Riemann Zeta Function at Even Positive Integers

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September 2023

Abstract

In this paper I derive a general formula for the Riemann zeta function at positive even integers using Fourier series. I also give some results of related summations.

1 Introduction

The Riemann zeta function is defined as the following,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + ...$$

where $Re(s) > 1$.

It is of significant mathematical importance, for example in the Riemann hypothesis, which states that the zeroes of the Riemann zeta function and its analytic continuation are even negative integers or have real part equal to $\frac{1}{2}$.

It is well known that $\zeta(1)$ does not exist, since it represents the harmonic series which is known to diverge.

A famous example is what is known as the Basel problem: calculating the value of $\zeta(2)$. Interestingly, $\zeta(2) = \frac{\pi^2}{6}$. One could also find that $\zeta(4) = \frac{\pi^4}{90}$. In fact, $\zeta(2k) = q\pi^{2k}$, for all positive integers k, and for some rational q. The same result does not hold for odd positive integers, in fact the exact value of these is unknown.

The first part of this paper focuses on deriving a general formula for $\zeta(2k)$, where k is a positive integer.

2 Main Theorem

The focus of this paper is to derive a general formula for $\zeta(2k)$, where k is a positive integer. I do so in this section.

Lemma 2.1. *The Fourier series for $f(x) = x^{2k}$ on $(0, 2\pi)$ with period π is*

$$\frac{(2\pi)^{2k}}{2k+1} + \sum_{n=1}^{\infty} I(n,k)\cos(nx) + J(n,k)\sin(nx)$$

where $I(n,k) = \frac{1}{\pi}\int_0^{2\pi} x^{2k}\cos(nx)dx$, and $J(n,k) = \frac{1}{\pi}\int_0^{2\pi} x^{2k}\sin(nx)dx$.

Proof.

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} x^{2k}dx = \frac{1}{2\pi}[\frac{x^{2k+1}}{2k+1}]\Big|_0^{2\pi} = \frac{1}{2\pi}\frac{(2\pi)^{2k+1}}{2k+1} = \frac{(2\pi)^{2k}}{2k+1}$$

$a_n = I(n,k)$ and $b_n = J(n,k)$ by definition of the Fourier series. □

We can see that $f(x) = x^{2k}$ and $f'(x) = 2kx^{2k-1}$ are defined everywhere in $(0, 2\pi)$ and are both piece-wise continuous, so $f(x)$ satisfies the Dirichlet conditions. Thus, the Fourier series converges to $f(x)$ when x is a point of continuity, and to $\frac{1}{2}(f(x+0) + f(x-0))$ when x is a point of discontinuity.

Lemma 2.2.

$$\sum_{n=1}^{\infty} I(n,k) = 2^{2k-1}\pi^{2k}\frac{2k-1}{2k+1}$$

Proof.

$$\frac{1}{2}(f(0+0) + f(0-0)) = 2^{2k-1}\pi^{2k}$$

Hence, the Fourier series converges to $2^{2k-1}\pi^{2k}$ at $x = 0$. Substituting this into the Fourier series in 2.1 gives,

$$2^{2k-1}\pi^{2k} = \frac{(2\pi)^{2k}}{2k+1} + \sum_{n=1}^{\infty} I(n,k)$$

$$\sum_{n=1}^{\infty} I(n,k) = 2^{2k-1}\pi^{2k} - \frac{(2\pi)^{2k}}{2k+1} = 2^{2k-1}\pi^{2k}\frac{2k-1}{2k+1}$$

□

Lemma 2.3.

$$I(n,k) = \frac{1}{n^2}(2^{2k}\pi^{2k-1}k - 2k(2k-1)I(n,k-1))$$

Proof.

$$I(n,k) = \frac{1}{\pi}\int_0^{2\pi} x^{2k}\cos(nx)dx$$

$$I(n,k) = \frac{1}{\pi}[\frac{1}{n}x^{2k}\sin(nx)]\Big|_0^{2\pi} - \frac{2k}{n\pi}\int_0^{2\pi} x^{2k-1}\sin(nx)dx$$

$$I(n,k) = -\frac{2k}{n\pi}\int_0^{2\pi} x^{2k-1}\sin(nx)dx$$

$$I(n,k) = -\frac{2k}{n\pi}[-\frac{1}{n}x^{2k-1}\cos(nx)]\Big|_0^{2\pi} + \frac{2k}{n\pi}\frac{2k-1}{n}\int_0^{2\pi} x^{2k-2}\cos(nx)dx$$

$$I(n,k) = \frac{1}{n^2}(2^{2k}\pi^{2k-1}k - 2k(2k-1)I(n,k-1))$$

□

Proposition 2.4. *We can write $I(n,k)$ in the following form, for some constants $C(k,m)$,*

$$I(n,k) = \sum_{m=1}^{k}\frac{C(k,m)}{n^{2m}} = \frac{C(k,1)}{n^2} + \frac{C(k,2)}{n^4} + ... + \frac{C(k,k)}{n^{2k}}$$

