ARMA (1,1)

- ARMA means autoregressive moving average.
- The time series $\{X_t\}$ is an ARMA (1,1) process if it is stationary and satisfies

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}, \qquad \text{for all } t,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$.

- In terms of backshift or Lag operator B, we can express X_t above as,

$$\phi(B) X_t = \theta(B) Z_t, \quad \text{where}$$

$$\phi(B) = 1 - \phi B \quad \text{and} \quad \theta(B) = 1 + \theta B.$$

Stationarity, Causality, invertibility:

- First question is to find it a stationary solution of X_t exists or not.

- If $|\phi| < 1$, we can show that the $MA(\infty)$ process,

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

is the unique stationary solution. This is a causal solution since X_t can be expressed in terms of Z_s, $s \leq t$.

- If $|\phi| > 1$, similarly it can be shown that,

$$X_t = -\theta/\phi \, Z_t - (\theta + \phi) \sum_{j=1}^{\infty} \phi^{-(j+1)} Z_{t+j}.$$

is the unique stationary solution. This solution is non-causal since X_t depends on future values. That is X_t is a function of Z_s for $s > t$.

- There is no stationary solution if $|\phi| = 1$.

Invertibility:

- Just as X_t can be expressed ~~as~~ in terms of Z_t, ~~Z_t~~ in some cases Z_t can be expressed in terms of X_t.
- Invertibility is a concept dual to causality which means that Z_t can be expressed solely in terms of X_s, $s \le t$.
- First we show that an ARMA(1,1) process is invertible if $|\theta| < 1$.
- Let $\xi(z)$ denote the power series expansion of $\frac{1}{\theta(z)} = \sum_{j=0}^{\infty} (-\theta)^j z^j$, which exists if $|\theta| < 1$.
- Now $Z_t = \xi(B)\phi(B)X_t = X_t - (\phi+\theta)\sum_{j=1}^{\infty}(-\theta)^{j-1}X_{t-j}$.

  Thus an ARMA(1,1) process is invertible if $|\theta| < 1$.
- Similarly if $|\theta| > 1$ we can show that
$$Z_t = -\phi/\theta \, X_t + (\theta+\phi)\sum_{j=1}^{\infty}(-\theta)^{-j-1}X_{t+j}.$$
- Note that Z_t now depends on X_s with $s \geq t$. So this ARMA(1,1) process is non-invertible.

ACF

- We assume that the ARMA(1,1) process is both causal and **invertible**. that is $|\phi| < 1$ and $|\theta| < 1$.
- The model is

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}.$$

or

$$X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}.$$

- Now $X_t X_{t-h} = \phi X_{t-1} X_{t-h} + Z_t X_{t-h} + \theta Z_{t-1} X_{t-h}$.
- Taking ~~E~~expectation on ~~bo~~both sides.

$$\gamma_x(h) = \phi \gamma_x(h-1) + E[Z_t X_{t-h}] + \theta E[Z_{t-1} X_{t-h}].$$

- For $h = 0$.

$$\gamma_x(0) = \phi \gamma_x(1) + E[Z_t X_t] + \theta E[Z_{t-1} X_t].$$

- $E[Z_t X_t] = E[\phi X_{t-1} Z_t] + E[Z_t^2] + \theta E[Z_{t-1} Z_t]$

$= \sigma^2$ (by causality)

ACF (2)

- $E[Z_{t-1} X_t] = \phi E[Z_{t-1} X_{t-1}] + E[Z_{t-1} Z_t] + \theta E[Z_t^2]$.
$$= \phi\sigma^2 + \theta\sigma^2 = (\phi+\theta)\sigma^2.$$

- so $\gamma_x(0) = \phi\gamma_x(1) + \sigma^2 + \theta(\phi+\theta)\sigma^2$.

