G H PATEL COLLEGE OF ENGINEERING & TECHNOLOGY, V V NAGAR

A.Y.2021-22: EVEN SEMESTER

102040405: DISCRETE MATHEMATICS

Assignment 2: Functions

1. Determine whether f is a function from Z to R if
   (a) $f(n) = \pm n$ (b) $f(n) = \sqrt{n^2+1}$ (c) $f(n) = 1/(n^2-4)$

No Yes No

2. Determine whether f is a function from the set of all bit strings to the set of integers if
   (a) $f(S)$ is the position of a 0 bit in S.
   (b) $f(S)$ is the number of a 1 bits in S.
   (c) $f(S)$ is the smallest integer i such that the ith bit of S is 1 and $f(S) = 0$ when S is the empty string, the string with no bits.

(a) The function value f(S) can't be determined if S does not contain any zero bit so f(S) is not a function

(b) f(S) is a function

(c) f(S) is not a function

3. Find the domain and range of each of the following functions that assigns:
   (a) to each nonnegative integer its last digit
   (b) the next largest integer to a positive integer
   (c) to a bit string the number of one bits in the string
   (d) to each bit string the number of ones minus the number of zeros
   (e) to each bit string twice the number of zeros in that string
   (f) to each positive integer the largest perfect square not exceeding this integer
   (g) the number of bits left over when a bit string is split into bytes

(a) Domain: N
Range $= \{0, 1, 2, \dots, 9\}$

(b) Domain: $N - \{0\}$
Range: $N - \{0, 1\}$

(c) Domain: Set of all bit string
Range: N

(d) Domain: Set of all bit string
Range: Z

(e) Domain: Set of all bit string or $N \cup \{0\}$
Range: Set of non-negative even integers

(f) Domain: N
Range: $\{x \mid x = n^2, n \in N\}$

(g) Domain: Set of all bit string
Range $= \{0, 1, 2, 3, 4, 5, 6, 7\}$

4. Find the domain and range of each of the following functions that assigns to:

(a) each pair of positive integers the first integer of the pair
(b) each positive integer its largest decimal digit
(c) a bit string the longest string of ones in the string
(d) each positive integer the largest integer not exceeding the square root of the integer
(e) each pair of positive integers the maximum of these two integers
(f) each positive integer the number of the digits that do not appear as decimal digits of the integer
(g) a bit string the number of times the block 11 appears
(h) a bit string the numerical position of the first 1 in the string and the value 0 to a bit string consisting of all 0s.

(a) Domain $= \{x / x \in N\} = \{1, 2, 3, \dots\}$ $Z^+ \times Z^+$
Range $= N$ Z^+

(b) Domain: $N - \{0\}$
Range $= \{1, 2, 3, \dots, 9\}$

(c) Domain: set of all bit string
Range: N

(d) Domain: $N - \{0\}$
Range $= Z$ $N - \{0\}$

(e) Domain: $N - \{0\} \times N - \{0\}$
Range: $N - \{0\}$

(f) Domain: $N - \{0\}$
Range: $\{0, 1, 2, 3, \dots, 9\}$

(g) Domain: Set of all bit string
Range: N

(h) Domain: Set of all bits string
Range: N

5. Find the values:

(a) $\left\lceil \left\lfloor \frac{1}{2} \right\rfloor + \left\lceil \frac{1}{2} \right\rceil + \frac{1}{2} \right\rceil$ (b) $\left\lfloor \frac{1}{2} + \left\lceil \frac{3}{2} \right\rceil \right\rfloor$ (c) $\left\lfloor \frac{1}{2} \cdot \left\lfloor \frac{5}{2} \right\rfloor \right\rfloor$

$\lceil 0 + 1 + \frac{1}{2} \rceil$

$\lceil 1.5 \rceil$

(a) 2 (b) 2 (c) 1

6. Determine whether each of these functions from Z to Z is one-to-one and onto.
(a) $f(n) = n - 1$ (b) $f(n) = n^2 + 1$ (c) $f(n) = n^3$ (d) $f(n) = \lceil n/2 \rceil$

