ENGR 3301-001

RLC CIRCUITS - Parallel & Series

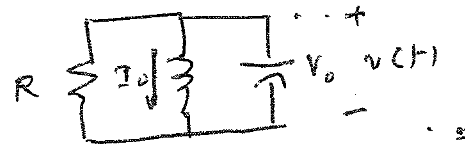
* Helpful Summary
* Examples.

<u>Disclaimer:</u> There are some known & possibly unknown arithmetic errors in the examples that I have not had the time to correct! 10/16/13

PARALLEL RLC CIRCUIT

Natural Response

- No energy source (voltage/current) present
- Response of Initial Conditions.



$$\boxed{\begin{aligned} i_L(0^+) &= I_0 \\ v(0^+) &= V_0 \end{aligned}}$$

$$\Rightarrow \frac{di(0^+)}{dt} = \frac{v(0^+)}{L}$$

ODE: $$\frac{d^2 i}{dt^2} + \frac{1}{RC} \cdot \frac{di}{dt} + \frac{i}{LC} = 0, \quad t \geq 0.$$

Parameters: $\alpha = \frac{1}{2RC}$ $\quad \omega_0 = \frac{1}{\sqrt{LC}}$

$$s_{1,2} = -\alpha \pm (\alpha^2 - \omega_0^2)^{1/2}$$

Three types of Response:

- **overdamped**, $\alpha^2 - \omega_0^2 > 0$

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, \quad t \geq 0.$$

$A_1 + A_2 = I_0$. $\quad s_1 A_1 + s_2 A_2 = \frac{di(0^+)}{dt}$

- **underdamped**, $\alpha^2 - \omega_0^2 < 0 \Rightarrow \omega_d = (\omega_0^2 - \alpha^2)^{1/2}$

$$i(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t)$$

$B_1 = i(0^+) = I_0$. $\quad -\alpha B_1 + \omega_d B_2 = \frac{di(0^+)}{dt}$.

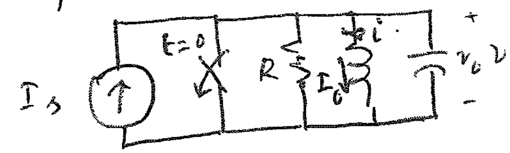
- **critically damped**, $\alpha^2 = \omega_0^2 \Rightarrow s_1 = -\alpha,\ s_2 = -\alpha$

$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}, \quad t \geq 0.$$

$D_2 = i(0^+) = I_0$, $\quad D_1 - \alpha D_2 = \frac{di(0^+)}{dt}$

Step Response

- Energy source(s) is (are) present
- Response due to energy source(s), and depends on the initial conditions.



$$\boxed{\begin{aligned} i(0^+) &= I_0 \\ v(0^+) &= V_0 \end{aligned}}$$

$$\frac{di(0^+)}{dt} = \frac{v(0^+)}{L}$$

ODE: $$\frac{d^2 i}{dt^2} + \frac{1}{RC} \frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}, \quad t \geq 0.$$

Parameters: See natural Response.

Final value: $I_f = i_L(\infty)$.

$$i(t) = I_f + \left\{ \begin{array}{l} \text{function of the} \\ \text{form of natural Response} \end{array} \right\}$$

Step Response:

Three types of Response:

- **overdamped:**

$$i(t) = I_f + A_1 e^{s_1 t} + A_2 e^{s_2 t}, \quad t \geq 0$$

- **Underdamped**

$$i(t) = I_f + B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t), \quad t \geq 0$$

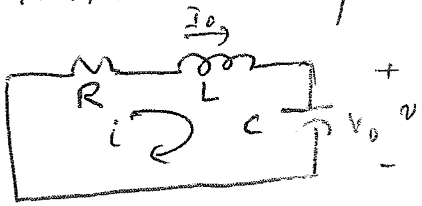
- **Critically damped**

$$i(t) = I_f + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}, \quad t \geq 0$$

CAUTION: values of the "undetermined" coefficients (A's, B's, D's) are <u>different</u> from natural Response, and are determined from initial conditions.

SERIES RLC CIRCUIT

Natural Response



$\boxed{i(0) = I_0 \quad v(0) = V_0}$

ODE: $\boxed{\dfrac{d^2 v}{dt^2} + \dfrac{R}{L}\dfrac{dv}{dt} + \dfrac{v}{LC} = 0}$

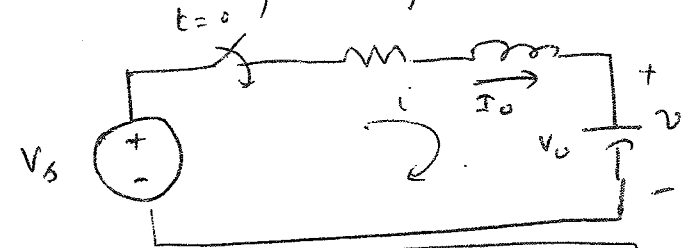
$$\frac{dv(0^+)}{dt} = \frac{i(0^+)}{C} = \frac{I_0}{C}$$

Parameters:

$$\alpha = \frac{R}{2L}, \quad \omega_0^2 = \frac{1}{LC}.$$

* Rest of this material – Similar to Parallel RLC
* use initial conditions to solve for the unknown coefficients.

Step Response



$\boxed{i(0) = I_0 \qquad v(0) = V_0.}$

$$\frac{dv(0)}{dt} = \frac{i_C(0^+)}{C} = \frac{i(0^+)}{C} = \frac{I_0}{C}$$

$\boxed{\text{ODE}}$ $\dfrac{d^2 v}{dt^2} + \dfrac{R}{L}\dfrac{dv}{dt} + \dfrac{1}{LC}v = \dfrac{V_s}{LC}, \quad t \geq 0.$

Need to establish the final value:

$$\boxed{V_f = v(\infty)}$$

* Rest of the material is similar to parallel RLC.

Note: For parallel RLC circuit,
if $i(t)$ is determined, $v(t)$, the voltage across the capacitor can be determined easily from $v(t) = L\dfrac{di}{dt}$.

Similarly, for series RLC circuit if $v(t)$ is determined, $i(t)$, the current through the inductor can be obtained from $i(t) = C\dfrac{dv}{dt}$.

Chapter 8

Natural and Step Response of Second-Order (RLC) Circuits

Here we review and then practice the techniques that enable us to analyze a limited group of circuits. These are circuits containing one equivalent resistor, one equivalent inductor, and one equivalent capacitor. The resistor, inductor, and capacitor can be connect in series or in parallel. Both the inductor and the capacitor may have initial stored energy. The use of the phrase "one equivalent" means that if the circuit contains two or more resistors, for example, they must be arranged in such a way that they can be combined in series and in parallel to form one single equivalent resistor. The same holds for circuits that contain two or more inductors, or two or more capacitors. These circuits are referred to as RLC circuits, and are also called **second-order circuits**, because their describing equation is a second-order differential equation.

