Assignment 4

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Download the python code, latex file and the pdf document from

https://github.com/Rishab9991/EE5609/tree/master/Assignments/Assignment4

Question:
Trace the central conic,

$$2x^2 - 2xy + y^2 + 2x - 2y = 0 \tag{1}$$

Solution: The general equation of a second degree (In vector form) can be expressed as,

$$\mathbf{x}^\mathbf{T}\mathbf{V}\mathbf{x} + 2\mathbf{u}^\mathbf{T}\mathbf{x} + f = 0 \tag{2}$$

The general equation of a second degree (In algebraic form) can be expressed as,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{3}$$

Comparing (1) with (3) , we get,

$$a = 2,\ b = -1,\ c = 1,\ d = 1,\ e = -1\ and\ f = 0 \tag{4}$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \mathbf{V}^\mathbf{T} \tag{5}$$

$$\implies \mathbf{V} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \tag{6}$$

and

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{7}$$

Finding the determinant of V we obtain,

$$|\mathbf{V}| = 1 > 0 \tag{8}$$

which means the given central conic is an ellipse which can be proven more effectively using,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\mathbf{T} \tag{9}$$

where $\mathbf{P}$ is a matrix of Eigen vectors and $\mathbf{D}$ is a diagonal matrix of Eigen values which will be computed subsequently.

Computing Eigen values for $\mathbf{V}$ using the characteristic equation of the matrix, we get the following quadratic equation in terms of λ

$$\lambda^2 - 3\lambda + 1 = 0 \tag{10}$$

$$\implies \lambda_1 = \frac{3 + \sqrt{5}}{2}\ and\ \lambda_2 = \frac{3 - \sqrt{5}}{2} \tag{11}$$

Eigen vectors can be computed using the following equation,

$$(\lambda\mathbf{I} - \mathbf{V}) = 0 \tag{12}$$

Solving this for λ_1 and λ_2 respectively and normalizing them we obtain,

$$\mathbf{p_1} = \sqrt{\frac{10 + 2\sqrt{5}}{\sqrt{10}}}\begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \tag{13}$$

$$\mathbf{p_2} = \sqrt{\frac{10 + 2\sqrt{5}}{\sqrt{10}}}\begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \tag{14}$$

$$\implies \mathbf{P} = \sqrt{\frac{10 + 2\sqrt{5}}{\sqrt{10}}}\begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \tag{15}$$

$$\mathbf{D} = \begin{pmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix} \tag{16}$$

Using (9) can verify that it holds which means that the given central conic is an ellipse. The center of the ellipse can be computed using,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{17}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{18}$$

The parameters of the ellipse are computed as follows,

$$\sqrt{\frac{\mathbf{u}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{3 - \sqrt{5}}{2}} \tag{19}$$

$$\sqrt{\frac{\mathbf{u}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{3 + \sqrt{5}}{2}} \tag{20}$$

The equation of the ellipse in vector form can be written as,

$$\mathbf{y}^{\mathbf{T}}\mathbf{D}\mathbf{y} = \mathbf{u}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{u} - f \tag{21}$$

Using the center and the parameters the ellipse can be plotted as follows.