Time-Invariant Linear filters

- The process $\{Y_t\}$ is the output of a linear filter $C = \{c_{t,k}, t,k = 0, \pm 1, \cdots\}$ applied to an input process $\{X_t\}$ if.

$$Y_t = \sum_{k=-\infty}^{\infty} c_{t,k} X_k , \quad t = 0, \pm 1, \cdots\cdots.$$

- The filter is said to be time-invariant if the weights $c_{t,t-k}$ are independent of t ie., if

$$c_{t,t-k} = \psi_k .$$

- If the filter is time-invariant then $Y_t = \sum_{k=-\infty}^{\infty} \psi_k X_{t-k}$ and $Y_{t-s} = \sum_{k=-\infty}^{\infty} \psi_k X_{t-s-k}$. So, the time-shifted process Y_{t-s} is obtained from X_{t-s} by using the ~~several~~ same filter which produces Y_t from X_t.

- The Time-Invariant Linear filter is said to be causal if $\psi_j = 0$ for $j < 0$. In this case Y_t is expressible in terms only of X_s, $s \le t$.

Spectral decomposition ~~..~~ with time-Invariant Linear filters.

- Theorem 3. Let $\{X_t\}$ be a stationary time series with mean zero and spectral density $f_x(\lambda)$. Suppose that $\psi = \{\psi_j, j = 0, \pm 1, \cdots\}$ is an absolutely summable time-invariant Linear filter. Then the time series

$$Y_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j}$$

is stationary with mean zero and spectral density.

$$f_Y(\lambda) = \psi(e^{i\lambda})\psi(e^{-i\lambda}) f_X(\lambda) = |\psi(e^{-i\lambda})|^2 f_X(\lambda),$$

where $\psi(e^{-i\lambda}) = \sum_{k=-\infty}^{\infty} \psi_k e^{-ik\lambda}$.

- Proof:- We know that, ~~..~~ $\{Y_t\}$ is stationary with mean 0 and auto covariance function

$$\gamma_X(h) = \sum_{j,k=-\infty}^{\infty} \psi_j \psi_k \gamma_X(h+k-j).$$

Since $\{X_t\}$ has spectral density $f_X(\lambda)$, we have.

$$\gamma_X(h+k-j) = \int_{-\pi}^{\pi} e^{i(h-j+k)\lambda} f(\lambda)\,d\lambda,$$

Now, we get,

$$\gamma_y(h) = \sum_{j,k=-\infty}^{\infty} \varphi_j \varphi_k \int_{-\pi}^{\pi} e^{i(h-j+k)\lambda} f_X(\lambda)\, d\lambda .$$

$$= \int_{-\pi}^{\pi} e^{ih\lambda} \left(\sum_{j=-\infty}^{\infty} \varphi_j e^{-ij\lambda} \right) \left(\sum_{k=-\infty}^{\infty} \varphi_k e^{ik\lambda} \right) f_x(\lambda)\, d\lambda .$$

$$= \int_{-\pi}^{\pi} e^{ih\lambda} \varphi(e^{-i\lambda}) \varphi(e^{i\lambda}) f_x(\lambda)\, d\lambda .$$

Now it follows that

$$f_y(\lambda) = \varphi(e^{-i\lambda}) \varphi(e^{i\lambda}) f_x(\lambda) .$$

$$= |\varphi(e^{-i\lambda})|^2 f_x(\lambda) . \qquad \square$$

- The function $\varphi(e^{-i\cdot})$ is called the transfer function of the filter. The squared modulus $|\varphi(e^{-i\cdot})|^2$ is ~~refered~~ called the power transfer function of the filter.

Spectral distribution of an ARMA process.

- Let $\{X_t\}$ be an ARMA(p,q) process given by

$$\phi(B)X_t = \theta(B)Z_t.$$

- Theorem 4. If $\{X_t\}$ is a causal ARMA(p,q) process satisfying $\phi(B)X_t = \theta(B)Z_t$, then

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}\cdot\frac{\theta(e^{-i\lambda})\,\theta(e^{i\lambda})}{\phi(e^{-i\lambda})\,\phi(e^{i\lambda})} = \frac{\sigma^2}{2\pi}\cdot\frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}.$$

Proof:- Note that $\{X_t\}$ can be obtained from $\{Z_t\}$ by application of the time-invariant linear transfer function; $\psi(e^{-i\lambda}) = \theta(e^{-i\lambda})/\phi(e^{-i\lambda})$.

