, we look at the simplest observable, the propagator/2-point function:

$$G_W(x)_{\alpha\beta} = \langle 0|\psi_\alpha(x)\bar{\psi}_\beta(0)|0\rangle \tag{5.1}$$

$$G_F(x)_{\alpha\beta} = \langle 0|\mathcal{T}\psi_\alpha(x)\bar{\psi}_\beta(0)|0\rangle \tag{5.2}$$

Where for fermions:

$$\mathcal{T}\psi_\alpha(x)\psi_\beta(0) = \begin{cases}\psi_\alpha(x)\bar{\psi}_\beta(0) & \text{if } x^0 > 0\\ -\bar{\psi}_\beta(0)\psi_\alpha(0) & \text{if } x^0 < 0\end{cases} \tag{5.3}$$

## 5.1 Computing the 2-point function

We compute the 2-point function by plugging in our expression for $\psi(x)$ that we found last week:

$$\psi(x) = \sum_{s=\pm}\int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right] \tag{5.4}$$

Therein:

$$\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(0)|0\rangle = \sum_{ss'}\int_{pp'} u_s(\mathbf{p})e^{ipx}\langle 0|b_s(\mathbf{p})b^\dagger_{s'}(\mathbf{p}')|0\rangle\bar{u}_{s',\beta}(\mathbf{p}')e^{-ip'\cdot 0} \tag{5.5}$$

where we use that the annihilation operator annihilates the vacuum, making the $d$-pieces of the expression zero. We can, for free, replace $b_s(\mathbf{p})b^\dagger_{s'}(\mathbf{p}')$ by an anticommutator since $b_s$ annihilates the vacuum. Then, since $\left\{b_s(\mathbf{p}), b^\dagger_{s'}(\mathbf{p}')\right\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')2\epsilon_{\mathbf{p}}\delta_{ss'}$, this allows us to get rid of the $\mathbf{p}'$ integral and the $s'$ sum, leaving us with:

$$\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(0)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\sum_{s=\pm} u_{s,\alpha}(\mathbf{p})\bar{u}_{s,\beta}(\mathbf{p})e^{ipx} \tag{5.6}$$

This simplifies via the completeness relation:

$$\sum_{s=\pm} u_{s,\alpha}(\mathbf{p})\bar{u}_{s,\beta}(\mathbf{p}) = (-\not{p} + m)_{\alpha\beta} \tag{5.7}$$

Which can be seen by computing it in the rest frame with $p^{\text{rest}}_\mu = (-m, \mathbf{0})$:

$$u_+ = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad u_- = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \quad \bar{u} = \gamma^0 u = \begin{pmatrix}0 & \mathbb{I}\\ \mathbb{I} & 0\end{pmatrix}u \tag{5.8}$$

$$u_+u_+^\dagger = m\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix} = m\begin{pmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix} \tag{5.9}$$

$$u_-u_-^\dagger = m\begin{pmatrix}0&0&0&0\\0&1&0&1\\0&0&0&0\\0&1&0&1\end{pmatrix} \tag{5.10}$$

thus:

$$\sum_{s=\pm} u_s u_s^\dagger = m\begin{pmatrix}\mathbb{I} & \mathbb{I}\\ \mathbb{I} & \mathbb{I}\end{pmatrix} = m\mathbb{I} + m\gamma^0 = m\mathbb{I} - p_0^{\text{rest}}\gamma^0 \tag{5.11}$$

Then we can act with our boost generators on the left and right to get the expression in a general frame:

$$\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = m\mathbb{I} - p_\mu\gamma^\mu \tag{5.12}$$

which gives a simple final expression:

$$\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(0)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}(m - \not{p})_{\alpha\beta}e^{ipx} \tag{5.13}$$

Since we are interested in the time-ordered correlation function, we should also look at the correlation function with the two field operators swapped:

$$\langle 0|\bar{\psi}_\beta(0)\psi_\alpha(x)|0\rangle = \sum_{ss'}\int_{pp'} \bar{v}_{s,\beta}(\mathbf{p})\langle 0|d_s(\mathbf{p})d_{s'}(\mathbf{p}')|0\rangle v_{s'\alpha}(\mathbf{p})e^{-ip'x} \tag{5.14}$$

We again replace $d_s(\mathbf{p})d_{s'}(\mathbf{p}')$ by the anticommutator, which kills the $s'$ and $p'$ sums:

$$\langle 0|\bar{\psi}_\beta(0)\psi_\alpha(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\sum_s \bar{v}_{s,\beta}(\mathbf{p})v_{s,\alpha}(\mathbf{p})e^{-ipx} = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}(-\not{p} - m)_{\alpha\beta}e^{-ipx} \tag{5.15}$$

where we have used the analogous completeness relation in the last equality. The bottom line is that the time-ordered Green's function looks like:

$$G_F(x) = \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}}\Theta(x^0)(m - \not{p})e^{ipx} + \Theta(-x^0)(\not{p} + m)e^{-ipx} \tag{5.16}$$

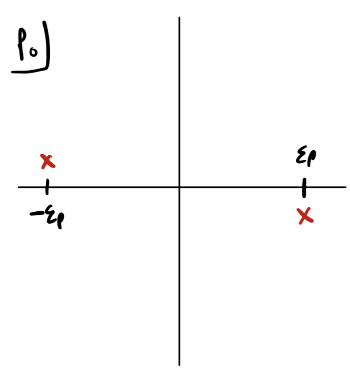
## 5.2 Lorentz-Covariant 2-point function

but there is a slightly nicer way to write this in a way that makes the expression manifestly Lorentz covariant. We will simplify it via an identity involving the integral:

$$\int \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{p^2 + m^2 - i0^+}f(p^\mu) \tag{5.17}$$

where we note that $px = -p^0x^0 + \mathbf{p}\cdot\mathbf{x}$, where $p^0$ is a free variable that is integrated over. The expression (as in the case of a free scalar) has two poles, at:

$$p^0 = \pm(\epsilon_{\mathbf{p}} - i0^+) \tag{5.18}$$



This will allow us to carry out the integral over $p^0$, leaving us with a integral over spatial momentum; reversing the process we will obtain an expression for $G_F(x)$. If $x^0 > 0$, we will want to close the contour in the lower half plane, and vise versa if $x^0 < 0$ (in order for the integral to converge):

$$\begin{aligned}\int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2+m^2-i0^+} f(p^\mu) &= -\Theta(x^0) \int \frac{d^3p}{(2\pi)^4}(-2\pi i) \frac{e^{i(\epsilon_p x^0 + \mathbf{p}\cdot\mathbf{x})}}{2\epsilon_{\mathbf{p}}} f(\epsilon_{\mathbf{p}}, \mathbf{p}) - \Theta(-x^0) \int \frac{d^3p}{(2\pi)^4}(2\pi i) \frac{e^{i(-\epsilon_p x^0 + \mathbf{p}\cdot\mathbf{x})}}{-2\epsilon_p} f(-\epsilon_{\mathbf{p}}, \mathbf{p}) \\ &= i \int \frac{d^3p}{(2\pi)^3 2\epsilon_{\mathbf{p}}} \left[\Theta(x^0) e^{ipx} f(\epsilon_{\mathbf{p}}, \mathbf{p}) + \Theta(-x^0) e^{-ipx} f(-\epsilon_{\mathbf{p}}, -\mathbf{p})\right]\end{aligned} \tag{5.19}$$

Now we have something that looks exactly like what we have for $G_F(x)$, so using this identity:

$$\boxed{G_F(x) = -i \int \frac{d^4p}{(2\pi)^4} e^{ipx} \frac{m - \not{p}}{p^2+m^2-i0^+}} \tag{5.20}$$

Fourier transforming, we have:

$$\boxed{G_F(p) = \int d^4x e^{-ipx} G_F(x) = -i \frac{m - \not{p}}{p^2+m^2-i0^+}} \tag{5.21}$$

This looks pretty similar to the scalar field propagator, up to the extra factor $m - \not{p}$. It's not so surprising that it looks similar, because the fermionic fields also satisfy the Klein-Gordon equation.

Note that:

$$\begin{aligned}(m+\not{p})(m-\not{p}) &= m^2 - \not{p}^2 \\ &= m^2 - p_\mu p_\nu \gamma^\mu \gamma^\nu \\ &= m^2 - p_\mu p_\nu \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\right) \\ &= m^2 - p_\mu p_\nu (-\eta^{\mu\nu}) \\ &= m^2 + p^2\end{aligned} \tag{5.22}$$

Which allows us to write the inverse:

$$(m+\not{p})^{-1} = \frac{m-\not{p}}{m^2+p^2} \tag{5.23}$$

Thus we may also write the Feynman Green's function as:

$$G_F^\psi(p) = \frac{-i}{m+\not{p}} \tag{5.24}$$

This is interesting! Remember for scalars that:

$$G_F^\phi(p) = \frac{-i}{p^2+m^2} \tag{5.25}$$

This was the "inverse of the equation of motion", and indeed we find the same for $G_F^\psi$:

$$-(m+\not{p})G_F(p) = i \implies \int \frac{d^4p}{(2\pi)^4} e^{ipx}(-\not{p}-m)G_F(p) = i\delta^4(x) \tag{5.26}$$

Note that we can write the $\not{p}$ as a derivative:

$$(i\not{p} - m) \int \frac{d^4p}{(2\pi)^4} e^{ipx} G_F(p) = i\delta^4(x) \tag{5.27}$$

and thus:

$$\boxed{(i\not{p} - m)G_F(x) = i\delta^4(x)} \tag{5.28}$$

which is the formal notion in which the Green's function is the inverse of the EoM/Dirac equation in this case.

