Question:

If A and B are languages, define

$$A \diamond B = \{xy \mid x \in A \text{ and } y \in B \text{ and } |x| = |y|\}.$$

Answer:

----SETP1----

Consider the two regular languages *A* and *B* over the input alphabet Σ. The language $A \diamond B$ is defined as, $A \diamond B = \{xy \mid x \in A \text{ and } y \in B \text{ and } |x| = |y|\}$. For the language $A \diamond B$, if PDA is constructed then it can be said that $A \diamond B$ is in CFL.

----SETP2----

Consider the DFA $D_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$ and $D_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$ for the languages *A* and *B* respectively. The strings of the language $A \diamond B$ are formed by concatenating equal length strings from *A* and *B*. Construct the PDA $M = (Q, \Sigma, \Gamma, \delta, q_{start}, F)$ for the language $A \diamond B$ as,

• From the start state q_{start}, push the symbol $ into the stack to know the bottom of the stack.

• For every symbol from the language *A*, push 1 into the stack. It guesses the end of the string that belongs to *A* when it reaches to the final state of D_A.

• Then, for every symbol from the language *B*, pop 1 from the stack. When the string reaches the final state of D_B, it moves to the final state *F* if and only if the top of the stack is $.

The PDA can be constructed informally as above described.

----SETP3----

The formal description of the PDA $M = (Q, \Sigma, \Gamma, \delta, q_{start}, F)$ is as follows:

• $Q = Q_A \cup Q_B \cup \{q_{start}, F\}$

• Σ is the input alphabet for *A* and *B*

• $\Gamma = \{\$, 1\}$ where $ is the symbol used to know the bottom of the stack and 1 is pushed every time into the stack when the symbol read from the language *A*.

• The transition function is defined as,

$$\delta(q_{start}, \varepsilon, \varepsilon) = \{q_A, \$\}$$
$$\delta(q, a, \varepsilon) = \{\delta_A(q, a), 1\} \qquad \text{if } q \in Q_A, \quad a \in \Sigma$$
$$\delta(q, \varepsilon, \varepsilon) = \{q_B, \varepsilon\} \qquad \text{if } q \in F_A$$
$$\delta(q, a, 1) = \{\delta_B(q, a), \varepsilon\} \qquad \text{if } q \in Q_B, \quad a \in \Sigma$$
$$\delta(q, \varepsilon, \$) = \{F, \varepsilon\} \qquad \text{if } q \in F_B$$

Any other transitions apart from this will not be accepted.

- q_{start} is the start state.
- *F* is the final state.

The PDA non deterministically guesses the end of the string from D_A and transitions to the start symbol of D_B if it is in a final state of D_A. The PDA *M* accepts the string when it is in accepting state of D_B while hitting the empty stack. This shows that $L(M) = A \diamond B$.

Therefore, for any two regular languages *A* and *B*, $A \diamond B$ is in CFL.