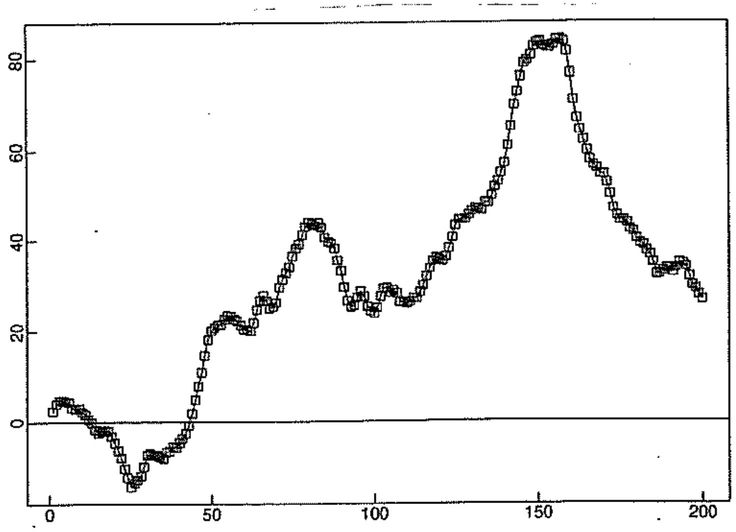
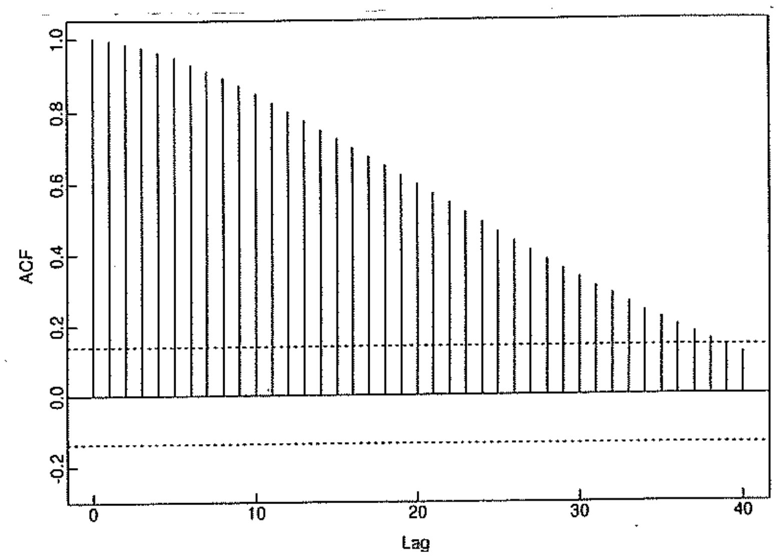
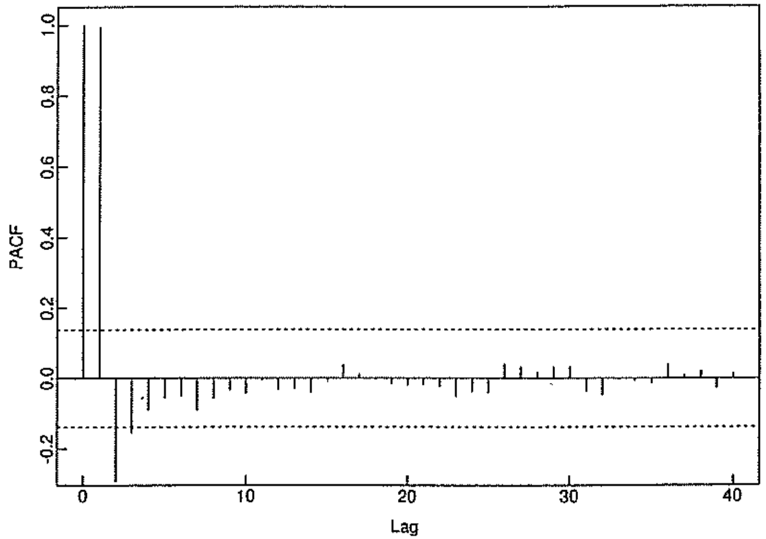
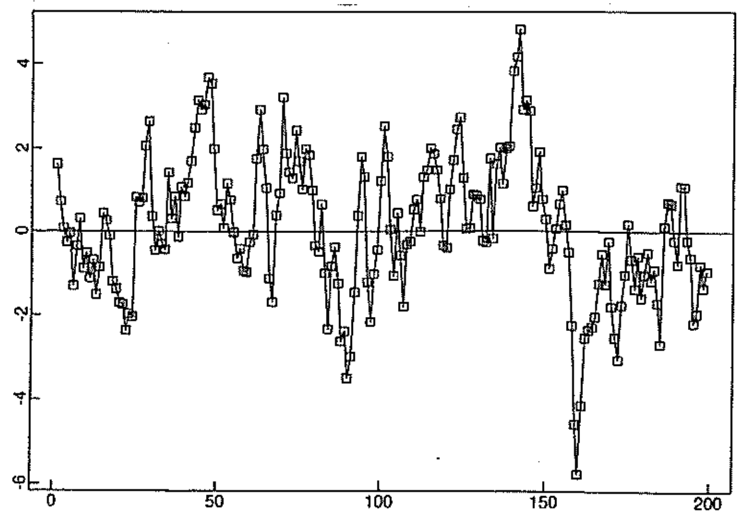
Integrated Processes.

