Time-Invariant Linear filters

- The process $\{Y_t\}$ is the output of a linear filter $C = \{c_{t,k}, t,k = 0, \pm 1, \cdots\}$ applied to an input process $\{X_t\}$ if.

$$Y_t = \sum_{k=-\infty}^{\infty} c_{t,k} X_k, \quad t = 0, \pm 1, \cdots .$$

- The filter is said to be time-invariant if the weights $c_{t,t-k}$ are independent of t i.e., if

$$c_{t,t-k} = \psi_k .$$

- If the filter is time-invariant then $Y_t = \sum_{k=-\infty}^{\infty} \psi_k X_{t-k}$ and $Y_{t-s} = \sum_{k=-\infty}^{\infty} \psi_k X_{t-s-k}$. So, the time-shifted process Y_{t-s} is obtained from X_{t-s} by using the ~~several~~ same filter which produces Y_t from X_t.

- The Time-Invariant Linear filter is said to be causal if $\psi_j = 0$ for $j < 0$. In this case Y_t is expressible in terms only of X_s, $s \le t$.

Spectral decomposition with time-Invariant Linear filters.

- Theorem 3. Let $\{X_t\}$ be a stationary time series with mean zero and spectral density $f_X(\lambda)$. Suppose that $\psi = \{\psi_j, j = 0, \pm 1, \cdots\}$ is an absolutely summable time-invariant linear filter. Then the time series

$$Y_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j}$$

is stationary with mean zero and spectral density.

$$f_Y(\lambda) = \psi(e^{i\lambda})\psi(e^{-i\lambda}) f_X(\lambda) = |\psi(e^{-i\lambda})|^2 f_X(\lambda),$$

where $\psi(e^{-i\lambda}) = \sum_{k=-\infty}^{\infty} \psi_k e^{-ik\lambda}$.

- Proof:- We know that, $\{Y_t\}$ is stationary with mean 0 and auto covariance function

$$\gamma_X(h) = \sum_{j,k=-\infty}^{\infty} \psi_j \psi_k \gamma_X(h+k-j).$$

Since $\{X_t\}$ has spectral density $f_X(\lambda)$, we have.

$$\gamma_X(h+k-j) = \int_{-\pi}^{\pi} e^{i(h-j+k)\lambda} f(\lambda)\, d\lambda,$$

Now, we get,

$$\gamma_y(h) = \sum_{j,k=-\infty}^{\infty} \psi_j \psi_k \int_{-\pi}^{\pi} e^{i(h-j+k)\lambda} f_X(\lambda)\, d\lambda .$$

$$= \int_{-\pi}^{\pi} e^{ih\lambda} \left(\sum_{j=-\infty}^{\infty} \psi_j \, e^{-ij\lambda} \right) \left(\sum_{k=-\infty}^{\infty} \psi_k \, e^{ik\lambda} \right) f_x(\lambda)\, d\lambda .$$

$$= \int_{-\pi}^{\pi} e^{ih\lambda} \, \psi(e^{-i\lambda}) \, \psi(e^{ik}) \, f_x(\lambda)\, d\lambda .$$

Now it follows that

$$f_y(\lambda) = \psi(e^{-i\lambda}) \, \psi(e^{i\lambda}) \, f_x(\lambda) .$$

$$= |\psi(e^{-i\lambda})|^2 f_x(\lambda) . \qquad \square$$

- The function $\psi(e^{-i\cdot})$ is called the transfer function of the filter. the squared modulus $|\psi(e^{-i\cdot})|^2$ is ~~refered~~ called the power transfer function of the filter.

Spectral distribution of an ARMA process.

- Let $\{X_t\}$ be an ARMA(p,q) process given by.

$$\phi(B)X_t = \theta(B)Z_t.$$

- Theorem 4. If $\{X_t\}$ is a causal ARMA(p,q) process satisfying $\phi(B)X_t = \theta(B)Z_t$, then.