Proof. Firstly see that $I(n,0) = 0$. Using this as our base case for induction, we can use 2.3 for the inductive step. Thus, all $I(n,k)$ must have this form. □

Lemma 2.5.

$$C(k,m) = (-1)^{m+1}2^{2k-2m+1}\pi^{2(k-m)}\frac{(2k)!}{(2k-2m+1)!}$$

Proof. Substituting $I(n,k) = \sum_{m=1}^{k}\frac{C(k,m)}{n^{2m}}$ into the reduction formula from 2.3 gives,

$$\sum_{m=1}^{k}\frac{C(k,m)}{n^{2m}} = \frac{1}{n^2}(2^{2k}\pi^{2k-2}k - 2k(2k-1)\sum_{m=1}^{k-1}\frac{C(k-1,m)}{n^{2m}})$$

Comparing coefficients of $\frac{1}{n^2}$ gives us that $C(k,1) = 2^{2k}\pi^{2k-2}k$.

Comparing coefficients of $\frac{1}{n^{2m}}$ gives,

$$C(k,m) = -2k(2k-1)C(k-1,m-1)$$

$$C(k,m) = C(k-m+1,1)\prod_{r=1}^{m-1} -2(k-r+1)(2(k-r+1)-1)$$

$$C(k,m) = (-1)^{m-1}2^{2k-2m+1}\pi^{2(k-m)}2(k-m+1)\prod_{r=1}^{m-1} 2(k-r+1)(2(k-r+1)-1)$$

$$C(k,m) = (-1)^{m+1}2^{2k-2m+1}\pi^{2(k-m)}\frac{(2k)!}{(2k-2m+1)!}$$

□

Lemma 2.6.

$$\zeta(2k) = \frac{(-1)^{k+1}2^{2k-1}\pi^{2k}}{(2k)!}(\frac{2k-1}{2(2k+1)} + \sum_{m=1}^{k-1}\frac{(-1)^m}{2^{2m-1}\pi^{2m}}\frac{(2k)!\zeta(2m)}{(2k-2m+1)!})$$

Proof. Substituting the form of $I(n,k)$ from 2.4 into the result from 2.2,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{k}\frac{C(k,m)}{n^{2m}} = 2^{2k-1}\pi^{2k}\frac{2k-1}{2k+1}$$

$$\sum_{m=1}^{k}C(k,m)\sum_{n=1}^{\infty}\frac{1}{n^{2m}} = 2^{2k-1}\pi^{2k}\frac{2k-1}{2k+1}$$

$$\sum_{m=1}^{k}C(k,m)\zeta(2m) = 2^{2k-1}\pi^{2k}\frac{2k-1}{2k+1}$$

$$C(k,k)\zeta(2k) = 2^{2k-1}\pi^{2k}\frac{2k-1}{2k+1} - \sum_{m=1}^{k-1}C(k,m)\zeta(2m)$$

$$\zeta(2k) = \frac{(-1)^{k+1}}{2(2k)!}(2^{2k-1}\pi^{2k}\frac{2k-1}{2k+1} - \sum_{m=1}^{k-1}(-1)^m 2^{2k-2m+1}\pi^{2k-2m}\frac{(2k)!\zeta(2m)}{(2k-2m+1)!}$$

$$\zeta(2k) = \frac{(-1)^{k+1}2^{2k-1}\pi^{2k}}{(2k)!}(\frac{2k-1}{2(2k+1)} + \sum_{m=1}^{k-1}\frac{(-1)^m}{2^{2m-1}\pi^{2m}}\frac{(2k)!\zeta(2m)}{(2k-2m+1)!})$$

□

Theorem 2.7.
$$\zeta(2k) = \frac{(-1)^{k+1}2^{2k-1}\pi^{2k}B_{2k}}{(2k)!}$$

where B_{2k} is the $2k^{th}$ Bernoulli number, which are defined as $B_n = -\sum_{r=0}^{n-1}\binom{n}{r}\frac{B_r}{n-r+1}$, and $B_0 = 1$.

Proof. We prove this by strong mathematical induction.

For the base case we use $k = 1$,

$$\zeta(2) = \frac{2\pi^2 B_2}{2} = \frac{\pi^2}{6}$$

So the formula works for $k = 1$, since this is the solution to the Basel problem (see appendix 4.1).

