3.3 THE SECANT METHOD

In this method, we choose two points x_n and x_{n-1} such that $f\left(x_n\right)$ and $f\left(x_{n-1}\right)$ are of opposite signs. Now, the equation of the chord joining the points $f\left(x_n\right)$ and $f\left(x_{n-1}\right)$ is from the graph on next page

$$\frac{y-f\left(x_n\right)}{f\left(x_{n-1}\right)-f\left(x_n\right)}=\frac{x-x_n}{x_{n-1}-x_n} \qquad (1)$$

Setting $y=0$ in Equation (1), we get

$$x=x_n-\frac{x_n-x_{n-1}}{f\left(x_n\right)-f\left(x_{n-1}\right)}f\left(x_n\right)$$

Hence, the first approximation to the root of $f\left(x\right)=0$ is given by

$$x_{n+1}=x_n-\frac{x_n-x_{n-1}}{f\left(x_n\right)-f\left(x_{n-1}\right)}f\left(x_n\right) \qquad \left(n\geq 1\right) \qquad (2)$$

(continued)

3.3 THE SECANT METHOD

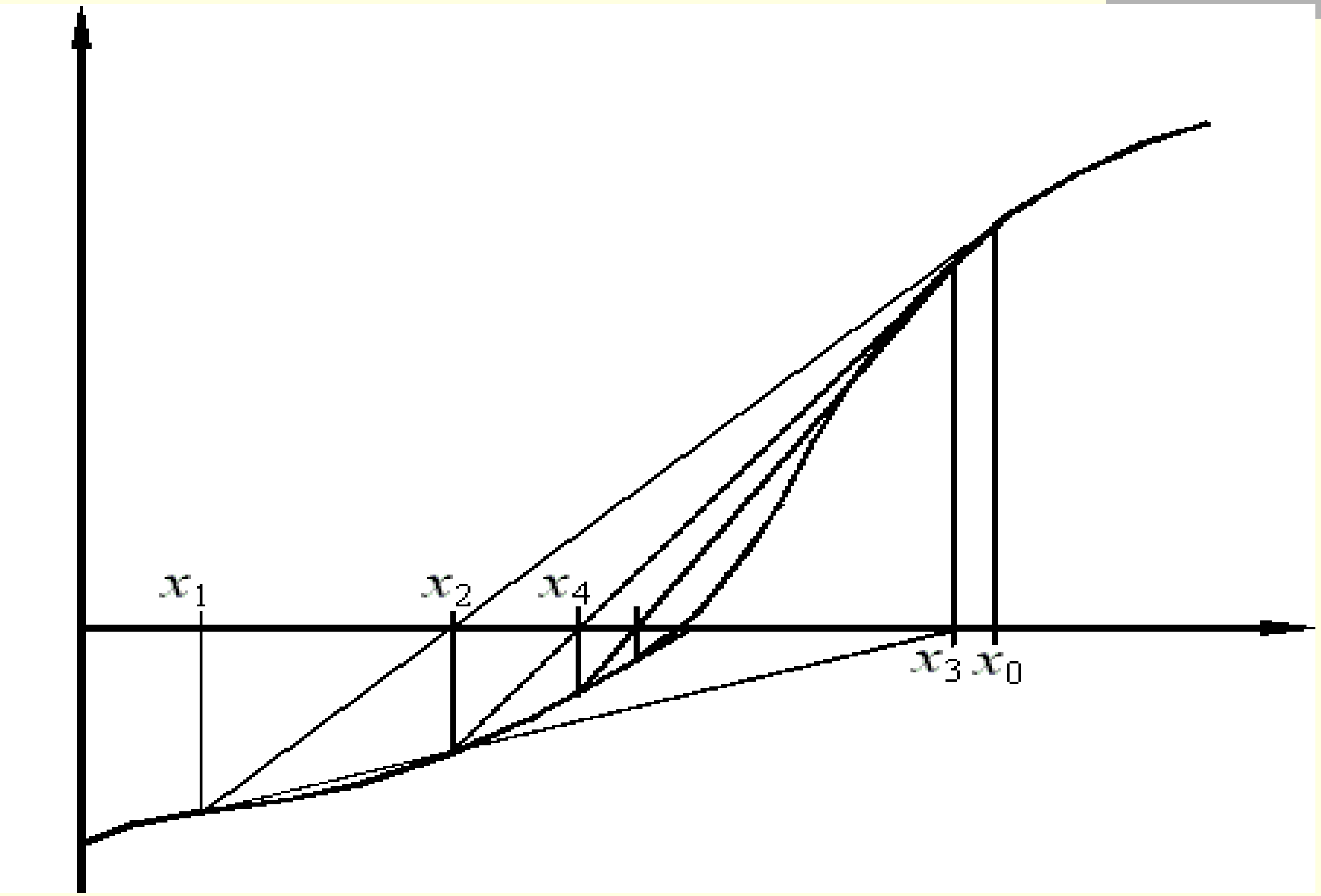


Figure 1 Graphical illustration of the secant method.

3.3 THE SECANT METHOD

We observe that $f\left(x_{n-1}\right)$ and $f\left(x_n\right)$ are of opposite signs. Thus, it is possible to apply to Equation (2). Hence , the successive approximations to the root of $f\left(x\right)=0$ is given by Equation (2). This method can best be understood through the following example.

3.3 THE SECANT METHOD

EXAMPLE 1: Use the Secant method to find a real root of the equation $x^3 - 9x + 1 = 0$, if the root lies between 2 and 4.

Solution: Let $f(x) = x^3 - 9x + 1.$

$f(2) =$ -9 and $f(4) = 29$ are of opposite signs, the root lies between 2 and 4. Taking, $x_1 = 2,\ x_2 = 4$ and using the Secant method, the first approximation is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 4 - \frac{2 \times 29}{38} = 2.47368$$

(continued)

3.3 THE SECANT METHOD