## 5.3 Path Integral for Fermions - Desiderata

We may now ask if there would have been a simpler/more direct way to get to this result - for the scalar field there indeed was, via the path integral formalism. The same will be true for fermions!

Recall for the scalar field that:

$$\begin{aligned}
\langle \mathcal{T}\phi(x)\phi(0)\rangle &= \int \mathcal{D}\phi\phi(x)\phi(0)\frac{e^{i\int \phi(\partial^2 - m^2)\phi}}{\int \mathcal{D}\phi e^{iS}} \\
&= \frac{1}{Z}\frac{\delta^2}{i\delta J(x) i\delta J(0)} \int \mathcal{D}\phi e^{i\int \phi(\partial^2-m^2)\phi + J\phi}\Big|_{J=0} \\
&= \frac{1}{Z}\frac{\delta^2}{i\delta J(x) i\delta J(0)} \int \mathcal{D}\phi \exp(\frac{i}{2}\int (\phi + D^{-1}J)D(\phi + D^{-1}J) - JD^{-1}J)\Big|_{J=0} \\
&= frac\delta^2 i\delta J(x) i\delta J(0) e^{-\frac{i}{2}\int JD^{-1}J}\Big|_{J=0} \\
&= \frac{-i}{-\partial^2 + m^2}
\end{aligned} \tag{5.29}$$

We now want a similar formalism for fermions; we would like:

$$\begin{aligned}
\langle 0|\mathcal{T}\psi_\alpha(x)\bar{\psi}_\beta(0)|0\rangle &= \int \mathcal{D}\psi\psi_\alpha(x)\bar{\psi}_\beta(0)e^{i\int d^4x\bar{\psi}(i\not{p}-m)\psi} / \int \mathcal{D}\psi e^{iS} \\
&= \frac{\delta^2}{\delta i\bar{\eta}_\alpha(x)\delta i\eta_\beta(x)} \int \mathcal{D}\psi e^{i\int \bar{\psi}D\psi + \bar{\psi}\eta + \bar{\eta}\psi}\Big|_{\eta=0} / \int \mathcal{D}\psi e^{iS} \\
&= \frac{\delta^2}{\delta i\bar{\eta}_\alpha(x)\delta i\eta_\beta(x)} e^{i\int \bar{\eta}D^{-1}\eta}\Big|_{\eta=0} / \int \mathcal{D}\psi e^{iS} \\
&= -\frac{i}{i\not{p} - m}
\end{aligned} \tag{5.30}$$

But for this to work, we require that the variable of integration $\psi$, as well as the source $\eta$, have to anticommute; they are Grassman numbers, where $\psi_1\psi_2 = -\psi_2\psi_1$. The approach that Srednicki takes in Section 44 is that one can define numbers with the desired property and the formalism carries through - i.e. he mostly says that "it works". But here, we will try to actually derive it.

## 5.4 Reviewing the Bosonic SHO Path Integral Derivation

Recall the Bosonic SHO; we had the action and Hamiltonian:

$$S = \int dt \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \implies \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{q}^2 \tag{5.31}$$

Then we could consider the evolution via $e^{-i\hat{H}t} = \prod_{i=1}^N e^{-i\hat{H}\delta t}$:

$$\langle q_f|e^{-i\hat{H}t}|q_i\rangle = \int \prod_{j=1}^N dq_j dp_j \ldots \langle q_2|e^{-i\hat{H}\delta t}|p_1\rangle\langle p_1|q_1\rangle \tag{5.32}$$

then using that:

$$\langle q_2|e^{-i\hat{H}\delta t}|p_1\rangle = e^{-i\left(\frac{p_1^2}{2m}+\frac{1}{2}kq_2^2\right)\delta t}\langle q_2|p_1\rangle\langle p_1|q_1\rangle = e^{-i\left(\frac{p_1^2}{2m}+\frac{1}{2}kq_2^2\right)\delta t}e^{ip_1(q_2-q_1)} \tag{5.33}$$

So then carrying out the Gaussian integral over $p_1$:

$$\langle q_f|e^{-i\hat{H}t}|q_i\rangle = \int\prod_{j=1}^{N}dq_jdp_j\dots e^{i\left(\frac{1}{2}m\frac{(q_2-q_1)^2}{\delta t^2}-\frac{1}{2}kq_2^2\right)\delta t} = \int\prod_{j=1}^{N}dq_jdp_j\dots e^{i\left(\frac{1}{2}m\dot{q}^2-\frac{1}{2}kq_2^2\right)\delta t} \tag{5.34}$$

then recognizing that what is left in the exponential is just the action, we could write:

$$\langle q_f|e^{-i\hat{H}t}|q_i\rangle = \int\mathcal{D}qe^{iS} \tag{5.35}$$

## 5.5 Deriving the Fermionic Path Integral

Now consider:

$$S = \int dt\psi^\dagger(i\partial_t - m)\psi \implies \hat{H} = m\hat{\psi}^\dagger\hat{\psi} \tag{5.36}$$

where $\hat{\psi}^2 = 0$. There is the ground state $|0\rangle$ annihilated by $\hat{\psi}$, and the state with one particle, $\hat{\psi}^\dagger|0\rangle = |1\rangle$. And then - we're done. We can't keep acting on this state with $\hat{\psi}^\dagger$, because this would give us zero. There are only two states in the Hilbert space, defined by:

$$\hat{\psi}|0\rangle = 0 \tag{5.37}$$

$$\hat{\psi}|1\rangle = \hat{\psi}\hat{\psi}^\dagger|0\rangle = \left\{\hat{\psi},\hat{\psi}^\dagger\right\}|0\rangle = |0\rangle \tag{5.38}$$

What was crucial in the bosonic QHO derivation was the eigenstates of the annihilation/creation operators - the coherent states. Here, we look at our defining relations, and $\hat{\psi}$ annihilates $|0\rangle$ and sends $|1\rangle$ to $|0\rangle$, which makes constructing an eigenstate... impossible, if we consider regular coefficients. But if we allow for coefficients which are Grassman numbers, then we can do this! We can define:

$$|\eta\rangle = |0\rangle + |1\rangle\eta \tag{5.39}$$

where then:

$$\hat{\psi}|\eta\rangle = 0 + |0\rangle\eta = |\eta\rangle\eta \tag{5.40}$$

if we let $\eta^2 = 0$. So, with these interesting objects of Grassman numbers, we can build eigenstates of $\hat{\psi}$. Similarly, we define:

$$\langle\eta^\dagger| = \langle 0| + \eta^\dagger\langle 1| \implies \langle\eta^\dagger|\hat{\psi}^\dagger = \eta^\dagger\langle\eta^\dagger| \tag{5.41}$$

What is the overlap between the fermionic coherent states? We calculate:

$$\langle\eta_1|\eta_2\rangle = (\langle 0| + \eta_1^\dagger\langle 1|)(|0\rangle + |1\rangle\eta_2) = 1 + \eta_1^\dagger\eta_2 = e^{\eta_1^\dagger\eta_2} \tag{5.42}$$

Writing this as an exponential seems strange, but it is completely correct. If we taylor expand, all second order terms and higher vanish as Grassman numbers square to zero. But this form does make the analogy with the bosonic case more clear. With this, we can now define the fermionic path integral:

$$
\begin{aligned}
\langle\psi_f|e^{-i\hat{H}t}|\psi_i\rangle &= \int\prod_{j=1}^{N} d\psi_j d\psi_j^\dagger \ldots \langle\psi_2|e^{-iH\delta t}|\psi_1^\dagger\rangle\langle\psi_1^\dagger|\psi_1\rangle \\
&= \int\prod_{j=1}^{N} d\psi_j d\psi_j^\dagger \ldots e^{-im\psi^\dagger\psi_2\delta t}\langle\psi_2|\psi_1^\dagger\rangle\langle\psi_1^\dagger|\psi_2\rangle \\
&= \int\prod_{j=1}^{N} d\psi_j d\psi_j^\dagger \ldots e^{-im\psi_1^\dagger\psi_2\delta t}e^{-\eta_1^\dagger(\eta_2-\eta_1)} \\
&= \int\prod_{i=1}^{N} d\psi_j \ldots e^{i(\psi_1^\dagger i(\psi_2-\psi_1)-m\psi_1^\dagger\psi_2)\delta t} \\
&\overset{N\to\infty}{\to} \int D\psi D\psi^\dagger e^{i\int dt\psi^\dagger(i\partial_t-m)\psi}
\end{aligned}
\tag{5.43}
$$

and in fact we are done; in the scalar case we integrated out the momenta ($p$), here we keep it ($\psi^\dagger$). So, we have our expression for a single fermion; on Thursday we will generalize this to the Dirac fermion.

