

Assignment 3.2

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Download the python code, latex file and the pdf doc from

<https://github.com/Rishab9991/EE5609/tree/master/Assignments/Assignment3.2>

Question:

Prove that the line

$$(1 \ 1)\mathbf{x} = 1 \quad (1)$$

touches the circle

$$\mathbf{x}^T \mathbf{x} - (8 \ 6)\mathbf{x} + 7 = 0 \quad (2)$$

and find the equations of the parallel and perpendicular tangents.

Solution: The vector equation of a line can be expressed as,

$$\mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad (3)$$

The general equation of a second degree can be expressed as.

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4)$$

Comparing (2) and (4) we get,

$$\mathbf{u} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \quad (5)$$

$$f = 7 \quad (6)$$

If \mathbf{n} is the normal vector of a line, equation of the given line can be written as,

$$\mathbf{n}^T \mathbf{x} = c \quad (7)$$

Comparing (1) and (7) we get,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (8)$$

The point of contact \mathbf{q} , of a line with a normal vector

\mathbf{n} to the conic in (4) is given by,

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) \quad (9)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (10)$$

For a circle,

$$\mathbf{V} = \mathbf{I} \quad (11)$$

where \mathbf{I} is the Identity matrix.

Solving for κ using (10) we get,

$$\kappa = \pm 3 \quad (12)$$

$$i.e. \mathbf{q}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } \kappa = -3 \quad (13)$$

and

$$\mathbf{q}_2 = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \text{ for } \kappa = 3 \quad (14)$$

To prove that the line touches the circle at \mathbf{q} need to check that

$$\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) = 0 \quad (15)$$

We know that,

$$\mathbf{m}^T \mathbf{n} = 0 \quad (16)$$

$$\Rightarrow m = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (17)$$

Using (13), (14) and (17), we can say that (15) holds true for both \mathbf{q}_1 and \mathbf{q}_2 which means that both those points lie on the circle i.e. there will be a tangent passing through each of them which can be found out using (7)

$$i.e. \mathbf{n}^T \mathbf{q}_1 = c_1 \quad (18)$$

$$\mathbf{n}^T \mathbf{q}_2 = c_2 \quad (19)$$

Using (18) and (19) we get,

$$c_1 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \quad (20)$$

$$c_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \end{pmatrix} = 13 \quad (21)$$

Using (18) the given line in the question is obtained. (1)

Therefore, the line parallel to the given line is,

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 13 \quad (22)$$

And the line perpendicular to the given line is the line passing through the normal vector \mathbf{n} found out in (8) and its equation will be of the form,

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = k \quad (23)$$

where k is a constant.

Consider Fig. 1

\mathbf{C} is the center of the circle. \mathbf{q}_1 and \mathbf{q}_2 are points of contact with the circle.

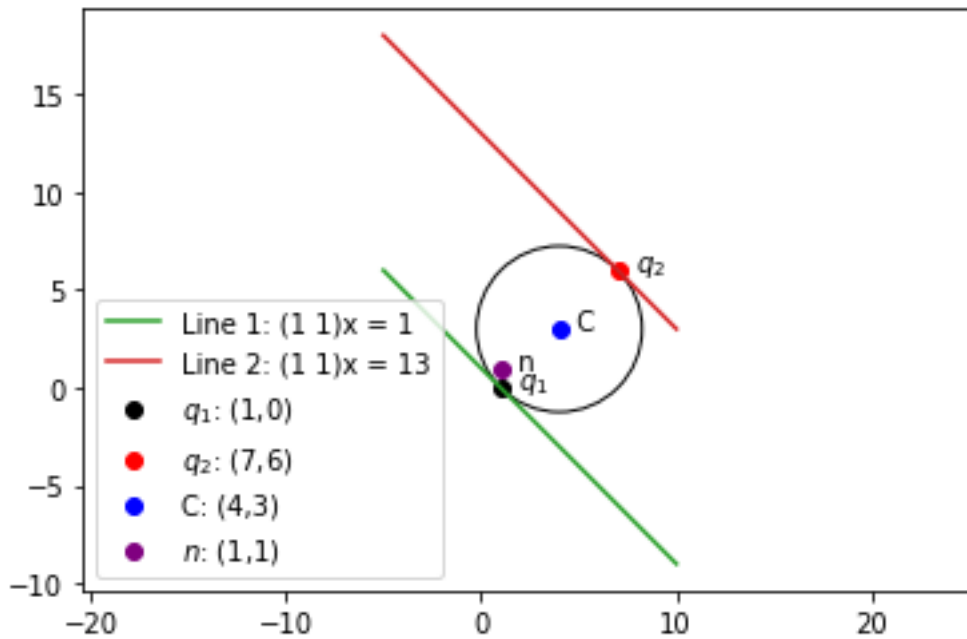


Fig. 1: Tangents parallel and perpendicular to the given line on the Circle