Summary of last session

-Chapter 1. Mathematical Preliminaries
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Chapter 1. Taylor's Theorem

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Chapter 4
Solving Systems of Linear Equations

4.0 Introduction

In this chapter we discuss the numerical aspects of solving systems of linear equations having the following form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{cases}$$

The above system has n equations in n unknowns $x_1, x_2, \cdots, x_n$, and the elements a_{ij} and b_j are assumed to be real numbers.

4.0 Introduction

In this chapter, we shall

- construct a general-purpose algorithm for solving the problem

$$\mathbf{Ax} = \mathbf{b}$$

- analyze the errors that are associated with the computer solution
- study methods for controlling and reducing them
- introduce the important topic of iterative algorithms for this problem.

4.1 Matrix Algebra

- a review of the basic concepts - Matrix

A matrix is a rectangular array of numbers such as

$$\begin{bmatrix} 3.0 & 1.1 & -0.12 \\ 6.2 & 0.0 & 0.15 \\ 0.6 & -4.0 & 1.3 \\ 9.3 & 2.1 & 8.2 \end{bmatrix}, \begin{bmatrix} 3 & 6 & \frac{11}{7} & -17 \end{bmatrix}, \begin{bmatrix} 3.2 \\ -4.7 \\ 0.11 \end{bmatrix}$$

The number of rows and columns is called the <u>dimension</u> of the matrix. Therefore, the aboves are, a $\underline{4 \times 3 \text{ matrix}}$, a $\underline{1 \times 4 \text{ matrix}}$ and a $\underline{3 \times 1 \text{ matrix}}$ respectively.

4.1 Matrix Algebra

- a review of the basic concepts - Transpose

The **transpose** of a matrix is denoted by $\mathbf{A}^{\mathrm{T}}$ and defined by $(\mathbf{A}^{\mathrm{T}})_{ij} = a_{ji}$. For example, if $\mathbf{A}$ denotes

$$\mathbf{A} = \begin{bmatrix} 3.0 & 1.1 & -0.12 \\ 6.2 & 0.0 & 0.15 \\ 0.6 & -4.0 & 1.3 \\ 9.3 & 2.1 & 8.2 \end{bmatrix}$$

then $\mathbf{A}^{\mathrm{T}}$ is

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 3.0 & 6.2 & 0.6 & 9.3 \\ 1.1 & 0.0 & -4.0 & 2.1 \\ -0.12 & 0.15 & 1.3 & 8.2 \end{bmatrix}$$

4.1 Matrix Algebra

- a review of the basic concepts

1. A matrix is called a symmetric matrix, if $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$;

2. A matrix is called a skew-symmetric matrix, if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$.

3. If $\mathbf{A}$ is matrix and λ is a scalar, then $\lambda\mathbf{A}$ is defined by

$$(\lambda\mathbf{A})_{\mathrm{ij}} = \lambda a_{ij}.$$

4. If $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are $m \times n$ matrices, then $\mathbf{A} + \mathbf{B}$ is defined by

$$(\mathbf{A} + \mathbf{B})_{ij} = a_{ij} + b_{ij}.$$

5. $-\mathbf{A}$ means $(-1)\mathbf{A}$.

6. If $\mathbf{A}$ is an $m \times p$ matrix and $\mathbf{B}$ is a $p \times n$ matrix, then $\mathbf{AB}$ is an $m \times n$ matrix defined by

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{p} a_{ik} b_{k\,j} \qquad (1 \le i \le m,\ 1 \le j \le n)$$

4.1 Matrix Algebra

- a review of the basic concepts - **Equivalence**

As we deal with the systems of linear equations, a concept of equivalence is important. Let us consider two systems, each of them consisting of n equations with n unknowns expressed by

$$\mathbf{Ax} = \mathbf{b} \quad \text{and} \quad \mathbf{Bx} = \mathbf{d}$$

If the two systems have precisely the same solutions, we call them **equivalent systems**. Thus, to solve a system of equations, we can solve any equivalent system instead; Given a system of equations to be solved, we transform it by certain elementry operations into a simpler equivalent system, which we then solve instead. This simple idea is at the heart of our numerical procedures.

