Solving Recurrence Relations.

1] Substitution Method

1) $T(n) = 2T(n-1) + c$, $k = 1$

$\Rightarrow k = 1$ $\quad T(n) = 2T(n-1) + c$ ---- (1)

$$\boxed{\begin{aligned} T(n-1) &= 2T(n-1-1) + c \\ &= 2T(n-2) + c \end{aligned}}$$

$\Rightarrow$ $\quad T(n) = 2 \times (2T(n-2) + c) + c$

$k = 2$ $\quad = 2^2 T(n-2) + 3c$ ---- (2)

$$\boxed{\begin{aligned} T(n-2) &= 2T(n-2-1) + c \\ &= 2T(n-3) + c \end{aligned}}$$

$\Rightarrow$ $\quad T(n) = 2^2(2T(n-3) + c) + 3c$

$k = 3$ $\quad = 2^3 T(n-3) + 7c$ ---- (3)

Now analyzing the pattern we get.

$T(n) = 2^k T(n-k) + (2^k - 1)c$

let $n - k = 0$

$\Rightarrow n = k$

we get,

$T(n) = 2^n T(0) + (2^n - 1)c$

$\Rightarrow T(0) = 1$

$= 2^n + 2^n c - c$

$$\boxed{\Rightarrow 2^n}$$

2) $T(n) = \begin{cases} 1 & n = 0 \\ T(n-1)+1 & n > 0 \end{cases}$

$$T(n) = T(n-1)+1 \qquad \text{------ } (1)$$

$$\boxed{\begin{aligned} T(n-1) &= T(n-1-1)+1 \\ &= T(n-2)+1 \end{aligned}}$$

$$\Rightarrow T(n) = \big(T(n-2)+1\big)+1$$

$$= T(n-2)+2 \qquad \text{------ } (2)$$

$$\boxed{\begin{aligned} T(n-2) &= T(n-2-1)+1 \\ &= T(n-3)+1 \end{aligned}}$$

$$\Rightarrow T(n) = \big(T(n-3)+1\big)+2$$

$$= T(n-3)+3 \qquad \text{------ } (3)$$

Analyzing it, we get.

$$T(n) = T(n-k)+k$$

let $n-k = 0$

$\therefore n = k$

$$\therefore T(n) = T(n-n)+n$$

$$= T(0)+n$$

Now $T(0) = 1$

$$\boxed{\therefore T(n) = n}$$

3) $T(n) = T(n-1) + n$

$\Rightarrow T(n) = T(n-1) + n$ ——— (1)

$$\boxed{\begin{aligned} T(n-1) &= (T(n-1-1) + n-1) \\ &= T(n-2) + (n-1) \end{aligned}}$$

$\Rightarrow T(n) = [T(n-2) + (n-1)] + n$

$= T(n-2) + (n-1) + n$ ——— (2)

$$\boxed{\begin{aligned} T(n-2) &= (T(n-2) - 1) + n-2 \\ &= T(n-3) + (n-2) \end{aligned}}$$

$\Rightarrow T(n) = [T(n-3) + (n-2)] + (n-1) + n$

$T(n) = T(n-3) + (n-2) + (n-1) + n$ —— (3)

we get,

$T(n) = T(n-k) + (n-(k-1)) + (n-(k-2)) +$
$\quad ---- + (n-1) + n$

Assume $n-k=0$

$T(n) = T(0) + (n-n+1) + (n-n+2) + --- + (n-1)$
$\quad + n$

$= T(0) + 1 + 2 + 3 + -- + (n-1) + n$

$T(n) = 1 + \frac{n(n+1)}{2}$

$\therefore T(n) = 1 + \frac{n^2+n}{2}$

$$\boxed{T(n) = n^2}$$

4) $T(n) = T(n-1) + \log n$

$\Rightarrow T(n) = T(n-1) + \log n$ ---------- (1)

$$\boxed{T(n-1) = T(n-2) + \log(n-1)}$$

$\Rightarrow T(n) = \left(T(n-2) + \log(n-1)\right) + \log n$

$= T(n-2) + \log(n-1) + \log n$ ---------- (2)

$$\boxed{T(n-2) = T(n-3) + \log(n-2)}$$

$\Rightarrow T(n) = \left(T(n-3) + \log(n-2)\right) + \log(n-1) + \log n$ ---------- (3)

we get,

$T(n) = T(n-k) + \log 1 + \log 2 + \log 3 + \ldots + \log(n-1) + \log(n)$

$\therefore n-k = 0$

$\therefore n = k$

$= T(0) + \log n!$

$T(n) \Rightarrow 1 + \log n!$

$$\boxed{T(n) = O(n\log n)}$$

5) $T(n) = 2T(n-1) + 1$

$\Rightarrow T(n) = 2T(n-1) + 1 \quad \text{------} \quad (1)$

$$\boxed{\begin{aligned} T(n-1) &= 2T(n-1-1) + 1 \\ &= 2T(n-2) + 1 \end{aligned}} \quad \text{------} \quad (2)$$

$\Rightarrow T(n) = 2\left[2T(n-2) + 1\right] + 1$

$= 2^2 T(n-2) + 2 \quad \text{------} \quad (2)$

$$\boxed{\begin{aligned} T(n-2) &= 2T(n-2-1) + 1 \\ &= 2T(n-3) + 1 \end{aligned}}$$

$\Rightarrow T(n) = 2^2\left[2T(n-3) + 2\right] + 1$

$= 2^3 T(n-3) + 3 \quad \text{------} \quad (3)$

we get,

$T(n) = 2^k T(n-k) + k$

Assume $n - k = 0$

$n = k$

$= 2^n T(n-n) + n$

$= 2^n T(0) + n$

$= 2^n (1) + n$

$= 2^n \cdot n$

$$\boxed{T(n) = 2^n}$$