More general autoregressive processes. [AR(p)]

- The p-th order autoregressive process denoted AR(p) is given by

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots\cdots + \phi_p X_{t-p} + Z_t ,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and Z_t is uncorrelated with X_s, $\forall s < t$.

- <u>ACF function</u>. The autocorrelation function of a general AR(p) is a bit complicated. It is best expressed in terms of certain recursive equations.

Note that for $h > 0$.

$$X_{t-h} X_t = \phi_1 X_{t-h} X_{t-1} + \phi_2 X_{t-h} X_{t-2} + \cdots\cdots + \phi_p X_{t-h} X_{t-p} + X_{t-h} Z_t .$$

Taking expectation on both sides we get.

$$\gamma_x(h) = \phi_1 \gamma_x(h-1) + \phi_2 \gamma_x(h-2) + \cdots + \phi_p \gamma_x(h-p).$$

So we get a recursive equation interms of the autocorrelations as

$$\rho_x(h) = \phi_1 \rho_x(h-1) + \phi_2 \rho_x(h-2) + \cdots + \phi_p \rho_x(h-p). \qquad h > 0.$$

ACF of AR(p).

Suppose we define a polynomial of the form,

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3 - \cdots\cdots - \phi_p z^p .$$

It can be shown that if $\phi(z) = 0$ has distinct roots $\xi_1, \xi_2, \cdots, \xi_p$,

then the solution of the recursive equations involving ρ_x has solution of the form

$$\rho_x(h) = \alpha_1 \xi_1^{-h} + \alpha_2 \xi_2^{-h} + \cdots + \alpha_p \xi_p^{-h} ,$$

where $\alpha_1, \alpha_2, \cdots, \alpha_p$ are arbitrary constants.

Now by substituting this solutions in the recursive equations above one can uniquely find $\alpha_1, \cdots, \alpha_p$.

- <u>Upshot</u>: The ACF ρ_x for an AR(p) process tails off as a mixture of exponentials decay or damped sine waves depending on the roots of $\phi(z) = 0$.

PACF of AR(p).

- Recall that by definition a AR(p) process is given by.

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t .$$

- Note that if we fix $X_{t-1}, X_{t-2}, \ldots, X_{t-p}$, the variables X_t and X_{t-p-1} are independent. So by definition $\alpha_x(h) = 0 \quad \forall\ h > p$.

- For $h \geq p$, we can show that the linear best predictor of X_{h+1} in terms of $X_1, \ldots, X_h$ is given by.

$$\hat{X}_{h+1} = \phi_1 X_h + \phi_2 X_{h-1} + \cdots + \phi_p X_{h-p+1} .$$

So for $h = p$, we get ϕ_{hh} the coefficient of X_1 as ϕ_p.
So $\alpha_x(p) = \phi_p$.

- For $h < p$ we need to compute $\alpha_x(h)$ numerically from the formula above.

AR(2) process or Yule process

- The process is generated by the relation.

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t ,$$

$\{Z_t\} \sim WN(0, \sigma^2)$ and $Z_t \perp\!\!\!\perp X_s \quad \forall s < t$.

- In order to find the ACF function, let us write.

$$\phi_1 = \frac{1}{\xi_1} + \frac{1}{\xi_2} \quad \text{and} \quad \phi_2 = -\frac{1}{\xi_1 \xi_2} .$$

Then it can be shown that.

$$\gamma(h) = \frac{\sigma^2 \xi_1^2 \xi_2^2}{(\xi_1 \xi_2 - 1)(\xi_2 - \xi_1)} \left[(\xi_1^2 - 1)^{-1} \xi_1^{1-h} - (\xi_2^2 - 1)^{-1} \xi_2^{1-h} \right] .$$

Note that both ξ_1 and ξ_2 may be complex.

- The $\rho_x(\cdot)$ can be computed directly from the ~~pair of~~ recursive equations

$$\rho_x(h) = \phi_1 \rho_x(h-1) + \phi_2 \rho_x(h) . \qquad h \geq 1 .$$

AR (2) (contd.)

Thus we have a pair of equations

$$\rho_x(1) = \phi_1 + \phi_2 \rho_x(1)$$

$$\rho_x(2) = \phi_1 \rho_x(1) + \phi_2 .$$

By solving them we get.

$$\rho_x(1) = \frac{\phi_1}{1-\phi_2} .$$

$$\rho_x(2) = \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1-\phi_2} . .$$

Now $\rho_x(h)$ can be computed recursively from the previous equation..

