Spectral distribution for processes with $\sum_{h=-\infty}^{\infty} |\gamma_x(h)| < \infty$

- Suppose that $\{X_t\}$ is a zero-mean stationary time series with autocovariance function $\gamma_X(h)$ satisfying $\sum_{h=-\infty}^{\infty} |\gamma_x(h)| < \infty$. The spectral density of $\{X_t\}$ is given by.

$$f_x(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_x(h), \quad -\infty < \lambda < \infty. \quad \cdots \cdots (1)$$

- Here $i = \sqrt{-1}$, $e^{i\lambda} = \cos\lambda + i\sin\lambda$.
- Since $\sum_{h=-\infty}^{\infty} |\gamma_x(h)| < \infty$, the sum in (1) converges absolutely.
- Note that, cosine and sine functions have period 2π, so f will also be periodic with the same period. So it is sufficient to restrict $\lambda \in (-\pi, \pi]$.
- Note that the left end is open.

Basic properties of f

- Theorem 1: A spectral density f_x satisfies the following properties.

  i) f_x is even, thus $f_x(\lambda) = f_x(-\lambda)$.

  ii) $f_x(\lambda) \geq 0 \quad \forall \lambda \in (-\pi, \pi]$.

  iii) $\gamma_x(h) = \int_{-\pi}^{\pi} e^{ik\lambda} f_x(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \cos(k\lambda) f_x(\lambda)\, d\lambda$.

Proof :- i) Consider $f_x(-\lambda) = \sum_{h=-\infty}^{\infty} e^{i\lambda h} \gamma_x(h) = \sum_{h=-\infty}^{\infty} \{\cos(\lambda h) + i\sin(\lambda h)\}\gamma_x(h)$

$$= \sum_{h=-\infty}^{\infty} \cos(\lambda h)\gamma_x(h) + \sum_{h=-\infty}^{-1} i\sin(\lambda h)\gamma_x(h) + \sum_{h=1}^{\infty} i\sin(\lambda h)\gamma_x(h).$$

Now $\sin(-x) = -\sin(x)$, ie it is odd, but $\gamma_x(h)$ is even. Thus

$$f_x(-\lambda) = \sum_{h=-\infty}^{\infty} \cos(\lambda h)\gamma_x(h) - \sum_{h=1}^{\infty} i\sin(\lambda h)\gamma_x(h) + \sum_{h=1}^{\infty} i\sin(\lambda h)\gamma_x(h).$$

$$= \sum_{h=-\infty}^{\infty} \cos(\lambda h)\gamma_x(h).$$

Clearly from the same steps we can show that $f_x(\lambda) = \sum_{h=-\infty}^{\infty} \cos(-\lambda h)\gamma_x(h) = f_x(-\lambda)$.

Proof contd.

ii) In order to show that $f_x(\lambda) \geq 0 \quad \forall \lambda \in (-\pi, \pi]$ ~~consider,~~ fix an integer N and consider the sum.

$$f_N(\lambda) = \frac{1}{2\pi N} E\left[\left|\sum_{r=1}^{N} X_r e^{-ir\lambda}\right|^2\right]$$

$$= \frac{1}{2\pi N} E\left[\left(\sum_{r=1}^{N} X_r e^{-ir\lambda}\right)\left(\sum_{s=1}^{N} X_s e^{is\lambda}\right)\right].$$

$$= \frac{1}{2\pi N} \left[\sum_{|h|<N} (N - |h|)\, e^{-ih\lambda} \gamma_x(h)\right].$$

$$= \frac{1}{2\pi N} \left[\sum_{|h|<N} (N-|h|) \{\cos(\lambda h) + i \sin(\lambda h)\} \gamma_x(h)\right].$$

$$= \frac{1}{2\pi} \left[\sum_{|h|<N} \left(1 - \frac{|h|}{N}\right) \{\cos(\lambda h)\} \gamma_x(h)\right].$$

Note that $f_N(\lambda) \geq 0$ and as $N \to \infty$, $f_N(\lambda) \longrightarrow \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_x(h)$
$= f_x(\lambda)$.

So $f_x(\lambda)$ is nonnegative. Absolute summability is required in the last step.

Proof. (contd. (2))

iii)
$$\int_{-\pi}^{\pi} e^{ik\lambda} f_x(\lambda)\, d\lambda = \int_{-\pi}^{\pi} e^{ik\lambda} \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_x(h)\, d\lambda.$$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{i(k-h)\lambda} \gamma_x(h)\, d\lambda.$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma_x(h) \int_{-\pi}^{\pi} e^{i(k-h)\lambda}\, d\lambda.$$

Now note that
$$\int_{-\pi}^{\pi} e^{i(k-h)\lambda}\, d\lambda = \begin{cases} 2\pi & \text{if } h=k \\ 0 & \text{if } h \neq k. \end{cases}$$

So it follows that
$$\int_{-\pi}^{\pi} e^{ik\lambda} f_x(\lambda)\, d\lambda = \gamma_x(k). \qquad \square$$

- Note that we have defined spectral density in (1) only for $\gamma_x(\cdot)$ such that $\sum_{h=-\infty}^{\infty} |\gamma_x(h)| < \infty$. This is not strictly required.
- We can define the spectral density by avoiding this condition.