4.1 Matrix Algebra

- a review of the basic concepts – Elementary Operations

There are the following three types of **elementry operations** :

Note : Here, ε_i denotes the ith equation in the system.

1. Interchanging two equations in the system : $\varepsilon_i \leftrightarrow \varepsilon_j$.
2. Multiplying an equation by a nonzero number : $\lambda\varepsilon_i \to \varepsilon_i$.
3. Adding to an equation a multiple of some other equation :

$$\varepsilon_i + \lambda\varepsilon_j \to \varepsilon_i.$$

Theorem 1 - on Equivalent Systems

If one system of equations is obtained from another by a finite squence of elementry operations, then the two systems are equivalent.

4.1 Matrix Algebra

- Matrix Properties – Identity Matrix

The $n \times n$ matrix $$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is called the **identity matrix**. It has the property that

$$\mathbf{I}\,\mathbf{A} = \mathbf{A} = \mathbf{A}\,\mathbf{I}$$

for any matrix **A** of size $n \times n$.

If **A** and **B** are two matrices such that $\mathbf{AB} = \mathbf{I}$, then we say that **B** is a **right inverse** of **A** and that **A** is a **left inverse** of **B**.

For example, $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.1 Matrix Algebra

- Matrix Properties – Matrix Inverse

Theorem 2 : (Theorem on right inverse)

A square matrix can possess at most one right inverse.

Theorem 3 : (Theorem on matrix inverse)

If $\mathbf{A}$ and $\mathbf{B}$ are square matrices such that $\mathbf{AB}=\mathbf{I}$, then $\mathbf{BA}=\mathbf{I}.$

Example :
$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.1 Matrix Algebra

- Matrix Properties - Matrix Inverse

If a square matrix **A** has a right inverse **B**, then **B** is unique and $\mathbf{BA} = \mathbf{AB} = \mathbf{I}$. Then **B** is called the **inverse** of **A** and say that **A** is **invertible** or **nonsingular** and of course **B** is therefore invertible and **A** is its inverse, we write $\mathbf{B} = \mathbf{A}^{-1}$ and $\mathbf{A} = \mathbf{B}^{-1}$.

For example,

$$\begin{bmatrix} -2 & 1 \\ \frac{2}{3} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{2}{3} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.1 Matrix Algebra

- Matrix Properties – Elementary Matrix

If $\mathbf{A}$ is invertible, then the system of equations $\mathbf{Ax} = \mathbf{b}$ has the solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

If $\mathbf{A}^{-1}$ is already available, this equation provides a good method of computing $\mathbf{x}$. The elementary operations discussed earlier can be carried out with matrix multiplications. An **elementry matrix** is defined to be an $n \times n$ matrix that arise when an elementry operation is applied to the $n \times n$ identity matrix. The elemenetry operations, expressed in terms of the rows of a matrix $\mathbf{A}$, are:

1. The interchange of two rows in $\mathbf{A}$, $\mathbf{A}_s \leftrightarrow \mathbf{A}_t$.
2. Multiplying one row by a nonzero constant: $\lambda\mathbf{A}_s \rightarrow \mathbf{A}_s$.
3. Adding to one row a multiple of another: $\mathbf{A}_s + \lambda\mathbf{A}_t \rightarrow \mathbf{A}_s$

4.1 Matrix Algebra

- Matrix Properties – Finding the inverse of a matrix

To find $\mathbf{A}^{-1}$, think about to apply a succession of elementary row operations to $\mathbf{A}$ such that

$$\mathbf{E}_m\mathbf{E}_{m\text{-}1}\ldots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$$

If $\mathbf{A}$ is invertible, such a sequence of elementary row operation can be applied to it and reducing it to $\mathbf{I}$, i.e.

$$\mathbf{E}_m\mathbf{E}_{m\text{-}1}\ldots\mathbf{E}_2\mathbf{E}_1\mathbf{A}=\mathbf{I}$$

This follows that

$$\mathbf{A}^{-1}=\mathbf{E}_m\mathbf{E}_{m\text{-}1}\ldots\mathbf{E}_2\mathbf{E}_1$$

$\mathbf{A}^{-1}$ is therefore obtained by applying the same sequence of elementary row operations to $\mathbf{I}$.