More generally, consider the equation.

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = 0 .$$

AR (2) (contd.)

The quadratic has two solutions

$$\xi_1 = -\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2} \quad \text{and} \quad \xi_2 = -\frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}$$

Clearly $\xi_1 \xi_2 = -1/\phi_2$. Thus $\frac{1}{\xi_1} = -\xi_2\phi_2 = \frac{\phi_1 - \sqrt{\phi_1^2+4\phi_2}}{2\cancel{\phi_2}}$

and $\frac{1}{\xi_2} = -\xi_1\phi_2 = \frac{\phi_1 + \sqrt{\phi_1^2+4\phi_2}}{2\cancel{\phi_2}}$.

So by the formula mentioned before.

$$\rho_z(h) = b_1 \left\{ \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2\cancel{\phi_2}} \right\}^h + b_2 \left\{ \frac{\phi_1 - \sqrt{\phi_1^2+4\phi_2}}{2} \right\}^h .$$

If $\phi_1^2 + 4\phi_2 = 0$

$$\rho_z(h) = (b_1 + b_2)\left[\frac{\phi_1}{2}\right]^h .$$

b_1 and b_2 are arbitrary constant to be solved from previous two equations.

PACF for AR (2)

- PACF for AR(2) is simpler to calculate.

- We know that

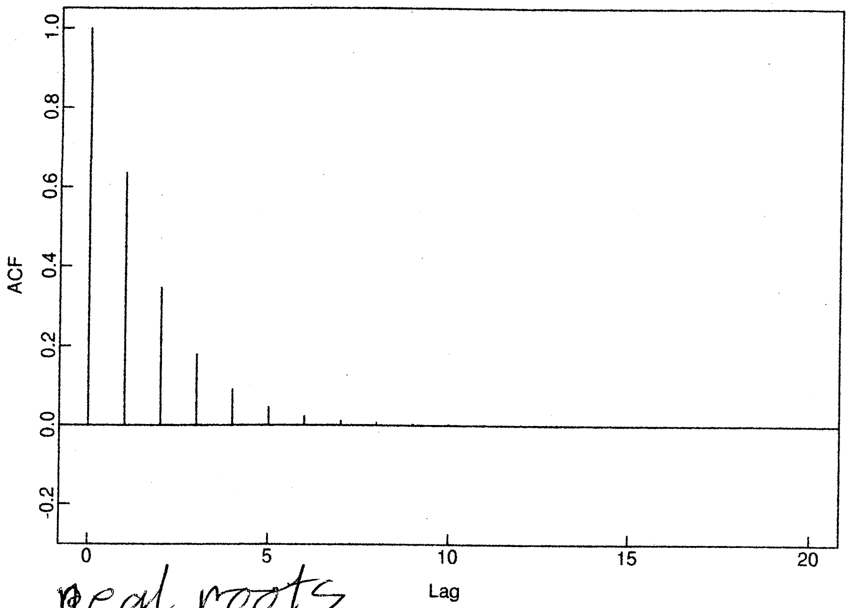
$$\alpha_x(h) = 0 \qquad \forall\ h > 2,$$

$$\alpha_x(2) = \phi_2$$

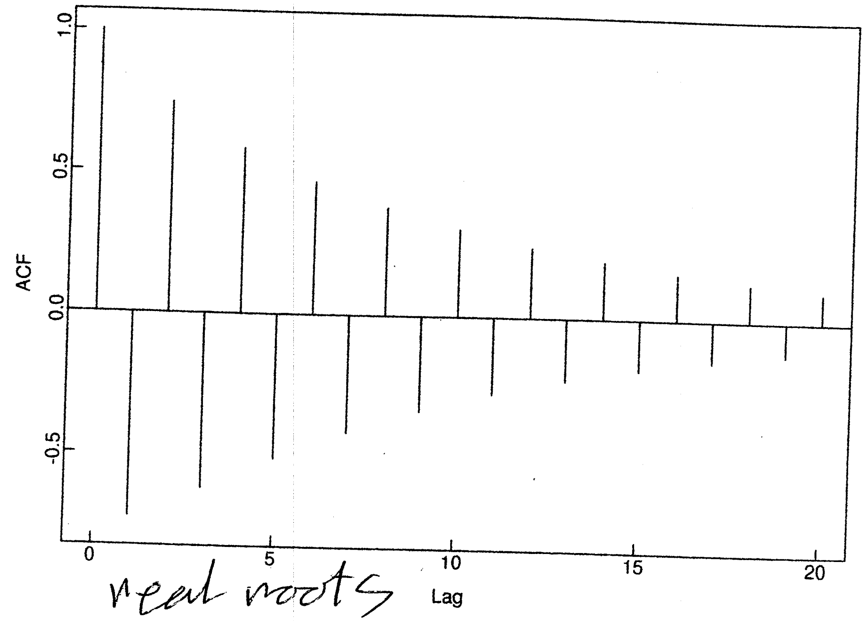
- By definition of PACF.