(a) ~~one~~ one to one & onto both

(b) not one-one & not onto
∴ because 1 & -1 have same image
∴ since -1 has no preimage, f is not onto

(c) one one & not onto
Since 2 has no preimage, f is not onto

(d) not one-one & ~~~~ onto
$f(1) = f(2) = 1$
∴ not one-one

7. Determine whether $f: Z \times Z \to Z$ is one-to-one and onto if

(a) $f(m,n) = 2m - n$ (b) $f(m,n) = m^2 - n^2$ (c) $f(m,n) = |m| - |n|$

(d) $f(m,n) = m + n$ (e) $f(m,n) = m$ (f) $f(m,n) = |n|$

(a) not one-one
onto
$f(1,1) = f(2,3) = 1$

(b) One-one ×
onto ×
$f(1,0) = f(-1,0) = 1$
Since 2 does not have preimage

(C) One-One ×
Onto ✓
$f(-1,2) = f(1,2) = -1$

(d) One-One ×
onto ✓
$f(1,2) = f(6,-3) = 3$

(e) One One ✓
onto ✓

(f) One One ×
onto ×
$f(0,-1) = f(0,1) = 1$
-2 does not have preimage

8. Determine whether each of these functions is a bijection from **R** to **R**.

(a) $f(x) = -3x + 4$ (b) $f(x) = 3x^2 + 7$ (c) $f(x) = \dfrac{x+1}{x+2}$ (d) $f(x) = \dfrac{x^2+1}{x^2+2}$

(a) let $x, x_2 \in R$
$f(x_1) = f(x_2)$
$-3x_1 + 4 = -3x_2 + 4$
$x_1 = x_2$
So function is One-One
$-\infty < x < \infty$
$-\infty < -3x < \infty$
$-\infty < -3x + 4 < \infty$
∴ bijective

(b) $3x_1^2 + 7 = 3x_2^2 + 7$
$x_1^2 = x_2^2$
$x_1 \neq x_2$
$-\infty < x < \infty$
$0 \leq x^2 < \infty$
$0 \leq 3x^2 < \infty$
$7 \leq 3x^2 + 7 < \infty$
range ≠ co-domain
∴ not bijective

(C) $\dfrac{x_1+1}{x_1+2} = \dfrac{x_2+1}{x_2+2}$
$x_1 = x_2$
∴ One One
but for $x = -2$
func^n is not bjective defined
∴ func^n is not byective

(d) $\dfrac{x_1^2+1}{x_1^2+2} = \dfrac{x_2^2+1}{x_2^2+2}$
$x_1 \neq x_2$
∴ func^n is not One-One
∴ function is not byjective

9. Show that the function $f: R \to R$ defined by $f(x) = e^x$ is not invertible. Modify the domain or codomain of f so that it becomes invertible.

If the co-domain is R then any real number that is almost 0 has no preimage so func^n is not invertible
- If codomain is R^+ then func^n is invertible.

10. Show that the function $f: R \to R^+ \cup \{0\}$ defined by $f(x) = |x|$ is not invertible. Modify the domain or codomain of f so that it becomes invertible.

Here $f(x) = f(-x) = x$ so function is not one-one so not invertible
∴ Domain $= R^+ \cup \{0\}$ Range $= R^+ \cup \{0\}$

11. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from **R** to **R**.

