Context Free languages

a) Given the languages are

$A=\{a^m b^n c^n \mid m,n \geq 0\}$ and

$B=\{a^n b^n c^m \mid m,n \geq 0\}$

Now we will show that both A and B are context-free languages.

In order to show, let us construct grammar that recognizes A.

$S \to UT$
$U \to aU \mid \varepsilon$
$T \to bTc \mid \varepsilon$

Observing the above grammar we can say that the language A is a context-free language.

Let us construct grammar that recognizes B.

$S \to TU$
$T \to aTb \mid \varepsilon$
$U \to cU \mid \varepsilon$

Observing the above grammar we can say that the language B is a context-free language.

Hence both A and B are context-free languages.

Consider $A \cap B=\{a^n b^n c^n \mid n \geq 0\}$.

Now check whether the language $A \cap B$ is a context-free or not using pumping lemma.

Let us assume that $A \cap B$ is a context-free language.

Pumping lemma states that every context-free language has a special value called *pumping length* such that all longer strings in the language can be "pumped",

let p be the pumping length for $A \cap B$.

Consider a string $s=a^p b^p c^p$.

Clearly s is a member of $A \cap B$ and of length at least p.

Now we prove that one condition of pumping lemma violated by proving s cannot be pumped.

If we divide s into $wxyz$, condition 2 stipulates that either v or y is non-empty.

Now consider one of the two cases, depending on whether substring v and y contains more than one type of alphabet symbol.

1. If both v and y contain only one type of symbol, v doesn't contain both a's and b's or both b's and c's, and the same holds for y. Here the string uv^2xy^2z cannot contain equal number of a's, b's and c's. Therefore it cannot be a member of $A \cap B$ which violates the first condition of the pumping lemma and thus is a contradiction to our hypothesis.

2. If either v or y contain more than one type of symbol uv^2xy^2z may contain equal number of the three alphabet symbols but not in the correct order. Hence it cannot be a member of $A \cap B$ and thus is a contradiction to our hypothesis.

One of the above two case must occur. However, both the cases raised contradiction. This is because of our assumption $A \cap B$ is a context-free language.

Hence our assumption is false and $A \cap B$ is not a context-free language.

Hence, we have A and B are context-free languages and $A \cap B$ is not a context-free language. So we can say that the language obtained by intersection of two context-free languages A and B is not a context a context-free language.

Therefore, the languages A and B are not closed under intersection.

b) Using DeMorgan's law we will show that the languages A and B is not closed under complementation.

DeMorgan's law states that for any two sets A and B, $\overline{A \cup B}=\overline{A} \cap \overline{B}$

We have A and B are two arbitrary context-free languages.

Let these languages are represented in 4-tuple form as $A=(V_1,\Sigma,R_1,S_1)$ and $B=(V_2,\Sigma,R_2,S_2)$ where

- V_1,V_2 are finite set of variables of A and B respectively.
- Σ is finite set, disjoint from V_1,V_2 are terminals of A and B respectively.
- R_1,R_2 are finite set of rules of A and B respectively.
- $S_1\in V_1,S_2\in V_2$ are the start variables of A and B respectively.

Now construct a grammar G that recognizes $A\cup B$.

So $G=(V,\Sigma,R,S)$ where

- $V=V_1\cup V_2$
- $R=R_1\cup R_2\cup\{S\to S_1,S\to S_2\}$. Here, R_1 and R_2 are disjoint.

Now we have to show that A and B are not closed under complementation.

Let us assume that A and B are closed under complementation.

Since, A and B are context-free languages, then $\overline{A}$ and $\overline{B}$ are also context-free-languages. We know that the context-free-languages are closed under union.

So, $\overline{A}\cup\overline{B}$ is closed. Hence $\overline{A}\cup\overline{B}$ is a context-free-language.

Since, $\overline{A}\cup\overline{B}$ is a context-free-language, we have $\overline{\overline{A}\cup\overline{B}}$ is a context-free-language.

Applying DeMorgan's law we get $\overline{\overline{A}\cup\overline{B}}=A\cap B$.

Hence $A\cap B$ is a context-free-language which is a contradiction to part(a).

This contradiction occurred because our assumption is wrong.

Hence A and B are not closed under complementation.

Therefore, class of context-free-languages is not closed under complementation.