$$\alpha_x(1) = \rho_1 = \frac{\phi_1}{1 - \phi_2}.$$

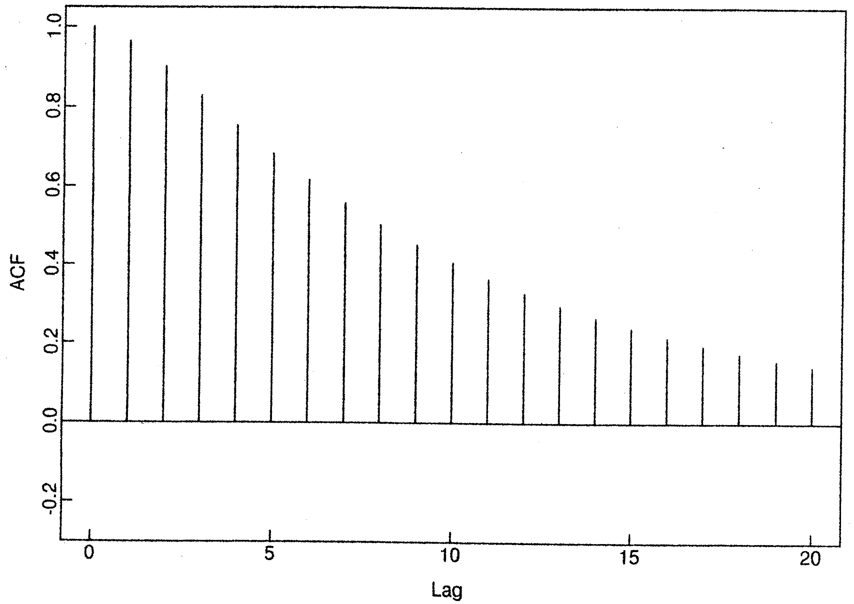
ACF of AR(2)



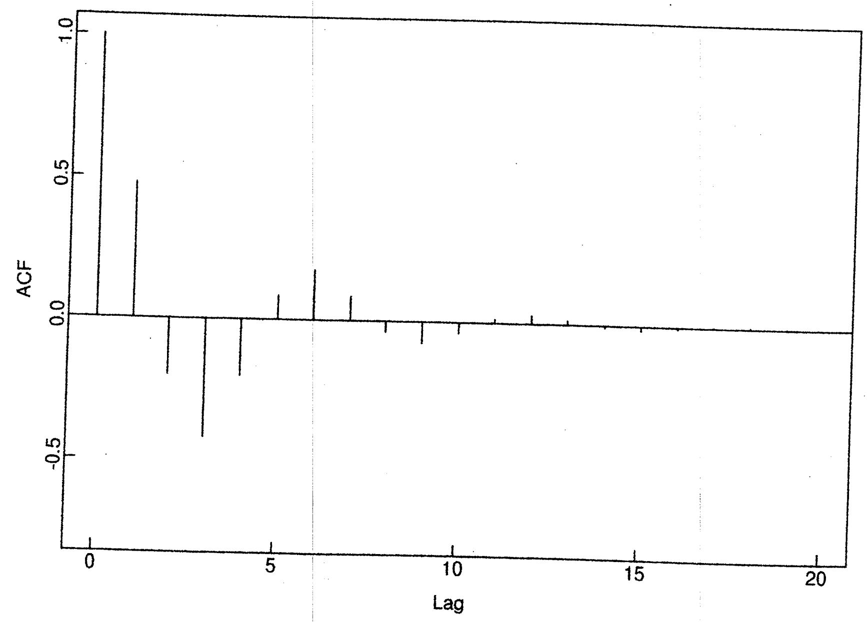
real roots



real roots



real roots



complex roots.

