Linear processes.

- Definition :- The time series $\{X_t\}$ is a linear process if it has the representation
$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$
for all t, where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\{\psi_j\}$ is a sequence of constants with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$.

- In terms of the lag operator or backward shift operator, we can write $X_t = \psi(B) Z_t$, where $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$.

- A linear process is called MA(∞) if $\psi_j = 0 \ \forall j < 0$, ie if
$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

- Every second-order stationary process is either a linear process or can be transformed to a linear process by subtracting a deterministic component.

Linear processes.

- Note that, the condition $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ ensures that the infinite sum in the definition converges almost everywhere.

- This follows since $E|Z_t| \leq \sigma$ and
$$E|X_t| \leq \sum_{j=-\infty}^{\infty} (|\psi_j| \, E|Z_{t-j}|) \leq \sigma \sum_{j=-\infty}^{\infty} |\psi_j| < \infty \, .$$

- Theorem :- Let $\{Y_t\}$ be a stationary time series with mean 0 and autocovariance function γ_Y. If $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, then the time series,
$$X_j = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} = \psi(B) Y_t \, ,$$
is stationary with mean 0 and autocovariance function
$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h+k-j) \, .$$

Construction of AR(1).

- We have defined an AR(1) process as a stationary solution $\{X_t\}$ of equations

$$X_t - \phi X_{t-1} = Z_t, \quad \cdots \cdots (1)$$

where $\{Z_t\} \sim WN(0, \sigma^2)$, $|\phi| < 1$ and Z_t is uncorrelated with X_s for all $s < t$.

- We need to show that such a ~~process~~ solution exists and is unique.

- Consider the linear process defined by

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j} \quad \cdots \cdots (2)$$

- Clearly $\sum_{j=0}^{\infty} |\phi|^j < \infty$ since $|\phi| < 1$.

- Also it follows that X_t ~~is a soln~~ in (2) is a solution of (1).

Construction of AR(1).

- To show that (2) is the only stationary solution, let $\{Y_t\}$ be any other stationary solution. Then from (1) we get.

$$Y_t = \phi Y_{t-1} + Z_t = Z_t + \phi Z_{t-1} + \phi^2 Y_{t-2}$$
$$= Z_t + \phi Z_{t-1} + \cdots + \phi^k Z_{t-k} + \phi^{k+1} Y_{t-k-1}.$$

If $\{Y_t\}$ is stationary $E[Y_t^2] \leq \gamma_y(0) \quad \forall t$.

Now Consider $E\left[\left(Y_t - \sum_{j=0}^{k} \phi^j Z_{t-j}\right)^2\right] = \phi^{2k+2} E[Y_{t-k-1}^2]$.

$$\leq \phi^{2k+2} \gamma_y(0)$$
$$\to 0 \quad \text{as } k \to \infty.$$

So $Y_t \longrightarrow X_t$ in mean square as $k \to \infty$. So X_t defined in (2) is the unique solution.

The case $|\phi| > 1$.

- If $|\phi| > 1$, notice that $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$ does not converge.
- However the equation (1) can be re-written as.

$$X_t = -\frac{1}{\phi} Z_{t+1} + \frac{1}{\phi} X_{t+1} \quad \cdots\cdots \quad (3)$$

$$= -1/\phi \, Z_{t+1} - \frac{1}{\phi^2} Z_{t+2} - \cdots + \frac{1}{\phi^{k+1}} X_{t+k+1}.$$

- This shows that $X_t = -\sum_{j=1}^{\infty} \frac{1}{\phi^j} Z_{t+j}$ is the unique stationary solution of (1).
- The solution in (3) is strange because X_t is correlated with Z_s for $s>t$, which does not happen in (2), where X_t is uncorrelated with Z_s, $\forall s>t$.
- The solution in (2) is called "causal" since it is not dependent on the future. $\{X_t\}$ in (2) is a causal autoregressive process.