These circuits usually contain a switch that is in one position for $t < 0$, switches positions at $t = 0$, and remains at that second position indefinitely. When the switch is in its first position, there may be an independent current or voltage source in the circuit as well, used to generate the energy that the inductor or capacitor will have stored at $t = 0$. When the switch moves to its second position, there may or may not be an independent current or voltage source in the circuit. If there is, it continues to supply energy to the circuit indefinitely, and we call the analysis a **step response** problem. If there is not an independent source in the circuit for $t \geq 0$, then the energy initially stored is dissipated to the resistor and we call the analysis a **natural response** problem. Fortunately the natural response problem and the step response problem are closely related, so we can use the same circuit analysis technique for both problems.

Analyzing RLC circuits connected in series is very similar to analyzing RLC circuits connected in parallel so we can also use the same circuit analysis technique for both circuits. There are basically five steps in the analysis: find the initial conditions, which consist of the initial current in the inductor and the initial voltage drop across the capacitor; find the final values, which are the final current in the inductor and the final voltage drop across the capacitor; find the **neper frequency**, α, which equals $1/2RC$ for the parallel circuit and equals $R/2L$ for the series circuit and the **resonant radian frequency**, ω_o, which is $\sqrt{1/LC}$, compare α^2 and ω_o^2 to determine whether the response type is **overdamped**, **underdamped**, or **critically damped** and write down the form of the response; use the response form to determine the value of the response at $t = 0$ and the value of the first derivative of the response at $t = 0$, then use the circuit to determine the same two quantities, providing enough information to solve for the unknown coefficients in the response form; and finally, use the calculated response to determine the values of any other requested voltages and currents in the circuit. For the natural response of the parallel RLC circuit the response we calculate is the voltage drop across the parallel elements. For the natural response of the series RLC circuit the response we calculate is the current through the series elements. For the step response problems, we will calculate the only quantity that has a non-zero final value — that is the voltage drop across the capacitor in the series RLC circuit and the current through the inductor in the parallel RLC circuit.

The analysis method for RLC circuits can be broken into the following steps:

1. Redraw the circuit as it appears for $t < 0$, replacing the switch with an open circuit if it is open, and with a short circuit if it is closed. Since it is assumed that the switch has been in this position for a long time, this places any inductors and capacitors in the presence of a constant source. Therefore, an inductor should be replaced by a short circuit and a capacitor should be replaced by an open circuit. Using this circuit for $t < 0$, calculate the current through the short circuit, which is the initial current I_o and the voltage drop across the open circuit, which is the initial voltage drop V_o.

2. Redraw the circuit as it appears for $t \geq 0$, replacing the switch with an open circuit if it is open and with a short circuit if it is closed. If there are no independent sources in the circuit, this is the natural response problem. The final value of the

voltage drop across the capacitor $V_f = 0$ and the final value of the current through the inductor $I_f = 0$, since all of the initially stored energy in the inductor and capacitor will be dissipated by the resistor as $t \to \infty$.

If there is an independent source in the circuit for $t \geq 0$ this is the step response problem. The inductor and capacitor will have been in the presence of this independent source for a long time as $t \to \infty$ so replace the inductor with a short circuit and the capacitor with an open circuit. Calculate the current in the short circuit, which is the final current, I_f, and the voltage drop across the open circuit, which is the final voltage V_f. Make sure the direction of the current arrow is the same when computing I_o in Step 1 and when computing I_f in this step. Make sure that the polarity of the voltage when you are calculating the initial voltage V_o is the same as the polarity of the voltage when you are calculating the final value V_f.

3. Determine the response type by calculating ω_o and α. For both series and parallel RLC circuits,

$$\omega_o = \sqrt{\frac{1}{LC}}$$

The computation of α depends on the configuration of the circuit:

$$\text{For series-connected } RLC \text{ circuits} \quad \alpha = \frac{R}{2L};$$

$$\text{For parallel-connected } RLC \text{ circuits} \quad \alpha = \frac{1}{2RC}$$

Then compare α^2 and ω_o^2 to determine the form of the response:

- If $\alpha^2 > \omega_o^2$, the response type is overdamped and of the form $X_f + A_1 e^{-s_1 t} + A_2 e^{-s_2 t}$.
- If $\alpha^2 < \omega_o^2$, the response type is underdamped and of the form $X_f + (B_1 \cos \omega_d t + B_2 \sin \omega_d t)e^{-\alpha t}$.
- If $\alpha^2 = \omega_o^2$, the response type is critically damped and of the form $X_f + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$.

In the above equations, X_f is the final value of the voltage or the current, depending on whether you are determining the response form for a voltage or a current. For the natural response problem $X_f = 0$ and you will determine the voltage drop for the parallel RLC circuit and the current for the series RLC circuit. For the step response problem, the non-zero final value will be the inductor current for the parallel RLC circuit and the capacitor voltage drop form the series RLC circuit.

4. Write the equation describing the response you are calculating. If the response form is overdamped, you will need to calculate s_1 and s_2 from the equation

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

If the response form is underdamped, you will need to calculate ω_d from the equation

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

If the response form is critically damped, you need make no additional calculations. Regardless of the response form you will have two unspecified coefficients whose values will be used to satisfy the initial conditions. Evaluate the initial value of the response (at $t = 0$) and the initial value of the first derivative of the response. These equations will involve the unknown coefficients. Then use the circuit to determine the initial value of the response, which will be I_o or V_o determined in Step 1, and the initial value of the first derivative of the response, whose value will also involve I_o or V_o. Then equate the initial values from the equation and its first derivative with the initial values from the circuit quantities. This provides two equations which, when solved simultaneously, will yield the values of the unknown coefficients. Complete this step by writing the response using the values of the unknown coefficients.

5. If a voltage or current other than the one you calculated in Step 4 was requested for this circuit, use the calculated value to determine the requested value. If the current or voltage you calculated was the one sought for the circuit, you are finished.

The following three examples illustrate the process of analyzing second-order circuits. There is one example for each of the three different response types. The examples illustrate both the series RLC circuit and the parallel RLC circuit and also illustrate both a natural response problem and a step response problem.

Example 8.1

Find $v_R(t)$ for the circuit in Fig. 8.1 for $t \geq 0$.

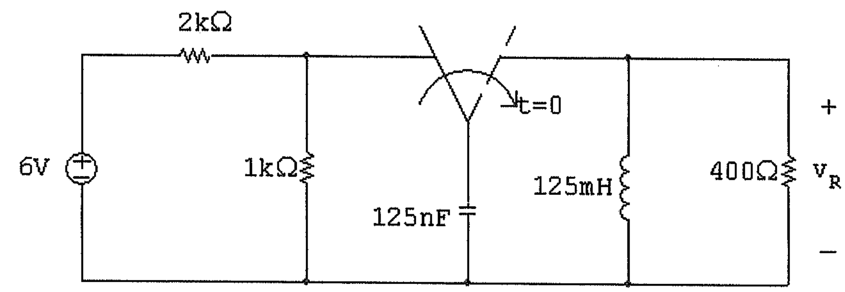


Figure 8.1: The circuit for Example 8.1

Solution

1. Redraw the circuit in Fig. 8.1 with the switch in its left hand position. This is the circuit for $t < 0$ and is used to establish the initial conditions. It is assumed that the switch has been in this position for a long time, so the inductor is replaced with a short circuit with current I_o and the capacitor is replaced by an open circuit with voltage drop V_o. The resulting circuit is shown in Fig. 8.2.