Now, assume that the formula holds $\forall m < k$. Then, from 2.6,

$$\zeta(2k) = \frac{(-1)^{k+1}2^{2k-1}\pi^{2k}}{(2k)!}\left(\frac{2k-1}{2(2k+1)} + \sum_{m=1}^{k-1}\frac{(-1)^m}{2^{2m-1}\pi^{2m}}\frac{(2k)!}{(2k-2m+1)!}\frac{(-1)^{m+1}2^{2m-1}\pi^{2m}B_{2m}}{(2m)!}\right)$$

$$\zeta(2k) = \frac{(-1)^{k+1}2^{2k-1}\pi^{2k}}{(2k)!}\left(\frac{2k-1}{2(2k+1)} - \sum_{m=1}^{k-1}\frac{(2k)!}{(2m)!(2k-2m+1)!}B_{2m}\right)$$

$$\zeta(2k) = \frac{(-1)^{k+1}2^{2k-1}\pi^{2k}}{(2k)!}\left(\frac{2k-1}{2(2k+1)} - \sum_{m=1}^{k-1}\binom{2k}{2m}\frac{B_{2m}}{2k-2m+1}\right)$$

So it remains to show that,

$$B_{2k} = \frac{2k-1}{2(2k+1)} - \sum_{m=1}^{k-1}\binom{2k}{2m}\frac{B_{2m}}{2k-2m+1}$$

It is well known that $B_n = 0$ for odd n, except for $B_1 = -\frac{1}{2}$ (see appendix 4.2).

So from the recurrence relation defining the Bernoulli numbers,

$$B_{2k} = -\sum_{m=0}^{2k-1}\binom{2k}{m}\frac{B_m}{2k-m+1}$$

$$B_{2k} = \frac{1}{2} - \sum_{m=1}^{k-1}\binom{2k}{2m}\frac{B_{2m}}{2k-2m+1}$$

$$B_{2k} = \frac{2k-1}{2(2k+1)} - \sum_{m=1}^{k-1} \binom{2k}{2m} \frac{B_{2m}}{2k-2m+1}$$

Which is the required result.

Hence, the theorem is proven by mathematical induction. $\square$

3 Related Results

Using the theorem from the previous chapter, we can derive results for some related summations.

Theorem 3.1.

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1} 2^{2k-1} \pi^{2k} B_{2k}}{(2k)!}$$

where B_{2k} is the $2k^{th}$ Bernoulli number, which are defined as $B_n = -\sum_{r=0}^{n-1} \binom{n}{r} \frac{B_r}{n-r+1}$, and $B_0 = 1$.

Corollary 3.1.1.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = \frac{(-1)^{k+1} (2^{2k}-1) \pi^{2k} B_{2k}}{2(2k)!}$$

Proof.

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = T$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^{2k}} = \frac{T}{2^{2k}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \sum_{n=1}^{\infty} \frac{1}{(2n)^{2k}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = T - \frac{T}{2^{2k}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = \frac{2^{2k}-1}{2^{2k}} T$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = \frac{(-1)^{k+1} (2^{2k}-1) \pi^{2k} B_{2k}}{2(2k)!}$$

□

Corollary 3.1.2.

$$\eta(2k) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2k}} = \frac{(-1)^{k+1}(2^{2k-1}-1)\pi^{2k}B_{2k}}{(2k)!}$$

Proof.

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = T$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^{2k}} = \frac{T}{2^{2k}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = T - \frac{T}{2^{2k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2k}} = (T - \frac{T}{2^{2k}}) - \frac{T}{2^{2k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2k}} = \frac{2^{2k-1}-1}{2^{2k-1}}T$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2k}} = \frac{(-1)^{k+1}(2^{2k-1}-1)\pi^{2k}B_{2k}}{(2k)!}$$

$\square$

4 Appendix

Appendix 4.1 The Basel problem states that $\zeta(2) = \frac{\pi^2}{6}$.

To see this, note that from 2.3, $I(n, 1) = \frac{4}{n^2}$.

Substituting this into 2.2 gives,

$$\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Giving the solution to the Basel problem.

Appendix 4.2 The Bernoulli numbers are defined by $B_0 = 1$, and

$$B_n = -\sum_{r=0}^{n-1} \binom{n}{r} \frac{B_r}{n - r + 1}$$

Apart from $B_1 = -\frac{1}{2}$, $B_n = 0$ for odd n.

Proofs of this fact require a different formulation of the Bernoulli numbers, and so are not presented in this paper.

For a proof of this fact, see https://planetmath.org/theoddbernoullinumbersarezero