A general definition of spectral density:

- A function f_x is the spectral density of a stationary time series $\{X_t\}$ with autocovariance function $\gamma_x(\cdot)$ if.

  i) $f_x(\lambda) \geq 0 \quad \forall \lambda \in (0, \pi]$.

  ii) $\gamma_x(h) = \int_{-\infty}^{\infty} e^{ih\lambda} f(\lambda)\, d\lambda$ for all integers h.

- Note that f_x does not need to be real. It may be complex. Infact there is no ~~need~~ requirement that $\{X_t\}$ is real. However, complex time series are rare.

- Theorem 2:- A real valued function f on $[-\pi, \pi]$ is the spectral density of a (real valued) stationary process if and only if

  i) $f(\lambda) = f(-\lambda)$ i.e. it is even.

  ii) $f(\lambda) \geq 0$ i.e. it is non-negative.

  iii) $\int_{-\pi}^{\pi} f(\lambda)\, d\lambda < \infty$.

General case contd:

Proof of Theorem 2:- Suppose ~~$\{X_t\}$~~ f is the spectral density of a real stationary process $\{X_t\}$ with real autocovariance function $\gamma_X(\cdot)$. Now if γ_X is absolutely summable ~~we have~~ i) and ii) follows from Theorem 1. For iii) note that

$$\int_{-\pi}^{\pi} f(\lambda)\, d\lambda = \int_{-\pi}^{\pi} e^{i \times 0 \times \lambda} f(\lambda)\, d\lambda = \gamma_X(0) < \infty \text{ by definition.}$$

If γ_X is not absolutely summable, ~~from γ_X~~ since γ_X is real it can be shown that $f_X(\lambda) = f_X(-\lambda)$. That is f_X is symmetric. In fact one can define f_X as a density from which the rest follows.

To show the converse note that $\gamma_X(-h) = \int_{-\pi}^{\pi} e^{-ih\lambda} f(\lambda)\, d\lambda$. Now by substituting $-\lambda = t$ we get $\gamma_X(-h) = -\int_{\pi}^{-\pi} e^{iht} f(-t)\, dt = \int_{-\pi}^{\pi} e^{iht} f(t)\, dt$

$$= \gamma_X(h).$$

Proof

To show that γ_X thus defined is n.n.d. let a_r, $r=1,2,\cdots,n$ are real numbers.

$$\sum_{r,s=1}^{n} a_r \gamma_x(r-s) a_s = \int_{-\pi}^{\pi} \sum_{r,s=1}^{n} a_r a_s e^{i\lambda(r-s)} f(\lambda)\, d\lambda .$$

$$= \int_{-\pi}^{\pi} \left| \sum_{r=1}^{n} a_r e^{i\lambda r} \right|^2 f(\lambda)\, d\lambda \geq 0 .$$

So γ_X defined is a autocovariance function.

- The following corollary is useful.
- Corollary 1: An absolutely summable function $\gamma_{\cdot}$ is the autocovariance function of a stationary time series if and only if it is even and

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) \geq 0 .$$

If so f is the spectral density of γ.

Example.

- Recall that we once tried to show that the function defined as

$$\gamma(h) = \begin{cases} 1, & \text{if } h=0 \\ \rho, & \text{if } h=1 \\ 0 & \text{o.w.} \end{cases}$$

is a valid autocovariance function iff $|\rho| < 1/2$.

The direct proof was tedious. Now we show how to prove the same result using spectral methods.

Notice that, $f(\lambda) = 1/2\pi \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_X(h) = 1/2\pi (1 + 2\rho \cos \lambda)$.

From Corollary 1 we need $f(\lambda) \geq 0 \ \forall \ \lambda \in (-\pi, \pi]$. This only happens iff $|\rho| < 1/2$.

Herglotz's theorem.

- We now state a result about the existence of spectral distributions.

- Theorem :- A complex valued function $\gamma_x(\cdot)$ defined on the integers is non-negative definite if and only if

$$\gamma_x(h) = \int_{[-\pi,\pi]} e^{ih\lambda}\, dF(\lambda) \quad \text{for all } h = 0, \pm 1, \pm 2, \cdots$$

where F is a right-continuous, non-decreasing, bounded function, on $[-\pi,\pi]$ and $F(-\pi) = 0$.

- The function F is called the spectral distribution function.

- Note that $F(\pi)$ ~~is~~ not necessarily 1, it is finite.

- So F is a generalised distribution function on $[-\pi,\pi]$. $G(\lambda) = F(\lambda)/F(\pi)$ is a probability distribution function on $[-\pi,\pi]$.

- Further note that $\gamma_x(0) = \int_{-\pi}^{\pi} dF(\lambda) = F(\pi)$, so

$$\rho_x(h) = \int_{[-\pi,\pi]} e^{ih\lambda}\, dG(\lambda).$$

Properties of F

- If F can be expressed as $F(\lambda) = \int_{-\pi}^{\lambda} f(y)\,dy$, $\forall \lambda \in [-\pi, \pi]$, then f is called the spectral density. Note that f may not exist.

