MAT-MEK-4270 Mandatory Assignment I

Vasin Phumimas

1.2.3 Exact solution
The two-dimensional wave equation is given by

$$u_{tt} = c^2\nabla^2 u, u \in \Omega \times [0, T] \tag{1}$$

where $\Omega \subseteq \mathbb{R}^2$.
We wish to show that $u(t, x, y) = \exp(i(k_x x + k_y y - \omega t))$ solves (1), this is done by simply inserting u into the wave equation and check that the left-hand side and equal to the right. The detailed computation is showed below:

$$\begin{aligned} u_{tt} &= D_t(-i\omega u) = -\omega^2 u \\ c^2\nabla^2 u &= c^2(u_{xx} + u_{yy}) \\ &= c^2(D_x(ik_x) + D_y(ik_y))u \\ &= -c^2(k_x^2 + k_y^2)u \end{aligned}$$

This implies that $u(t, x, y)$ satisfies the wave equation if and only if $\omega^2 = c^2\|\mathbf{k}\|^2$, with $\mathbf{k}$ being the wave number vector.
1.2.4 Dispersion coefficient
Discretizing (1) using finite differences gives the discrete wave equation

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^n}{\Delta t^2} = c^2\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2}\right) \tag{2}$$

If all components in the wave number vector are equal to k then the solution of (2) reads

$$u_{i,j}^n = \exp\{\mathbf{i}(kh(i+j) - \tilde{\omega} n \Delta t)\}$$

Plugging $u_{i,j}^n$ into (2) and dividing both sides of the equation by $u_{i,j}^n$ yields

$$\frac{1}{\Delta t^2}\left(\exp(-\mathbf{i}\tilde{\omega}\Delta t) - 2 + \exp(\mathbf{i}\tilde{\omega}\Delta t)\right) = \frac{c^2}{h^2}\left(2\exp(\mathbf{i}kh) - 2 + 2\exp(-\mathbf{i}kh)\right) \tag{3}$$

We may greatly simplify (3) by using the fact that $\exp(\mathbf{i}a) = \cos(a) + \mathbf{i}\sin(a)$, that cosine and sine are even and odd functions, respectively. The resulting simplified equation is

$$\frac{1}{\Delta t^2}(\cos(\tilde{\omega}\Delta t) - 1) = \frac{2c^2}{h^2}(\cos(kh) - 1)$$

We can simplify the equation even further by noting that $c^2 = \frac{h^2}{2\Delta t^2}$. This leads to

$$\begin{aligned}
\cos(\tilde{\omega}\Delta t) &= \cos(kh) \\
\Rightarrow \tilde{\omega}\Delta t &= kh \\
\Rightarrow \tilde{\omega} &= \frac{kh}{\Delta t}
\end{aligned}$$

From the previous exercise we have the dispersion relation $\omega^2 = c^2\|\mathbf{k}\|^2$. We replace c^2 by $\frac{C^2h^2}{\Delta t^2}$ to obtain

$$\begin{aligned}
\omega^2 &= c^2\|\mathbf{k}\|^2 \\
&= \frac{C^2h^2}{\Delta t^2}(2k^2) \\
&= \frac{h^2}{(2^{\frac{1}{2}})^2\Delta t^2}(2k^2) \\
&= \frac{k^2h^2}{\Delta t^2}
\end{aligned}$$

This shows that $\omega = \tilde{\omega}$.