STATIONARY MODELS

- A time series $\{X_t,\ t = 0, \pm 1, \ldots\}$ is said to be stationary if its statistical properties are similar to those of the "time shifted" series $\{X_{t+h},\ t = 0, \pm 1, \ldots\}$ for any integer h.

- In other words the starting point does not matter.

- The concept of stationarity is important, it allows us to "create" replications of short sequences from a given time series data.

- This in turn facilitates statistical analysis.

AUTOCOVARIANCE

- Autocovariance is an important concept in time series analysis.

- **Definition** (Autocovariance):- If $\{X_t, t \in T\}$ is a process such that $Var[X_t] < \infty$ for each $t \in T$, then the autocovariance function of $\{X_t\}$, denoted $\gamma_x(\cdot,\cdot)$ is defined by

$$\gamma_x(r,s) = Cov[X_r, X_s] = E[(X_r - E[X_r])(X_s - E[X_s]), \quad r, s \in T.$$

- The autocovariance measures the dependence between X_r and X_s.

- In a time series since the observations are dependent, $\gamma_x(r,s) \neq 0$ for some $r, s \in T$.

- A autocorrelation function can be defined as

$$\gamma_x(r,s)/\sqrt{\gamma_x(r,r)\gamma_x(s,s)},$$

however such a concept is mostly useful when the series is stationary.

Weak Stationarity

- **Definition** (Stationarity):- A time series $\{X_t\}$ is (weakly) stationary if
  1. $Var[X_t] < \infty$, for all $t \in \mathbb{Z}$,
  2. $E[X_t] = m$, for all $t \in \mathbb{Z}$,
  3. $\gamma_x(r,s) = \gamma_x(r+t, s+t)$, for all $r, s, t \in \mathbb{Z}$.

- From the definition we see that $\{X_t,\ t \in \mathbb{Z}\}$ is weakly stationary if it has a finite variance and the mean and autocovariance does not depend on t.

- Condition 3 implies $\gamma_x(t+h, t)$ is independent of t, for each h. That is for a weakly stationary series the "lag" h matters not the point t.

- Thus in context of a stationary X_t, one can define autocovariance with only the lag. That is, it is sufficient to say

$$\gamma_x(h) := \gamma_x(h, 0) = \gamma_x(t+h, t), \quad \forall t \in \mathbb{Z}.$$

Strict Stationarity

- Weak stationarity defined before only requires moments up to second order to be independent of t.

- This does not necessarily imply the joint distribution of subsequences of equal length of $\{X_t,\ t \in \mathbb{Z}\}$ would be same.

- **Definition** (Strict stationarity):- A time series $\{X_t,\ t \in \mathbb{Z}\}$ is said to be strictly stationary if the joint distributions of $(X_{t_1}, \ldots, X_{t_k})^T$ and $(X_{t_1+h}, \ldots, X_{t_k+h})^T$ are the same for all positive integers k and for all $t_1, t_2, \ldots, t_k,\ h \in \mathbb{Z}$.

- Clearly strict stationarity with finite second moments implies weak stationarity. Intuitively it implies that the graphs over two time intervals of equal length of a realisation of the time series should exhibit similar statistical characteristics. For example, the proportion of ordinates exceeding a given level x should roughly be the same.

Weak Stationarity does not imply strict stationarity

- **Example**:- Let $\{Z_t\}$ be a sequence of i.i.d. $N(0,1)$ random variables.
- Define a sequence of random variables $\{X_t\}$:

$$X_t = \begin{cases} Z_t & \text{if } t \text{ is odd,} \\ (Z_t^2 - 1)/2 & \text{if } t \text{ is even.} \end{cases}$$

- Note that $E[X_t] = 0$ and $Var[X_t] = 1$ for all t. So $\{X_t\}$ is weakly stationary.
- However, the distribution of X_t differs when t is odd from that when t is even.
- Thus $\{X_t\}$ is not strictly stationary.

AUTOCORRELATION (ASSUMING STATIONARITY)

- Let $\{X_t\}$ be a stationary time series with autocovariance function:

$$\gamma_x(h) = Cov[X_{t+h}, X_t] \text{ for any } t \in \mathbb{Z}.$$

The autocorrelation function (ACF) of $\{X_t\}$ is defined by

$$\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = Cor(X_{t+h}, X_t).$$

- An autocorrelation function $\rho_x(h)$ has all the properties of an autocovariance function and additionally satisfies $\rho_x(0) = 1$. In particular $\rho(\cdot)$ is an autocorrelation function of a stationary process iff it is an autocovariance function with $\rho(0) = 1$.

