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Institute of Engineering & Technology

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Discrete mathematics

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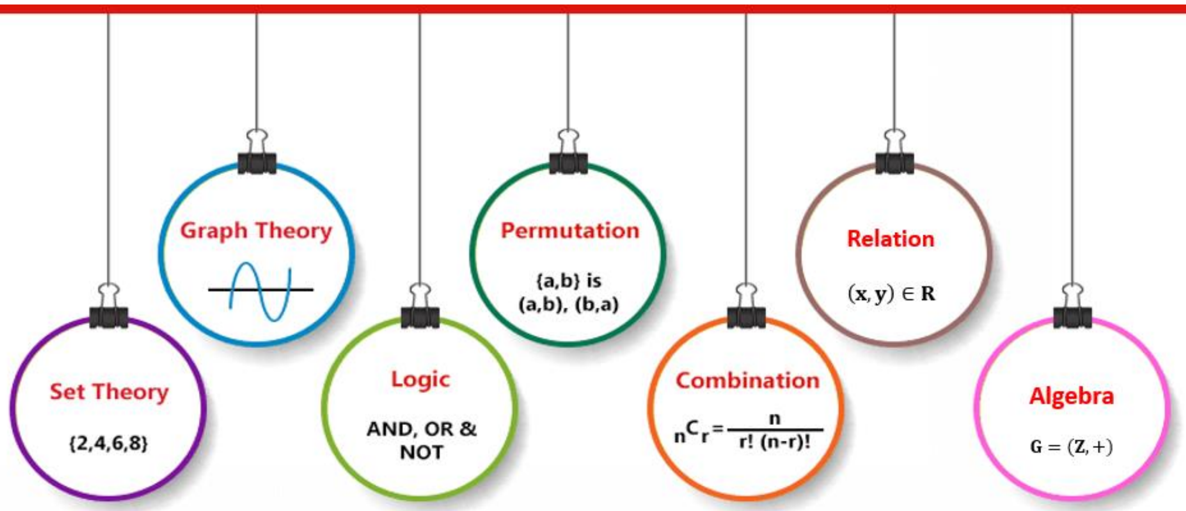
(Computer Engineering)



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DARSHAN INSTITUTE OF ENGINEERING & TECHNOLOGY

I N D E X

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UNIT 1 » SET THEORY, FUNCTIONS AND COUNTING

PART-I SET THEORY

❖ INTRODUCTION

- ✓ The theory of set was developed by German mathematician Georg Cantor (1845 - 1918). He first encountered sets while working on 'problems on trigonometric series'.
- ✓ The concept of set serves as a fundamental part of the present day mathematics. Today this concept is being used in almost every branch of mathematics. Sets are used to define the concepts of relation and function. The study of geometry, sequences, probability, etc. requires knowledge of sets. In this unit, we will discuss basic definitions and operation involving sets.

❖ SET

- ✓ A set is collection of well-defined objects.
- ✓ Each object in the set is called an element of the set.
- ✓ Elements of a set are usually denoted by lower case letter (a, b, c ...). While sets are denoted by capital letters (A, B, C ...).
- ✓ The symbol '$\in$ (is belongs to)' indicates the membership in a set. While the symbol '$\notin$ (is not belongs to)' is used to indicate that an element is not in the set.
- ✓ Example: If $A = \{1, 2, 3, a, b\}$, then we can write that $a \in A$, $1 \in A$ but $4 \notin A$.

❖ REPRESENTATION OF A SET

- ✓ Listing method (or Tabular form)
  - ➢ In listing method, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }. For example, the set of all natural numbers less than 6 is described in listing method as $\{1, 2, 3, 4, 5\}$.
- ✓ Property method (or Set-builder form)
  - ➢ In property method, all the elements of a set possess a single common property which is not possessed any element outside the set. For example, the set of all natural number between 4 & 9 is described in property method as

$$\{x : x \text{ is a natural number } \& \ 4 < x < 9\}$$

❖ SOME DEFINITIONS

- ✓ Empty set (null set) does not contain any element and It is denoted as { } or ϕ.
- ✓ A set which contain at least one element is called Non-empty set.
- ✓ A set which contain exactly one element is called Singleton set.
- ✓ A set which contain finite number of elements is called Finite set.
- ✓ A set which contain infinite number of elements is called Infinite set.
- ✓ Two set A and B are said to be Equal if they have exactly the same elements and we write $A = B$. Otherwise, the sets are said to be unequal and we write $A \neq B$.
- ✓ In any discussion in set theory, there always happens to be a set that contains all set under consideration i.e. it is a super set of each of the given sets. Such a set is called the universal set and is denoted by U.

❖ SUBSET

- ✓ If every element of a set A is an element of a set B then we can say that A is a subset of B.

$$\text{i.e. } A \subset B \text{ if } \forall a \in A \implies a \in B$$

- ✓ Example: If $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$, then $A \subset B$.
- ✓ Note: If A is a subset of B, then B is a superset of A.
- ✓ Let A be any non-empty set. Then
  - ➢ ϕ & A are improper subsets of A and remaining all are called proper subsets of A.
  - ➢ If set A has n elements then there are total 2^n subsets of A.

❖ OPERATIONS ON SETS

- ✓ Let U be a universal set and A & B are any two sets. Then
  - ➢ The Union of A & B is the set which consists of all the elements of A and all the elements of B, the common elements being taken only once. Symbolically, we write $A \cup B$ and usually read as 'A union B'.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

  - ➢ The Intersection of A & B is the set of all elements which are common to both A and B. Symbolically, we write $A \cap B$ and usually read as 'A intersection B'.

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

- The Difference of the sets A & B in this order is the set of elements which belong to A but not to B. Symbolically, we write $A - B$ and usually read as 'A minus B'.

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

- The Symmetric difference of sets A & B is the set of elements which belong to $A \cup B$ but not to $A \cap B$. Symbolically, we write $A\Delta B$ and usually read as 'A delta B'.

$$A\Delta B = \{x : x \in A \cup B \text{ and } x \notin A \cap B\}$$

- The Complement of A is the set of all elements which belong to universal set U but not to A. Symbolically, we write A'.

$$A' = \{x : x \in U \text{ and } x \notin A\}$$

✓ Example: Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$ & $B = \{4, 5, 6,7, 8\}$.

Find $A \cup B, A \cap B, A - B, A\Delta B, A', B'$.

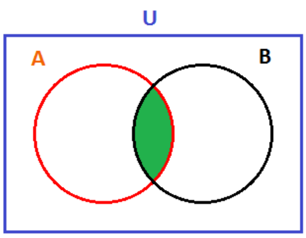
Solution: $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A \cap B = \{4, 5\}$, $A - B = \{1, 2, 3\}$, $A' = \{6, 7, 8, 9, 10\}$,

$B' = \{1, 2, 3, 9, 10\}$, $A\Delta B = (A \cup B) - (A \cap B) = \{1, 2, 3, 6, 7, 8\}$.

❖ VENN DIAGRAM

✓ The relationship between sets can be represented by means of diagrams which are known as Venn diagrams. Venn diagrams are named after the English logician, John Venn (1834 - 1883).

✓ This diagrams consist of rectangle and closed curves usually circles. The universal set is represented by rectangle and its subsets by circles.



Figure(1)

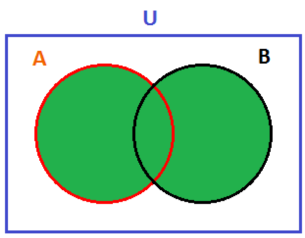


Figure (2)

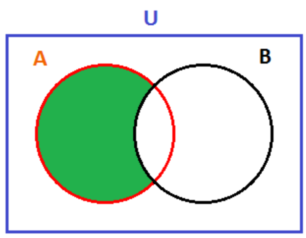


Figure (3)

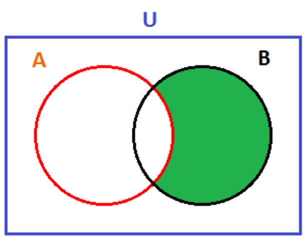


Figure (4)

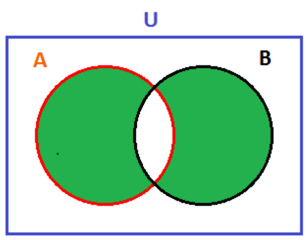


Figure (5)

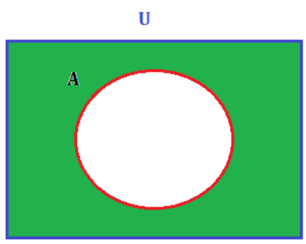


Figure (6)

✓ The shaded area in figure (1) is $A \cap B$ and the shaded area in figure (2) is $A \cup B$.

✓ The shaded area in figure (3) is $A - B$ and the shaded area in figure (4) is $B - A$.

✓ The shaded area in figure (5) is $A \Delta B$ and the shaded area in figure (6) is A'.

METHOD-1: EXAMPLES ON BASIC SET THEORY

H	1	Give another description of the following sets. a) $A = \{x : x \text{ is an integer and } 5 \le x \le 12\}$ b) $B = \{2, 4, 6, 8, \dots\}$ c) $C = \{x : x \in \mathbb{N}, 4 < x^2 < 40\}$ d) $D = \{x : x^3 - x = 0, x \in \mathbb{Z}\}$ **Answer** : $\{\mathbf{5, 6, 7, 8, 9, 10, 11, 12}\}, \{\mathbf{x : x \text{ is an even number}} \; \& \; \mathbf{x} \in \mathbb{N}\},$ $\{\mathbf{3, 4, 5, 6}\}, \{\mathbf{-1, 0, 1}\}$	
C	2	Give another description of the following sets. a) $A = \{x : x \in \mathbb{N}, x \text{ is a multiple of } 7 \; \& \; x \le 49\}$ b) $B = \{1, 8, 27, 64, 125, 216\}$ c) $C = \{x : x \in \mathbb{R}, \sin x = 0\}$ d) $D = \{x : x \text{ is a divisor of } 36, x \in \mathbb{N}\}$ **Answer** : $\{\mathbf{7, 14, 21, 28, 35, 42, 49}\}, \{\mathbf{x^3 : x} \in \mathbb{N} \; \mathbf{and} \; \mathbf{x \le 6}\}, \{\mathbf{k\pi : k} \in \mathbb{Z}\},$ $\{\mathbf{1, 2, 3, 4, 6, 9, 12, 18, 36}\}$	
C	3	Give another description of the following sets. a) $A = \{(x, y) : x, y \in \mathbb{N}, \lvert x - y \rvert \text{ is even number } \& \; x, y \le 6\}$ b) $B = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$ c) $C = \{(x, y) : x, y \in \mathbb{N}, \; x \text{ divides } y \; \& \; y \le 9\}$ **Answer** : $\{\mathbf{(1,3), (1,5), (2,4), (2,6), (3,1), (3,5), (4,2), (4,6), (5,1), (5,3),}$ $\mathbf{(6,2), (6,4)}\}$, $\{\mathbf{(x, y) : y = x^2} \; \& \; \mathbf{1 \le x \le 5}\},$ $\{\mathbf{(1,1), (1,2), (2,2), (1,3), (3,3), (1,4), (2,4), (4,4), (1,5), (5,5),}$ $\mathbf{(1,6), (2,6), (3,6), (6,6), (1,7), (7,7), (1,8), (2,8), (4,8), (8,8),}$ $\mathbf{(1,9), (3,9), (9,9)}\}$	

H	4	Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{2,3,4,5,6,9\}$, $B = \{1,4,5,6,7,8\}$. Find $A \cup B, A \cap B, A - B, B - A, A \,\Delta\, B, A', B'$. **Answer** : $\{1,2,\ldots,9\}, \{4,5,6\}, \{2,3,9\}, \{1,7,8\}, \{1,2,3,7,8,9\}, \{1,7,8,10\}$, $\{2,3,9,10\}$	
C	5	Let $A = \{4k+1 : k \in \mathbb{Z}\}$, $B = \{6k-1 : k \in \mathbb{Z}\}$. Find $A \cap B$. **Answer** : $A \cap B = \{12k+5 : k \in \mathbb{Z}\}$	
C	6	If $A = \{x : x \in \mathbb{R}, x^2 - 3x - 4 = 0\}$, $B = \{x : x \in \mathbb{Z}, x^2 = x\}$, then find $A \cup B, A \cap B$, $A - B, A\Delta B$. **Answer** : $\{-1,0,1,4\}, \{\emptyset\}, \{-1,4\}, \{-1,0,1,4\}$	
C	7	Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{1,3,5,7,9\}$, $B = \{1,5,6,8\}$, $C = \{1,4,6,7\}$. Verify (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, (b) $(A \cup B)' = A' \cap B'$, (c) $A - B = A \cap B'$, (d) $A\Delta B = B\Delta A$, (e) $A - C = A - (A \cap C)$.	
H	8	Let $U = \{1,2,3,4,5,6,7,8\}$, $A = \{1,2,3,4,5\}$, $B = \{1,3,5,6\}$, $C = \{1,2,3\}$. Verify (a) $(A - B) \cup B = A \cup B$, (b) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$, (c) $A - (B - C) = (A - B) \cup (A \cap C)$, (d) $A\Delta A = \phi$, (e) $A\Delta\phi = A$.	
C	9	Draw the Venn diagram for the following. $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{2,3,4,5,6,9\}$, $B = \{1,4,5,6,7,8\}$	
H	10	Draw the Venn diagram for the following. $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{1,3,5,7,9\}$, $B = \{1,5,6,8\}$, $C = \{1,4,6,7\}$	
H	11	Prove the following statements using Venn diagram. (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, (b) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$	
C	12	Prove the following statements using Venn diagram. (a) $(A \cup B)' = A' \cap B'$, (b) $A - (B \cup C) = (A - B) \cap (A - C)$	

❖ CARDINAL NUMBER

- ✓ The number of elements in a finite set A is called the cardinal number of set A and is denoted by $n(A)$ or $|A|$.
- ✓ Example: If $A = \{1,2,3,5,8\}$, then $n(A) = 5$.

❖ IMPORTANT RESULTS OF CARDINAL NUMBER

✓ Let A, B, C are finite sets in a finite universal set U. Then

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ (The Inclusion-Exclusion Principle)
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A \Delta B) = n(A) + n(B) - 2\, n(A \cap B)$
- $n(A') = n(U) - n(A)$
- $n(A' \cap B') = n(U) - n(A \cup B)$
- $n(A' \cup B') = n(U) - n(A \cap B)$

METHOD-2: EXAMPLES ON CARDINAL NUMBER

C	1	Let A and B be sets, such that $n(A) = 50, n(B) = 50, n(A \cup B) = 75$. Find $n(A \cap B)$. **Answer : 25**	
H	2	Let A and B be sets, such that $n(A) = 12, n(A \cup B) = 36, n(A \cap B) = 8$. Find $n(B)$. **Answer : 32**	
C	3	Let A and B are two subsets of universal set U, such that $n(A) = 20, n(B) = 30, n(U) = 80, n(A \cap B) = 10$. Find $n(A' \cap B')$. **Answer : 40**	
C	4	Among 50 patients admitted to a hospital, 25 are diagnosed with pneumonia, 30 with bronchitis, and 10 with both pneumonia and bronchitis. Determine (a) The number of patients diagnosed with pneumonia or bronchitis (or both), (b) The number of patients not diagnosed with pneumonia or bronchitis. **Answer : 45, 5**	

H	**5**	A survey in 1986 asked households whether they had a VCR, a CD player or cable TV. 40 had a VCR, 60 had a CD player and 50 had cable TV. 25 owned VCR and CD player. 30 owned a CD player and had cable TV. 35 owned a VCR and had cable TV. 10 households had all three. How many households had at least one of the three? **Answer : 70**	
C	**6**	Among 18 students in a room, 7 study mathematics, 10 study science and 10 study computer programming. Also, 3 study mathematics and science, 4 study mathematics and computer programming, and 5 study science and computer programming. We know that 1 student studies all three subjects. How many of these students study none of the three subjects? **Answer : 2**	

❖ POWER SET

- ✓ The set or family of all the subsets of a given set A is said to be the power set of A and is denoted by $P(A)$. Symbolically, $P(A) = \{X : X \subseteq A\}$.
- ✓ It means, if $X \in P(A) \Longrightarrow X \subseteq A$. Further, $\phi \in P(A)$ & $A \in P(A)$.
- ✓ Example: If $A = \{a, b, c\}$, then $P(A) = \{\phi, A, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$
- ✓ Note: If A has n elements, then its power set $P(A)$ has 2^n elements.