4.1 Matrix Algebra

- Matrix Properties – Positive definiteness

An important fundamental concept is the **positive definiteness** of a matrix. A matrix $\mathbf{A}$ is possitive definite if $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} > 0$ for any nonzero vector $\mathbf{x}$. For example,

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

is postive definite because

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1 + x_2)^2 + x_1{}^2 + x_2{}^2 > 0$$

for all x_1 and x_2 except $x_1 = x_2 = 0$.

4.2 *LU* and Cholesky Factorizations

- Easy to Solve System

Let us consider a system of n linear equations in n unknowns $x_1,\ x_2, \cdots, x_n$. It can be written in the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

The matrices in this equation are denoted by $\mathbf{A}, \mathbf{x},$ and $\mathbf{b}$. Thus, our system is simply

$$\mathbf{Ax} = \mathbf{b} \qquad (1)$$

4.2 *LU* and Cholesky Factorizations

- Easy to Solve System

We look for special types of systems that can be easily solved. For example, suppose that the $n \times n$ matrix $\mathbf{A}$ has a diagonal structure. This means that all the nonzero elements of $\mathbf{A}$ are on the main diagonal, and system (1) is

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{x} = \begin{bmatrix} b_1/a_{11} \\ b_2/a_{22} \\ b_3/a_{33} \\ \vdots \\ b_n/a_{nn} \end{bmatrix}$$

In this case our system collapses to n simple equations, and the solution is shown above. If $a_{ii} = 0$ for some index i, and if $b_i = 0$ also, then x_i can be any real number. If $a_{ii} = 0$ and $b_i \neq 0$, no solution of the system exists.

4.2 *LU* and Cholesky Factorizations

- Easy to Solve System

We are searching for easy solutions of system (1), we assume a lower triangular structure for **A**. This means that all the nonzero elements of **A** are situated on or below the main diagonal, and system (1) is

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

To solve this, we assume that $a_{ii} \neq 0$ for all i; then obtain x_1 from the first equation. With the known value of x_1 substituted into the second equation, solve the second equation for x_2. We proceed in the same way, obtaining $x_1, x_2, \cdots, x_n$, one at a time and in this order.

4.2 *LU* and Cholesky Factorizations

- *LU*-Factorizations

Suppose $\mathbf{A}$ can be factored into the product of a lower triangular matrix $\mathbf{L}$ and an upper triangular matrix $\mathbf{U}$, that is $\mathbf{A} = \mathbf{LU}$.

Then, to solve the system of equations

$$\mathbf{Ax} = \mathbf{b}$$

it is to solve this problem in two stages:

$$\mathbf{L\,z} = \mathbf{b}, \qquad \text{solve for } \mathbf{z}$$

$$\mathbf{Ux} = \mathbf{z} \qquad \text{solve for } \mathbf{x}$$

Our previous analysis indicates that solving these two triangular systems is simple. We shall show how the factorization $\mathbf{A} = \mathbf{LU}$ can be carried out, provided that in certain steps of the computation 0 divisors are not encountered. (Note: not every matrix has such a factorization).

4.2 *LU* and Cholesky Factorizations

We begin with an $n \times n$ matrix $\mathbf{A}$ and search for matrices

$$\mathbf{L} = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

such that
$$\mathbf{A} = \mathbf{L}\mathbf{U} \tag{2}$$
when this is possible, we say that $\mathbf{A}$ has an **LU**-*decomposition*. It turns out that $\mathbf{L}$ and $\mathbf{U}$ are not uniquely determined by (2). In fact for each i, we can assign a nonzero value to either l_{ii} or u_{ii} (but not both).

For example, one simple choice is to set

$$l_{ii} = 1 \qquad \text{for } i = 1, 2, 3, \cdots, n,$$

thus making $\mathbf{L}$ unit lower triangular. Another obvious choice is to make $\mathbf{U}$ unit upper triangular ($u_{ii} = 1$ for each i).