Some Basic Properties of $\gamma_x(\cdot)$

- **Theorem**:- For any stationary $\{X_t,\ t \in \mathbb{Z}\}$ the following relations hold:

  1. $\gamma_x(0) \geq 0$,
  2. $|\gamma_x(h)| \leq \gamma_x(0)$,
  3. $\gamma_x(h) = \gamma_x(-h)$ ie. γ_x is an even function.

  Proof:- 1. $\gamma_x(0) = Var[X_t] \geq 0$, by definition.

  2. Clearly $|\rho_x(h)| \leq 1$, so $|\gamma_x(h)| \leq \gamma_x(0)$.
  3. $\gamma_x(h) = Cov[X_{t+h}, X_t] = Cov[X_t, X_{t+h}] = \gamma_x(-h)$. $\square$

- **Definition**:- A real valued function κ defined on the integers is non-negative definite if $\sum_{i,j=1}^{n} a_i \kappa(i-j) a_j \geq 0$ for all positive integers n and vectors $a = (a_1, \ldots, a_n)^T$ with real-valued components a_i.

- **Theorem**:- A real-valued function defined on the integers is the autocovariance function of a stationary time series iff it is even and non-negative definite.

EXAMPLES

1. I.I.D. Noise: If $\{X_t\}$ is an IID Noise and $E[X_t^2] = \sigma^2 < \infty$, then clearly $\{X_t\}$ is stationary. Further by independence.

$$\gamma_x(h) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{o.w.} \end{cases}$$

   Also since for any k and h, $(X_1, X_2, \ldots, X_k)^T$ has the same distribution as $(X_{1+h}, X_{2+h}, \ldots, X_{k+h})^T$. So $\{X_t\}$ is strictly stationary.

2. White noise: If $\{X_t\}$ are uncorrelated random variables, each with zero mean and finite variance σ^2, then clearly $\{X_t\}$ is stationary. Such a sequence is referred to as white noise. White noise does not require X_t to be identically distributed. So $\{X_t\}$ may not be strictly stationary.

3. Random Walk: Suppose $\{X_t\}$ is an IID Noise. Let $S_t = X_1 + \cdots + X_t$, for $t = 1, 2, \ldots$. Clearly $E[X_t] = 0$, $Var[X_t] = E[X_t^2] = t\sigma^2 < \infty$, for all t.

   For any integer $h > 0$, $\gamma_S(h) = Cov[S_{t+h}, S_t] = Cov[S_t + X_{t+1} + \cdots + X_{t+h}, S_t]$.

   Now since S_t is independent of X_{t+h}, for all integer $h > 0$, $\gamma_S(h) = Cov[S_t, S_t] = t\sigma^2$. So the sequence $\{S_t\}$ is not weakly stationary.

FIRST-ORDER MOVING AVERAGE OR $MA(1)$ PROCESS

- Consider the series defined by the equation:
$$X_t = Z_t + \theta Z_{t-1}, \ t = 0, \pm 1, \ldots$$
where $\{Z_t\}$ is a $WN(0, \sigma^2)$ and θ is a real valued constant.

- Clearly, $E[X_t] = 0$.

- $Var[X_t^2] = Var[Z_t] + \theta^2 Var[Z_{t-1}] = \sigma^2 + \theta^2\sigma^2 = \sigma^2(1+\theta^2) < \infty$.

- For the auto-covariance function we not that:
$$\begin{aligned}\gamma_x(-1) &= Cov[X_t, X_{t+1}] = Cov[Z_t + \theta Z_{t-1}, Z_{t+1} + \theta Z_t] \\ &= Cov[Z_t, \theta Z_t] = \theta Var[Z_t] = \theta\sigma^2.\end{aligned}$$

- For any integer $|h| > 1$, $\gamma_x(-h) = Cov[X_t, X_{t+h}] = Cov[Z_t + \theta Z_{t-1}, Z_{t+h} + \theta Z_{t+h-1}] = 0$.

- Thus the ACF functions for $MA(1)$ process looks like:
$$\rho_x(h) = \begin{cases} 1, & \text{if } h = 0, \\ \theta/(1+\theta^2), & \text{if } h = \pm 1, \\ 0, & \text{if } |h| > 1. \end{cases}$$

FIRST-ORDER AUTOREGRESSION OR $AR(1)$ PROCESS

- Let $\{X_t\}$ is a stationary series satisfying the equations

$$X_t = \phi X_{t-1} + Z_t, \;\; t = 0, \pm 1, \ldots$$

where $\{Z_t\}$ is a $WN\left(0, \sigma^2\right)$, $|\phi| < 1$ and Z_t is uncorrelated with X_s for each $s < t$.