❖ CARTESIAN PRODUCT

- ✓ If A and B are any two non-empty sets, then cartesian product of A & B is defined and denoted as

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

- ✓ Example: If $A = \{a, b\}$ and $B = \{1, 2\}$, then cartesian product of A and B is written as

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

- ✓ Notes
  - ➢ If A & B are empty sets, then $A \times B = \phi$.
  - ➢ In general, $A \times B \neq B \times A$.

- If A has m elements and B has n elements, then $A \times B$ has $m \times n$ elements.
- Two dimensional cartesian plane is set of all ordered pair resulting from the product $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is a set of all real numbers.

METHOD-3: EXAMPLES ON POWER SET AND CARTESIAN PRODUCT

H	1	Give the power sets of $\{x, y, z\}$ and $\{1, 2, 3\}$. **Answer :** $\{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$, $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$	
C	2	Give the power sets of following. a) $A = \{x : x \text{ is multiple of } 4, x \in \mathbb{N} \text{ and } x \leq 16\}$ b) $B = \{x : x \text{ is a prime number and } x < 8\}$ **Answer :** $\{\emptyset, \{4\}, \{8\}, \{12\}, \{16\}, \{4, 8\}, \{4, 12\}, \{4, 16\}, \{8, 12\}, \{8, 16\}, \{12, 16\},$ $\{4, 8, 12\}, \{8, 12, 16\}, \{12, 16, 4\}, \{16, 4, 8\}, \{4, 8, 12, 16\}\}$, $\{\emptyset, \{2\}, \{3\}, \{5\}, \{7\}, \{2, 3\}, \{2, 5\}, \{2, 7\}, \{3, 5\}, \{3, 7\}, \{5, 7\}, \{2, 3, 5\},$ $\{3, 5, 7\}, \{5, 7, 2\}, \{7, 2, 3\}, \{2, 3, 5, 7\}\}$	
H	3	Let $A = \{a, b, c, d\}$ be a set. How many elements in P(A)? How many proper and improper subsets of A? **Answer :** **16, 14, 2**	
H	4	Let $A = \{a, b, c\}$ and $B = \{1, 2\}$ be two sets, then write $A \times B$. **Answer :** $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$	
H	5	If $A \times A = B \times B$, then prove that $A = B$.	
C	6	Let $A = \{\alpha, \beta\}$ and $B = \{1, 2, 3\}$. Then, write $B \times B, A \times A$ and $(A \times B) \cap (B \times A)$. **Answer :** $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$, $\{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}, \{\emptyset\}$	

❖ BASIC SET IDENTITIES

✓ Properties of the operation of union

- $A \cup B = B \cup A$ (Commutative law)

- $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)
- $A \cup \phi = A$ (Identity element)
- $A \cup A = A$ (Idempotent property)
- $U \cup A = U$ (Law of U)

✓ Properties of the operation of intersection

- $A \cap B = B \cap A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $A \cap \phi = \phi$
- $A \cap A = A$
- $U \cap A = A$

✓ Properties of the operation of difference, symmetric difference and complement

- $A - B \neq B - A$
- $(A')' = A$
- $U' = \phi$
- $\phi' = U$
- $A \Delta B = B \Delta A$
- $A \Delta (B \Delta C) = (A \Delta B) \Delta C$
- $A \Delta A = \phi$
- $A \Delta \phi = A$

✓ Some important results

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive law)
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive law)
- $(A \cup B) \cap A = A$
- $(A \cap B) \cup A = A$
- $(A \cup B) \cap B = B$
- $(A \cap B) \cup B = B$

- $A \cup A' = U$
- $A \cap A' = \phi$
- $(A \cup B)' = A' \cap B'$ (De Morgan's law)
- $(A \cap B)' = A' \cup B'$ (De Morgan's law)
- $A - (B \cup C) = (A - B) \cap (A - C)$ (De Morgan's law)
- $A - (B \cap C) = (A - B) \cup (A - C)$ (De Morgan's law)
- $A - B = A \cap B'$
- $B - A = B \cap A'$

✓ Some results on cartesian product

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B - C) = (A \times B) - (A \times C)$
- If $A \subseteq B$, then $(A \times C) \subseteq (B \times C)$

METHOD-4: EXAMPLES ON BASIC SET IDENTITIES

H	**1**	If $A = \{1, 3, 5, 7\}, B = \{2, 4, 7, 8\}, C = \{5, 6, 8, 9\}$, then check the identities. (a) $(A \cup B) \cup C = A \cup (B \cup C)$, (b) $A\Delta B = B\Delta A$, (c) $A - C \neq C - A$	
C	**2**	Let $A = \{1,2,3,6,7\}, B = \{2,3,4,8,9\}, C = \{3,4,5,10,11\}$, then check the identities. (a) $(A \cap B) \cap C = A \cap (B \cap C)$, (b) $A\Delta(B\Delta C) = (A\Delta B)\Delta C$, (c) $A\Delta A = \emptyset$	
H	**3**	If A={x : x is a divisor of 24}, B={x : x is a divisor of 18}, C={x : x is a divisor of 6}, then check the identities. a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ c) $A - (B \cap C) = (A - B) \cup (A - C)$ b) $A - (B \cup C) = (A - B) \cap (A - C)$	
C	**4**	Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, A = \{2, 3, 4, 5, 6, 9\}, B = \{1, 4, 5, 6, 7, 8\}, C = \{\emptyset\}$. Then check the identities. a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ c) $(A \cap B)' = A' \cup B'$ b) $(A \cup B)' = A' \cap B'$	

H	**5**	State and prove De Morgan's law.	
C	**6**	Do as directed. a) If $A \subset B$ & $C \subset D$, then prove that $(A \cup C) \subset (B \cup D)$. b) Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. c) Prove that $A - B = A \cap B'$.	
H	**7**	Do as directed. a) If $A \subset B$ & $C \subset D$, then prove that $(A \cap C) \subset (B \cap D)$. b) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. c) Prove that $B - A = B \cap A'$.	
H	**8**	Prove the following identities using Venn diagram. (a) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$, (b) $A - B = A - (A \cap B)$	
C	**9**	Prove the following identities using Venn diagram. (a) $A \cap (B - C) = (A \cap B) - (A \cap C)$, (b) $(A \cap B) \cup (A - B) = A$	
C	**10**	If $A \cap B = \emptyset, A \cup B = U$, then prove that $A' = B$.	

PART-II FUNCTIONS

❖ INTRODUCTION

- ✓ Function deals with linking pair of elements from two sets and then introduce relation between the two elements in the pair.
- ✓ Practically in every day of our lives, we pair the members of two sets of numbers.
- ✓ Examples
  - ➢ Each hour of the day is paired with the local temperature reading by T.V. Station's weatherman.
  - ➢ A teacher often pairs each set of score with the number of students receiving that score to see more clearly how well the class has understood the lesson.
- ✓ Finally, we shall learn about special relations called functions.
- ✓ The function is a special relation from one set to another set, in which every element of first set is in relation (uniquely) with the elements of another set.

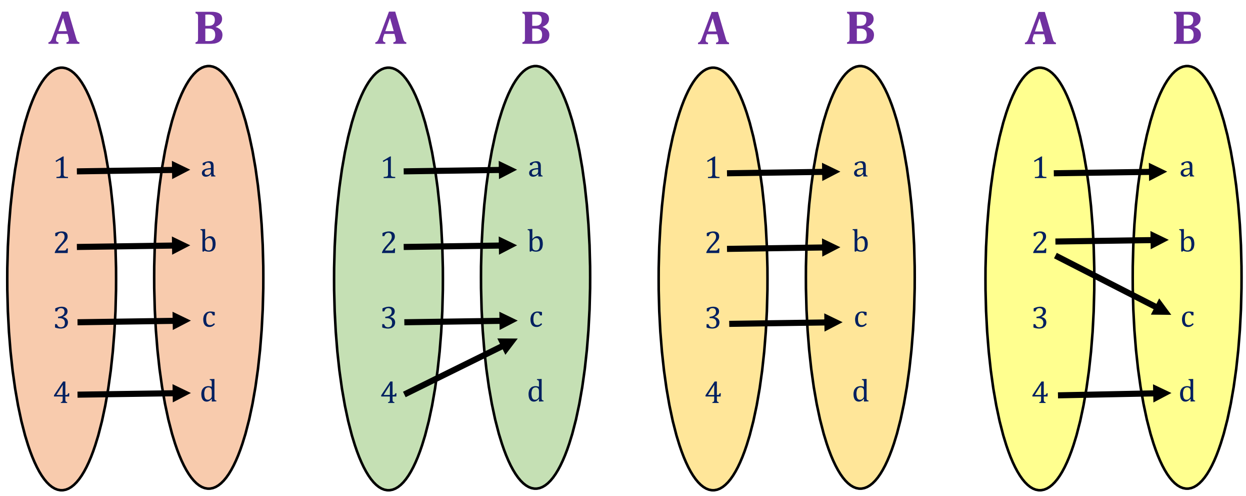


FIGURE (1) FIGURE (2) FIGURE (3) FIGURE (4)

- ✓ The relation, from A to B, shown in figure (1) and (2) are functions. But a relation, from A to B, shown in figure (3) is not a function because $4 \in A$ is not in relation to any element of B. Also a relation, from A to B, shown in figure (4) is not a function because $2 \in A$ is in relation with two elements of B. Now, we will define a function.

❖ FUNCTION

- ✓ Let A and B be two non-empty sets. Suppose that to each element of set A we assign unique element of set B, then collection of such assignments is called a function from A to B.
- ✓ Functions are ordinarily denoted by symbols. For example, let f denote a function from A to B. Then, we write $f : A \to B$.
- ✓ The set A is called domain of the function. (i. e. $D_f = A$)
- ✓ The set B is called co-domain of the function.
- ✓ Example: Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e\}$, then
  - ➢ $f = \{(1, a), (2, b), (3, c)\}$ is function from A to B.$[f : A \to B, f(1) = a, f(2) = b, f(3) = c]$.
  - ➢ $f = \{(1, a), (2, b), (3, b)\}$ is function from A to B.$[f : A \to B, f(1) = a, f(2) = b, f(3) = b]$.
  - ➢ $f = \{(1, a), (2, a), (3, a)\}$ is function from A to B.$[f : A \to B, f(1) = a, f(2) = a, f(3) = a]$.
- ✓ Notes
  - ➢ If $n(A) = m$ and $n(B) = n$, then we can create n^m different functions from A to B.
  - ➢ Any function from $\mathbb{R}$ to $\mathbb{R}$ is called a real function.
- ✓ A program written in a high-level language is mapped into a machine language by a compiler. Similarly, the output from a computer is a function of its input.

❖ RANGE OF A FUNCTION

- ✓ The set of images of all elements of a domain is called range of a given function.
- ✓ Example: Let $f : N \to N, f(x) = 7x$ i.e. $f(1) = 7, f(2) = 14, f(3) = 21, \dots \dots$
  - ➢ Here, 7 is an image of 1, 14 is an image of 2, … etc.
  - ➢ In above example range of f is

$$R_f = \{7, 14, 21, 28, \dots\}$$

❖ SOME STANDARD FUNCTIONS

- ✓ Identity function
  - ➢ Let A be a non-empty set. The function $f : A \to A$ define by $f(x) = x, \forall x \in A$ is called an identity function on A. Also, an identity function on A is denoted by I_A.

- This function maps any element of A onto itself. For this function, the range is entire co-domain.

✓ Constant function

- A function whose range is a singleton set is called a constant function.
- Thus, a function $f : A \to B, f(x) = c, \forall x \in A$ where c is a fixed element of B, is called a constant function.

✓ Even function

- A function $f : A \to B$ is said to be an even function if $f(-x) = f(x), \forall x \in A$.
- $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ is an even function.

✓ Odd function

- A function $f : A \to B$ is said to be an odd function if $f(-x) = -f(x), \forall x \in A$.
- $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ is an odd function.

✓ Modulus function

- A function $f : \mathbb{R} \to \mathbb{R}, f(x) = |x|$ is called a modulus function or absolute value function defined as,

$$|x| = \begin{cases} x\,; & x \geq 0 \\ -x\,; & x < 0 \end{cases}$$

- The range of this function is $\mathbb{R}^+ \cup \{0\}$.

✓ Peano's successor function

- A function $f : \mathbb{N} \to \mathbb{N}, f(n) = n + 1, \forall n \in \mathbb{N}$ is called Peano's successor function.
- Obviously, $f(1) = 2, f(2) = 3, f(3) = 4, \dots \dots$ then, the range of this function is $\mathbb{N} - \{1\}$.

✓ Greatest integer function (Floor function)

- A function $f : \mathbb{R} \to \mathbb{R}, f(x) = [x]$ or $\lfloor x \rfloor, \forall x \in \mathbb{R}$ is called a greatest integer function defined as,

$$[x] \text{ or } \lfloor x \rfloor = \text{greatest integer not exceeding } x$$

- $\lfloor 1.2 \rfloor = 1, \lfloor 2 \rfloor = 2, \lfloor -1.7 \rfloor = -2, \dots \dots$ etc.
- The range of this function is $\mathbb{Z}$.

- ✓ Least integer function (Ceiling function)
  - ➢ A function $f : \mathbb{R} \to \mathbb{R}, f(x) = \lceil x \rceil, \forall x \in \mathbb{R}$ is called a ceiling function defined as,

$$\lceil x \rceil = \text{least integer not less than } x$$

  - ➢ $\lceil 1.2 \rceil = 2, \lceil 2 \rceil = 2, \lceil -1.7 \rceil = -1, \dots \dots$ etc.
  - ➢ The range of this function is $\mathbb{Z}$.

❖ ALGEBRA OF REAL FUNCTIONS

- ✓ Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are two functions with $A \cap B \neq \phi$. Then,
  - ➢ The addition of functions is defined as

$$(f+g) : (A \cap B) \to \mathbb{R}, (f+g)(x) = f(x) + g(x), \forall x \in A \cap B$$

  - ➢ The subtraction of functions is defined as

$$(f-g) : (A \cap B) \to \mathbb{R}, (f-g)(x) = f(x) - g(x), \forall x \in A \cap B$$

  - ➢ The multiplication of functions is defined as

$$(f \cdot g) : (A \cap B) \to \mathbb{R}, (f \cdot g)(x) = f(x) \cdot g(x), \forall x \in A \cap B$$

  - ➢ The division (quotient) of functions is defined as

$$\left(\frac{f}{g}\right) : (A \cap B) - \{x : g(x) = 0\} \to \mathbb{R}, \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \forall x \in (A \cap B) - \{x : g(x) = 0\}$$

METHOD-5: BASIC EXAMPLES ON FUNCTION

H	1	Define: Function, Range of a function.	
H	2	Define: Identity function, Constant function, Even and Odd function, Modulus function, Peano's successor function, Greatest integer function(Floor function), Least integer function(Ceiling function).	