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}\cdot\frac{\theta(e^{-i\lambda})\,\theta(e^{i\lambda})}{\phi(e^{-i\lambda})\,\phi(e^{i\lambda})} = \frac{\sigma^2}{2\pi}\cdot\frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}.$$

Proof:- Note that $\{X_t\}$ can be obtained from $\{Z_t\}$ by application of the time-invariant linear transfer function; $\psi(e^{-i\lambda}) = \theta(e^{-i\lambda})/\phi(e^{-i\lambda})$.

Now the spectral density of $\{Z_t\}$ is $f_Z(\lambda) = \sigma^2/2\pi$.

So using theorem 3. we get $f_X(\lambda) = \frac{\sigma^2}{2\pi}\,\frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}$. □

- Note that, the spectral density is a ratio of two trigonometric polynomials.
- This spectral density is called rational.

Example: AR(2) and ARMA(1,1).

- For AR(2) process the spectral density is immediate.

$$f_x(\lambda) = \frac{\sigma^2}{2\pi} \left(1-\phi_1 e^{-i\lambda} - \phi_2 e^{-2i\lambda}\right)^{-1} \left(1-\phi_1 e^{i\lambda} - \phi_2 e^{2i\lambda}\right)^{-1}.$$

$$= \frac{\sigma^2}{2\pi} \left(1 + \phi_1^2 + 2\phi_2 + \phi_2^2 + 2(\phi_1\phi_2 - \phi_1)\cos\lambda - 4\phi_2 \cos^2\lambda\right)^{-1}.$$

- For ARMA(1,1) process we know.

$$\psi(B) = \frac{1+\theta B}{1-\phi B}.$$

- So the spectral density is given that.

$$f_x(\lambda) = \frac{\sigma^2}{2\pi} \cdot \frac{(1+\theta e^{-i\lambda})(1+\theta e^{i\lambda})}{(1-\phi e^{-i\lambda})(1-\phi e^{i\lambda})} = \frac{\sigma^2}{2\pi} \frac{1+\theta^2+2\theta\cos\lambda}{1+\phi^2+2\phi\cos\lambda}.$$

Spectral distribution of linear combination of sinusoids..

- For any arbitrary ~~complex~~ real stationary process $\{X_t\}$, one can write.

$$X_t = \sum_{j=1}^{n} A(\lambda_j)\, e^{it\lambda_j} ,$$

where $-\pi < \lambda_1 < \lambda_2 < \cdots < \lambda_n = \pi$ and $A(\lambda_1), \cdots, A(\lambda_n)$ are uncorrelated complex random variables such that.

$$E[A(\lambda_j)] = 0, \quad E[A(\lambda_j)\overline{A(\lambda_j)}] = \sigma_j^2 .$$

- For $\{X_t\}$ to be real-valued one can show that $A(\lambda_n)$ is real, $\lambda_j = -\lambda_{n-j}$ and $A(\lambda_j) = \overline{A(\lambda_{n-j})}$ for $j = 1, \cdots, n-1$.
- In particular we can write.

$$X_t = \sum_{j=1}^{n} \left(C(\lambda_j) \cos t\lambda_j - D(\lambda_j) \sin t\lambda_j \right) ,$$

where $A(\lambda_j) = C(\lambda_j) + i D(\lambda_j)$, $j = 1, 2, \cdots, n$ and $D(\lambda_j) = 0$.

Connection to spectral distribution.

- The real-valued process is stationary implies
$$E[X_t]=0 \quad \text{and} \quad E[X_{t+h} X_t] = \gamma_x(h) = \sum_{j=1}^{n} \sigma_j^2 e^{ih\lambda_j}.$$
Note that the ~~latter~~ RHS is independent of t.

- Define a distribution function
$$F(\lambda) = \sum_{j:\lambda_j \le \lambda} \sigma_j^2 .$$

- We can write $\quad \gamma_x(h) = \sum_{j=1}^{n} \sigma_j^2 e^{ih\lambda_j} = \int_{(-\pi,\pi]} e^{ih\lambda} dF(\lambda)$.