Now the spectral density of $\{Z_t\}$ is $f_Z(\lambda) = \sigma^2/2\pi$.

So using theorem 3. we get $f_X(\lambda) = \frac{\sigma^2}{2\pi}\,\frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}$. $\square$

- Note that, the spectral density is a ratio of two trigonometric polynomials.
- The spectral density is called rational.

Example: AR(2) and ARMA(1,1)

- For AR(2) process the spectral density is immediate.

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}\left(1-\phi_1 e^{-i\lambda}-\phi_2 e^{-2i\lambda}\right)^{-1}\left(1-\phi_1 e^{i\lambda}-\phi_1 e^{2i\lambda}\right)^{-1}.$$

$$= \frac{\sigma^2}{2\pi}\left(1+\phi_1^2+2\phi_2+\phi_2^2+2(\phi_1\phi_2-\phi_1)\cos\lambda-4\phi_2\cos^2\lambda\right)^{-1}.$$

- For ARMA(1,1) process we know.

$$\psi(B) = \frac{1+\theta B}{1-\phi B}.$$

- So the spectral density is given that.

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}\cdot\frac{(1+\theta e^{-i\lambda})(1+\theta e^{i\lambda})}{(1-\phi e^{-i\lambda})(1-\phi e^{i\lambda})} = \frac{\sigma^2}{2\pi}\;\frac{1+\theta^2+2\theta\cos\lambda}{1+\phi^2+2\phi\cos\lambda}.$$

Spectral distribution of linear combination of sinusoids.

- For any arbitrary ~~complex~~ real stationary process, $\{X_t\}$ one can write.

$$X_t = \sum_{j=1}^{n} A(\lambda_j)\, e^{it\lambda_j} \quad ,$$

where $-\pi < \lambda_1 < \lambda_2 < \cdots < \lambda_n = \pi$ and $A(\lambda_1), \cdots, A(\lambda_n)$ are uncorrelated complex random variables such that.

$$E[A(\lambda_j)] = 0, \quad E[A(\lambda_j)\overline{A(\lambda_j)}] = \sigma_j^2 .$$

- For $\{X_t\}$ to be real-valued one can show that $A(\lambda_n)$ is real, $\lambda_j = -\lambda_{n-j}$ and $A(\lambda_j) = \overline{A(\lambda_{n-j})}$ for $j = 1, \cdots, n-1$.
- In particular we can write.

$$X_t = \sum_{j=1}^{n} \left(C(\lambda_j)\cos t\lambda_j - D(\lambda_j)\sin t\lambda_j \right) ,$$

where $A(\lambda_j) = C(\lambda_j) + i D(\lambda_j)$, $j = 1, 2, \cdots, n$ and $\cancel{D(\lambda_j) = 0}$. $D(\lambda_n) = 0$.

Connection to ~~spe~~ spectral distribution.

- The real-valued process is stationary implies

$$E[X_t]=0 \quad \text{and} \quad E[X_{t+h}\bar{X}_t] = \gamma_X(h) = \sum_{j=1}^{n} \sigma_j^2 e^{ih\lambda_j}.$$

Note that the ~~latter~~ RHS is independent of t.

- Define a distribution function

$$F(\lambda) = \sum_{j:\lambda_j \le \lambda} \sigma_j^2 .$$

- We can write $\gamma_X(h) = \sum_{j=1}^{n} \sigma_j^2 e^{ih\lambda_j} = \int_{(-\pi,\pi]} e^{ih\lambda} dF(\lambda)$.

- Note that $F(\lambda)$ as we have defined it is a discrete ~~jump~~ step function, so $dF(\lambda)$ has to be looked at carefully.
- $dF(\lambda)$ would be zero everywhere except at λ_j, where it is σ_j^2. That is, it would be ~~spik~~ full of spikes.
- The integral ~~can~~ can be ~~mote~~ explained as a Riemann-Stieltjes integral.