C	3	Which of the following relation is a function? If yes, determine domain and range. a) $\{(2,1),(5,1),(8,1),(11,1),(14,1),(17,1)\}$ b) $\{(2,1),(4,2),(6,3),(8,4),(10,5),(12,6),(14,7)\}$ c) $\{(1,3),(1,5),(2,5)\}$ **Answer :** $\mathbf{yes, \{2,5,8,11,14,17\}, \{1\},}$ $\mathbf{yes, \{2,4,6,8,10,12,14\}, \{1,2,3,4,5,6,7\}, no}$	
H	4	Which of the following relation is a function? If yes, determine domain and range. a) $\{(1,1),(3,1),(5,1),(7,1),(9,1)\}$ b) $\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7)\}$ c) $\{(2,3),(2,5),(3,5)\}$ **Answer :** $\mathbf{yes, \{1,3,5,7,9\}, \{1\}, yes, \{1,2,3,4,5,6,7\}, \{1,2,3,4,5,6,7\}, no}$	
H	5	Let $f(x) = x^2$ and $g(x) = 2x + 1$ be the real functions, then find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, $\left(\frac{f}{g}\right)(x)$. **Answer :** $\mathbf{x^2 + 2x + 1, x^2 - 2x - 1, x^2 \cdot (2x+1), \frac{x^2}{2x+1}}$	
C	6	Let $f(x) = \sqrt{x}$ and $g(x) = x$ be the real functions, then find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, $\left(\frac{f}{g}\right)(x)$. **Answer :** $\mathbf{\sqrt{x} + x, \sqrt{x} - x, \sqrt{x} \cdot (x), \frac{\sqrt{x}}{x}}$	
C	7	Find the domain and range of functions (a) $f(x) = \|x-1\|$, (b) $f(x) = \sqrt{9 - x^2}$. **Answer :** $\mathbb{R}, \mathbf{[0,\infty), [-3,3], [0,3]}$	
H	8	Find the domain and range of functions (a) $f(x) = \|x-2\|$, (b) $f(x) = \sqrt{4 - x^2}$. **Answer :** $\mathbb{R}, \mathbf{[0,\infty), [-2,2], [0,2]}$	
C	9	Draw the graph of functions (a) $f(x) = \|x-1\|$, (b) $f(x) = x^2$, (c) $f(x) = \sin x$.	
H	10	Draw the graph of functions (a) $f(x) = \log x$, (b) $f(x) = x^2 - 1$, (c) $f(x) = 3x + 1$.	
H	11	If $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2x + 1$, then find $f(1), f(2), f(-2), f(0)$. **Answer :** $\mathbf{0, 1, 9, 1}$	

C	**12**	If $f : \mathbb{R} \to \mathbb{R}, f(x) = 2\lvert x\rvert - \lvert -x\rvert$, then find $f(3), f\left(\frac{1}{2}\right), f(-3)$. **Answer :** $\mathbf{3, \frac{1}{2}, 3}$	

❖ INJECTIVE FUNCTION (ONE TO ONE FUNCTION)

- ✓ A function $f : A \to B$ is called a one to one (injective) function if distinct elements of A are mapped into distinct elements of B. In other words, f is one to one if $\forall x_1, x_2 \in A$

$$x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2) \quad \text{or} \quad f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$$

- ✓ Notes
  - ➢ If a given function is not a one to one function, then it is called a many to one function.

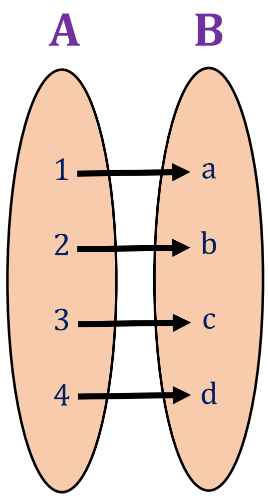


FIGURE (1)

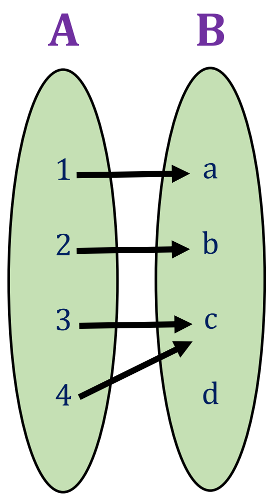


FIGURE (2)

  - ➢ A function, shown in figure (1), is a one to one function. But a function, shown in figure (2), is not a one to one function.
- ✓ Example: Let $A = \{1, 2, 3\}, B = \{1, 2, 3, 4, 5, 6\}$. Then,
  - ➢ $f : A \to B, f = \{(1, 2), (2, 4), (3, 6)\}$ is a one to one function.
  - ➢ $f : A \to B, f = \{(1, 2), (2, 2), (3, 6)\}$ is not a one to one function. $[f(1) = 2 \;\&\; f(2) = 2]$
  - ➢ $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + 3$ is a one to one function.
  - ➢ $f : \mathbb{R} \to [-1, 1], f(x) = \sin x$ is not a one to one function. $[f(0) = 0 \;\&\; f(\pi) = 0]$

❖ SURJECTIVE FUNCTION (ONTO FUNCTION)

- ✓ A function $f : A \to B$ is called an onto (surjective) if $R_f = B$(Range of f = Co-domain of f).
- ✓ Notes
  - ➢ If a given function is not an onto function, then it is called an into function.

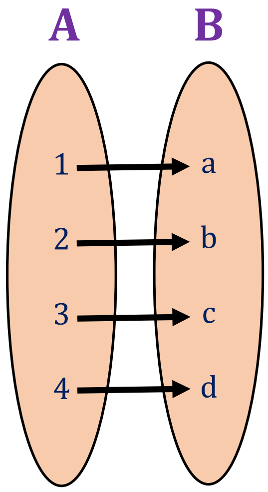


FIGURE (1)

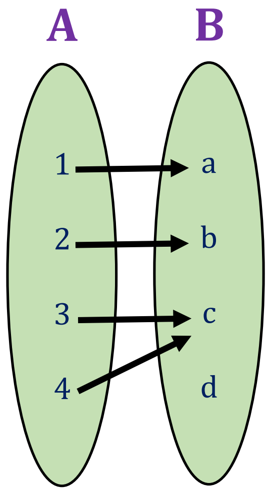


FIGURE (2)

  - ➢ A function, shown in figure (1), is an onto function. But a function, shown in figure (2), is not an onto function.
- ✓ Example: Let $A = \{1, 2, 3\}$, $B = \{1, 4, 9\}$. Then,
  - ➢ $f : A \to B, f = \{(1, 1), (2, 4), (3, 9)\}$ is an onto function.
  - ➢ $f : A \to B, f = \{(1, 4), (2, 9), (3, 4)\}$ is not an onto function. $[R_f = \{4, 9\} \neq B]$
  - ➢ $f : \mathbb{N} \to \mathbb{N}, f(n) = n + 2$ is not an onto function. $[R_f = \{3, 4, 5, 6, \dots\} \neq \mathbb{N}$(co-domain)$]$

❖ BIJECTIVE FUNCTION

- ✓ A function is said to be bijective function if it is one to one and onto both.

❖ COMPOSITION OF TWO FUNCTIONS

- ✓ Consider functions $f : A \to B$ and $g : C \to D$. Then, we may define $g \circ f : A \to D$, called the composition of f & g, as follows.

$$(g \circ f)(x) = g\big(f(x)\big), x \in A, \text{ where } B \subseteq C$$

- ✓ That is, we find the image of x under f and then find the image of $f(x)$ under g.
- ✓ Note: Similarly, we can define $f \circ g$.
- ✓ Example: Let $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 2$ and $g(x) = x - 2$, then

$$(g \circ f)(x) = g(f(x)) = g(x+2) = (x+2) - 2 = x$$

$$(f \circ g)(x) = f(g(x)) = f(x-2) = (x-2) + 2 = x$$

❖ PROPERTIES OF COMPOSITION OF FUNCTIONS

- ✓ Let $f : A \rightarrow B$, then $f \circ I_A = f = I_B \circ f$.
- ✓ Let the function $f : A \rightarrow B$ and $g : B \rightarrow C$ satisfy $g \circ f = I_A$ & $f \circ g = I_B$ then g is unique.
- ✓ In general, $f \circ g \neq g \circ f$.
- ✓ Let the function $f : A \rightarrow B$ and $g : B \rightarrow C$, then
  - ➢ If f and g are one to one functions, then $g \circ f$ is also one to one function.
  - ➢ If f and g are onto functions, then $g \circ f$ is also onto function.
- ✓ In usual notation, $(g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x)$.

❖ INVERSE OF A FUNCTION

- ✓ Let $f : A \rightarrow B$ be a function and if there exist a function $g : B \rightarrow A$ such that $g \circ f = I_A$ and $f \circ g = I_B$, then we say g is the inverse function of f and g is denoted by f^{-1}.
- ✓ Note: A function f has the inverse function g if and only if f is one to one and onto both.
- ✓ Examples
  - ➢ Let $A = \{1,2,3\}$, $B = \{1,4,9\}$ and $f : A \rightarrow B$, $f = \{(1,1),(2,4),(3,9)\}$, then

$$f^{-1} : B \rightarrow A, f^{-1} = \{(1,1),(4,2),(9,3)\}$$

  - ➢ Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x + 2$. For the inverse of f, let $y = x + 2$.

$$\Longrightarrow x = y - 2 \Longrightarrow f^{-1}(x) = x - 2$$

METHOD-6: EXAMPLES ON INVERSE AND COMPOSITION OF FUNCTIONS

H	1	Define: Injective function, surjective function, inverse function, bijective function.	
H	2	Let $f : \{2,3,4,5\} \rightarrow \{3,4,5,9\}$ and $g : \{3,4,5,9\} \rightarrow \{7,11,15\}$ be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$, $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find $g \circ f$. **Answer** : $\mathbf{g \circ f(x) = \{(2,7),(3,7),(4,11),(5,11)\}}$	

H	3	If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$, then show that $g \circ f \neq f \circ g$.	
C	4	Let $X = \{1, 2, 3\}$ and f, g, h and s be functions from X to X as given below. $f = \{(1,2),(2,3),(3,1)\}$ $g = \{(1,2),(2,1),(3,3)\}$ $h = \{(1,1),(2,2),(3,1)\}$ $s = \{(1,1),(2,2),(3,3)\}$ Then find $f \circ g, g \circ f, (g \circ f)^{-1}, f \circ h \circ g, s \circ g, g \circ s, s \circ s$, and $f \circ s$. **Answer :** $\{(1,3),(2,2),(3,1)\}, \{(1,1),(2,3),(3,2)\}, \{(1,1),(2,3),(3,2)\}$ $\{(1,3),(2,2),(3,2)\}, \{(1,2),(2,1),(3,3)\}, \{(1,2),(2,1),(3,3)\}$ $\{(1,1),(2,2),(3,3)\}, \{(1,2),(2,3),(3,1)\}$	
C	5	Let $f(x) = x + 2, g(x) = x - 2, h(x) = 3x$ for $x \in \mathbb{R}$. Find $g \circ f, f \circ g, (f \circ g)^{-1}, f \circ f, g \circ g, f \circ h, h \circ g, h \circ f$, and $f \circ h \circ g$. **Answer :** $\mathbf{x, x, x, x+4, x-4, 3x+2, 3x-6, 3x+6, 3x-4}$	
C	6	Let $f : \mathbb{R} \to \mathbb{R}, f(x) = -x^2$ and $g : \mathbb{R}^+ \to \mathbb{R}^+, g(x) = \sqrt{x}$. Find $f \circ g$ and $g \circ f$. **Answer :** $\mathbf{f \circ g(x) = -x}$ & $\mathbf{g \circ f(x)}$ **is not possible**	
H	7	Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 2$ and $g : \mathbb{R} \to \mathbb{R}, g(x) = x + 4$. Find $f \circ g$ and $g \circ f$. **Answer :** $\mathbf{f \circ g(x) = x^2 + 8x + 14}$ & $\mathbf{g \circ f(x) = x^2 + 2}$	
C	8	Are the following functions one to one and onto? If yes, find its inverse. a) $f : \mathbb{R} \to \mathbb{R}, f(x) = ax + b$ b) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 2$ c) $f : \mathbb{R} - \left\{-\frac{3}{2}\right\} \to \mathbb{R} - \left\{\frac{3}{2}\right\}, f(x) = \frac{3x+2}{2x+3}$ **Answer :** **yes,** $\mathbf{\frac{x-b}{a}}$**, yes,** $\mathbf{(x+2)^{\frac{1}{3}}}$**, yes,** $\mathbf{\frac{2-3x}{2x-3}}$	
H	9	Are the following functions one to one and onto? If yes, find its inverse. a) $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + 3$ b) $f : \mathbb{R}^+ \to \mathbb{R}^+, f(x) = x^2$ c) $f : \mathbb{R} - \{-1\} \to \mathbb{R} - \{-1\}, f(x) = \frac{1-x}{1+x}$ **Answer :** **yes,** $\mathbf{\frac{x-3}{2}}$**, yes,** $\mathbf{\sqrt{x}}$**, yes,** $\mathbf{\frac{1-x}{1+x}}$	

PART-III COUNTING

❖ INTRODUCTION

- ✓ Suppose you have a suitcase with a number lock. The number lock has 4 wheels each labelled with 10 digits from 0 to 9. The lock can be opened if 4 specific digits are arranged in a particular sequence with no repetition.
- ✓ Somehow, you have forgotten this specific sequence of digits. You remember only the first digit which is 7. In order to open the lock, how many sequences of 3-digits you may have to check with? To answer this question, you may, immediately, start listing all possible arrangements of 9 remaining digits taken 3 at a time.
- ✓ But, this method will be tedious, because the number of possible sequences may be large. Here, in this section, we shall learn some basic counting techniques which will enable us to answer this question without actually listing 3-digit arrangements.
- ✓ In fact, these techniques will be useful in determining the number of different ways of arranging and selecting objects without actually listing them. As a first step, we shall examine a principle which is most fundamental to the learning of counting techniques.

❖ FUNDAMENTAL PRINCIPLES OF COUNTING

- ✓ Fundamental principle of multiplication
  - ➢ Let there are two parts A and B of a certain event. The part A can be done in m different ways. If corresponding to each way of doing the part A of the event, the part B can be done in n ways, then there are $m \times n$ ways to complete the event.
- ✓ Example: How many two digits even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

  Solution: Here, the unit's place can be filled by 2 and 4(two different ways) and the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by multiplication principle, the required number of two digits even numbers is $5 \times 2 = 10$.
- ✓ Fundamental principle of addition
  - ➢ Suppose that A and B are two disjoint events (i.e. they never occur together). Further suppose that A occur in m ways and B in n ways. Then, A and B can occur in $m + n$ ways.

❖ PIGEONHOLE PRINCIPLE

- ✓ Statement: If n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

  Proof: We will give a proof by contraposition.

  - ➢ Suppose that none of the m containers has more than one item. Then, total number of items would be at most m. Which contradicts with our hypothesis that there are n items and $n > m$ (i.e. we have at least $n = m + 1$ items). Hence, we proved.
- ✓ This principle is also called Dirichlet's box principle or Dirichlet's drawer principle.
- ✓ A function f from a set with n+1 elements to a set with n elements is not one to one.
- ✓ Example: In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

❖ GENERALIZED PIGEONHOLE PRINCIPLE

- ✓ Statement: If N objects are placed into K boxes, then there is at least one box containing at least $\left\lceil \frac{N}{K} \right\rceil$ objects.
- ✓ Example: Given a group of 100 people, at minimum, how many people were born in the same month?

  Answer: $\left\lceil \frac{N}{K} \right\rceil = \left\lceil \frac{100}{12} \right\rceil = \lceil 8.33 \rceil = 9$
- ✓ Note: The principle just proves the existence of overlaps. It says nothing of the number of overlaps (which falls under the subject of probability distribution).

