Assignment 3.2

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Download the python code, latex file and the pdf doc from

https://github.com/Rishab9991/EE5609/tree/master/Assignments/Assignment3.2

Question:
Prove that the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1 \tag{1}$$

touches the circle

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} - \begin{pmatrix}8 & 6\end{pmatrix}\mathbf{x} + 7 = 0 \tag{2}$$

and find the equations of the parallel and perpendicular tangents.

Solution: The general equation of a second degree can be expressed as,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{3}$$

Comparing (2) and (3) we get,

$$\mathbf{u} = \begin{pmatrix}-4\\-3\end{pmatrix} \tag{4}$$

$$f = 7 \tag{5}$$

If $\mathbf{n}$ is the normal vector, $\mathbf{P}$ is a point on that line then equation of the line can be written as,

$$\mathbf{n}^{\mathbf{T}}(\mathbf{x} - \mathbf{P}) = 0 \tag{6}$$

$$\implies \mathbf{n}^{\mathbf{T}}\mathbf{x} = c \tag{7}$$

where c = $\mathbf{n}^{\mathbf{T}}\mathbf{P}$

Comparing (1) and (7) we get,

$$\mathbf{n} = \begin{pmatrix}1\\1\end{pmatrix} and \ c = 1 \tag{8}$$

The point of contact q, of a line with a normal vector $\mathbf{n}$ to the conic in (3) is given by,

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{9}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{n}}} \tag{10}$$

For a circle,

$$\mathbf{V} = \mathbf{I} \tag{11}$$

where I is the Identity matrix.

Solving for κ using (10) we get,

$$\kappa = \pm 3 \tag{12}$$

$$i.e. \ \mathbf{q_1} = \begin{pmatrix}1\\0\end{pmatrix} \ for \ \kappa = -3 \tag{13}$$

and

$$\mathbf{q_2} = \begin{pmatrix}7\\6\end{pmatrix} \ for \ \kappa = 3 \tag{14}$$

To prove that the line touches the circle at $\mathbf{q}$ need to check that

$$\mathbf{m}^{\mathbf{T}}(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{15}$$

We know that,

$$\mathbf{m}^{\mathbf{T}}\mathbf{n} = 0 \tag{16}$$

$$\implies m = \begin{pmatrix}-1\\1\end{pmatrix} \tag{17}$$

Using (13), (14) and (17), the expression in (15) holds true for both $\mathbf{q_1}$ and $\mathbf{q_2}$ which means that both those points lie on the circle i.e. there will be a tangent passing through each of them which can be found out using (7)

$$i.e. \ \mathbf{n}^{\mathbf{T}}\mathbf{q_1} = c_1 \tag{18}$$

$$\mathbf{n}^{\mathbf{T}}\mathbf{q_2} = c_2 \tag{19}$$

where,

$$c_1 = \mathbf{n}^{\mathbf{T}}\mathbf{q_1} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1 \tag{20}$$

which was already obtained in (8) and

$$c_2 = \mathbf{n}^{\mathbf{T}}\mathbf{q_2} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}7\\6\end{pmatrix} = 13 \tag{21}$$

Using (18) the given line in the question is obtained which is (1)

Therefore, the tangent parallel to (1) is,

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 13 \tag{22}$$

And the line(s) perpendicular to (1) can be found out using (8) and here the normal vector for this line will be **m** which was calculated using (17) and its equation(s) will be,

$$\mathbf{m}^{\mathbf{T}}\mathbf{x} = c_3 \tag{23}$$

$$\mathbf{m}^{\mathbf{T}}\mathbf{x} = c_4 \tag{24}$$

where,

$$c_3 = \mathbf{m}^{\mathbf{T}}\mathbf{q_1} = \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = -1 \tag{25}$$

$$c_4 = \mathbf{m}^{\mathbf{T}}\mathbf{q_2} = \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}7\\6\end{pmatrix} = -1 \tag{26}$$

Therefore, the line perpendicular to (1) and also to (22) is,

$$\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = -1 \tag{27}$$

In Fig. 1. **C** is the center of the circle. $\mathbf{q_1}$ and $\mathbf{q_2}$ are points of contact with the circle.

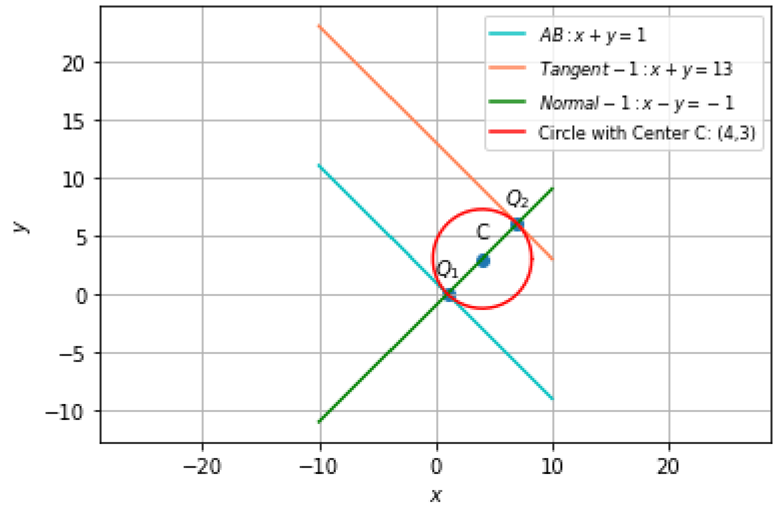

Fig. 1: Tangents and Normal on the Circle

Line 1 is (1), Line 2 is (22) and Line 3 is (27).