$fog = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5$
$gof = g(f(x)) = g(x^2+1) = (x^2+1) + 2 = x^2 + 3$

12. If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

$g(x) = g(y) \Rightarrow f(g(x)) = f(g(y))$ (By defination of function)

$f\circ g(x) = f\circ g(y)$ (By definition of composition

$x = y$ ($\because f\circ g$ is one to one)

13. If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

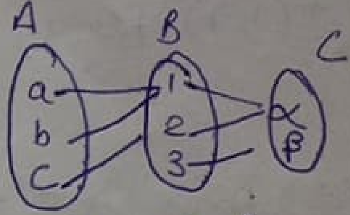
No, let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ & $C = \{\alpha, \beta\}$

Let $g: A \to B$ & $f: B \to C$ be defined as

$g(a) = g(b) = 1$ $g(c) = 2$

$f(1) = f(2) = \alpha$ $f(3) = \beta$

$f\circ g$ is onto but g is not onto because 3 doesn't have preimage



14. Suppose that g is a function from A to B and f is a function from B to C.
(a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.
(b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

a) Let $x, y \in A$, $(f(g(x))$ by the defination of composition, $f(g(x)) = (f\circ g)(x)$. Since f & g is one-one it follows that $g(x) = g(y)$. Then we have $f(g(y)) = f\circ g(y)$. $x = y$, since g is one-one.

15. Show that the function $f(x) = ax + b$ from $\boldsymbol{R}$ to $\boldsymbol{R}$ is invertible, where a and b are constants, with $a \neq 0$, and find the inverse of f.

→ Suppose that $f(x_1) = f(x_2)$ for two real no. x_1 & x_2, $ax_1 + b = ax_2 + b$ $\Rightarrow x_1 = x_2$, f is one-one

→ $y \in R$, the $y = f(x) = ax + b$ has a unique solⁿ $x = y - \frac{b}{a}$ ($\because a \neq 0$) $\therefore f$ is onto $\therefore$ Given function is invertible.

$f^{-1}(x) = \frac{y-b}{a}$

16. Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = x^2$. Find
(a) $f^{-1}(\{1\})$ (b) $f^{-1}(\{x \mid 0 < x < 1\})$ (c) $f^{-1}(\{x \mid x > 4\})$.

(a) $\Rightarrow \pm 1$
$\Rightarrow \{1, -1\}$

(b) $\Rightarrow \{x \mid 0 < x < 1\}$

(c) $\Rightarrow R - [-2, 2]$

17. Let $g(x) = \lfloor x \rfloor$. Find
(a) $g^{-1}(\{0\})$ (b) $g^{-1}(\{-1, 0, 1\})$ (c) $g^{-1}(\{x \mid 0 < x < 1\})$.

(a) $\Rightarrow [0, 1)$

(b) $\Rightarrow [-1, 2]$

(c) $\Rightarrow \phi$

18. Let $f: A \to B$ and $S, T \subseteq A$. Show that

(a) $f(S \cup T) = f(S) \cup f(T)$ (b) $f(S \cap T) \subseteq f(S) \cap f(T)$.

Give an example to show that the inclusion in part (b) may be proper.

(a) Let, $y \in f(S \cup T)$

$\exists\, x \in S \cup T$ such that $f(x) = y$

$\exists\, x \in S$ or $\exists\, x \in T$ such that $f(x) = y$

$f(x) \in f(S)$ or $f(x) \in f(T)$

$f(x) = y \in f(S) \cup f(T)$

$f(S \cup T) \subseteq f(S) \cup f(T)$ — ①

Let, $y \in f(S) \cup f(T)$

$y \in f(S)$ or $y \in f(T)$

($\exists\, x \in S$ such that $f(x) = y$) or

($\exists\, x \in S \cup T$ such that $f(x) = y$)

$\exists\, x \in S \cup T$ such that $f(x) = y$

$f(x) = y \in f(S \cup T)$ $\therefore f(S) \cup f(T) \subseteq f(S \cup T)$

(b) $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2$

Let $S = \{\ldots, -3, -2, -1, 0\}$

$= \{x \in \mathbb{Z} / x \leq 0\}$

$T = \{0, 1, 2, 3, \ldots\}$

$= \{x \in \mathbb{Z} / x \geq 0\}$

$S \cap T = \{0\}$

Also, $f(S) = \{0, 1, 4, 9, \ldots\} = f(T)$

$f(S) \cap f(T) = \{0, 1, 4, 9, \ldots\}$

$f(S \cap T) = \{0\}$.