❖ FACTORIAL

- ✓ The continued product of first n natural numbers is called 'n factorial' and is denoted by n!.

$$\text{i.e. } n! = 1 \times 2 \times 3 \times \dots \dots \times (n-1) \times n$$

- ✓ Example: $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$
- ✓ Notes
  - ➢ $n! = n \times (n-1)!$
  - ➢ $0! = 1$

❖ PERMUTATION

- ✓ Each of the arrangements which can be made by taking some or all of a number of things is called a permutation.
- ✓ Example: Six arrangements can be made with three distinct objects a, b, c taking two at a time are

$$ab, ba, bc, cb, ac, ca$$

  - ➢ Each of these arrangements is called a permutation.
- ✓ Note: It should be noted that in permutation the order of arrangement is taken into account, when the order is changed, a different permutation is obtained.
- ✓ The number of all permutations of n distinct things, taken r at a time is denoted by the symbol $P(n, r)$.

❖ DERIVATION OF THE FORMULA FOR $P(n, r)$

- ✓ Suppose that we are given 'n' distinct objects and wish to arrange 'r' of these objects in a line where $1 \leq r \leq n$.
- ✓ Since there are 'n' ways of choosing the 1^{st} object, after this is done '$n - 1$' ways of choosing the 2^{nd} object and finally $n - (r - 1)$ ways of choosing the r^{th} object.
- ✓ It follows by the fundamental principle of counting that the number of different arrangements (permutations) is given by

$$P(n, r) = n\,(n-1)\,(n-2) \ldots \ldots \ldots (n-r+1)$$

$$= \frac{n\,(n-1)\,(n-2) \ldots \ldots \ldots (n-r+1)(n-r)(n-r-1) \ldots \ldots \ldots 3.2.1}{(n-r)(n-r-1) \ldots \ldots \ldots 3.2.1}$$

$$= \frac{n!}{(n-r)!}$$

- ✓ Notes
  - ➢ $P(n, n) = n!$
  - ➢ $P(n, 0) = 1$
- ✓ Suppose that a set consists of 'n' objects of which n_1 are of one type, n_2 are of second type, $\ldots, n_k$ are of k^{th} type. Here $n = n_1 + n_2 + \cdots + n_k$. Then the number of different permutations of the objects is

$$\frac{n!}{n_1!\ n_2!\ \dots\ n_k!}$$

- ✓ Example: A number of different permutations of the letters of the word JISSISSITTI is

$$\frac{11!}{1!\ \ 4!\ \ \ 4!\ \ 2!} = 34650$$

- ✓ If ‘r’ objects are to be arranged out of ‘n’ objects and if repetition of an object is allowed then the total number of permutations is n^r.
- ✓ Example: Different numbers of three digits can be formed from the digits 1, 2, 3, 4, 5 is

$$5^3 = 125$$

- ✓ The number of ways to arrange n distinct objects along a fixed circle (i.e. cannot be picked up out of the plane and turned over) is $(n-1)!$.

❖ GENERATING PERMUTATION

- ✓ Any set with n elements can be placed in one to one correspondence with the set $\{1, 2, 3, \dots, n\}$. We can list the permutations of any set of n elements by generating the permutations of the n smallest positive integers and then replacing these integers with the corresponding elements.
- ✓ Many different algorithms have been developed to generate the n! permutations of this set. We will describe one of these that is based on the lexicographic (or dictionary) ordering of the set of permutations of $\{1, 2, 3, \dots, n\}$.

❖ PROCEDURE TO OBTAIN NEXT PERMUTATION (LEXICOGRAPHIC ORDERING)

- ✓ An algorithm for generating the permutations of $\{1, 2, \dots, n\}$ can be based on a procedure that constructs the next permutation in lexicographic order following a given permutation $a_1 a_2 \cdots a_n$. We will show how this can be done. First, suppose that $a_{n-1} < a_n$. Interchange a_{n-1} and a_n to obtain a larger permutation. No other permutation is both larger than the original per-mutation and smaller than the permutation obtained by interchanging a_{n-1} & a_n. For instance, the next larger permutation after 234156 is 234165.
- ✓ On the other hand, if $a_{n-I} > a_n$, then a larger permutation cannot be obtained by interchanging these last two terms in the permutation. Look at the last three integers in the permutation. If $a_{n-2} < a_{n-1}$, then the last three integers in the permutation can be rearranged to obtain the next largest permutation. Put the smaller of the two integers a_{n-1}

& a_n that is greater than a_{n-2} in position $n-2$. Then, place the remaining integer and a_{n-2} into the last two positions in increasing order. For instance, the next larger permutation after 234165 is 234516. And so on.

✓ Example: What is the next permutation in Lexicographic order after 362541?

Solution: 364125.

METHOD-7: EXAMPLES ON PERMUTATION

H	1	Explain the pigeonhole principle.	
C	2	Evaluate ${}^5P_2,\ {}^7P_3,\ {}^9P_4$. **Answer : 20, 210, 3024**	
H	3	Find n, if (a) ${}^nP_5 = 42 \cdot {}^nP_3$, (b) $3 \cdot P(n,4) = 5 \cdot P(n-1,4)$. **Answer : 10, 10**	
C	4	Find n, if (a) ${}^nP_4 = 42 \cdot {}^nP_2$, (b) $2 \cdot P(n,2) + 50 = P(2n,2)$. **Answer : 9, 5**	
C	5	Find r, if $5 \cdot {}^4P_r = 6 \cdot {}^5P_{r-1}$. **Answer : 3 or 8**	
H	6	Suppose repetition are not permitted. a) How many three digit numbers can be formed from six digits 2, 3, 5, 6, 7, 9? b) How many of these numbers are less than 400? c) How many are even? **Answer : 120, 40, 40**	
H	7	Find the number of ways that a party of seven persons can arrange themselves a) in a row of seven chairs. b) around a circular table. **Answer : 5040, 720**	
C	8	Find the number of distinct permutation that can be formed from all the letters of (a) RADAR, (b) UNUSUAL. **Answer : 30, 840**	

C	9	Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements, a) Do the words start with P? b) Do all the vowels always occur together? c) Do the vowels never occur together? d) Do the words begin with I and end in P? **Answer : 1663200, 138600, 16800, 1646400, 12600**	
C	10	In a certain programming language, the variable should be length three and should be made up of two letters followed by a digit or of length two made up of a letter followed by a digit. How many variables can be formed? What if letters are not to be repeated? **Answer : 7020, 6760**	
H	11	6 boys and 6 girls are to be seated in a raw, how many ways can they be seated if a) All boys are to be seated together and all girls are to be seated together? b) No two girls should be seated together? c) Boys occupy extreme position? **Answer : 1036800, 6220800, 108864000**	
H	12	How many ways can the letters in the word MISSISSIPPI can be arranged? What if P's are to be separated? **Answer : 34650, 28350**	
C	13	If repetitions are not allowed, how many four digit numbers can be formed from 1, 2, 3, 7, 8, 5? How many of these numbers are less than 5000? How many of these are even? How many of these are odd? How many of these containing 3 and 5? **Answer : 360, 180, 120, 240, 144**	
H	14	A word that reads the same when read in forward or backward is called as palindrome. How many seven letter palindromes can be form English alphabets? **Answer : 456976**	
H	15	Generate the permutations of the integers 1, 2, 3 in lexicographic order. **Answer : 123, 132, 213, 231, 312, 321**	

❖ COMBINATION

- ✓ In a permutation we are interested in the order of arrangement of the objects. For example, ABC is a different permutation from BCA. In many problems, however, we are interested only in selecting or choosing objects without regard to order. Such selections are called combination.

- ✓ The total number of combination (selections) of 'r' objects selected from 'n' objects is denoted and defined by

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

- ✓ Notes
  - ➢ $\binom{n}{0} = \binom{n}{n} = 1$
  - ➢ $\binom{n}{1} = n$
  - ➢ $\binom{n}{r} = \frac{1}{r!} \cdot {}^{n}P_{r}$
  - ➢ $\binom{n}{r} = \frac{n}{r} \times \binom{n-1}{r-1}$

- ✓ The number of r-combinations with repetition allowed (multi sets of size r) that can be selected from a set of n elements is $\binom{n+r-1}{r}$.

- ✓ Examples
  - ➢ The number of ways in which 3 card can be chosen from 8 cards is

$$\binom{8}{3} = \frac{8!}{3!\,(8-3)!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

  - ➢ A club has 10 male and 8 female members. A committee composed of 3 men and 4 women is formed. In how many ways this be done?

$$\binom{10}{3} \times \binom{8}{4} = 120 \times 70 = 8400$$

  - ➢ Out of 6 boys and 4 girls in how many ways a committee of five members can be formed in which there are at most 2 girls are included?

$$\binom{4}{2} \times \binom{6}{3} + \binom{4}{1} \times \binom{6}{4} + \binom{4}{0} \times \binom{6}{5} = 120 + 60 + 6 = 186$$

❖ GENERATING COMBINATION

- ✓ An algorithm for generating the r-combinations of the set $\{1, 2, 3, \dots, n\}$ will be given. An r-combination can be represented by a sequence containing the elements in the subset in increasing order. The r-combinations can be listed using lexicographic order on these sequences. The next combinations after $a_1 a_2 \dots a_r$ can be obtained in the following way.
    - ➢ First, locate the last element a_i in a sequence such that $a_i \neq n - r + i$.
    - ➢ Then, replace a_i with $a_i + 1$ and a_j with $a_i + j - i + 1$, for $j = i + 1, i + 2, \dots, r$.
    - ➢ Now, it is left to show that this produces the next larger combination in lexicographic order.
- ✓ Example: Find the next larger 4-combination of the set $\{1, 2, 3, 4, 5, 6\}$ after $\{1, 2, 5, 6\}$.

  Solution: $\{1, 3, 4, 5\}$.

❖ BINOMIAL COEFFICIENT

$$(a+b)^1 = a + b = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1$$

$$(a+b)^2 = a^2 + 2ab + b^2 = \binom{2}{0} a^2 b^0 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^0 b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = \binom{3}{0} a^3 b^0 + \binom{3}{1} a^2 b^1 + \binom{3}{2} a^1 b^2 + \binom{3}{3} a^0 b^3$$

$$\dots \dots \dots \dots \dots$$

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n} a^0 b^n = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^k$$

- ✓ Above expression is called binomial theorem, where n is positive integer and expression $\binom{n}{k}$ is often called binomial coefficient.

❖ PROPERTIES OF BINOMIAL COEFFICIENT

- ✓ $\binom{n}{k} = \binom{n}{n-k}$
- ✓ $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$
- ✓ If $\binom{n}{x} = \binom{n}{y}$ then either $x = y$ or $x + y = n$.
- ✓ $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ and $\sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} = 0$

❖ ARITHMETIC PROGRESSION

✓ Members a, b and c are said to be in arithmetic progression if

$$b = \frac{a + c}{2}$$

METHOD-8: EXAMPLES ON COMBINATION

H	1	Show that ${}^{14}C_4, {}^{14}C_5, {}^{14}C_6$ are in A.P.	
C	2	Prove that (a) ${}^{n}C_r = {}^{n}C_{n-r}$, (b) ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$	
C	3	Find the value of n if $C(n, 5) = C(n, 13)$ and hence find $C(n, 2)$. **Answer : 18, 153**	
H	4	Find the value of n if $\binom{2n}{3} = 11\binom{n}{3}$ and hence find $\binom{n}{2}$. **Answer : 6, 15**	
C	5	Find the value of r if (a) $\binom{8}{r} = 28$, (b) $\binom{12}{r} = \binom{12}{r+2}$. **Answer : 2 or 6, 5**	
C	6	If $C(n-1, 4), C(n-1, 5), C(n-1, 6)$ are in A.P., then find n. **Answer : 15 or 8**	
H	7	Show that $C(2n, 2) = 2 \cdot C(n, 2) + n^2$.	
H	8	In how many ways can a committee consisting of three men and two women be chosen from seven men and five women? **Answer : 350**	
H	9	A bag contains 6 white, 5 red marbles. Find the number of ways four marbles can be drawn from the bag if (a) they can be any color, (b) two must be white and two red, (c) they must all be of the same color. **Answer : 330, 150, 20**	

C	10	Out of 12 employees, group of four trainees is to be sent for “Software testing and QA” training of one month. (a) In how many ways can the four employees be selected? (b) What if there are two employees who refuse to go together? (c) What if there are two employees who want to go together? (d) What if there are two employees who want to go together and there are two employees who refuse to go together? **Answer : 495, 450, 255, 226**	
H	11	There are 2 white, 3 red, 4 green marbles. Three marbles are drawn. In how many ways can this be done so that at least one red marble is selected? **Answer : 64**	
H	12	3 cards are chosen from a pack of 52 cards. In how many a) ways can this be done? b) ways can you select three cards so that all of them are face cards? c) of the selections, all cards are of the same color? d) of them all cards are of the same suit? **Answer : 22100, 220, 5200, 1144**	
H	13	A reception committee consisting of 6 students for the annual function of a college is to be formed from 8 boys and 5 girls. In how many ways can we do it if the committee is to contain (a) exactly four girls, (b) at most two girls, (c) at least three girls? **Answer : 140, 1008, 708**	

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UNIT-2 » PROPOSITIONAL LOGIC AND PREDICATE LOGIC

PART-I PROPOSITIONAL LOGIC

❖ INTRODUCTION

- ✓ One of the main aim of logic is to provide rules by which one can determine whether any particular argument or reasoning is valid (correct).
- ✓ Logic is concerned with all kinds of reasoning, whether they be legal arguments or mathematical proofs or conclusions in a scientific theory based upon a set of hypothesis.
- ✓ Because of the diversity of their application, these rules, called rules of inference, must be stated in general terms and must be independent of any particular argument or discipline involved.

❖ STATEMENTS AND NOTIFICATION

- ✓ In this section we introduce certain basic units of our object language called primary (primitive or atomic) statements.
- ✓ We begin by assuming that the object language contains a set of declarative sentences which cannot be further broken down or analyzed into simpler sentences. These are primary statements.
- ✓ Only those declarative sentences will be admitted in the object language which have one and only one of two values called “truth values”.
- ✓ The two truth values are true and false denoted by T and F respectively. Occasionally, they are also denoted by the symbols 1 and 0.
- ✓ Declarative sentences in the object language are of two types. The first type includes those sentences which are considered to be primitive in the object language. These will be denoted by distinct symbols selected from the capital letters A, B, C, …, P, Q, …, while declarative sentences of the second type are obtained from the primitive ones by using certain symbols, called connectives, and certain punctuation marks, such as parentheses, to join primitive sentences.
- ✓ In any case, all the declarative sentences to which it is possible to assign one and only one of the two possible truth values are called statements. These statements which do not contain any of the connectives are called atomic (primary, primitive) statements.

- ✓ Examples
  - ➢ Canada is a country. (This statement has truth value true.)
  - ➢ Moscow is the capital of Spain. (This statement has truth value false.)
  - ➢ This statement is false. (This sentence has no truth value. So, it is not a statement.)
  - ➢ 1+101=110. (Its truth value depends upon context. In decimal system it is false. But, in binary it is true.)
  - ➢ Close the door. (This is not a statement because it is a command.)
  - ➢ Toronto is an old city. (This statement may have different truth values in the world.)

❖ CONNECTIVES

- ✓ It is possible to construct complicated statements from simple statements by using certain connecting words or expressions known as “sentential connectives”.
- ✓ The statements that we consider initially are simple statements, called atomic or primary (simple) statements. As already indicated, new statements can be formed from atomic statements through the use of sentential connectives. This resulting statements are called molecular or compound (composite) statements. Thus, the atomic statements are those which do not have any connectives.
- ✓ NEGATION: The negation of statement is generally formed by introducing the word “not” at a proper place in the statement or by prefixing the statement with the phrase “it is not the case that”. If “P” denotes a statement, then the negation of “P” is written as “$\urcorner P$” or $\sim P$ or $\overline{P}$ and read as “not P”. If the truth value of “P” is T, then the truth value of “$\urcorner P$” is F. Also, if the truth value of “P” is F, then the truth value of “$\urcorner P$” is T.
- ✓ Examples
  - ➢ P : London is a city. $\Longrightarrow$ $\urcorner P$: London is not a city.
  - ➢ P : I went to my class yesterday. $\Longrightarrow$ $\urcorner P$: I did not go to my class yesterday.
- ✓ Truth table for negation is as follows.