# 6 Path Integral for Dirac QFT, Interacting Fermions

Last lecture, we found a path integral representation for fermionic quantum mechanics (wherein we had to replace the variables being integrated over by Grassman numbers):

$$
Z = \int\mathcal{D}\psi\mathcal{D}\psi^\dagger e^{i\int dt\psi^\dagger(i\partial_t-m)\psi} \tag{6.1}
$$

## 6.1 Generating Functional + Two-Point Function for Dirac QFT

For the Dirac QFT, we can consider the generating functional:

$$
Z[\eta,\bar{\eta}] = \int\mathcal{D}\psi\mathcal{D}\psi^\dagger e^{i\int d^4x\bar{\psi}(i\not{\partial}-m)\psi+\bar{\psi}\eta+\bar{\eta}\psi} \tag{6.2}
$$

From this, we can obtain the time-ordered two-point function (keeping in mind that these are Grassman numbers, so introducing minus signs accordingly where we would need to move the derivative past a grassman number) via:

$$
\begin{aligned}
\langle 0|\mathcal{T}\psi(x)\bar{\psi}(0)|0\rangle &= \frac{1}{Z[0,0]}\int\mathcal{D}\psi\mathcal{D}\psi^\dagger\psi(x)\bar{\psi}(0)e^{iS} \\
&= \frac{1}{Z[0,0]}\int\mathcal{D}\psi\mathcal{D}\psi^\dagger\left(\frac{\delta}{i\delta\bar{\eta}(x)}\right)\left(-\frac{\delta}{i\delta\eta(0)}\right)e^{i\int\bar{\psi}(i\not{\partial}-m)\psi+\bar{\psi}\eta+\bar{\eta}\psi} \\
&= \frac{1}{Z[0,0]}\left.\frac{\delta^2}{\delta\bar{\eta}(x)\delta\eta(0)}Z[\eta,\bar{\eta}]\right|_{\eta=0} \\
&= \left.\frac{\delta^2}{\delta\bar{\eta}(x)\delta\eta(0)}\log Z[\eta,\bar{\eta}]\right|_{\eta=0}
\end{aligned}
\tag{6.3}
$$

We can thus compute fermionic correlation functions via a path integral (note that we compute the connected correlation function above, which coincides with the disconnected correlation function here as the one point functions vanish). We will find that we can exactly perform the integral for $Z[\eta,\bar{\eta}]$ and so this fully solves the theory; all correlation functions can then be obtained as functional derivatives of the generating functional. Let's carry out this integral:

$$Z\eta, \bar{\eta} = \int \mathcal{D}\psi\mathcal{D}\psi^\dagger e^{-i\int \frac{d^4p}{(2\pi)^4}\bar{\psi}_p(-\not{p}+m)\psi_p - \bar{\psi}_p\eta_p - \bar{\eta}_p\psi_p} \tag{6.4}$$

We complete the square; we want this to look like the Gaussian integral with argument $\bar{\chi}_p(\not{p}+m)\chi_p$, which motivates the definition of:

$$\begin{aligned} \chi_p &= \psi_p - (\not{p}+m)^{-1}\eta_p \\ \bar{\chi}_p &= \bar{\psi}_p - \eta_p^\dagger(\not{p}^\dagger + m)^{-1}\gamma^0 \end{aligned} \tag{6.5}$$

where we note that:

$$\gamma^0(\not{p})^\dagger\gamma^0 = \not{p} \tag{6.6}$$

which follows from:

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu \tag{6.7}$$

Of course with this substitution we end up with a term quadratic in the $\eta$s:

$$\bar{\psi}_p(-\not{p}+m)\psi_p - \bar{\psi}_p\eta_p - \bar{\eta}_p\psi_p = \bar{\chi}_p(\not{p}+m)\chi_p - \bar{\eta}_p(\not{p}+m)^{-1}\eta_p \tag{6.8}$$

so our generating functional becomes:

$$Z[\eta,\bar{\eta}] = \left(\int \mathcal{D}\chi\mathcal{D}\chi^\dagger e^{-i\int \frac{d^4p}{(2\pi)^4}\bar{\chi}_p(\not{p}+m)\chi_p}\right) e^{i\int \frac{d^4p}{(2\pi)^4}\bar{\eta}_p(\not{p}+m)^{-1}\eta_p} = Z[0,0]e^{i\int \frac{d^4p}{(2\pi)^4}\bar{\eta}_p(\not{p}+m)^{-1}\eta_p} \tag{6.9}$$

Thus:

$$\log Z[\eta,\bar{\eta}] = \log Z[0,0] + i\int \frac{d^4p}{(2\pi)^4}\bar{\eta}_p(\not{p}+m)^{-1}\eta_p \tag{6.10}$$

Writing this in position space (and leaving off the constant $\log Z[0,0]$):

$$\log Z[\eta,\bar{\eta}] = \int d^4x d^4x' \frac{d^4p}{(2\pi)^4}\bar{\eta}(x)\frac{e^{ip(x-x')}}{\not{p}+m}\eta(x') = -\int d^4x d^4x' \bar{\eta}(x) G_F(x-x')\eta(x') \tag{6.11}$$

Thus taking two derivatives, we find (as we did found in the canonical quantization calculation we did last class):

$$\langle 0|\mathcal{T}\psi(x)\bar{\psi}(0)|0\rangle = \frac{\delta^2}{\delta\bar{\eta}(x)\delta\eta(0)}\log Z[\eta,\bar{\eta}]\bigg|_{\eta=0} = G_F(x) \tag{6.12}$$

## 6.2 Wick's theorem for fermions

So we have rederived the expression for the two-point function, but we have done a lot more; we now have a simple way to compute arbitrary correlation functions. We have shown the generating functional is quadratic in $\eta$, implying a version of Wick's theorem for fermions. For example, looking at the connected four-point function (which vanishes):

$$
\begin{aligned}
0 = & \frac{\delta^4}{\delta\eta_1\delta\eta_2\delta\eta_3\delta\eta_4}\log Z\bigg|_{\eta=0} \\
&= \partial_{\eta_1}\partial_{\eta_2}\partial_{\eta_3}\left(\frac{\partial_{\eta_4}Z}{Z}\right) \\
&= \partial_{\eta_1}\partial_{\eta_2}\left(\frac{\partial_{\eta_3}\partial_{\eta_4}Z}{Z} - \frac{\partial_{\eta_3}Z\partial_{\eta_4}Z}{Z}\right) \\
&= \partial_{\eta_1}\left(\frac{\partial_{\eta_2}\partial_{\eta_3}\partial_{\eta_4}Z}{Z} - \frac{\partial_{\eta_3}\partial_{\eta_4}Z\partial_{\eta_2}Z}{Z^2} - \frac{\partial_{\eta_2}\partial_{\eta_3}Z\partial_{\eta_4}Z}{Z^2} + \frac{\partial_{\eta_3}Z\partial_{\eta_2}\partial_{\eta_4}Z}{Z^2}\right) \\
&= \frac{\partial_{\eta_1}\partial_{\eta_2}\partial_{\eta_3}\partial_{\eta_4}Z}{Z} - \frac{\partial_{\eta_3}\partial_{\eta_4}Z\partial_{\eta_1}\partial_{\eta_2}Z}{Z^2} - \frac{\partial_{\eta_2}\partial_{\eta_3}Z\partial_{\eta_1}\partial_{\eta_4}Z}{Z^2} + \frac{\partial_{\eta_1}\partial_{\eta_3}Z\partial_{\eta_2}\partial_{\eta_3}Z}{Z^2}
\end{aligned}
\tag{6.13}
$$

where in the fourth and fifth equality we use that the one-point function vanishes and hence we can discard any terms of the form of $\partial_\eta Z\Big|_{\eta=0} = 0$. Note the sign of the last term in the fourth equality, which arises from moving the $\partial_{\eta_2}$ derivative past the $\partial_{\eta_3}Z$, which results in an extra negative sign as these are anticommuting numbers - this is the only distinction from the analogous calculation we did for the scalar QFT. Thus, writing the four-point function in terms of two point functions:

$$\langle\psi_1\psi_2\psi_3\psi_4\rangle = \langle\psi_1\psi_2\rangle\langle\psi_3\psi_4\rangle + \langle\psi_2\psi_3\rangle\langle\psi_1\psi_4\rangle - \langle\psi_1\psi_3\rangle\langle\psi_2\psi_4\rangle \tag{6.14}$$

which we can see is a sum over all pairings, except every time we have to move fermionic variables across each other to pair them we introduce a minus sign. The above result hinges on the fact that:

$$\frac{\delta^4}{\delta\eta_1\delta\eta_2\delta\eta_3\delta\eta_4}\log Z\bigg|_{\eta=0} = 0 \tag{6.15}$$

which is true for free Dirac theories (for which the generating functional is quadratic in the sources).