$\therefore$ Thus, $f(S \cap T) = f(S) \cap f(T)$ is not true

19. Let $f: A \to B$ and $S, T \subseteq B$. Show that

(a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$ (b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

(c) $f^{-1}(\bar{S}) = \overline{f^{-1}(S)}$.

(a) Let $x \in f^{-1}(S \cup T) \iff f(x) \in S \cup T$

$\iff f(x) \in S$ or $f(x) \in T$

$\iff x \in f^{-1}(S)$ or $x \in f^{-1}(T)$

$\iff x \in f^{-1}(S) \cup f^{-1}(T)$

$\therefore f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

(b) $x \in f^{-1}(S \cap T) \iff f(x) \in S \cap T$

$\iff f(x) \in S$ and $f(x) \in T$

$\iff x \in f^{-1}(S)$ and $x \in f^{-1}(T)$

$\iff x \in f^{-1}(S) \cap f^{-1}(T)$

$\therefore f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

(c) Let $x \in f^{-1}(\bar{S}) \iff f(x) \in \bar{S}$

$\iff f(x) \notin S$

$\iff x \notin f^{-1}(S)$

~~$\iff x \in \overline{f^{-1}(S)}$~~

~~$f^{-1}(\bar{S}) = \overline{f^{-1}(S)}$~~

20. In asynchronous transfer mode (ATM) (a communications protocol used on backbone networks), data are organized into cells of 53 bytes. How many ATM cells can be transmitted in 10 seconds over a link operating at the following rates?

(a) 128 kilobits per second (b) 300 kilobits per second (c) 1 megabit per second

(a) total bits to be transmitted: $= 128000 \times 10$
$= 1280000$
Total ATM cells that can be transmitted: $\left\lceil \frac{1280000}{53\times 8} \right\rceil = 3018$

(b) total bits to be transmitted $= 30000 \times 10$
$= 300000$
Total cells $= \frac{300000}{53 \times 8} = 707.54$

(C) Total bits $= 1000000 \times 10$
$= 10000000$
Total cells $= \frac{10000000}{53 \times 8}$
$= 23,584.90$

21. Draw graphs of each of these functions.

$(a)\ f(x) = \lceil x \rceil + \lfloor x/2 \rfloor$ $(b)\ f(x) = \lceil 1/x \rceil$ $(c)\ f(x) = \lfloor 2x + 1 \rfloor$

$(d)\ f(x) = \lceil x/2 \rceil \lfloor x/2 \rfloor$ $(e)\ f(x) = \lfloor 2\lceil x/2 \rceil + 1/2 \rfloor$ $(f)\ f(x) = \lfloor x^2 \rfloor$

Use separate sheet of paper to draw the graphs.

For any real number x, it is possible to find an integer n such that

$$n \le x < n + 1.$$

In this case, $\lfloor x \rfloor = n$, and $\lceil x \rceil = n + 1$. It is clear that $\lfloor x \rfloor \le x \le \lceil x \rceil$. It is also clear that we can express x as $x = n + \varepsilon$, where $0 \le \varepsilon < 1$. Use this fact to prove the elementary results in the examples below.

22. Let x be a real number and m, n are integers. Prove that

$(a)\ \lceil x \rceil - \lfloor x \rfloor = \begin{cases} 1, & \text{if } x \text{ is not an ineger} \\ 0, & \text{if } x \text{ is an ineger} \end{cases}$.

$(b)\ \lceil x + m \rceil = \lceil x \rceil + m.$

$(c)\ x < n$ if and only if $\lfloor x \rfloor < n$. $(d)\ n < x$ if and only if $n < \lceil x \rceil$.

$(e)\ x \le n$ if and only if $\lceil x \rceil \le n$. $(f)\ n \le x$ if and only if $n \le \lfloor x \rfloor$.