- Taking expectation on both sides and using $E[Z_t] = 0$ we get,

$$E[X_t] = \phi E[X_{t-1}] + E[Z_t] \;\Rightarrow\; (1 - \phi)E[X_t] = E[Z_t] \;\Rightarrow\; E[X_t] = 0.$$

- Now for an integer $h > 0$ and since $\gamma_x(-h) = \gamma_x(h)$, we note that,

$$\begin{aligned}\gamma_x(-h) =& Cov[X_t, X_{t-h}] = Cov[\phi X_{t-1} + Z_t, X_{t-h}] \\ =& \phi Cov[X_{t-1}, X_{t-h}] + Cov[Z_t, X_{t-h}] = \phi\gamma_x(h-1) + 0 = \phi\gamma_x(h-1).\end{aligned}$$

- So for $h \neq 0$, it follows that $\gamma_x(h) = \phi^{|h|}\gamma_x(0)$. Thus the ACF is $\rho_x(h) = \phi^{|h|}$, $h = 0, \pm 1, \ldots$.

- Since Z_t is uncorrelated with X_{t-1}, it follows that $Cov[X_t, Z_t] = Cov[\phi X_{t-1} + Z_t, Z_t] = Var[Z_t] = \sigma^2$. We also get $\gamma_x(0) = \sigma^2/(1 - \phi^2)$.

ACF PLOTS

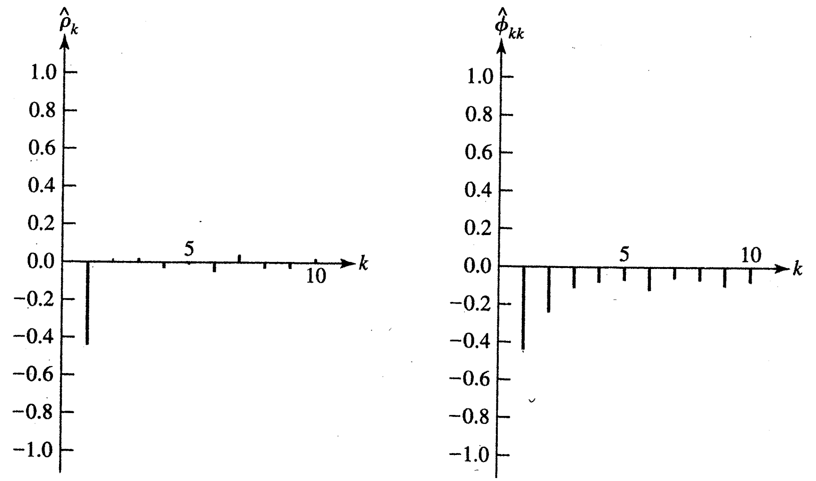

FIGURE 3.11 Sample ACF and sample PACF of a simulated MA(1)

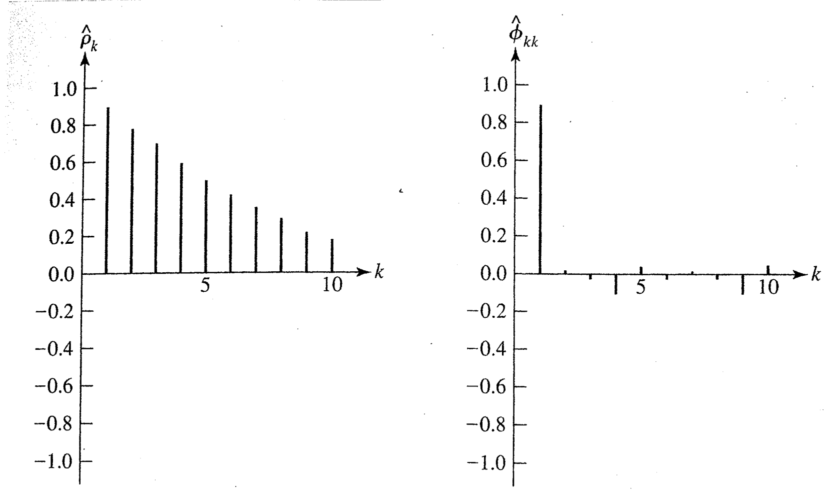

FIGURE 3.3 Sample ACF and sample PACF of a simulated AR(1)

Partial Autocorrelation (1)

- The autocovariance function is useful to determine the marginal dependence between the X_t and X_{t+h} for some integer t and h.

