Question:

Give an example of an undecidable language B, where $B \leq_{\mathrm{m}} \overline{B}$.

Answer:

----SETP1----

Undecidable language:

A language is an undecidable language, if it is not Turing-decidable. In other words, a language is undecidable language when there exists no Turing machine that can decide the language.

For example, let $B_{TM} = \{\langle M, w\rangle \mid M \text{ is a } TM \text{ and accepts the input 'w'}\}$ is undecidable.

----SETP2----

Proof by contradiction:

Assume that B_{TM} is decidable.

Assume that the Turing machine A decides B_{TM}. So, the decidabilty of the Turing machine *A* is defined as:

$$A\langle M, w\rangle = \begin{cases} accept & \text{if } M \text{ accepts input } w \\ reject & \text{if } M \text{ does not accept the input } w \end{cases}$$

Using the Turing machine *A*, construct another Turing machine *X* that decides whether a machine *M* accepts its own encoding $\langle M\rangle$ is:

1. Input is $\langle M\rangle$, where M is some Turing machine.

2. Run *A* on $\langle M, \langle M\rangle\rangle$.

3. If *A* accepts the language, *reject*. Otherwise, *accept*.

So, the decidabilty of the Turing machine *X* is defined as:

$$X\langle M\rangle = \begin{cases} accept & \text{if } M \text{ does not accept } \langle M\rangle \\ reject & \text{if } M \text{ accepts } \langle M\rangle \end{cases}$$

The above specification cannot be satisfied by the machine. The Turing decidability of X on its own encoding $\langle X\rangle$ is:

$$X\langle X\rangle = \begin{cases} accept & \text{if } D \text{ does not accept } \langle D\rangle \\ reject & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

Hence, neither *X* nor *A* can exist. That is, neither *X* nor *A* can accept the Turing machine *M*.

Thus, B_{TM} is undecidable.