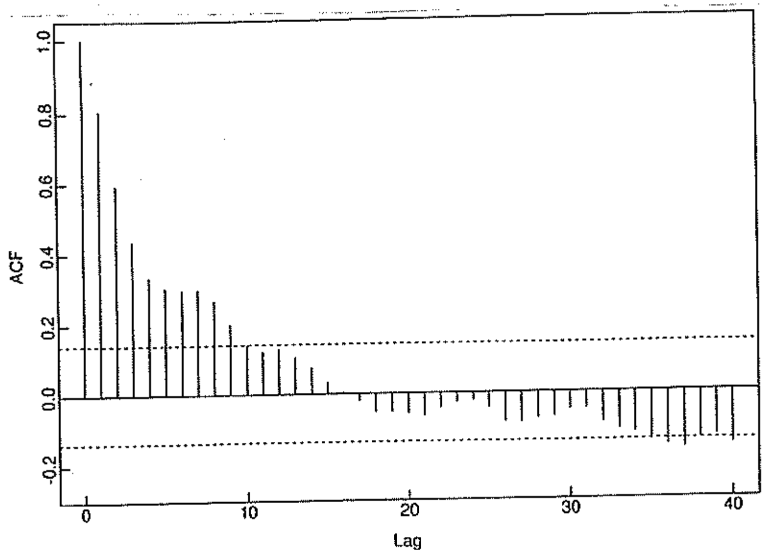


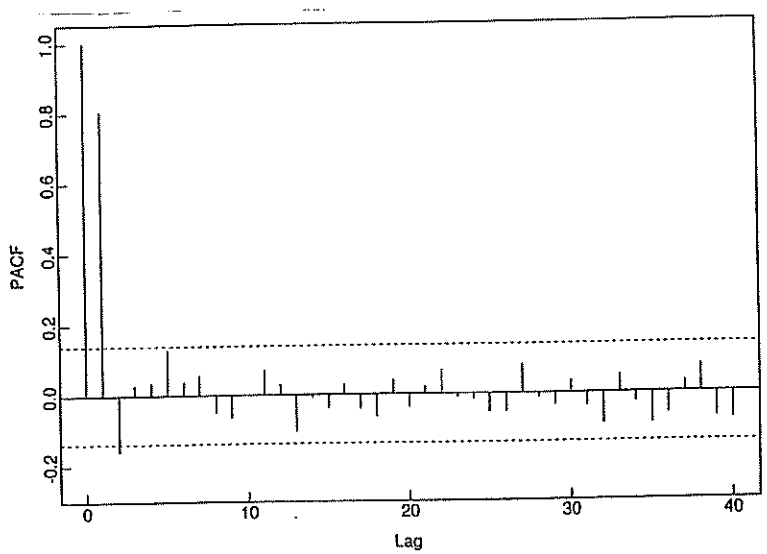


$\{x_t\}$.