- However, often That is not enough. One requires some idea about the conditional dependence between X_t and X_{t+h} as well.

- This is measured by the partial autocorrelation which we describe below.

- In simple words, the partial autocovariance between X_t and X_{t+h} is defined as

$$Cov[X_t, X_{t+h}|X_{t+1}, \ldots, X_{t+h-1}].$$

- It is the conditional covariance of X_t and X_{t+h} given the values of X_{t+1}, ..., X_{t+h-1}.

- The partial autocorrelation is similarly defined as

$$\begin{aligned} & Corr[X_t, X_{T+h}|X_{t+1}, \ldots, X_{t+h-1}] \\ = & \frac{Cov[X_t, X_{t+h}|X_{t+1}, \ldots, X_{t+h-1}]}{\sqrt{Var[X_t|X_{t+1}, \ldots, X_{t+h-1}]Var[X_{t+h}|X_{t+1}, \ldots, X_{t+h-1}]}}. \end{aligned}$$

Partial Autocorrelation (2)

- In whatever follows below we assume $\{X_t\}$ is stationary.
- There are several interpretation of Partial autocorrelations.
- The one which leads to easy computation can be obtained by considering the following regression problem.
- Suppose we regress X_{t+h} on X_{t+h-1}, ..., X_t. That is we consider,
$$X_{t+h} = \phi_{h1}X_{t+h-1} + \phi_{h2}X_{t+h-2} + \cdots + \phi_{hh}X_t + Z_{t+h},$$
where ϕ_{hi} denotes the ith regression coefficient and Z_{t+h} is a white noise with mean 0 and uncorrelated with X_{t+h-j}, for all $j = 1, 2, \ldots, h$.
- By multiplying both sides by X_{t+h-j} on both sides and taking expectation we get,
$$\gamma_x(j) = \phi_{h1}\gamma_x(j-1) + \phi_{h2}\gamma_x(j-2) + \cdots + \phi_{hh}\gamma_x(j-k).$$
Here we use that γ_x is even.
- Dividing both sides by $\gamma_x(0)$ we see that,
$$\rho_x(j) = \phi_{h1}\rho_x(j-1) + \phi_{h2}\rho_x(j-2) + \cdots + \phi_{hh}\rho_x(j-k).$$

- The equation holds for all $j = 1, 2, \ldots, k$. So we have the following system of equations:

$$\begin{aligned}
\gamma_x(1) &= \phi_{h1}\rho_x(0) + \phi_{h2}\rho_x(1) + \cdots + \phi_{hh}\rho_x(k-1) \\
\gamma_x(2) &= \phi_{h1}\rho_x(1) + \phi_{h2}\rho_x(2) + \cdots + \phi_{hh}\rho_x(k-2) \\
&\vdots \\
\gamma_x(k) &= \phi_{h1}\rho_x(k-1) + \phi_{h2}\rho_x(k-2) + \cdots + \phi_{hh}\rho_x(0).
\end{aligned}$$

- The partial autocorrelation function $\alpha_x(\cdot)$ of $\{X_t\}$ is given by

$$\alpha_x(h) = \begin{cases} 1, & \text{if } h = 0 \\ \phi_{hh}, & \text{if } h \geq 1, \end{cases}$$

  where ϕ_{hh} is the solution of the above set of simultaneous equations.

- So we need to solve the set of simultaneous equations above, which will give us ϕ_{h1}, ϕ_{h2}, ..., ϕ_{hh}.

- Intended $\alpha_x(h)$ is the value ϕ_{hh} we obtain.

PARTIAL AUTOCORRELATION (4)

- The solution can be obtained using the Cramer's rule:
- Thus we get

$$\phi_{hh} = \frac{\begin{vmatrix} 1 & \rho_x(1) & \rho_x(2) & \cdots & \rho_x(h-2) & \rho_x(1) \\ \rho_x(1) & 1 & \rho_x(1) & \cdots & \rho_x(h-3) & \rho_x(2) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_x(h-1) & \rho_x(h-2) & \rho_x(h-3) & \cdots & \rho_x(1) & \rho_x(h) \end{vmatrix}}{\begin{vmatrix} 1 & \rho_x(1) & \rho_x(2) & \cdots & \rho_x(h-2) & \rho_x(h-1) \\ \rho_x(1) & 1 & \rho_x(1) & \cdots & \rho_x(h-3) & \rho_x(h-2) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_x(h-1) & \rho_x(h-2) & \rho_x(h-3) & \cdots & \rho_x(1) & 1 \end{vmatrix}}.$$

- Here $|A|$ means the determinant of matrix A.
- Note that $\alpha_x(1)$ is obtained by regressing X_{t+1} on X_t. So the corresponding $\phi_{11} = \rho_x(1)$.
- So $\alpha_x(1) = \rho_x(1)$.

