exponential smoothing ① $\hat{X}_1 = X_1$ ② $\hat{X}_t = \alpha X_t + (1-\alpha)\hat{X}_{t-1}$ ③ $\hat{X}_t = (1-\alpha)^{t-1}X_1 + \sum_{j=1}^{t-2}\alpha(1-\alpha)^j X_{t-j}$

Holt-Winters 1.trend: $\hat{X}_{T+k} = \hat{a}_T + \sum_{i=1}^{k}\phi^i \hat{b}_T$ {$\phi>1$ exp, $\phi=1$ linear, $\phi<1$ damped}

2.seasonality ① $\hat{a}_{d+1} = X_{d+1}$, $\hat{b}_{t+1} = (X_{d+1}-X_1)/d$, $\hat{S}_i = X_i - \{X_1 - \hat{b}_{d+1}(i-1)\}$ for $i=1,\dots,d$;

加型 ② $\hat{a}_{t+1} = \alpha(X_{t+1} - \hat{S}_{t+1-d}) + (1-\alpha)(\hat{a}_t + \hat{b}_t)$, $\hat{b}_{t+1} = \beta(\hat{a}_{t+1} - \hat{a}_t) + (1-\beta)\hat{b}_t$, $\hat{S}_{t+1} = \gamma(X_{t+1} - \hat{a}_{t+1}) + (1-\gamma)\hat{S}_{t+1-d}$, $\hat{X}_{T+k} = \hat{a}_T + \hat{b}_T k + \hat{S}_{T+k}$

乘型 ② $\hat{a}_{t+1} = \alpha\frac{X_{t+1}}{\hat{S}_{t+1-d}} + (1-\alpha)(\hat{a}_t+\hat{b}_t)$, $\hat{b}_{t+1}$ 不变, $\hat{S}_{t+1} = \gamma\frac{X_{t+1}}{\hat{a}_{t+1}} + (1-\gamma)\hat{S}_{t+1-d}$, $\hat{X}_{T+k} = (\hat{a}_T + \hat{b}_T k)\hat{S}_{T+k}$

AR(1) $X_t = \phi X_{t-1} + Z_t$ 1. MA(∞) causal → $X_t = \sum_{j=0}^{\infty}\phi^j Z_{t-j}$ 2. $\gamma_X(h) = \frac{\phi^h\sigma^2}{1-\phi^2}$ 3. $\alpha_X(0)=1$, $\alpha_X(1)=\phi$

AR(2) 1. ACF: $\rho_X(1) = \frac{\phi_1}{1-\phi_2}$, $\rho_X(2) = \frac{\phi_1^2+\phi_2-\phi_2^2}{1-\phi_2}$, $\rho_X(h) = \phi_1\rho_X(h-1) + \phi_2\rho_X(h-2)$ 2. PACF: $\alpha_X(1) = \rho_X(1)$, $\alpha_X(2) = \phi_2$, $\alpha_X(h>2) = 0$ 3. $(1-\phi_1 B - \phi_2 B^2)(1+\psi_1 B + \psi_2 B^2 + \cdots) = 1 \to \psi_1 = \phi_1$, $\psi_2 = \psi_1\phi_1 + \phi_2$, $\psi_j = \psi_{j-1}\phi_1 + \psi_{j-2}\phi_2$

AR(p) 1. $X_t = \sum_{j=1}^{p}\phi_j X_{t-j} + Z_t \to \Phi(B)X_t = Z_t$ 2. ACF $\rho_X(h) = \sum_{j=1}^{p}\phi_j\rho_X(h-j)$

MA(1) $X_t = Z_t + \theta Z_{t-1}$ 1. $\gamma_X(0) = (1+\theta^2)\sigma^2$, $\gamma_X(1) = \theta\sigma^2$, $\gamma_X(h>1) = 0$ 2. $\alpha_X(h) = -\frac{(-\theta)^h}{1+\theta^2+\cdots+\theta^{2h}}$

MA(2) 1. ACF: $\rho_X(0)=1$, $\rho_X(1) = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}$, $\rho_X(2) = \frac{\theta_2}{1+\theta_1^2+\theta_2^2}$ 2. PACF $\alpha_X(1) = \rho_X(1)$, $\alpha_X(2) = \frac{\rho_X(2) - \rho_X^2(1)}{1-\rho_X^2(1)}$