and $f\left(x_3\right)=-6.12644$. Since $f\left(x_2\right)$ and $f\left(x_3\right)$ are of opposite signs, the root lies between x_2 and x_3. The second approximation to the root is given by

$$x_4 = x_3 - \frac{x_3 - x_2}{f\left(x_3\right) - f\left(x_2\right)} f\left(x_3\right)$$

$$= 2.73989$$

and $f\left(x_4\right)=-3.090707$ (continued)

3.3 THE SECANT METHOD

Now, since $f(x_2)$ and $f(x_4)$ are of opposite signs, the third approximation is obtained from

$$x_5 = x_4 - \frac{x_4 - x_2}{f(x_4) - f(x_2)} f(x_4) = 2.86125$$

and $f(x_5) = -1.32686$. This procedure can be continued till we get the desired result. The first three iterations are shown as follows.

n	x_{n+1}	$f(x_{n+1})$
3	2.47368	–6.12644
4	2.73989	– 3.090707
5	2.86125	–1.32686

3.3 THE SECANT METHOD

EXAMPLE 2: Use the Secant method to find a zero of the function

$$f(x) = x^3 - \sinh x + 4x^2 + 6x + 9$$

Solution: A rough plot suggests that there is a zero between 7 and 8. We take these two points as $x_0 = 7$ and $x_1 = 8$ in the algorithm, the following results were obtained:

n	x_n	$f(x_n)$
0	7.00000	0.417×10^{2}
1	8.00000	-0.665×10^{3}
2	7.05895	0.208×10^{2}
3	7.11764	-0.183×10^{1}
4	7.11289	0.710×10^{-1}
5	7.11306	0.244×10^{-3}
6	7.11306	0.191×10^{-4}

3.3 THE SECANT METHOD - Error Analysis

We consider only the simple case of no exchanging of endpoints in the following analysis. From the definition of the Secant method (2), we have, with $e_n = x_n - r$,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \quad (2)$$

$$e_{n+1} = x_{n+1} - r = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})} - \text{r}$$

$$= \frac{f(x_n)(x_{n-1} - r) - f(x_{n-1})(x_n - r)}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(x_n)e_{n-1} - f(x_{n-1})e_n}{f(x_n) - f(x_{n-1})}$$