- If G is a discrete probability distribution, then the series is said to have a discrete spectrum.

- If γ is real (ie the series is real), F is symmetric in the sense that $\int_{(a,b]} dF(z) = \int_{[-b,-a)} dF(z)$ $\forall a, b$ such that $0 < a < b$.

  Alternatively, ~~we can write~~. F would satisfy:

  $$F(\lambda) = F(\pi-) - F(-\lambda-). \quad -\pi < \lambda < \pi.$$

  Here $F(\lambda-)$ is the left limit ie $F(\lambda-) = \lim_{h \downarrow 0} F(\lambda - h)$
  $= F(x < \lambda)$.

Example:

- White noise :- If $\{x_t\} \sim WN(0,\sigma^2)$, then $\gamma(0)=\sigma^2$ and $\gamma(h)=0$ for all $|h|>0$.

$$f(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-i\lambda h}\gamma_x(h) = \sigma^2/2\pi .$$

So the spectrum is flat.

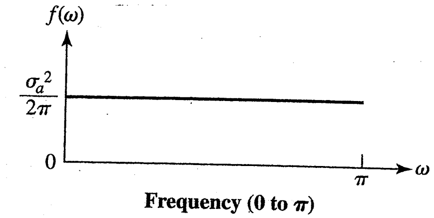

FIGURE 12.4 The spectrum of a white noise process.

- AR(1) process: suppose
$$x_t = \phi x_{t-1} + Z_t$$
be a standard AR(1) process.

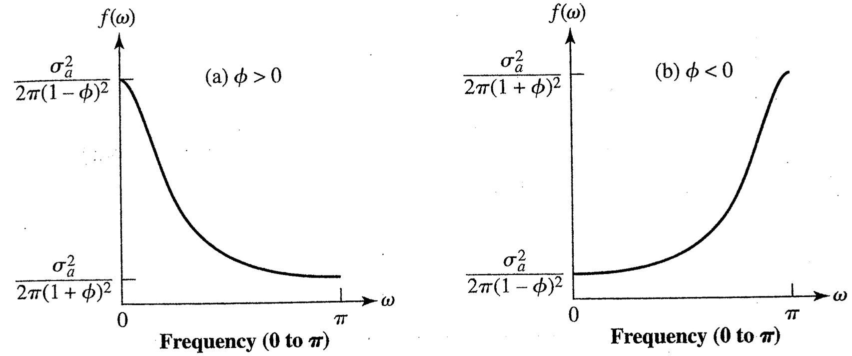

FIGURE 12.5 Spectrum for the AR(1) process.

We know that

$$\rho(h) = \phi^{|h|} \quad \forall |h|>0 .$$

$$\gamma(0) = \frac{\sigma^2}{1-\phi^2} .$$

For the spectral density we compute

$$f(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-i\lambda h}\gamma_x(h) = \frac{\gamma(0)}{2\pi}\left\{1+\phi^h\left(e^{i\lambda h}+e^{-i\lambda h}\right)\right\} = \frac{\sigma^2}{2\pi}\left(1-2\phi\cos\lambda+\phi^2\right)^{-1} .$$

Example

- MA(1) : Consider a MA(1) process with

$$X_t = Z_t + \theta Z_{t-1}, \quad Z_t \sim WN(0, \sigma^2).$$

Now

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \gamma_z(h).$$

$$= \frac{1}{2\pi} \left\{ \gamma_z(0) + \frac{\gamma_z(0)\,\theta}{1+\theta^2} \left(e^{-i\lambda} + e^{i\lambda}\right) \right\}$$

$$= \frac{\sigma^2}{2\pi} \left\{ 1 + \theta^2 + \theta\left(e^{-i\lambda} + e^{i\lambda}\right) \right\}$$

$$= \frac{\sigma^2}{2\pi} \left(1 + 2\theta \cos\lambda + \theta^2 \right).$$

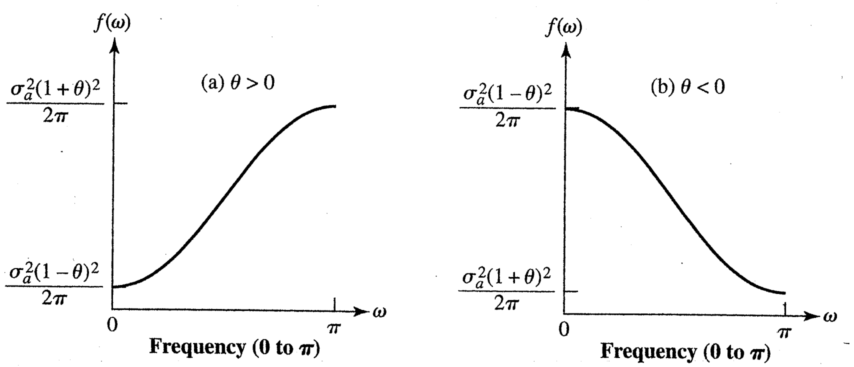

FIGURE 12.6 Spectrum for the MA(1) process.