Assignment 4

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Download the python code, latex file and the pdf document from

https://github.com/Rishab9991/EE5609/tree/master/Assignments/Assignment4

Question:
Trace the central conic,

$$2x^2 - 2xy + y^2 + 2x - 2y = 0 \tag{1}$$

Solution: The general equation of a second degree (In algebraic form) can be expressed as,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2}$$

The general equation of a second degree (In vector form) can be expressed as,

$$\mathbf{x^T V x} + 2\mathbf{u^T x} + f = 0 \tag{3}$$

Comparing (1) with (2) , we get,

$$a = 2,\ b = -1,\ c = 1,\ d = 1,\ e = -1 \ and\ f = 0 \tag{4}$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \mathbf{V^T} \tag{5}$$

$$\implies \mathbf{V} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \tag{6}$$

and

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{7}$$

Finding the determinant of V we obtain,

$$|\mathbf{V}| = 1 > 0 \tag{8}$$

which means the given central conic is an ellipse which can be proven more effectively using,

$$\mathbf{V} = \mathbf{PDP^T} \tag{9}$$

where **P** is a matrix of Eigen vectors and **D** is a diagonal matrix of Eigen values which will be computed subsequently.

Computing Eigen values for **V** using the characteristic equation of the matrix, we get the following quadratic equation in terms of λ

$$\lambda^2 - 3\lambda + 1 = 0 \tag{10}$$

$$\implies \lambda_1 = \frac{3+\sqrt{5}}{2} \ and\ \lambda_2 = \frac{3-\sqrt{5}}{2} \tag{11}$$

Eigen vectors can be computed using the following equation,

$$(\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{12}$$

Solving this for λ_1 and λ_2 respectively and normalizing them we obtain,

$$\mathbf{p_1} = \sqrt{\frac{2}{5-\sqrt{5}}} \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \tag{13}$$

$$\mathbf{p_2} = \sqrt{\frac{2}{5+\sqrt{5}}} \begin{pmatrix} 1 \\ \frac{\sqrt{5}+1}{2} \end{pmatrix} \tag{14}$$

Simplifying,

$$\implies \mathbf{P} = \begin{pmatrix} 0.850 & 0.525 \\ -0.525 & 0.850 \end{pmatrix} \tag{15}$$

$$\mathbf{D} = \begin{pmatrix} 2.618 & 0 \\ 0 & 0.381 \end{pmatrix} \tag{16}$$

Using (9) can verify that it holds which means that the given central conic is an ellipse. The center of the ellipse can be computed using,

$$\mathbf{c} = -\mathbf{V^{-1}u} \tag{17}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{18}$$

The parameters of the ellipse are computed as follows,

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{3 - \sqrt{5}}{2}} = 0.437 \tag{19}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{3 + \sqrt{5}}{2}} = 1.144 \tag{20}$$

The angle of Rotation can be obtained by equating $\mathbf{P}$ with the Rotation matrix as follows.

$$\begin{pmatrix} 0.850 & -0.525 \\ 0.525 & 0.850 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{21}$$

$$\implies \theta = \frac{\pi}{5.66} \tag{22}$$

Using the center and θ the actual ellipse can be plotted with the help of standard ellipse as follows. $\mathbf{c}$ is the center of the actual ellipse, $\mathbf{o}$ is the center of the standard ellipse.

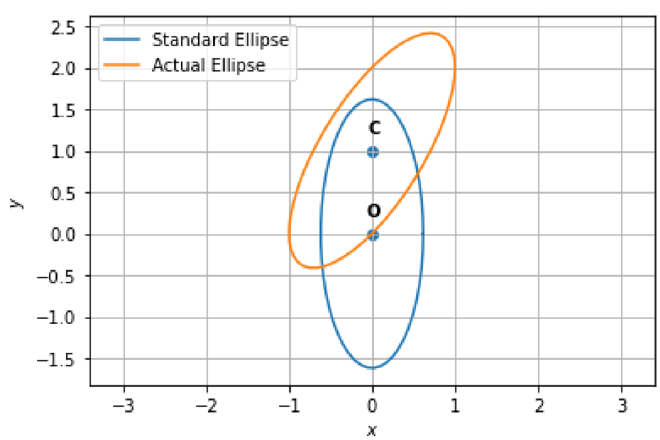

Fig. 1: Standard and Actual Ellipses