(a) Let $x = n + \varepsilon\ (0 \le \varepsilon < 1)$

$\lceil n+\varepsilon \rceil - \lfloor n+\varepsilon \rfloor$ $(\because 0 \le \varepsilon \le 1)$

if x is not an integer

then $0 < \varepsilon < 1$

$\lceil n+\varepsilon \rceil = n+1$ & $\lfloor n+\varepsilon \rfloor = n$

$\therefore n+1-n = 1$

when x is an integer

the $\varepsilon = 0$

$\lceil n+0 \rceil = n$ & $\lfloor n+0 \rfloor = n$

$\therefore n - n = 0$

(b) Let $x = n + \varepsilon\ (0 \le \varepsilon < 1)$

$\lceil n+\varepsilon+m \rceil = LHS$

Case 1: Let x is not integer then

$0 < \varepsilon < 1$

$\lceil n+\varepsilon+m \rceil = n+m+1$

$\lceil x \rceil + m = RHS$

$\lceil n+\varepsilon \rceil + m = n+1+m$

$= LHS$

Case 2: Let x is an integer then $\varepsilon = 0$,

LHS $=$ $\lceil n+\varepsilon+m \rceil = n+m$

RHS $= \lceil x \rceil + m$

$= \lceil n+\varepsilon \rceil + m$

$= \lceil n+0 \rceil + m$

$= n+m = LHS$

Hence $\lceil x+m \rceil = \lceil x \rceil + m$

(c) Let $x < n$

$\lfloor x \rfloor \le x$ so $\lfloor x \rfloor \le x < n$

hence $\lfloor x \rfloor < n$

Let $\lfloor x \rfloor < n$

$x - 1 < \lfloor x \rfloor \le \lceil x \rceil < x+1$

$x - 1 < \lfloor x \rfloor$, i.e. $x < \lfloor x \rfloor + 1$

but $\lfloor x \rfloor < n$, so $\lfloor x \rfloor + 1 \le n$

and $x < \lfloor x \rfloor + 1 \le n$

hence $x < n$

$x \le \lceil x \rceil$, it follows that $n \le \lceil x \rceil$

Suppose that $n \ge x$

It follows that $\lceil x \rceil \le n$. This means that of $n < \lceil x \rceil$, then $n < x$

(e) As $x \le \lceil x \rceil$

Let $x \le n$, since n is an integer no smaller than x and $\lceil x \rceil$ is by defination the smallest such integer is $\lceil x \rceil \le n$

23. Prove or disprove each of these statements about the floor and ceiling functions. Let x and y be real numbers.

(*a*) $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$ (*b*) $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$ *or* 1

(*c*) $\lfloor 2x \rfloor = 2\lfloor x \rfloor$ (*d*) $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$

(*e*) $\left\lceil \frac{x}{2} \right\rceil = \left\lfloor \frac{x+1}{2} \right\rfloor$ (*f*) $\left\lceil \frac{\lceil x/2 \rceil}{2} \right\rceil = \left\lceil \frac{x}{4} \right\rceil$

(*g*) $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \le \lfloor 2x \rfloor + \lfloor 2y \rfloor$

(a) Let $x = n + \varepsilon$

Case 1: $\varepsilon = 0$

$LHS = \lfloor \lceil x \rceil \rfloor = \lfloor \lceil n \rceil \rfloor$
$= \lfloor n \rfloor = n$

$RHS = \lceil x \rceil = \lceil n \rceil = n$

Case 2: $\varepsilon \neq 0$

$LHS = \lfloor \lceil x \rceil \rfloor = \lfloor \lceil n + \varepsilon \rceil \rfloor = \lfloor n+1 \rfloor$
$= n+1$

$RHS = \lceil x \rceil = \lceil n + \varepsilon \rceil$
$= n+1$

$LHS = RHS$

(b) Let $x = n_1 + \varepsilon_1$, $0 \le \varepsilon_1 < 1$
$y = n_2 + \varepsilon_2$, $0 \le \varepsilon_2 < 1$

