# Assignment 7

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Download the latex file and the pdf doc from

https://github.com/Rishab9991/EE5609/tree/master/Assignments/Assignment7

## Question

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Let m, n and r be natural numbers. Let $\mathbf{A}$ be an m $\times$ n matrix with real entries such that $(\mathbf{A}\mathbf{A}^{\mathbf{t}})^r = \mathbf{I}$, where $\mathbf{I}$ is the m $\times$ m identity matrix and $\mathbf{A}^{\mathbf{t}}$ is the transpose of the matrix $\mathbf{A}$. We can conclude that
**Options**

1) m = n
2) $\mathbf{A}\mathbf{A}^{\mathbf{t}}$ is invertible
3) $\mathbf{A}^{\mathbf{t}}\mathbf{A}$ is invertible
4) if m = n, then $\mathbf{A}$ is invertible

## Solution

| **Option 1** | To conclude that m = n |
|---|---|
| Assumptions | **For the example:** Without loss of generality, Let m = 2, n = 3 and $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ <br> $\implies \mathbf{A^t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ |
| Proof | We know that $(\mathbf{AA^t})^r = \mathbf{I}$ which is a square matrix of order m × m <br> For any natural value of r, a square matrix ($\mathbf{I}$) of order m × m is obtained <br> Hence, we cannot conclude that m = n because we get $\mathbf{I}$ of order m × m even if m ≠ n. To illustrate this, Consider the following example <br> $\mathbf{AA^t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$ ($\mathbf{A}$ and $\mathbf{A^t}$ from Assumptions) <br> $(\mathbf{AA^t})^r = \mathbf{I}$ <br> Here m ≠ n. Therefore, **Option 1** is incorrect |

TABLE I: Option 1

| **Option 2** | To conclude that $\mathbf{AA^t}$ is invertible |
|---|---|
| Assumptions | $\mathbf{AA^t}$ is not invertible |
| Proof | $\implies \left\|\mathbf{AA^t}\right\| = 0 \implies \left\|(\mathbf{AA^t})^r\right\| = 0$ <br> $\implies (\mathbf{AA^t})^r \neq \mathbf{I} \left(\left\|\mathbf{I}\right\| = 1\right)$ <br> Since, this is a contradiction to the assumption made we can conclude that $\mathbf{AA^t}$ is invertible. Therefore, **Option 2** is correct |

TABLE II: Option 2

| **Option 3** | To conclude that $\mathbf{A^tA}$ is invertible |
|---|---|
| Assumptions | Without loss of generality, Let m = 2, n = 3 and $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ <br> $\implies \mathbf{A^t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ |
| Proof | $\implies \mathbf{A^tA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies \left\|\mathbf{A^tA}\right\| = 0$ <br> This means that $\mathbf{A^tA}$ is not invertible. Therefore, **Option 3** is incorrect |

TABLE III: Option 3

| **Option 4** | To conclude that if m = n then $\mathbf{A}$ is invertible |
|---|---|
| Assumptions | Let m = n |
| Proof | Since $(\mathbf{AA^t})^r = \mathbf{I} \implies \left\|(\mathbf{AA^t})^r\right\| = \left\|\mathbf{I}\right\| = 1$ <br> $\implies \left(\left\|\mathbf{A}\right\|\left\|\mathbf{A^t}\right\|\right)^r = 1$ ($\mathbf{A}$ is a square matrix) <br> $\implies \left(\left\|\mathbf{A}\right\|\right)^{2r} = 1$ <br> Therefore, **Option 4** is correct |

TABLE IV: Option 4