4.2 *LU* and Cholesky Factorizations

- *LU*-Factorization

To derive an algorithm for the **LU**-factorization of **A**, we consider

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Comparing the elements in the first rows on both sides, we have

$$u_{1k} = a_{1k}, \quad k = 1, 2, \cdots, n;$$

Comparing the elements in the first column on both sides, we have

$$l_{k1} = a_{k1} / u_{11}, \quad k = 2, 3, \cdots, n;$$

Comparing the rest elements in the second row, we have

$$u_{2k} = a_{2k} - l_{21} u_{1k}, \quad k = 2, 3, \cdots, n;$$

Comparing the rest elements in the second column, we have

$$l_{k2} = (a_{k2} - l_{k1} u_{12}) / u_{22}, \quad k = 3, 4, \cdots, n;$$

4.2 *LU* and Cholesky Factorizations

– *LU*-Factorization

In general, after obtain u_{1k} and l_{k1} $(k = 1,2,\cdots,n)$, for $i = 2,3,\cdots,n$, we can get each element for **U** and **L** step by step using the following formula:

$$\begin{cases} u_{ik} = a_{ik} - \sum_{j=1}^{i-1} l_{ij} u_{jk}, \\ l_{ki} = \left[a_{ki} - \sum_{j=1}^{i-1} l_{kj} u_{ji} \right] / u_{ii} \end{cases} \qquad k = i,\ i+1,\cdots,\ n \qquad (3)$$

This is the procedure of the Doolittle's factorization.

The calculation order is such as

$$\begin{array}{ll} u_{11}, u_{12}, \cdots\cdots, u_{1n}; & l_{21}, l_{31}, \cdots\cdots, l_{n1}; \\ u_{22}, \cdots\cdots, u_{2n}; & l_{32}, \cdots\cdots, l_{n2}; \\ \vdots & \vdots \\ u_{n-1\,n-1}, u_{n-1\,n}; & l_{n\,n-1}; \\ u_{nn}; & . \end{array}$$

4.2 *LU* and Cholesky Factorizations

- *LU*-Factorization – memory/calculation time saving

We have made use of the fact that, for a fixed *i*, when u_{ik} has been calculated, a_{ik} is not needed any more in the subsequent calculations, therefore the value of u_{ik} can be stored in the memory element for a_{ik}; and similarly when l_{ki} is calculated, a_{ki} is also not needed any more and the calculated value of l_{ki} can be stored in the memory element for a_{ki}. In fact, the storing elements for **A** now for **U** and **L** becomes

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \to \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ l_{21} & u_{22} & u_{23} & \cdots & u_{2n} \\ l_{31} & l_{32} & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & u_{nn} \end{bmatrix}$$

It is noted that the calculations (3) for u_{ik} and l_{ki} can be carried out simultaneously. On some computers this can actually be done, with a considerable savings in execution time.

4.2 *LU* and Cholesky Factorizations

- *LU*-Factorization – *LDU* factorization

The computation of the *ith* row in **U** and the *i*th column in **L** as described completes the *ith* step in the algorithm. This decomposion of A is called **Doolittle's factorization**. If the upper matrix is a unit triangular matrix, then the decomposition is called **Crout's factorization**. We can have following further factorization

$$\mathbf{A} = \mathbf{LDU}$$

where **D** is a diagonal matrix, and **L**, **U** are unit lower and upper triangular matrices. This is the **LDU** factorization. It can be done through the further factorization of **U** or **L**. When $\mathbf{U} = \mathbf{L}^{\mathrm{T}}$, so that $l_{ii} = u_{ii}$ for $i = 1, 2, 3, \ldots, n$, that is $\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$, the algorithm is called **Cholesky's factorization**. We shall discuss the Cholesky method in more detail later in this section since this factoring requires the matrix **A** to have several special properties; namely, **A** should be real, symmetric, and positive definite.

