Strategies for designing geometric transformations on quantum images

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A R T I C L E   I N F O

A B S T R A C T

Three design strategies for constructing new geometric transformations on quantum images from other transformations are proposed. The strategies focus on the affected areas in the images, the separability, and smoothness of the transformations by exploiting a representation of images on quantum computers extensively. The complexity in terms of the number of basic gates and the depth of the corresponding circuits are analyzed to understand the advantages and disadvantages of each strategy. Examples to demonstrate the applicability of the proposed strategies are presented. The strategies provide high level tools to explore and analyze transformations which are necessary to build practical image processing applications on quantum computers.

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1. Introduction

Quantum computing has the ability to solve problems whose best classical solutions are considered inefficient [7]. Perhaps the best-known examples are Shor's polynomial time integer factorization algorithm [19,21] and Grover's quadratic speed-up database search algorithm [9].

Research on quantum image processing began with proposals on quantum image representations such as Qubit Lattice [23,24], Real Ket [12], and Flexible Representation of Quantum Images (FRQI) [15]. The quantum images are two dimensional arrays of qubits in Qubit Lattice [23,24], quantum states which have grey levels as coefficients of the states in Real Ket [11], and normalized states capturing information about both color and positions in FRQI [15]. Quantum versions of classical transformations which are related to image processing have been proposed such as quantum Fourier transform [19], quantum discrete cosine transform [11,22], and quantum Wavelet transform [8]. These image processing transforms are more efficient in quantum computation than classical ones [19]. Quantum algorithms have been used to speed up classical image processing problems because of their proven efficiency over the classical versions [2,4,5]. There are classical image processing operations that cannot be applied on quantum images, for example convolution and correlation [13], because all operations in quantum computation must be invertible. By exploiting FRQI representation for images, quantum transformations can be used to process quantum images directly [15].

Quantum computation is naturally formulated using quantum circuit model. Many quantum algorithms are expressed in terms of uniform special-purpose circuits which depend strongly on the problem at hand. There is considerable research on universal quantum gates which can efficiently simulate any quantum transformation [19]. These gates comprise various levels of abstraction and combinations of the universal gates; NOT (N), Controlled-NOT (C), and the Toffoli (T) gates. They combine to form what is often referred to in the literature as the NCT library [1,18,25]. Complexity theory on quantum computation has been studied to discover and analyze the transformations from the basic gates [1,19]. Realizations of the basic gates have been proposed in order to build large scale quantum computers [18,19].

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Our goal is to extend the use of quantum circuit models using the basic gates in the NCT library for quantum image representation and processing. Three strategies to extend the number of geometric transformations [16] on quantum images are the main contributions. To accomplish this, the FRQI representation of quantum images [15] is adopted for all our images. This enables us to encode all the essential information about the images such as color and position for further processing and manipulation. The strategies exploit the adopted representation of quantum images in terms of sub-blocks, separability, and control levels as follows:

- The first strategy considers transformations targeted at sub-blocks within quantum images. This can be achieved by adding more control to indicate information about the sub-blocks.
- The second strategy extends the separability of classical operations to quantum transformations. Using the FRQI representation, it is possible to define and examine separable and completely separable geometric transformations.
- The third strategy focuses on the smoothness of the transformations which may not be achieved using any of the earlier-mentioned strategies. Cyclic shift transformations, which use multi-level controls, are the main technique to obtain the smoothness.

Various remarks, lemmas, and theorems are presented to analyze the efficiency and limitations of each strategy based on the complexity of the transformations. Some simple applications on quantum image management are discussed to demonstrate the flexibility of the proposals in constructing new geometric transformations on quantum images. The proposed strategies extend a number of high level tools to expand processing operations which are the cornerstones for building practical image processing applications on quantum computers.

The rest of the paper starts with a brief overview of the background on quantum computation, the flexible representation of quantum images (FRQI), the general framework and classification of all geometric transformations on FRQI. In subsequent sections the various definitions, lemmas, theorems and proofs are presented for each of the above-mentioned strategies. Examples to show how the various design strategies can be applied are presented in Section 6.

2. Background on quantum computation, FRQI representation, and geometric transformations

We start by introducing the notations used in this paper which have been used in a wide range of quantum computation literature. The state of a quantum system is described as a vector in a Hilbert space called a ket in Dirac or quantum mechanical notation. The ket and its adjoint, bra, notations are defined as follows:

$$|u\rangle = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}, \quad u_i \in \mathbb{C}, \ i = 0, 1, \ldots, n-1,$$

$$\langle u| = |u\rangle^\dagger = \begin{bmatrix} u_0^\dagger u_1^\dagger \ldots u_{n-1}^\dagger \end{bmatrix}.$$

The notation for the tensor or Kronecker product, $\otimes$, is used to express the composition of quantum systems. The tensor product of two matrices A and B is defined as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1q} \\ b_{21} & b_{22} & \ldots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \ldots & b_{pq} \end{bmatrix},$$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1m}B \\ a_{21}B & a_{22}B & \ldots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \ldots & a_{nm}B \end{bmatrix},$$

where

$$a_{ij}B = \begin{bmatrix} a_{ij}b_{11} & a_{ij}b_{12} & \ldots & a_{ij}b_{1q} \\ a_{ij}b_{21} & a_{ij}b_{22} & \ldots & a_{ij}b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{ij}b_{p1} & a_{ij}b_{p2} & \ldots & a_{ij}b_{pq} \end{bmatrix}, \quad \forall i, j.$$

The short notation for tensor product $|u\rangle \otimes |v\rangle$ of two vectors or two kets, $|u\rangle$ and $|v\rangle$, is $|uv\rangle$ or $|u\rangle|v\rangle$ and we use $A^{\otimes n} = A \otimes A \otimes \cdots \otimes A$ to denote the tensor product of matrix A for n times.