3.3 THE SECANT METHOD
- Error Analysis

Factoring out $e_n e_{n-1}$ and inserting $(x_n - x_{n-1})/(x_n - x_{n-1})$, we obtain

$$e_{n+1} = \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]\left[\frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}}\right] e_n e_{n-1} \qquad (3)$$

By Taylor's Theorem,

$$f(x_n) = f(r + e_n) = f(r) + e_n f'(r) + \frac{1}{2} e_n^2 f''(r) + o(e_n^3)$$

Since $f(r) = 0$, this gives us

$$f(x_n)/e_n = f'(r) + \frac{1}{2} e_n f''(r) + o(e_n^2)$$

Changing the index to $n-1$ yields

$$f(x_{n-1})\Big/ e_{n-1} = f'(r) + \frac{1}{2} e_{n-1} f''(r) + o(e_{n-1}^2)$$

3.3 THE SECANT METHOD

- Error Analysis

By subtraction of these equations, we get

$$f(x_n)\big/e_n - f(x_{n-1})\big/e_{n-1} = \frac{1}{2}(e_n - e_{n-1})f''(r) + o(e_{n-1}^2)$$

Since $x_n - x_{n-1} = e_n - e_{n-1}$, we arrive at the equation

$$\frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}} \approx \frac{1}{2}f''(r)$$

The first bracketed expression in Equation (3) can be written as

$$\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \approx \frac{1}{f'(r)}$$

Hence, we have shown that

$$e_{n+1} = \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]\left[\frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}}\right]e_n e_{n-1} \approx \frac{1}{2}\frac{f''(r)}{f'(r)}e_n e_{n-1} = Ce_n e_{n-1}$$

(4)

3.3 THE SECANT METHOD

- Error Analysis

$$e_{n+1} \approx C e_n^2 \quad (2)$$

This equation is similar to the one encountered in the analysis of Newton's method-Equation (2). To discover the **order of convergence** of the Secant method, we postulate the following asymptotic relationship:

$$|e_{n+1}| \sim A |e_n|^{\alpha} \qquad (5)$$

where A is a positive constant. This means that the ratio $|e_{n+1}|/(A|e_n|^{\alpha})$ tends to 1 as $n \to \infty$ and implies α-order convergence. Hence,

$$|e_n| \sim A|e_{n-1}|^{\alpha} \quad \text{and} \quad |e_{n-1}| \sim (A^{-1}|e_n|)^{1/\alpha} \qquad (6)$$

In Equation (4), we substitute the asymptotic values of $|e_{n+1}|$ and $|e_{n-1}|$ from relation (5) & (6). The result is

$$e_{n+1} \approx C e_n e_{n-1} \quad (4)$$

$$e_{n+1} = C e_n e_{n-1} \quad \Rightarrow \quad A|e_n|^{\alpha} \sim |C||e_n| A^{-1/\alpha} |e_n|^{1/\alpha}$$

This can be written as

$$A^{1+1/\alpha} |C|^{-1} \sim |e_n|^{1-\alpha+1/\alpha} \qquad (7)$$

3.3 THE SECANT METHOD

- Error Analysis

$$A^{1+1/\alpha}|C|^{-1} \sim |e_n|^{1-\alpha+1/\alpha} = 1 \quad (7)$$

Since the left side of this relation is nonzero constant while $e_n \to 0$, we conclude that $1-\alpha+1/\alpha=0$ or $\alpha=(1+\sqrt{5})/2 \approx 1.62$, taking the positive root. Hence, the secant method's rate of convergence is superlinear (that is, better than linear). Now we can determine A since the right side of relation (7) is 1. Thus, using the equation $1+1/\alpha=\alpha$, we have

$$A=|C|^{1/(1+1/\alpha)}=|C|^{1/\alpha}=|C|^{\alpha-1}=|C|^{0.62}=\left|\frac{f''(r)}{2f'(r)}\right|^{0.62} \qquad 1+1/\alpha=\alpha,\ 1/\alpha=\alpha-1$$

With A as just given, we finally have for the secant method

$$|e_{n+1}| \approx A|e_n|^{(1+\sqrt{5})/2}$$

Since $(1+\sqrt{5})/2 \approx 1.62 < 2$, the rapidity of convergence of the secant method is not as good as Newton's method but is better than the bisection method.