- $Y_t = (1-B)X_t$

- We generated X_t as
  $(1-.8B)(1-B)X_t = Z_t$, $Z_t \sim WN(0,1)$

- If we fit an $AR(1)$ on $\{Y_t\}$ we get
  $(1-.808B)(1-B)X_t = Z_t$, $Z_t \sim WN(0,.978)$.

ARIMA

- Definition:- If d is a non-negative integer, then $\{X_t\}$ is an ARIMA(p,d,q) process if $Y_t = (1-B)^d X_t$ is a causal ARMA(p,q) process.

- From the definition ~~of~~ above ~~as~~ $\{X_t\}$ is a ARIMA(p,d,q) process if it satisfies ~~the~~ a difference equation of the form.

$$\phi^*(B)X_t \equiv \phi(B)(1-B)^d X_t = \theta(B)Z_t, \text{ where } \{Z_t\} \sim WN(0,\sigma^2),$$

where $\phi(z)$ and $\theta(z)$ are polynomials of degree p and q respectively, and $\phi(z) \neq 0 \;\forall\; |z| \leq 1$.

- Note that the polynomial $\phi^*(z)$ has a root of order d at $z=1$.
- If $d=0$, $\{X_t\}$ reduces to an ARMA(p,q) process.

ARIMA and the data with trend

- Example :- Let $\{x_t\}$ be a process defined by.

$$x_t = a_0 + a_1 t + z_t , \qquad z_t \sim WN(0, \sigma^2).$$

Let $y_t = x_t - x_{t-1} = a_0 + a_1 t + z_t - (a_0 + a_1(t-1) + z_{t-1})$

$$= a_1 + (z_t - z_{t-1}).$$

Further $y_t - y_{t-1} = x_t - 2x_{t-1} - x_{t-2} = a_1 + (z_t - z_{t-1}) - a_1 - (z_{t-1} - z_{t-2})$

$$= z_t - 2z_{t-1} + z_{t-2}.$$

- Notice that $\{x_t\}$ had a quadratic trend, but $\{y_t - y_{t-1}\}$ does not have any trend.
- So the process $(1-B)^2 x_t$ is trend less.
- More generally if $\{x_t\}$ has a polynomial trend of degree $d-1$, the series $(1-B)^d x_t$ will not have any trend.
- Thus an ARIMA process is useful when the process shows polynomial trend.

Homogeneous non-stationary processes.

- A somewhat related motivation of ~~a~~ an ARIMA process comes from what is sometimes refered to as homogeneous non-stationarity.
- This means that the process is very much similar in different time-points excepting its mean levels.
- That is the process in nonstationary, but that is mostly in its mean.
- Such a process $\{X_t\}$ would satisfy the equation.

$$\psi(B)(X_t + c) = \psi(B) X_t .$$

This equation has a ~~solt~~ solution as $\psi(B) = \phi(B)(1-B)^d$ for some ~~stationary auto polynomial $\phi(B)$~~ stationary autoregressive polynomial $\phi(B)$ and integer $d > 0$.

Example :

- The random walk model can be writen as.

$$(1-B)X_t = Z_t , \quad Z_t \sim WN(0,\sigma^2) .$$

- So $\{X_t\} \sim ARIMA(0,1,0)$.

