PHYS 142 Discussion Week 3 - Conductors and Method of Images

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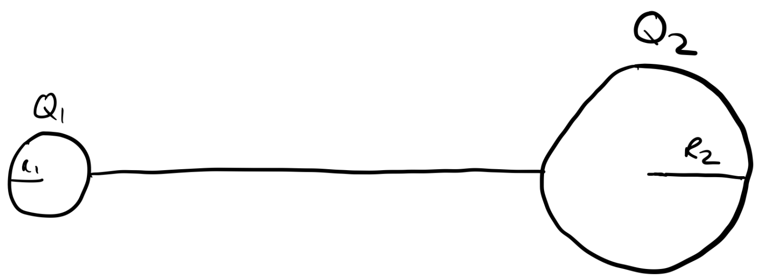
1 Warm-Up Problem; Connected Conductors

We have two charged metal balls, of radii R_1, R_2, connected by a thin metal rod. The balls are separated by an appreciable distance, and have total charge Q. What is the ratio of charges $\frac{Q_1}{Q_2}$ on each ball?



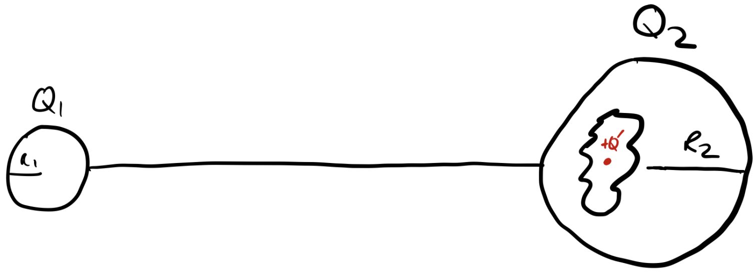
First, since $\mathbf{E} = \mathbf{0}$ in a conductor, we have that $V_1 = V_2$ as there is no potential difference between the two balls (follows from the definition of V based on the line integral of $\mathbf{E}$). Next, because the two charged metal balls are conducting, we note that $\rho = 0$ inside of them, so all of the charge resides on the surface of each ball. This implies that the potential at the surface of each ball is given by that for a spherical shell of charge, which equivalent (by Gauss' Law and the spherical symmetry) to that of a point charge at a given radius, i.e.:

$$V_1 = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R_1}, \quad V_2 = \frac{1}{4\pi\epsilon_0}\frac{Q_2}{R_2} \tag{1.1}$$

Note that we can neglect the effect of the one sphere on the potential of the other as the two are separated by a large distance. Equating the two potentials and solving for the ratio of charge, we find:

$$\boxed{\frac{Q_1}{Q_2} = \frac{R_1}{R_2}} \tag{1.2}$$

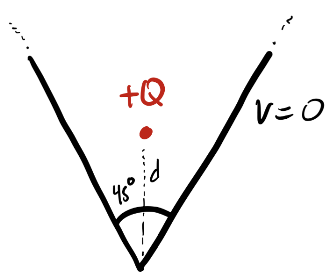
Follow-up question; what happens if we bore a hole inside one of the conductors and put in a $+Q'$ amount of charge? How does the ratio (if any) change? How does the total amount of charge on the outer surface of the two balls change?



The total amount of charge on the outer surface of the two balls goes up by Q', so a total outer surface charge of $Q + Q'$. $\mathbf{E} = 0$ in a conductor, so if we draw a Gaussian surface around the cavity inside the conductor we find that there must be zero enclosed charge - $-Q$ thus collects on the surface of the internal cavity to cancel out the field from the $+Q$ charge inside the cavity. The ratio of charges doesn't change. The easy way to see this is that the $V =$ const across the two balls doesn't change with the presence of the cavity + internal charge, and the Gauss' argument still goes through as before since all of the remaining charge collects on the surface of the balls as before, and someone outside of the conductor cannot "see" the hole + internal surface charge due to the screening.

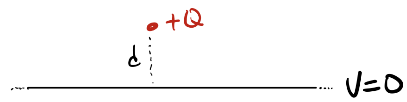
2 Hard Problem; Conducting Planes at an Angle

Consider two grounded ($\phi = 0$) conducting planes at 45 degrees, meeting at the origin. We now consider a $+q$ charge placed a distance d from the origin. What is the force on the proton? What is the energy in this configuration? Seems like a nearly impossible problem, as its very hard to think about the induced surface charge and the force that this applies to the proton/the energy stored in the electric field. Let's instead consider a sequence of simpler problems and build up our intuition.



3 The Classic Method of Images Problem

Let's first think about finding out the force on a charge $+q$ (and energy configuration) of a distance d above a single infinite, grounding conducting plane of charge (take this plane to lie in the xy-plane with $z = 0$)



Still seems like a hard problem. Let's consider solving it in the potential formulation, where we will see the power of this way of thinking about electrostatics. Essentially, need to solve:

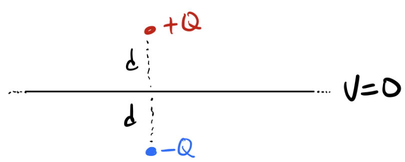
$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \tag{3.1}$$

Armed with the *uniqueness theorem*; Assuming there is a solution $\phi(\mathbf{r})$ for the electrostatic potential to Poisson's equation (Eq. (3.1)), with a given set of conductors with potentials ϕ_k specified on the boundaries of the system, the solution must be unique.

The proof is in Purcell, and you will likely discuss this in class. The argument is also given briefly in the appendix for your leisure. But armed with this, let's solve the problem - it will admit a beautiful solution via the method of images.

The uniqueness theorem tells us that to find the potential in the upper half plane (the region of interest), all we need to do is find an equivalent charge configuration which has the same boundary conditions as the provided problem, and find the potential of that charge configuration - then by uniqueness, we know that this gives the same potential (and thus electric field, force etc.) as the original problem!