- When $h = 1$.
$$\gamma_x(1) = \phi\gamma_x(0) + E[Z_t X_{t-1}] + \theta E[Z_{t-1} X_{t-1}]$$
$$= \phi\gamma_x(0) + \theta\sigma^2$$
$E[Z_t X_{t-1}] = 0$ [by causality]

- $\gamma_x(0) = \phi(\phi\gamma_x(0) + \theta\sigma^2) + \sigma^2 + \theta(\phi+\theta)\sigma^2$.
$$= \phi^2\gamma_x(0) + \theta\phi\sigma^2 + \sigma^2 + \theta(\phi+\theta)\sigma^2.$$

so $\gamma_x(0) = \dfrac{(1 + 2\theta\phi + \theta^2)\sigma^2}{1-\phi^2}$

Note that $\gamma_x(0)$ is not defined if $\phi^2 = 1$.

- $\gamma_x(1) = \dfrac{\phi(1+2\theta\phi+\theta^2)}{1-\phi^2}\sigma^2 + \theta\sigma^2 = \dfrac{(\phi+\theta)(1+\theta\phi)}{1-\phi^2}\sigma^2$.

ACF and PACF.

- For $h=2$.

$$\gamma_z(2) = \phi\,\gamma_z(1) + E[Z_t X_{t-2}] + \theta E[Z_{t-1} X_{t-2}].$$
$$= \phi\,\gamma_z(1).$$

- For $h>2$.

$$\gamma_z(h) = \phi\,\gamma_z(h-1) + E[Z_t X_{t-h}] + \theta E[Z_{t-1} X_{t-h}].$$
$$= \phi\,\gamma_z(h-1).$$

- To summurise

$$\gamma_z(h) = \begin{cases} \dfrac{1+2\theta\phi+\theta^2}{1-\phi^2}\sigma^2 & \text{if } h=0 \\ \dfrac{(\phi+\theta)(1+\theta\phi)}{1-\phi^2}\sigma^2 & \text{if } h=1 \\ \phi\,\gamma_z(h-1) & \text{if } h\geq 2. \end{cases}$$

- Thus the ACF is given by.

$$\rho_z(h) = \begin{cases} 1 & \text{if } h=0 \\ \dfrac{(\phi+\theta)(1+\theta\phi)}{1+2\theta\phi+\theta^2} & \text{if } h=1 \\ \phi\,\rho_z(h-1) & \text{if } h\geq 2 \end{cases}$$

PACF

- PACF of an ARMA(1,1) is complicated, It can be calculated from the general formula.
- However since ~~AR(1) or~~ MA(1) is a special case of ARMA(1,1), the PACF function of the latter will not become zero after any finite lag.
- From the ACF function it is also clear that it won't become zero for any finite h.
- In general the nature of ACF and PACF plots are more complicated ~~than~~ for ARMA(1,1) than AR(1) or MA(1).