- In this model clearly.

$$X_t = X_{t-1} + Z_t .$$

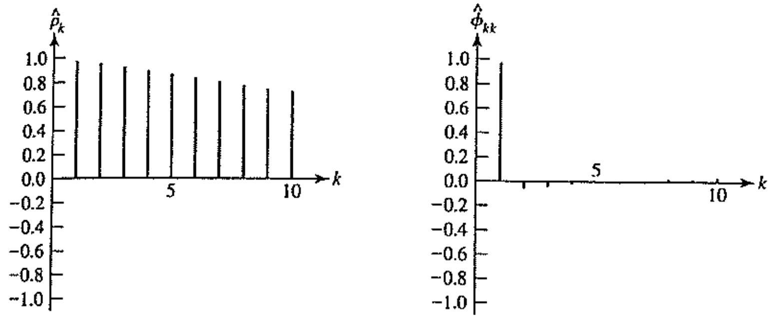
- Thus given the past information, the level of the time series model is given by

$$E[X_t \mid X_{t-1}, X_{t-2}, \cdots, X_0] = X_{t-1} .$$

- It can also be viewed as a limit of the AR(1) process with $\phi \to 1$.
- Now since for an AR(1) process $\rho_X(h) = \phi^{|h|}$, X_t would have large ~~non-var~~ spikes in its ACF plot. However, after differencing $Y_t = X_t - X_{t-1} = Z_t$, the ACF plot would not show many spikes.
- The process $\{X_t\}$ satisfying $(1-B)X_t = \theta_0 + Z_t$, with $\theta_0 \neq 0$, is called a random walk with drift.

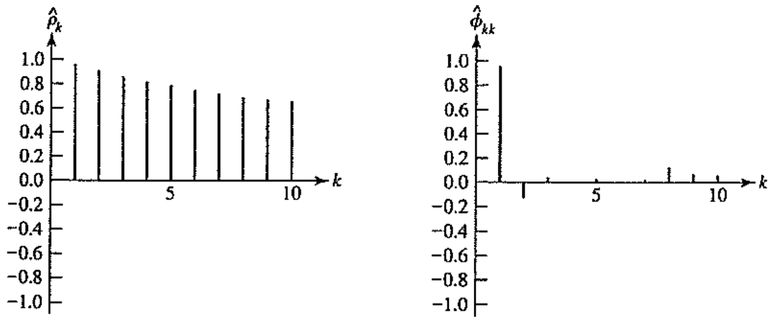
Example (contd.)

(a)



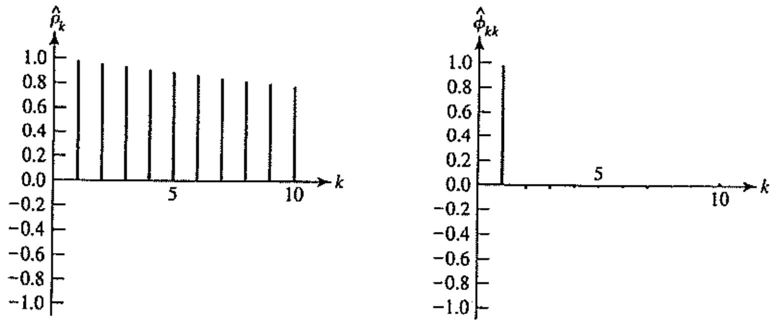
$(1-.8B)(1-B)X_t = Z_t$

(b)



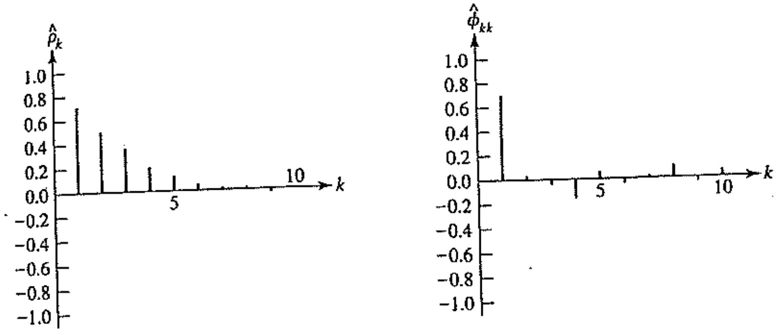
$(1-B)X_t = (1-.75B)Z_t$

(c)