- For $\alpha_x(2)$ we consider the regression problem

$$X_{t+2} = \phi_{21} X_{t+2-1} + \phi_{22} X_t + Z_{t+2}.$$

- By following the same way as before, it follows that

$$\alpha_x(2) = \phi_{22} = \frac{\begin{vmatrix} 1 & \rho_x(1) \\ \rho_x(1) & \rho_x(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho_x(1) \\ \rho_x(1) & 1 \end{vmatrix}}.$$

- Similarly we get

$$\alpha_x(3) = \phi_{33} = \frac{\begin{vmatrix} 1 & \rho_x(1) & \rho_x(1) \\ \rho_x(1) & 1 & \rho_x(2) \\ \rho_x(2) & \rho_x(1) & \rho_x(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho_x(1) & \rho_x(2) \\ \rho_x(1) & 1 & \rho_x(1) \\ \rho_x(2) & \rho_x(1) & 1 \end{vmatrix}}.$$

Partial Autocorrelation (6)

- The interpretation of the partial autocovariance that follows from the definition is that it is the covariance between X_t and X_{t+h} when we take the effect of X_{t+1}, ..., X_{t+h-1} away from both of them.

- Suppose we define

$$\hat{X}_{t+h} = a_1 X_{t+h-1} + a_2 X_{t+h-2} + \cdots + a_{h-1} X_{t+1}.$$

and

$$\hat{X}_t = b_1 X_{t+1} + b_2 X_{t+2} + \cdots + b_{t+h-1} X_{t+h-1}.$$

where a_i and b_i are the coefficients obtained by minimising $E[X_{t+h} - \hat{X}_{t+h}]^2$ and $E[X_t - \hat{X}_t]^2$ respectively.

- By definition

$$\alpha_x(h) = \frac{Cov[(X_{t+h} - \hat{X}_{t+h})(X_t - \hat{X}_t)]}{\sqrt{Var[X_{t+h} - \hat{X}_{t+h}]Var[X_t - \hat{X}_t]}}.$$

- It can be shown that $\alpha_x(h)$ is equal to ϕ_{hh} from the previous slides.

First-order moving average or $AR(1)$ Process

- Recall that we defined an $AR(1)$ process as
$$X_t = \phi X_{t+1} + Z_t,$$
where Z_t is $WN(0, \sigma^2)$.

- To find $\alpha_x(1)$ we consider the regression problem:
$$\widehat{X}_{t+1} = a_1 X_t$$

- Clearly the best value of a_1 would be ϕ. So $\alpha(1) = \phi$. We could have also used $\alpha(1) = \gamma_x(1) = \phi$.

- To find $\alpha_x(2)$ we note that if we fix the value of X_{t+1}, the value of X_{t+2} does not depend on X_t.

- Thus given X_{t+1}, X_{t+2} is independent of X_t. So by definition $\alpha_x(2) = 0$.

- By the same argument for any integer $h > 1$, $\alpha_x(h) = 0$.

- Thus for an $AR(1)$ process the PACF $\alpha_x(h)$ has a spike at $h = 1$. Otherwise it is 0.

FIRST-ORDER AUTOREGRESSION OR $MA(1)$ PROCESS

- For a $MA(1)$ process
$$X_t = Z_t + \theta Z_{t-1},$$
we can show that the PACF at lag h has the form
$$\alpha_x(h) = -(-\theta)^h/(1 + \theta^2 + \cdots + \theta^{2h}).$$

- So the PACF for a $MA(1)$ process decreases exponentially with lag h.

- It does not drop abruptly after lag 1 as for the ACF.