The FRQI representation of quantum images, which enables the application of unitary transformations on the images, was proposed in [14]. This proposal integrates information about colors and their corresponding positions in an image into

Fig. 1. Vertical and horizontal coordinates encoded in qubits.

a quantum state having its formula as in (1)

$$|I(\theta)\rangle = \frac{1}{2^n} \sum_{k=0}^{2^{2n}-1} |c_k\rangle \otimes |k\rangle, \tag{1}$$

$$|c_k\rangle = \cos\theta_k |0\rangle + \sin\theta_k |1\rangle, \tag{2}$$

$$\theta_k \in \left[0, \frac{\pi}{2}\right], \quad k = 0, 1, \ldots, 2^{2n}-1, \tag{3}$$

where $\otimes$ is the tensor product notation, $|0\rangle$, $|1\rangle$ are 2-D computational basis quantum states, $|k\rangle$, $k = 0, 1, \ldots, 2^{2n}-1$ are 2^{2n}-D computational basis quantum state and $\theta = (\theta_0, \theta_1, \ldots, \theta_{2^{2n}-1})$ is the vector of angles encoding colors. There are two parts in the FRQI representation of an image; $|c_k\rangle$ and $|k\rangle$ which encode information about the colors and their corresponding positions in the image, respectively. For the 2-D images, the position information $|k\rangle$ encoded in position qubits includes two parts, the vertical and horizontal coordinates. In $2n$-qubit systems for preparing quantum images, or n-size images, the vector $|k\rangle$

$$|k\rangle = |y\rangle|x\rangle = |y_{n-1}y_{n-2}\ldots y_0\rangle|x_{n-1}x_{n-2}\ldots x_0\rangle,$$
$$x_i, y_i \in \{0, 1\},$$

for every $i = 0, 1, \ldots, n$, which encodes the first n-qubit $y_{n-1}, y_{n-2}, \ldots, y_0$ the vertical location and the second n-qubit $x_{n-1}, x_{n-2}, \ldots, x_0$ encodes the horizontal location information as shown in Fig. 1.

Geometric transformations on quantum images (GTQI) [16] are the operations which are performed based on the geometric information of images, i.e., information about position of every point in the image. These transformations are akin to *"shuffling"* the image content point-by-point but their global effect being a transformation on the entire image content geometrically as dictated by the gate sequence needed to accomplish the desired transformation. This we refer to as geometric transformations, G_I, which on FRQI quantum images can be defined as in (4),

$$G_I(|I(\theta)\rangle) = \frac{1}{2^n} \sum_{k=0}^{2^{2n}-1} |c_k\rangle \otimes G(|k\rangle), \tag{4}$$

where $G(|k\rangle)$ for $k = 0, 1, \ldots, 2^{2n}-1$ are the unitary transformations performing geometric exchanges based on the vertical and horizontal locations. The performance of the geometric transformations on quantum images, G_I, is based on the function, G, on the computational basis vectors. The general structure of circuits for geometric transformations on FRQI images is shown in Fig. 2.

An advantage of FRQI representation is that we can use the color qubit as an extra space to reduce the complexity of all geometric transformations. This property is shown by Remark 1.

Remark 1. If on a n-size FRQI image, $n \geq 3$, the geometric transformations in the form of $C^{2n-1}(\sigma_x)$, which have $2n-1$ controls on NOT gates, can be constructed by $8(2n-4)$ Toffoli gates.

Proof. In FRQI representation the color qubit is not used in the geometric transformations, $C^{2n-1}(\sigma_x)$. Therefore, we can use the qubit as the extra bit to reduce the number of Toffoli gates used to construct $C^{2n-1}(\sigma_x)$ gates by applying Corollary 7.4 in [1] on FRQI representation. □
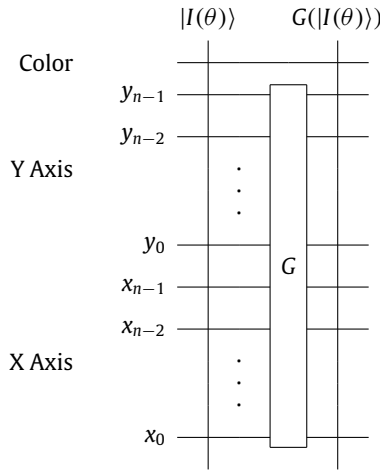
Fig. 2. General circuit design for geometric transformations on quantum images.

The structure of the set of all geometric transformations on FRQI images can be studied from the algebraic theory view point. Each geometric transformation can be considered as a permutation of positions. Therefore, all geometric transformations on FRQI images form a group under the operation of cascading two geometric transformations. After setting the isomorphism between the group of all geometric transformations and a subgroup of permutations, the group theory can be applied to classify these geometric transformations. The classification is based on the content of the set of generators, or the gate library, used in the corresponding circuits [3,6,14,17,20,25]. There are three gate libraries related to NOT, CNOT and Toffoli gates as follows:

- the library, N, contains only NOT gate.
- the library, NC, contains NOT and CNOT gates.
- the library, NCT, contains NOT, CNOT, and Toffoli gates.

The circuits which contain only NOT gates perform the bit translations [20] as follows:

$$f(x) = x \oplus b, \quad b \in \mathbb{Z}_n^2. \tag{5}$$

The circuits which contain only CNOT gates perform the linear transformations [3,20] as follows:

$$f(x \oplus y) = f(x) \oplus f(y), \quad x, y \in \mathbb{Z}_n^2. \tag{6}$$

If we put a linear transformation, f, after a bit translation indicated by b then we produce an affine transformation g [6]

$$g(x) = f(x \oplus b) = f(x) \oplus f(b), \quad x, b \in \mathbb{Z}_n^2. \tag{7}$$

These circuits are comprised of CNOT and NOT gates to perform affine transformations. Using the NCT library, we can generate all geometric transformations on FRQI images. The following three parameters are usually used to analyze the complexity of quantum circuits, C,

- the number of basic gates, $|C|$, used in the circuit,
- the width, $W(C)$, of the circuit or the number of qubits involved in the circuit,
- the depth, $D(C)$, of the circuit or the minimum number of layers that the circuit can be partitioned into.

Designing an image processing operation related to geometric transformations, however, is not as straight forward as the above equations suggest. For example, constructing the reflection and rotation operations in Fig. 3 is not obvious using the NC library. Fig. 4 shows the construction of 90° rotations on FRQI images by using NOT and SWAP gates, wherein each of the SWAP gates can be constructed from 3 CNOT gates. It means that designing new geometric transformations for image processing, for example quantum image watermarking [10], is difficult. This difficulty can be overcome by using high level tools to design and analyze new transformations on FRQI images. The transformations provide more flexibility and enable the designers to create new image processing applications on quantum computers rather than being constrained to using lower level operations, i.e. basic gates.