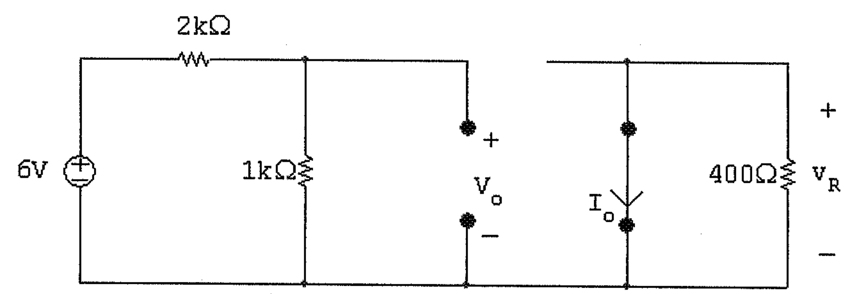


Figure 8.2: The circuit for Example 8.1, for $t < 0$, used to establish the initial conditions.

We must analyze this circuit to find I_o and V_o. I_o is easy because there is no current flowing in the right hand side of the circuit, due to the position of the switch. Therefore,

$$I_o = 0 \text{ A}.$$

To find V_o we use voltage division as follows:

$$V_o = \frac{1000}{1000 + 2000}(6) = 2 \text{ V}.$$

2. Redraw the circuit in Fig. 8.1 with the switch in its right hand position. This is the circuit for $t \geq 0$ and is used to establish the final values, since the circuit will be in this configuration as $t \rightarrow \infty$. It is assumed that the switch has been in this position for a long time, so the inductor is replaced with a short circuit with current I_f and the capacitor is replaced with an open circuit with voltage drop V_f. The resulting circuit is shown in Fig. 8.3.

   As you can see, there are no independent sources in this circuit, so the stored energy in the capacitor will dissipate in the resistor leaving no energy in the capacitor. There was never any stored energy in the inductor. Thus, the final voltage $V_f = 0$V and the final current $I_f = 0$A.

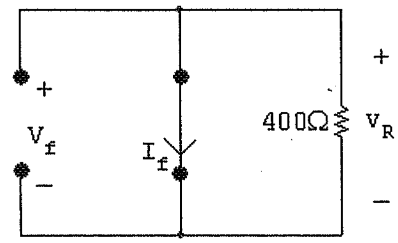


Figure 8.3: The circuit for Example 8.1, for $t \geq 0$, used to establish the final values.

3. To find α and ω_o we consider the values of the resistor, the inductor, and the capacitor in the circuit for $t \geq 0$, as shown in Fig. 8.4.

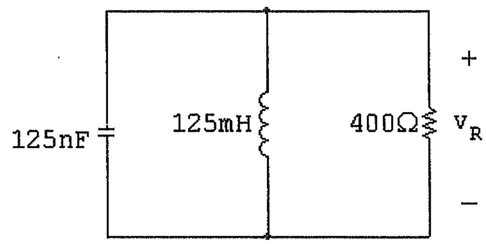


Figure 8.4: The circuit for Example 8.1, for $t \geq 0$, used to calculate α and ω_o.

We substitute these values into the equations appropriate for the parallel RLC circuit:

$$\alpha = \frac{1}{2RC} = \frac{1}{2(400)(125 \times 10^{-9})} = 10{,}000 \text{ rad/sec;}$$

$$\omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(0.125)(125 \times 10^{-9})}} = 8000 \text{ rad/sec.}$$

Now we compare the values of α^2 and ω_o^2 to determine the response type. Since $\alpha^2 > \omega_o^2$, the response is overdamped, and since this is the parallel RLC natural response problem, the response we determine is the voltage across the parallel components, $v_R(t)$.

4. In order to specify an overdamped response, we need to calculate the values of the complex frequencies s_1 and s_2:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} = -10{,}000 \pm \sqrt{10{,}000^2 - 8000^2} = -10{,}000 \pm 6000$$

Thus,

$$s_1 = -4000 \text{ rad/sec} \qquad \text{and} \qquad s_2 = -16{,}000 \text{ rad/sec.}$$

Therefore, the response is

$$v_R(t) = A_1 e^{-4000t} + A_2 e^{-16{,}000t} \text{ V.}$$

To calculate the coefficients A_1 and A_2, we need two equations. The first equation is the result of evaluating the response $v_R(t)$ at $t = 0$ and settling the result equal to the initial value of the voltage from the circuit, V_o:

$$v_R(0) = A_1 + A_2 = V_o = \overset{2}{\cancel{6}} \text{ V.}$$

The second equation is the result of evaluating the first derivative of the response $v_R(t)$ at $t = 0$ and setting the result equal to the initial value of the first derivative of the voltage from the circuit. The first derivative of the response $v_R(t)$ at $t = 0$ is

$$\frac{dv_R(0)}{dt} = -4000A_1 - 16{,}000A_2$$

The first derivative of the $v_R(t)$ from the circuit is the same as the first derivative of the voltage across the capacitor, since the circuit components are in parallel. For the capacitor we know that

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

so

$$i_C(0) = C\frac{dv_C(0)}{dt}$$

and

$$\frac{dv_C(0)}{dt} = \frac{1}{C}i_C(0).$$

We don't know the value of the initial current in the capacitor, $i_C(0)$, but we do know that the sum of the capacitor current, the inductor current, and the resistor current must be zero, from KCL. Therefore,

$$\begin{aligned}
\frac{dv_C(0)}{dt} &= \frac{1}{C}i_C(0) = \frac{1}{C}[-i_L(0) - i_R(0)] \\
&= \frac{1}{C}\left[-I_o - \frac{V_o}{R}\right] = \frac{1}{125 \times 10^{-9}}\left[0 - \frac{2}{400}\right] \\
&= -40,000 \text{ V/s}.
\end{aligned}$$

To summarize, the two equations used to solve for A_1 and A_2 are

$$\begin{aligned}
A_1 + A_2 &= 2 \\
-4000A_1 - 16{,}000A_2 &= -40{,}000
\end{aligned}$$

Since these equations are already in standard form, we can use the calculator to solve them:

$$A_1 = -0.667; \qquad A_2 = 2.667.$$

Thus,

$$v_R(t) = -0.667e^{-4000t} + 2.667e^{-16{,}000t} \text{ V}, \quad t \geq 0.$$

5. Since the voltage drop across the parallel components was the only quantity requested in the original circuit shown in Fig. 8.1, no further analysis is required.

Example 8.2

Find $v_C(t)$ for the circuit in Fig. 8.5 for $t \geq 0$.

