

Assignment 4

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Download the python code, latex file and the pdf document from

<https://github.com/Rishab9991/EE5609/tree/master/Assignments/Assignment4>

Question:

Trace the central conic,

$$2x^2 - 2xy + y^2 + 2x - 2y = 0 \quad (1)$$

Solution: The general equation of a second degree (In algebraic form) can be expressed as,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2)$$

The general equation of a second degree (In vector form) can be expressed as,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3)$$

Comparing (1) with (2), we get,

$$a = 2, b = -1, c = 1, d = 1, e = -1 \text{ and } f = 0 \quad (4)$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \mathbf{V}^T \quad (5)$$

$$\Rightarrow \mathbf{V} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad (6)$$

and

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (7)$$

Finding the determinant of V we obtain,

$$|\mathbf{V}| = 1 > 0 \quad (8)$$

which means the given central conic is an ellipse which can be proven more effectively using,

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (9)$$

where \mathbf{P} is a matrix of Eigen vectors and \mathbf{D} is a diagonal matrix of Eigen values which will be computed subsequently.

Computing Eigen values for \mathbf{V} using the characteristic equation of the matrix, we get the following quadratic equation in terms of λ

$$\lambda^2 - 3\lambda + 1 = 0 \quad (10)$$

$$\Rightarrow \lambda_1 = \frac{3 + \sqrt{5}}{2} \text{ and } \lambda_2 = \frac{3 - \sqrt{5}}{2} \quad (11)$$

Eigen vectors can be computed using the following equation,

$$(\lambda \mathbf{I} - \mathbf{V}) = 0 \quad (12)$$

Solving this for λ_1 and λ_2 respectively and normalizing them we obtain,

$$\mathbf{p}_1 = \sqrt{\frac{10 + 2\sqrt{5}}{\sqrt{10}}} \begin{pmatrix} 1 \\ \frac{1 - \sqrt{5}}{2} \end{pmatrix} \quad (13)$$

$$\mathbf{p}_2 = \sqrt{\frac{10 + 2\sqrt{5}}{\sqrt{10}}} \begin{pmatrix} 1 \\ \frac{1 + \sqrt{5}}{2} \end{pmatrix} \quad (14)$$

$$\Rightarrow \mathbf{P} = \sqrt{\frac{10 + 2\sqrt{5}}{\sqrt{10}}} \begin{pmatrix} 1 & \frac{1 - \sqrt{5}}{2} \\ 1 & \frac{1 + \sqrt{5}}{2} \end{pmatrix} \quad (15)$$

$$\mathbf{D} = \begin{pmatrix} \frac{3 + \sqrt{5}}{2} & 0 \\ 0 & \frac{3 - \sqrt{5}}{2} \end{pmatrix} \quad (16)$$

Using (9) can verify that it holds which means that the given central conic is an ellipse. The center of the ellipse can be computed using,

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (17)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (18)$$

The parameters of the ellipse are computed as

follows,

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{3 - \sqrt{5}}{2}} \quad (19)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{3 + \sqrt{5}}{2}} \quad (20)$$

Using the center and the parameters the ellipse can be plotted as follows. In Fig. 1. \mathbf{c} is the center of the ellipse, a and b are the lengths of the semi-minor and major axes respectively.

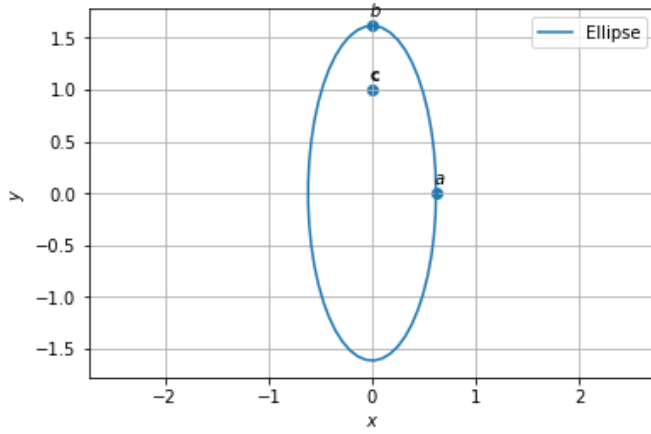


Fig. 1: Ellipse