3. Application of geometric transformations to sub-blocks in quantum images

When geometric transformations are well-understood, often designers of new operations would want to use smaller versions of the transformations as the main components in the larger operations. Therefore, a major strategy is the
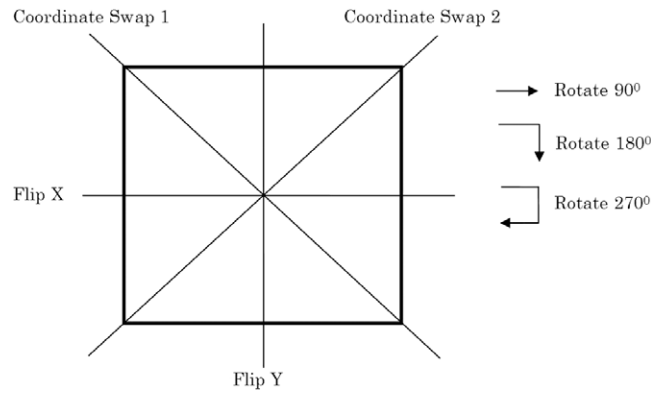
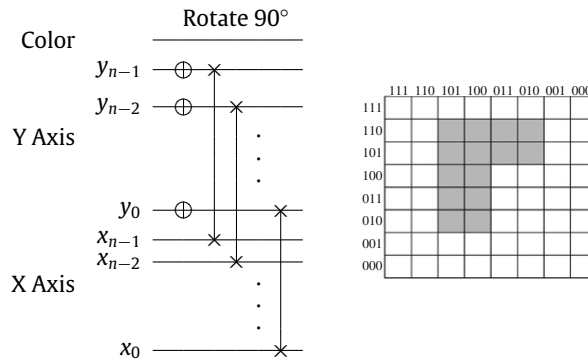**Fig. 3.** Reflections and rotations on FRQI images can be constructed by *NC* library.

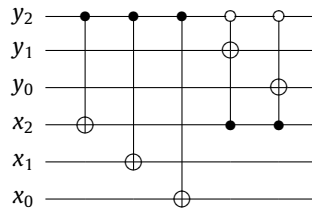**Fig. 4.** 90° rotation circuit for FRQI images and the output of the image in Fig. 1.

Fig. 5. Circuit for the application of flip along *Y* axis on upper half and flip along *X* axis on lower left quadrant of an 8×8 FRQI image.

application of a transformation to a sub-block in an image. Fig. 5 shows an example of using the flip operation [16] along the *X* and *Y* axes on sub-blocks of an 8×8 image. In the FRQI representation, the realization of these kind of transformations are simplified by using additional control over the original transformation. In doing so, the complexity of the circuit increases in comparison with the original transformation in terms of both the depth and number of basic gates in the circuit. Alternatively, the designer can analyze the operations on the whole image by removing the conditions which were used to indicate the sub-blocks.

To illustrate a simple method of how using more control helps to focus the effect of a transformation on a sub-block, it is necessary to look closely at the FRQI representation. When each of the position qubits become the control information, it will divide the operating area of the original transformation into two sub-blocks. Fig. 6 shows two examples of using a single control to divide the 8×8 image into two halves depending on the type of control and the control qubit. Therefore, by providing more controls for an operation the effect of the original transformation will be targeted on a smaller area.

In order to analyze the change in complexity of the circuit caused by applying more control on the original operations, some properties of FRQI related to the number of controls and the size of affected sub-blocks are analyzed.

Remark 2. On a $2^n \times 2^n$ image in FRQI representation, $C^m(\sigma_x)$ gates have affect on 2^{2n-m} sub-blocks where m is the number of controls on the NOT gate and $1 \leq m \leq 2n - 1$.
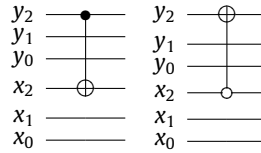
Fig. 6. The control on the y_2 qubit in the left circuit divides the whole 8×8 image into upper and lower halves. Meanwhile the $\circ$ control on the x_2 qubit in the other circuit focuses the effect of the NOT gate on the right half of the 8×8 FRQI images. Two NOT gates and a CNOT gate are required to decompose this $\circ$ control gate.

Remark 2 shows the relationship between the size of the sub-block and the number of controls on the $C^m(\sigma_x)$ gates. Thus, allowing a deeper probe into the image content where the GTQI operation will be performed. The more the number of controls the transformations have, the less the size of the affected areas. Specifying the area in which the transformation will be applied increases the complexity of the new transformation in terms of the depth and the number of basic gates in the corresponding circuit.

Lemma 1. *If the original transformation includes a NOT, b CNOT, and c Toffoli gates, then new transformation, which is produced by adding a single control to the original transformation, contains a CNOT, $b+4c$ Toffoli gates and can be partitioned into $a+b+4c$ layers.*

Proof. By adding a single control to the original transformation, the NOT, CNOT, and Toffoli gates become CNOT, Toffoli, and $C^3(\sigma_x)$ gates, respectively. It is known that a $C^3(\sigma_x)$ gate is decomposed into 4 Toffoli gates. Therefore, the new circuit contains a CNOT and $b + 4c$ Toffoli gates. All the basic gates can be partitioned in to $a + b + 4c$ layers because they share the added controlled qubit. □

It is trivial to extend Lemma 1 to the case of adding two controls to the original transformation. The general case for adding more than two controls is presented in Theorem 1.