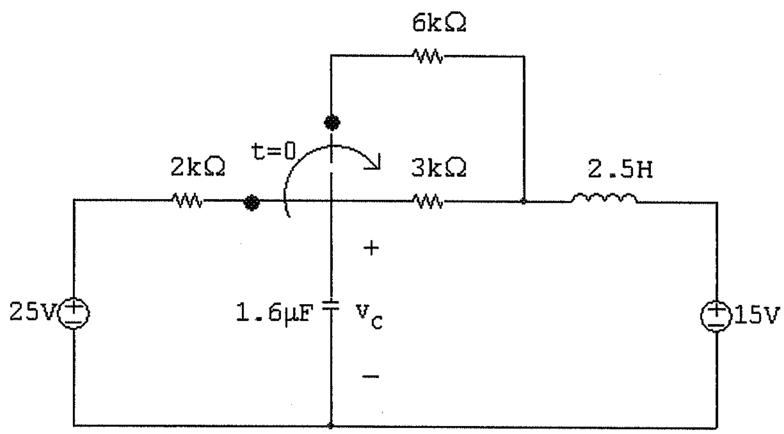


Figure 8.5: The circuit for Example 8.2

Solution

1. Redraw the circuit in Fig. 8.5 with the switch in its down position. This is the circuit for $t < 0$ and is used to establish the initial conditions. It is assumed that the switch has been in this position for a long time, so the inductor is replaced with a short circuit with current I_o and the capacitor is replaced by an open circuit with voltage drop V_o. The resulting circuit is shown in Fig. 8.6.

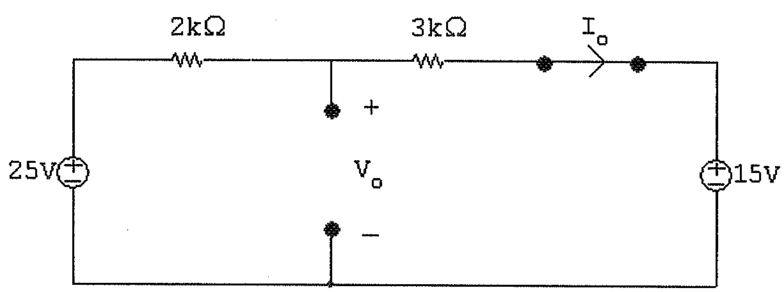


Figure 8.6: The circuit for Example 8.2, for $t < 0$, used to establish the initial conditions.

We must analyze this circuit to find I_o and V_o. I_o can be found by applying Ohm's law. Therefore,

$$I_o = \frac{25 - 15}{2000 + 3000} = \overset{2}{\cancel{5}}\,\text{mA}.$$

To find V_o we write a node voltage equation as follows:

$$\frac{V_o - 25}{2000} + \frac{V_o - 15}{3000} = 0.$$

Solving for V_o we get

$$V_o = 21\ \text{V}.$$

2. Redraw the circuit in Fig. 8.5 with the switch in its up position. This is the circuit for $t \geq 0$ and is used to establish the final values, since the circuit will be in this configuration as $t \to \infty$. It is assumed that the switch has been in this position for a long time, so the inductor is replaced with a short circuit with current I_f and the capacitor is replaced with an open circuit with voltage drop V_f. Notice that we have also combined the parallel resistors into a single resistor with the value $3000 \| 6000 = 2$ kΩ. The resulting circuit is shown in Fig. 8.7.

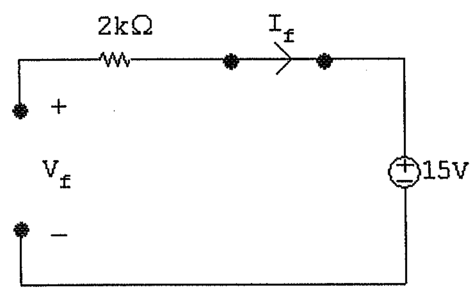


Figure 8.7: The circuit for Example 8.2, for $t \geq 0$, used to establish the final values.

As you can see, there is an independent source in this circuit, so this is a step response problem. Because of the open circuit created by the capacitor, there is no current, so

$$I_f = 0 \text{ A} \qquad \text{and} \qquad V_f = 15 \text{ V}.$$

3. To find α and ω_o we consider the values of the resistor, the inductor, and the capacitor in the circuit for $t \geq 0$, as shown in Fig. 8.8.

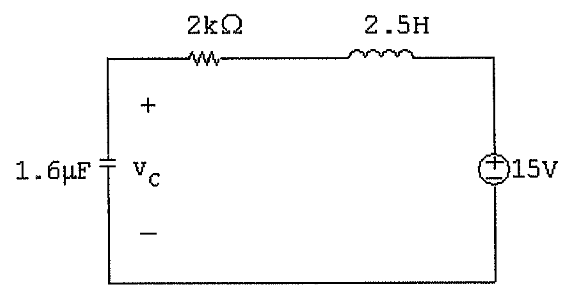


Figure 8.8: The circuit for Example 8.2, for $t \geq 0$, used to calculate α and ω_o.

We substitute these values into the equations appropriate for the series RLC circuit:

$$\alpha = \frac{R}{2L} = \frac{3000}{2(2.5)} = 400 \text{ rad/sec};$$

$$\omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(2.5)(1.6 \times 10^{-6})}} = 500 \text{ rad/sec}.$$

Now we compare the values of α^2 and ω_o^2 to determine the response type. Since $\alpha^2 < \omega_o^2$, the response is underdamped, and since this is the series RLC step response problem and the only non-zero final value is the voltage drop across the capacitor, the response we determine is the voltage across the capacitor, $v_C(t)$.

4. In order to specify an underdamped response, we need to calculate the values of the damped radian frequency ω_d:

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2} = \sqrt{500^2 - 400^2} = 300 \text{ rad/sec}.$$

Therefore, the response is

$$v_C(t) = V_f + (B_1 \cos 300t + B_2 \sin 300t)e^{-400t} \text{ V}.$$

To calculate the coefficients B_1 and B_2, we need two equations. The first equation is the result of evaluating the response $v_C(t)$ at $t = 0$ and settling the result equal to the initial value of the voltage from the circuit, V_o:

$$v_C(0) = V_f + B_1 = V_o = 21 \text{ V}.$$

The second equation is the result of evaluating the first derivative of the response $v_C(t)$ at $t = 0$ and setting the result equal to the initial value of the first derivative of the voltage from the circuit. The first derivative of $v_C(t)$ at $t = 0$ is

$$\frac{dv_C(0)}{dt} = -400B_1 + 500B_2$$

The first derivative of the $v_C(t)$ from the circuit can be found from the describing equation for the voltage and current in a capacitor:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

so

$$i_C(0) = C\frac{dv_C(0)}{dt}$$

and

$$\frac{dv_C(0)}{dt} = \frac{1}{C}i_C(0).$$

The initial current in the capacitor, $i_C(0)$, has the same value as the initial value of the current in the inductor, since they are in series, but the opposite sign. Therefore,

$$\frac{dv_C(0)}{dt} = \frac{1}{C}i_C(0) = \frac{1}{C}I_o = \frac{1}{1.6 \times 10^{-6}}(-0.005) = -3125.$$

To summarize, the two equations used to solve for A_1 and A_2 are

$$\begin{aligned} B_1 &= 21 - 15 = 6 \\ -400B_1 + 300B_2 &= -3125 \end{aligned}$$

Since these equations are already in standard form, we can use the calculator to solve them:

$$B_1 = 6; \qquad B_2 = -2.417.$$

Thus,

$$v_C(t) = 15 + (6\cos 300t - 2.417\sin 300t)e^{-400t} \text{ V}, \quad t \geq 0.$$

5. Since the voltage drop across the capacitor was the only quantity requested in the original circuit shown in Fig. 8.5, no further analysis is required.

