Assignment 4

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Download the python code, latex file and the pdf document from

https://github.com/Rishab9991/EE5609/tree/master/Assignments/Assignment4

Question:
(Loney Pg 346, Question 17) Trace the central conic,

$$2x^2 - 2xy + y^2 + 2x - 2y = 0 \tag{1}$$

Solution: The general equation of a second degree (In algebraic form) can be expressed as,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2}$$

The general equation of a second degree (In vector form) can be expressed as,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{3}$$

Comparing (1) with (2) , we get,

$$a = 2,\ b = -1,\ c = 1,\ d = 1,\ e = -1\ and\ f = 0 \tag{4}$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \mathbf{V}^{\mathbf{T}} \tag{5}$$

$$\implies \mathbf{V} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \tag{6}$$

and

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{7}$$

Finding the determinant of V we obtain,

$$|\mathbf{V}| = 1 > 0 \tag{8}$$

which means the given central conic is an ellipse which can be proven more effectively using,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathbf{T}} \tag{9}$$

where $\mathbf{P}$ is a matrix of Eigen vectors and $\mathbf{D}$ is a diagonal matrix of Eigen values which will be computed subsequently.
Computing Eigen values for $\mathbf{V}$ using the characteristic equation of the matrix, we get the following quadratic equation in terms of λ

$$\lambda^2 - 3\lambda + 1 = 0 \tag{10}$$

$$\implies \lambda_1 = \frac{3 + \sqrt{5}}{2}\ and\ \lambda_2 = \frac{3 - \sqrt{5}}{2} \tag{11}$$

Eigen vectors can be computed using the following equation,

$$(\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{12}$$

Solving this for λ_1 and λ_2 respectively and normalizing them we obtain,

$$\mathbf{p_1} = \sqrt{\frac{2}{5 - \sqrt{5}}}\begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \tag{13}$$

$$\mathbf{p_2} = \sqrt{\frac{2}{5 + \sqrt{5}}}\begin{pmatrix} 1 \\ \frac{\sqrt{5}+1}{2} \end{pmatrix} \tag{14}$$

Simplifying,

$$\implies \mathbf{P} = \begin{pmatrix} \sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}} \\ \frac{1-\sqrt{5}}{\sqrt{5\sqrt{2}-\sqrt{10}}} & \frac{1+\sqrt{5}}{\sqrt{5\sqrt{2}+\sqrt{10}}} \end{pmatrix} \tag{15}$$

$$\mathbf{D} = \begin{pmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix} \tag{16}$$

Using (9) can verify that it holds which means that the given central conic is an ellipse. The center of the ellipse can be computed using,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{17}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{18}$$

The parameters of the ellipse are computed as follows,

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{3 - \sqrt{5}}{2}} \tag{19}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{3 + \sqrt{5}}{2}} \tag{20}$$

The angle of Rotation can be obtained by equating **P** with the Rotation matrix which is,

$$\mathbf{P} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & cos\theta \end{pmatrix} \tag{21}$$

Comparing (15) and (21) we get,

$$\theta = \frac{\pi}{5.66} \tag{22}$$

Using the Affine transformation we find out the actual ellipse,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} + \mathbf{c} \tag{23}$$

which means the actual ellipse is obtained by translating and rotating the standard ellipse w.r.t center, **c** from (18) and angle of rotation, θ from (22) respectively.
Using the above data along with **o** (Origin), the center of the standard ellipse, the actual ellipse is plotted as follows.

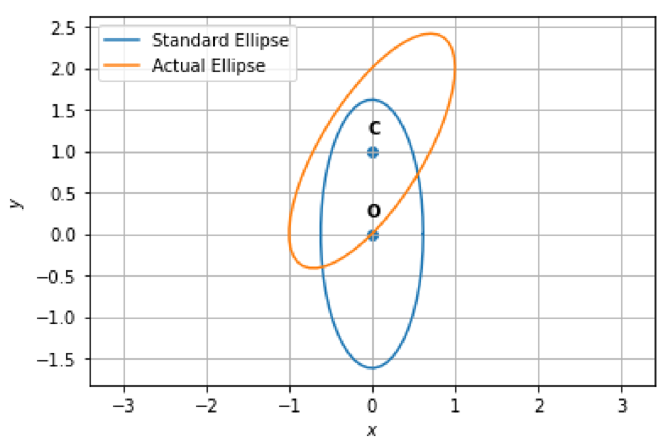

Fig. 1: Standard and Actual Ellipses