Theorem 1. *If the original transformation on an n-sized ($n \geq 2$) image includes a NOT, b CNOT, and c Toffoli gates, then new transformation, which is produced by adding m, $3 \leq m \leq 2n - 3$, control to the original transformation, contains a $C^m(\sigma_x)$, b $C^{m+1}(\sigma_x)$, and c $C^{m+2}(\sigma_x)$ gates and the circuit can be decomposed into M, $(a+b+c).4(m-2) < M < (a+b+c).4(2n-4)$, Toffoli gates.*

Proof. By adding m controls to the original transformation, the NOT, CNOT, and Toffoli gates become $C^m(\sigma_x)$, $C^{m+1}(\sigma_x)$, and $C^{m+2}(\sigma_x)$ gates, respectively. In the case $3 \leq m \leq n$, we can see that $(a + b + c).4(m - 2) < M < (a + b + c).8(m - 5)$ by using Lemmas 7.2 and 7.4 in [1]. In the case $n < m \leq 2n - 3$, using Remark 1 in Section 2 we can show that $(a+b+c).8(2m-4) < M < (a+b+c).8(2n-4)$. Therefore, the total number of Toffoli gates is M and $(a+b+c).4(m-2) < M < (a + b + c).4(2n - 4)$. □

It is easy to imply from Theorem 1 that the number of Toffoli gates, M, is equal to the number of layers the circuit can be partitioned into because they share m controls.

Theorem 1 provides the guidelines for applying geometric transformations to image sub-blocks as follows:

- the number of controls used to indicate the sub-blocks should be small, i.e. the size of the sub-blocks should be large.
- the number of basic gates in the original transformation should be small.

4. Separable geometric transformations

In designing new transformations for quantum computation, the main resource can be chosen from the basic gates as the lowest level or from complex transformations combining various basic gates. In order to design practical and flexible transformations, the designers often choose higher level approaches. An example of such strategies was shown in the previous section, the starting point is to apply only one transformation on a sub-block of an image. Consequently, the cost of the new transformations increases in terms of circuit depth and the number of basic gates. In classical image processing, the concept of separable transformations plays an important role because these transformations can be implemented at lower costs. In quantum computation, the separability of the transformations have not been studied extensively because the complexity of a transformation mainly focuses on the number of basic gates not on the depth of the circuit of these basic gates. The separable geometric transformation in the FRQI representation is a natural step because the separability of X and Y axes is inherent in the representation.

Definition 1. A geometric transformation, G is called separable if there are two geometric transformations G_1 and G_2 such that

$$G(|i\rangle) = G_1(|i_1\rangle) \otimes G_2(|i_2\rangle), \tag{8}$$

where $|i\rangle = |i_1\rangle \otimes |i_2\rangle$ in which i_1 and i_2 are two disjoint subsets of the set of position qubits in the FRQI image.

The separable transformations are easy to recognize from the structure of corresponding FRQI circuits. The circuit C_G of a separable transformation G can be partitioned into two parallelly independent sub-circuits C_{G_1} and C_{G_1} of the transformations G_1 and G_2, respectively. Fig. 7 shows examples of separable transformations.
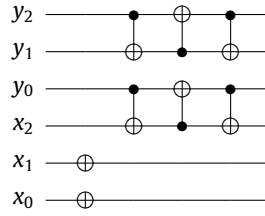
Fig. 7. The first layer of the circuit can be separated into 2 NOT gates. The next three layers have 6 CNOT gates to swap two separable pairs of position qubits (y_2, y_1) and (y_0, x_2).
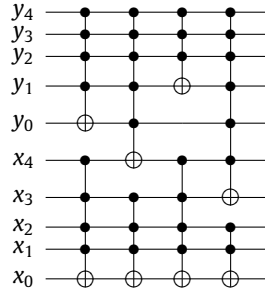

Fig. 8. The transformations in the first two layers are separable but not completely separable while the other transformations are completely separable.

It is obvious to see that the complexity $|C_G|$ of the separable circuit C_G can be calculated through $|C_{G_1}|$ and $|C_{G_2}|$ as follows:

$$|C_G| = |C_{G_1}| + |C_{G_2}|. \tag{9}$$

However, finding the relation between the depth of the circuit C_G and the depth of the circuits C_{G_1} and C_{G_2} is not that easy. When C_G is constructed from C_{G_1} and C_{G_2}, the depth of C_G, $D(C_G)$, should not increase in order to optimize the space of the available circuit. The completely separable transformations are defined as follows:

Definition 2. A geometric transformation G is called completely separable if G is separable into G_1 and G_2 as mentioned in Definition 1 and

$$D(C_G) = \max\{D(C_{G_1}), D(C_{G_2})\}. \tag{10}$$

The set of completely separable transformations is a proper subset of the set of all separable transformations because there are separable transformations that are not completely separable as shown in Fig. 8. The following theorem is the sufficient condition for completely separable transformations.

Theorem 2. *If G is a separable transformation into G_1 and G_2 as in Definition 1 on an n-sized FRQI image and G_1 and G_2 do not include $C^p(\sigma_x)$, $C^q(\sigma_x)$ ($p, q > 2$ and $p + q = 2n - 2$), respectively, in a same layer then G is completely separable.*

Proof. By inspection and using Remark 1. □

An example of a completely separable transformation is introduced here to show the relationship between the above-mentioned definitions on FRQI images. The example shows the way to construct the 180° rotation operation in the clockwise direction by combining the flip operations along X and Y axes. On n-sized FRQI images, each flip operation comprises n NOT gates, which are put on Y or X position qubits, in only one layer. Therefore, the rotation is constructed by $2n$ NOT gates on one layer as shown in Fig. 9.

5. Multi-level controlled geometric transformations

The new transformations discussed in 3 and 4 have effect on sub-blocks in larger images. For that reason, the changes on the images are easy to observe. In classical image processing, there are operations which produce smooth changes in the images. It is necessary to find transformations having similar capabilities for quantum image processing. To do so, the designers can start with transformations in cyclic groups. If the order of the cyclic group is large enough then the effect of a transformation in that group on an image can be regarded as smooth. In addition, the cyclic transformations can be efficiently realized based on the NCT library. Position shifting transformations defined in Definition 3 can be good candidates for such operations.
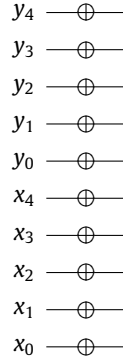
Fig. 9. The rotation 180° clockwise is constructed by $2n$ NOT gates for $n = 5$. The rotation operation is completely separable transformation.
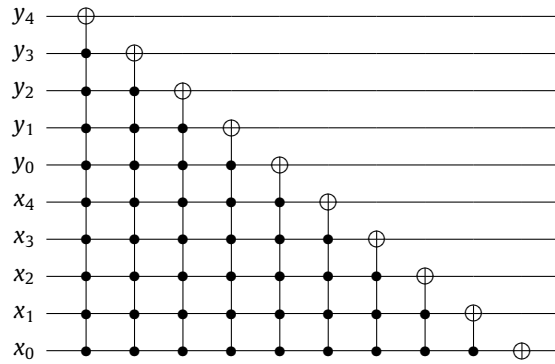

Fig. 10. The cyclic shift transformation for the case $c = 1$ and $n = 5$.