ACF and PACF for AR(2)

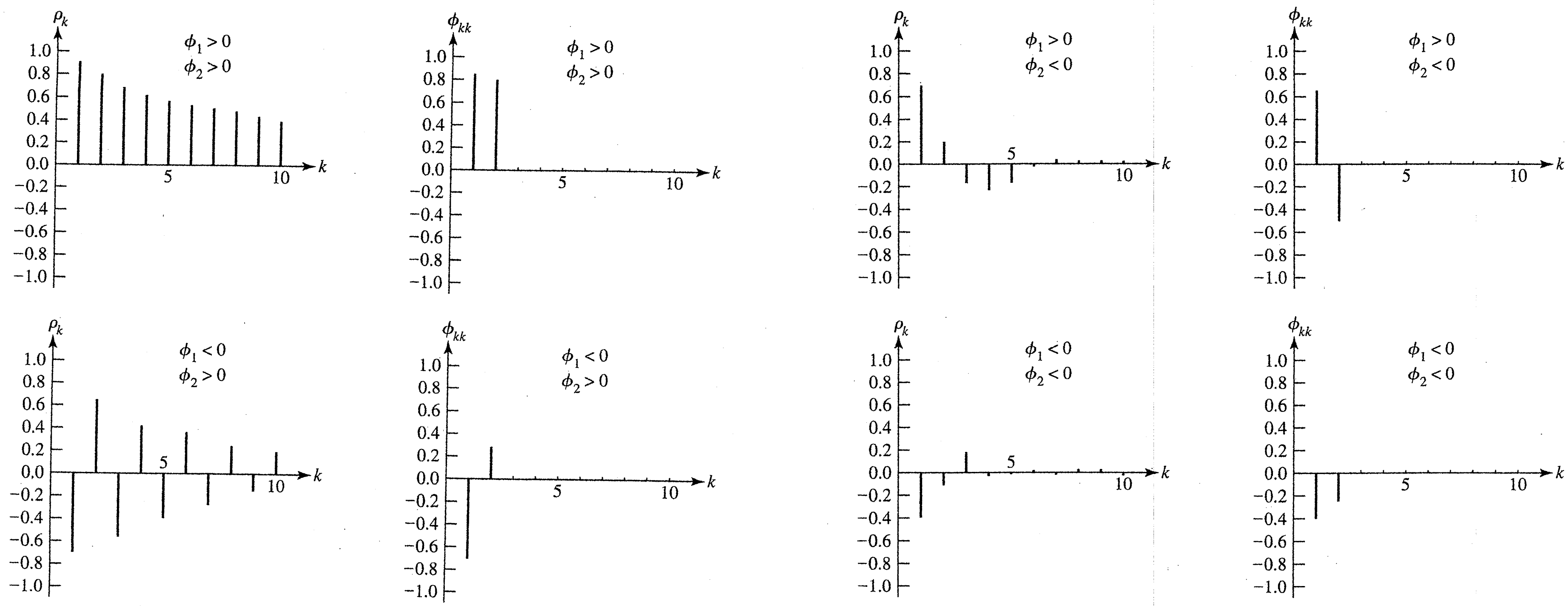


FIGURE 3.7 ACF and PACF of AR(2) process: $(1 - \phi_1 B - \phi_2 B^2)\dot{Z}_t = a_t$.

General Moving average process or MA(q) process.

- A moving average process of order q ie MA(q) process is defined by

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q},$$

where $\{Z_t\} \sim WN(0, \sigma^2)$.

- It clearly follows that:

$$Var[X_t] = \gamma_z(0) = \sigma^2\left(1 + \sum_{j=1}^{q} \theta_j^2\right).$$

- For $\gamma_z(h)$, $h > 0$, we consider.

$$X_t X_{t-h} = Z_t X_{t-h} + \theta_1 Z_{t-1} X_{t-h} + \cdots + \theta_q Z_{t-q} X_{t-h}.$$

Now $E[Z_{t-i} X_{t-h}] = E[Z_{t-i} Z_{t-h} + \theta_1 Z_{t-i} Z_{t-h-1} + \cdots + \theta_q Z_{t-i} Z_{t-h-q}$

$$= \begin{cases} \sigma^2 & \text{if } i = h. \\ \theta_{i-h}\sigma^2 & \text{if } i < h \le q. \\ 0 & \text{o.w.} \end{cases}$$

MA(q) contd.

