Let $N=(Q,\Sigma,\delta,q_0,F)$ be an NFA with *K* states that recognizes some language *A*

(a) Suppose *A* is non empty

• Then there must be an accept state $q\in F$ that can be reached from the start state q_0.

• Let *w* be the string that can be accepted by *N* when traveling along the shortest path q_0 to q.

• Let *n* be the length of *w*.

• Then, the sequence of state $q_0,q_1...q$ in the shortest path from q_0 to *q* has length $n+1$.

• Note that all the states from q_1 to q in this sequence must be distinct; otherwise we would find a shorter path from q_0 to q by removing the repeated states.

• Since there are only *K* states in N and there are *n* distinct states in the shortest path from q_0 to q, we have $n\le K$.

• Clearly, *w* is accepted by *N* because *q* is an accept state

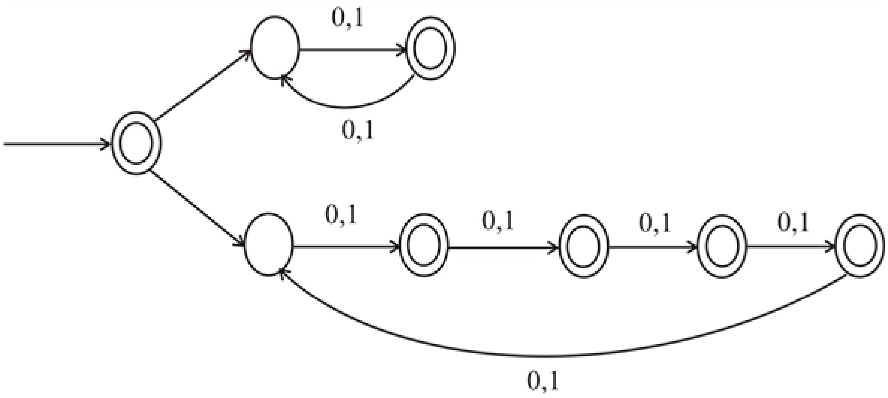
• So *A* contains a string of length at most *K*.

(b) Example:

Suppose $\Sigma=\{0,1\}$ and N be the NFA with $K=\delta$ states.

Let *A* be the language recognized by *N*.

The State diagram of *N* is as follows



• Clearly *N* accepts the empty string.

• For any nonempty string *w*, *N* will reject *w* if and only if the length of *w* is divisible by 2 and 5.

• Thus $\overline{A}$ consists of all non empty strings of length divisible by 10

• So $\overline{A}$ is non-empty and the shortest string in $\overline{A}$ has length $10>K$

• Hence we got the contradiction of part (a) when we replace *A* by $\overline{A}$.

(c) We know that " Every non deterministic finite automaton has an equivalent

deterministic finite automaton"

• So we convert *N* into a DFA *M* that also recognizes *A*, where the set of states in *M* is the set of subsets of *Q*.

• Then we swap the accept and non accept states of *M* to obtain a DFA $\overline{M}$ that recognizes $\overline{A}$.

• Note that $\overline{M}$ has 2^K states.

• Applying part (a) by replacing *N* with $\overline{M}$, we can conclude that if $\overline{A}$ is non-empties, then $\overline{A}$ contains a string of length at most 2^K.

(d) The idea used in part (b) can be generalized to obtain a bound close to an exponential form.

• Let $2\le P_1<P_2<...<P_n\le\sqrt{K}$ be all the primes in $\left[1,\sqrt{K}\right]$

• For each P_i, construct a DFA M_i of P_i states that rejects only strings of length divisible of P_i.

• Finally, construct an NFA *M* by union all these M_i machines in the following ways:

→ create a separate starting state q_0 add an $\in$ transition from q_0 to the starting states of each M_i, and also designate q_0 as an accepting state.

→ This machine M will have $1+P_1+P_2+...+P_m \leq K$ states.

→ On the other hand, *M* rejects a string *w* if and only if a is non empty and $|w|$ is divisible by $P_1P_2...P_m$.

→Hence, the shortest string is rejected by *M* has length $P_1P_2...P_m$.

→ Now coming to the analysis of the bound,

By the prime Number theorem, there are approximately $\ln$ *n* prime numbers in $[0,n]$ for *n* sufficiently large.

→ Hence, $m \approx \frac{1}{2}\ln K$.

→ Since there are about $\frac{1}{4}\ln K$ primes in $\left[0,\sqrt[4]{K}\right]$, there are about $m-\frac{1}{4}\ln K \approx \frac{1}{4}\ln K$ primes in $\left[\sqrt[4]{K},\sqrt{K}\right]$

Thus $P_1P_2...P_m \geq \left(\sqrt[4]{K}\right)^{\frac{1}{4}\ln K} = K^{\frac{1}{16}\ln K}$