Definition 3. Position shifting transformations on n-sized FRQI are defined as follows:

$$P(|i\rangle) = |i'\rangle, \quad i \in 0, 1, \ldots, 2^{2n} - 1, \tag{11}$$

where $i' = i + c \mod 2^{2n}$, and $c \in \{1, \ldots, 2^{2n} - 1\}$.

Theorem 3. *The complexity of position shifting transformations on n-sized FRQI in the worst case is $O(n^2)$.*

Proof. We start the proof with the case $c = 1 = 2^0$ by introducing the cyclic shift circuit [3]. The cyclic shift as shown in Fig. 10 is the realization of the position shifting transformation for the case $c = 1$. Using Lemmas 7.2 and 7.4 in [1] we can see that the complexity of the cyclic shift is $O(n^2)$. In the next step, we consider the case $c = 2^k$ where $1 < k < 2n$. We can express the index i by $i_1.2^k + i_2$. It is trivial that

$$(i + c) \mod 2^{2n} = ((i_1 + 1).2^k + i_2) \mod 2^{2n} = (i_1 + 1) \mod 2^{2n-k} + i_2. \tag{12}$$

From (12) we see that the circuit for the case $c = 2^k$ can be implemented by applying the circuit in the previous case on the $2n - k$ highest qubits as shown in Fig. 11 for the case $k = 3$. For arbitrary c, we can express c and the index i by

$$c = c_0 2^0 + c_1 2^1 + \cdots + c_{2n-1} 2^{2n-1}, \tag{13}$$

$$i = i_0 2^0 + i_1 2^1 + \cdots + i_{2n-1} 2^{2n-1}. \tag{14}$$

Thus, it is easy to see that

$$(c + i) \mod 2^{2n} = \sum_{k=0}^{k=2n-1} (c_k + i_k) 2^k \mod 2^{k+1}. \tag{15}$$

That means we can realize the circuit for the shift transformation by putting a group of gates as explained in the case $c = 2^k$ for each k such that $c_k + i_k \mod 2 = 1$. An example for the case $c = 24$ on 5 qubits FRQI images is shown in Fig. 12.

Using the above-mentioned method to construct the circuit for arbitrary c, we can see that the worst case is when $c = 2^{2n} - 1$, i.e. the circuit when the position shifting transformation is most the complex. In the worst case, the circuit includes $k + 1\ C^{2n-k-1}(\sigma_x)$ gates where $k = 0, 1, \ldots, 2n - 1$. Thus, the complexity of position shifting transformation in the worst case is $O(n^2)$. $\square$
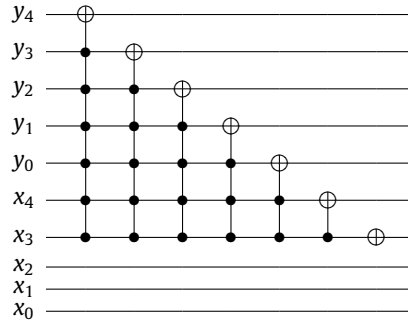
Fig. 11. The position shifting transformation for the case $c = 2^3$ and $n = 5$.
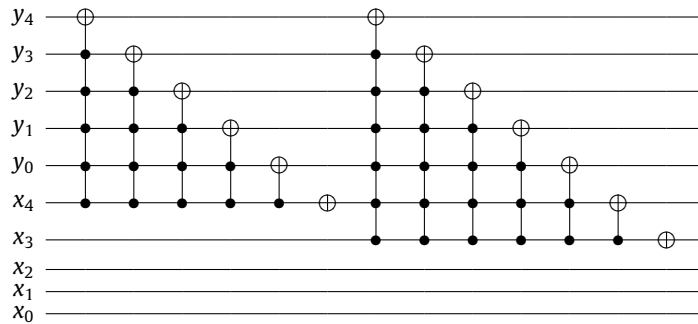

Fig. 12. The position shifting transformation for the case $c = 24$ and $n = 5$.

Multi-level controlled geometric transformations are used to build position shifting transformations. This strategy uses only one gate specifically, the NOT gate, as the main component and is applied on areas in the images from the smallest comprising the two points, until the largest, which targets the whole image. Other transformations can be constructed by the strategy based on different gates or group of gates. The new transformations are inseparable because layers in the transformations are interdependent. Combining the multi-level control strategy and other strategies presented in previous sections creates a huge number of ways to build new geometric transformations. Designers can add more control to restrict the performance of the transformations to smaller areas or to use two or more transformations as components to build separable ones.

6. Applications for the proposed design strategies

Each of the proposed design strategies can be deployed to accomplish certain image processing tasks on quantum computers. More complex tasks might require a combination of two or more design strategies to realize. In this section two simple applications employing these proposed strategies are demonstrated.

The second strategy whereby GTQI operations are confined to certain sub-blocks of an image were explored in [10] to accomplish a watermarking and authentication scheme on FRQI images. Therein the scheme was based on using the classical content of the image and watermark signal to pre-determine

- sub-blocks in the original image that can withstand tolerable distortion,
- the type of restricted GTQI operation to assign to each sub-block.

Here we demonstrate a slightly more crude version of that application. In determining the type of GTQI operations to perform on each sub-block, the available GTQI gate library is limited to the flip along horizontal axis, coordinate swap, and 180° rotation operations.

In doing this, we consider the 256 × 256 version of the popular Lena image as the original image. Our aim is to obtain a gate sequence comprising various GTQI operations whose effects are confined to smaller sub-blocks in the original image. This way the global effect of applying this gate sequence on the original image is an output image which shows high fidelity in terms of visual quality compared to the original image.