MA(q) 1. $X_t = Z_t + \sum_{j=1}^{q}\theta_j Z_{t-j}$ 2. $\gamma_X(h\le q) = \sigma^2\sum_{j=0}^{q-h}\theta_j\theta_{j+h}$, $\gamma_X(h>q) = 0$

ARMA(1,1) stationary solution: $|\phi|<1 \to X_t = Z_t + (\phi+\theta)\sum_{j=1}^{\infty}\phi^{j-1}Z_{t-j}$, causal ◁

$|\phi|>1 \to X_t = -\theta\phi^{-1}Z_t - (\phi+\theta)\sum_{j=1}^{\infty}\phi^{-j-1}Z_{t+j}$, non-causal ◁

$\to Z_t = X_t - (\phi+\theta)\sum_{j=1}^{\infty}(-\theta)^{j-1}X_{t-j}$, invertible

$\gamma(h) = \sigma^2\sum_{j=0}^{\infty}\psi_j\psi_{j+h}$ where $\psi_j = (\phi+\theta)\phi^{j-1}$

ARMA(p,q) 1. $(1-\phi_1 B - \cdots - \phi_p B^p)X_t = (1+\theta_1 B + \cdots + \theta_q B^q)Z_t$ ① stationary → $|\Phi \text{ roots}| \neq 1$ ② causal → $|\Phi \text{ roots}| > 1$ ③ invertible → $|\Theta \text{ roots}| > 1$

BLP 一. one-step 1. $\phi_{t+1|t} = (\phi_{t1},\dots,\phi_{tt})$, matrix $\Gamma_t \to (\Gamma_t)_{ij} = \gamma(i-j)$, $\gamma_t = (\gamma(1),\dots,\gamma(t))$ 2. $\phi_{t+1|t} = \Gamma_t^{-1}\gamma_{1:t}$ 3. residual variance $V_{t+1|t} = \gamma(0) - \gamma_t^T\Gamma_t^{-1}\gamma_t$

二. h-step 1. $\gamma_{h:t+h-1} = (\gamma(h),\gamma(h+1),\dots,\gamma(t+h-1))^T$ 2. $\phi_{t+h|t} = \Gamma_t^{-1}\gamma_{h:t+h-1}$ 3. $V_{t+h|t} = \gamma(0) - \gamma_{h:t+h-1}^T\Gamma_t^{-1}\gamma_{h:t+h-1}$

Innovations algo for MA(q) 1. $\hat{X}_{t+1} = \sum_{j=1}^{q}\theta_j Z_{t+1-j}$ 2. approx Z by U: $U_t = X_t - \hat{X}_{t|t-1}$, $\mathrm{Cov}(U_t,U_s) = 0$ 3. BLP $\hat{X}_{t+1|t} = \theta_{t1}U_t + \cdots + \theta_{tt}U_1$

4. $\theta_{t,t-j} = \frac{\gamma(t-j) - \sum_{k=0}^{j-1}\theta_{j,j-k}\theta_{t,t-k}V_{k+1|k}}{V_{j+1|j}}$, for $j=0,\dots,t-1$

$V_{t+1|t} = \gamma(0) - \sum_{j=0}^{t-1}\theta_{t,t-j}^2 V_{j+1|j}$, $\hat{X}_{t+1|t} = \sum_{j=1}^{t}\theta_{tj}U_{t+1-j}$, $\begin{cases}\hat{X}_{1|0} = 0\\ V_{1|0} = \gamma(0)\end{cases}$

For MA(1) $\theta_{t1} = \frac{\theta\sigma^2}{V_{t|t-1}}$, $V_{t+1|t} = (1+\theta^2-\theta\theta_{t1})\sigma^2$, $\hat{X}_{t+1|t} = \frac{\theta(X_t - \hat{X}_{t|t-1})\sigma^2}{V_{t|t-1}}$

BLP for ARMA(p,q) $\hat{X}_{t+h|t} = \begin{cases}\sum_{j=1}^{t}\theta_{t+h-1,j}(X_{t+h-j} - \hat{X}_{t+h-j|t+h-j-1}) & 1\le h\le \max(p,q)\\ \sum_{k=1}^{p}\phi_k\hat{X}_{t+h-k|t} + \sum_{j=h}^{q}\theta_{t+h-1,j}(X_{t+h-j} - \hat{X}_{t+h-j|t+h-j-1})\end{cases}$

