From the problem statement assume a language A is context free and a language B is regular.

From the theorem 2.20: A language is a context free language if and only if some push down automaton recognizes it.

From the corollary 1.40: A language is regular if and only if some nondeterministic finite automaton (NFA) recognizes it.

To prove that A/B is context free then it must be recognized by some push down automaton.

The proof is as follows:

Firstly construct a push down automata P_A for the context free language A

Consider the following pushdown automata P_A:

$P_A = (Q_A, \Sigma, \Gamma, \delta_A, q_A, F_A)$

and a NFA M for the regular language B.

Consider the following NFA *M*:

$M = (Q_B, \Sigma, \delta_B, q_B, F_B)$

Assume $P_{A/B}$ is the push down automaton that recognizes the Context free language *A/B*.

$P_{A/B} = (Q^{A/B}, \Sigma, \Gamma^{A/B}, \delta_{A/B}, q_{A/B}, F_{A/B})$

- Now $P_{A/B}$ will read the prefix **w** of the input string.
- $P_{A/B}$ will guess that it has reached to the end of input string **w** at a non-deterministically chosen point;
- $P_{A/B}$ will behave like P_A and *M* running concurrently, except that it will guess the input string **x,** rather than actually reading it as input.
- If it is feasible in this way to concurrently to reach an accepting state of both P_A and *M* then $P_{A/B}$ accepts.
- Note that there is no reason why the stack would have to be empty at the point where $P_{A/B}$ begins the guessing phase.
- So it is essential for $P_{A/B}$ to carry on *modeling* P_A in order to properly account for the stack contents.

$P_{A/B}$ is defined as follows:

- $Q^{A/B} = Q_A\ (Q_A \times Q_B)$
- $\Gamma^{A/B} = \Gamma$
- $q_{A/B} = q_0 P_A$ where $q_0 = q_A = q_B$
- $F_{A/B} = F_A \times F_B$
- $\delta_{A/B}$ is defined as follows: For $Q_A \in Q_A$ (*i.e. if* $P_{A/B}$ is the initial phase):

$$\delta_{A/B}(q_A, a, u) = \begin{cases} \delta_A(q_A, a, u), & \text{if } a \in \Sigma, \\ \delta_A(q_A, \in, u) \cup \{(q_A, q_{B,0}), \in\} & \text{if } a = \in. \end{cases}$$

For $(q_A, q_B) \in Q_A \times Q_B$) (is the guessing phase):

$$\delta_{A/B}((q_A, q_B), a, u) = \begin{cases} \phi, & \text{if } a \in \Sigma, \\ \cup_{b \in \Sigma_\epsilon} \{((r_A, r_B), v) : (r_A, v) \in \delta_A(q_A, b, u) \text{ and } r_A \in \delta_B(q_B, b)\}, & \text{if } a = \in. \end{cases}$$

- Therefore it can be claimed that $P_{A/B}$ accepts **w** if and only if there occurs a string **x** such that P_A accepts **wx** and *M* accepts **x**.
- For instance an acceptance calculation of $P_{A/B}$ on input **w**, all of **w** must be read during the 1st stage.
- The input symbols b that are predicted through the 2nd stage determine a string **x** that is recognized M and is such that **wx** is recognized by P_A.
- Contrariwise, if w is a string with the property that **wx** belongs to A for some x belong to B, then there is an acceptance calculation of $P_{A/B}$ in which w is read through the 1st stage, and the input x is predicted in the 2nd stage.
- In this instance the P_A components of the states determine an acceptance calculation on P_A on input **wx** and the *M*- components of the states determine an acceptance calculation of B on input **x**.

Hence, it is proved A/B is context free language by using $P_{A/B}$ from the above discussion.