Question:

Rice’s theorem. Let P be any nontrivial property of the language of a Turing machine. Prove that the problem of determining whether a given Turingmachine’s language has property P is undecidable.

In more formal terms, let P be a language consisting of Turing machine descriptions where P fulfills two conditions. First, P is nontrivial—it contains some, but not all, TM descriptions. Second, P is a property of the TM’s language whenever $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in P \text{ iff } \langle M_2 \rangle \in P.$ Here, M_1 and M_2 are any TMs. Prove that P is an undecidable language.

Answer:

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Rice's theorem

Given P non trivial property of language of Turing machine, it is required to prove that P is un-decidable. Consider on the contrary that P is decidable language that satisfies the properties. Consider R_P be a Turing machine that decides P. Now it is required to show that how to decide A_{TM} using R_P by constructing Turing machine S.

First, let T_θ be a Turing machine that always reject, so $L(T_\theta) = \phi$. It can be consider that $\langle T_\theta \rangle \notin P$ without loss of generality, because it is possible to proceed with $\bar{P}$ instead of P if $\langle T_\theta \rangle \in P$. Because P is non-trivial, there exist a Turing machine T with $\langle T \rangle \in P$. Now construct S based on T and R_P as follows:

$S ="$ On input $\langle \text{M,w} \rangle$:
1. Use M and w to construct the following Turing machine M_w.

$M_w ="$ On input x:
1. Simulate M on w. If it halts and rejects,reject.
2. Simulate T on x. If T accepts x, accept."

2. Use TM R_p to determine if $\langle M_w \rangle \in P$. If YES, accept,else reject."

Note that TM M_w has property that (1) if M accept w, $L(M_w) = L(T)$, and (2) if M does not accept w, $L(M) = \phi = L(T_\phi)$.

In other words, $\langle M_w \rangle \in P$ if and only if M accept w.

Since the construction of M_w from T, M and w takes finite steps, the TM S is decider for A_{TM}. This creates a contradiction since A_{TM} is an un-decidable language. In conclusion, P is un-decidable.