**TM equality:**

The TM equality is represented as follows:

$EQ_{TM} = \{(\langle M\rangle, \langle N\rangle)\ where\ M\ and\ N\ are\ Turing\ machines\ and\ L(M) = L(N)\}$.

$EQ'_{TM} = \{(\langle M'\rangle, \langle N'\rangle)\ where\ M'\ and\ N'\ are\ Turing\ machines\ and\ L(M') = L(N')\}$

$EQ_{TM} \not\leq_m \overline{EQ_{TM}}$ means that $EQ_{TM}$ is not mapping reducible to $\overline{EQ_{TM}}$. This means that $EQ_{TM}$ is not mapping reducible to its complement.

**Proof:**

In order prove that $EQ_{TM} \not\leq_m \overline{EQ_{TM}}$, first prove that $EQ_{TM}$ is not Turing-recognizable.

According to Theorem 5.28 and Corollary 5.29, $A \leq_m B$ only if both $A$ and $B$ are Turing recognizable or not Turing recognizable.

$\overline{EQ_{TM}}$ is complement of $EQ_{TM}$. So, if $EQ_{TM}$ is not Turing-recognizable then, $\overline{EQ_{TM}}$ is Turing-recognizable and vice-versa. This, result in not mapping reducibility between $EQ_{TM}$ and $\overline{EQ_{TM}}$.

**Example:**

Assume $A_{TM}$ is a Turing machine and is mapping is mapping reducible to $\overline{EQ_{TM}}$ that is $A_{TM} \leq_m \overline{EQ_{TM}}$.

The function $f_2 : A_{TM} \rightarrow \overline{EQ_{TM}}$ is defined as follows:

$$\begin{aligned}
&f_2 : On\ input\ \langle M, w\rangle \\
&\quad Construct\ machine\ M_3 : on\ any\ input,\ reject. \\
&\quad Construct\ machine\ M_4 : on\ any\ input\ x, run\ M\ on\ w. \\
&\qquad\qquad\qquad\qquad If\ it\ accepts,\ accept\ x. \\
&\quad Output\ \langle M_3, M_4\rangle
\end{aligned}$$

**Explanation:**

- The machine $M_1$ accepts nothing.
- If $M$ accepts $w$, then $M_2$ accepts everything. Otherwise it accepts nothing.
- So, $\langle M, w\rangle \in A_{TM} \Leftrightarrow \langle M_3, M_4\rangle \in \overline{EQ_{TM}}$ and $f_2$ is clearly computable. Thus, it is a reduction from $A_{TM}$ to $\overline{EQ_{TM}}$.
- So, $EQ_{TM}$ is not Turing-recognizable.

Thus, if $EQ_{TM}$ is not Turing-recognizable and $\overline{EQ_{TM}}$ is Turing-recognizable then $EQ_{TM}$ is not mapping reducible to $\overline{EQ_{TM}}$. That is, $EQ_{TM} \not\leq_m \overline{EQ_{TM}}$.

**Hence, proved.**