P	$\urcorner P$
T	F
F	T

- ✓ CONJUNCTION: The conjunction of two statements P and Q is the statement $P \wedge Q$ which is read as “P and Q”. The statement $P \wedge Q$ has the truth value T whenever both P and Q have the truth value T, otherwise it has the truth value F.
- ✓ Example

  P : It is raining today.

  Q : There is a wind storm.

  $P \wedge Q$: It is raining today and there is a wind storm.
- ✓ Truth table for conjunction is as follows.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

- ✓ Note: The truth values of $P \wedge Q$ and $Q \wedge P$ are same.
- ✓ DISJUNCTION: The disjunction of two statements P and Q is the statement $P \vee Q$ which is read as “P or Q”. The statement $P \vee Q$ has the truth value F only when both P and Q have the truth value F, otherwise it has the truth value T.
- ✓ Example

  P : There is something wrong with the bulb.

  Q : There is something wrong with the wiring.

  $P \vee Q$: There is something wrong with the bulb or with the wiring.
- ✓ Truth table for disjunction is as follows.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

- ✓ Note: The statement “Twenty or thirty animals were killed in the fire today.” is not disjunction because here “or” is used to indicate approximate numbers.

❖ STATEMENT FORMULAS

- ✓ Let P and Q be any two statements. Then, some of compound statements formed by using P and Q are $\urcorner P, P \wedge Q, (P \vee Q) \wedge (\urcorner P)$ etc. The compound statements given here are statement formulas derived from the statement variables P and Q. Therefore, P and Q may be called the components of the statement formulas.

❖ CONDITIONAL

- ✓ If P and Q are any two statements, then the statement $P \rightarrow Q$ which is read as "If P, then Q" is called a conditional statement.
- ✓ The statement $P \rightarrow Q$ has a truth value F, "when Q has truth value F and P has T ", otherwise it has the truth value T.
- ✓ Truth table for conditional is as follows.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- ✓ Here, the statement P is called the antecedent and Q the consequent.

❖ BICONDITIONAL

- ✓ If P and Q are any two statements, then the statement $P \rightleftarrows Q$ which is read as "P if and only if Q"(sometimes written as "P iff Q") is called a biconditional statement.
- ✓ The statement $P \rightleftarrows Q$ has the truth value T whenever both P and Q have identical truth value.
- ✓ Truth table for biconditional is as follows.

P	Q	$P \rightleftarrows Q$
T	T	T
T	F	F
F	T	F
F	F	T

METHOD-1: BASIC EXAMPLES ON PROPOSITIONAL LOGIC

H	1	Write truth table for Negation, Conjuction, Disjunction, Conditional and Biconditional.	
H	2	Form the conjunction of P : It is raining today. and Q : There are 20 tables in this room.	
C	3	Translate the statement "Jack and Jill went up the hill." into symbolic form.	
C	4	Consider the statements P : Mark is rich. and Q : Mark is happy. Write the following statements into symbolic form. a) Mark is poor but happy. b) Mark is rich or unhappy. c) Mark is neither rich nor happy. d) Mark is poor or he is both rich and unhappy.	
H	5	Construct the truth table for the statement formula $P \vee \neg Q$.	
H	6	Construct the truth table for $P \wedge \neg P$.	
C	7	Construct the truth table for $(P \vee Q) \wedge R$.	
H	8	Construct the truth table for the following formulas. a) $\neg(\neg P \vee \neg Q)$ b) $\neg(\neg P \wedge \neg Q)$ c) $P \wedge (P \vee Q)$ d) $P \wedge (Q \wedge P)$ e) $\neg P \wedge (\neg Q \wedge R) \vee (Q \wedge R) \vee (P \wedge R)$ f) $(P \wedge Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$	
C	9	Given the truth values of P and Q as T and those of R and S as F, find the truth values of the following. a) $P \vee (Q \wedge R)$ b) $\big(P \wedge (Q \wedge R)\big) \vee \neg\big((P \vee Q) \wedge (R \vee S)\big)$ c) $\big(\neg(P \wedge Q) \vee (\neg R)\big) \vee \Big(\big((\neg P \wedge Q) \vee (\neg R)\big) \wedge S\Big)$	
C	10	Write the statement "If either Jerry takes Calculus or Ken takes Sociology, then Larry will take English." in symbolic form.	

H	11	Write the statement "The crop will be destroyed if there is a flood." in symbolic form.	
C	12	Construct the truth table for $(P \to Q) \wedge (Q \to P)$.	
C	13	Construct the truth table for $\urcorner(P \wedge Q) \rightleftarrows (\urcorner P \vee \urcorner Q)$.	
H	14	Construct the truth table for the following formulas. a) $\big(Q \wedge (P \to Q)\big) \to P$ b) $\urcorner\big(P \vee (Q \wedge R)\big) \rightleftarrows (P \vee Q) \wedge (P \vee R)$	
H	15	Given the truth values of P and Q as T and those of R and S as F, find the truth values of the following. a) $\big(\urcorner(P \wedge Q) \vee (\urcorner R)\big) \vee \big((Q \rightleftarrows \urcorner P) \to (R \vee \urcorner S)\big)$ b) $(P \rightleftarrows R) \wedge (\urcorner Q \to S)$ c) $\Big(P \vee \big(Q \to (R \wedge \urcorner P)\big)\Big) \rightleftarrows (Q \vee \urcorner S)$	

❖ WELL-FORMED FORMULA

- ✓ A statement formula is an expression which is a string consisting of variables (capital letters with or without subscripts), parentheses and connective symbols. Not every string of these symbol is a formula. We shall now give a recursive definition of a statement formula, often called a well-formed formula.
- ✓ A well-formed formula can be generated by the following rules.
  - ➢ A statement variable standing alone is a well-formed formula.
  - ➢ If A is a well-formed formula, then $\urcorner A$ is a well-formed formula.
  - ➢ If A and B are well-formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$ and $(A \rightleftarrows B)$ are well-fromed formulas.
- ✓ A string of symbols containing the statement variables, connectives, and parentheses is a well-formed formula, if and only if it can be obtained by finitely many applications of these rules.
- ✓ The following are well-formed formulas.
  - ➢ $\urcorner(P \wedge Q)$

- $\urcorner(P \vee Q)$
- $\big(P \rightarrow (P \wedge Q)\big)$
- $\Big(\big((P \rightarrow Q) \wedge (Q \rightarrow R)\big) \rightleftarrows (P \rightarrow R)\Big)$

✓ The following are not well-formed formulas.

- $\urcorner P \wedge Q$
- $(P \rightarrow Q) \rightarrow (\wedge Q)$
- $(P \rightarrow Q$
- $(P \rightarrow Q) \rightarrow Q)$

❖ TAUTOLOGIES

✓ If the final column of a truth table of a given statement formula is true regardless then it is called a universally valid formula or a tautology or a logical truth.

✓ If the final column of a truth table of a given statement formula is false regardless then it is called a contradiction.

✓ So, the negation of a contradiction is a tautology.

✓ We may say that a statement formula which is tautology, is identically true and a formula which is a contradiction, is identically false.

✓ Example: $\urcorner P \vee P$ is a tautology.

❖ EQUIVALENCE FORMULA

✓ Let A and B be two statement formulas. If truth values of A are equal to the truth values of B, then A and B are said to be equivalent.

✓ It is denoted by $A \Leftrightarrow B$.

✓ Example: $\urcorner\urcorner P$ is equivalent to P.

✓ Equivalence is a symmetric relation. (i.e. A is equivalence to B, then B is equivalence to A.)

❖ DUALITY LAW

✓ The formula A and A^* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$ (also T by F and F by T). The connectives $\wedge$ and $\vee$ are also called duals of each other.

❖ TAUTOLOGICAL IMPLICATIONS

- ✓ Statement A is said to tautologically imply statement B if and only if $A \rightarrow B$ is a tautology.
- ✓ It is denoted by $A \Rightarrow B$.

METHOD-2: EXAMPLES ON TAUTOLOGIES AND EQUIVALENCES

C	1	Check whether the given formulas are well-formed formulas. Also, indicate which ones are tautologies or contradictions. a) $\left(P \rightarrow (P \vee Q)\right)$ b) $\left(\left(P \rightarrow (\urcorner P)\right) \rightarrow \urcorner P\right)$ c) $\left((\urcorner Q \wedge P) \wedge Q\right)$ d) $\left((P \rightarrow (Q \rightarrow R)\right) \rightarrow \left((P \rightarrow Q) \rightarrow (P \rightarrow R)\right)$ e) $\left((\urcorner P \rightarrow Q) \rightarrow (Q \rightarrow P)\right))$ f) $\left((P \wedge Q) \leftrightarrows P\right)$	
H	2	Show that $\left((P \vee Q) \wedge \urcorner\left(\urcorner P \wedge (\urcorner Q \vee \urcorner R)\right)\right) \vee (\urcorner P \wedge \urcorner Q) \vee (\urcorner P \wedge \urcorner R)$ is a tautology.	
C	3	Show the following implications. a) $(P \wedge Q) \Rightarrow (P \rightarrow Q)$ b) $P \Rightarrow (Q \rightarrow P)$ c) $\left(P \rightarrow (Q \rightarrow R)\right) \Rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$	
C	4	Prove that $(P \rightarrow Q) \Leftrightarrow (\urcorner P \vee Q)$.	
H	5	Show that $(P \vee Q) \wedge \left(\urcorner P \wedge (\urcorner P \wedge Q)\right) \Leftrightarrow (\urcorner P \wedge Q)$.	
H	6	Show that $\left(\urcorner P \wedge (\urcorner Q \wedge R)\right) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$.	
H	7	Show that $\urcorner(P \wedge Q) \rightarrow \left(\urcorner P \vee (\urcorner P \vee Q)\right) \Leftrightarrow (\urcorner P \vee Q)$.	
C	8	Show that $P \rightarrow (Q \rightarrow R) \Leftrightarrow P \rightarrow (\urcorner Q \vee R) \Leftrightarrow (P \wedge Q) \rightarrow R$.	
C	9	If $A(P, Q, R)$ is $\urcorner P \wedge \urcorner(Q \vee R)$, then verify $\urcorner A(P, Q, R) \Leftrightarrow A^*(\urcorner P, \urcorner Q, \urcorner R)$.	

H	**10**	Show the following equivalences. a) $P \rightarrow (Q \rightarrow P) \Leftrightarrow \neg P \rightarrow (P \rightarrow Q)$ b) $P \rightarrow (Q \vee R) \Leftrightarrow (P \rightarrow Q) \vee (P \rightarrow R)$ c) $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (P \vee R) \rightarrow Q$ d) $\neg(P \rightleftarrows Q) \Leftrightarrow (P \vee Q) \wedge \neg(P \wedge Q)$	
H	**11**	Show that P is equivalent to the following formulas. a) $\neg\neg P$ b) $P \wedge P$ c) $P \vee P$ d) $P \vee (P \wedge Q)$ e) $P \wedge (P \vee Q)$ f) $(P \wedge Q) \vee (P \wedge \neg Q)$ g) $(P \vee Q) \wedge (P \vee \neg Q)$	
H	**12**	Show the following equivalences. a) $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ b) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ c) $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$ d) $\neg(P \rightleftarrows Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$	

PART-II PREDICATE LOGIC

❖ PREDICATES

- ✓ Let us consider two statements.
  - ➢ John is a bachelor.
  - ➢ Smith is a bachelor.
- ✓ Obviously, if we express these statements by symbols, we require two different symbols to denote them. Such symbols do not reveal the common features of these two statements; viz, both are statements about two different individuals who are bachelors.
- ✓ If we introduce some symbol to denote, "is a bachelor" and a method to join it with symbols denoting the names of individuals, then we will have a symbolism to denote statements about any individual's being a bachelor. The part "is a bachelor" is called a predicate.
- ✓ Now, denote the predicate "is a bachelor" symbolically by the predicate letter B, "John" by j, and "Smith" by s. Then, statements 1 and 2 can be written as $B(\,j\,)$ and $B(\,s\,)$ respectively. In general, any statement of the type "P is Q" where Q is a predicate and P is the subject can be denoted by Q(P).
- ✓ Example: "Jack is taller than Jill."
  - ➢ For this example, we consider G symbolizes "is taller than", j_1 denotes "Jack", and j_2 denotes "Jill", then statement can be translated as $G(\,j_1, j_2)$.
- ✓ "Canada is to the north of the United states" can be translated by $N(C, S)$, where N denotes the predicate "is to the north of ", C for "Canada " and S for "United states".

❖ STATEMENT FUNCTION, VARIABLES, AND QUANTIFIERS

- ✓ Let H be the predicate "is a mortal", b the name "Jack", c "Canada", and s "A shirt". Then, $H(b), H(c)$ and $H(s)$ all denote statements. In fact, these statements have a common form. If we write $H(x)$ for "x is a mortal", then $H(b), H(c)$ and $H(s)$ and others having the same form can be obtained from $H(x)$ by replacing x by an appropriate name. The letter x used here is a placeholder, called statement variable.
- ✓ A simple statement function of one variable is defined to be an expression consisting of a predicate symbol and an individual variable. Such a statement function becomes a statement when the variable is replaced by the name of any object. The statement resulting

from a replacement is called a substitution instance of the statement function and is a formula of statement calculus.

- ✓ Example: Let $G(x, y)$ denotes "x is taller than y.".
  - ➢ Now, if x and y both are replaced by the names of the objects, we get a statement. If m represents Jack and n Jill, then we have

    $G(m, n)$: Jack is taller than Jill.

    $G(n, m)$: Jill is taller than Jack.
- ✓ $x + 2 = 5$ is a statement function for universe of real numbers.
- ✓ $x^2 + 1 = 0$ is a statement function for universe of complex numbers not real numbers.
- ✓ Note: $x^2 - 1 = (x - 1)\,(x + 1)$ is not a statement function, in fact it is a statement because it is true for all x. Similarly, following are also such examples.
  - ➢ "All men are mortal."
  - ➢ "Any integer is either positive or negative."
- ✓ Now, for the statement "All men are mortal." the symbolize statement is

$$(x)\big(M(x) \to H(x)\big) \text{ or } (\forall x)\big(M(x) \to H(x)\big)$$

  Where $M(x)$: x is a man. & $H(x)$: x is a mortal.
- ✓ Here, the symbols (x) or $(\forall x)$ are called universal quantifiers. Strictly, the quantification symbol is "( )" or "($\forall$)", and it contains the variable which is to be quantified.
- ✓ Also, for the statement "Some real numbers are rational." the symbolize statement is

$$(\exists x)\big(M(x) \land H(x)\big)$$

  Where $M(x)$: x is a real number. & $H(x)$: x is a rational.
- ✓ Then, the symbol "$(\exists x)$" is called the existential quantifier, which symbolizes expressions such as "there is at least one x such that" or "there exists an x such that" or "for some x".

❖ PREDICATE FORMULAS

- ✓ Consider the n-place predicate, capital letter is followed by n individual variables which are enclosed in parentheses and separated by commas.

- ✓ Example: $P(x_1, x_2, \ldots, x_n)$ denotes an n-place predicate formula in which the letter P is an n-place predicate and $x_1, x_2, \ldots, x_n$ are individual variables. In general, $P(x_1, x_2, \ldots, x_n)$ will be called an atomic formula of predicate calculus. It may be noted that our symbolism includes the atomic formulas of the statement calculus as special cases $(n = 0)$.
- ✓ A well-formed formula of predicate calculus is obtained by using the following rules.
  - ➢ An atomic formula is a well-formed formula.
  - ➢ If A is a well formed formula, then ⅂A is a well-formed formula.
  - ➢ If A and B are well-formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \rightleftarrows B)$ are also well-formed formulas.
  - ➢ If A is a well-formed formula and x is any variable, then $(x)A$ and $(\exists x)A$ are well-formed formulas.
- ✓ Only those formulas obtained by using these rules are well-formed formulas.