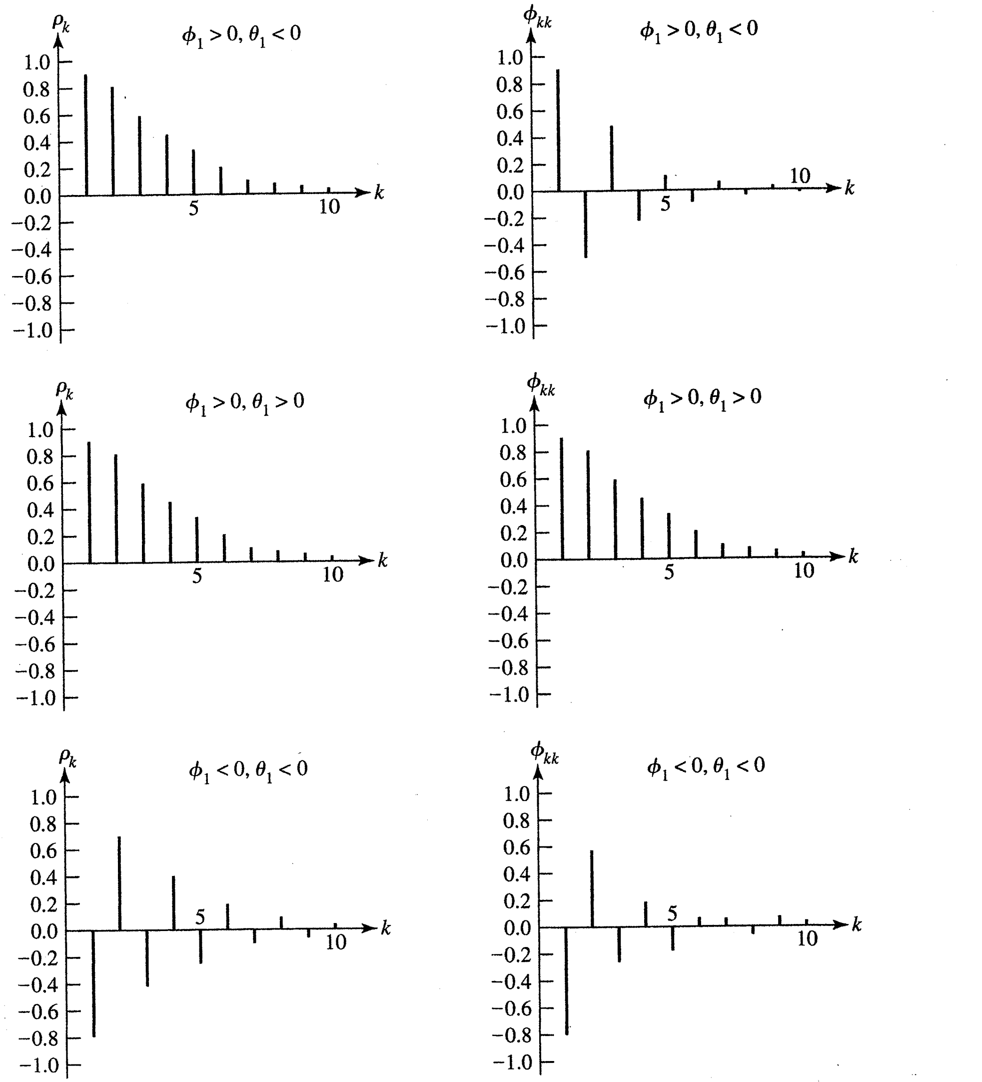

FIGURE 3.14 ACF and PACF of ARMA(1, 1) model $(1 - \phi_1 B)\dot{Z}_t = (1 - \theta_1 B)a_t$.