Method of moments 1. WN sample ACF $\hat{\rho}_{1:h} \xrightarrow{d} N(0,\frac{1}{n})$

AR(p) Yule-Walker $\hat{\phi}_{YW} = \hat{\gamma}_p\hat{\Gamma}_p^{-1}$, $\hat{\sigma}^2_{YW} = \hat{\gamma}(0) - \hat{\gamma}_p^T\hat{\Gamma}_p^{-1}\hat{\gamma}_p$

for causal AR(p): $(\hat{\phi}_{n+1|h} - \phi_{n+1|h}) \xrightarrow{d} N(0, \frac{\sigma^2\Gamma_p^{-1}}{n})$ ☆, $\hat{\sigma}^2 \xrightarrow{P} \sigma^2$, $\hat{\alpha}(h) \xrightarrow{d} N(0,\frac{1}{n})$

asymptotic dist fit AR(1)→AR(1) $\hat{\phi} \sim N(\phi, \frac{1-\phi^2}{n})$, AR(2)→AR(1) $\phi_1 = \phi$, $\phi_2 = 0$

$\begin{pmatrix}\hat{\phi}_1\\ \hat{\phi}_2\end{pmatrix} \sim N\left[\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \frac{1}{n}\begin{pmatrix}1-\phi_2^2 & -\phi_1(1+\phi_2)\\ -\phi_1(1+\phi_2) & 1-\phi_2^2\end{pmatrix}\right]$

MoM for MA(1) $\hat{\theta} = \frac{1-\sqrt{1-4\hat{\rho}(1)^2}}{2\hat{\rho}(1)}$ $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, \frac{1+\theta^2+4\theta^4+\theta^6+\theta^8}{(1-\theta^2)^2})$

MLE-conditional for AR(1) ① $S_c(\phi) = \sum_{k=2}^{n}(X_k - \phi X_{k-1})^2$ ② $\hat{\phi}_{MLE-CSS} = \arg\min S_c(\phi)$ ③ $\hat{\sigma}^2_{MLE-CSS} = S_c(\phi)/(n-1)$ unconditional MLE $S(\phi) =$ $L(\phi,\sigma^2; X_{1:n}) = (2\pi\sigma^2)^a \cdot b \cdot \exp\{-\frac{1}{2\sigma^2}S(\phi)\}$, $\hat{\sigma}^2_{MLE} = \frac{S(\hat{\phi})}{n}$

Ljung-Box 1. 检测space是否过小 2. one-step residual $U_k = X_k - \hat{X}_{k|k-1}$ $\hat{\rho}_{\hat{U},1:h} \xrightarrow{d} N(0,\frac{1}{n})$ 3. $Q = n(n+2)\sum_{k=1}^{h}\frac{\hat{\rho}_{\hat{U}}(k)^2}{n-k}$ 4. $Q \sim \chi^2_{h-p-q}$ ARMA(p,q)

AIC, AICc 1. 检测space是否过大 2. $AIC = -2\log lik(\hat{\beta}_{MLE}, \hat{\sigma}^2_{MLE}; X_{1:n}) + 2(p+q+1)$ 3. $AICc = -2\log lik(\hat{\beta}_{MLE}, \hat{\sigma}^2_{MLE}; X_{1:n}) + \frac{2(p+q+1)n}{n-p-q-2}$

Fisher info: $I(\theta) = E[(\frac{\partial}{\partial\theta}\log f(X;\theta))^2|\theta] = \int_R (\frac{\partial}{\partial\theta}\log f(x;\theta))^2 f(x;\theta)dx = -E[\frac{\partial^2}{\partial^2\theta}\log f(X;\theta)|\theta]$

$\hat{\beta}_{MLE}$ asymptotic $(\hat{\beta}_{MLE} - \beta_0) \xrightarrow{d} N(0, \frac{1}{nI(\beta_0)})$