Example 8.3

There is no initial energy stored in the circuit in Fig. 8.9. Find $i_L(t)$ for this circuit for $t \geq 0$.

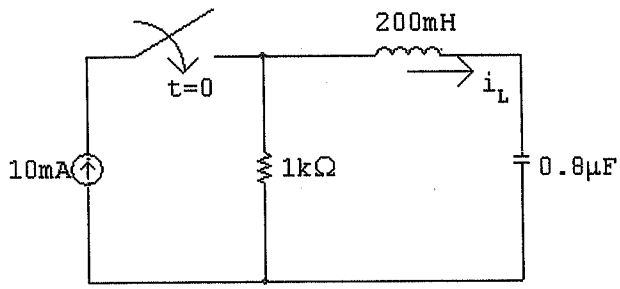


Figure 8.9: The circuit for Example 8.3

Solution

1. Since we have already been told that there is no initial stored energy in the circuit, we don't need to analyze the circuit for $t < 0$ to find the initial conditions, since they are both zero. Therefore,

$$I_o = 0 \text{ A} \qquad \text{and} \qquad V_o = 0 \text{ V}.$$

2. Redraw the circuit in Fig. 8.9 with the switch closed. This is the circuit for $t \geq 0$ and is used to establish the final values, since the circuit will be in this configuration as $t \to \infty$. It is assumed that the switch has been in this position for a long time, so the inductor is replaced with a short circuit with current I_f and the capacitor is replaced with an open circuit with voltage drop V_f. We have also performed a source transformation to turn the parallel combination of the current source and resistor into a series combination of a voltage source and the same resistor. The resulting circuit is shown in Fig. 8.10.

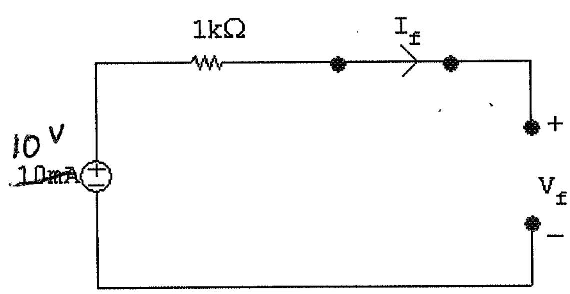


Figure 8.10: The circuit for Example 8.3, for $t \geq 0$, used to establish the final values.

As you can see, there is an independent source in this circuit, so this is a step response problem. Because of the open circuit created by the capacitor, there is no current, so

$$I_f = 0 \text{ A} \qquad \text{and} \qquad V_f = 10 \text{ V}.$$

3. To find α and ω_o we consider the values of the resistor, the inductor, and the capacitor in the circuit for $t \geq 0$, as shown in Fig. 8.11.

We substitute these values into the equations appropriate for the series RLC circuit:

$$\alpha = \frac{R}{2L} = \frac{1000}{2(0.2)} = 2500 \text{ rad/sec};$$

$$\omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(0.2)(0.8 \times 10^{-6})}} = 2500 \text{ rad/sec}.$$

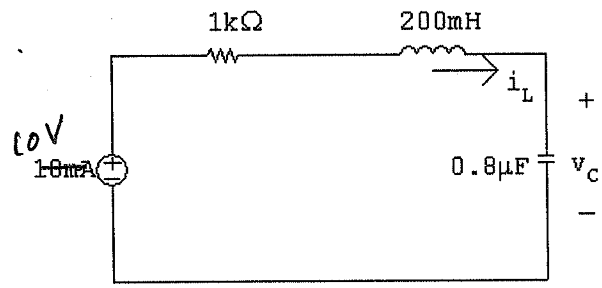


Figure 8.11: The circuit for Example 8.3, for $t \geq 0$, used to calculate α and ω_o.

Now we compare the values of α^2 and ω_O^2 to determine the response type. Since $\alpha^2 = \omega_o^2$, the response is critically damped, and since this is the series RLC step response problem and the only non-zero final value is the voltage drop across the capacitor, the response we determine is the voltage across the capacitor, $v_C(t)$.

4. We don't need to make any additional calculations for the critically damped response type. Therefore, the response is

$$v_C(t) = V_f + D_1te^{-2500t} + D_2e^{-2500t} \text{ V}.$$

To calculate the coefficients D_1 and D_2, we need two equations. The first equation is the result of evaluating the response $v_C(t)$ at $t = 0$ and settling the result equal to the initial value of the voltage from the circuit, V_o:

$$v_C(0) = V_f + D_2 = V_o = 0 \text{ V}.$$

The second equation is the result of evaluating the first derivative of the response $v_C(t)$ at $t = 0$ and setting the result equal to the initial value of the first derivative of the voltage from the circuit. The first derivative of the $v_C(t)$ at $t = 0$ is

$$\frac{dv_C(0)}{dt} = D_1 - 2500D_2$$

The first derivative of $v_C(t)$ from the circuit can be found from the describing equation for the voltage and current in a capacitor:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

so

$$i_C(0) = C\frac{dv_C(0)}{dt}$$

and

$$\frac{dv_C(0)}{dt} = \frac{1}{C}i_C(0).$$

The initial current in the capacitor, $i_C(0)$, has the same value as the initial value of the current in the inductor, since they are in series. Therefore,

$$\frac{dv_C(0)}{dt} = \frac{1}{C}i_C(0) = \frac{1}{C}I_o = \frac{1}{1.6 \times 10^{-6}}(0) = 0.$$

To summarize, the two equations used to solve for D_1 and D_2 are

$$\begin{aligned} D_2 &= 0 - 10 = -10 \\ D_1 - 2500D_2 &= 0 \end{aligned}$$

Since these equations are already in standard form, we can use the calculator to solve them:

$$D_1 = -25{,}000; \qquad D_2 = -10.$$

Thus,

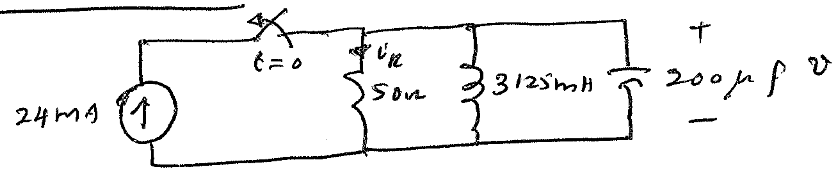
$$v_C(t) = 10 - 25{,}000te^{-2500t} - 10e^{-2500t} \text{ V}, \quad t \geq 0.$$