Of course, its worth noting that throughout we have suppressed the indices above, but each of the $\psi$s are spinors, so these are really all matrix equations.

## 6.3 Interacting Fermions - Yukawa Theory

There are many theories of interacting fermions in nature - the main focus of our study will be QED, but we will look at a simpler theory first; namely coupling a scalar and a Dirac fermion.

To review, we have the Lagrangian for the free scalar:

$$\mathcal{L}_\phi = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 \tag{6.16}$$

and the Lagrangian for the free Dirac fermion:

$$\mathcal{L}_\psi = \bar{\psi}(i\not{p} - m)\psi \tag{6.17}$$

We can of course have these theories as independent/non-interacting, but let's make them talk to each other! How can we couple the two? The simplest way we may couple the two is:

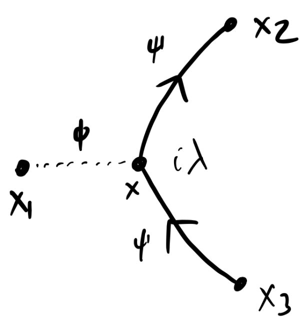
$$\mathcal{L}_{\text{Yukawa}} = \lambda\phi\bar{\psi}\psi \tag{6.18}$$

which is indeed a Lorentz-invariant scalar, as we require. You will explore this theory in PS3. When building a theory, it is good to start off with the simplest interactions possible (i.e. contain the least number of derivatives possible), as these are the most relevant interactions (as we can ascertain via dimensional analysis).

This interaction will lead to a correlation, which we can compute perturbatively in the usual way using the path integral formalism; to first order:

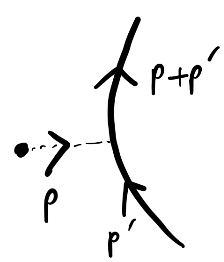
$$\begin{aligned}
\langle 0|\phi(x_1)\psi(x_2)\bar{\psi}(x_3)|0\rangle &\cong \left\langle iS_{\text{int}}\phi(x_1)\phi(x_2)\bar{\psi}(x_3)\right\rangle_{\text{free}} \\
&= i\lambda\int d^4x\,\langle\phi(x)\bar{\psi}(x)\psi(x)\phi(x_1)\psi(x_2)\bar{\psi}(x_3)\rangle_{\text{free}} \\
&= i\lambda\int d^4x G_F^{\phi}(x-x_1)\,\langle\psi(x_2)\bar{\psi}(x)\psi(x)\bar{\psi}(x_3)\rangle \\
&= i\lambda\int d^4x G_{\phi}^{F}(x-x_1)G_{\psi}^{F}(x_2-x)G_{\psi}^{F}(x-x_3)
\end{aligned} \tag{6.19}$$

We have contracted the two scalar fields together (the only nonvanishing contraction in the free theory), as well as $\psi(x_2)$ with $\bar{\psi}(x)$ and $\psi(x)$ with $\bar{\psi}(x_3)$ such that the Feynman diagram is connected (and hence nonvanishing). In position space, we can imagine this Feynman diagram as:

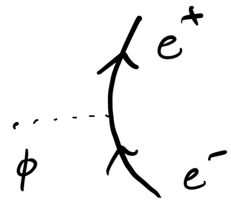



Or in momentum space:

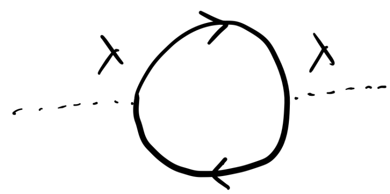
$$\left\langle\phi_p\psi\bar{\psi}_{p'}\right\rangle = i\lambda G_{\phi}(p)G_{\psi}(p+p')G_{\psi}(p') = i\lambda\frac{-i}{p^2+M^2}\frac{-i}{\not{p}+\not{p}'+m}\frac{-i}{\not{p}'+m} \tag{6.20}$$




The amputated correlator $\left\langle\phi_p\psi\bar{\psi}_{p'}\right\rangle_{\text{amp}} = i\lambda\mathbb{I}$ will be related through LSZ to the 1→2 S-matrix, namely the decay of a scalar particle into an electron/positron pair. When is the scalar particle unstable? If we imagine this in the rest frame of the scalar particle, this process requires $M > 2m$ (to produce the fermion and its antiparticle pair, plus giving the two kinetic energy) - when this condition is met $\phi$ is unstable, allowing for $\phi \to e^-e^+$, with decay rate $\Gamma \sim \lambda^2$ (which you will compute in PS4).




The decay rate also enters into the self-energy of the scalar, from which it is evident that $\Gamma \sim \lambda^2$.




## 6.4 Heavy Scalar Limit of Yukawa Theory - Fermi EFT

Taking the limit where $M$ is huge (larger beyond all other energy scales), we can study its effects by integrating it out. Our full QFT has the generating functional:

$$Z = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \mathcal{D}\phi e^{i\int \bar{\psi}(i\not{\partial}-m)\psi + \lambda\phi\bar{\psi}\psi - \frac{1}{2}\phi(\partial^2+M^2)\phi} \tag{6.21}$$

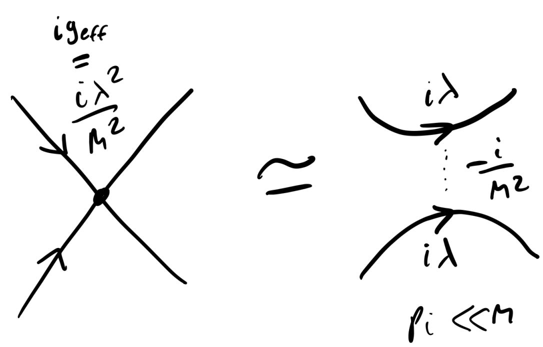
Note that if we just look at the $\phi$s (i.e. the $\phi$-sector), this is actually just a Gaussian theory! In our ultra-heavy limit, the mass energy is much greater than the kinetic term, so we neglect $\frac{1}{2}\phi(\partial^2)\phi$. Thus, writing the $\phi$-part as:

$$-\frac{1}{2}(\phi - \frac{\lambda}{M^2}\bar{\psi}\psi)M^2\left(\phi - \frac{\lambda}{M^2}\bar{\psi}\psi\right) + \frac{1}{2}\frac{\lambda}{M^2}\bar{\psi}\psi\bar{\psi}\psi \tag{6.22}$$

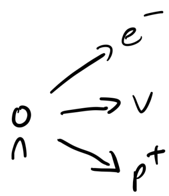
Thus writing $\tilde{\phi} = (\phi - \frac{\lambda^2}{M^2}\bar{\psi}\psi)$, the above becomes:

$$Z = \left(\int \mathcal{D}\tilde{\phi} e^{-\frac{1}{2}\tilde{\phi}M\tilde{\phi}}\right) \int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{i\int \bar{\psi}(i\not{\partial}-m)\psi + \frac{1}{2}\frac{\lambda^2}{M^2}\bar{\psi}\psi\bar{\psi}\psi} \tag{6.23}$$

So we can see that the $\phi$ generated an effective 4-fermion interaction! Free fermions do not scatter, but we have the emergence of an interaction vertex, mediated by the $\phi$ particle (which creates an effective interaction $ig_{\text{eff}} = i\frac{\lambda^2}{M^2}$); we can picture this as:



This was the first effective theory considered - this is Fermi's effective field theory (EFT) of weak interactions, to explain $\beta$-decay. Radioactive materials emit electrons from time to time, with more specifically a neutron decaying into an electron, proton, and neutrino. This requires a 4-fermion interaction $g\bar{\psi}_1\psi_2\bar{\psi}_3\psi_4$. The microscopics of the theory came later, but the first try at explaining this phenomena was this EFT.




This $\psi\bar{\psi}\psi\bar{\psi}$ interaction is irrelevant, or non-renormalizable. This is an energy that shrinks at small energies/IR and grows at high energies/UV. It is thus not UV complete, and cannot be the microscopic description of the theory. For us, the "UV completion" is the Yukawa theory (in Fermi's case it was a little more complicated, e.g. with Gauge fields). This effective interaction can be thought of as a force between particles (here the fermions $e^-, e^+$) that is mediated by the mysterious heavy particle. We can literally write this interaction as a potential:

$$\begin{aligned} H_{\text{int}} &= -\frac{\lambda^2}{2M^2}\int d^4x\psi(\bar{x})\psi(x)\bar{\psi}(x)\psi(x) \\ &= \int d^4xd^4y\psi(\bar{x})\psi(x)V(x-y)\bar{\psi}(y)\psi(y) \end{aligned} \tag{6.24}$$

where:

$$V(x-y) = -\frac{\lambda^2}{2M^2}\delta^4(x-y) \tag{6.25}$$

is a local interaction. In the problem set, you will find that this interaction is not a delta function (it is just sharply peaked), but in this extreme limit we currently consider, it is completely local. Given a particle at $x'$, because $V < 0$, it is energetically favourable to place another particle nearby. The Yukawa potential allows scalar particles to generate attractive interactions between fermions.