ARIMA(p,d,q) $\nabla = I - B$ 1. if X_t is stationary, so is $\nabla^k X_t$ 2. $\Phi(B)(1-B)^d X_t = c + \Theta(B)Z_t$, $c = (1-\phi_1-\cdots-\phi_p)E[\nabla^d X_t]$ 3. $\nabla^h X_t = \sum_{k=0}^{h}(-1)^k\binom{h}{k}B^k X_t$

Stationary test AR(1) Dicky-fuller

1. 假设 $X_t = \phi X_{t-1} + Z_t$ 2. $\nabla X_t = (\phi-1)X_{t-1} + Z_t$ 3. $H_0: \phi=1$ vs $H_1: |\phi|<1$

4. $n(\hat{\phi}-1) = \frac{n\sum_{t=1}^{n}(X_t - X_{t-1})X_{t-1}}{\sum_{t=1}^{n}X_{t-1}^2} \xrightarrow{d} \frac{\frac{1}{2}(W(1)^2-1)}{\int_0^1 W(t)^2dt}$, $W(t)$: 标准布朗运动

Augmented Dicky-fuller AR(p)

1. 假设 $X_t = \sum_{j=1}^{p}\phi_j X_{t-j} + Z_t$ 2. $\nabla X_t = \gamma X_{t-1} + \sum_{j=1}^{p}\psi_j\nabla X_{t-j} + Z_t$ where $\gamma = \sum_{j=1}^{p}\phi_j - 1$ and $\psi_j = -\sum_{i=j+1}^{p}\phi_i$ for $j=1,\dots,p-1$

3. $\gamma = -\Phi(1)$ hence test $H_0: \gamma=0 \to$ non-stationary 4. $\frac{\hat{\gamma}}{se(\hat{\gamma})} \sim$ Wald

Stationary test KPSS

1. $X_t = R_t + Y_t$, Y_t stationary, $R_t = R_{t-1} + Z_t$, $Var(Z_t) = \sigma^2$

2. $H_0: \sigma^2 = 0$ vs $H_1: \sigma^2>0$, H_0等效于 X_t stationary

3. $S_t = \sum_{j=1}^{t}(X_j - \bar{X})$, $\hat{V}$ estimate $V = Var\{S_n/\sqrt{n}\}$

4. $H_0 \to \frac{\sum_{t=1}^{n}S_t^2}{n^2\hat{V}} \xrightarrow{d} \int_0^1 B(t)^2dt$, $B(t) := W(t) - tW(1)$

SARIMA(p,d,q)(P,D,Q) $\Phi^S(B^S)\Phi(B)(I-B^S)^D(I-B)^d X_t = c + \Theta^S(B^S)\Theta(B)Z_t$

Box-Cox transform $f_\lambda(x) = \begin{cases}\log(x), & \lambda=0\\ \frac{\mathrm{sign}(x)|x|^\lambda - 1}{\lambda}, & \lambda\neq 0\end{cases}$

Forecast ARIMA(p,d,q) X_t ① let $Y_t = (1-B)^d X_t$, stationary ② $X_t = Y_t - \sum_{j=1}^{d}(-1)^j\binom{d}{j}X_{t-j}$ ③ $\hat{X}_{t+h|t} = \hat{Y}_{t+h|t} - \sum_{j=1}^{d}(-1)^j\binom{d}{j}\hat{X}_{t+h-j|t}$ ↳ by BLP

Modern time series analysis

一. 分train-test集 ① 时序版CV $\begin{cases}\text{train: 前}n\text{个}, n=6,7,\cdots m\text{个}\\ \text{test: 第}n+h\text{个}\end{cases}$

二. point-forecast error ① $e_{n+h} = X_{n+h} - \hat{X}_{n+h|n}$ (N: train+test, n: train) ② $MSE = \frac{1}{N-n}\sum_{h=1}^{N-n}e_{n+h}^2$ ③ $MAE = \frac{1}{N-n}\sum_{h=1}^{N-n}|e_{n+h}|$

比较不同时序模型,需normalize forecast e: $\frac{1}{n-1}\sum_{t=2}^{n}|X_t - X_{t-1}|$

三. distributional-forecast:

1. Monte-carlo 应用在 innovations $U_1,\dots,U_n$ 上 → U_k distribution

2. $X_n^* = X_n$, $X_{n+h}^* = X_{n+h-1}^* + U_{n+h}^*$ where U_{n+h}^* drawn uniformly from $\{U_1\cdots U_n\}$ → get prediction interval by quantile of U_k dist

3. quantile loss / pinball loss

$$Q_{p,t} = \begin{cases}2(1-p)(f_{p,t} - X_t), & X_t < f_{p,t}\\ 2p(X_t - f_{p,t}), & X_t > f_{p,t}\end{cases}$$

$f_{p,t}$ is p-th quantile of distributional forecast at time t.

