HOLT-WINTERS ALGORITHMS

- So far we have assumed that the series has a flat trend and no seasonality.

- For such a series prediction is simple. That is for $k \geq 1$ the forecast value at time $T + k$ is the smoothed value at time T.

- If the series has trend and seasonality our forecasting method should correct for them as well.

- Using Holt-Winters algorithms we can make forecasts which corrects for the trend and seasonality.

- There are two Holt-Winters methods. The first one corrects for trend and the second corrects for trend and seasonality.

EXHIBIT 8.8
Commonly Used Exponential Smoothing Techniques

Model type	Trend profile	Seasonal profile
Simple (single)	None	None
Holt	Linear	None
Holt-Winters	Linear	Additive or multiplicative

- *Current level*: The current level serves as the starting point for the forecast. It is an exponential weighted average of the time series at time T. The last observation cannot be used as a starting point because it may set the forecasts off at the wrong level.

- *Current Trend*: The current trend represents the amount by which we expect the time series to grow or decline per time period in to the future. It is often calculated as an exponentially weighted average of past period to period changes in the level of the series. In this way, recent growth of decline in the time series is given more weight than changes farther back in time.

- *Current Seasonal Index*: The current seasonal index is the amount or degree by which the season's value tends to exceed or fall short of the norm.

- If there is evidence of a linear trend in the series, a natural way to forecast the value of Y_{t+k}, for $k \geq 1$ is to use the formula:

$$\widehat{X}_{T+k} = \widehat{a}_T + \widehat{b}_T k.$$

  Here $\widehat{a}_T$ is the current level at time T and $\widehat{b}_T$ is the current trend at time T.

- Holt(1957) postulated an exponential smoothing based method to find $\widehat{a}_T$ and $\widehat{b}_T$.

- The algorithm can be describe as follows:

  Step 1 Take $\widehat{a}_2 = X_2$ and $\widehat{b}_2 = X_2 - X_1$.

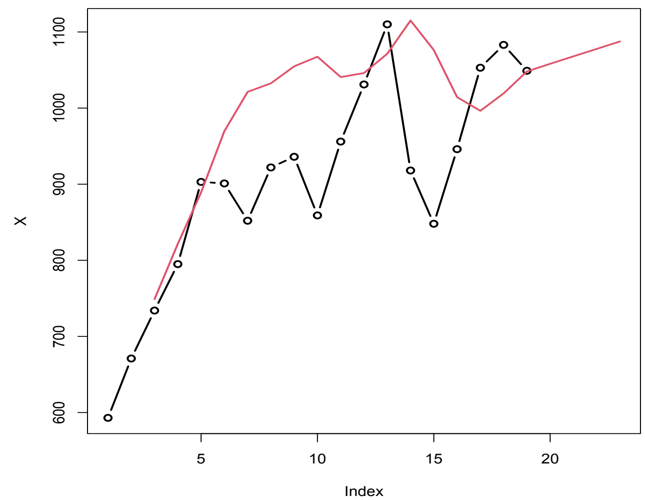
  Step 2 Specify $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. Now for $t = 2, 3, \ldots, T-1$,

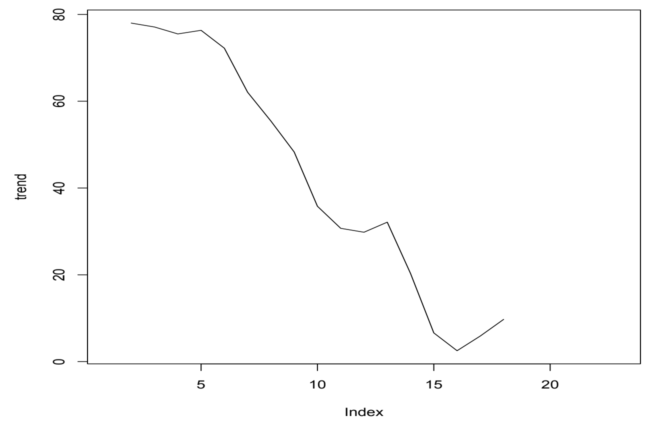
$$\begin{aligned}
\widehat{X}_{t+1} &= \widehat{a}_t + \widehat{b}_t, \\
\widehat{a}_{t+1} &= \alpha X_{t+1} + (1-\alpha)(\widehat{a}_t + \widehat{b}_t), \\
\widehat{b}_{t+1} &= \beta(\widehat{a}_{t+1} - \widehat{a}_t) + (1-\beta)\widehat{b}_t.
\end{aligned}$$

- The parameters α and β may be user specified or may be determined by minimising the total squared forecasting loss.

