Regular and Non Regular Expression

Assume $\begin{aligned} A &= \{2n \mid \text{n is a natural number}\} \\ &= \{2,\ 4,\ 8,\ 16,\ 18,\ 20\ldots\} \end{aligned}$

So $B_2(A)=\{10,\ 100,\ 1000, 10000\ldots\}$ should be regular, because $B_2(A)$ is recognized by regular expression $10*$.

Now $B_3(A)=\{2,\ 11,\ 22,\ 121\ldots\}$

$B_3(A)$ is non regular.

It can be proved by contradiction that $B_3(A)$ is non regular.

Take on the contrary that $B_3(A)$ is regular.

Now p will be pumping length according to pumping lemma.

Choose u as element of $B_3(A)$ and the length of u should at least $p+1$.

Since $u \in B_3(A)$ and $i>1$, by pumping lemma u can be divided into three part, $u=xyz$ where $\forall i \geq 0$ the string $xy^iz \in B_3(A)$

According to condition three of pumping lemma, $|xy| \leq p$, and $|z|>0$.

If rightmost digit of z is 0, then u will be the power of 3. But u is power of 2, hence it is impossible. Hence the rightmost digit of 0 will must be 1 or may be 2.

For $i>1$, $u' = xy^iz$ is power of 2 which is greater than u, so it is easy to generate it after u is added to itself few numbers of times. That is, if u is added at least 3 times to itself, there must be carry from the right to left column of z. When i will increase, the carries will affect more columns. For very large i, the carries will bleed on y, as pumping lemma condition two says that y should not be empty.

Power of 2 will be generated for very large value of i which cannot be generated by copying y to i times. The carries will force y as well as z to change. Hence, it is showing that $B_3(A)$ does not satisfy pumping lemma. Hence, the initial assumption that $B_3(A)$ is regular, is wrong.

Hence, $B_3(A)$ is non-regular.