- Note that $F(\lambda)$ as we have defined it is a discrete ~~jump~~ step function, so $dF(\lambda)$ has to be looked at carefully.

- $dF(\lambda)$ would be zero everywhere except at λ_j, where it is σ_j^2. That is, it would be ~~spik~~ full of spikes.

- The integral ~~is~~ can be ~~made~~ explained as a Riemann-Stieltjes integral.

A bit more.

- Every zero-mean stationary process has a represtation as.

$$X_t = \int_{(-\pi,\pi]} e^{it\lambda}\, dZ(\lambda).$$

This integral is a stochastic integral.

- The corresponding auto-covariance function γ_X can be expressed as.

$$\gamma_X(h) = \int_{(-\pi,\pi]} e^{ih\lambda}\, dF(\lambda),$$

where F is a distribution function with $F(-\pi)=0$ and $F(\pi)=\gamma_X(0)$.

The periodogram

- If $\{X_t\}$ is a stationary time series with autocovariance function $\gamma_X(\cdot)$ and spectral density $f_X(\lambda)$.
- $\hat{\gamma}_X(\cdot)$ of the observation $\{x_1, \cdots, x_n\}$ is an estimate of $\gamma_X(\cdot)$.
- Similarly, $2\pi \hat{f}_X(\lambda)$ can be estimated using a periodogram $I_n(\cdot)$ of the observations.
- Suppose we define $\omega_k = \frac{2\pi k}{n}$, $k = -\left[\frac{n-1}{2}\right], \cdots, \left[\frac{n}{2}\right]$, where $[y]$ denotes the largest integer less than or equal to y.
- Let F_n be the set of such ω_k and call it the Fourier frequencies associated with sample size n.
- Note that each $\omega_k \in (-\pi, \pi]$.

Periodogram (contd.)

- Let $e_k = \frac{1}{\sqrt{n}} [e^{i\omega_k}, \dots, e^{ni\omega_k}]^T$, $k = -[\frac{n-1}{2}], \dots, [\frac{n}{2}]$.

- Note that $e_j^* e_k = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j \neq k. \end{cases}$

- ~~Also $e_j^* e_j = 1$~~ Thus e_k, $k \in \{-[\frac{n-1}{2}], \dots, [\frac{n}{2}]\}$ forms a basis of the n-dimensional complex plane.

- Thus any n-dimensional complex number x we can write

$$x = \sum_{k=-[\frac{n-1}{2}]}^{[n/2]} a_k e_k. \qquad (2)$$

- In order to find a_k we note the orthogonality of the basis vectors. Then

$$e_k^* x = a_k = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{-it\omega_k}.$$

Discrete Fourier transform and Periodogram

- The sequence of numbers $\{a_k\}$ is called the discrete Fourier transform of the sequence $\{x_1, \cdots, x_n\}$.

- The t^{th} component in ② we note that,

$$X_t = \sum_{k=-[(n-1)/2]}^{[n/2]} a_k \left[\cos(\omega_k t) + i \sin(\omega_k t)\right], \quad t = 1, 2, \cdots, n.$$

- The periodogram of $\{x_1, x_2, \cdots, x_n\}$ is the function.

$$I_n(\lambda) = \frac{1}{n}\left|\sum_{t=1}^{n} x_t e^{-it\lambda}\right|^2.$$

- Note that if λ is one of the Fourier frequencies ω_k, $I_n(\omega_k) = |a_k|^2$.

- Further, $\displaystyle\sum_{t=1}^{n} |x_t|^2 = \sum_{k=-[(n-1)/2]}^{[n/2]} |a_k|^2 = \sum_{k=-[(n-1)/2]}^{[n/2]} I_n(\omega_k)$.

- The total variation in the observations is the total of periodogram.

Connection between $I_n(\lambda)$ and $f_x(\lambda)$.