四. Multivariate → forecast based on other time series

暴力美学法: regress on 其它时序和它们的lag. i.e. given ts $(W_{1t}),\dots,(W_{kt}) \to X_t = W_{1t} + \cdots W_{kt} + W_{1,t-1} + \cdots + Y_t$ ARIMA

Geometric sum $b + ba + ba^2 + ba^3 + \cdots = \frac{b}{1-a}$

Q1: $\nabla^3 X_t$ look like $WN \sim \mu$, given $X_{2017}=a$, $X_{2016}=b$, $X_{2015}=c$, 求 X_{2018}?

$\nabla^3 X_t = (I-B)^3 X_t = \sum_{k=0}^{3}(-1)^k \binom{3}{k} B^k X_t = X_t - 3X_{t-1} + 3X_{t-2} - X_{t-3} + Z_t$

$\rightarrow X_t = 3X_{t-1} - 3X_{t-2} + X_{t-3} + \nabla^3 X_t \rightarrow X_{2018} = 3a - 3b + c + \mu$

Q2: WN: $X_1,\dots,X_n$; n大; 前100个 $\hat{\rho}_1 \cdots \hat{\rho}_{100}$ 中至少3个大于5% CI概率

即 $|\rho_k| \geq 1.96/\sqrt{n}$?

$\hat{\rho}_{1:n} \xrightarrow{d} N(0, \frac{1}{n})$; 5% CI → p=0.05 超越 → Binomial (p=0.05, n=100)

$P(\sum_{k=1}^{100} 1B_k \geq 3) = 1 - P(\Sigma=0) - P(\Sigma=1) - P(\Sigma=2) = 1 - 0.95^{100} - \binom{100}{1} 0.95^{99} 0.05$

$- \binom{100}{2} 0.95^{98} 0.05^2 = 88.17$

Q3. Let X_t be weakly stationary, $Y_t = \sum_{j=a}^{b} c_j X_{t+j}$, show Y_t is weakly stationary?

$E(Y_t) = E(\sum_{j=a}^{b} c_j X_{t+j}) = \sum_{j=a}^{b} c_j E(X_{t+j}) = \mu \sum_{j=a}^{b} c_j$

$cov(Y_{t+k}, Y_t) = cov(\sum_{j=a}^{b} c_j X_{t+k+j}, \sum_{m=a}^{b} c_m X_{t+m})$

$= \sum_{j=a}^{b}\sum_{m=a}^{b} c_j c_m \, cov(X_{t+k+j}, X_{t+m}) = \sum_{j=a}^{b}\sum_{m=a}^{b} c_j c_m \gamma_X(k+j-m)$

Q4 $(1-\frac{1}{2}B)X_t = (1+B^2)Z_t$

(a) show X_t causal, write its MA(∞)

Φ has root 2 → causal

$\psi = \frac{1}{1-\frac{1}{2}B}$ by geometric series $= \sum_{j\geq 0} (\frac{B}{2})^j$ →

$X_t = \psi(Z_t + Z_{t-2}) = \sum_{j\geq 0}(\frac{B}{2})^j (Z_t + Z_{t-2}) = \sum_{j\geq 0}(\frac{B}{2})^j Z_t + \sum_{j\geq 0}(\frac{B}{2})^j Z_{t-2}$

$= \sum_{j\geq 0} \frac{Z_{t-j}}{2^j} + \sum_{j\geq 0} \frac{Z_{t-2-j}}{2^j} = Z_t + \frac{Z_{t-1}}{2} + \sum_{j\geq 2} (\frac{Z_{t-j}}{2^j} + \frac{Z_{t-j}}{2^{j-2}})$

$= Z_t + \frac{Z_{t-1}}{2} + 5\sum_{j\geq 2} \frac{Z_{t-j}}{2^j}$

(b) $\gamma(h) = cov(X_t, X_{t+h})$? $\gamma(0) = Var(X_t) = \sigma^2(1 + \frac{1}{4} + 25\sum_{j\geq 2}\frac{1}{4^j})$