A bit more.

- Every zero-mean stationary process has a represtation as.

$$X_t = \int_{(-\pi,\pi]} e^{it\lambda} \, dZ(\lambda).$$

This integral is a stochastic integral.

- The corresponding auto-covariance function γ_X can be expressed as.

$$\gamma_X(h) = \int_{(-\pi,\pi]} e^{ih\lambda} \, dF(\lambda),$$

where F is a distribution function with $F(-\pi) = 0$ and $F(\pi) = \gamma_X(0)$.

The periodogram

- If $\{X_t\}$ is a stationary time series with autocovariance function $\gamma_X(\cdot)$ and spectral density $f_X(\lambda)$.
- $\hat{\gamma}_X(\cdot)$ of the observation $\{X_1, \cdots, X_n\}$ is an estimate of $\gamma_X(\cdot)$.
- Similarly, $2\pi \widehat{f_X(\lambda)}$ can be estimated using a periodogram $I_n(\cdot)$ of the observations.
- Suppose we define $\omega_k = \frac{2\pi k}{n}$, $k = -\left[\frac{n-1}{2}\right], \cdots, \left[\frac{n}{2}\right]$, where $[y]$ denotes the largest integer less than or equal to y.
- Let F_n be the set of such ω_k and call it the Fourier frequencies associated with sample size n.
- Note that each $\omega_k \in (-\pi, \pi]$.

Periodogram (contd.)

- Let $e_k = \frac{1}{\sqrt{n}} [e^{i\omega_k}, \ldots, e^{ni\omega_k}]^T$, $k = -[\frac{n-1}{2}], \ldots, [\frac{n}{2}]$.

- Note that $e_j^* e_k = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j \neq k. \end{cases}$

- ~~Also $e_j^* e_j = 1$~~ Thus e_k, $k \in \{-[\frac{n-1}{2}], \ldots, [\frac{n}{2}]\}$ forms a basis of the n-dimensional complex plane.

- Thus any n-dimensional complex number x we can write.

$$x = \sum_{k=-[\frac{n-1}{2}]}^{[n/2]} a_k e_k. \qquad \cdots\cdots \quad (2)$$

- In order to find a_k we note the orthogonality of the basis vectors. Then

$$e_k^* x = a_k = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{-it\omega_k}.$$

Discrete Fourier transform and Periodogram

- The sequence of numbers $\{a_k\}$ is called the discrete Fourier transform of the sequence $\{x_1, \cdots, x_n\}$.

- The t^{th} component in ② we note that,

$$X_t = \sum_{k=-[(n-1)/2]}^{[n/2]} a_k \left[\cos(\omega_k t) + i \sin(\omega_k t)\right], \quad t = 1, 2, \cdots, n.$$

- The periodogram of $\{x_1, x_2, \cdots, x_n\}$ is the function.

$$I_n(\lambda) = \frac{1}{n}\left|\sum_{t=1}^{n} x_t e^{-it\lambda}\right|^2.$$

- Note that if λ is one of the Fourier frequencies ω_k, $I_n(\omega_k) = |a_k|^2$.

- Further, $\displaystyle\sum_{t=1}^{n} |x_t|^2 = \sum_{k=-[(n-1)/2]}^{[n/2]} |a_k|^2 = \sum_{k=-[(n-1)/2]}^{[n/2]} I_n(\omega_k)$.

- The total variation in the observations is the total of periodogram.

Connection between $I_n(\lambda)$ and $f_x(\lambda)$.

Theorem 5. If $x_1, \cdots, x_n$ are ~~any~~ real numbers and ω_k is any of the non zero Fourier frequencies $2\pi k/n$ in $(-\pi, \pi]$ then.

$$I_n(\omega_k) = \sum_{|h|<n} \hat{\gamma}_x(h) e^{-ih\omega_k},$$

where $\hat{\gamma}_x(h)$ is the sample autocovariance function of $x_1, \cdots, x_n$.