Theorem 5. If $x_1, \cdots, x_n$ are ~~any~~ real numbers and ω_k is any of the non zero Fourier frequencies $2\pi k/n$ in $(-\pi, \pi]$ then.

$$I_n(\omega_k) = \sum_{|h|<n} \hat{\gamma}_x(h) e^{-ih\omega_k},$$

where $\hat{\gamma}_x(h)$ is the sample autocovariance function of $x_1, \cdots, x_n$.

Proof:- If $\omega_k \neq 0$, $\sum_{t=1}^{n} e^{-it\omega_k} = 0$. So we can centre the observations and consider $x_t - \bar{x}$. Now from the definition

$$I_n(\omega_k) = \frac{1}{n} \sum_{s=1}^{n} \sum_{t=1}^{n} (x_s - \bar{x})(x_t - \bar{x}) e^{-i(t-s)\omega_k}.$$

$$= \sum_{|h|<n} \hat{\gamma}_x(h) e^{-ih\omega_k}. \qquad \square$$

Estimated spectrum of a white noise.

- ~~Re~~ If $Z_t \sim WN(0, \sigma^2)$, we have shown that
  $f_Z(\lambda) = \sigma^2/2\pi$, which is uniform.

- the sample versions look like this.

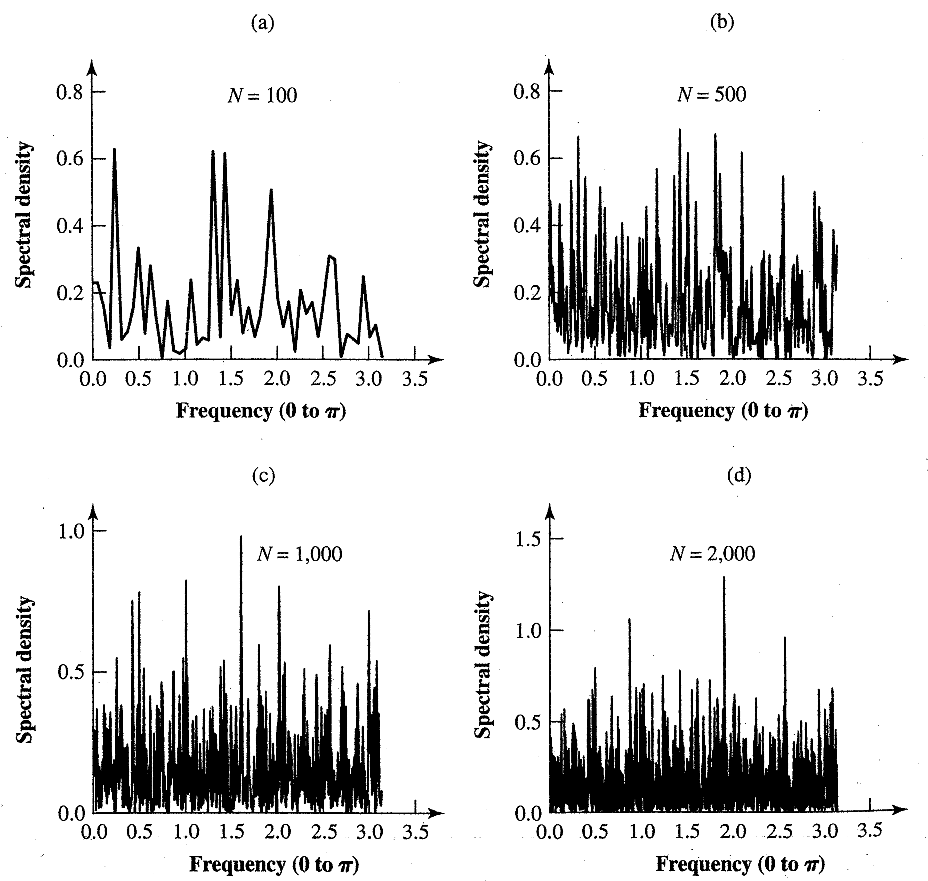

FIGURE 13.2 Sample spectrum of a white noise process.