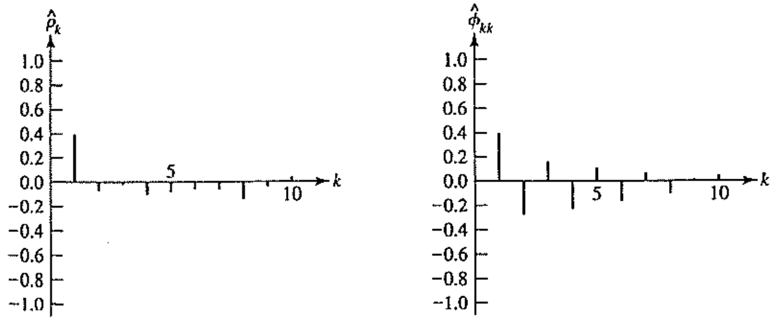
$(1-.9B)(1-B)X_t = (1-.5B)Z_t$

(a)



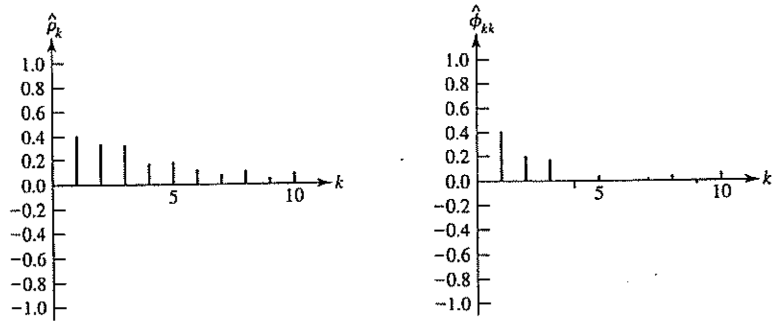
$Y_t = (1-B)X_t$

(b)



$Y_t = (1-B)X_t$

(c)



$Y_t = (1-B)X_t$

Some other properties.

- An $ARIMA(0,d,q)$ model is sometimes refered to as an $IMA(d,q)$ model.

- In order to fit an $ARIMA(p,d,q)$ model, one first needs to guess the value of d. This can be guessed by considering the ACF plot of the differenced series. The number of spikes will reduce upto a level, ~~till th~~ then it won't change.

- Once this d is found, we can fit an $ARMA(p,q)$ model to the differenced data.

Seasonal ARIMA (SARIMA)

- Definition:- If d and D are non-negative integers, then $\{X_t\}$ is a seasonal ARIMA $(p,d,q)\times(P,D,Q)_s$ process with period s if the differenced series $Y_t = (1-B)^d(1-B^s)^D X_t$ is a causal ARMA process of the form

$$\phi(B)\,\Phi(B^s)\,Y_t = \theta(B)\,\Theta(B^s)\,Z_t\,, \quad Z_t \sim WN(0,\sigma^2),$$

where $\phi(z) = 1-\phi_1 z - \cdots - \phi_p z^p$, $\Phi(z) = 1-\Phi_1 z - \cdots - \Phi_P z^P$,

$$\cancel{\theta(z) = 1+\theta_1 z + \cdots + \theta_p z^p \text{ and } \Theta(z) = 1+\Theta_1 z+\cdots+\Theta_Q z^Q.}$$

$$\theta(z) = 1+\theta_1 z+\cdots+\theta_q z^q \quad \text{and} \quad \Theta(z) = 1+\Theta_1 z+\cdots+\Theta_Q z^Q.$$

- Note that Y_t is causal if and only if $\phi(z)\neq 0$ and $\Phi(z)\neq 0$ for $|z|\leq 1$. In practice D is rarely larger than 1 and P and Q are usually less than 3.

Example.

- Airline model :- The following model have been used ~~in~~ to represent many seasonal time serieses. This include airline data, trade data etc..