5. Now we must calculate the quantity requested in the original circuit shown in Fig. 8.5, which is the current in the inductor, $i_L(t)$. Since this is a series RLC circuit, the current in the inductor is the same as the current in the capacitor, whose value we can calculate from the derivative of the voltage drop across the capacitor:

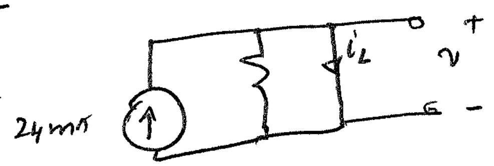
$$\begin{aligned} i_L(t) &= i_C(t) = C\frac{dv(t)}{dt} \\ &= (0.8 \times 10^{-6})[-25{,}000e - 2500t + (-2500)(-25{,}000)te^{-2500t} \\ &\qquad +(-2500)(-10)e^{-2500t}] \\ &= 50te^{-2500t}\ \text{A}, \quad t \geq 0. \end{aligned}$$

Now try using the second-order circuit analysis method for each of the practice problems below.

Workbook Problem 8.2



$t < 0$



$i_L(0^-) = 24\,mA$
$i_R(0^-) = 0\ A$
$v(0^-) = 0\ V$

$t \geq 0$

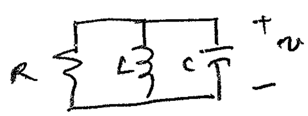
$i_L(0^+) = 24\,mA \qquad v(0^+) = 0\ V$

$$C\frac{dv(t)}{dt} = i_C(t) = -i_L(t) - i_R(t)$$

$$\Rightarrow \frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = -\frac{i_L(0^+)}{C} - \frac{\overset{0}{\cancel{i_R(0^+)}}}{C} = \frac{-24\times10^{-3}}{200\times10^{-6}} = -120.$$

Initial conditions:

$$\boxed{\begin{aligned} v(0^+) &= 0\,V \\ \frac{dv(0^+)}{dt} &= -120 \end{aligned}}$$



$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \qquad t \geq 0.$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times50\times200\times10^{-6}} = \frac{1}{2\times10^{-2}} = 50$$

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{3125\times10^{-3}\times200\times10^{-6}} = \frac{10^9}{3125\times200}$$

$$= \frac{16\,\cancel{10^4}\times10^5}{\cancel{3125}\times \cancel{200}\,100} = \boxed{1600}$$

$$s_{1,2} = -\alpha \pm (\alpha^2 - \omega_0^2)^{1/2}$$
$$= -50 \pm (900)^{1/2} \Rightarrow \boxed{s_1 = -80,\ s_2 = -20}$$ over damped

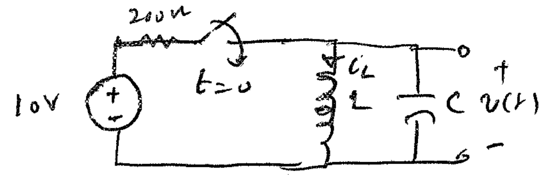
$$v(t) = A_1 e^{-80t} + A_2 e^{-20t}, \quad t \geq 0.$$

$$\left.\begin{aligned} v(0) &= 0 = A_1 + A_2 \\ \frac{dv(0)}{dt} &= -120 = -80A_1 - 20A_2 \end{aligned}\right] \Rightarrow \boxed{A_1 = 2,\ A_2 = -2}\,V$$

$$\Rightarrow \boxed{v(t) = \left(2e^{-80t} - 2e^{-20t}\right)V}, \quad t \geq 0.$$

$$i_R(t) = \frac{v(t)}{R} \Rightarrow \boxed{i_R(t) = 40e^{-80t} - 40e^{-20t}\ mA, \quad t \geq 0}$$

Workbook Problem 8.6



$R = 200\,\Omega$
$L = 400\,mH$
$C = 2.5\,\mu H$

* No prior energy in L & C before switch closes
* Determine $i_L(t)$.

$\underline{t<0}$ $\quad i_L(0^-) = 0\ ,\quad v(0^-) = 0$

$\underline{t \geq 0}$ $\quad \boxed{i_L(0^+) = i_L(0^-) = 0} \quad L\dfrac{di_L(0)}{dt} = v(0) = 0 \Rightarrow \boxed{\dfrac{di_L(0^+)}{dt} = 0}$

$\longrightarrow$ Initial Conditions

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 200 \times 2.5 \times 10^{-6}} = \boxed{1000\ rad/sec}$$

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{400 \times 10^{-3} \times 2.5 \times 10^{-6}} = 10^6 \Rightarrow \boxed{\omega_0 = 1000\ rad/sec}$$

$$\alpha^2 = \omega_0^2 \Rightarrow \boxed{\text{Critically Damped}}$$

$$i_L(t) = I_f + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t},\quad t \geq 0$$

$$I_f = \frac{10V}{200\Omega} = \boxed{50\ mA}$$

$$i_L(0) = 50 + D_2\ mA \Rightarrow \boxed{D_2 = -50\ mA}$$

$$\frac{di_L(0)}{dt} = 0 = D_1 - 1000 D_2 \Rightarrow \boxed{D_1 = -50{,}000\ mA}$$

$$\Rightarrow \boxed{i_L(t) = 50 - 50{,}000 t e^{-1000t} - 50 e^{-1000t}\ \underline{mA},\quad t \geq 0}$$

8.8

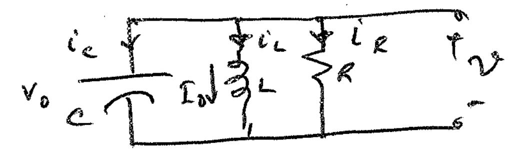


fig. 8.1.

Given:

$$v(t) = -8e^{-250t} + 32e^{-1000t} \text{ V}, \quad t \geq 0 \quad — ①$$

$$C = 0.1\,\mu F, \; V_0 = 24 V, \; I_0 = 0 A.$$

Determine:

(a) R, L, α and ω_0.

(b) $i_R(t)$, $i_L(t)$, $i_c(t)$, $t \geq 0^+$.