EXAMPLE

	Data	Fitted values	level	Trend
1	593.00			
2	671.00		671.00	78.00
3	734.00	749.00	744.50	77.10
4	795.00	821.60	813.62	75.50
5	903.00	889.12	893.29	76.34
6	901.00	969.62	949.04	72.22
7	852.00	1021.26	970.48	62.06
8	922.00	1032.54	999.38	55.43
9	936.00	1054.81	1019.17	48.30
10	859.00	1067.47	1004.93	35.79
11	956.00	1040.72	1015.31	30.71
12	1031.00	1046.02	1041.51	29.81
13	1110.00	1071.32	1082.93	32.13
14	918.00	1115.06	1055.94	20.31
15	848.00	1076.25	1007.77	6.61
16	946.00	1014.38	993.87	2.51
17	1053.00	996.38	1013.37	5.91
18	1083.00	1019.27	1038.39	9.73
19	1049.00	1048.12		
20		1058.17		
21		1067.95		
22		1077.73		
23		1087.52		





- Both damped and exponential trend can be represented in a single forecasting equation, given by:

$$\widehat{Y}_{T+k} = \widehat{a}_T + \sum_{i=1}^{k} \phi^i \widehat{b}_t.$$

Here ϕ is usually pre-specified is called the trend modification parameter.

- This parameter ϕ determines the forecast profile.

$\phi > 1$	the trend is exponential,
$\phi = 1$	the trend is linear,
$\phi < 1$	the trend is damped,
$\phi = 0$	there is no trend,

- $\widehat{a}_T$ and $\widehat{b}_T$ can be computed from the Holt's algorithm.

DETAILS

1. Suppose that the time series $y_1, y_2, \ldots, y_n$ exhibits a linear trend for which the level and growth rate are changing somewhat with no seasonal pattern. Furthermore, suppose that we question whether the growth rate at the end of the time series will continue into the future. Then the estimate ℓ_T for the **level** and the estimate b_T for the **growth rate** are given by the smoothing equations

$$\ell_T = \alpha y_T + (1-\alpha)(\ell_{T-1} + \phi b_{T-1})$$
$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)\phi b_{T-1}$$

where α and γ are **smoothing constants** between 0 and 1, and ϕ is a **damping factor** between 0 and 1.

2. A **point forecast made in time period T for $y_{T+\tau}$** is

$$\hat{y}_{T+\tau}(T) = \ell_T + (\phi b_T + \phi^2 b_T + \cdots + \phi^\tau b_T)$$

3. If $\tau = 1$, then a **95% prediction interval computed in time period T for y_{T+1}** is

$$[\hat{y}_{T+1}(T) \pm z_{[.025]} s]$$

If $\tau = 2$, then a **95% prediction interval computed in time period T for y_{T+2}** is

$$\left[\hat{y}_{T+2} \pm z_{[.025]} s\sqrt{1 + \alpha^2(1+\phi\gamma)^2}\right]$$

If $\tau = 3$, then a **95% prediction interval computed in time period T for y_{T+3}** is

$$\left[\hat{y}_{T+3}(T) \pm z_{[.025]} s\sqrt{1 + \alpha^2(1+\phi\gamma)^2 + \alpha^2(1+\phi\gamma+\phi^2\gamma)^2}\right]$$

If $\tau \geq 4$, then a **95% prediction interval computed in time period T for $y_{T+\tau}$** is

$$\left[\hat{y}_{T+\tau}(T) \pm z_{[.025]} s\sqrt{1 + \sum_{j=1}^{\tau-1} \alpha^2(1+\phi_j\gamma)^2}\right]$$

where $\phi_j = \phi + \phi^2 + \cdots + \phi^j$.

- Seasonality can be additive or multiplicative.

- If seasonality is present, the following formulae may be used for forecasting.

$$\text{Additive: } \widehat{Y}_{T+k} = \widehat{a}_T + \widehat{b}_T k + \widehat{s}_{T+k},$$
$$\text{Multiplicative: } \widehat{Y}_{T+k} = (\widehat{a}_T + \widehat{b}_T k)\widehat{s}_{T+k},$$

  where $\widehat{s}_{T+k}$ is the seasonality effect at time $T+k$.

- Suppose that the seasonal component has period d. That is for all $k > 1$, $\widehat{s}_{T+k} = \widehat{s}_{T+k-d}$

- One can use the Holt-Winters algorithm for forecasting.

- For additive seasonal models the algorithm looks like

  Step 1. Initiate $\widehat{a}_{d+1} = X_{d+1}$, $\widehat{b}_{d+1} = (X_{d+1} - X_1)/d$ and $\widehat{s}_i = X_i - \{X_1 + \widehat{b}_{d+1}(i-1)\}$, for $i = 1, 2, \ldots, d$.

Step 2. For given $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq 1$, for $t = 2, 3, \ldots, T-1$, we use the recursion,

$$\begin{aligned}
\widehat{a}_{t+1} &= \alpha(X_{t+1} - \widehat{s}_{t+1-d}) + (1-\alpha)(\widehat{a}_t + \widehat{b}_t), \\
\widehat{b}_{t+1} &= \beta(\widehat{a}_{t+1} - \widehat{a}_t) + (1-\beta)\widehat{b}_t, \\
\widehat{s}_{t+1} &= \gamma(X_{t+1} - \widehat{a}_{t+1}) + (1-\gamma)\widehat{s}_{t+1-d}.
\end{aligned}$$

- Notice that the smoothing puts more weights on the recent past than the distant one.
- The period d of the seasonality component can be obtained from the background knowledge.