3.3 THE SECANT METHOD - Error Analysis

However, each step of the secant method requires only *one* new function evaluation, whereas each step of the Newton algorithm requires *two* function evaluations: $f(x)$ and $f'(x)$. Since function evaluations constitute the principal computational burden in these algorithms, a *pair* of steps in the secant method is comparable to one step in the Newton method. For two steps of the secant method, we have

$$|e_{n+2}| \sim A|e_{n+1}|^{\alpha} \sim A^{1+\alpha}|e_n|^{\alpha^2} = A^{1+\alpha}|e_n|^{(3+\sqrt{5})/2}$$

This is considerably *better* than the quadratic convergence of the Newton method since $(3+\sqrt{5})/2 \approx 2.62$. Of course, two steps of the secant method would require more work per iteration.

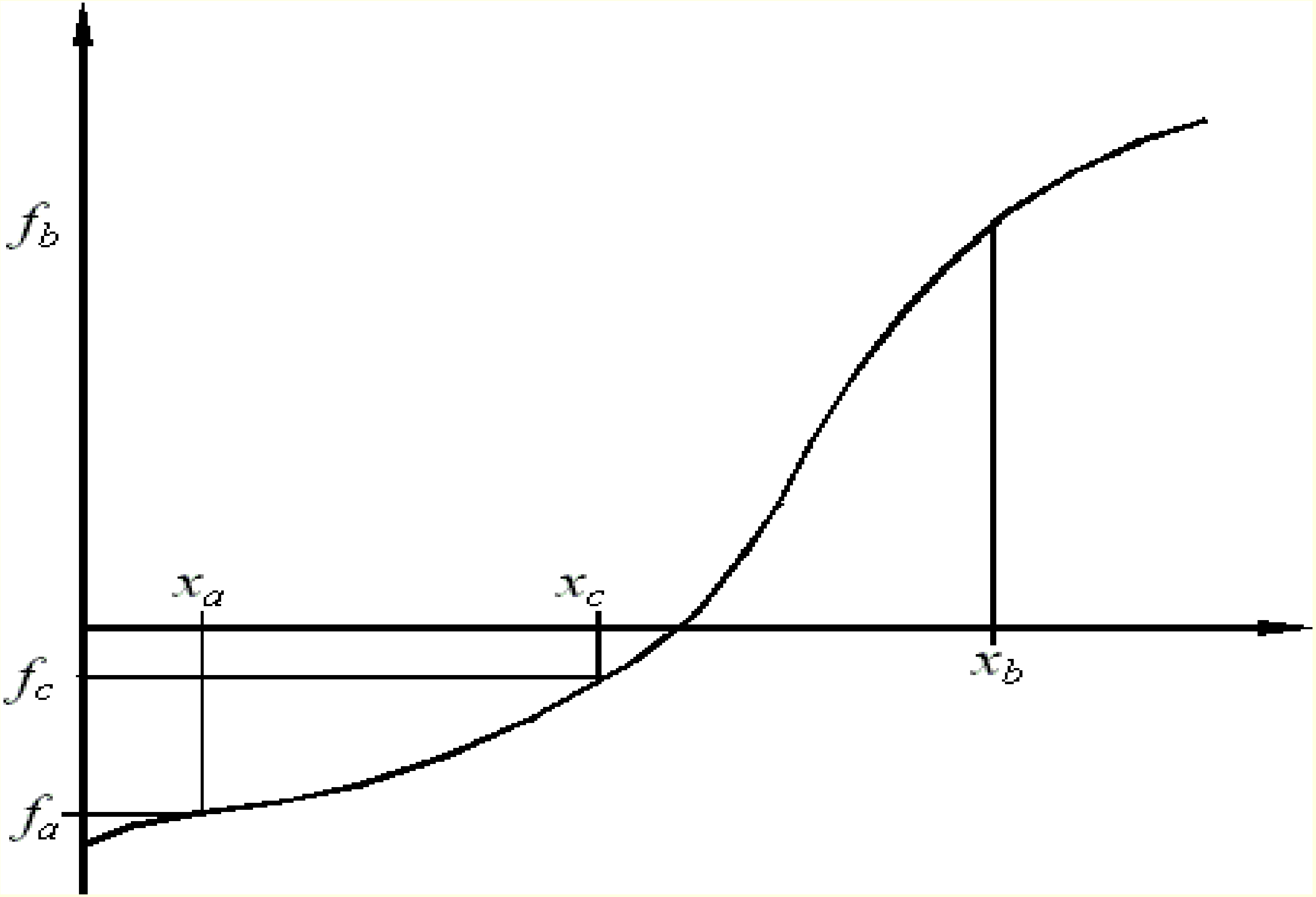
Summary of Bisection Method, Newton-Raphson Method and Secant Method:

Bisection Method:

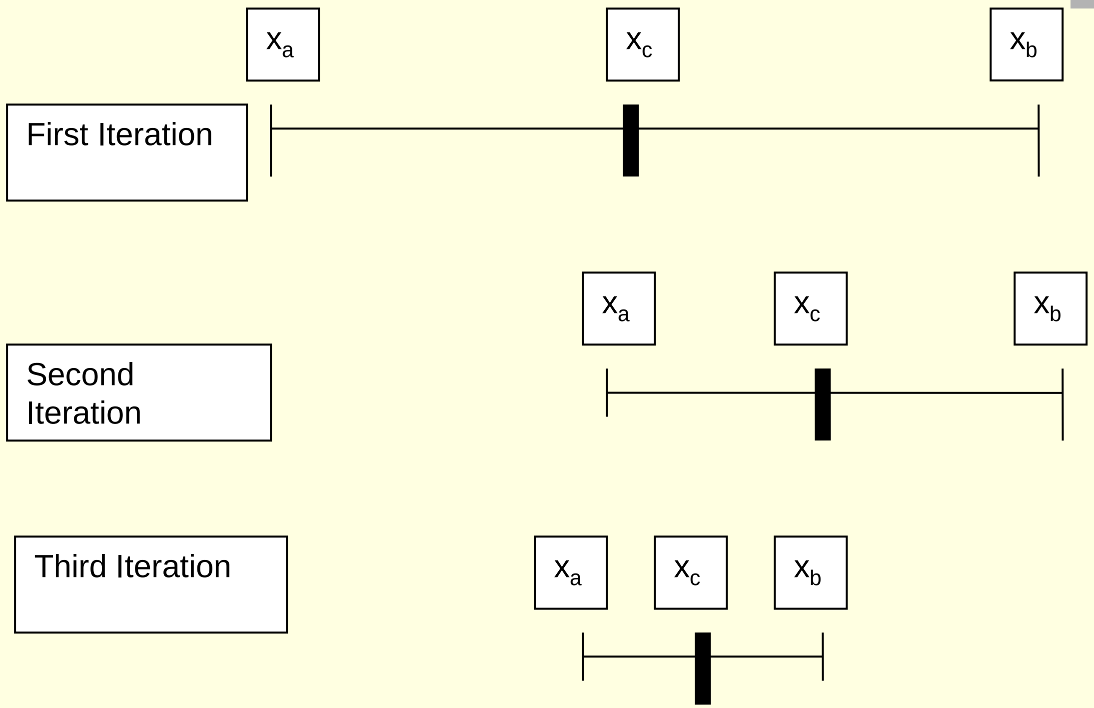
This is the simplest method for finding a root to an equation. One of the main drawbacks is that we need two initial guesses x_a and x_b which bracket the root: let $f_a = f\left(x_a\right)$ and $f_b = f\left(x_b\right)$ such that $f_a f_b \leq 0$.

The below graphical representation of the bisection method showing two initial guesses $\left(x_a \text{ and } x_b \text{ bracketting the root}\right)$.