Thus

$$\gamma_x(h) = \begin{cases} \sigma^2(\theta_h + \theta_{h+1}\theta_1 + \cdots + \theta_q\theta_{q-h}) & \text{if } h = 1, 2, \cdots, q. \\ 0 & \text{if } h > q. \end{cases}$$

Deviding by $\gamma_x(0)$ we get.

$$\cancel{B}\rho_x(h) = \begin{cases} \dfrac{\theta_h + \theta_1\theta_{h+1} + \cdots + \theta_q\theta_{q-h}}{1 + \theta_1^2 + \cdots + \theta_h^2}, & h = 1, 2, \cdots, q \\ 0 & \text{o.w.} \end{cases}$$

- It can be shown that the PACF of a MA(q) process will tail of exponentially ~~oo~~ and/or in damped sign waves depending on the nature of the roots of the polynomial.

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q.$$

MA(2) process

- The process is defined as

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}.$$

where $\{Z_t\} \sim WN(0, \sigma^2)$.

- From the formula the ACF turns out to be.

$$\rho_X(h) = \begin{cases} 1 & \text{if } h=0 \\ \dfrac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} & \text{if } h=1 \\ \dfrac{\theta_2}{1+\theta_1^2+\theta_2^2} & \text{if } h=0 \\ 0 & \text{if } h>2. \end{cases}$$

- The PACF can be shown to have the form.

$$\alpha_X(1) = \rho_X(1) = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}.$$

$$\alpha_X(2) = \frac{\rho_X(2) - \rho_X(1)^2}{1 - \rho_X(1)^2} \quad \text{etc} \ldots$$

ACF and PACF for MA(2).

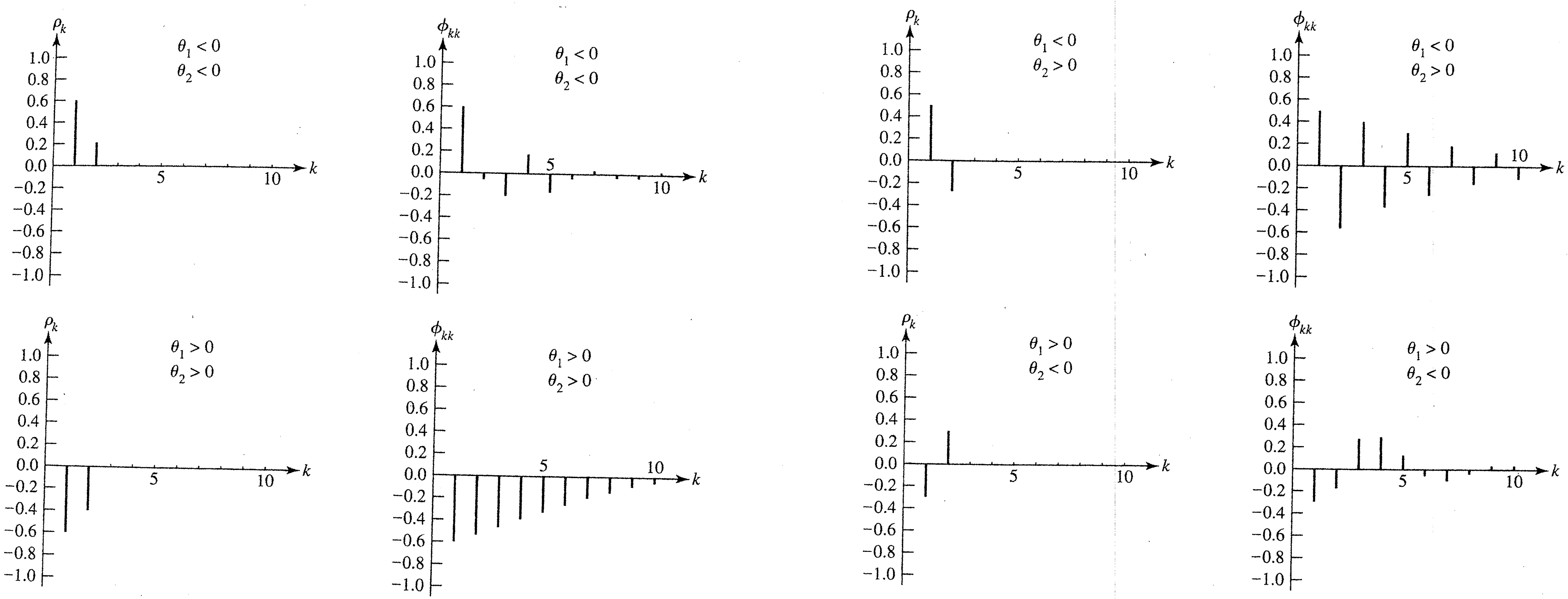


FIGURE 3.12 ACF and PACF of MA(2) processes: $Z_t = (1 - \theta_1 B - \theta_2 B^2)a_t$.

