Assignment 7

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Download the python code, latex file and the pdf doc from

https://github.com/Rishab9991/EE5609/tree/master/Assignments/Assignment7

Question

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Let m, n and r be natural numbers. Let **A** be an m x n matrix with real entries such that $(AA^t)^r = \mathbf{I}$, where **I** is the m x m identity matrix and $\mathbf{A^t}$ is the transpose of the matrix **A**. We can conclude that
Options:

1) m = n
2) $\mathbf{AA^t}$ is invertible
3) $\mathbf{A^tA}$ is invertible
4) if m = n, then **A** is invertible

Solution

Given **A** be m x n matrix with real entries such that,

$$(AA^t)^r = \mathbf{I} \tag{1}$$

Option 1
Let m $\neq$ n.
Without loss of generality, Consider m = 2, n = 3 and,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2}$$

$$\implies \mathbf{A^t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3}$$

Finding $\mathbf{AA^t}$ we get,

$$\mathbf{AA^t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{4}$$

$$\implies (AA^t)^r = \mathbf{I} \tag{5}$$

This was not possible if m = n. Therefore, Option 1 is incorrect.
Option 2
Assume $\mathbf{AA^t}$ is not invertible.

$$\implies \left|\mathbf{AA^t}\right| = 0 \tag{6}$$

$$\implies \left|(AA^t)^r\right| = 0 \tag{7}$$

But (7) contradicts (1) as $(AA^t)^r$ will not be a identity matrix if it has a determinant of 0. Hence, $\mathbf{AA^t}$ is invertible. Therefore, Option **2** is correct.
Option 3
Similar to option 1, the same matrix **A** (2) is considered. Finding $\mathbf{A^tA}$ we get,

$$\mathbf{A^tA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

$$\implies \left|\mathbf{A^tA}\right| = 0 \tag{9}$$

$\implies$ $\mathbf{A^tA}$ is not invertible. Therefore, Option 3 is incorrect.
Option 4
Let m = n. From (1) we can say that,

$$\left|(AA^t)^r\right| = \left|\mathbf{I}\right| = 1 \tag{10}$$

$$\implies \left(\left|\mathbf{A}\right|\left|A^t\right|\right)^r = 1 \tag{11}$$

$$\implies \left|\mathbf{A}\right|^{2r} = 1 \tag{12}$$

From (12), we can say that,

$$\left|\mathbf{A}\right| \neq 0 \tag{13}$$

From (13), we can say that **A** is invertible when m = n. Therefore, Option **4** is correct.
We can conclude that $\mathbf{AA^t}$ is invertible and if m = n **A** is invertible. (Options **2** and **4** are valid)