$= \sigma^2(\frac{5}{4} + \frac{25}{12})$ $\quad \gamma(h) = \sigma^2 (\frac{5}{2^h} + \frac{5}{2^{h+1}} + \frac{5}{2^h}\sum_{j\geq 2}\frac{1}{4^j})$

Q5. MA(1) $X_t = Z_t + \theta Z_{t-1}$

(a) 求 min-MSE one-step $\tilde{X}_{n+1} = E(X_{n+1} | X_n, X_{n-1}, \cdots)$

MA(1) as AR(∞) → $\sum_{j=0}^{\infty}(-\theta)^j X_{t-j} = Z_t$ → $X_t = -\sum_{j=1}^{\infty}(-\theta)^j X_{t-j} + Z_t$

$\tilde{X}_{n+1} = E(-\sum_{j=1}^{\infty}(-\theta)^j X_{n+1-j} + Z_{n+1} | X_n, X_{n-1}, \cdots)$

$= -\sum_{j=1}^{\infty}(-\theta)^j X_{n+1-j}$

(b). truncated $\tilde{X}_{n+1}^n$ equals $\tilde{X}_{n+1}$ but $0 = X_0 = X_{-1} = \cdots$

show $E[(X_{n+1} - \tilde{X}_{n+1}^n)^2] = \sigma^2(1 + \theta^{2+2n})$

$X_{n+1} = -\sum_{j=1}^{\infty}(-\theta)^j X_{n+1-j} + Z_{n+1}$; $\tilde{X}_{n+1}^n = -\sum_{j=1}^{n}(-\theta)^j X_{n+1-j}$

$X_{n+1} - \tilde{X}_{n+1}^n = Z_{n+1} - \sum_{j=n+1}^{\infty}(-\theta)^j X_{n+1-j}$

let $k = j-n-1$ → $= -\sum_{k=0}^{\infty}(-\theta)^{k+n+1} X_{-k} = (-\theta)^{n+1} \underbrace{\sum_{k=0}^{\infty}(-\theta)^k X_k}_{=Z_0}$

$E[(\,)^2] = Var(\,) = \sigma^2 + \theta^{2+2n}\sigma^2 = \sigma^2(1 + \theta^{2+2n})$

Q6. AR(1) $X_t - \phi X_{t-1} = Z_t$, 求 BLP $\tilde{X}_{n+m}$

$\hat{X}_{n+1} = \phi X_n + \cancel{Z_t} \to 0$

$\hat{X}_{n+2} = \phi \hat{X}_{n+1} + \cancel{Z_t} \to 0$ $\quad \Rightarrow \hat{X}_{n+m} = \phi^m X_n$

$\hat{X}_{n+3} = \phi \hat{X}_{n+2} + \cancel{Z_t} \to 0$

show $E(X_{n+m} - \hat{X}_{n+m})^2 = \sigma^2 \frac{1-\phi^{2m}}{1-\phi^2}$

① $= \gamma(0) - \gamma_{m:m+n-1} \Gamma_n^{-1} \gamma_{m:m+n-1}$ $\quad = \begin{pmatrix} \phi^m \\ \vdots \\ 0 \end{pmatrix}$

$= \gamma(0) - \gamma(m)\phi^m$

$= \frac{\sigma^2}{1-\phi^2} - \frac{\sigma^2\phi^m}{1-\phi^2}\phi^m$

② $X_{n+m} = \phi^m X_n = \phi^m \sum_{k=0}^{\infty}\phi^k Z_{n-k}$, let $k = -m + j$

$X_{n+m} = \sum_{j=0}^{\infty}\phi^j Z_{n+m-j}$ ⇒ 差 $= \sum_{j=m}^{\infty}\phi^j Z_{n+m-j}$

$E(\,)^2 = \sigma^2 \sum_{j=0}^{m-1}\phi^{2j}$

Q7 $(1-0.2B)X_t = (1-0.1B)Z_t$, 求 ACVF $\gamma(h)$? ARMA(1,1)

$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty}\psi_j\psi_{j+h}$ where $\psi_j = (\phi+\theta)\phi^{j-1}$

