



# Cholesky分解法(平方根法)

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- i) 记 $D=\text{diag}(u_{11}, u_{22}, \dots, u_{nn})$ ,  $\hat{U} = D^{-1}U$

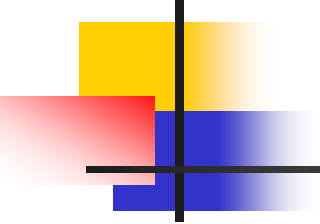
$$A=LU=(LD)(D^{-1}U)=LD\hat{U} \quad (\text{分解惟一})$$

其中 $L$ 是单位下三角阵,  $\hat{U}$ 是单位上三角阵.

- 如果 $A$ 是对称正定的, 则 $\hat{U}=L^T$ ,

$$A = LDL^T = (LD^{\frac{1}{2}})(LD^{\frac{1}{2}})^T = \bar{L}\bar{L}^T$$

其中 $\bar{L}$ 是下三角阵



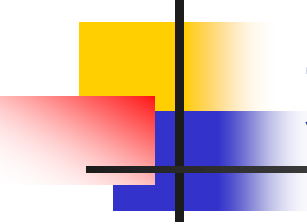
$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \bar{l}_{11} & 0 & \cdots & 0 \\ \bar{l}_{21} & \bar{l}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{l}_{n1} & \bar{l}_{n2} & \cdots & \bar{l}_{nn} \end{pmatrix} \begin{pmatrix} \bar{l}_{11} & \bar{l}_{21} & \cdots & \bar{l}_{n1} \\ 0 & \bar{l}_{22} & \cdots & \bar{l}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{l}_{nn} \end{pmatrix} = \bar{\mathbf{L}}\bar{\mathbf{L}}^T$$

称为矩阵的Cholesky分解.

利用矩阵运算比较两边得其分解公式: 对 $k=1,2,3,\dots,n$ ,

$$\begin{cases} \bar{l}_{kk} = (a_{kk} - \sum_{r=1}^{k-1} \bar{l}_{kr}^2)^{1/2} \\ \bar{l}_{ik} = (a_{ik} - \sum_{r=1}^{k-1} \bar{l}_{ir} \bar{l}_{kr}) / \bar{l}_{kk} \quad (i = k+1, k+2, \dots, n) \end{cases}$$

其中  $\sum_1^0 = 0$



求解  $Ax=b \Leftrightarrow$  求解

$$\begin{cases} \bar{L}y = b \\ \bar{L}^T x = y \end{cases}$$

■ ii) 求解方程组  $\bar{L}y=b$ , 即

$$\begin{pmatrix} \bar{l}_{11} & 0 & \cdots & 0 \\ \bar{l}_{21} & \bar{l}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{l}_{n1} & \bar{l}_{n2} & \cdots & \bar{l}_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

其计算公式为

$$y_k = \left( b_k - \sum_{r=1}^{k-1} \bar{l}_{kr} y_r \right) / \bar{l}_{kk} \quad (k = 1, 2, \cdots, n)$$

其中  $\sum_{r=1}^0 = 0$

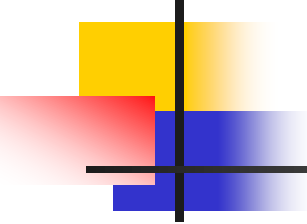
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- iii) 求解方程组  $\bar{L}^T x = y$ , 即

$$\begin{pmatrix} \bar{l}_{11} & \bar{l}_{21} & \cdots & \bar{l}_{n1} \\ 0 & \bar{l}_{22} & \cdots & \bar{l}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{l}_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

其计算公式为

$$x_k = (y_k - \sum_{r=k+1}^n \bar{l}_{rk} x_r) / \bar{l}_{kk} \quad (k = n, n-1, \dots, 2, 1)$$

其中  $\sum_{r=n+1}^n = 0$

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- Cholesky分解法的存储可利用原系数矩阵(对称)的存储单元, 它没有增加储存单元, 存储形式.

$$\begin{pmatrix} \bar{l}_{11} & & \cdots & \\ \bar{l}_{21} & \bar{l}_{22} & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ \bar{l}_{n1} & \bar{l}_{n2} & \cdots & \bar{l}_{nn} \end{pmatrix} \rightarrow A$$

- 平方根法是用于解系数矩阵为正定矩阵的线性方程组.

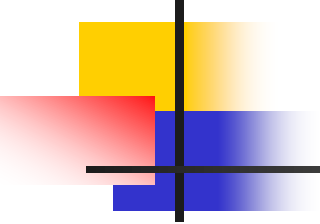
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- 例 用平方根法 (Cholesky分解) 解方程组

$$\begin{bmatrix} 3 & 2 & 3 \\ 2 & 2 & 0 \\ 3 & 0 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$$

解 由于系数矩阵 $A$ 对称正定, 故一定有分解形式  $A = \bar{L}\bar{L}^T$   
其中 $\bar{L}$ 为下三角阵.