A final comment; if we spell out $\bar{\psi}\psi$, we find:

$$\bar{\psi}\psi \sim b^\dagger b + d^\dagger d. \tag{6.26}$$

Thus, all the particles - both fermions and anti-fermions attract each other. Phrased another way:

$$H_{\text{int}} \sim \int_{xx'}(n_b+n_d)_x V(x-x')(n_b+n_d)_{x'} \tag{6.27}$$

This is *not* how we see electrons behave (which of course repulse), which indicates to us that the scalar particle is not what mediates the force between electrons. Instead, in QED we have that the force between electrons is mediated by the photon/gauge field[4] $A_\mu$. This does not couple $\bar{\psi}, \psi$, but rather couples the current $j^\mu = \bar{\psi}\gamma^\mu\psi$, and in particular:

$$\bar{\psi}\gamma^0\psi = b^\dagger b - d^\dagger d \tag{6.28}$$

So in QED:

$$H_{\text{int}} \sim -\int_{xx'}(n_b-n_d)_x V(x-x')(n_b-n_d)_{x'} \tag{6.29}$$

which means particles of the same charge repel, and particles of opposite charge attract. We will return to this soon!

[4] This wins out over the scalar mediation, as the photon is massles (and hence can mediate a long-range interaction), compared to the scalar mediated interaction which is short-ranged

Q (from me): Is it possible to look at the opposite limit where $2m \gg M$ (i.e. where the fermions are much heavier than the scalar), and integrate out the fermions, since the fermion subsector of the Yukawa Lagrangian is also Gaussian?

A (Luca): Yes; but note that this generates diagrams to all powers/orders of $\phi$, so is quite complicated. But, this is what people do, e.g., when they do quantum monte carlo simulations, because fermions are in general difficult to simulate (presumably due to sign problems).

# 7 Vector Fields and QED I - Classical E&M, Maxwell Equations, Gauge Invariance, Coupling to Matter

We move to the next nontrivial representation of the Lorentz group! Namely the $(\frac{1}{2}, \frac{1}{2})$ representation with spin-1, which transforms like a 4-vector. You have already encountered this previously in the form of the photon in E&M. Today we will start from classical electromagnetism, and derive the Maxwell equations from an action principle.

## 7.1 The Maxwell Equations

We can write Maxwell's equations compactly in a Lorentz covariant way:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{7.1}$$

$$\epsilon^{\mu\nu\lambda\rho}\partial_\nu F_{\lambda\rho} = 0 \tag{7.2}$$

with $F_{\mu\nu} = -F_{\nu\mu}$ the antisymmetric Maxwell stress tensor, with $\frac{4\cdot 3}{2} = 6$ components; 3 for the electric and 3 for the magnetic field. In particular:

$$\begin{aligned} F_{0i} &= E_i \\ F_{ij} &= -\epsilon_{ijk}B^k \end{aligned} \tag{7.3}$$

Indeed, let's see that this compact form of the Maxwell equations indeed reproduces the more familiar form. Looking at Eq. (7.1) for $\nu = 0$:

$$\rho = j^0 = \partial_i F^{i0} = -\partial_i F^{0i} = (-1)^2\partial_i F_{01} = \boldsymbol{\nabla}\cdot\mathbf{E} \tag{7.4}$$

which is Gauss' Law. For $\nu = i$:

$$j^i = \partial_0 F^{0i} + p_j F^{ji} = -\partial_0 E_i - \epsilon_{ijk}\partial_j B_k = -\dot{E}_i + \epsilon_{ijk}\partial_j B_k \tag{7.5}$$

or if we look at all components:

$$\mathbf{j} = -\dot{\mathbf{E}} + \boldsymbol{\nabla}\times\mathbf{E} \tag{7.6}$$

which is Ampere's Law.

Now looking at Eq. (7.2) for $\mu = 0$:

$$0 = \epsilon^{ijk}\partial_i F_{jk} = -\epsilon^{ijk}\epsilon_{jkl}\partial_i B^l = -\delta^i_l\partial_i B^l \propto \boldsymbol{\nabla}\cdot\mathbf{B} \implies 0 = \boldsymbol{\nabla}\cdot\mathbf{B} \tag{7.7}$$

which gives the Magnetic Gauss' law. Finally, for $\mu = i$:

$$0 = \epsilon^{i\nu\lambda\rho}\partial_\nu F_{\lambda\rho} = \epsilon^{i0jk}\partial_0 F_{jk} + \epsilon^{ij0k}\partial_j F_{0k} = \epsilon^{ijk}\epsilon_{jkl}\partial_0 B^l + 2\epsilon^{ijk}\partial_j E_k = 2\delta^i_l\partial_0 B^l + 2\epsilon^{ijk}\partial_j E_k \implies \dot{\mathbf{B}} + \boldsymbol{\nabla}\times\mathbf{E} = 0 \tag{7.8}$$

note in the second equality we have used the requirement that one of the indices be time (as the Levi-Civita tensor is only nonzero when all four indices are different.) The last equation is Faraday's Law, and so we have reproduced the four familiar versions of Maxwell's Law.

One can check that $F_{\mu\nu}$ transforms like a tensor under Lorentz transformations:

$$F_{\mu\nu}(x) \to \Lambda_\mu^{\ \alpha} \Lambda_\nu^{\ \mu} F_{\alpha\beta}(\Lambda^{-1}x) \tag{7.9}$$

Intuitively, under a boost charges become currents and so we transform between the tensor components. Writing things in terms of the electromagnetic tensor makes lorentz invariance manifest, provided $j^\mu$ is a Lorentz 4-vector. Another feature - observe that the current $j^\mu$ also has to be conserved:

$$\partial_\mu j^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = 0 \tag{7.10}$$

where we conclude things are zero because the derivatives are symmetric while $F^{\mu\nu}$ is antisymmetric. This property is very reminiscent of Noether's theorem - it gives us the intuition that the electric/magnetic fields will want to couple to field theories with a $U(1)$ conservation law.

## 7.2 Action Principle for Free Maxwell Theory

Writing things in terms of $\mathbf{E}, \mathbf{B}$ is not great because they transform into each other under Lorentz boosts (the equations thus do not look Lorentz invariance). There is a further issue. The Magnetic Gauss' Law $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ implies that, locally, $\mathbf{B}$ is a curl:

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{7.11}$$

or equivalently:

$$F_{ij} = \partial_i A_j - \partial_j A_i. \tag{7.12}$$

Similarly, Faraday's Law says:

$$0 = \boldsymbol{\nabla} \times (\mathbf{E} + \dot{\mathbf{A}}) \tag{7.13}$$

which tells us that locally, $\mathbf{E} + \dot{\mathbf{A}}$ is a gradient:

$$\mathbf{E} + \dot{\mathbf{A}} = \boldsymbol{\nabla} A_0 \tag{7.14}$$

or:

$$F_{i0} = \partial_i A_0 - \partial_0 A_i \tag{7.15}$$

In summary:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7.16}$$

Writing the field strength in this way, Eq. (7.2) is now automatic:

$$\epsilon^{\mu\nu\lambda\rho} \partial_\nu F_{\lambda\rho} = 2\epsilon^{\mu\nu\lambda\rho} \partial_\nu \partial_\lambda A_\rho = 0 \tag{7.17}$$

where we conclude that this vanishes as $\epsilon^{\mu\nu\lambda\rho}$ is antisymmetric while $\partial_\nu \partial_\lambda$ is symmetric. Where does Eq. (7.1) come from in this perspective? We view it as arising as the equation of motion of an action. So, let's try to find an action principle for Maxwell's equations. Let's start without charged matter; $\partial_\mu F^{\mu\nu} = 0$. Schematically, it looks like:

$$0 = \partial_\mu F^{\mu\nu} \sim \partial\partial A \sim \text{Klein-Gordon} \implies \mathcal{L} = \frac{1}{2} A \partial\partial A \tag{7.18}$$

Let's be a little more precise. There are only two possible index contractions:

(a) $\partial_\mu A_\nu \partial^\mu A^\nu$

(b) $\partial_\mu A^\mu \partial_\nu A^\nu$

(c) There is an apparent final contender $\partial_\mu A_\nu \partial^\nu A^\mu$, but we can integrate by parts to swap derivatives, which makes this term equivalent to (b).