In particular, finding a charge configuration with $V = 0$ for $z = 0$ is quite easy. Place an equal and opposite charge the same distance away below the plane:



The force on the $+Q$ charge is just that from the image charge configuration:

$$\boxed{\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{Q^2}{(2d)^2}\hat{\mathbf{z}}} \tag{3.2}$$

The total energy of the (image) charge configuration is then simply the work required to assemble the configuration:

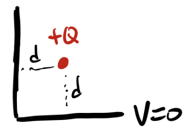
$$U_{\text{tot}} = W = -\frac{1}{4\pi\epsilon_0}\frac{Q^2}{2d} \tag{3.3}$$

The total energy associated to the upper half plane/physical charge configuration is thus half of this (consider that both above/below the xy-plane have equal contributions to the energy), so:

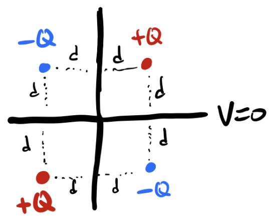
$$\boxed{U = -\frac{1}{8\pi\epsilon_0}\frac{Q^2}{2d}} \tag{3.4}$$

4 One step up from the classic problem - two plates

Now consider the force applied to the charge and the energy stored in the following configuration, with two infinite conducting planes extending from the origin, boxing in a point charge $+Q$ a distance d away from both:



First, note that we are free to extend the conducting planes out into the negative x and y axes; this changes nothing about the boundary conditions for the region of interest (the first quadrant) which is the only condition required to specify the solution. Now, we can simply place image charges across the "mirrors" of each of the conducting planes:



which we can verify indeed satisfies $V = 0$ on each of the conducting planes via pairing up $+Q$ charges with $-Q$ charges across each plane. The force on the physical $+Q$ charge can be obtained by superposition of the forces applied from each of the other charges:

$$\boxed{\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{Q^2}{(2d)^2}\hat{\mathbf{x}} - \frac{1}{4\pi\epsilon_0}\frac{Q^2}{(2d)^2}\hat{\mathbf{y}} + \frac{1}{4\pi\epsilon_0}\frac{Q^2}{(2\sqrt{2}d)^2}(\sqrt{2}\hat{\mathbf{x}} + \sqrt{2}\hat{\mathbf{y}})} \tag{4.1}$$

The total potential energy in the configuration is:

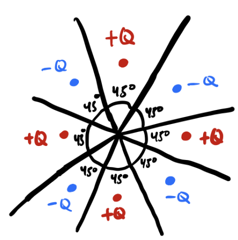
$$U_{\text{tot}} = W = 4 \cdot -\frac{1}{4\pi\epsilon_0}\frac{Q^2}{2d} + 2 \cdot \frac{1}{4\pi\epsilon_0}\frac{Q^2}{2\sqrt{2}d} \tag{4.2}$$

And the potential energy of the physical charge configuration will be a quarter of this total:

$$\boxed{U = \frac{1}{4}\left(-\frac{1}{\pi\epsilon_0}\frac{Q^2}{2d} + \frac{1}{2\pi\epsilon_0}\frac{Q^2}{2\sqrt{2}d}\right)} \tag{4.3}$$

5 Back to the problem

Now let's turn to our original, impossible seeming problem. Using what we have learned, we see that the image charge configuration that produces the desired $V = 0$ boundary conditions is as follows:



From which the force and energy can be (tediously) obtained by adding up the contributions from each of the image charges. I'll leave this as an exercise. Another thing you can check; this image charge argument works for any angle θ for which $\frac{360}{\theta} = 2N$ for $N \in \mathbb{Z}$. Can you explain why it would not work if $\frac{360}{\theta}$ was an odd integer?

6 Appendix: Uniqueness Theorem

Again, the statement of the theorem:

Assuming there is a solution $\phi(\mathbf{r})$ for the electrostatic potential to Poisson's equation (Eq. (3.1)), with a given set of conductors with potentials ϕ_k specified on the boundaries of the system, the solution must be unique.

We solve the simpler case of $\rho = 0$, so for Laplace's equation:

$$\nabla^2 \phi = 0 \tag{6.1}$$

The ingredient we require is the "averaging" property of solutions to Laplace's equation. Namely, you saw in class how the average over a sphere of a function that satisfies Laplace's equation is equal to the value of the function at its center[1]. This implies that no solutions to Laplace's equation can have extrema (except on the boundaries).

With the above fact about solutions, suppose we have two solutions $\phi_1(\mathbf{r}), \phi_2(\mathbf{r})$ for a given set of boundary conditions. We now show that $\phi_1(\mathbf{r}) = \phi_2(\mathbf{r})$. Define the difference:

$$D(\mathbf{r}) = \phi_1(\mathbf{r}) - \phi_2(\mathbf{r}) \tag{6.2}$$

which also solves Laplace's equation;

$$\nabla^2 D(\mathbf{r}) = \nabla^2 \phi_1(\mathbf{r}) - \nabla^2 \phi_2(\mathbf{r}) = 0 - 0 = 0. \tag{6.3}$$

Since ϕ_1, ϕ_2 have the same boundary condition ϕ_k on the boundaries of the system, it follows that D must be zero on such boundaries. But since D as a solution to Laplace's equation cannot have extrema (save for the boundaries), it being zero on the boundaries implies it is zero everywhere (else there would be an extrema somewhere). Thus $D = 0$ and hence $\phi_1 = \phi_2$, proving the uniqueness of solution.

[1] For further intuition, consider the 1D case of $\dfrac{\partial^2 \phi}{\partial x^2} = 0$ which has solutions $Ax + B$/straight line, which is clearly the average of any two points symmetrically away from it