Summary of Bisection Method, Newton-Raphson Method and Secant Method:



Interval is successively divided in 1/2



We repeat this interval halving until either the exact root has been found or the interval is smaller than some specific tolerance.

$$\text{linear convergence:} \qquad |\varepsilon_{n+1}| \sim \frac{|\varepsilon_n|}{2}$$

Summary of Bisection Method, Newton-Raphson Method and Secant Method:

Newton - Raphson Method :

Consider the Taylor Series expansion of $f(x)$ about some point $x = x_0$.

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + O\left(|x - x_0|^3\right) \qquad (1)$$

Setting the quadratic and higher terms to zero and solving the linear approximation of $f(x) = 0$ for x gives

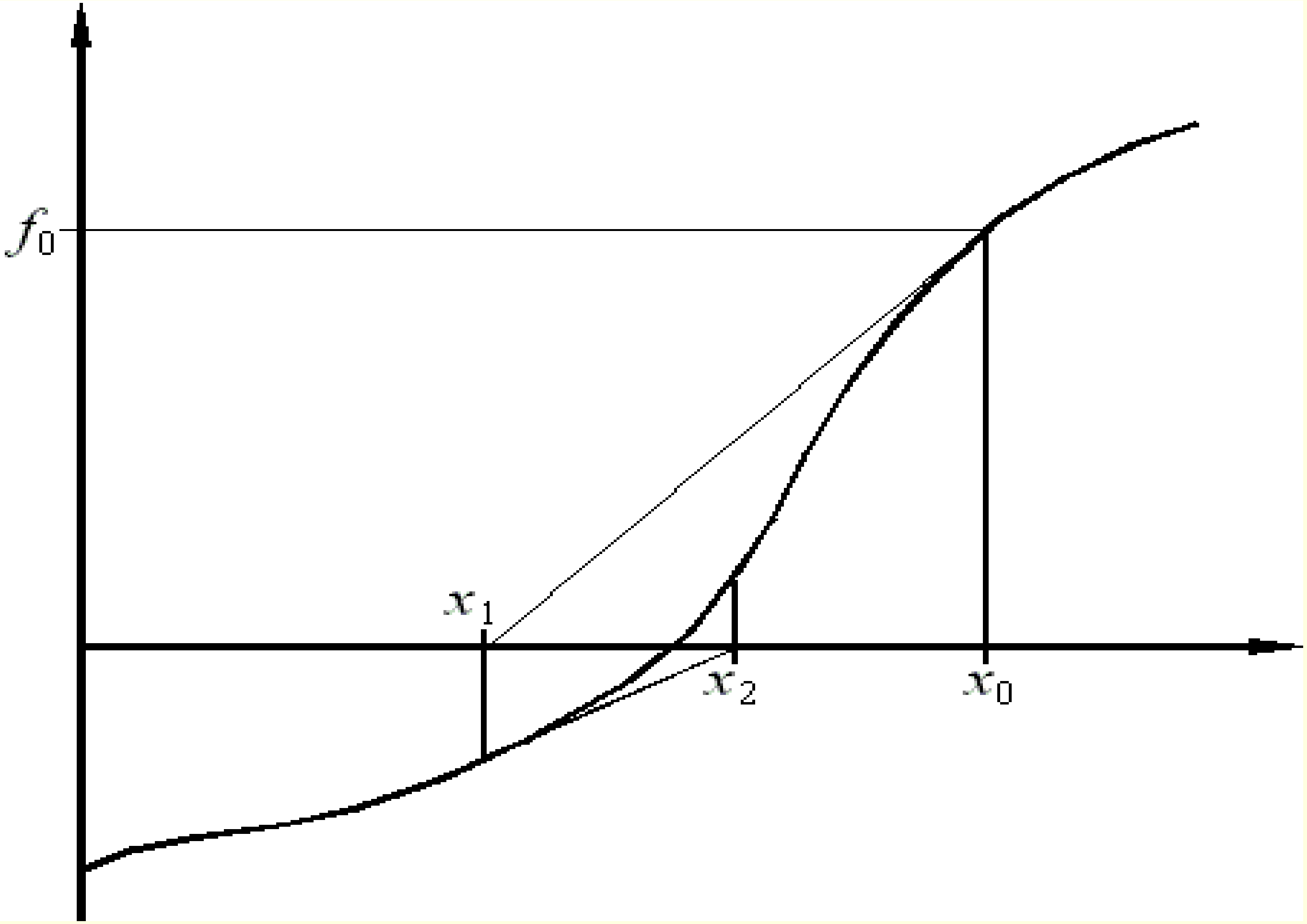
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad (2)$$

Subsequent iterations are defined in a similar manner as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (3)$$

The below figure provides a graphical interpretation of this.