The lag operator.

- As we have noted so far all stationary models try to predict the present or the future depending on the past.
- Such models with "lags" are more conveniently handled with a lag operator.
- Suppose we define

$$B(X_t) =: B X_t := X_{t-1}$$

- B is a function which takes X_t to X_{t-1}. So $B X_{t+h} = B X_{t+h-1}$, $B^2 X_t = B(B(X_t)) = B(X_{t-1}) = X_{t-2}$, and so on.
- Thus an MA(1) model is.

$$X_t = Z_t + \theta_1 Z_{t-1} = (1 + \theta_1 B) Z_t.$$

Lag operator.

- An AR(1) will be.

$$X_t = \phi_1 X_{t-1} + Z_t.$$

$$\Rightarrow \quad X_t - \phi_1 X_{t-1} = Z_t.$$

$$\Rightarrow \quad (1 - \phi_1 B) X_t = Z_t.$$

- MA(2) can be written as.

$$X_t = (1 + \theta_1 B + \theta_2 B^2) Z_t.$$

- AR(2) should be.

$$(1 - \phi_1 B - \phi_2 B^2) X_t = Z_t$$

- Notice that we are expressing everything in terms of a polynomial in B.
- So if $\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q$, a MA(q) process is simply $X_t = \theta(B) Z_t$.
- Similarly if $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$, an AR(p) process is simply $\phi(B) X_t = Z_t$.

The Dual Relationship between AR(p) and MA(q) processes.

- Polynomials in Lag operators can be handled like polynomial in "x" (recall high school math, power series etc.).

- In particular given a $\phi(B)$ we sometimes can find an expression for $1/\phi(B)$ as another polynomial in B say $\psi(B)$ such that

$$\psi(B)\,\phi(B) = 1 .$$

- Now an AR(p) process is given by

$$\phi_p(B)\,X_t = Z_t .$$

- So $$X_t = \frac{1}{\phi_p(B)} Z_t .$$

- If $\psi(B)$ is such that $\psi(B)\,\phi_p(B) = 1$, then

$$X_t = \psi(B)\, Z_t .$$

- Thus if we can find $\psi(B)$, we can write X_t as a MA pr.

AR(p) and MA(∞).

- Suppose $\psi(B)$ is a polynomial and lets restrict to AR(2), ie $\phi(B) = 1 - \phi_1 B - \phi_2 B^2$
- So we want

$$(1 - \phi_1 B - \phi_2 B^2)(1 + \psi_1 B + \psi_2 B^2 + \cdots\cdots) = 1.$$

- So

$$\begin{aligned} & 1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots \\ & - \phi_1 B - \phi_1 \psi_1 B^2 - \phi_1 \psi_2 B^3 + \cdots \\ & \qquad - \phi_2 B^2 - \phi_2 \psi_1 B^3 + \cdots\cdots = 1. \end{aligned}$$

- Now equating coeffients of B^j with 0 we get.

$$\psi_1 - \phi_1 = 0 \quad \Rightarrow \quad \psi_1 = \phi_1$$

$$\psi_2 - \phi_1 \psi_1 - \phi_2 = 0 \Rightarrow \psi_2 = \psi_1 \phi_1 + \phi_2 = \phi_1^2 + \phi_2$$

.... and so on.

- In general we can show that for $j \geq 2$.

$$\psi_j = \psi_{j-1} \phi_1 + \psi_{j-2} \phi_2.$$

AR(P) and MA(∞). also MA(q) and AR(∞)

- Note that ψ_j may not be zero in general.
- So we have a $\psi(B)$ which is like a MA(∞) process. That is an MA process which depends on infinite lag.
- We need one more thing. To be a meaningful polynomial the series $\sum_{j=0}^{\infty} \psi_j Z_{t-j}$ has to converge.
- This happens for instance if $\phi_2 = 0$ and $|\phi_1| < 1$.
- In this case we can show that $\psi_j = \phi_1^j$ and we get an MA(∞) process.
- Similarly it is possible to show that some MA(q) process can be equivalent to a AR(∞) process.
- In particular a MA(1) process with $|\theta_1| < 1$ is equivalent to an AR(∞).