

# MAT-MEK-4270 Mandatory Assignment I

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## 1.2.3 Exact solution

The two-dimensional wave equation is given by

$$u_{tt} = c^2 \nabla^2 u, u \in \Omega \times [0, T] \quad (1)$$

where  $\Omega \subseteq \mathbb{R}^2$ .

We wish to show that  $u(t, x, y) = \exp(i(k_x x + k_y y - \omega t))$  solves (1), this is done by simply inserting  $u$  into the wave equation and verify that both sides of the equation are equal. The detailed computation is showed below:

$$\begin{aligned} u_{tt} &= D_t(-i\omega u) = -\omega^2 u \\ c^2 \nabla^2 u &= c^2(u_{xx} + u_{yy}) \\ &= c^2(D_x(ik_x) + D_y(ik_y))u \\ &= -c^2(k_x^2 + k_y^2)u \end{aligned}$$

This implies that  $u(t, x, y)$  satisfies the wave equation if and only if  $\omega^2 = c^2 \|\mathbf{k}\|^2$ , with  $\mathbf{k}$  being the wave number vector.

## 1.2.4 Dispersion coefficient

Discretizing (1) using finite differences gives the discrete wave equation

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2 \left( \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2} \right) \quad (2)$$

If all components in the wave number vector are equal to  $k$  then the solution of (2) reads

$$u_{i,j}^n = \exp\{\mathbf{i}(kh(i+j) - \tilde{\omega}n\Delta t)\}$$

Plugging  $u_{i,j}^n$  into (2) and dividing both sides of the equation by  $u_{i,j}^n$  yields

$$\frac{1}{\Delta t^2} (\exp(-\mathbf{i}\tilde{\omega}\Delta t) - 2 + \exp(\mathbf{i}\tilde{\omega}\Delta t)) = \frac{c^2}{h^2} (2\exp(\mathbf{i}kh) - 2 + 2\exp(-\mathbf{i}kh)) \quad (3)$$

We may greatly simplify (3) by using the fact that  $\exp(\mathbf{i}a) = \cos(a) + \mathbf{i}\sin(a)$ , that cosine and sine are even and odd functions, respectively. The resulting simplified equation is

$$\frac{1}{\Delta t^2} (\cos(\tilde{\omega}\Delta t) - 1) = \frac{2c^2}{h^2} (\cos(kh) - 1)$$

We can simplify the equation even further by noting that  $c^2 = \frac{h^2}{2\Delta t^2}$ . This leads to

$$\begin{aligned} \cos(\tilde{\omega}\Delta t) &= \cos(kh) \\ \Rightarrow \tilde{\omega}\Delta t &= kh \\ \Rightarrow \tilde{\omega} &= \frac{kh}{\Delta t} \end{aligned}$$

From the previous exercise we have the dispersion relation  $\omega^2 = c^2\|\mathbf{k}\|^2$ . We replace  $c^2$  by  $\frac{C^2h^2}{\Delta t^2}$  to obtain

$$\begin{aligned} \omega^2 &= c^2\|\mathbf{k}\|^2 \\ &= \frac{C^2h^2}{\Delta t^2} (2k^2) \\ &= \frac{h^2}{(2^{\frac{1}{2}})^2\Delta t^2} (2k^2) \\ &= \frac{k^2h^2}{\Delta t^2} \end{aligned}$$

This shows that  $\omega = \tilde{\omega}$ .