Summary of Bisection Method, Newton-Raphson Method and Secant Method:



Summary of Bisection Method, Newton-Raphson Method and Secant Method:

Newton - Raphson converges much more rapidly than the bisection method. However, if f' vanishes at an iteration point, or indeed even between the current estimate and the root, then the method will fail to converge.

Error: $$e_{n+1} = Ce_n^2$$

Summary of Bisection Method, Newton-Raphson Method and Secant Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (3)$$

Secant Method:

This method is essentially the same as Newton-Raphson except that the derivative $f'(x)$ is approximated by a finite difference based on the current and the preceding estimate for the root , i.e.

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \quad (4)$$

and this is substituted into the Newton-Raphson formula (3) to give

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n) \quad (5)$$

A graphical representation of the method working is shown in Figure 1.

Summary of Bisection Method, Newton-Raphson Method and Secant Method:

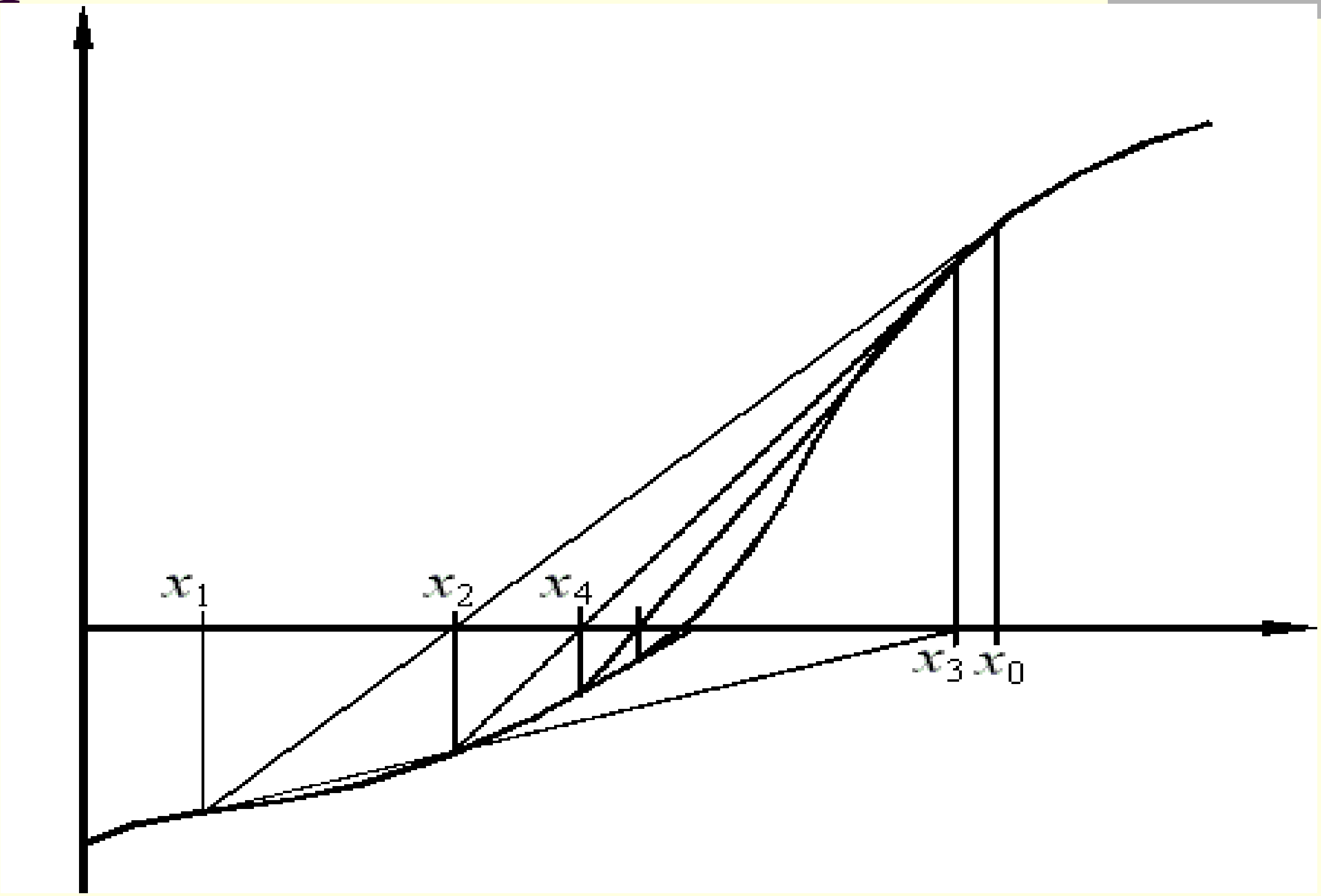


Figure 1 Graphical illustration of the secant method.

Summary of Bisection Method, Newton-Raphson Method and Secant Method:

Secant Method:

Error: $$|e_{n+1}| \approx A|e_n|^{(1+\sqrt{5})/2}$$

EXAMPLE:

Consider the equation $f\left(x\right)=\cos\left(x\right)-\frac{1}{2}$ (6)

Bisection Method:

- Initial guesses $x=0$ and $x=\frac{\pi}{2}$.
- Expect linear convergence : $\left|\varepsilon_{n+1}\right|\sim\frac{\left|\varepsilon_n\right|}{2}$.

Iteration	Error	$\frac{e_{n+1}}{e_n}$
0	–0.261799	–0.500001909862
1	0.130900	–0.4999984721161
2	–0.0654498	–0.5000015278886