$= \sigma^2 \sum_{j=0}^{\infty}(\phi+\theta)\phi^{j-1}(\phi+\theta)\phi^{j+h-1}$

$= \sigma^2(\phi+\theta)^2 \sum_{j=0}^{\infty}\phi^{2j+h-2}$

$\hat{X}_3 = \phi_{21}(X_2-\mu) + \phi_{22}(X_1-\mu) + \mu$

where (ϕ_{21}, ϕ_{22}) is solution to $(\phi_{21}, \phi_{22})^T = \Gamma_2^{-1}[\gamma(1), \gamma(2)]^T$

Q8 X_t is causal AR(1) with seasonality 2 $[X_t = \phi X_{t-2} + Z_t]$

1. MA(∞)? $X_t = \sum_{j=0}^{\infty}\phi^j Z_{t-2j}$

2. ACVF? $\gamma(h) = \frac{\phi^{\frac{h}{2}}\sigma^2}{1-\phi^2}$, h even $\quad = 0$, h odd

$\gamma(h) = E\{\sum_{j=0}^{\infty}\phi^j Z_{t-2j} \sum_{j=0}^{\infty}\phi^j Z_{t-h-2j}\}$

$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\phi^{j+k} E\{Z_{t-2j} Z_{t-h-2k}\}$

$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\phi^{j+k}\sigma^2 1\{2j = h+2k\}$ ← $k = \frac{2j+h}{2}$

$= \sum_{j=0}^{\infty}\phi^{2j+\frac{h}{2}}\sigma^2 = \frac{\phi^{\frac{h}{2}}}{1-\phi^2}\sigma^2$

3. Likelihood function of (X_1, X_2, X_3) $\quad$ AR(1) → $\gamma(0) = \frac{\sigma^2}{1-\phi^2}$; $X_3 = \phi X_1 + Z_3$

$p_1 = p(X_1; \phi, \sigma^2) = \frac{1}{\sqrt{2\pi\gamma(0)}} e^{-\frac{X_1^2}{2\gamma(0)}}$

$p_2 = p(X_2 | X_1; \phi, \sigma^2) = \frac{1}{\sqrt{2\pi\gamma(0)}} e^{-\frac{X_2^2}{2\gamma(0)}}$

$p_3 = p(X_3 | X_2, X_1; \phi, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_3-\phi X_1)^2}{2\sigma^2}}$

X_3 = 常数 + Z_t, X_1已知

$L = p_1 p_2 p_3 = (2\pi\sigma^2)^{-\frac{3}{2}}(1-\phi^2) e^{-\frac{1}{2\sigma^2}[(1-\phi^2)(X_1^2+X_2^2) + (X_3-\phi X_1)^2]}$

unconditional $S(\phi)$

4. AIC for model? expand $\hat{\sigma}^2_{MLE}$ $\quad \hat{\sigma}^2_{MLE} = \frac{S(\phi)}{n} = \frac{S(\phi)}{3}$

$AIC = -2L(\hat{\phi}_{MLE}, \hat{\sigma}^2_{MLE}; X_{1:n}) + \underbrace{2(p+q+1)}_{=4}$

$= -2(-\frac{3}{2})\log(2\pi) + 3\log(\hat{\sigma}^2_{MLE}) - 2\log(1-\hat{\phi}^2_{MLE}) + \frac{S(\hat{\phi}_{MLE})}{\hat{\sigma}^2_{MLE}} + 4$

5. BLP $\hat{X}_{n+1|n}$ for n=1,2,3? given $X_1 \cdots X_n$

$\hat{X}_{2|1} = 0$ by convention; $\hat{X}_{3|2} = \phi X_1$; $\hat{X}_{4|3} = \phi X_2$

6. $Var(X_{n+1} - \hat{X}_{n+1|n})$? n=1, 2, 3

$Var(X_2 - \hat{X}_{2|1}) = Var(X_2) = \gamma(0) = \frac{\sigma^2}{1-\phi^2}$

$Var(X_3 - \hat{X}_{3|2}) = Var(X_3 - X_3 + \phi X_1) = Var(X_4 - \hat{X}_{4|3}) = Var(X_4 - X_4 + \phi X_2)$

known, treat as constant

$= Var(Z_t) = \sigma^2$