Question:

Let J =
$$\{w \mid \text{either } w = 0x \text{ for some } x \in A_{\mathsf{TM}}, \text{ or } w = 1y \text{ for some } y \in \overline{A_{\mathsf{TM}}}\ \}.$$
is Turing-recognizable.

Answer:

----SETP1----

Turing-recognizable

Firstly demonstrate the reduction $f : \Sigma^* \rightarrow \Sigma^*$ of $\overline{A_{TM}}$ to J.

Assume a string $z \in \Sigma^*$. So that $f(z) = 1z$.

By definition of J, $z \in \overline{A_{TM}}$ iff $1z \in J$.

Hence f is a reduction of $\overline{A_{TM}}$ to J, Thus $\overline{A_{TM}} \leq_m J$.

By using the Corollary:

"If $\overline{A_{TM}} \leq_m B$, A is not a Turing-recognizable, then B is not Turing-recognizable."

Because $\overline{A_{TM}}$ is not Turing-recognizable, by Corollary J is not Turing-recognizable.

----SETP2----

Now demonstrate the reduction $f : \Sigma^* \rightarrow \Sigma^*$ of A_{TM} to J.

Assume a string $t \in \Sigma^*$. So that $g(t) = 0t$.

By definition of J, $t \in A_{TM}$ iff $0t \in J$.

Hence g is reduction of A_{TM} to J, Thus $A_{TM} \leq_m J$.

A function which reduces language L_1 to language L_2 also reduces $\overline{L_1}$ to language $\overline{L_2}$. Hence, g is reduction from $\overline{A_{TM}}$ to $\overline{J}$, Thus $\overline{A_{TM}} \leq_m \overline{J}$.

By using the Corollary:

"If $\overline{A_{TM}} \leq_m B$, A is not a Turing-recognizable, then B is not Turing-recognizable."

Because $\overline{A_{TM}}$ is not Turing-recognizable, by Corollary $\overline{J}$ is also not Turing-recognizable.

Therefore neither J nor $\overline{J}$ is Turing-recognizable.