3	0.0327250	–0.4999969442
4	–0.0163624	–0.500003669437
5	0.00818126	–0.4999951107776
6	–0.00409059	–0.5000110008581
7	0.00204534	–0.4999755541866
8	–0.00102262	–0.5000449824959
9	0.000511356	–0.4999139542706
10	–0.000255634	–0.5001721210794
11	0.000127861	–0.4996574405018
12	–0.0000638867	–0.5006848060707
13	0.0000319871	–0.4986322611303
14	–0.0000159498	–0.5027410020.188
15	0.00000801862	

Newton - Raphson Method:

- Initial guess: $x = \frac{\pi}{2}$, (note that can not use $x = 0$ as derivative vanishes here).
- Expect quadratic convergence: $\varepsilon_{n+1} \sim C\varepsilon_n^{\ 2}$.

Iteration	Error	$\frac{e_{n+1}}{e_n}$	$\frac{e_{n+1}}{e_n^{\ 2}}$
0	0.0235988	0.00653855280777	0.2770714107399
1	0.000154302	0.0000445311143083	0.2885971297087
2	0.0000000687124	0.000000014553	–
3	$1.0E-15$		
4	Machine Accuracy		

- Solution found to roundoff error $\left(O\left(10^{-15}\right)\right)$ in three iterations.

Secant Method:

- Initial guesses $x = 0$ and $x = \frac{\pi}{2}$; Expect convergence : $\varepsilon_{n+1} \sim C\varepsilon_n^{1.618}$.

Iteration	Error	$\frac{e_{n+1}}{e_n}$	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert^{1.618}}$
0	–0.261799	0.1213205550823	0.2777
1	–0.0317616	–0.09730712558561	0.8203
2	0.00309063	–0.009399086917554	0.3344
3	–0.0000290491	0.0008898244696049	0.5664
4	–0.0000000258486	–0.00000838405174748	0.4098
5	0.00000000000216716		
6	Machine accuracy		

- The clear winner is the Newton-Raphson method.

Exercises

*Ex*1. Use Newton method and secant method to find solution near $x_0 = 2$ for function $f(x) = x^3 - 3x - 1 = 0$, for an accuracy of 10^{-4}.

(For the secant method, $x_0 = 2, x_1 = 2.5$)