Assignment 3.2

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Download the python code, latex file and the pdf doc from

https://github.com/Rishab9991/EE5609/tree/master/Assignments/Assignment3.2

Question:
Prove that the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1 \tag{1}$$

touches the circle

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} - \begin{pmatrix}8 & 6\end{pmatrix}\mathbf{x} + 7 = 0 \tag{2}$$

and find the equations of the parallel and perpendicular tangents.

Solution: The vector equation of a line can be expressed as,

$$\mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{3}$$

The general equation of a second degree can be expressed as.

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{4}$$

Comparing (2) and (4) we get,

$$\mathbf{u} = \begin{pmatrix}-4\\-3\end{pmatrix} \tag{5}$$

$$f = 7 \tag{6}$$

If $\mathbf{n}$ is the normal vector of a line, equation of the given line can be written as,

$$\mathbf{n}^{\mathbf{T}}\mathbf{x} = c \tag{7}$$

Comparing (1) and (7) we get,

$$\mathbf{n} = \begin{pmatrix}1\\1\end{pmatrix} \tag{8}$$

The point of contact q, of a line with a normal vector $\mathbf{n}$ to the conic in (4) is given by,

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) \tag{9}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{n}}} \tag{10}$$

For a circle,

$$\mathbf{V} = \mathbf{I} \tag{11}$$

where I is the Identity matrix.

Solving for κ using (10) we get,

$$\kappa = \pm 3 \tag{12}$$

$$i.e.\ \mathbf{q_1} = \begin{pmatrix}1\\0\end{pmatrix}\ for\ \kappa = -3 \tag{13}$$

and

$$\mathbf{q_2} = \begin{pmatrix}7\\6\end{pmatrix}\ for\ \kappa = 3 \tag{14}$$

To prove that the line touches the circle at $\mathbf{q}$ need to check that

$$\mathbf{m}^{\mathbf{T}}\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) = 0 \tag{15}$$

We know that,

$$\mathbf{m}^{\mathbf{T}}\mathbf{n} = 0 \tag{16}$$

$$\implies m = \begin{pmatrix}-1\\1\end{pmatrix} \tag{17}$$

Using (13), (14) and (17), the expression in (15) holds true for both $\mathbf{q_1}$ and $\mathbf{q_2}$ which means that both those points lie on the circle i.e. there will be a tangent passing through each of them which can be found out using (7)

$$i.e.\ \mathbf{n}^{\mathbf{T}}\mathbf{q_1} = c_1 \tag{18}$$

$$\mathbf{n}^{\mathrm{T}}\mathbf{q_2} = c_2 \tag{19}$$

Using (18) and (19) we get,

$$c_1 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{20}$$

$$c_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \end{pmatrix} = 13 \tag{21}$$

Using (18) the given line in the question is obtained. (1)

Therefore, the line parallel to the given line is,

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 13 \tag{22}$$

And the line perpendicular to the given line is the line passing through the normal vector $\mathbf{n}$ found out in (8) and its equation will be of the form,

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = k \tag{23}$$

where k is a constant.

Consider Fig. 1

C is the center of the circle. **q1** and **q2** are points of contact with the circle.

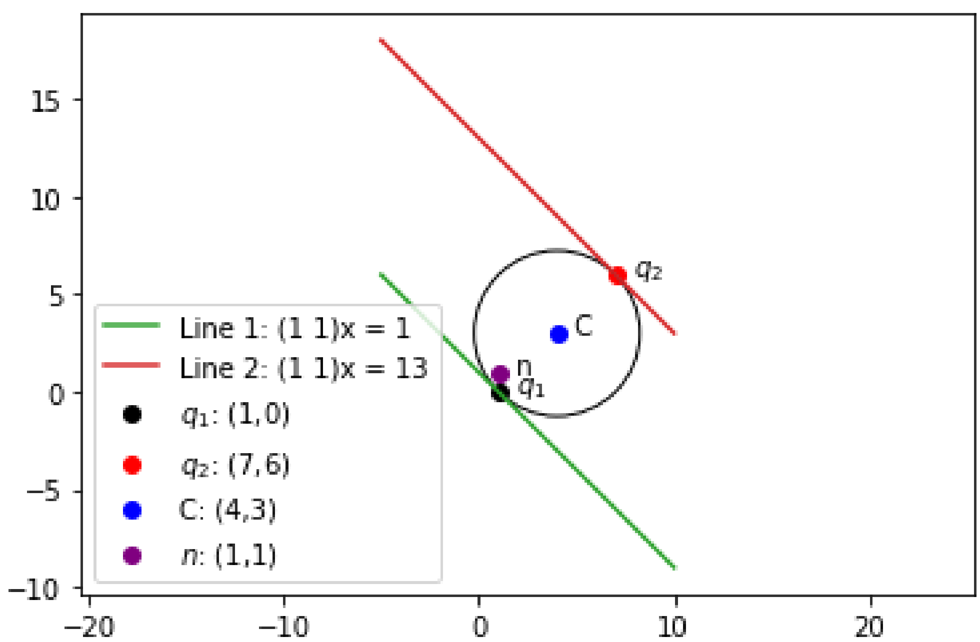

Fig. 1: Tangents parallel and perpendicular to the given line on the Circle