Soln:

a From ①, we identify:

$$s_1 = -\alpha + (\alpha^2 - \omega_0^2)^{1/2} = -250$$

$$s_2 = -\alpha - (\alpha^2 - \omega_0^2)^{1/2} = -1000$$

Add $\Rightarrow$ $-2\alpha = -1250$

$\Rightarrow$ $\boxed{\alpha = 625}$ rad/sec

But $\alpha = \dfrac{1}{2RC} \Rightarrow 625 = \dfrac{1}{2R \times 0.1 \times 10^{-6}} \Rightarrow \boxed{R = 8 k\Omega}$

$s_1 - s_2 = 2(\alpha^2 - \omega_0^2)^{1/2} = 750 \Rightarrow 4(\alpha^2 - \omega_0^2) = (750)^2$

with $\alpha = 625$, we obtain $\boxed{\omega_0 = 500 \text{ rad/sec}}$

But $\omega_0^2 = \dfrac{1}{LC} \Rightarrow (500)^2 = \dfrac{1}{L \times 0.2 \times 10^{-6}} \Rightarrow \boxed{L = 40 H}$

$\boxed{R = 8 k\Omega, \; L = 40 H, \; \alpha = 625 \text{ rad/sec}, \; \omega_0 = 500 \text{ rad/sec}}$ ← Answer (a)

b

$$i_R(t) = \frac{v(t)}{R} = -1.\,e^{-250t} + 4e^{-1000t} \text{ mA}, \quad t \geq 0^+$$

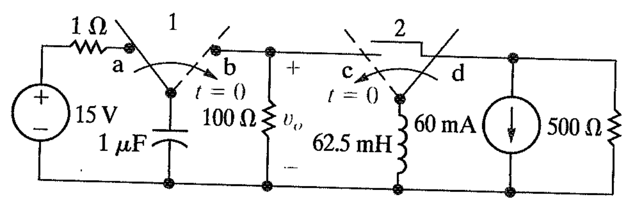
$$i_c(t) = C\frac{dv(t)}{dt} = 0.2e^{-250t} - 3.2e^{-1000t} \text{ mA}, \quad t \geq 0^+$$

$$i_L(t) = -(i_R + i_c) = 0.8e^{-250t} - 0.8e^{-1000t} \text{ mA}, \quad t \geq 0.$$

8.18

8.18 PSPICE MULTISIM The two switches in the circuit seen in Fig. P8.18 operate synchronously. When switch 1 is in position a, switch 2 is in position d. When switch 1 moves to position b, switch 2 moves to position c. Switch 1 has been in position a for a long time. At $t = 0$, the switches move to their alternate positions. Find $v_o(t)$ for $t \geq 0$.

Figure P8.18



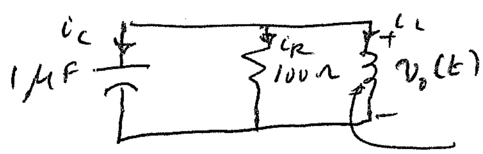
$t < 0$:

* Capacitor in open ckt and charged to 15 V. ⟹ $V_o = 15\,V$.

* Inductor in short ckt ⟹ $I_o = -60\,mA$

← Initial Condition information

$t \geq 0$



$i_L(0) = I_o = -60\,mA$

$i_R(0) = \frac{V_o}{R} = \frac{15\,V}{100\,\Omega} = 150\,mA$

$i_c(0) = -(i_L(0) + i_R(0))$

$= -90\,mA.$

Note that $v_o(t)$ is the voltage across the capacitor, Resistor as well as the inductor.

$$\frac{d^2 v_o}{dt^2} + \frac{1}{RC}\frac{dv_o}{dt} + \frac{1}{LC}v_o = 0,$$

Initial Condition Info.:

$$\boxed{\begin{aligned} &v_o(0) = V_o = 15\,V \\ &\frac{dv_o(0)}{dt} = \frac{i_c(0)}{C} = \frac{-90 \times 10^{-3}}{1 \times 10^{-6}} \\ &= -90{,}000. \end{aligned}}$$ — (1)

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 100 \times 1 \times 10^{-6}} = 500\,rad/sec$$

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{62.5 \times 10^{-3} \times 1 \times 10^{-6}} = 16 \times 10^6 \Rightarrow \boxed{\omega_o = 4000\,rad/sec}$$

$$s_{1,2} = -\alpha \pm (\alpha^2 - \omega_o^2)^{1/2} \Rightarrow \boxed{\begin{aligned} s_1 &= -2000\,rad/sec \\ s_2 &= -8000\,rad/sec \end{aligned}}$$

Response, $v_o(t)$, $t \geq 0$

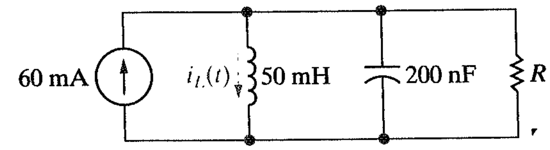
$$v_o(t) = A_1 e^{-s_1 t} + A_2 e^{-s_2 t}, \quad t \geq 0. \quad \text{— (2)}$$

$$\left.\begin{aligned} v_o(0) &= V_o = 15 = A_1 + A_2 \\ \frac{dv_o(0)}{dt} &= -90{,}000 = -2000 A_1 - 8000 A_2 \end{aligned}\right\} \Rightarrow \boxed{A_1 = 5, \quad A_2 = 10}$$

$$\Rightarrow \boxed{v_o(t) = (5e^{-2000t} + 10e^{-8000t})\,V, \quad t \geq 0}$$

8.29 PSPICE MULTISIM Assume that at the instant the 60 mA dc current source is applied to the circuit in Fig. P8.29, the initial current in the 50 mH inductor is −45 mA, and the initial voltage on the capacitor is 15 V (positive at the upper terminal). Find the expression for $i_L(t)$ for $t \geq 0$ if R equals 200 Ω.

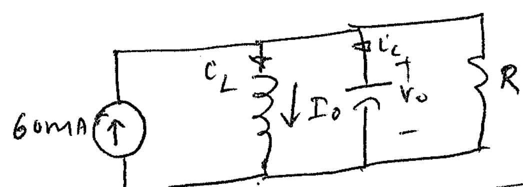
Figure P8.29



$\underline{t<0}$

$I_0 = -45\,mA \qquad V_0 = 15\,V.$

$\underline{t \geq 0}$



$L = 50\,mH$
$C = 200\,nF$
$R = 200\,\Omega$

Initial Condition:

$$\boxed{i_L(0) = I_0 = -45\,mA \qquad \frac{di_L(0)}{dt} = \frac{V_0}{L} = \frac{15}{50\times10^{-3}} = 300}$$

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\cdot\frac{di_L(t)}{dt} + \frac{1}{LC}\cdot i_L(t) = \frac{I_S}{LC}, \quad I_S = 60\,mA.$$

$$\boxed{I_f = i_L(\infty) = I_S = 60\,mA}$$

$$i_L(t) = I_f + \left[\text{Function of the same form as Natural Response}\right]. \quad \text{—} (1)$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times200\times200\times10^{-9}} = \frac{1}{8\times10^{-5}} = \frac{10^2\times10^3}{8} = \boxed{12500\ \text{rad/sec}}$$