❖ FREE AND BOUND VARIABLES

- ✓ Given a formula containing a part of the form $(x)P(x)$ and $(\exists x)P(x)$, such a part is called an x-bound part of the formula. Any occurrence of x in x-bound part of a formula is called bound occurrence of x, while any occurrence of x or of any variable that is not a bound occurrence is called free occurrence.
- ✓ Further, the formula $P(x)$ either in $(x)P(x)$ or in $(\exists x)P(x)$ is described as the scope of the quantifier. In other word, the scope of the quantifier is the formula immediately following the quantifier.
- ✓ Examples
  - ➢ In $(x)P(x, y)$, $P(x, y)$ is the scope of the quantifier, occurrence of x is bound occurrence and occurrence of y is free occurrence.
  - ➢ In $(x)\big(P(x) \rightarrow Q(x)\big)$, the scope of the universal quantifier is $P(x) \rightarrow Q(x)$, and all occurrence of x are bound.
  - ➢ Also, in $(x)\big(P(x) \rightarrow (\exists y)R(x, y)\big)$, the scope of (x) is $P(x) \rightarrow (\exists y)R(x, y)$, while scope of $(\exists y)$ is $R(x, y)$. All occurrence of both x and y are bound occurrences.

❖ THE UNIVERSE OF DISCOURSE

- ✓ The process of symbolizing a statement in predicate calculus can be quite complicated. However, some specification can be introduced by limiting the class of individuals or objects under consideration.
- ✓ This limitation means that the variables which are quantified stand for only those objects which are members of a particular set or class. Such a restricted class is called the universe of discourse or the domain of individuals or simply the universe.
- ✓ If the discussion refers to human beings only, then the universe of discourse is the class of human beings.
- ✓ In elementary algebra or number theory, the universe of discourse could be numbers (real, complex, rational, etc.).
- ✓ Examples
  - ➢ Let us symbolize the statement "all men are giants.".
  - ➢ Let $G(x)$: x is a giant and $M(x)$: x is a man. Then, given statement can be symbolized as $(x)\big(M(x) \rightarrow G(x)\big)$. Now, if we restrict the variable x to the universe which is class of men, then the statement is $(x)G(x)$.
  - ➢ Let us symbolize the statement "all cats are animals.".
  - ➢ Then, it is true for any universe of discourse. Let $E = \{\text{cuddle}, \text{ginger}, 0, 1\}$ (first two elements are name of the cats). Obviously, statement is true over E.
  - ➢ Now, if we take $C(x)$: x is a cat. and $A(x)$: x is an animal. Then, the statement $(x)C(x) \rightarrow A(x)$ is rue over E. But, the statement $(x)\big(C(x) \wedge A(x)\big)$ is false over E because statement is not true for elements 0 and 1 of the set E.

❖ VALID FORMULAS AND EQUIVALENCES

- ✓ Let A and B be any two predicate formulas defined over a common universe denoted by the symbol E. If, for every assignment of object names from the universe of discourse E to each of the variables appearing in A and B, the resulting statements have the same truth values, then the predicate formulas A and B are said to be equivalent to each other over E.
- ✓ This idea is symbolized by writing $A \Leftrightarrow B$ over E. The definition of implication can extend in the same way. It is assumed that the same object names are assigned to the same variables throughout both A and B.

- ✓ Similarly, a formula A is said to be valid in E written ‡ A in E if, for every assignment of object names of E to the corresponding variables in A and for every assignment of statements to statement variables, the resulting statement have the truth value T.
- ✓ As before, if a formula is valid for an arbitrary E, then it is written as ‡ A.
- ✓ Note: $A \Leftrightarrow B$ requires that the equivalence of A and B be examined over all universe, and the same is true for ‡ A, since these statements are made for any arbitrary universe.

METHOD-3: BASIC EXAMPLES ON PREDICATE LOGIC

C	**1**	Let $P(x): x$ is a person. $F(x,y): x$ is the father of y. $M(x,y): x$ is the mother of y. Write the predicate "x is the father of the mother of y.".	
C	**2**	Symbolize the expression "All the world loves a lover.".	
H	**3**	Which of the following are statements? a) $(x)\big(P(x) \lor Q(x)\big) \land R$ b) $(x)\big(P(x) \lor Q(x)\big) \land (\exists x)S(x)$ c) $(x)\big(P(x) \lor Q(x)\big) \land S(x)$	
C	**4**	Indicate the variables that are free and bound. Also, show the scope of the quantifiers. a) $(x)\big(P(x) \land R(x)\big) \to (x)P(x) \land Q(x)$ b) $(x)\big(P(x) \land (\exists x)Q(x)\big) \lor \big((x)P(x) \to Q(x)\big)$ c) $(x)\big(P(x) \rightleftarrows Q(x) \land (\exists x)R(x)\big) \land S(x)$	
C	**5**	Find the truth value of a) $(x)\big(P(x) \lor Q(x)\big)$, where $P(x): x = 1,\ Q(x): x = 2$ and the universe of discourse is $\{1, 2\}$. b) $(x)\big(P \to Q(x)\big) \lor R(a)$, where $P:\ 2 > 1, Q(x): x \le 3, R(x): x > 5,$ and $a: 5,$ with the universe being $\{-2, 3, 6\}$. c) $(\exists x)\big(P(x) \to Q(x)\big) \land T$, Where $P(x): x > 2,\ Q(x): x = 0$, and T is any tautology, with the universe of discourse as $\{1\}$.	

H	**6**	Show that $(\exists z)\big(Q(z) \wedge R(z)\big)$ is not implied by the formulas $(\exists x)\big(P(x) \wedge Q(x)\big)$ and $(\exists y)\big(P(y) \wedge R(y)\big)$, by assuming the universe of discourse which has two elements.	

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UNIT-3 » RELATIONS, PARTIAL ORDERING AND RECURSION

PART-I RELATIONS

❖ INTRODUCTION

- ✓ Much of mathematics is about finding a pattern – a recognizable link between quantities that change. In our daily life, we come across many patterns that characterize relations such as brother and sister, father and son, teacher and student.
- ✓ In mathematics, also we come across many relations such as number m is less than number n, line l is parallel to line m, set A is a subset of set B. In all these, we notice that a relation involves pairs of objects in certain order.
- ✓ In this chapter, we will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair.

❖ RELATION

- ✓ Let A and B be non-empty sets. A binary relation or simply relation from A to B is a subset of $A \times B$.
- ✓ Suppose that R is a relation from A to B. Then, R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair, $x \in A$ and $y \in B$, exactly one of the following is true.
  - ➢ $(x, y) \in R$, then we say x is R-related to y, written as $x\, R\, y$.
  - ➢ $(x, y) \notin R$, then we say x is not R-related to y.
- ✓ Note: If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say that R is a relation on A.

❖ DOMAIN

- ✓ Let R be a binary relation from A to B. Then, the domain is denoted and defined as,

$$D_R = \{x \in A : (x, y) \in R\}$$

❖ RANGE

- ✓ Let R be a binary relation from A to B. Then, the range is denoted and defined as,

$$R_R = \{y \in B : (x, y) \in R\}$$

- Examples
  - A familiar example is relation "greater than" for real numbers. This relation is denoted by " $>$ ". In fact, " $>$ " should be considered as the name of a set whose elements are ordered pairs.

$$>= \{(x, y) : x \text{ and } y \text{ are real numbers and } x > y\}$$

  - The definition of relation permits any set of ordered pairs to define a relation like

$$S = \{(2,4), (1,3), (x, 6), (veer, *)\}$$

Here, $D_R = \{2,1, x, veer\}$ & $R_R = \{4, 3, 6, *\}$

UNIVERSAL RELATION

- Let A and B be two non-empty sets. Then, $A \times B$, subset of itself, is called universal relation from A to B.
- Example: Let $A = \{1,2\}$ and $B = \{x, y\}$.

$$A \times B = \{\{1, x\}, \{1, y\}, \{2, x\}, \{2, y\}\}$$

VOID RELATION

- Let A and B be two non-empty sets. Then, the empty set $\{\phi\} \subset A \times B$ is called void (null) relation from A to B.
- Example: Let $A = \{1,2\}$ and $B = \{x, y\}$. Then, the relation presented by the set $\{\emptyset\}$ is called void (null) relation.

UNION, INTERSECTION AND COMPLEMENT OPERATIONS ON RELATIONS

- Let $A = \{1,2,3,4\}$ and

$$R = \{(x, y) : \text{for } x, y \in A, x - y \text{ is an integral multiple of } 2\} = \{(1,3), (3,1), (2,4), (4,2)\}$$

$$S = \{(x, y) : \text{for } x, y \in A, x - y \text{ is an integral multiple of } 3\} = \{(1,4), (4,1)\}$$

  - $R \cup S = \{(1,3), (3,1), (2,4), (4,2), (1,4), (4,1)\}$
  - $R \cap S = \emptyset$
- For complement, let $A = \{a, b\}$ and $B = \{1,2\}$, then $A \times B = \{\{a, 1\}, \{b, 1\}, \{a, 2\}, \{b, 2\}\}$. Also, let $R = \{\{a, 1\}, \{a, 2\}, \{b, 2\}\}$.
  - Then, complement relation for R is $\{\{b, 1\}\}$.

❖ PROPERTIES OF BINARY RELATIONS IN A SET

- ✓ Reflexive
  - ➢ A binary relation R in a set A is said to be reflexive if, for every $x \in A$,

$$(x, x) \in R$$

- ✓ Irreflexive
  - ➢ A binary relation R in a set A is said to be irreflexive if, for every $x \in A$,

$$(x, x) \notin R$$

- ✓ Symmetric
  - ➢ A binary relation R in a set A is said to be symmetric if, for every $x, y \in A$,

$$\text{whenever } (x, y) \in R, \text{then } (y, x) \in R$$

- ✓ Antisymmetric
  - ➢ A binary relation R in a set A is said to be antisymmetric if, for every $x, y \in A$,

$$\text{whenever } (x, y) \in R \ \ \& \ \ (y, x) \in R, \text{then } x = y$$

- ✓ Transitive
  - ➢ A binary relation R in a set A is said to be transitive if, for every $x, y, z \in A$,

$$\text{whenever } (x, y) \in R \ \ \& \ \ (y, z) \in R, \text{then } (x, z) \in R$$

- ✓ Notes
  - ➢ If relations R and S both are reflexive, then $R \cup S$ and $R \cap S$ are also reflexive.
  - ➢ If relations R and S are symmetric and transitive, then $R \cap S$ is also symmetric and transitive.

METHOD-1: BASIC EXAMPLES ON RELATION

H	1	Define the following terms with example. Binary relation, Domain, Range, Universal relation, Void relation, Union, Antisymmetric, Intersection, Complement relation, Reflexive, Irreflexive, Symmetric, Transitive.	

H	2	Let $A = \{(1,2),(2,4),(3,3)\}$ and $B = \{(1,3),(2,4),(4,2)\}$. Find $A \cup B, A \cap B, D(A), D(B), D(A \cup B), R(A), R(B)$ and $R(A \cap B)$. **Answer :** $\mathbf{A \cup B = \{(1,2),(2,4),(3,3),(1,3),(4,2)\}, A \cap B = \{(2,4)\},}$ $\mathbf{D(A) = \{1,2,3\}, D(B) = \{1,2,4\}, D(A \cup B) = \{1,2,3,4\},}$ $\mathbf{R(A) = \{2,4,3\}, R(B) = \{3,4,2\}, R(A \cap B) = \{4\}}$	
C	3	What are the ranges of the relations $S = \{(x,x^2) : x \in N\}$ and $T = \{(x,2x) : x \in N\}$,where $N = \{0,1,2,3,\dots\}$? Also, find $S \cap T$ and $S \cup T$. **Answer :** $\mathbf{R(S) = \{x^2 : x \in N\}, R(T) = \{2x : x \in N\}, S \cap T = \{(0,0),(2,4)\},}$ $\mathbf{S \cup T = \{(0,0),(1,1),(1,2),(2,4),(3,9),(3,6),(4,16),(4,8),\dots\dots\}}$	
C	4	Let L denotes the relation "less than or equal to" and D denotes the relation "divides", where x D y means "x divides y", defined on a set $\{1,2,3,6\}$. Write L and D as a sets, and find $L \cap D$. **Answer :** $\mathbf{D = \{(1,1),(1,2),(1,3),(1,6),(2,2),(2,6),(3,3),(3,6),(6,6)\}}$ $\mathbf{L = \{(1,1),(1,2),(1,3),(1,6),(2,2),(2,3),(2,6),(3,3),(3,6),(6,6)\}}$ $\mathbf{L \cap D = \{(1,1),(1,2),(1,3),(1,6),(2,2),(2,6),(3,3),(3,6),(6,6)\}}$	
H	5	Give an example of a relation which is a) neither reflexive nor irreflexive? b) both symmetric and antisymmetric? c) reflexive but not symmetric? d) symmetric but not reflexive or transitive? e) transitive but not reflexive or symmetric? **Answer :** $\mathbf{A = \{(1,1),(2,3),(3,2),(3,3)\}, B = \{(x,y) : x,y \in \mathbb{N}, x = y\},}$ $\mathbf{C = \{(1,1),(1,2),(2,2)\}, D = \{(1,2),(2,1)\},}$ $\mathbf{E = \{(1,2),(2,3),(1,3)\}}$	
H	6	Check whether the following relations are transitive or not. $R_1 = \{(1,1)\}$, $R_2 = \{(1,2),(2,2)\}$, $R_3 = \{(1,2),(2,3),(1,3),(2,1)\}$ **Answer : yes, yes, no**	
C	7	Let L denotes the relation "less than or equal to" and D denotes the relation "divides", where x D y means "x divides y", defined on a set $\{1,2,3,6\}$. Show that both L and D are reflexive, antisymmetric and transitive.	

C	8	Given $S = \{1,2,3,4, \dots ,10\}$ and a relation R on S, where $R = \{(x, y) : x + y = 10\}$, what are the properties of the relation R? **Answer : symmetric but not reflexive, irreflexive, antisymmetric or transitive.**	

❖ RELATION MATRIX

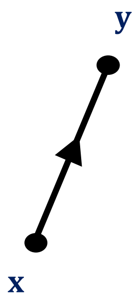
- ✓ A relation R from a set A to a set B can be represented by a matrix called relation matrix of R.
- ✓ The relation matrix of R can be represented by constructing a table whose columns are successive elements of B & rows are successive elements of A. i.e. if $(x_i, y_j) \in R$, then we enter 1 in i^{th} row and j^{th} column. Similarly, if $(x_i, y_j) \notin R$, then we enter 0 in i^{th} row and j^{th} column.
- ✓ Let $A = \{x_1, x_2, x_3\}$ and $B = \{y_1, y_2\}$. Also, let $R = \{(x_1, y_1), (x_2, y_1), (x_3, y_2), (x_2, y_2)\}$. Then, the table representation looks like,

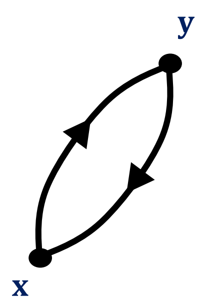
	y_1	y_2
x_1	1	0
x_2	1	1
x_3	0	1

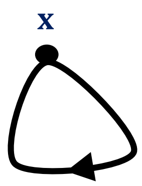
- ✓ Hence, the relation matrix is $M_R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

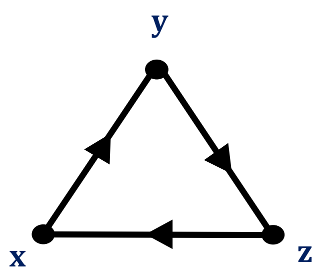
❖ GRAPH OF A RELATION

- ✓ A relation can also be represented pictorially by drawing its graph.
- ✓ Let R be a relation in $A = \{x_1, x_2, \dots, x_m\}$. The elements of A are represented by points or circles called nodes. The nodes may also be called vertices.
- ✓ Now, if $(x_i, x_j) \in R$, then we connect nodes x_i and x_j by an arc and put an arrow on the arc in the direction from x_i to x_j. Thus, when all the nodes corresponding to the ordered pairs in R are connected by arcs with proper arrows, we get a graph of the relation R.
- ✓ If $(x_i, x_j) \in R$ and $(x_j, x_i) \in R$, then we draw two arcs between x_i and x_j, which is called a loop.