ACF AND PACF PLOTS FOR $MA(1)$ AND $AR(1)$ PROCESSES

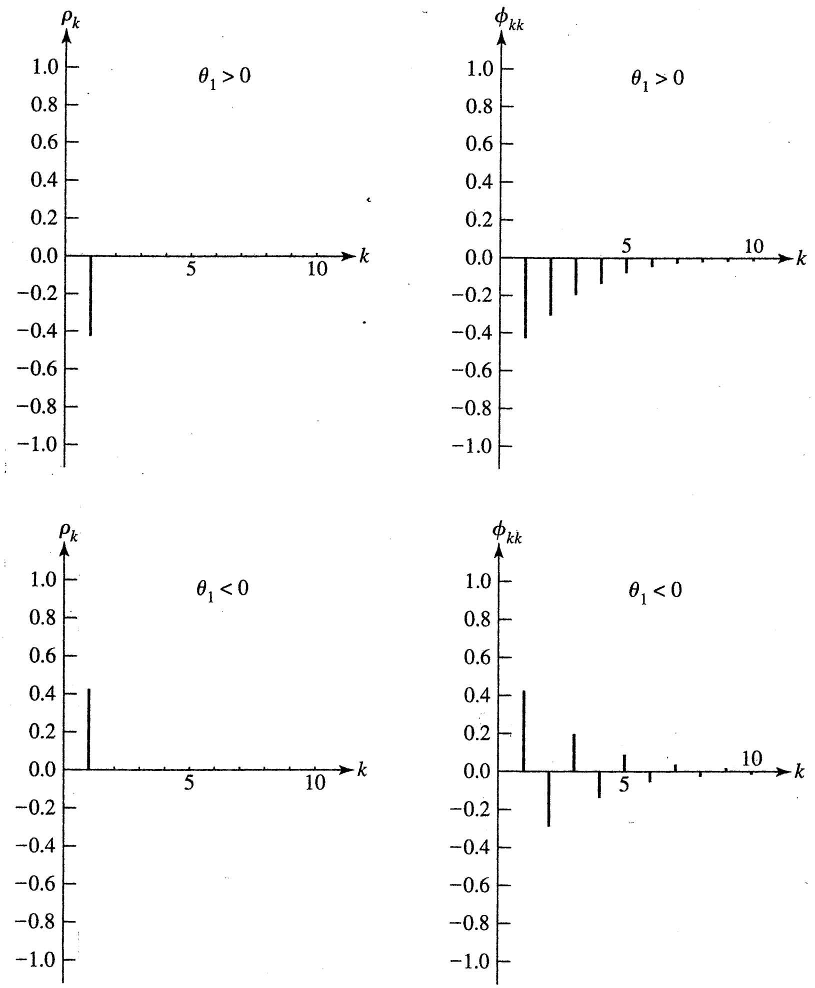

FIGURE 3.10 ACF and PACF of MA(1) processes: $\dot{Z}_t = (1 - \theta_1 B)a_t$.