Guided by [10] and Theorem 1 in 3, we limit the size of each sub-block to 32 × 32 pixels and choose from the classical version of the input image five sub-areas that may yield less obvious distortions on the output image. The Lena image with these five sub-areas labelled from 1 to 5 are shown in Fig. 13.
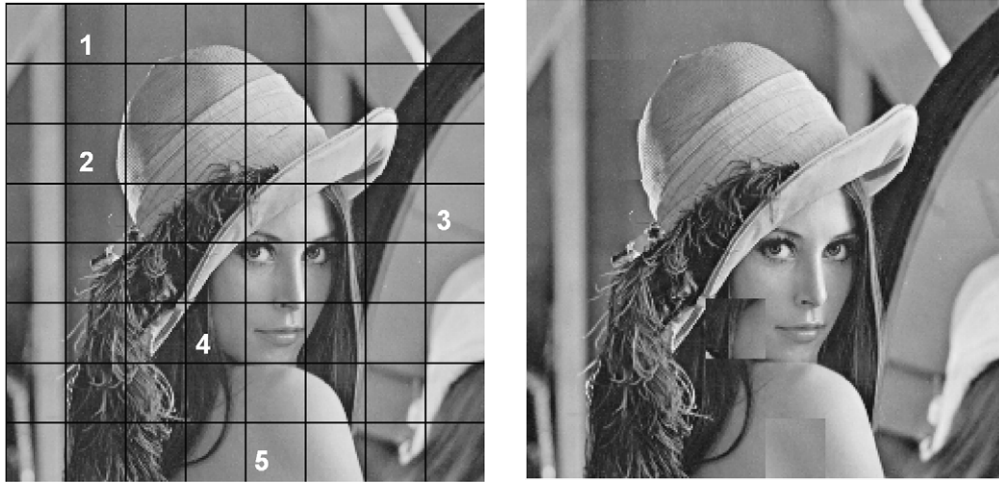
Fig. 13. The original Lena image to demonstrate the strategy presented in 3.
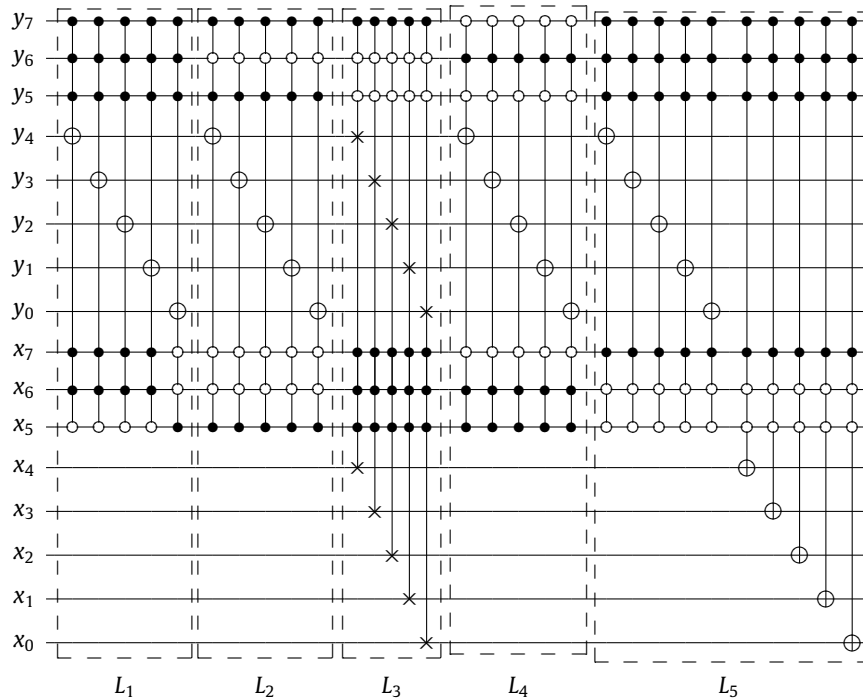

Fig. 14. The quantum circuit to realize the output Lena image.

In order to obtain the output image as shown in Fig. 13, we assign the horizontal flip operations to sub-blocks 1, 2, and 4; coordinate swap on sub-block 3; and 180° rotation on sub-block 5, albeit arbitrarily. Using the circuits for these operations as presented in 3, each pre-assigned GTQI operation is restricted to its pre-determined sub-block. Accordingly, the circuit to realize the transformed version of the Lena image comprises the sequence of operations targeting these respective sub-blocks. This circuit is shown in Fig. 14 and the resulting image is shown on the right of Fig. 13. In Fig. 14, the layers L_1–L_5 each indicates the sub-circuits to perform the pre-determined GTQI operations on the chosen sub-block with the corresponding label. Each layer L_1–L_4 consists of 5 sub-layers while L_5 has 10 sub-layers which shows that 180° rotation consists of two flip operations as discussed in Section 2 and [16]. In L_3, each sub-layer is a controlled swap operation. All the layers in the circuit can be decomposed in terms of NCT gate library [1].

From the output image we can see that transformations on sub-blocks 4 and 5 have produced somewhat conspicuous distortions. This was intentional and the purpose is to show that realizing an imperceptible version of the original image depends not only on the size of the sub-blocks but also on the appropriate choice of the GTQI operations for each sub-block. In [10], however, these issues are addressed in the watermark embedment procedure wherein the combined gate sequences to realize the transformations called the watermark embedment circuit depends on the classical content of the image and
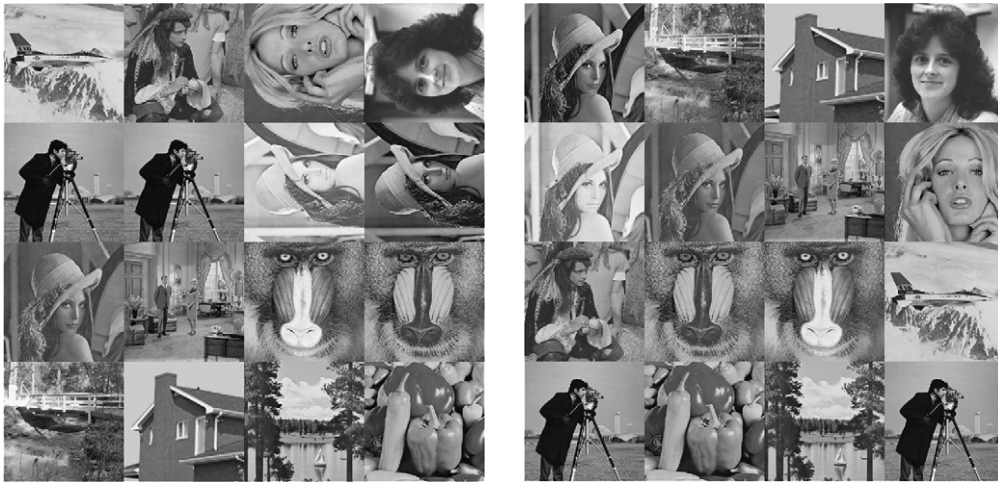
Fig. 15. The input for the image management application on the left side is a 1024 × 1024. It consists of 16 smaller 256 × 256 images. The image on the right side is the output image in which some of the smaller images have been moved and (or) rotated.