→ Let $\varepsilon_1 = 0$ & $\varepsilon_2 \neq 0$

$\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil$
$= \lceil n_1 \rceil + \lceil n_2 + \varepsilon_2 \rceil - \lceil n_1 + n_2 + \varepsilon_2 \rceil$
$= \lceil n_1 \rceil + \lceil n_2 + 1 \rceil - (n_1 + n_2 + 1) = 0$

→ Let $\varepsilon_1 \neq 0$ & $\varepsilon_2 = 0$

$\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil =$
$\lceil n_1 + \varepsilon_1 \rceil + \lceil n_2 \rceil - \lceil n_1 + n_2 + \varepsilon_1 \rceil$
$= (n_1 + 1) - n_2 - (n_1 + n_2 + 1)$
$= 0$

→ Let $\varepsilon_x = 0$ & $\varepsilon_2 = 0$

$\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$

→ Let $\varepsilon_1 \neq 0$, $\varepsilon_2 \neq 0$

$\lceil x \rceil = \lceil n_1 + \varepsilon_1 \rceil = n_1 + 1$

$\lceil y \rceil = \lceil n_2 + \varepsilon_2 \rceil = n_2 + 1$

$\lceil x + y \rceil = \lceil n_1 + \varepsilon_1 + n_2 + \varepsilon_2 \rceil$
$= \lceil n_1 + n_2 + (\varepsilon_1 + \varepsilon_2) \rceil$

Case 1: $0 \le \varepsilon_1 + \varepsilon_2 \le 2$
$0 \le \varepsilon_1 + \varepsilon_2 \le 1$

$\lceil n_1 + n_2 + (\varepsilon_1 + \varepsilon_2) \rceil$
$= n_1 + n_2 + 1$

$\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 1$

Case 2: $0 \le \varepsilon_1 + \varepsilon_2 \le 2$

$\lceil n_1 + n_2 + (\varepsilon_1 + \varepsilon_2) \rceil = n_1 + n_2 + 2$

$\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$

24. Prove that if x is a positive real number, then

(*a*) $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$ (*b*) $\left\lceil \sqrt{\lceil x \rceil} \right\rceil = \lceil \sqrt{x} \rceil$.

In these examples, square roots of floor and ceiling functions are involved. So, instead of assuming $x = n + \varepsilon$, we will represent x as $x = n^2 + m + \varepsilon$, where n^2 is the nearest perfect square less than or equal to x. For example, 12.2 can be written as $12 = 9 + 3 + \varepsilon$.

Proof of (*a*):

Let $x = n^2 + m + \varepsilon$. Then $\lfloor x \rfloor = n^2 + m$.

So, $n \le \sqrt{\lfloor x \rfloor} < n + 1$. (*Why?*)

$\therefore L.H.S. = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = n$.

Also, $n \le \sqrt{x} < n + 1$. (*Why?*)

$\therefore R.H.S. = \lfloor \sqrt{x} \rfloor = n$.

Thus, (*a*) is proved.

In a similar way, try to prove the second.

(b) $\left\lceil \sqrt{\lceil x \rceil} \right\rceil = \lceil \sqrt{x} \rceil$

$\lceil x \rceil = n^2 + m + 1$

$\therefore n \le \sqrt{\lceil x \rceil} \le n+1$

$LHS = \left\lceil \sqrt{\lceil x \rceil} \right\rceil = n+1$

$n \le \sqrt{x} < n+1$

→ $RHS = \lceil \sqrt{x} \rceil = n+1$

$\left\lceil \sqrt{\lceil x \rceil} \right\rceil = \lceil \sqrt{x} \rceil$

$LHS = RHS$

25. Let a and b be real numbers with $a < b$. Use the floor and/or ceiling functions to express the number of integers n that satisfy the inequality $a \le n \le b$.

$\lfloor b \rfloor - \lceil a \rceil$

let's take one eg.

