Assignment 7

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Download the python code, latex file and the pdf doc from

https://github.com/Rishab9991/EE5609/tree/master/Assignments/Assignment7

Question

(Q.No.72, UGC June 2017)
Let m, n and r be natural numbers. Let $\mathbf{A}$ be an m $\times$ n matrix with real entries such that $(\mathbf{AA^t})^r = \mathbf{I}$, where $\mathbf{I}$ is the m $\times$ m identity matrix and $\mathbf{A^t}$ is the transpose of the matrix $\mathbf{A}$. We can conclude that
Options

1) m = n
2) $\mathbf{AA^t}$ is invertible
3) $\mathbf{A^tA}$ is invertible
4) if m = n, then $\mathbf{A}$ is invertible

Solution

Option 1	To conclude that m = n
Assumptions	**For the example:** Without loss of generality, Let m = 2, n = 3 and $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\implies \mathbf{A^t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$
Proof	We know that $(\mathbf{AA^t})^r = \mathbf{I}$ which is a square matrix of order m × m For any natural value of r, a square matrix ($\mathbf{I}$) of order m × m is obtained Hence, we cannot conclude that m = n because we get $\mathbf{I}$ of order m × m even if m ≠ n. To illustrate this, Consider the following example $\mathbf{AA^t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$ ($\mathbf{A}$ and $\mathbf{A^t}$ from Assumptions) $(\mathbf{AA^t})^r = \mathbf{I}$ Here m ≠ n. Therefore, **Option 1** is incorrect

TABLE I: Option 1

Option 2	To conclude that $\mathbf{AA^t}$ is invertible
Assumptions	$\mathbf{AA^t}$ is not invertible
Proof	$\implies \left\|\mathbf{AA^t}\right\| = 0 \implies \left\|(\mathbf{AA^t})^r\right\| = 0$ $\implies (\mathbf{AA^t})^r \neq \mathbf{I} \left(\left\|\mathbf{I}\right\| = 1\right)$ Since, this is a contradiction to the assumption made we can conclude that $\mathbf{AA^t}$ is invertible. Therefore, **Option 2** is correct

TABLE II: Option 2

Option 3	To conclude that $\mathbf{A^tA}$ is invertible
Assumptions	Without loss of generality, Let m = 2, n = 3 and $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\implies \mathbf{A^t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$
Proof	$\implies \mathbf{A^tA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies \left\|\mathbf{A^tA}\right\| = 0$ This means that $\mathbf{A^tA}$ is not invertible. Therefore, **Option 3** is incorrect

TABLE III: Option 3

Option 4	To conclude that if m = n then $\mathbf{A}$ is invertible
Assumptions	Let m = n
Proof	Since $(\mathbf{AA^t})^r = \mathbf{I} \implies \left\|(\mathbf{AA^t})^r\right\| = \left\|\mathbf{I}\right\| = 1$ $\implies \left(\left\|\mathbf{A}\right\|\left\|\mathbf{A^t}\right\|\right)^r = 1$ ($\mathbf{A}$ is a square matrix) $\implies \left(\left\|\mathbf{A}\right\|\right)^{2r} = 1$ Therefore, **Option 4** is correct

TABLE IV: Option 4