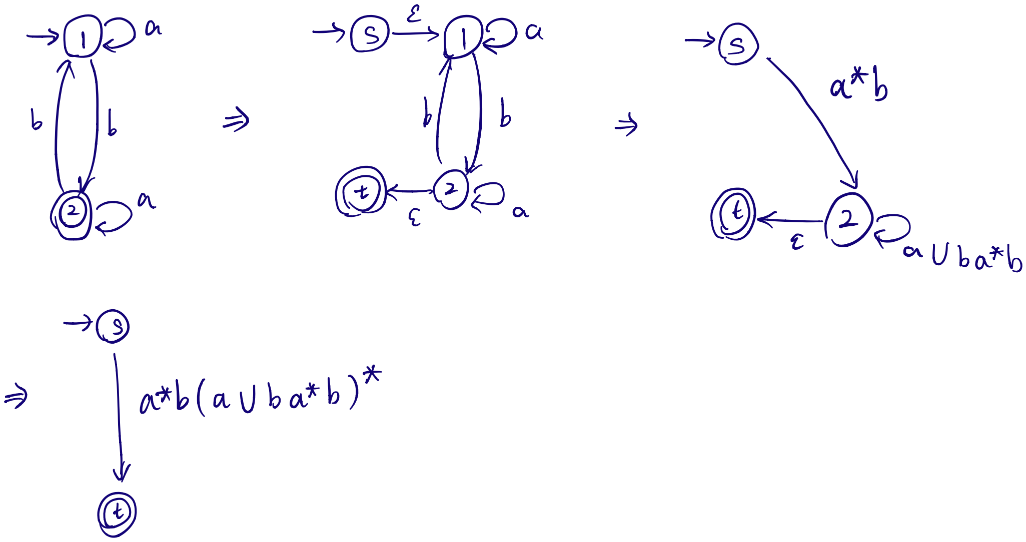
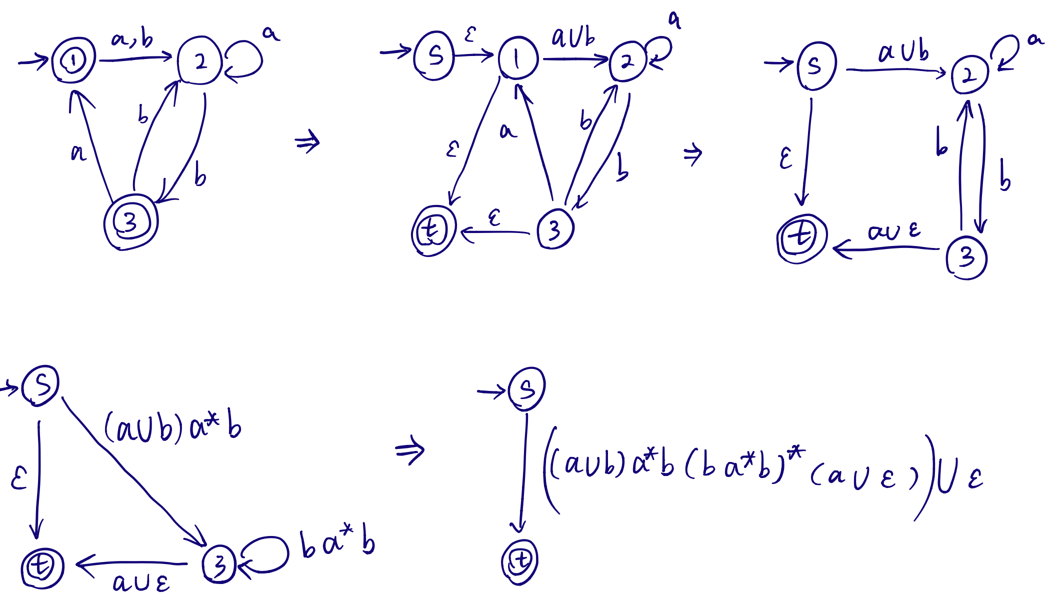
1. (a)


a
b
b
a
S
ε
1
2
t
a*b
a ∪ b a*b
a*b(a ∪ b a*b)*

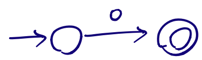
(b)


a,b
a
b
b
aUb
ε
a ∪ ε
(aUb)a*b
b a*b
((aUb)a*b(ba*b)*(a ∪ ε)) ∪ ε

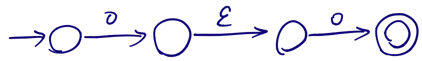
2.