Holt-Winters Algorithm for Multiplicative Seasonality

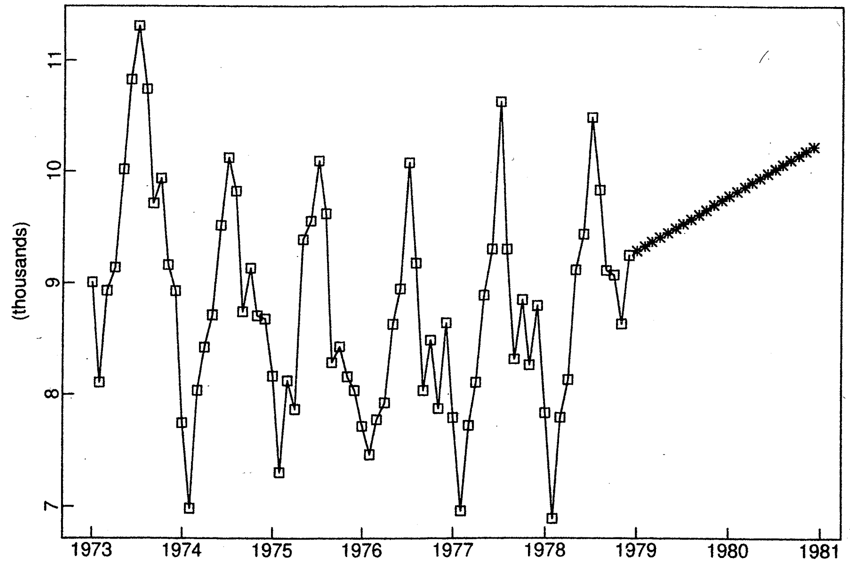
- When the seasonality effect is multiplicative, the Holt-Winters algorithm is easily modified.
- The recursion steps are as follows:

$$\begin{aligned}\widehat{a}_{t+1} &= \alpha \frac{X_{t+1}}{\widehat{s}_{t+1-d}} + (1-\alpha)(\widehat{a}_t + \widehat{b}_t),\\ \widehat{b}_{t+1} &= \beta(\widehat{a}_{t+1} - \widehat{a}_t) + (1-\beta)\widehat{b}_t,\\ \widehat{s}_{t+1} &= \gamma \frac{X_{t+1}}{\widehat{a}_{t+1}} + (1-\gamma)\widehat{s}_{t+1-d}.\end{aligned}$$

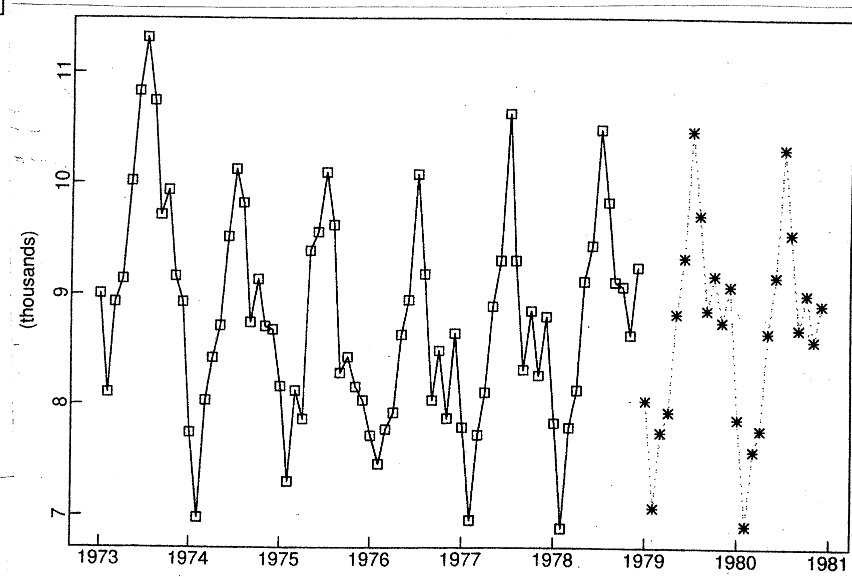
- There are several choices for initial values. The following are suggested by several authors.

$$\begin{aligned}\widehat{a}_{d+1} &= \frac{1}{d}\sum_{t=1}^{d} X_t,\\ \widehat{b}_{d+1} &= \frac{1}{d}\left\{\sum_{t=1}^{d} X_t/d - \sum_{t=d+1}^{2d} X_t/d\right\},\\ \widehat{s}_t &= \left\{X_t - (t-1)\widehat{b}_{d+1}/2\right\}/\widehat{a}_{d+1} \text{ for } t = 1, 2, \ldots, d.\end{aligned}$$

EXAMPLE



The data set DEATHS.DAT with 24 values predicted by the nonseasonal Holt-Winters algorithm.



The data set DEATHS.DAT with 24 values predicted by the seasonal Holt-Winters algorithm.