Polynomial Interpolation and Sums of Powers

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Abstract

Analysis of my investigation into the construction of an algorithm to find a formula for the sum of integers raised to some power.

Contents

1 Introduction

This paper will discuss a method for computing a closed form formula for $\sum_{r=1}^{n}(r^k)$, where $k \in \mathbb{N}$.

The formulae for when $k \in 1, 2, 3$ are generally well-known, but it begs to question how to generate the formula for large k.

$$\sum_{r=1}^{n}(r^1) = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n}(r^2) = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n}(r^3) = \frac{1}{4}n^2(n+1)^2$$

$$\sum_{r=1}^{n}(r^4) = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

Perhaps the most well-known way of generating these polynomials is Faulhaber's formula, which gives an efficient and simple $O(k^2)$ algorithm, however it depends on the generation of Bernoulli numbers, and so formulae can only be found for sufficiently small k.

$$\sum_{r=1}^{n}(r^k) = \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p + \sum_{r=2}^{k}(\frac{B_r}{r!}k^{\underline{r-1}}n^{k-r+1})$$

Where B_k denotes the kth Bernoulli number, and $r^{\underline{k}}$ is the falling factorial.

Such methods have been thoroughly researched by mathematicians including Pierre de Fermat, Blaise Pascal and Jakob Bernoulli. I present a different method which follows the work of Leonhard Euler and his research into differences of sequences.

2 Differences of Sequences

As a build up to the polynomial interpolation outlined in the next chapter, we look at the nth difference of a sequence of numbers a_k, which is denoted by $\Delta^n a_k$. For example, consider the following sequence of numbers,

$$1, 1, 2, 3, 5, 8$$

Then the nth difference is calculated as,

$$1, 1, 2, 3, 5, 8$$
$$0, 1, 1, 2, 3$$
$$1, 0, 1, 1$$
$$-1, 1, 0$$

Where each element is the difference between the terms above it, a sort of "subtracting" Pascal's triangle.

This yields the relation, $\Delta^{n+1} a_k = \Delta^n a_{k+1} - \Delta^n a_k$. We now aim to find a formula for calculating $\Delta^n a_k$ using this relation and Pascal's triangle.

Consider the sequence, a_1, a_2, a_3, a_4, a_5. The differences are shown below,

$$a_1, a_2, a_3, a_4, a_5$$
$$a_2 - a_1, a_3 - a_2, a_4 - a_3, a_5 - a_4$$
$$a_3 - 2a_2 + a_1, a_4 - 2a_3 + a_2, a_5 - 2a_4 + a_3$$
$$a_4 - 3a_3 + 3a_2 - a_1, a_5 - 3a_4 + 3a_3 - a_2$$
$$a_5 - 4a_4 + 6a_3 - 4a_2 + a_1$$

It seems as though the coefficients of a_k in the nth difference are the coefficients in the binomial expansion of $(x-1)^n$, which makes sense given its similarity to Pascal's triangle. From this, we give the following proposition,

Proposition 2.1 $$\Delta^n a_k = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a_{k+i}$$

Now we prove this formula by induction.

When $n = 1$, $\Delta^n a_k = a_{k+1} - a_k$ from our formula earlier. The formula gives,

$$\Delta^1 a_k = \sum_{i=0}^{1} \binom{1}{i} (-1)^{1-i} a_{k+i} = \binom{1}{0} (-1)^1 a_k + \binom{1}{1} (-1)^0 a_{k+1} = a_{k+1} - a_k$$

So our formula holds when $n = 1$, which serves as our base case.

Now assume for our inductive hypothesis that,

$$\Delta^n a_k = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a_{k+i}$$

.

Now for the inductive step,

$$\Delta^{n+1} a_k = \Delta^n a_{k+1} - \Delta^n a_k$$

$$\Delta^{n+1} a_k = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a_{k+1+i} - \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a_{k+i}$$

$$\Delta^{n+1} a_k = \sum_{i=1}^{n+1} \binom{n}{i-1} (-1)^{n+1-i} a_{k+i} - \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a_{k+i}$$

$$\Delta^{n+1} a_k = \sum_{i=1}^{n} \binom{n}{i-1} (-1)^{n+1-i} a_{k+i} + \binom{n}{n} (-1)^0 a_{k+n+1} - \sum_{i=1}^{n} \binom{n}{i} (-1)^{n-i} a_{k+i} - \binom{n}{0} (-1)^n a_k$$