0:

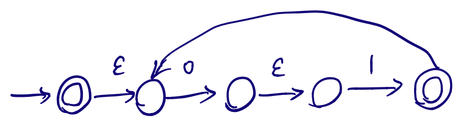
0:



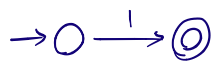
00:



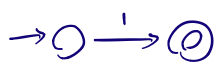
$(00)^*$:



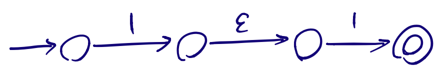
1:



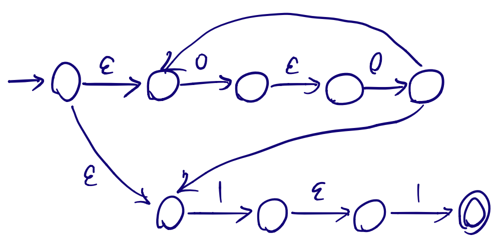
1:



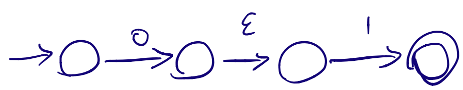
11:



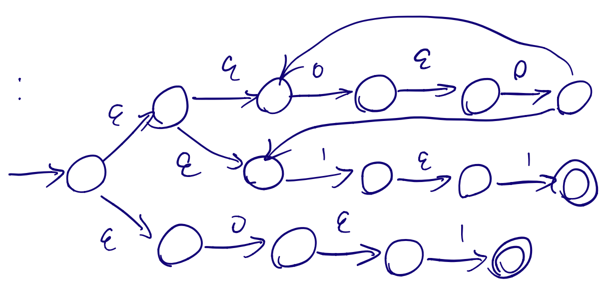
$(00)^*(11)$:



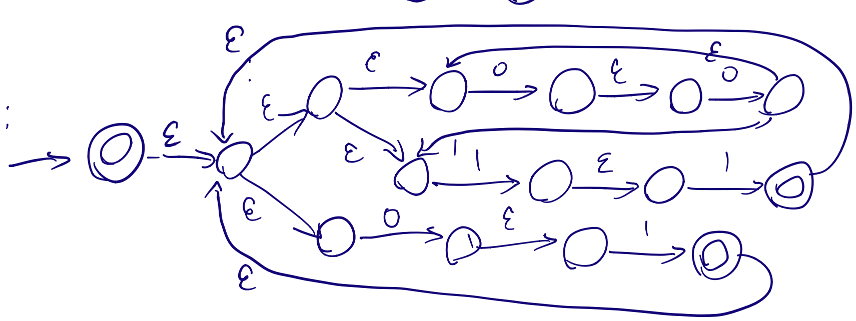
01:



$((00)^*(11) \cup 01)$:



$((00)^*(11) \cup 01)^*$:



3. $A_2 = \{ w \in \{a, b\}^* \mid w = w^R \}$ is not regular.

proof: Assume that A_2 is regular. Let p be the pumping length, and then let string s is " $a^p b a^p$ ". According to pumping lemma, because s is in A_2 and the length of s is bigger than p, s could be devided into 3 segments $s = xyz$

① According to Condition 3: $|xy| \leq p$; so y only contains "a". we can suppose that there are q "a"s in y;

② According to Condition 2, $|y| > 0$, so $q > 0$

③ According to Condition 1: $\forall i \geq 0$, $xy^iz \in A_2$.