4.2 *LU* and Cholesky Factorizations

- *LU*-Factorization

Example :

Find the Doolittle, Crout, and Cholesky factorizations of **A**

$$A = \begin{bmatrix} 60 & 30 & 20 \\ 30 & 20 & 15 \\ 20 & 15 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{cases} u_{ik} = a_{ik} - \sum_{j=1}^{i-1} l_{ij} u_{jk}, \\ l_{ki} = \left[a_{ki} - \sum_{j=1}^{i-1} l_{kj} u_{ji} \right] / u_{ii} \end{cases} \qquad k = i,\ i+1, \ldots, n$$

4.2 *LU* and Cholesky Factorizations

- *LU*-Factorization

$$i=1,\ \ u_{11}=a_{11}=60;\ u_{12}=a_{12}=30;\ u_{13}=a_{13}=20;$$

$$l_{21}=a_{21}/u_{11}=30/60=1/2;\ l_{31}=a_{31}/u_{11}=20/60=1/3;$$

$$i=2,\ u_{22}=a_{22}-l_{21}u_{12}=20-\frac{1}{2}\times 30=5$$

$$u_{23}=a_{23}-l_{21}u_{13}=15-\frac{1}{2}\times 20=5$$

$$l_{32}=(a_{32}-l_{31}u_{12})/u_{22}=(15-\frac{1}{3}\times 30)/5=1$$

$$i=3,\ u_{33}=a_{33}-(l_{31}u_{13}+l_{32}u_{23})=12-(\frac{1}{3}\times 20+1\times 5)=\frac{1}{3}$$

$$\mathbf{A}=\begin{bmatrix}60&30&20\\30&20&15\\20&15&12\end{bmatrix}=\begin{bmatrix}1&0&0\\1/2&1&0\\1/3&1&1\end{bmatrix}\begin{bmatrix}60&30&20\\0&5&5\\0&0&1/3\end{bmatrix}=\mathbf{LU}$$

4.2 *LU* and Cholesky Factorizations

- Cholesky Factorizations

We put the diagonal elements of $\mathbf{U}$ into a diagonal matrix $\mathbf{D}$, and obtain

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 60 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{LD\hat{U}}$$

By putting $\mathbf{\hat{L}} = \mathbf{LD}$, we obtain the Crout factorization.

$$\mathbf{A} = \begin{bmatrix} 60 & 0 & 0 \\ 30 & 5 & 0 \\ 20 & 5 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{\hat{L}\hat{U}}$$

The Cholesky factorization is obtained by splitting $\mathbf{D}$ into the form $\mathbf{D}^{1/2}\mathbf{D}^{1/2}$ in the $\mathbf{LDU}$-factorization and associating one factor with $\mathbf{L}$ and the other with $\mathbf{\hat{U}}$.

4.2 *LU* and Cholesky Factorizations

- Cholesky Factorizations

That is

$$\mathbf{A}=\begin{bmatrix}1&0&0\\1/2&1&0\\1/3&1&1\end{bmatrix}\begin{bmatrix}\sqrt{60}&0&0\\0&\sqrt{5}&0\\0&0&\frac{1}{3}\sqrt{3}\end{bmatrix}\begin{bmatrix}\sqrt{60}&0&0\\0&\sqrt{5}&0\\0&0&\frac{1}{3}\sqrt{3}\end{bmatrix}\begin{bmatrix}1&1/2&1/3\\0&1&1\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}\sqrt{60}&0&0\\\frac{1}{2}\sqrt{60}&\sqrt{5}&0\\\frac{1}{3}\sqrt{60}&\sqrt{5}&\frac{1}{3}\sqrt{3}\end{bmatrix}\begin{bmatrix}\sqrt{60}&\frac{1}{2}\sqrt{60}&\frac{1}{3}\sqrt{60}\\0&\sqrt{5}&\sqrt{5}\\0&0&\frac{1}{3}\sqrt{3}\end{bmatrix}=\widetilde{\mathbf{L}}\widetilde{\mathbf{L}}^{\mathrm{T}}$$

4.2 *LU* and Cholesky Factorizations

- Cholesky Factorization

Theorem on LU - Decomposition

Theorem 1: If all n leading principal minors of $n \times n$ matrix A are nonsingular, then A has an LU decomposition.

Proof : Ignore.

Cholesky Theorem on $\mathbf{LL^T}$ - Factorization

Theorem 2 : If **A** is a real, symmetric, and positive definite matrix, then it has a unique factorization, $\mathbf{A} = \mathbf{LL}^T$, in which **L** is lower triangular with a positive diagonal.