Proof:- If $\omega_k \neq 0$, $\sum_{t=1}^{n} e^{-it\omega_k} = 0$. So we can centre the observations and consider $x_t - \bar{x}$. Now from the definition

$$I_n(\omega_k) = \frac{1}{n} \sum_{s=1}^{n} \sum_{t=1}^{n} (x_s - \bar{x})(x_t - \bar{x}) e^{-i(t-s)\omega_k}.$$

$$= \sum_{|h|<n} \hat{\gamma}_x(h) e^{-ih\omega_k}. \quad \square$$

Estimated spectrum of a white noise.

- If $Z_t \sim WN(0, \sigma^2)$, we have shown that
  $f_Z(\lambda) = \sigma^2 / 2\pi$, which is uniform.

- The sample versions look like this.

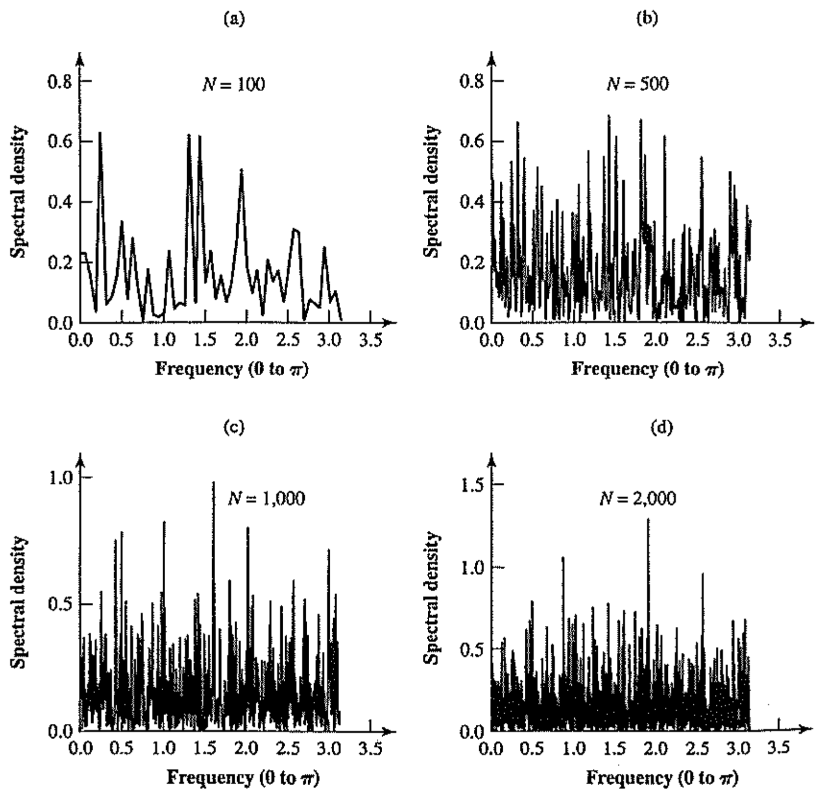

FIGURE 13.2 Sample spectrum of a white noise process.

Note that, the sample version of the spectrum is noisy and more importantly the noise does not reduce with the sample size.

The original spectrum is uniform.

Distribution of the Periodogram Ordinates.

- Result:- If $X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$, $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, where $Z_t \sim WN(0, \sigma^2)$,

and $\sum_{h=-\infty}^{\infty} |h| |\gamma_X(h)| < \infty$ holds, then for any collection of m distinct frequencies

$\omega_j \in (0, 1/2)$ with $\omega_{j:n} \longrightarrow \omega_j$

$$\frac{2\, I(\omega_{j:n})}{f(\omega_j)} \xrightarrow{d} \text{iid } \chi^2_2 .$$

provided $f(\omega_j) > 0$, for $j = 1, \cdots, m$.

- Note that for large n, $E[I(\omega_{j:n})] \approx f(\omega_j)$. However, $Var[I(\omega_{j:n})] \approx [f(\omega_j)]^2$ or $sd[I(\omega_{j:n})] \approx f(\omega_j)$. That is, the expectation and the standard deviations are of same order.
- Thus the noise swamps the signal.
- In order to estimate the spectrum, we need to smooth the spectrum using some smoother.