Let $i = 2$, $s = xyyz = a^{p+q} b a^p$ should be a palindrome, but $p+q \neq p$, so $s \notin A_2$.

Hence, A_2 is not regular.

4. $B_n = \{a^k \mid k$ is a multiple of $n\}$ is regular.

Proof: Let alphabet Σ be $\{a\}$, according to the definition of regular expression:

① if R_1 and R_2 are regular, then $R_1 \circ R_2$ is regular.

② if R is regular, then R^* is regular.

So, a is regular, $a^2 = a \circ a$ is also regular, we can get:

$a^n = a \circ a \circ \cdots \circ a$ is regular for each $n \geq 1$

Then, $(a^n)^*$ is also regular, and the regular expression $(a^n)^*$ represent all the multiples k of n, which k is from 0 to infinite.

To sum up, $B_n = (a^n)^*$ is regular.

5. (a) $B = \{1^k y \mid y \in \{0,1\}^*$, and y contains at least k 1s, for $k \geq 1\}$.

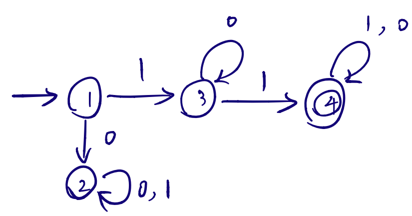
proof:

Let R represent a regular expression, and R^k represent $R \circ R \circ \cdots \circ R$, which contains k Rs, so R^k is also a regular expression. Trivially, $R_1 \cup R_2$ and R^* are also regular expressions. We can generate the regular expression of B according to the conditions above.

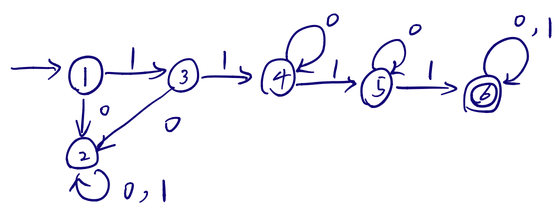
$$RE(B) = 1^k (0^* 1)^k (0 \cup 1)^*$$

For more intuitive, we can draw a diagram, for example, the diagrams with $k=1,2$ are drawn as following:

$k=1$,



$k=2$,



Then we prove $y' = \{y \mid y \in \{0,1\}^*$ and y contains at least k 1s$\}$ and $y'' = (0^* 1)^k (0 \cup 1)^*$ are isovalent.

For any $y' = (0 \cup 1)^*$, and y' contains at leats k 1s, we can devide y' into $k+1$ segments. In the first k segment(s), each segment ends with 1, like: 1, 01, 0....01, so they can be sum up to (0^*1), and can be connected to $(0^*1)^k$, And the last segment of y' is still a $(0\cup1)^*$, so y' can rewrite to $(0^*1)^k(0\cup1)^*$, in other word, y''. QED

(b) $C = \{1^k y \mid y \in \{0,1\}^*$, and y contains at most k 1s, for $k \geq 1\}$

proof: Assume that C is regular. Let p be the pumping length, and select $\underline{1}^p 0 \underline{1}^p$ as s, which satisfies the difinite of C. The length of s is bigger than p, so according to pumping lemma, $s = xwz$, $|xw| \leq p$, and $xw^iz \in C$, $\forall i \geq 0$

Let $i=0$, then $s = xz = \underline{1}^{p-q} 0 \underline{1}^p = \underline{1}^k y$ $(|y|>0)$

From this, ① y must contain the substring "$0\underline{1}^p$" because $\underline{1}^k$ can only contains 1, so ② $k \leq p-q$. But ③ y contains more than p 1s because of ①

so, sum up with ②③ $|\underline{1}^k| = k \leq p-q < p \leq |\{p \mid p=1 \,\&\, p \in y\}|$

in other word, y contains more than k 1s, so $s \notin C$.

Hence, C is not regular. QED.