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Assignment 3.2

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Download the python code, latex file and the pdf doc from

https://github.com/Rishab9991/EE5609/tree/master/ Assignments/Assignment3.2

Question:

Prove that the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 1 \tag{1}$$

touches the circle

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} - \begin{pmatrix} 8 & 6 \end{pmatrix} \mathbf{x} + 7 = 0 \tag{2}$$

and find the equations of the parallel and perpendicular tangents.

Solution: The general equation of a second degree can be expressed as,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{3}$$

Comparing (2) and (3) we get,

$$\mathbf{u} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{4}$$

$$f = 7 \tag{5}$$

If n is the normal vector, P is a point on that line then equation of the line can be written as,

$$\mathbf{n}^{\mathbf{T}}\left(\mathbf{x} - \mathbf{P}\right) = 0 \tag{6}$$

$$\implies \mathbf{n}^{\mathsf{T}}\mathbf{x} = c \tag{7}$$

where $c = \mathbf{n}^{T} \mathbf{P}$

Comparing (1) and (7) we get,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} and \ c = 1 \tag{8}$$

The point of contact q, of a line with a normal vector **n** to the conic in (3) is given by,

$$\mathbf{q} = \mathbf{V}^{-1} \left(\kappa \mathbf{n} - \mathbf{u} \right) \tag{9}$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{n}}}$$
 (10)

For a circle,

$$\mathbf{V} = \mathbf{I} \tag{11}$$

where I is the Identity matrix.

Solving for κ using (10) we get,

$$\kappa = \pm 3 \tag{12}$$

i.e.
$$\mathbf{q_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
 for $\kappa = -3$ (13)

and

$$\mathbf{q_2} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \text{ for } \kappa = 3 \tag{14}$$

To prove that the line touches the circle at \mathbf{q} need to check that

$$\mathbf{m}^{\mathbf{T}} \left(\mathbf{V} \mathbf{q} + \mathbf{u} \right) = 0 \tag{15}$$

We know that,

$$\mathbf{m}^{\mathbf{T}}\mathbf{n} = 0 \tag{16}$$

$$\implies m = \begin{pmatrix} -1\\1 \end{pmatrix} \tag{17}$$

Using (13), (14) and (17), the expression in (15) holds true for both $\mathbf{q_1}$ and $\mathbf{q_2}$ which means that both those points lie on the circle i.e. there will be a tangent passing through each of them which can be found out using (7)

$$i.e. \mathbf{n}^{\mathrm{T}} \mathbf{q}_1 = c_1 \tag{18}$$

$$\mathbf{n}^{\mathbf{T}}\mathbf{q}_{2} = c_{2} \tag{19}$$

where,

$$c_1 = \mathbf{n}^{\mathsf{T}} \mathbf{q_1} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{20}$$

which was already obtained in (8) and

$$c_2 = \mathbf{n}^{\mathbf{T}} \mathbf{q_2} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \end{pmatrix} = 13 \tag{21}$$

Using (18) the given line in the question is obtained which is (1)

Therefore, the tangent parallel to (1) is,

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 13 \tag{22}$$

And the line(s) perpendicular to (1) can be found out using (8) and here the normal vector for this line will be **m** which was calculated using (17) and its equation(s) will be,

$$\mathbf{m}^{\mathbf{T}}\mathbf{x} = c_3 \tag{23}$$

$$\mathbf{m}^{\mathsf{T}}\mathbf{x} = c_4 \tag{24}$$

where,

$$c_3 = \mathbf{m}^{\mathbf{T}} \mathbf{q_1} = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 \tag{25}$$

$$c_4 = \mathbf{m}^{\mathsf{T}} \mathbf{q}_2 = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \end{pmatrix} = -1 \tag{26}$$

Therefore, the line perpendicular to (1) and also to (22) is,

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = -1 \tag{27}$$

In Fig. 1. C is the center of the circle. $\mathbf{q_1}$ and $\mathbf{q_2}$ are points of contact with the circle.

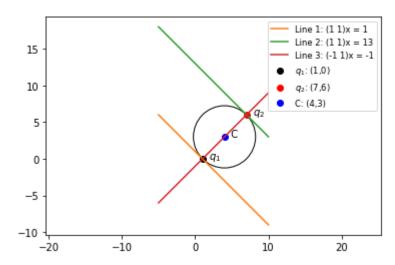


Fig. 1: Tangents and Normal on the Circle

Line 1 is (1), Line 2 is (22) and Line 3 is (27).