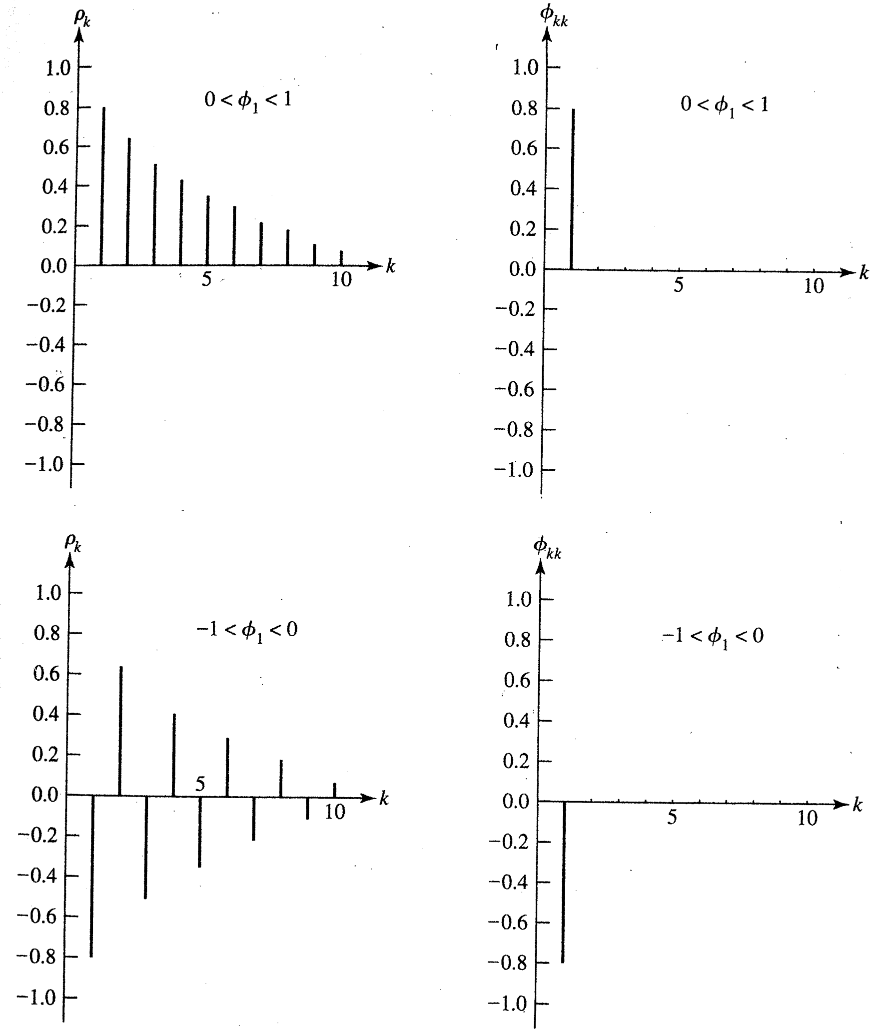

FIGURE 3.1 ACF and PACF of the AR(1) process: $(1 - \phi_1 B)\dot{Z}_t = a_t$.

PARTIAL AUTOCORRELATION (EXTRA 1)

- We show that the two definitions of PACF, ie $\alpha_Z(h)$ we defined before are the same.
- Recall that we considered two predictive equations for $\hat{X}_{t+h}$ and $\hat{X}_t$ depending on $X_{t+1}, \ldots, X_{t+h-1}$ as follows.

$$\hat{X}_{t+h} = a_1 X_{t+h-1} + a_2 X_{t+h-2} + \cdots + a_{h-1} X_{t+1}.$$

$$\hat{X}_t = b_1 X_{t+1} + b_2 X_{t+2} + \cdots + b_{t+h-1} X_{t+h-1}.$$

- The coefficients a, and b are determined by minimising the least square errors.
- That is we minimise:

$$E[(\hat{X}_{t+h} - X_{t+h})^2] = E[(X_{t+h} - a_1 X_{t+h-1} - a_2 X_{t+h-2} - \cdots - a_{h-1} X_{t+1})^2].$$

and $E[(X_t - \hat{X}_t)^2] = E[(X_t - b_1 X_{t+1} - b_2 X_{t+2} - \cdots - b_{t+h-1} X_{t+h-1})^2]$.

PARTIAL AUTOCORRELATION (EXTRA 2)

- We first consider $E[(\hat{x}_{t+h} - x_{t+h})^2]$.
- By differentiating w.r.t a_i, $i = 1, 2, \cdots, h-1$ and equating the expectation to 0, we get the following set of equations.

$$\gamma_x(i) = a_1 \gamma_x(i-1) + a_2 \gamma_x(i-2) + \cdots + a_{h-1}\gamma_x(i-h+1).$$

or

$$\rho_x(i) = a_1 \rho_x(i-1) + a_2 \rho_x(i-2) + \cdots + a_{h-1} \rho_x(i-h+1).$$

the $1 \le i \le k-1$.

- Thus we solve the equation.

$$\begin{bmatrix} \rho_x(1) \\ \rho_x(2) \\ \vdots \\ \rho_x(k-1) \end{bmatrix} = \begin{bmatrix} 1 & \rho_x(1) & \rho_x(2) & \cdots & \rho_x(k-2) \\ \rho_x(1) & 1 & \rho_x(1) & \cdots & \rho_x(k-3) \\ \vdots & & & & \vdots \\ \rho_x(k-2) & \rho_x(k-3) & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k-1} \end{bmatrix}.$$

PARTIAL AUTOCORRELATION (EXTRA 3)

- Similarly for $E\left[(X_t - \hat{X}_t)^2\right]$ we get.
- $$\begin{bmatrix} \rho_Z(1) \\ \rho_Z(2) \\ \vdots \\ \rho_Z(k-1) \end{bmatrix} = \begin{bmatrix} 1 & \rho_Z(1) & \rho_Z(2) & \cdots & \rho_Z(k-2) \\ \rho_Z(1) & 1 & \rho_Z(1) & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \rho_Z(k-2) & \rho_Z(k-3) & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_{k-1} \end{bmatrix}.$$
- This means $a_i = b_i \quad \forall i \in \{1, 2, \cdots, k-1\}$.
- Thus the optimal regression coefficients are same.
- Now we need to compute the variance and the covariance.
- First note that since we equated the derivative of $E\left[(\hat{X}_{t+h} - \hat{X}_t)^2\right] = 0$, we get.

$$E\left[X_{t+h-i}\left(X_{t+h} - a_1 X_{t+h-1} - a_2 X_{t+h-2} - \cdots - a_{h-1} X_{t+1}\right)\right] = 0$$

$\forall i = 1, 2, \cdots, h-1$. ..... (A).