$$\Delta^{n+1} a_k = \sum_{i=1}^{n} (-1)^{n+1-i} a_{k+i} (\binom{n}{i-1} + \binom{n}{i}) + \binom{n}{n} (-1)^0 a_{k+n+1} - \binom{n}{0} (-1)^n a_k$$

$$\Delta^{n+1} a_k = \sum_{i=1}^{n} (-1)^{n+1-i} a_{k+i} \binom{n+1}{i} + a_{k+n+1} - (-1)^n a_k$$

$$\Delta^{n+1} a_k = \sum_{i=0}^{n+1} (-1)^{n+1-i} a_{k+i} \binom{n+1}{i}$$

Which has the same form as the formula for $n+1$. We have shown that if the formula is true for n then it is true for $n+1$. Since it holds for $n = 1$, the formula therefore holds for all positive integers n.

This approach was originally used by Leonhard Euler to prove that all numbers of the form $4m + 1$ could be written as the sum of two squares, but we show in the next chapter that it gives us a formula for the nth difference of a polynomial sequence.

3 Polynomial Sequences

A polynomial sequence is any infinite sequence, $f(1), f(2), f(3), ...$ where,

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + ... + a_1 x + a_0$$

The goal of this to chapter is to find a method to find this polynomial, starting with the polynomial sequence. We now make the claim,

Proposition 3.1 A polynomial of degree d is uniquely defined by $d+1$ points.

We prove this by contradiction. Assume that there exist two non-identical polynomials, $f(x)$ and $g(x)$, both of degree d, which pass through the same $d+1$ points. Then $f(x) - g(x)$ is a polynomial with degree $\leq d$, with $d+1$ roots. However a degree d polynomial has at most d roots. Contradiction. Hence $f(x) - g(x)$ must be the zero polynomial, and so $f(x)$ is identical to $g(x)$.

Hence, given a polynomial sequence of degree d and $d+1$ terms we know there exists a unique polynomial which produces the sequence. Consider the polynomial,

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + ... + a_1 x + a_0$$

Then we can create a polynomial sequence with $f(x)$. Let us only consider the leading coefficient of $f(x)$ in this sequence. Since the coefficients are effectively independent of each other, then the coefficient of x^d in the nth difference is given by proposition 3.1 as,

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} (i+k)^d$$

The nth difference of the whole polynomial is simply the sum of the nth differences for each coefficient. The formula above tells us that the $n+1$th difference of x^n is 0 because the two terms above are constant. And so there are $d+1-n$ terms in the nth difference of a degree d polynomial. This tells us that in the case where $k = 1$,

$$\Delta^n f(x) = \sum_{j=1}^{d+1-n} (a_{d+1-j} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} (i+1)^{d+1-j})$$

Because $k = 1$, these values are equal to the leftmost column of the triangle of differences. If we know these values, we can calculate the values of the coefficients by back-substitution, since the corresponding matrix is in row-echelon form.

4 Sums of Powers

As mentioned in chapter 1, our aim is to find a formula for $\sum_{r=1}^{n}(r^k)$. Looking at the first 4 formulae suggests that a polynomial of degree $k+1$ should suffice.

Proposition 4.1 $\sum_{r=1}^{n}(r^k) = f(n)$, where $deg(f) = k+1$.

https://math.stackexchange.com/questions/1893693/elementary-proof-of-polynomial-degree-of-sum-of-pth-powers?rq=1

Hence, our method of polynomial interpolation via proposition 3.1 can be used to find our polynomial formula. The value of the nth difference can be computed via calculating the first $k+2$ terms since they are the sum of powers, and then applying the formula from proposition 3.1. This method doesn't seem to yield an explicit formula for $f(n)$, however it can be easily computed via back-substitution in $O(n^3)$ by a computer algorithm. A proof of concept program is available on my GitHub page.

As an example, the formula for the sum of the first n powers of 12 is computed by the algorithm to be,

$$\sum_{r=1}^{n}(r^{12}) = \frac{1}{13}n^{13} + \frac{1}{2}n^{12} + n^{11} - \frac{11}{6}n^9 + \frac{22}{7}n^7 - \frac{33}{10}n^5 + \frac{5}{3}n^3 - \frac{691}{2730}n$$

Plotting the graph of $(RunTime)^{1/4}$ against k verifies that the algorithm is $O(k^4)$.