CMPT 726 Assignment 0

1. a) $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{3-0}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \underline{\begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}}$

$B^{-1} = \frac{1}{10-12} = -\frac{1}{2}\begin{pmatrix} 2 & -4 \\ -3 & 5 \end{pmatrix} = \underline{\begin{pmatrix} -1 & 2 \\ 3/2 & -5/2 \end{pmatrix}}$

$C^{-1} = \frac{1}{4-9} = -\frac{1}{5}\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = \underline{\begin{pmatrix} -2/5 & -3/5 \\ -3/5 & -2/5 \end{pmatrix}}$

b) $BC = \begin{pmatrix} 5\cdot 2 + 4\cdot -3 & 5\cdot -3 + 4\cdot 2 \\ 3\cdot 2 + 2\cdot -3 & 3\cdot -3 + 2\cdot 2 \end{pmatrix} = \underline{\begin{pmatrix} -2 & -7 \\ 0 & -5 \end{pmatrix}}$

$CB = \begin{pmatrix} 2\cdot 5 + -3\cdot 3 & 2\cdot 4 + -3\cdot 2 \\ -3\cdot 5 + 2\cdot 3 & -3\cdot 4 + 2\cdot 2 \end{pmatrix} = \underline{\begin{pmatrix} 1 & 2 \\ -9 & -8 \end{pmatrix}}$

c) Find when $\det(C-\lambda I)=0$

$\det\left(\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)=0 \Leftrightarrow \det\begin{pmatrix} 2-\lambda & -3 \\ -3 & 2-\lambda \end{pmatrix} = 0$

$= (2-\lambda)(2-\lambda) - (-3)(-3) = 4-2\lambda-2\lambda+\lambda^2-9$

$= \lambda^2-4\lambda-5=0 \Leftrightarrow (\lambda-5)(\lambda+1)$

$= \underline{\lambda=5, \lambda=-1}$
eigen values

eigen vector for $\lambda=5$: $(C-\lambda I)\vec{v}=0 \to C-5I = \begin{pmatrix} 2-5 & -3 \\ -3 & 2-5 \end{pmatrix} = \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}$

$\begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3v_1 & -3v_2 \\ -3v_1 & -3v_2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $= -3v_1-3v_2=0$, $= -3v_1-3v_2=0$ Same equations, so we remove one

$-3v_1-3v_2=0 \to v_1+v_2=0 \to v_1=-v_2 \to \underline{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \xrightarrow{\text{normalizing}} \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \underline{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}}$

For $\lambda=-1$: $\begin{pmatrix} 2-(-1) & -3 \\ -3 & 2-(-1) \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{matrix} 3v_1-3v_2=0 \\ -3v_1+3v_2=0 \end{matrix} \to \begin{matrix} v_1=v_2 \\ v_1=v_2 \end{matrix} \xrightarrow{\text{normalizing}} \underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \to \underline{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}$

normalized eigenvectors

2. a) $f(\vec{x}) = (x_1\ x_2)\begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1\ x_2)\begin{pmatrix} 6x_1+3x_2 \\ 3x_1 \end{pmatrix} = 6x_1^2+3x_1x_2+3x_1x_2 = 6x_1^2+6x_1x_2$

$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}(6x_1^2+6x_1x_2) = 12x_1+6x_2 = 12x_1+6x_2$ sub $x_1=2$, $x_2=4$ $\to 12(2)+6(4) = \underline{48}$

$\frac{\partial^2 f}{\partial x_2 x_1} = \frac{\partial}{\partial x_2}(12x_1+6x_2) = \underline{6}$

b) Gradient Vector: $\nabla f(\vec{x}) = \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{pmatrix} = \begin{pmatrix} 12x_1 + 6x_2 \\ 6x_1 \end{pmatrix}$

Hessian: need 2nd partials & mixed partials

$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}(12x_1 + 6x_2) = 12$

$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}(6x_1) = 0$

$\frac{\partial^2 f}{\partial x_1 x_2} = \frac{\partial}{\partial x_2}(12x_1 + 6x_2) = 6$

$\frac{\partial^2 f}{\partial x_2 x_1} = \frac{\partial}{\partial x_1}(6x_1) = 6$

$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \underline{\begin{pmatrix} 12 & 6 \\ 6 & 0 \end{pmatrix}}$

3. a) PMF -> $\underline{P(Y=y) = 1/8 \text{ for } y \in \{1, 2, \ldots, 8\}}$

CDF -> $F_Y(y) = P(Y \leq y)$

$F_Y(1) = P(Y \leq 1) = 1/8$, $F_Y(2) = P(Y \leq 2) = 2/8 \ldots F_Y(8) = P(Y \leq 8) = 1$

-> $\underline{F_Y(y) = y/8 \text{ for } y \in \{1, 2, \ldots, 8\}}$

b) $P(Y=2 \mid Y \text{ is even}) = \frac{P(Y=2 \text{ and } Y \text{ is even})}{P(Y \text{ is even})}$

$= \frac{1/8}{4/8} = \frac{1/8}{1/2} = \underline{\frac{1}{4}}$

c) $E(Y) = \frac{1}{8}(1+2+\ldots+8) = \frac{1}{8} \cdot 36 = 4.5$

$T = Y_1 + Y_2 + \ldots + Y_n \Rightarrow E[T] = E[Y_1] + E[Y_2] + \ldots + E[Y_n] = n \cdot E[Y] = \underline{4.5n}$

$Var(Y) = E[Y^2] - (E[Y])^2$

$E[Y^2] = \frac{1}{8}(1^2 + \ldots + 8^2) = \frac{1}{8}(204) = 25.5$

$(E[Y])^2 = 4.5^2 = 20.25$

$Var(Y) = 25.5 - 20.25 = 5.25$

$Var(T) = Var(Y_1) + \ldots + Var(Y_n) = n \cdot Var(Y) = \underline{5.25n}$