Proof : Ignore.

4.2 *LU* and Cholesky Factorizations

- LL^T Cholesky Factorization

Cholesky factorization is a special case of the general **LU** factorization. If **A** is real, symmetric, and positive definite, then by Theorem 2 it has a unique factorization of the form $\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$, in which **L** is lower trigangular and has positive diagonal. Comparing both sides of $\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & \cdots & l_{n1} \\ 0 & l_{22} & \cdots & l_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & l_{nn} \end{bmatrix}$$

we have

$$a_{jj} = l_{j1}^2 + \cdots + l_{jj}^2$$

4.2 *LU* and Cholesky Factorizations

- LL^T Cholesky Factorization

(we have $a_{jj} = l_{j1}^2 + \cdots + l_{jj}^2$)

Thus, the diagonal entry is computed by

$$l_{jj} = (a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2)^{1/2}$$

and other elements are related to a_{ij} by

$$a_{ij} = l_{i1}l_{j1} + \cdots + l_{ij}l_{jj} \qquad j < i$$

and calculated by $l_{ij} = (a_{ij} - \sum_{k=1}^{j-1} l_{ik}l_{jk})/l_{jj}$

Thus the formula of the **square root** method is

$$\left\{\begin{aligned} & l_{jj} = (a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2)^{1/2} \\ & l_{ij} = (a_{ij} - \sum_{k=1}^{j-1} l_{ik}l_{jk})/l_{jj}, (i = j+1, \cdots, n) \end{aligned}\right. \qquad j = 1, 2, \cdots, n$$

It is seen that for any $k \le j$, we have

$$|l_{jk}| \le \sqrt{a_{jj}}$$

4.2 *LU* and Cholesky Factorizations

- Example

Use the Gauss elimination method to find solutions for a system

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 6 \\ x_1 - x_2 + 5x_3 = 0 \\ 4x_1 + x_2 - 2x_3 = 2 \end{cases}$$

To eliminate x_1 from the second and third equations, we substract $\frac{1}{2}$ times the first equation from the second, and substract 2 times the first equation from the third, and get an equivelent system as

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 6 \\ -3x_2 + 6x_3 = -3 \\ -7x_2 + 2x_3 = -10 \end{cases}$$

4.2 *LU* and Cholesky Factorizations

- Example

Similarly, we substract $\frac{7}{3}$ times the second equation from the third to remove x_2 from the third equation, and get

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 6 \\ -3x_2 + 6x_3 = -3 \\ -12x_3 = -3 \end{cases}$$

We then start back substitution and easily get the solution

$$\begin{cases} x_3 = \dfrac{1}{4} \\ x_2 = \dfrac{3}{2} \\ x_1 = \dfrac{1}{4} \end{cases}$$

4.2 *LU* and Cholesky Factorizations

- Example

Find the Doolittle, Crout and **LDU** factorizations for **A**

$$\mathbf{Ax} = \begin{bmatrix} 2 & 4 & -2 \\ 1 & -1 & 5 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} = \mathbf{b}$$

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & -1 & 5 \\ 4 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & 7/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & -3 & 6 \\ 0 & 0 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & 7/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -12 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \\ 4 & -7 & -12 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercises

*Ex*1. Using the block form of matrix to prove: Doing elementary row operations for $A_{m\times n}$ is equivalent to multiplying on left by a corresponding elementary matrix. Similarly, Doing elementary column operations for $A_{m\times n}$ is equivalent to multiplying on righ by a corresponding elementary matrix.

*Ex*2. Let $E(p,q,\lambda)$ be the matrix that results from the $n\times n$ identity matrix when λ times row q is added to row p.(Assume that $p\neq q$) Prove that

$$E^{-1}(p,q,\lambda)=E(p,q,-\lambda)$$

*Ex*3. Prove that if A is skew-symetric, then the diagonal elements are 0. Moreover, if the order of the matrix is odd, then the matrix is singular.

Ex4. Whether the matrix $\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$ is positive definite?

习题

$Ex5$. Find the LU-factorization of the matrix

$$A=\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}$$

$Ex6$. Factor the matrix $A=\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, so that $A=LL^T$, where L is lower triangular.