- The model is a SARIMA $(0,1,1)\times(0,1,1)_{12}$ that is

$$(1-B)(1-B^{12})X_t = (1+\theta B)(1+\Theta B^{12}) Z_t, \quad Z_t \sim WN(0,\sigma^2).$$

- It can be easily shown that

$$\gamma_z(0) = (1+\theta^2)(1+\Theta^2)\sigma^2$$

$$\left.\gamma_z(1) = \theta(1+\Theta^2)\sigma^2 \right\} MA(1)$$

$$\left.\begin{aligned} \gamma_z(11) &= \theta\Theta\sigma^2 \\ \gamma_z(12) &= \Theta(1+\theta^2)\sigma^2 \\ \gamma_z(13) &= \theta\Theta\sigma^2 \end{aligned}\right\} SMA(1)$$

$$\gamma_z(j) = 0 \quad \text{o.w.}$$

$$\rho_z(1) = \frac{\theta}{1+\theta^2}$$

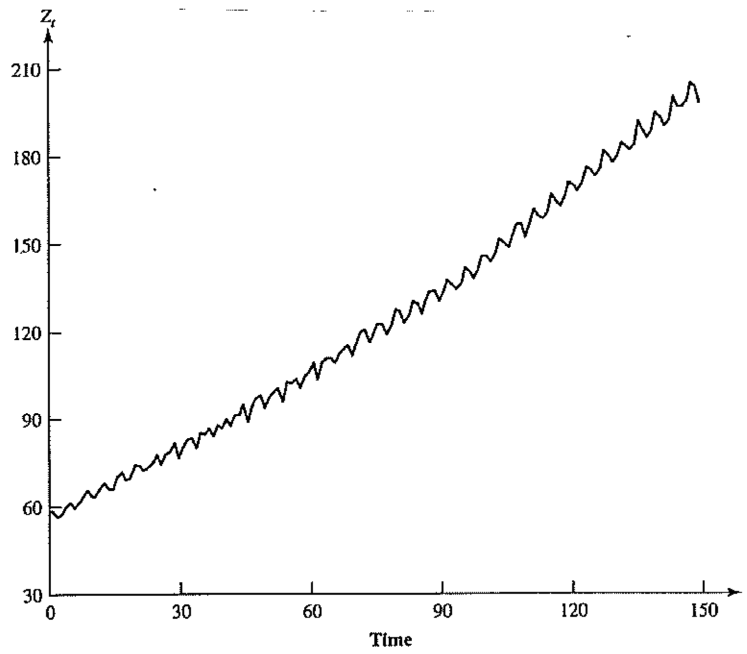
$$\rho_z(11) = \rho_z(13) = \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)}.$$

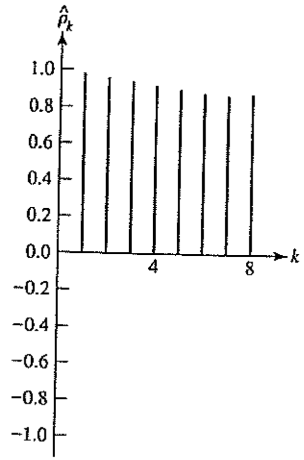
$$\rho_z(12) = \frac{\Theta}{1+\Theta^2}.$$

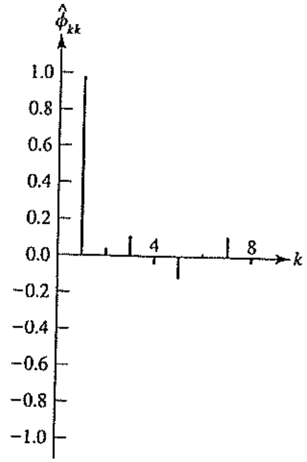
$$\rho_z(j) = 0 \quad \text{o.w.}$$

Example.

- Consider the following time series.



