Question:

Give an example of a language that is not context free but that acts like a CFL in the pumping lemma. Prove that your example works. (See the analogous example for regular languages in Problem 1.54.)

Problem 1.54

Consider the language

$$F = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}.$$

a. Show that F is not regular.

b. Show that F acts like a regular language in the pumping lemma. In other words, give a pumping length p and demonstrate that F satisfies the three conditions of the pumping lemma for this value of p.

c. Explain why parts (a) and (b) do not contradict the pumping lemma.

Answer:

----SETP1----

Consider the following details:

Let $F = \{a^i b^j c^k d^m \mid i, j, k, m \geq 0 \textit{ and if } i = 1 \textit{ then } j = k = m\}$.

----SETP2----

Proof that F is not a context free Language (by contradiction):

• Suppose F is context free, then $F \cap \{ab^i c^j d^k \mid i, j, k \geq 0\} = \{ab^i c^i d^i \mid i \geq 0\} = G$ is context free since $\{ab^i c^j d^k \mid i, j, k \geq 0\}$ is the language of the regular expression $ab^*c^*d^*$ which is regular, and also the intersection of a context free language with a regular language is context free.

• By showing that G cannot be context free using pumping lemma, it will contradict the fact that F is context free.

----SETP3----

Suppose the pumping length of G is p and take $s = ab^p c^p d^p \in G$ with $|s| > p$.

There exists *u*, *v*, *x*, *y*, *z* such that $s = uvxyz$ and,

(1) $uv^n xy^n z \in G$ for all n ≥ 0,

(2) $|vy| > 0$ and

(3) $|vxy| \leq p$.

----SETP4----

For any valid choice of *uvxyz* that $uv^2xy^2z \notin G$, take i = 0. Then,

Case 1: If *v* or *y* contains *a*, then $uv^2xy^2\ z$ will have more than one a and thus is not in *G*.

Case 2: If *v* and *y* do not contain a, then from $|vxy| \leq p$, *vxy* can have at most two other symbols from *b*, *c* or *d*.

Therefore, $uv^2xy^2\ z$ will not have the same number of the three symbols.

Thus, G is not a CFL and therefore F is also not a CFL.

----SETP5----

Proof that F is a context free language using the pumping lemma:

Let $p = 2$. Now for any string $s \in F$, with $|s| \geq 2$, can be written as $uvxyz$ such that

(1) $uv^n xy^n z \in G$ for all $n \geq 0$,

(2) $|vy| > 0$

(3) $|vxy| \leq p$

Now,

Case 1: $s = a^i b^j c^k d^m$ with $i \neq 2$ and $i + j + k + m \geq 2$.

In this case, let $u = v = x = \in$, *y* is the first symbol in *s* and *z* be the remaining symbols.

Then (2) and (3) hold, and $uv^n xy^n z$ will have either zero as (*i* = 0 or *i* = 1 and *n* = 0) or more than one a followed by a string of the form $b^j c^k d^m$, so it will remain in *F* and therefore (1) holds.

Case 2: s = $a^2 b^j c^k d^m \in F$ for some *j*, *k*, $m \geq 0$ (so $|s| \geq 2$). Take $u = v = x = \in$, $y = a^2$ (in this case *a* would not work if any *j*, *k*, *m* > 0 because there is pumping down, but if a would be taken as the first symbol then it will work), and $z = b^j c^k d^m$. Then (2) and (3) hold and $xy^i z\ =\ a^{2+2(i-1)} b^j c^k d^m \in F$, therefore, (1) also holds.

Thus, in either cases, the conditions of the pumping lemma hold.

----SETP6----

Therefore, F , which is not a Context Free Language, satisfies the pumping lemma.