20/1/22

Beta function:-

for $m, n > 0$, the definite integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$ is called Beta function and it is denoted by $B(m,n)$

i.e. $$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

properties of Beta function.

(1) $B(m,n) = B(n,m)$

(2) $$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

(3) $$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

(4) $$B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$$

Q1) Evaluate

$$I = \int_0^{\pi/2} \sqrt{\tan x}\,dx$$

→ let $$I = \int_0^{\pi/2} \sqrt{\tan x}\,dx$$

$$= \int_0^{\pi/2} \sqrt{\frac{\sin x}{\cos x}}\,dx$$

$$= \int_0^{\pi/2} (\sin x)^{1/2}\cdot(\cos x)^{-1/2}\,dx$$

$\frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$ = By property 4

$$= \frac{1}{2} B\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{\frac{1}{2}\,\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$$

$$= \frac{\frac{1}{2}\,\Gamma\frac{3}{4}\cdot\Gamma\frac{1}{4}}{\Gamma 1}$$

$$= \frac{1}{2}\,\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$$

$$= \frac{1}{2}\,\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2}\,\frac{\pi}{\sin\left[\left(\frac{1}{4}\right)\pi\right]}$$ using $\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$

$$= \frac{1}{2}\,\frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$= \frac{\sqrt{2}}{2}\cdot\pi$$

$$= \frac{\pi}{\sqrt{2}}$$

Page No.	
Date	

eg.2] Evaluate $\int_0^2 x^4 \cdot (8-x^3)^{-1/3}\, dx$

$\rightarrow$ let

$$I = \int_0^2 x^4 (8-x^3)^{-1/3} dx$$

$$= \int_0^2 x^4 (8-x^3)^{-1/3} dx$$

put $x^3 = 8t$

$x = (8t)^{1/3}$

$x = 2 \cdot t^{1/3}$

$dx = \frac{2}{3} t^{-2/3} dt$

$dx = \frac{2}{3} t^{1-1/3} dt$

for $x = 0, t = 0$

$x = 2 ; t = 1$

$$I = \int_0^1 \cancel{x^4} (2t^{1/3})^4 \cdot (8-8t)^{-1/3} \cdot \frac{2}{3} t^{1/3 - 1} dt$$

$$= \int \frac{2}{3} \times 16 \cdot t^{4/3} \cdot t^{-2/3} \cdot 8^{-1/3} (1-t)^{-1/3} \cdot dt$$

$$= \frac{32}{3} \cdot \int_0^1 t^{5/3 - 1} \cdot 8^{-1/3} (1-t)^{-1/3} dt$$

$$= \frac{32}{3} \cdot \frac{1}{8^{1/3}} \int_0^1 t^{(5/3 - 1)} (1-t)^{2/3 - 1} dt$$

$$= \frac{16}{3} \int_0^1 B\left(\frac{5}{3}, \frac{2}{3}\right)$$

$$= \frac{16}{3} \frac{\Gamma\frac{5}{3} \, \Gamma\frac{2}{3}}{\Gamma\left(\frac{5}{3} + \frac{2}{3}\right)}$$

$$= \frac{16}{3} \cdot \frac{\Gamma\frac{5}{3} \cdot \Gamma\frac{2}{3}}{\Gamma\frac{7}{3}}$$

$$= \frac{16}{3} \frac{\Gamma\left(\frac{2}{3}+1\right) \Gamma\frac{2}{3}}{\Gamma\left(\frac{4}{3}+1\right)}$$

$$= \frac{16}{3} \frac{\frac{2}{3} \cdot \Gamma\frac{2}{3} \, \Gamma\frac{2}{3}}{\frac{4}{3} \Gamma\frac{4}{3}} \quad \text{by } \Gamma(n+1) = n\Gamma n$$

$$= \frac{16}{3} \times \frac{2}{4} \frac{\left(\Gamma\frac{2}{3}\right)^2}{\Gamma\left(\frac{1}{3}+1\right)}$$

$$= \frac{8}{3} \frac{\left(\Gamma\frac{2}{3}\right)^2}{\frac{1}{3} \Gamma\frac{1}{3}}$$

$$= 8 \frac{\left(\Gamma\frac{2}{3}\right)^2}{\Gamma\frac{1}{3}} \quad //$$

Q.3) Show that

$$\int_0^1 \sqrt{1-\sqrt{x}} \, dx = \int_0^{1/2} \sqrt{2y-(2y)^2} \, dy = \pi/30$$

$\rightarrow$ let

$$I_1 = \int_0^1 \sqrt{1-\sqrt{x}} \, dx \quad \text{put } \boxed{\sqrt{x} = t} \text{ i.e. } x = t^2$$

$$dx = 2t \, dt$$

$x=0,\ t=0$

$x=1,\ t=1$

$$= \int_0^1 \sqrt{1-t} \; 2t \, dt$$

$$= 2 \int_0^1 t \sqrt{1-t} \, dt$$

$$= 2\int_0^1 t^1 (1-t)^{1/2}$$

$$= 2B\left(2, \tfrac{3}{2}\right) \quad \text{— (I)}$$

~~Sol2.~~

~~Sol~~ Let $I_2 = \int_0^{1/2} \sqrt{2y - (2y)^2}\, dy$

$2y = t$

$y = t/2$ when $y = 0,\ t = 0$

$dy = \dfrac{dt}{2}$ $y = 1/2,\ t = 1$

$$= \int_0^1 \sqrt{t - t^2}\, \frac{dt}{2}$$

$$= \frac{1}{2}\int_0^1 \sqrt{t(1-t)}\, dt$$

$$= \frac{1}{2}\int_0^1 t^{1/2} (1-t)^{1/2}\, dt$$

$$= \frac{1}{2} B\left(\tfrac{3}{2}, \tfrac{3}{2}\right) \quad \text{— (II)}$$

$\therefore$

$$I = I_1 \cdot I_2$$

$$= 2B\left(2, \tfrac{3}{2}\right) \cdot \frac{1}{2} B\left(\tfrac{3}{2}, \tfrac{3}{2}\right)$$

$$= B\left(2, \tfrac{3}{2}\right) \cdot B\left(\tfrac{3}{2}, \tfrac{3}{2}\right)$$

$$= \frac{\Gamma 2\, \Gamma \frac{3}{2}}{\Gamma\left(2 + \frac{3}{2}\right)} \cdot \frac{\Gamma \frac{3}{2} \cdot \Gamma \frac{3}{2}}{\Gamma\left(\frac{3}{2} + \frac{3}{2}\right)}$$

Page No. | Date

$$= \frac{1!\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} \cdot \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}$$

$$= \frac{\cancel{\Gamma\left(\frac{3}{2}\right)}}{\frac{5}{2}\,\frac{3}{2}\,\cancel{\Gamma\left(\frac{3}{2}\right)}} \cdot \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2!}$$

$$= \frac{1}{\frac{15}{4}} \cdot \frac{\frac{1}{\cancel{4}}\sqrt{\pi}\,\sqrt{\pi}}{2}$$

$$= \frac{\pi}{30}$$ hence proved.