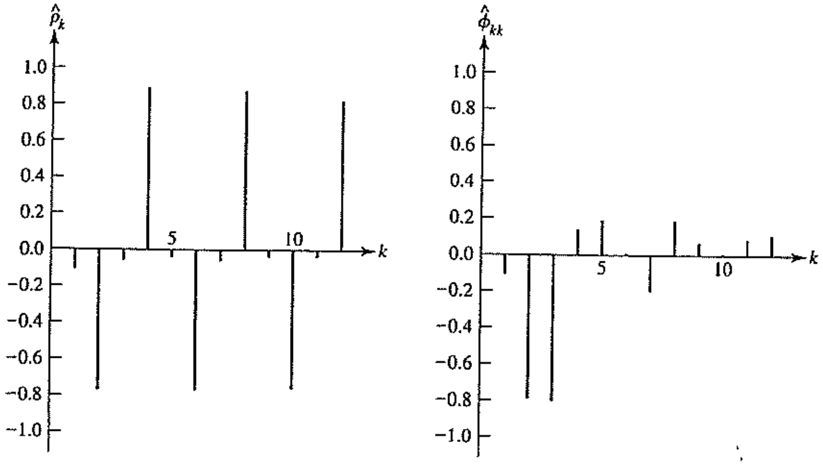




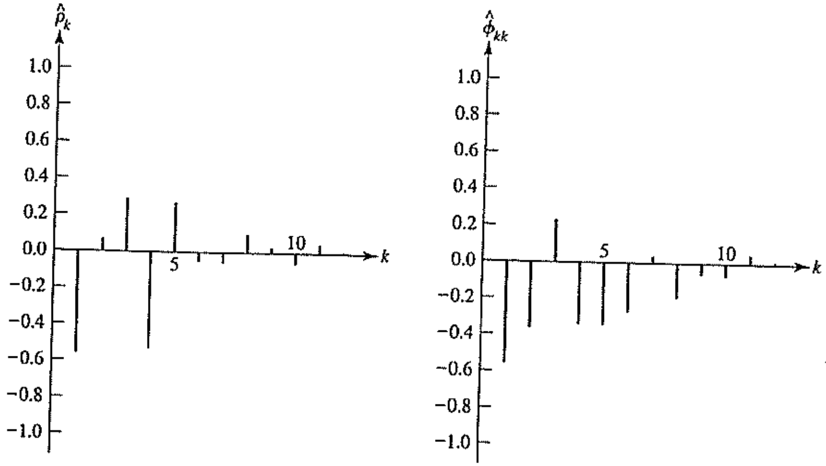
- The trend looks linear, so one differencing seems reasonable.

Example (contd.).

(a)



(b)



- $Y_t = (1-B)X_t$
- The ACF of Y_t shows a ~~cycle~~ slowly decreasing cycle of length 4. So $s=4$ and a seasonal differencing makes sense.
- $W_t = (1-B^4)Y_t$.

- The ACF of W_t has spike at $h=1,3,4,5$ and not at 2.
- So may be W_t is a seasonal ARMA with $p=0, P=0, q=1, Q=1$.
- That is $X_t \sim SARIMA(0,1,1)\times(0,1,1)_4$

or $(1-B)(1-B^4)X_t = (1+\theta B)(1+\Theta B)Z_t$

- The original series was generated from $(1-B)(1-B^4)X_t = (1-.8B)(1-.6B)Z_t$.

Connection of ARIMA(0,1,1) to Exponential Smoothing.

- Consider the model

$$(1-B)\,x_t = (1-\theta B)\,Z_t \qquad |\theta|<1 \text{ and } Z_t \sim WN(0,\sigma^2).$$

- Note that

$$
\begin{aligned}
Z_t &= \frac{1-B}{(1-\theta B)}\,x_t. \\
&= (1-B)(1+\theta B+\theta^2B^2+\dots)\,x_t \\
&= (1+\theta B+\theta^2B^2+\dots-B-\theta B^2-\theta^2B^3\dots)\,x_t \\
&= (1-(1-\theta)B-(1-\theta)\theta B^2-(1-\theta)\theta^2B^3-\dots)\,x_t \\
&= (1-\alpha B-\alpha(1-\alpha)B^2-\alpha(1-\alpha)^2B^3-\dots)\,x_t \quad (\alpha=(1-\theta)).
\end{aligned}
$$

- So $\underline{x_t} = \alpha \underline{\sum_{j=1}^{\infty}(1-\alpha)^{j-1}x_{t-j}} + Z_t.$

- That is $\hat{x}_t = \alpha\sum_{j=1}^{\infty}(1-\alpha)^{j-1}x_{t-j}.$

- You can show that $\hat{x}_{t+1} = \alpha x_t + (1-\alpha)\hat{x}_t.$

- This is exponential smoothing.