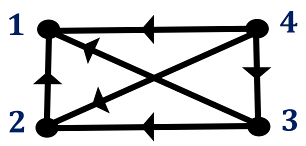
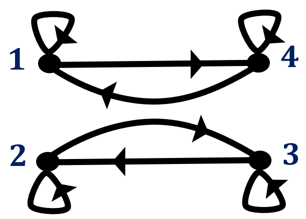






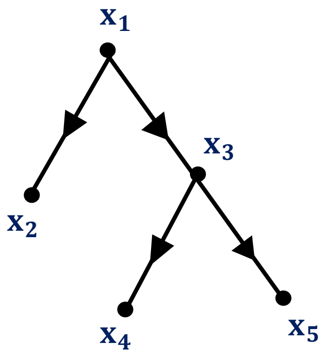
x R y | **x R y** and **y R x** | **x R x** | **x R y** , **y R z** and **z R x**

METHOD-2: EXAMPLES ON RELATION MATRIX AND GRAPH OF A RELATION

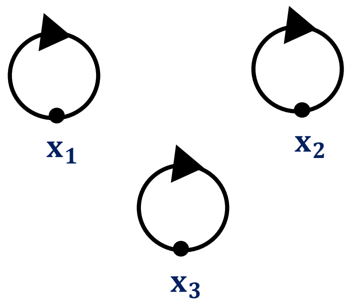
C	1	Let $X = \{1,2,3,4\}$ and $R = \{(x, y) : x > y\}$. Draw the graph of R and give its matrix. **Answer :** $M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ 	
H	2	Let $A = \{a, b, c\}$ and denote the subsets of A by $B_0, B_1, \dots, B_7$ given as $B_0 = \phi$, $B_1 = \{c\}$, $B_2 = \{b\}$, $B_3 = \{b, c\}$, $B_4 = \{a\}$, $B_5 = \{a, c\}$, $B_6 = \{a, b\}$, $B_7 = \{a, b, c\}$. If R is the relation of proper subset on these subsets, then give the matrix of the relation. **Answer :** $M_R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	
H	3	Let $X = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,4), (4,1), (4,4), (2,2), (2,3), (3,2), (3,3)\}$. Write matrix of R and sketch its graph. **Answer :** $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ 	

C | **4** | Determine the properties of the relations given by the graphs as given below. Also, write corresponding relation matrices.

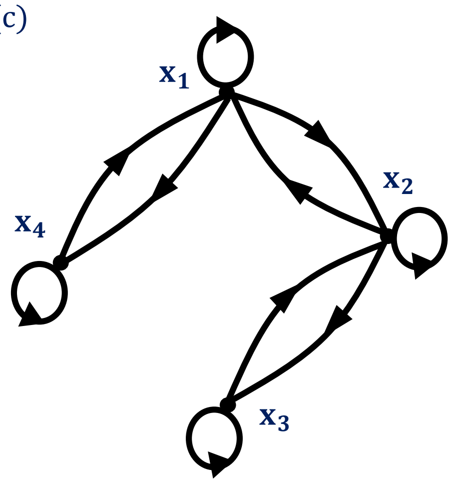
(a)



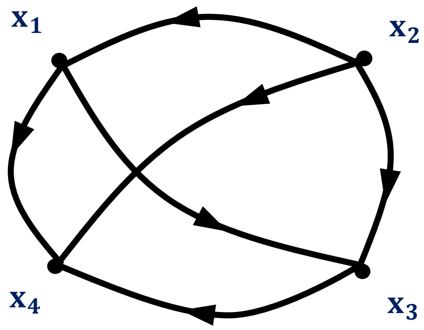
(b)



(c)



(d)



Answer : relation (a) is antisymmetric, relation (b) is reflexive, relation (c) is reflexive and symmetric and relation (d) is transitive.

$$M_a = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, M_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_c = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$M_d = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

❖ PARTITION AND COVERING OF A SET

✓ Let S be a given set and $A = \{A_1, A_2, A_3, \dots, A_m\}$ where each A_i ; $i = 1,2,\dots,m$ is a subset of S and

$$\bigcup_{i=1}^{m} A_i = S$$

then, the set A is called a covering of S and the sets $A_1, A_2, \dots, A_m$ are said to be covers of S.

- ✓ Also, if the elements of A, which are subsets of S, are mutually disjoint, then A is called a partition of S and sets $A_1, A_2, \dots, A_m$ are called the blocks of the partition.
- ✓ Example: Let $S = \{x, y, z\}$. Then
  - ➢ $A = \{\{x, y\}, \{y, z\}\}$ is a covering of S.
  - ➢ $B = \{\{x\}, \{y, z\}\}$ is a partition of S.
  - ➢ $C = \{\{x, y, z\}\}$ is a partition of S.
  - ➢ $D = \{\{x\}, \{y\}, \{z\}\}$ is a partition of S.
  - ➢ $E = \{\{x\}, \{x, y\}, \{x, z\}\}$ is a covering of S.
- ✓ Note: Every partition is a cover, but a cover may not be a partition.

❖ EQUIVALENCE RELATION

- ✓ A relation R on A is called an equivalence relation if it is reflexive, symmetric and transitive.
- ✓ Example: Let $A = \{1, 2, \dots, 7\}$ and $R = \{(x, y) : x - y \text{ is divisible by } 3\}$.
  - ➢ For any $a \in A$, $a - a$ is divisible by 3. Hence, $a\,R\,a$. So, R is reflexive.
  - ➢ For any $a, b \in A$, let $a\,R\,b$. i.e. $a - b$ is divisible by 3. So, $b - a$ is also divisible by 3. Hence, $b\,R\,a$. So, R is symmetric.
  - ➢ For any $a, b, c \in A$, let $a\,R\,b$ and $b\,R\,c$. i.e. $a - b$ and $b - c$ are divisible by 3. So, $a - c = (a - b) + (b - c)$ is also divisible by 3. Hence, $a\,R\,c$. So, R is transitive.
  - ➢ Hence, R is an equivalence relation on A.

❖ EQUIVALENCE CLASS

- ✓ Let R be an equivalence relation on a set A. For any $x \in A$, the set $[x]_R \subseteq A$ given by

$$[x]_R = \{y : y \in A \text{ and } x\,R\,y\}$$

is called an R-equivalence class generated by $x \in A$.

❖ PROPERTIES OF EQUIVALENCE CLASS

- ✓ For any $x \in A$, we have $x\,R\,x$ because R is reflexive, therefore $x \in [x]_R$.
- ✓ Let $y \in A$ be any element such that $x\,R\,y$, then we have $[x]_R = [y]_R$.
- ✓ If $(x, y) \notin R$, then,$[x]_R \neq [y]_R$. Because, if $[x]_R = [y]_R$, then there exist at least one $z \in [x]_R$ and $z \in [y]_R$, that gives $x\,R\,z$ and $y\,R\,z$, i.e. $x\,R\,y$, which contradicts to $(x, y) \notin R$.

❖ COMPATIBILITY RELATION

- ✓ A relation R in A is said to be compatibility relation if it is reflexive and symmetric.
- ✓ Note: Every equivalence relation is a compatibility relation. But, reverse may not be true.

❖ MAXIMAL COMPATIBILITY BLOCK

- ✓ Let A be a set and R be a compatibility relation on A. Then, a subset $C \subseteq A$ is called a maximal compatibility block if any element of C is compatible to every other element of C and no element of $A - C$ is compatible to all the elements of C.
- ✓ Let $R = \{(x, y) : x, y \in X \text{ and } x \text{ and } y \text{ contain some common letter}\}$ be a relation on $X = \{ball, bed, dog, let, egg\}$. Then, R is a compatibility relation denoted by " $\approx$ ". Also, note that R is not equivalence relation. If we denote the elements of X by x_1, x_2, x_3, x_4, x_5 then the graph is as shown here in figure (1).

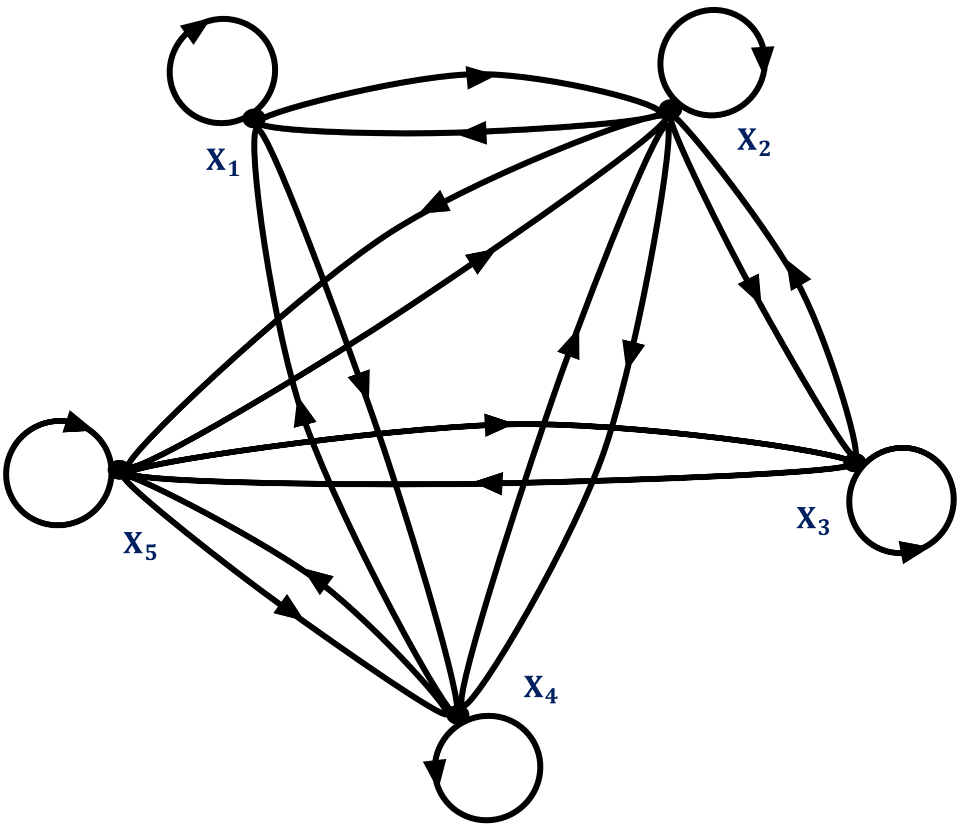


Figure (1)

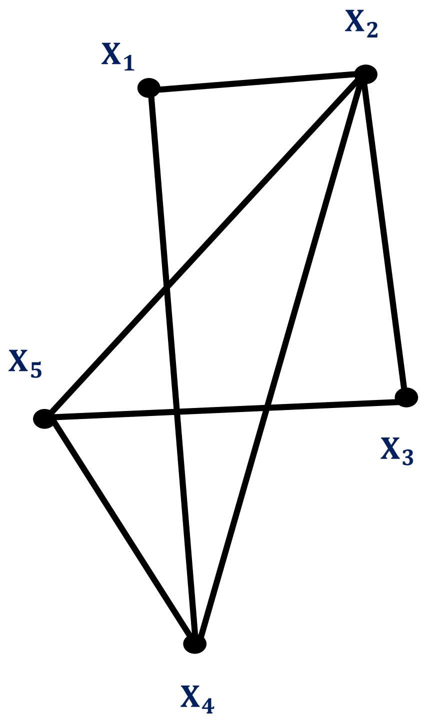


Figure (2)

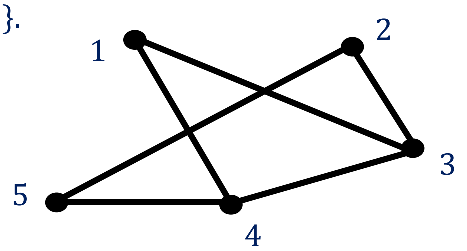
- ✓ Since, relation is compatibility relation, it is not necessary to draw loops at each element nor it is necessary to draw both x R y and y R x. So, we can simplify graph as shown in figure (2).
- ✓ The relation matrix here is symmetric and has its diagonal elements unity. Therefore, it is sufficient to give only the elements of the lower triangular part only as shown as below.

x_2	1			
x_3	0	1		
x_4	1	1	0	
x_5	0	1	1	1
	x_1	x_2	x_3	x_4

- ✓ It is clear that the subsets $\{x_1, x_2, x_4\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}$ are maximal compatibility blocks.

❖ HOW TO FIND MAXIMAL COMPATIBILITY BLOCK AND MATRIX

- ✓ For this first we draw a simplified graph of the compatibility relation and pick from this graph the largest complete polygons i.e. a polygon in which any vertex is connected to every other vertex. In addition to this any element which is related only to itself forms a maximal compatibility block. Similarly, any two elements which are compatible to one another but to no other elements also form a maximal compatibility block. For example, triangle.
- ✓ Example: The maximal compatibility blocks of a compatibility relation R with simple graph as given below are $\{1,3,4\}, \{2,3\}, \{4,5\}, \{2,5\}$.



- ➢ Also, the matrix for this compatibility relation is as given below.

2	0			
3	1	1		
4	1	0	1	
5	0	1	0	1
	1	2	3	4

❖ COMPOSITE RELATION

- ✓ Let R be a relation from A to B and S be a relation from B to C. Then, a relation written as $R \circ S$ is called a composite relation of R and S, defined by

$$R \circ S = \{(x, z) : \text{for } x \in A \text{ and } z \in C \text{ there exist } y \in B \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}$$

- ✓ The operation of obtaining $R \circ S$ is called composition of relations.

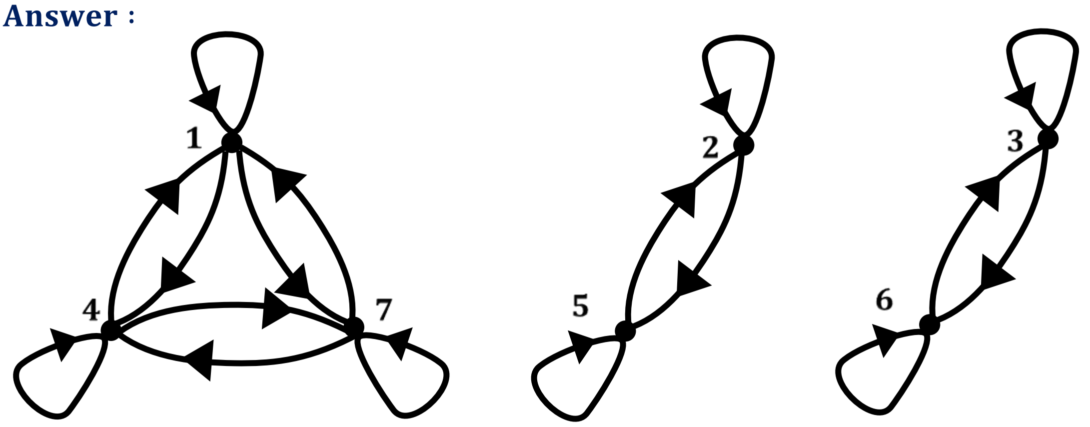
❖ CONVERSE OF A RELATION

- ✓ Given a relation R from A to B, a relation $\widetilde{R}$ from B to A is called the converse of R, where the ordered pairs of $\widetilde{R}$ are obtained by interchanging the members in each of the ordered pairs of R. This means, for $x \in A$ and $y \in B$, $x\,R\,y \Leftrightarrow y\,\widetilde{R}\,x$.