Thus, we have:

$$S = \frac{1}{2}a(\partial_\mu A_\nu)^2 + b(\partial_\mu A^\mu)^2 \tag{7.19}$$

so then the variation becomes:

$$\delta S = -\int a\partial_\mu A_\nu \partial^\mu \delta A^\nu + b\partial_\mu A^\mu \partial_\nu \delta A^\nu \tag{7.20}$$

now integrating by parts to isolate the variation:

$$\delta S = \int \delta A_\nu [a\partial_\mu \partial^\mu A_\nu + b\partial^\mu \partial_\nu A_\mu] \tag{7.21}$$

Hence $\delta S = 0$ for any variation $\delta A_\nu$ forces the term in brackets to be zero, i.e.:

$$a\partial_\mu(\partial^\mu A^\nu) + b\partial_\mu(\partial^\nu A^\mu) = 0 \tag{7.22}$$

This indeed yields the equation of motion:

$$0 = \partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{7.23}$$

in the case that $b = -a$. Thus, the action that produces Eq. (7.1) is:

$$S = -\frac{1}{2}\int (\partial_\mu A_\nu)^2 - (\partial_\mu A^\mu)^2 \tag{7.24}$$

Integrating by parts the second term to swap the derivatives $\partial_\mu A^\mu \partial_\nu A^\nu = \partial_\nu A^\mu \partial_\mu A^\nu$, we obtain:

$$S = -\frac{1}{2}\int \partial^\mu A^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) = -\frac{1}{2}\int \partial^\mu A^\nu F_{\mu\nu} \tag{7.25}$$

Since $\partial^\mu A^\nu$ is contracted with an antisymmetric tensor, we can replace it with its antisymmetric part, which is just the maxwell tensor again:

$$\partial^\mu A^\nu \overset{\text{antisymmetric}}{=} \frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{2}F^{\mu\nu} \tag{7.26}$$

and thus:

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} \tag{7.27}$$

we have thus obtained an action principle for the Maxwell equations with no matter.

## 7.3 Adding Charged Matter

What we really want is not $\partial_\mu F^{\mu\nu} = 0$, but rather $\partial_\mu F^{\mu\nu} = j^\nu$, or equivalently:

$$\partial_\mu F^{\mu\nu} - j^\nu = 0 \tag{7.28}$$

We already have the first term from $\delta S$ that we constructed above, we just need another term in the action for which $\delta S'/\delta A_\nu$ gives the current. We need:

$$\delta S' = -\int \delta A_\nu j^\nu \implies S' = -\int A_\nu j^\nu \tag{7.29}$$

and thus the total action is:

$$\boxed{S[A] = -\int d^4x \frac{1}{4}(F_{\mu\nu})^2 + A_\mu j^\mu} \tag{7.30}$$

with the additional equation:

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{7.31}$$

from these we can reproduce everything from classical electromagnetism.

## 7.4 Gauge Invariance

In addition to Lorentz and translation symmetry, the above theory has a strange "symmetry" known as Gauge invariance (not a true symmetry, just an invariance of the action). This symmetry acts on the vector field as follows:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x) \tag{7.32}$$

where $\lambda(x)$ is an *arbitrary* scalar function of spacetime. How do we see that this indeed leaves the action invariant? Indeed:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu - \partial_\mu A_\mu - (\partial_\mu \partial_\nu \lambda - \partial_\nu \partial_\mu \lambda) = \partial_\mu A_\nu - \partial_\mu A_\mu = F_{\mu\nu} \tag{7.33}$$

where the term in brackets vanishes due to the symmetry of the derivatives. Writing the field strength as a curl hence picks up a kind of redundancy. Moreover, we can see that the second part of the action is also invariant:

$$-\int A_\mu j^\mu \to -\int A_\mu j^\mu + \partial_\mu \lambda j^\mu \overset{\text{IBP}}{=} -\int A_\mu j^\mu - \lambda \partial_\mu j^\mu = -\int A_\mu j^\mu \tag{7.34}$$

where in the last equality we use that $\partial_\mu j^\mu = 0$, as we couple to conserved currents.

This is a bit different from a symmetry - it should be thought of as a redundancy of our description. Several different choices of $A_\mu$ can give rise to the same physics. Loosely, we can think of it as one "component" of $A_\mu$ does not appear in the action. One can fix this redundancy by choosing/fixing a gauge.

(a) $A_0 = 0$ (Temporal gauge)

(b) $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ (Coloumb gauge)

(c) $\partial_\mu A^\mu = 0$ (Lorenz gauge)

To understand what is left of the dynamics of our system, let us pick one gauge and then study what is left. For now, we will choose the Coloumb gauge, which is one that you may have already played with in your E&M class (it is a useful choice for working with electrostatics). Let's see the impact of this choice on the action.

$$\begin{aligned} S = -\frac{1}{4}\int F^2 &= -\frac{1}{4}\int -2F_{0i}F^{0i} - F_{ij}F^{ij} \\ &= \int \frac{1}{2}(\partial_0 A_i - \partial_i A_0)(\partial_0 A_i - \partial_i A_0) + \frac{1}{2}\partial_i A_j(\partial_i A_j - \partial_j A_i) \\ &= \int \frac{1}{2}(\partial_0 A_i)^2 + \frac{1}{2}(\partial_i A_0)^2 - \frac{1}{2}(\partial_i A_j)^2 \\ &= \int \frac{1}{2}A_i(-\partial_0^2 + \partial_j^2)A_i - \frac{1}{2}A_0 \boldsymbol{\nabla}^2 A_0 \end{aligned} \tag{7.35}$$

where in the third line we note that the use of IBP gives $\partial_i A_i = \boldsymbol{\nabla} \cdot \mathbf{A} = 0$ by our choice of gauge for some of the terms. We see here that the equation of motion for $A_0$ is:

$$\boldsymbol{\nabla}^2 A^0 = j^0 \tag{7.36}$$

i.e. $A^0$ in this gauge is not a propagating degree of freedom. It's fixed by the charge density profile.

To review; we started with $A_\mu$ which has 4 degrees of freedom. This was a bit redundant, because the action was invariant under a transformation $A_\mu \to A_\mu + \partial_\mu \lambda$. We fixed a gauge (Coloumb) so that we had $A_\mu, \boldsymbol{\nabla} \cdot \mathbf{A} = 0$, i.e. we now have 3 degrees of freedom. This gave an additional constraint, namely $A^0$ being fixed by $j^0$. Hence by the end we were left with 2 degrees of freedom. This is what we expected; 2 is the DoFs for a massless field (the 2 helicities), as you saw on a former homework.

## 7.5 Coupling Maxwell to Dirac

So, we have our Maxwell theory:

$$S = -\int d^4x \frac{1}{4e^2}(F_{\mu\nu})^2 + A_\mu j^\mu \tag{7.37}$$

where we have renormalized $A \to eA$ implying $\partial_\mu F^{\mu\nu} = ej^\nu$. It needs to be coupled to a conserved current $\partial_\mu j^\mu = 0$. Dirac fermions have precisely such a current. Namely, with the action:

$$S = \int \bar{\psi}(i\not{\partial} - m)\psi \tag{7.38}$$

we have the symmetry $\psi \to e^{i\lambda}\psi$ which the Noether procedure gives the current:

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{7.39}$$

The full Maxwell + matter (Dirac) action is thus:

$$S = \int \bar{\psi}(i\not{\partial} - m)\psi - A_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{4e^2}(F_{\mu\nu})^2 \tag{7.40}$$

Something we can do in this theory (and something that one often does) is to package the interaction $A_\mu\bar{\psi}\gamma^\mu\psi$ into the derivative:

$$\boxed{S = \int \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4e^2}(F_{\mu\nu})^2} \tag{7.41}$$

where we have defined a covariant derivative:

$$D_\mu = \partial_\mu + iA_\mu \tag{7.42}$$

so then:

$$\not{D}\psi = \gamma^\mu(\partial_\mu + iA_\mu)\psi = \not{\partial}\psi + iA_\mu\gamma^\mu\psi \tag{7.43}$$

So, we have our action for QED! We have our Dirac theory, our Maxwell theory, and then our interaction, which leads to many interesting effects. The presence of this interaction makes the theory unsolvable, but quite interesting and subtle.

We can ask why did we package the derivative in this strange way. We do this because this coupled theory still retains the gauge invariance of Maxwell theory. Namely, the action is invariant under the transformations:

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu\lambda(x) \\ \psi &\to e^{i\lambda(x)}\psi \end{aligned} \tag{7.44}$$

Interestingly, this is still an invariance of the action when $\lambda(x)$ is a fully arbitrary (real) scalar function of spacetime. Let's check that this is true. The $F^2$ invariance carries over from the maxwell theory, and $m\bar{\psi}\psi$ is also easy to see (the $e^{\pm i\lambda}$s cancel). What about the kinetic term $\bar{\psi}i\not{D}\psi$? Well, looking at how the covariant derivative transforms:

$$\begin{aligned} D_\mu\psi = \partial_\mu\psi + iA_\mu\psi &\to \partial_\mu(e^{i\lambda(x)}\psi) + iA_\mu e^{i\lambda(x)}\psi - i\partial_\mu\lambda e^{i\lambda(x)}\psi \\ &= i\partial_\mu\lambda e^{i\lambda}\psi - i\partial_\mu\lambda e^{i\lambda}\psi + e^{i\lambda}\partial_\mu\psi + e^{i\lambda}iA_\mu\psi = e^{i\lambda(x)}D_\mu\psi \end{aligned} \tag{7.45}$$

Thus we see that $D_\mu\psi \to e^{i\lambda(x)}D_\mu\psi$, which is why we wanted to package things this way to get this easy transformation rule. Thus the kinetic term is easily seen to be invariant:

$$\bar{\psi}\gamma^\mu D_\mu\psi \to \bar{\psi}e^{-i\lambda}e^{i\lambda}\gamma^\mu D_\mu\psi = \bar{\psi}\gamma^\mu D_\mu\psi \tag{7.46}$$

Thus - as in the case with pure Maxwell theory, it is still true that the photon carries less degrees of freedom than as it first appears. The degrees of freedom available here will be the 2 helicities of the photon and the Dirac field.