Partial Autocorrelation (extra 4)

- Now $\text{Var}[(\hat{x}_{t+h} - x_{t+h})] = E[(x_{t+h} - a_1 x_{t+h-1} - \cdots - a_{h-1} x_{t+1})^2]$.

$$= E[x_{t+h}(x_{t+h} - a_1 x_{t+h-1} - \cdots - a_{h-1} x_{t+1})]$$

$$- a_1 E[x_{t+h-1}(x_{t+h} - a_1 x_{t+h-1} - \cdots - a_{h-1} x_{t+1})]$$

$$- \cdots - a_{h-1} E[x_{t+1}(x_{t+h} - a_1 x_{t+h-1} - \cdots - a_{h-1} x_{t+1})] .$$

$$= E[x_{t+h}(x_{t+h} - a_1 x_{t+h-1} - \cdots - a_{h-1} x_{t+1})] + 0 \cdots \text{by (A)}.$$

$$= \gamma_x(0) - a_1 \gamma_x(1) - \cdots - a_{h-1} \gamma_x(h-1) .$$

- By stationarity $\text{Var}[(\hat{x}_t - x_t)] = \gamma_x(0) - a_1 \gamma_x(1) - \cdots - a_{h-1} \gamma_x(h-1)$.
- $\text{Cov}[(\hat{x}_{t+h} - x_{t+h})(\hat{x}_t - x_t)]$

$$= E[x_{t+h}(x_t - a_1 x_{t+1} - \cdots - a_{h-1} x_{t+h-1})] \cdots\cdots (*)$$

$$= \gamma_x(h) - a_1 \gamma_x(h-1) - \cdots a_{h-1} \gamma_x(1) .$$

- the equality of $(*)$ follows from the relation anologous to (A).

- So by definition.

$$\alpha_x(h) = \frac{\gamma_x(h) - a_1\gamma_x(h-1) - \cdots - a_{h-1}\gamma_x(1)}{\gamma_x(0) - a_1\gamma_x(1) - \cdots - a_{h-1}\gamma_x(h-1)} = \frac{\rho_x(h) - a_1\rho_x(h-1) - \cdots - a_{h-1}\rho_x(1)}{1 - a_1\rho_x(1) - \cdots - a_{h-1}\rho_x(h-1)}.$$

- Now we have to plug in $a_1, a_2, \cdots, a_{h-1}$.
- From the equations determining a_i, using Cramer's rule we get.

$$a_i = \frac{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_{i-2} & \rho_1 & \rho_i & \cdots & \rho_{h-2} \\ \rho_1 & 1 & \cdots & \rho_{i-3} & \rho_2 & \rho_2 & \cdots & \rho_{h-3} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ \rho_{h-2} & \rho_{h-3} & \cdots & \rho_{h-i} & \rho_{h-1} & \rho_{h-i-2} & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_{i-2} & \rho_{i-1} & \rho_i & \cdots & \rho_{h-2} \\ \rho_1 & 1 & \cdots & \rho_{i-3} & \rho_{i-2} & \rho_{i-1} & \cdots & \rho_{h-3} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \rho_{h-2} & \rho_{h-3} & \cdots & \rho_{h-i} & \rho_{h-i-1} & \rho_{h-i-2} & \cdots & 1 \end{vmatrix}}.$$

- Now by substituting a_i in the expression of $\alpha_z(h)$ and multiplying the ~~deter~~ denominator and numerator with the determinant

$$\begin{vmatrix} 1 & \rho_z(1) & \cdots & \rho_z(h-2) \\ \rho_z(1) & 1 & \cdots & \rho_z(h-3) \\ \vdots & & & \vdots \\ \rho_z(h-2) & \rho_z(h-3) & \cdots & 1 \end{vmatrix}$$

we get

$$\alpha_z(h) = \frac{\begin{vmatrix} 1 & \rho_z(1) & \rho_z(2) & \cdots & \rho_z(h-2) & \rho_z(1) \\ \rho_z(1) & & & & \vdots & \vdots \\ \vdots & & & & & \\ \rho_z(h-1) & \cdots & & \cdots & \rho_z(1) & \rho_z(h) \end{vmatrix}}{\begin{vmatrix} 1 & \rho_z(1) & \cdots & \rho_z(h-1) \\ \rho_z(1) & 1 & & \vdots \\ \vdots & & & \vdots \\ \rho_z(h-1) & \rho_z(h-2) & \cdots \rho_z(1) & 1 \end{vmatrix}}.$$