Fig. 16. Labeling relative positions of smaller images in the input image of Fig. 15.

watermark signal as a pair while the size of each sub-block to assign the restricted GTQI operations depends on the size of the original pair.

The second application is a simple image management application which seeks to move and rotate some small images within a larger image. This is a common task on classical computers. On quantum computers, however, the task is still a challenge because the methods to perform the movement and rotations of images are not well-understood. Some simplifications are needed in order to simulate the image management application. We can specify conditions on the input images and define the movement and rotations of the images as follows:

- There is only one input image at a time, i.e the large 1024 × 1024. In Fig. 15, the input image is the larger 1024 × 1024 image which comprises 16 smaller 256 × 256 images.
- The movement of images infers to changes in relative positions of the smaller images caused by a transformation on the larger image.
- Image rotation means the rotation of the smaller images caused by a transformation on the larger image.

This 16-image group can be represented by labels from 1 to 16, i.e one label for each small image, for convenience sake. This representation is shown in Fig. 16.

To perform the image management application on quantum computers, the large 1024 × 1024 image comprising 16 smaller 256 × 256 images is prepared in FRQI representation with 20 qubits, i.e. 10 qubits for each coordinate. To do the same task classically is inefficient because of the two point exchanges needed, which operate exponentially on the number of qubits. The quantum version of the application can be implemented more efficiently based on the FRQI representation.

The circuit in Fig. 18 is used to realize the output image on the right in Fig. 15. To further understand the procedure for obtaining this output image from the circuit, the position labeling representation for the smaller images shown in Fig. 16 is used. This enables a step by step breakdown of the circuit as follows:

1. To swap between the content of the upper and lower left half as shown in Fig. 17(a), we use the first layer in the circuit. This is indicated by the control on the qubit x_9 to target qubit y_9.
2. Swapping between the content of the upper and lower right half along the vertical axis as shown in Fig. 17(a) is achieved by imposing a control on x_9 as seen in the second layer of the circuit in Fig. 18.
3. The third layer of the circuit swaps the content of the 2 × 2 sub-blocks in the upper half along the horizontal. To do this the control qubit is assigned on y_9 to target y_8.
4. As shown in step 4 in Fig. 17(b) the content of the 2 × 2 sub-blocks in the lower half are swapped by adding a control on y_9 which targets x_9.
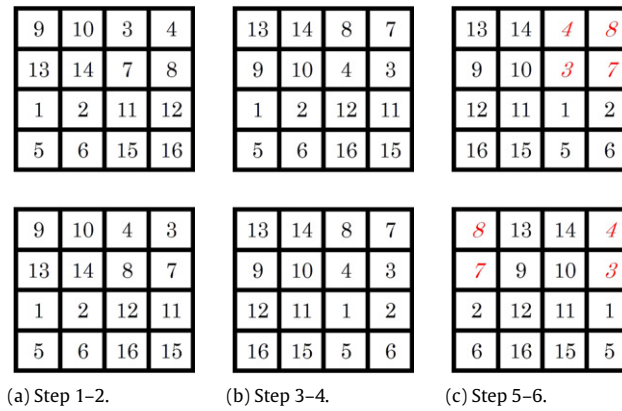
Fig. 17. Step by step transformations for the image management application to produce the output image in Fig. 15 based on the labeling in Fig. 16. The italicized labels indicate the rotated smaller images.
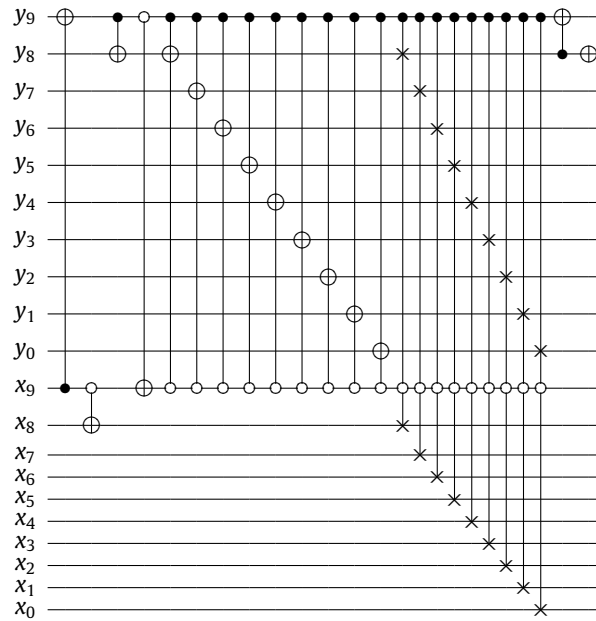

Fig. 18. The quantum circuit for the image management application.

5. Imposing controls on y_9 and x_9 in the next 18 layers, the content upper left of the 2×2 sub-block is rotated 90° clockwise.
6. Finally, the last two layers result a forward shift in all positions along the horizontal. This is shown in step 6 in Fig. 17(c).

This procedure, and the transition of the position labels in Fig. 17 indicate how the circuit utilizes the strategies discussed in previous sections to realize various transformations on the input image resulting in an entirely transformed output image. The swap operations in steps 2 and 3 show a good example of separable transformations and can be merged into a single transformation to reduce space.