Note; here we have chosen to couple Dirac with Maxwell, but we can choose any theory with a conserved current; for example the scalar theory:

$$S = \int (\partial\phi)^2 + (\partial\phi)^4 \tag{7.47}$$

has the shift symmetry $\phi \to \phi + c$. In this model the covariant derivative will look different. We have the gauge transformations:

$$\begin{aligned} \phi &\to \phi + \lambda(x) \\ A_\mu &\to A_\mu - \partial_\mu \lambda(x) \end{aligned} \tag{7.48}$$

So then the correct way to make a covariant derivative is:

$$D_\mu \phi = \partial_\mu \phi - A_\mu \to \partial_\mu \phi - A_\mu \tag{7.49}$$

So we have the action:

$$S = -\int (D\phi)^2 + (D\phi)^4 + \frac{1}{4e^2}F^2 \tag{7.50}$$

which actually turns out to describe a superconductor! We won't discuss this in lecture in too much detail, but you may see it show up on a future homework.

# 8 Electron Dipole Moment, Path Integral for Gauge Fields

Last lecture, we found the QED action:

$$S = \int d^4x \bar{\psi}(i\not{\partial} - m)\psi + A_\mu \bar{\psi}\gamma^\mu \psi + \frac{1}{4e^2}(F_{\mu\nu})^2 = \int d^4x \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4e^2}F^2 \tag{8.1}$$

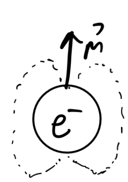
where in the second equality we introduce the covariant derivative:

$$D_\mu \psi = \partial_\mu \psi + iA_\mu \psi \tag{8.2}$$

which is useful because it makes the action manifestly gauge invariant (you can observe that under a $U(1)$ transformation, the covariant derivative only picks up a phase).

## 8.1 Magnetic dipole of the electron

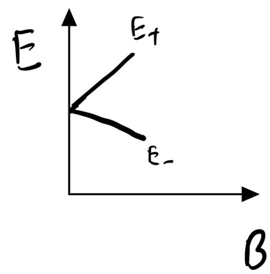
We do not yet know how to treat the gauge fields at a quantum-mechanical level. But even before going there, we can already make some predictions, just via treating the gauge fields classically. Namely, the electron is not just a point charge, but it also couples to the magnetic field, and we will see the prediction for its magnetic moment come out of the QED action.



For a non-relativistic particle, we could consider a Zeeman coupling:

$$H = \frac{(\mathbf{p} + \mathbf{A})^2}{2m} + \mu \mathbf{B} \cdot \sigma \tag{8.3}$$

wherein the particle eigenstates $|\mathbf{p}, s = \pm\rangle$ are based on the particle momenta $\mathbf{p}$ and spin $s$. $\mu$ is the free parameter, which tells us the energy splitting between the two spin states (as a function of $B$):




Let's study the modified/gauged Dirac equation (the equation of motion in QED):

$$0 = (i\not{D} + m)\psi \tag{8.4}$$

and square it, in the way we derived Klein-Gordon equation from the regular dirac equation (This is how we derived the spectrum of free dirac fermions to be $E = \sqrt{p^2 + m^2}$):

$$\begin{aligned} 0 &= (i\not{D} + m)(-i\not{D} + m)\psi \\ &= (\not{D}\not{D} + m^2)\psi \\ &= (\gamma^\mu \gamma^\nu D_\mu D_\nu + m^2)\psi \end{aligned} \tag{8.5}$$

Now we write:

$$\gamma^\mu \gamma^\nu = \frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = -\eta^{\mu\nu} - 2iS^{\mu\nu} \tag{8.6}$$

with:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{8.7}$$

Thus:

$$0 = (-D^2 - iS^{\mu\nu} 2D_\mu D_\nu + m^2)\psi \tag{8.8}$$

If we had no background ($A \to 0$), then the derivatives of $D_\mu D_\nu$ commute (hence symmetric), and its multiplied by an antisymmetric $S^{\mu\nu}$ and so the entire term vanishes. We are just left with the Klein-Gordon equation, as we expect. Since $2D_\mu D_\nu$ is contracted with $S^{\mu\nu}$, let us write it as:

$$2D_\mu D_\nu \psi = (D_\mu D_\nu - D_\nu D_\mu)\psi \tag{8.9}$$

where:

$$D_\mu D_\nu \psi = (\partial_\mu + iA_\mu)(\partial_\nu \psi + iA_\nu \psi) = \partial_\mu \partial_\nu \psi + iA_\nu \partial_\mu \psi + i\partial_\mu A_\nu \psi + iA_\mu \partial_\nu \psi - A_\mu A_\nu \psi \tag{8.10}$$

So the difference becomes (notice that most of the terms here are symmetric and cancel, with the exception of the $i\partial_\mu A_\nu \psi$):

$$[D_\mu, D_\nu]\psi = i(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = iF_{\mu\nu}\psi \tag{8.11}$$

This is nice! Covariant derivatives are gauge invariant, and the thing we get out here is indeed a gauge invariant quantity.

Thus, returning to our equation of motion:

$$0 = (-D^2 + S^{\mu\nu}F_{\mu\nu} + m^2)\psi \tag{8.12}$$

Now, we want to study the energies of solutions to this equation when there is a magnetic field. We set $A_0 = 0$ and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. Only $F_{ij}$ are activated. What are the $S^{ij}$?:

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{i}{4}\begin{pmatrix} 0 & \sigma^i \\ \bar{\sigma}^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ \bar{\sigma}^j & 0 \end{pmatrix} - (i \leftrightarrow j) = \frac{i}{4}\begin{pmatrix} [\sigma^i, \bar{\sigma}^j] & 0 \\ 0 & [\bar{\sigma}^i, \sigma^j] \end{pmatrix} \tag{8.13}$$

Then looking at the relations that the Paulis satisfy:

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma_k \tag{8.14}$$

we conclude:

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{8.15}$$

Thus:

$$S^{ij}F_{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}(\partial_i A_j - \partial_j A_i) = \begin{pmatrix} \sigma^k B_k & 0 \\ 0 & \sigma^k B_k \end{pmatrix} \tag{8.16}$$

So we have the full differential equation - to get the energies now all we need to do is fourier transform:

$$E^2 = (\mathbf{p} + \mathbf{A})^2 + m^2 + \begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{B} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \mathbf{B} \end{pmatrix} \tag{8.17}$$

So the energy in the non-relativistic limit (wherein $m^2 \gg (\mathbf{p} + \mathbf{A})^2$) becomes:

$$E = m\sqrt{1 + \frac{(\mathbf{p} + \mathbf{A})^2}{m^2} + \frac{1}{m^2}\begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{B} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \mathbf{B} \end{pmatrix}} \approx m + \frac{(\mathbf{p} + \mathbf{A})^2}{2m} + \frac{1}{2m}\begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{B} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \mathbf{B} \end{pmatrix} \tag{8.18}$$

With the normalization of $A \to eA$ (the more familiar normalization from E&M):

$$E - m = \frac{(\mathbf{p} + e\mathbf{A})^2}{2m} + \frac{e}{2m}\boldsymbol{\sigma} \cdot \mathbf{B} \tag{8.19}$$

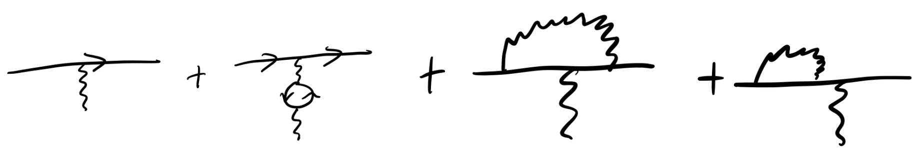
and thus:

$$\boxed{\mu = \frac{e}{2m}} \tag{8.20}$$

This is a nice and simple prediction, and one can compare it to experiments; we find:

$$\mu = \frac{e}{2m} \cdot 1.0011597\ldots \tag{8.21}$$

which is very close, but off! Why is it off? We treat the photon classically here, and there are quantum corrections. Pictorially, we can imagine that these corrections may look like (with the leftmost diagram the classical prediction):



+ 2-loop

The 1-loop corrections give the 0.00115, and if you are braver and have a lot more time you can keep going, and the current agreement of theory and experiment is $10^{-13}$ (wow!) The dimensionless coupling in QED turns out to be small, which is why the perturbative expansion is useful here.