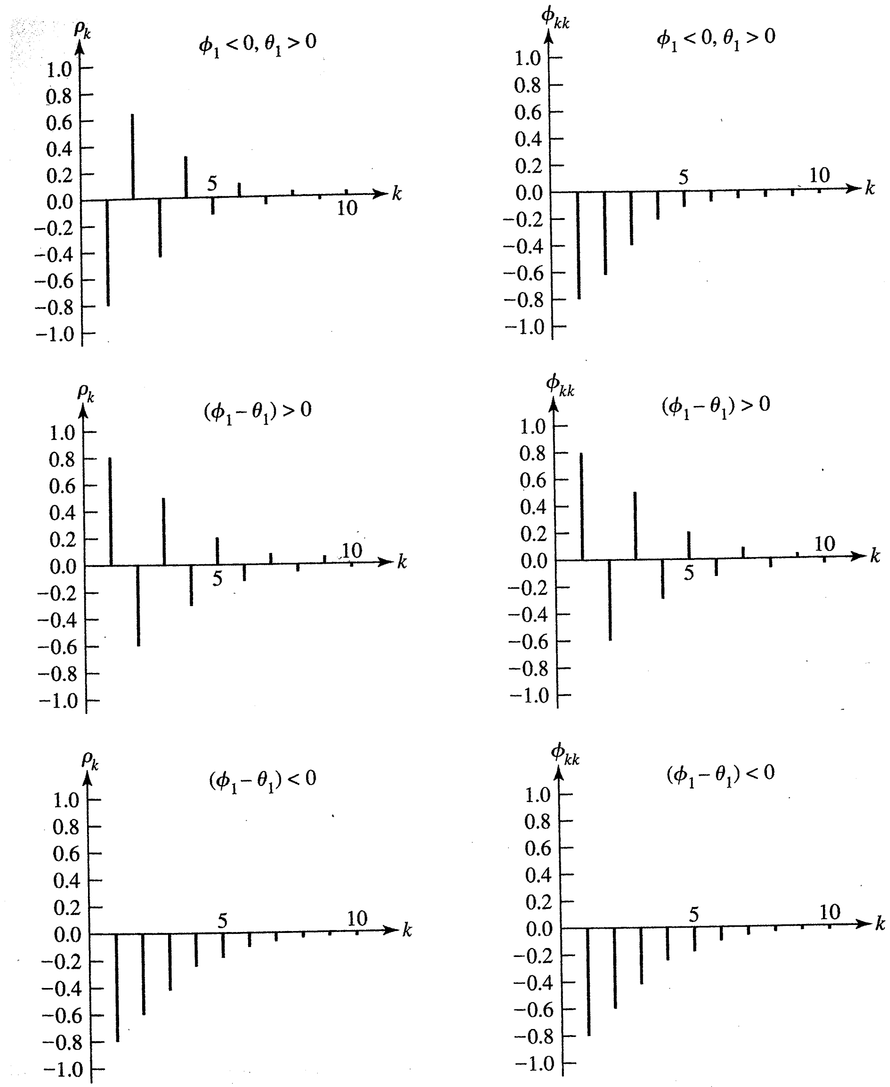

FIGURE 3.14 (*Continued*)

ARMA (p,q) process

- $\{X_t\}$ is an ARMA(p,q) process if $\{X_t\}$ is stationary and if for every t,

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q},$$

where $\{Z_t\} \sim WN(0,\sigma^2)$.

- In terms of backward shift operator B it can be expressed as

$$\phi(B) X_t = \theta(B) Z_t,$$

where $\phi(z)$ and $\theta(z)$ are the pth and qth degree polynomials,

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p.$$

and

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q.$$

- A stationary solution of $\{X_t\}$ exists and is unique if and only if

$$\phi(z) \neq 0 \quad \text{for all } |z| = 1.$$

- The solution may be complex.

Causality and Invertibility:

- An ARMA(p,q) process $\{X_t\}$ is causal, or a causal function of $\{Z_t\}$, if there exist constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and
$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \quad \text{for all } t.$$
Causality is equivalent to the condition
$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0 \text{ for all } |z| \leq 1.$$

- An ARMA(p,q) process $\{X_t\}$ is invertible if there exist constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and
$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \quad \text{for all } t.$$
Invertibility is equivalent to the condition
$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0 \text{ for all } |z| \leq 1.$$

- It can be shown that $\pi_j + \sum_{k=1}^{q} \theta_k \pi_{j-k} = -\phi_j$, $j = 0, 1, \ldots$
where $\phi_0 := -1$, $\phi_j := 0$ $\forall j > p$ and $\pi_j := 0$ for $j < 0$.

Example

- Consider the ARMA(1,1) process $\{X_t\}$ satisfying the equations

$$X_t - .5X_{t-1} = Z_t + .4Z_{t-1}, \quad \{Z_t\} \sim WN(0,\sigma^2).$$

Here $\phi(z) = 1 - .5z$. Now $\phi(z)=0$ has solution $z=2$.
There is no solution in the unit interval. So the process is stationary and causal.

The corresponding $MA(\infty)$ representation of $\{X_t\}$ are ~~found~~ as follows:

$$\psi_0 = 1, \quad \psi_1 = .4+.5, \quad \psi_2 = .5(.4+.5), \quad \psi_j = .5^{j-1}(.4+.5), \; j=1,2,\cdots$$

Also $\theta(z) = 1+.4z$, $\theta(z)=0$ has a solution $z = -1/.4 = -2.5$, which is located outside the unit interval.

The corresponding $\{\pi_j\}$ is given by.

$$\pi_0 = 1, \quad \pi_1 = -(.4+.5), \quad \pi_2 = -(.4+.5)(-.4), \quad \pi_j = -(.4+.5)(-.4)^{j-1},$$
$$j = 1,2,\cdots$$

Example 2

- ARMA (2, 1) process defined by .

$$X_t - .75\, X_{t-1} + .5625\, X_{t-2} = Z_t + 1.25\, Z_{t-1} \quad , \{Z_t\} \sim WN(0, \sigma^2)$$

$\phi(z) = 1 - .75 z + .5625 z^2$ has zeroes at $z = 2(1 \pm i\sqrt{3})/3$, which lie outside the unit circle. So the process is causal.

$\theta(z) = 1 + 1.25 z$ has a zero $z = -.8$ which is inside the unit interval, so this process is not invertible.