These simple yet flexible applications demonstrate the applicability of the proposed strategies. This is important in extending classical-like geometric transformations for various image processing tasks on quantum computers.

7. Conclusion

Various strategies that allow for designing geometric transformations on quantum images have been proposed. These strategies focus on the affected areas in the images to facilitate the separable and smooth transformations by exploiting the FRQI representation for the images. The first strategy considers transformations that target sub-blocks within the large image. It was achieved by adding extra controls to specify the necessary information about the sub-blocks. The separability of classical operations are extended to quantum versions in the second strategy. Based on the extension, separable and completely separable geometric transformations on quantum images were examined. The third strategy focused on the

smoothness of the transformations using multi-level controls. Simple applications were used to demonstrate both the feasibility and applicability of proposed strategies.

The results in this paper can be extended towards the following directions. In terms of designing quantum circuits with low complexity for arbitrary geometric transformations, further investigation in terms of the group theory view point can be explored. The strategies presented in this paper provide high level tools to expand the number of transformations which are necessary to build practical image processing applications on quantum computers. Exploiting quantum based signal processing operations already available in the literature such as discrete cosine transformation [11,22], wavelet transformation [8], etc. the quantum watermarking scheme in Section 6 and [10] can be improved upon. Combining the proposed strategies, specifically the third one with the color related operations [15], these operations can be used as major components to depict motion, thus opening the door towards quantum movie representation.

References

[1] A. Barenco, C.H. Bennett, R. Cleve, D.P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J.A. Smolin, H. Weinfurter, Elementary gates for quantum computation, Physical Review A 52 (1995).
[2] G. Beach, C. Lomont, C. Cohen, Quantum image processing (quip), in: Proc. of Applied Imagery Pattern Recognition Workshop, 2003, pp. 39–44.
[3] T. Beth, M. Rötteler, Quantum Algorithms: Applicable Algebra and Quantum Physics, in: Springer Tracts in Modern Physics, vol. 173, Springer-Verlag, 2001, pp. 96–150.
[4] S. Caraiman, V.I. Manta, New applications of quantum algorithms to computer graphics: the quantum random sample consensus algorithm, in: Proc. of the 6th ACM Conference on Computing Frontier, 2009, pp. 81–88.
[5] D. Curtis, D.A. Meyer, Towards quantum template matching, in: Proc. of the SPIE, vol. 5161, 2004, pp. 134–141.
[6] T.G. Draper, Nonlinear complexity of boolean permutations, Ph.D. Thesis, Dept. of Mathematics, Univ. of Maryland, 2009. URL: http://www.lib.umd.edu/drum/bitstream/1903/9449/1/Draper_umd_0117E_10256.pdf.
[7] R.P. Feynman, Simulating physics with computers, International Journal of Theoretical Physics 21 (6/7) (1982) 467–488.
[8] A. Fijany, C.P. Williams, Quantum wavelet transform: fast algorithm and complete circuits, 1998. URL: arXiv:quantph/9809004.
[9] L. Grover, A fast quantum mechanical algorithm for database search, in: Proc. of the 28th Ann. ACM Symp. on the Theory of Computing, 1996, pp. 212–219.
[10] A.M. Iliyasu, P.Q. Le, F. Dong, K. Hirota, Restricted geometric transformations and their applications for quantum image watermarking and authentication, in: Proc. of Asian conf. on Quantum Information Science, 2010, pp. 96–97.
[11] A. Klappenecker, M. Rötteler, Discrete cosine transforms on quantum computers, in: Proc. of the 2nd Inter. Symp. on Image and Signal Processing and Analysis, 2001, pp. 464–468.
[12] J.I. Latorre, Image compression and entanglement, 2005. arXiv:quant-ph/0510031.
[13] C. Lomont, Quantum convolution and quantum correlation algorithms are physically impossible, 2003. arXiv:quantph/0309070.
[14] C. Lomont, Quantum circuit identities, 2003. arXiv:quantph/0307111.
[15] P.Q. Le, F. Dong, K. Hirota, A flexible representation of quantum images for polynomial preparation, image compression, and processing operations, Quantum Information Processing 10 (1) (2011) 63–84.
[16] P.Q. Le, A.M. Iliyasu, F. Dong, K. Hirota, Fast geometric transformations on quantum images, IAENG International Journal of Applied Mathematics 40 (3) (2010) 113–123. URL: http://www.iaeng.org/IJAM/issues_v40/issue_3/IJAM_40_3_02.pdf.
[17] D. Maslov, G.W. Dueck, D.M. Miller, N. Camille, Quantum circuit simplication and level compaction, IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems 27 (3) (2008) 436–444.
[18] T. Monz, K. Kim, W. Hansel, M. Riebe, A.S. Villar, P. Schindler, M. Chwalla, M. Hennrich, R. Blatt, Realization of the quantum Toffoli gate with trapped ions, Physical Review Letters 102 (2009).
[19] M. Nielsen, I. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, New York, 2000.
[20] V.V. Shende, A.K. Prasad, I.L. Markov, J.P. Hayes, Synthesis of reversible logic circuits, IEEE Transactions on Computer Aided Design 22 (6) (2003) 710–722.
[21] P.W. Shor, Algorithms for quantum computation: discrete logarithms and factoring, in: Proc. 35th Ann. Symp. Foundations of Computer Science, IEEE Computer Soc. Press, 1994, pp. 124–134.
[22] C.C. Tseng, T.M. Hwang, Quantum circuit design of 8 × 8 discrete cosine transforms using its fast computation on graph, in: Proc. ISCAS 2005, 2005, pp. 828–831.
[23] S.E. Venegas-Andraca, J.L. Ball, Processing images in entangled quantum systems, Quantum Information Processing 9 (1) (2010) 1–11.
[24] S.E. Venegas-Andraca, S. Bose, Storing, processing and retrieving an image using quantum mechanics, in: Proc. of the SPIE Conf. Quantum Information and Computation, 2003, pp. 137–147.
[25] A.D. Vos, B. Raa, L. Storme, Generating the group of reversible logic gates, Journal of Physics A: Mathematical and General 35 (2002) 7063–7078.