## 8.2 Path Integral for Gauge Fields - First attempt

Quantizing gauge fields is important not just for a more accurate prediction of the electron dipole, but in various aspects of particle and condensed matter physics. To do this, we jump straight to the path integral formulation. A good reference for this is Peskin 9.4, as well as Fradkin 9.5. We follow the approach by Fadeev and Popov approach. There are several advantages:

- Path integrals! Much easier to calculate things
- Lorentz invariance is manifest
- Generalizes easily to quantization of non-abelian gauge fields

There are however alternatives you can read about, e.g. Gupta-Bleuler.

The naive guess for the path integral would be:

$$Z = \int \mathcal{D}A_0 \mathcal{D}A_1 \mathcal{D}A_2 \mathcal{D}A_3 e^{iS_M[A]} \tag{8.22}$$

with:

$$S_M[A] = -\frac{1}{4} \int d^4x F^2 \tag{8.23}$$

and indeed this is not too far off. But it has some issues - namely due to gauge invariance. Loosely, gauge invariance tells us that there is a component of the gauge fields that does not enter into the action. So, one of the four integrals is over nothing. If this was it, it wouldn't be a huge problem, but there is a related larger problem; namely the kinetic term $\sim A\partial\partial A \sim ADA$ is not invertible (which is what we usually do to get the propagator). How do we see this? Well, looking at the Maxwell action:

$$S_M[A] = -\frac{1}{4}\int_x F^2 = -\frac{1}{2}\int_x \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A_\mu) = \int_x A_\nu (\partial^2 \eta^{\mu\nu} - \partial^\nu \partial^\mu) A_\mu \tag{8.24}$$

and in momentum space this becomes:

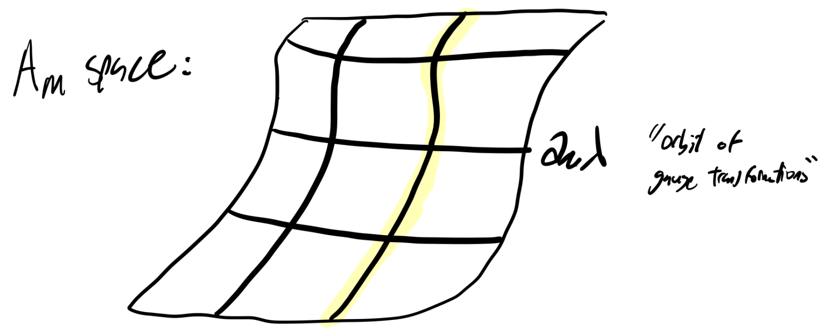
$$S_M[A] = -\int \frac{d^4p}{(2\pi)^4} A_\mu(-p)(p^2\eta^{\mu\nu} - p^\mu p^\nu) A_\nu(p) \tag{8.25}$$

with "$D$" $= (p^2\eta^{\mu\nu} - p^\mu p^\nu)$. Usually, we solve theories in the path integral formalism by coupling the theory to sources (say, $A_\mu J^\mu$) and complete the square to get something of the form $JD^{-1}J$. But here, $D$ is not invertible; indeed, we see this by acting $D$ on $p^\mu$:

$$(p^2\eta^{\mu\nu} - p^\mu p^\nu)p^\mu = p^2 p^\mu - p^2 p^\mu = 0 \tag{8.26}$$

i.e. $p^\mu$ is in the kernel of this matrix, and hence the matrix is not invertible. The underlying reason for this is due to the gauge invariance - "Pure gauge configurations" $A_\mu(x) = \partial_\mu \lambda(x)$ have zero action and we should remove them/not be integrating over them.

A visual intuition; the space of gauge fields $A_\mu$ is large, but many of these fields are related via gauge transformations, and we integrate over these as well. The resolution is to choose a slice satisfying gauge fixing.




## 8.3 Gauge Fixing in the Path Integral

Consider a gauge-fixing condition:

$$g(A_\mu) = \nabla \cdot \mathbf{A} \text{ (Coloumb)}, \quad g(A_\mu) = \partial_\mu A^\mu \text{ (Lorenz)}, \ldots \tag{8.27}$$

After we fix a gauge, we can no longer relate different fields via gauge transformations, and we only integrate over "true" degrees of freedom. To this end would like to introduce something like $\delta(g(A))$ in the path integral. This will enforce that only configurations along a slice would enter the path integral.

To not mess things up, we will insert:

$$1 = \int \mathcal{D}\lambda \delta(g(A_\lambda)) \det \frac{\delta g(A_\lambda)}{\delta \lambda} \tag{8.28}$$

where given $A_\mu$, $A_\mu^\lambda = A_\mu + \partial_\mu \lambda$. The above expression is in analog to the single-variable function case, where $1 = \int dx \delta(f(x))|f'(x)|$. So, our path integral then becomes:

$$Z = \int \mathcal{D}\lambda \mathcal{D}A \delta(g(A^\lambda)) \det \frac{\delta g(A_\lambda)}{\delta \lambda} e^{iS_M[A]} \tag{8.29}$$

A comments; the path integral we started with is independent of any $g$, but since we insert 1, we don't introduce $g$ dependence (as should be the case, since we shouldn't expect fixing a gauge to change any of the physics). The introduction of this gauge fixing path integral will allow us to isolate the infinity/redundancy from the original path integral. Carrying on, note that the Maxwell equation is gauge invariant, so we can replace $A \to A_\lambda$. Therein only $A_\lambda$ appears in the integrand, so we can instead replace all variables with $A$, leaving the integrand $\lambda$-independent (note that the derivative term is also $\lambda$ independent, as we evaluate it at $A$ after taking the functional derivative):

$$Z = \left( \int \mathcal{D}\lambda \right) \int \mathcal{D}A \delta(g(A)) \left. \frac{\delta g(A^\lambda)}{\delta \lambda} \right|_A e^{iS_M[A]} \tag{8.30}$$

Thus we have isolated the integral $\int \mathcal{D}\lambda$ over gauge orbits (that gives a factor of infinity) in a way that we can manifestly see it does nothing. Now we have an integral over $A$ with a delta function, which picks out a "slice" in gauge field space.

Now, this doesn't quite look like our regular Gaussian path integral, so we may have cause for worry. We will deal with this $\delta(g(A))$ by averaging over carefully chosen gauges. We consider:

$$g_\omega(A) = \partial_\mu A^\mu - \omega(x) \tag{8.31}$$

the Lorenz gauge plus a function of spacetime. We average over these gauges with Gaussian weight:

$$e^{i\int d^4x \frac{\omega^2(x)}{2\xi}} \tag{8.32}$$

This trick will produce a Gaussian path integral!

$$Z = \int \mathcal{D}A\mathcal{D}\omega e^{-i\int \frac{\omega^2(x)}{2\xi}} \delta(\partial_\mu A^\mu - \omega) \det(\partial^2) e^{-\frac{i}{4}\int_x F^2} \tag{8.33}$$

An aside; the $\det(\partial^2)$ term becomes much more interesting in the context of Non-Abelian gauge theories, as it then retains some dependence on $A$. Now, the $\omega$ integral is easy to evaluate, because we have a $\delta$ function fixing it:

$$Z = \int \mathcal{D}Ae^{-i\int_x \frac{1}{4}F^2 + \frac{1}{2\xi}(\partial_\mu A^\mu)^2} \tag{8.34}$$

So what was the effect of all of this? We get an updated action:

$$S = -\int d^4x \frac{1}{4}F^2 + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{8.35}$$

What's the point of doing this? Now, the kinetic term turns out to be invertible! Fourier transforming:

$$S = -\int \frac{d^4p}{(2\pi)^4} A_\mu(-p)(\eta^{\mu\nu}p^2 - p^\mu p^\nu(1 - \frac{1}{\xi}))A_\nu(p) \tag{8.36}$$

Let's find an inverse of:

$$D^{\mu\nu} = \eta^{\mu\nu}p^2 - p^\mu p^\nu(1 - \frac{1}{\xi}) \tag{8.37}$$

via guessing; we want units of inverse $p^2$ and we want it to be Lorentz covariant with two indices, so our general guess is:

$$(D^{-1})_{\mu\nu} = \frac{1}{p^2}(\eta_{\mu\nu} + \alpha\frac{p_\mu p_\nu}{p^2}) \tag{8.38}$$

Multiplying this out:

$$(D^{-1})^{\mu\nu}D_{\nu\lambda} = (\eta^{\mu\nu} + \alpha\frac{p^\mu p^\nu}{p^2})(\eta_{\nu\lambda} - \frac{p_\nu p_\lambda}{p^2}(1 - \frac{1}{xi})) = \delta^\mu_\lambda\left(\alpha - \left(1 - \frac{1}{\xi}\right)\right)\frac{p^\mu p_\lambda}{p^2} - \alpha(1 - \frac{1}{\xi})\frac{p^\mu p_\lambda}{p^2} \tag{8.39}$$

and we want the last terms to cancel which yields $\alpha = \xi - 1$. This gives a propagator we can invert, we can get the two-point function of the photon from this, and so on. We shall continue the discussion next week.