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{50\times10^{-3}\times200\times10^{-9}} = 10^8\ (\text{rad/sec})^2 \Rightarrow \boxed{\omega_0 = 10^4\ \text{rad/sec}}$$

$$s_{1,2} = -\alpha \pm (\alpha^2-\omega_0^2)^{1/2} \Rightarrow \boxed{\begin{aligned} s_1 &= -5{,}000\ \text{rad/sec} \\ s_2 &= -20{,}000\ \text{rad/sec}\end{aligned}}$$

(1) Can be written as

$$i_L(t) = 60 + A_1 e^{-5000t} + A_2 e^{-20{,}000t}\ mA, \quad t \geq 0$$

with the initial Conditions $i_L(0)$ and $\frac{di_L(0)}{dt}$.

$$\Rightarrow i_L(0) = -45 = 60 + A_1 + A_2 \Rightarrow A_1 + A_2 = -60\times10^{-3}$$

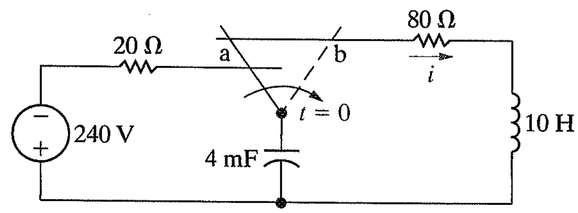
$$\frac{di_L(0)}{dt} = 300 = -5000A_1 - 20{,}000A_2$$

$$\Rightarrow \boxed{\begin{aligned} A_1 &= -120\ mA \\ A_2 &= 15\ mA\end{aligned}}$$

$$\boxed{i_L = 60 - 120e^{-5{,}000t} + 15e^{-20{,}000t}\ mA}, \quad t \geq 0.$$

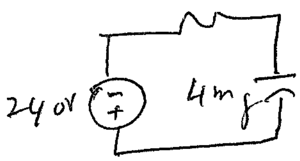
8.45 PSPICE MULTISIM The switch in the circuit shown in Fig. P8.45 has been in position a for a long time. At $t = 0$, the switch is moved instantaneously to position b. Find $i(t)$ for $t \geq 0$.

Figure P8.45



$\underline{t < 0}$

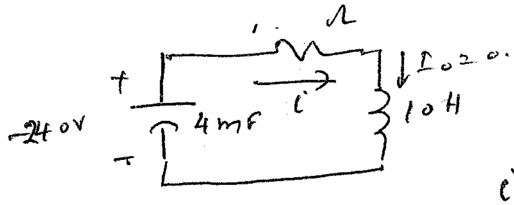
Capacitor is open ckt → fully charged.



$\boxed{V_0 = -240\ V}$

$\boxed{i_L(0^-) = I_0 = 0\ A}$

$\underline{t \geq 0}$



$$\boxed{\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0.}$$

$i(0^+) = i(0^-) = I_0 = 0\ A$ (Current cannot change suddenly in inductor)

$$\frac{di(0^+)}{dt} = \frac{V_0}{L} = \frac{-240}{10} = -24.$$

$$\alpha = \frac{R}{2L} = \frac{80}{2 \times 10} = \boxed{4}\ \text{rad/sec}$$

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{10 \times 4 \times 10^{-3}} = 25 \Rightarrow \boxed{\omega_0 = 5\ \text{rad/sec}}$$

(underdamped.)

$$\alpha^2 - \omega_0^2 = -9 \Rightarrow \omega_d = 3\ \text{rad/sec}.$$

$$s_1 = -4 + j3, \quad s_2 = -4 - j3.$$

$$i(t) = B_1 e^{-4t} \cdot \cos(3t) + B_2 e^{-4t} \sin(3t)$$

$$i(0) = 0 = B_1$$

$$\left.\frac{di(t)}{dt}\right|_{t=0} = -4B_1 + 3B_2 = -24$$

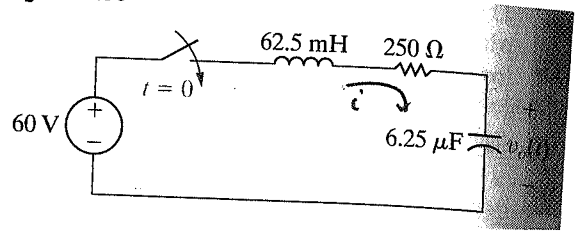
$$\Rightarrow \boxed{B_2 = 8\ A.}$$

$$\Rightarrow \boxed{i(t) = 8 e^{-4t} \sin(3t), \quad t \geq 0}$$

8.50 The initial energy stored in the circuit in Fig. P8.50 is zero. Find $v_o(t)$ for $t \geq 0$.

PSPICE
MULTISIM

Figure P8.50

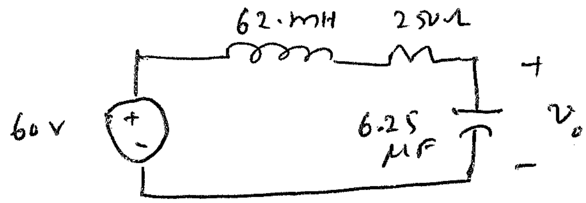


$t<0$, No energy in the ckt.

$\Rightarrow v_o(0^-) = 0. \qquad i(0^-) = 0.$

$t \geq 0$ Initial Condition:

$$\boxed{v_o(0^+) = 0, \quad i(0^+) = 0 \Rightarrow \frac{dv(0^+)}{dt} = \frac{I_o}{C} = 0}$$



Final value

$$\boxed{V_f = 60V}$$

$$\boxed{\frac{d^2v_o}{dt^2} + \frac{R}{L}\frac{dv_o}{dt} + \frac{v_o}{LC} = \frac{60}{LC}}, \quad t \geq 0$$

$$\alpha = \frac{R}{2L} = \frac{250}{2 \times 62.5 \times 10^{-3}} = \boxed{2{,}000 \text{ rad/sec}}$$

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{62.5 \times 10^{-3} \times 6.25 \times 10^{-3}} = 256 \times 10^4 \Rightarrow \boxed{\omega_o = 1{,}600 \text{ rad/sec}}$$

$$\alpha^2 - \omega_o^2 > 0 \Rightarrow \boxed{\text{over damped}}$$

$$s_{1,2} = -\alpha \pm (\alpha^2 - \omega_o^2)^{1/2} \Rightarrow \begin{array}{l} s_1 = -800 \text{ rad/sec} \\ s_2 = -3200 \text{ rad/sec} \end{array}$$

$v_o(t) = V_f$ + (Natural Response Type eqn.)

$$\Rightarrow v_o(t) = 60 + A_1 e^{-800t} + A_2 e^{-3200t}, \quad t \geq 0.$$

$$v_o(0^+) = 0 = 60 + A_1 + A_2 \Rightarrow A_1 + A_2 = -60$$

$$\frac{dv_o(0^+)}{dt} = 0 = -800A_1 - 3200A_2 \Rightarrow A_1 + 4A_2 = 0.$$

$$\Rightarrow \boxed{A_1 = -80V, \quad A_2 = 20V}$$

$$\Rightarrow \boxed{v_o(t) = 60 - 80e^{-800t} + 20e^{-3200t}}$$