$a = 3.1$, $b = 9.4$ then $\lfloor 9.4 \rfloor - \lceil 3.1 \rceil = 9 - 4 = 5$

$\therefore$ So, no. of integers betn a & b is 5

$3.1 \le 5 \le 9.4$

26. Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

$$\begin{aligned}(f \circ g) \circ (g^{-1} \circ f^{-1}) &= f \circ (g \circ g^{-1}) \circ f^{-1} \\ &= f \circ I_g \circ f^{-1} \\ &= f \circ f^{-1} \\ &= I\end{aligned}$$

$$\begin{aligned}(g^{-1} \circ f^{-1}) \circ (f \circ g) &= g^{-1} \circ (f^{-1} \circ f) \circ g \\ &= g^{-1} \circ I_f \circ g \\ &= I\end{aligned}$$

$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

27. Let S be subset of a universal set U. The characteristic function f_S of S is the function from U to the set $\{0, 1\}$ such that $f_S(x) = 1$ if x belongs to S and $f_S(x) = 0$ if x does not belong to S. Let A and B be sets. Show that for all x,

(a) $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$
(b) $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$
(c) $f_{\bar{A}}(x) = 1 - f_A(x)$
(d) $f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x) \cdot f_B(x)$

(a)

f_A	f_B	$f_{A \cap B}$	$f_A.f_B$
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

hence: $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$

(b)

f_A	f_B	$f_{(A \cup B)}$ = LHS	$f_A.f_B$	RHS
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	1

8

hence $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x).f_B(x)$

(c)

f_A	LHS = $f_{\bar{A}}$	RHS
0	1.	0
1	0	1

hence: $f_{\bar{A}}(x) = 1 - f_A(x)$

(d)

f_A	f_B	LHS	$f_A.f_B$	RHS
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	0	1	0

Hence, $f_{(A \oplus B)}(x) = f_A(1) + f(x) - 2f_A(x).f_B(x)$

Q 22

(f)

As $\lfloor x \rfloor \leq x$

- Let $n \leq x$. Since n is an integer not exceeding x and $\lfloor x \rfloor$ is by definitions the largest such integer clearly $n \leq \lfloor x \rfloor$

Q23

(c) $\lfloor 2x \rfloor = 2\lfloor x \rfloor$

let $x = 0.5$

$LHS = \lfloor 2 \times 0.5 \rfloor = 1$

$RHS = 2\lfloor 0.5 \rfloor = 0$

$RHS \neq LHS$

(d) $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$

let $x = 0.5$ & $y = 1.5$

$LHS = \lceil 0.5 \times 1.5 \rceil = 1$

$RHS = \lceil 0.5 \rceil \lceil 1.5 \rceil = 2$

$RHS \neq LHS$

(e) $\lceil x/2 \rceil = \lceil \frac{x+2}{2} \rceil$

Let $x = 2$

$LHS = \lceil 2/2 \rceil = 1$

$RHS = \lceil \frac{2+1}{2} \rceil = 2$

$LHS \neq RHS$

(f) $\left\lceil \frac{\lceil x/2 \rceil}{2} \right\rceil = \left\lceil \frac{x}{4} \right\rceil$

$\left\lceil \frac{\lceil \frac{1}{2} \rceil}{2} \right\rceil = \left\lceil \frac{\frac{1}{2}}{4} \right\rceil$

$\left\lceil \frac{1}{2} \right\rceil \neq \left\lceil \frac{1}{8} \right\rceil$

It is false

(g) $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

Let $x = n + \varepsilon$ and $y = m + f$
where n & m are integers
and ε and $f < 1$

$\therefore$ LHS $= n+m+(n+m)$ or ~~m~~ $= n+m+(m+n+1)$

RHS = Sum of two quantities

$\rightarrow$ The first is either

$2n$ (if $f < \frac{1}{2}$) or $(2n+1)$ (if $f \geq \frac{1}{2}$)

The second is either

$2n$ (if $f < \frac{1}{2}$) or

$2m+1$ (if $f \geq \frac{1}{2}$)

Res