Assignment 8

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Download the latex file and the pdf doc from

https://github.com/Rishab9991/EE5609/tree/master/Assignments/Assignment8

Question

(Q.No.27, UGC Dec 2015)
Let $\mathbf{A} \neq \mathbf{I_n}$ be an n × n matrix such that $\mathbf{A^2} = \mathbf{A}$, where $\mathbf{I_n}$ is the Identity matrix of order n. Which of the following is false?

Options

1) $(\mathbf{I_n} - \mathbf{A})^2 = \mathbf{I_n} - \mathbf{A}$
2) Trace($\mathbf{A}$) = Rank($\mathbf{A}$)
3) Rank($\mathbf{A}$) + Rank($\mathbf{I_n} - \mathbf{A}$) = n
4) The eigenvalues of $\mathbf{A}$ are each equal to 1

Solution

Option 1	$(\mathbf{I_n} - \mathbf{A})^2 = \mathbf{I_n} - \mathbf{A}$
Assumptions	Made from the Question
Proof	We know that $(\mathbf{A} - \mathbf{B})^2 = \mathbf{A^2} + \mathbf{B^2}$ - $\mathbf{AB}$ - $\mathbf{BA}$ ($\mathbf{A}$ and $\mathbf{B}$ are matrices of same order) $\implies (\mathbf{I_n} - \mathbf{A})^2 = \mathbf{I_n^2} + \mathbf{A^2} - \mathbf{I_n A} - \mathbf{A I_n}$ $\implies (\mathbf{I_n} - \mathbf{A})^2 = \mathbf{I_n} + \mathbf{A} - \mathbf{2A}$ $\implies (\mathbf{I_n} - \mathbf{A})^2 = \mathbf{I_n} - \mathbf{A}$. Therefore, **Option 1** is true.

TABLE I: Option 1

Option 2	Trace($\mathbf{A}$) = Rank($\mathbf{A}$)
Assumptions	Made from the Question
Proof	Using Rank Factorisation, $\mathbf{A} = \mathbf{BC}$ where $\mathbf{B}$ is a full column rank matrix of order $n \times r$ and $\mathbf{C}$ is a full row rank matrix of order $r \times n$ then $\mathbf{B}$ has left inverse and $\mathbf{C}$ has right inverse. Since $\mathbf{A^2} = \mathbf{A}$, $(\mathbf{BC})(\mathbf{BC}) = \mathbf{BC}$ $\implies \mathbf{CBC} = \mathbf{C} \implies \mathbf{CB} = \mathbf{I}$ Here, $\mathbf{I}$ is a $r \times r$ matrix $\implies$ Trace($\mathbf{A}$) = Trace($\mathbf{BC}$) = Trace($\mathbf{CB}$) = Trace($\mathbf{I}$) = r = Rank($\mathbf{A}$). Therefore, **Option 2** is true.

TABLE II: Option 2

Option 3	Rank($\mathbf{A}$) + Rank($\mathbf{I_n} - \mathbf{A}$) = n
Assumptions	Made from the Question
Proof	Using Rank-Nullity Theorem, Rank($\mathbf{A}$) + Rank($\mathbf{I_n} - \mathbf{A}$) = n Therefore, **Option 3** is true.

TABLE III: Option 3

Option 4	The eigenvalues of $\mathbf{A}$ are each equal to 1
Assumptions	Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Proof by Counter Example	$\implies \left\lvert \mathbf{A} \right\rvert = 0 \implies \lambda_1 \lambda_2 = 0$ which means that either λ_1 or λ_2 are 0 $\implies$ Each eigenvalue need not be 1 Therefore, **Option 4** is the only false option.

TABLE IV: Option 4