(1) 对系数矩阵进行Cholesky分解 $A = \bar{L}\bar{L}^T$ , 即

$$\begin{bmatrix} 3 & 2 & 3 \\ 2 & 2 & 0 \\ 3 & 0 & 12 \end{bmatrix} = \begin{bmatrix} \bar{l}_{11} & & \\ \bar{l}_{21} & \bar{l}_{22} & \\ \bar{l}_{31} & \bar{l}_{32} & \bar{l}_{33} \end{bmatrix} \begin{bmatrix} \bar{l}_{11} & \bar{l}_{21} & \bar{l}_{31} \\ & \bar{l}_{22} & \bar{l}_{32} \\ & & \bar{l}_{33} \end{bmatrix}$$



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由公式得  $\bar{l}_{11} = \sqrt{3}, \quad \bar{l}_{21} = \frac{2}{\sqrt{3}}, \quad \bar{l}_{31} = \sqrt{3}$

$$\bar{l}_{22} = \sqrt{\frac{2}{3}}, \quad \bar{l}_{32} = -\sqrt{6}, \quad \bar{l}_{33} = \sqrt{3}$$

解方程组  $\bar{L}\mathbf{y}=\mathbf{b}$  , 即

$$\begin{bmatrix} \sqrt{3} & & \\ \frac{2}{\sqrt{3}} & \sqrt{\frac{2}{3}} & \\ \sqrt{3} & -\sqrt{6} & \sqrt{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$$

得  $\mathbf{y} = (\frac{5}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}})^T$

解方程组  $\bar{L}^T \mathbf{x}=\mathbf{y}$  , 即

$$\begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \sqrt{3} \\ & \sqrt{\frac{2}{3}} & -\sqrt{6} \\ & & \sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

得  $\mathbf{x} = (1, \frac{1}{2}, \frac{1}{3})^T$



## 改进的Cholesky分解法(改进的平方根法)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \begin{pmatrix} 1 & l_{21} & \cdots & l_{n1} \\ 0 & 1 & \cdots & l_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = LDL^T$$

利用矩阵运算比较两边得其分解公式:对 $k=1,2,3,\dots,n$ ,

$$\begin{cases} d_k = a_{kk} - \sum_{r=1}^{k-1} l_{kr}^2 d_r \\ l_{ik} = (a_{ik} - \sum_{r=1}^{k-1} l_{ir} d_r l_{kr}) / d_k \quad (i = k+1, k+2, \dots, n) \end{cases}$$

其中  $\sum_{r=n+1}^n = 0$



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- ii) 求解方程组  $\mathbf{L}\mathbf{y}=\mathbf{b}$

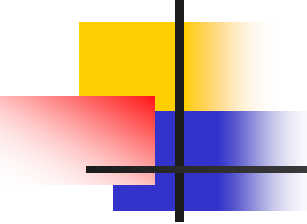
$$y_k = b_k - \sum_{r=1}^{k-1} l_{kr} y_r \quad (k=1, 2, \dots, n)$$

其中  $\sum_{r=1}^0 = 0$

- iii) 求解方程组  $\mathbf{L}^T \mathbf{x} = \mathbf{D}^{-1} \mathbf{y}$

$$x_k = y_k / d_k - \sum_{r=k+1}^n l_{rk} x_r \quad (k=n, n-1, \dots, 2, 1)$$

其中  $\sum_{r=n+1}^n = 0$

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- 改进的Cholesky分解法的存储可利用原系数矩阵(对称)的存储单元, 它没有增加储存单元, 存储形式.

$$\begin{pmatrix} d_1 & & \cdots & \\ l_{21} & d_2 & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & d_n \end{pmatrix} \rightarrow A$$

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- 例 用改进的平方根法解方程组

$$\begin{bmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \\ 30 \end{bmatrix}$$

- 解系数矩阵是对称的, 故可分解为 $LDL^T$ , 设有分解

$$\begin{bmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix} = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ & 1 & l_{32} \\ & & 1 \end{bmatrix}$$

- 由公式得

$$d_1 = 3, \quad l_{21} = 1, \quad l_{31} = \frac{5}{3}$$

$$d_2 = 2, \quad l_{32} = 2, \quad d_3 = \frac{2}{3}$$



解方程组 $Ly=b$ ，即

$$\begin{bmatrix} 1 & & \\ 1 & 1 & \\ \frac{5}{3} & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \\ 30 \end{bmatrix}$$

得  $y = (10, 6, \frac{4}{3})^T$   
解方程组 $L^T x = D^{-1}y$ ，即

$$\begin{bmatrix} 1 & 1 & \frac{5}{3} \\ & 1 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & & \\ & \frac{1}{2} & \\ & & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 10 \\ 6 \\ \frac{4}{3} \end{bmatrix}$$

得  $x = (1, -1, 2)^T$