❖ TRANSITIVE CLOSURE OF A RELATION

- ✓ Let R be relation in a finite set A. The relation $R^{+} = R \cup R^{2} \cup R^{3} \cup \dots$ in A is called the transitive closure of R in A.

METHOD-3: EXAMPLES ON EQUIVALENCE AND COMPATIBILITY RELATION

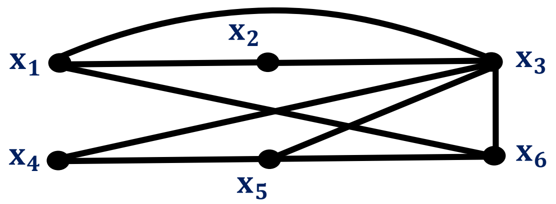
H	1	Define with examples: Partition, Covering, Equivalence relation, Equivalence class, Compatibility relation, Maximal compatibility block, Composite relation, Converse of a relation, Transitive closure.	
C	2	Let $X = \{1, 2, \dots \dots, 7\}$ and $R = \{(x, y) : x - y \text{ is divisible by } 3\}$. Show that R is an equivalence relation. Also, draw the graph of R. **Answer :**  	
H	3	Let $X = \{1, 2, \dots \dots, 7\}$ and $R = \{(x, y) : x - y \text{ is even}\}$. Show that R is an equivalence relation.	

H	**4**	Let $Z = A_1 \cup A_2 \cup A_3$. Where $A_1 = \{\dots, 1, 4, 7, \dots\}, A_2 = \{\dots, 2, 5, 8, \dots\}$ and $A_3 = \{\dots, 3, 6, 9, \dots\}$. Then, define equivalence relation whose equivalence classes are A_1, A_2, A_3. **Answer : $\mathbf{R = \{(x, y) : x - y}$ is devisible by 3$\}$ over the set of integers**	
C	**5**	Let $\mathbb{Z}$ be the set of integers and R be the relation called "congruence modulo 3" defined by $R = \{(x, y) : x - y \text{ is divisible by } 3\}$. Determine equivalence classes generated by the elements of $\mathbb{Z}$. **Answer : $\mathbb{Z}/\mathbf{R} = \{[\mathbf{0}]_{\mathbf{R}}, [\mathbf{1}]_{\mathbf{R}}, [\mathbf{2}]_{\mathbf{R}}\}$**	
H	**6**	Prove that relation "congruence modulo m" given by $R = \{(x, y) : x - y \text{ is divisible by m}\}$, over the set of positive integers, is an equivalence relation.	
H	**7**	Let S be the set of all statement functions in n variables and let R be the relation given by $R = \{(x, y) ; x \Leftrightarrow y\}$. Discuss the equivalence classes generated by the elements of S. **Answer : there are $\mathbf{2^{2^n}}$ R-equivalence classes**	
H	**8**	Let $X = \{a, b, c, d, e\}$ and let $C = \{\{a, b\}, \{c\}, \{d, e\}\}$. Show that the partition C defines an equivalence relation on X.	
C	**9**	Let R denote a relation on the set of ordered pairs of positive integers such that $(x, y)\ R\ (u, v)$ iff $xv = yu$. Show that R is an equivalence relation.	
C	**10**	Let $R = \{(x, y) : x, y \in X$ and x and y contain some common letter$\}$ be relation on X where $X = \{ball, bed, dog, let, egg\}$. Then, show that R is a compatibility relation.	

C **11** Let the compatibility relation on a set $\{x_1, x_2, x_3, \dots, x_6\}$ be given by the following matrix. Draw the graph. Also, find maximal compatibility blocks of the relation.

$$
\begin{array}{c|ccccc}
x_2 & 1 & & & & \\
x_3 & 1 & 1 & & & \\
x_4 & 0 & 0 & 1 & & \\
x_5 & 0 & 0 & 1 & 1 & \\
x_6 & 1 & 0 & 1 & 0 & 1 \\
\hline
 & x_1 & x_2 & x_3 & x_4 & x_5
\end{array}
$$

Answer : $\mathbf{\{x_1, x_2, x_3\}, \{x_1, x_3, x_6\}, \{x_3, x_4, x_5\}, \{x_3, x_5, x_6\}}$



H **12** Let $R = \{(1,2), (3,4), (2,2)\}$ and $S = \{(4,2), (2,5), (3,1), (1,3)\}$. Find $R \circ S$, $S \circ R$, $R \circ (S \circ R)$, $(R \circ S) \circ R$, $R \circ R$, $S \circ S$, and $R \circ R \circ R$.

Answer : $\mathbf{R \circ S = \{(1,5), (3,2), (2,5)\}, S \circ R = \{(4,2), (3,2), (1,4)\}}$

$\mathbf{R \circ R = \{(1,2), (2,2)\}, S \circ S = \{(4,5), (3,3), (1,1)\}}$

$\mathbf{R \circ R \circ R = \{(1,2), (2,2)\}, R \circ (S \circ R) = (R \circ S) \circ R = \{(3,2)\}}$

C **13** Let R and S be two relations on a set of positive integers A, where $R = \{(x, 2x) : x \in A\}$ and $S = \{(x, 7x) : x \in A\}$. Find $R \circ S$, $R \circ R$, $R \circ R \circ R$ and $R \circ S \circ R$.

Answer : $\mathbf{R \circ S = \{(x, 14x) : x \in A\}, R \circ R = \{(x, 4x) : x \in A\}}$

$\mathbf{R \circ R \circ R = \{(x, 8x) : x \in A\}, R \circ S \circ R = \{(x, 28x) : x \in A\}}$

C **14** Let $R = \{(1,2), (3,4), (2,2)\}$ and $S = \{(4,2), (2,5), (3,1), (1,3)\}$ be relations on a set $A = \{1, 2, 3, 4, 5\}$. Obtain relation matrices for $R \circ S$ and $S \circ R$.

Answer : $\mathbf{M_{R \circ S} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, M_{S \circ R} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$

H	15	Given the relation matrix $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ of a relation R on a set $\{a, b, c\}$, find the relation matrices of $\widetilde{R}, R^2 = R \circ R, R^3 = R \circ R \circ R$, and $R \circ \widetilde{R}$. **Answer :** $\mathbf{M_{\widetilde{R}}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \mathbf{M_{R^2}} = \mathbf{M_{R^3}} = \mathbf{M_{R \circ \widetilde{R}}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$	
C	16	Given the relation matrices M_R and M_S, find $M_{R \circ S}, M_{\widetilde{R}}, M_{\widetilde{S}}, M_{R \,\widetilde{\circ}\, S}$, and show that $M_{R \,\widetilde{\circ}\, S} = M_{\widetilde{S} \circ \widetilde{R}}$. $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ **Answer :** $\mathbf{M_{R \circ S}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}, \mathbf{M_{\widetilde{R}}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \mathbf{M_{\widetilde{S}}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix},$ $\mathbf{M_{R \,\widetilde{\circ}\, S}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$	

PART-II PARTIAL ORDERING

❖ PARTIAL ORDERING

- ✓ A binary relation R in a set P is called a partial order relation or partial ordering in P iff R is reflexive, antisymmetric, and transitive. Also, it is denoted by $\leq$ and the ordered pair $(P, \leq)$ is called a partially ordered set or a poset.
- ✓ Examples
  - ➢ Let P be the set of real numbers. The relation $\leq$ (less than or equal to) is a partial ordering on P.
  - ➢ The relation inclusion $\subseteq$ on $P = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ is a partial ordering.
  - ➢ Let $P = \{2, 3, 6, 8\}$ and $\leq$ be a relation "divides", then $(P, \leq)$ is a poset.

❖ TOTALLY ORDERED SET

- ✓ Let $(P, \leq)$ be a partially ordered set. If for every $x, y \in P$ we have either $x \leq y$ or $y \leq x$, then $\leq$ is called a simple ordering or linear ordering on P, and $(P, \leq)$ is called a totally ordered or simply ordered set or a chain.
- ✓ Example: Set $I_n = \{1,2,3 \dots, n\}$ with natural ordering "less than or equal to".

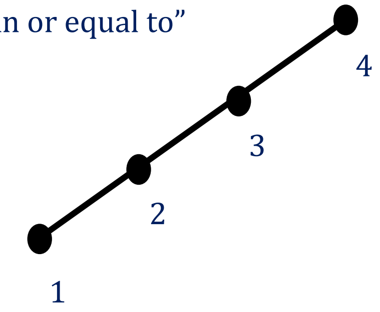
❖ COVER

- ✓ In a poset $(P, \leq)$ an element $y \in P$ is said to cover an element $x \in P$ if $x \leq y$ and if there does not exist an element $z \in P$ such that $x \leq z$ and $z \leq y$.

❖ HASSE DIAGRAM

- ✓ A poset $(P, \leq)$ can be represented by means of a diagram known as a Hasse diagram or partial ordered set diagram. In such a diagram, each element is represented by a small circle or a dot.
- ✓ The circle for $x \in P$ drawn below the circle for $y \in P$ if $x \leq y$, and a line is drawn between x and y.
- ✓ If $x \leq y$, but y does not cover x, then x and y are not connected directly by a single line. However, they are connected through one or more elements of P.

- ✓ Example: Let $P = \{1, 2, 3, 4\}$ and $\leq$ be a relation "less than or equal to" then the Hasse diagram is as follows.

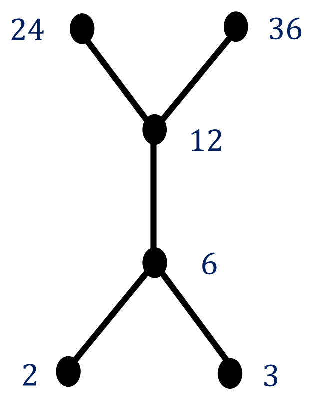


❖ LEAST AND GREATEST MEMBER

- ✓ Let $(P, \leq)$ be a poset. If there exist an element $y \in P$ such that $y \leq x$ for all $x \in P$, then y is called the least member in P relative to the partial ordering $\leq$. Similarly, if there exists an element $y \in P$ such that $x \leq y$ for all $x \in P$, then y is called greatest member in P relative to $\leq$.
- ✓ The least member is usually denoted by 0 and greatest member by 1.
- ✓ Example: Let $P = \{1, 2, 3, 4\}$ and $\leq$ be a relation "less than or equal to".
  - ➢ Here, least member is 1 and greatest member is 4.
- ✓ Note: For any poset least and greatest member, if exists, are unique. It may happen that the least or the greatest member does not exist.

❖ MINIMAL AND MAXIMAL MEMBER

- ✓ Let $(P, \leq)$ be a poset. If there does not exist an element $x \in P$ such that $x \leq y$ for $y \in P$, then y is called the minimal member in P relative to the partial ordering $\leq$. If there does not exist an element $x \in P$ such that $y \leq x$ for $y \in P$, then y is called the maximal member in P relative to the partial ordering $\leq$.
- ✓ Example: Let $P = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be for divide.



  - ➢ Here, there are two minimal members 2 and 3.
  - ➢ Also, there are two maximal members 24 and 36.

❖ UPPER BOUND AND LOWER BOUND

- ✓ Let $(P, \leq)$ be a poset and let $A \subseteq P$. Any element $x \in P$ is an upper bound for A if for all $a \in A$, $a \leq x$. Similarly, any element $x \in P$ is a lower bound for A if for all $a \in A$, $x \leq a$.

❖ LEAST UPPER BOUND AND GREATEST LOWER BOUND

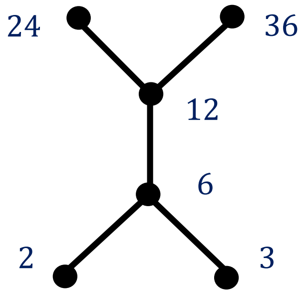
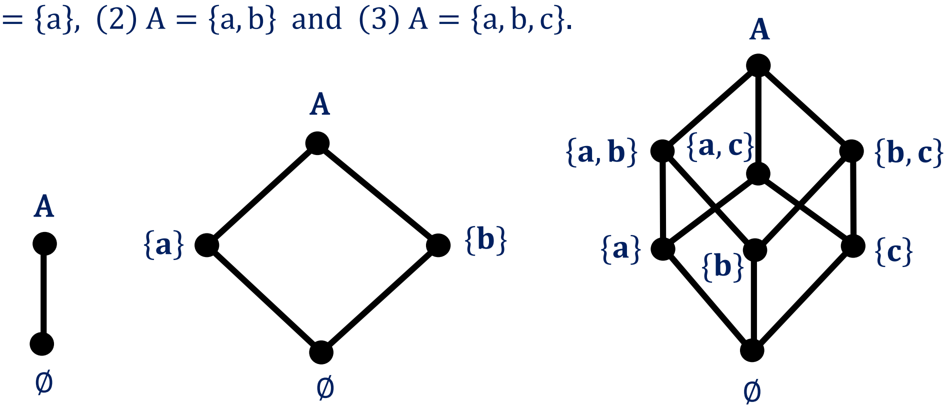
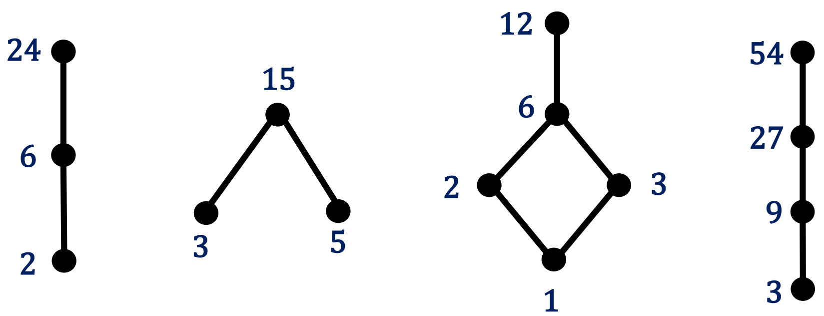
- ✓ Let $(P, \leq)$ be a partially ordered set and let $A \subseteq P$. An element $x \in P$ is a least upper bound or supremum for A if x is an upper bound for A and $x \leq y$ where y is any upper bound for A. Similarly, an element $x \in P$ is a greatest lower bound or infimum for A if x is an lower bound for A and $y \leq x$ where y is any lower bound for A.
- ✓ Least upper bound is denoted by "LUB" or "SUP", and greatest lower bound is denoted by "GLB" or "INF".
- ✓ Example: Let $P = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be for divide. Also, let $A = \{6, 12\}$.
  - ➢ Here, lower bounds are 2, 3 and 6. But, the greatest lower bound is 6.
  - ➢ Similarly, upper bounds are 12, 24 and 36. But, the least upper bound is 12.
- ✓ Note: Both "GLB" and "LUB" are unique if exists.

❖ WELL-ORDERED SET

- ✓ A partially ordered set is called well-ordered if every non-empty subset of it has a least member.
- ✓ Example: A simplest well-ordered set is $I_n = \{1, 2, \dots, n\}$ with natural ordering "less than or equal to".

METHOD-4: EXAMPLES ON POSET AND HASSE DIAGRAM

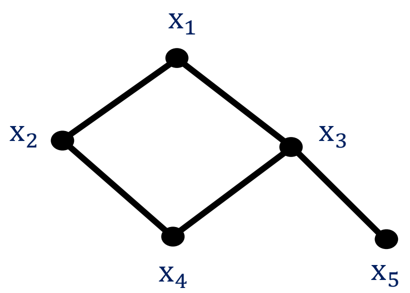
H	1	Define the following terms with example: Partial order relation, Partially ordered set (poset), Simple (linear) ordering, Totally ordered set (chain or simply ordered set), Cover, Least member, Greatest member, Minimal member, Maximal member, Upper bound, Lower bound, Least upper bound (supremum), Greatest lower bound (infimum), Well-ordered set.	
C	2	Show that the relation $\subseteq$(inclusion) on a set $P(A)$, i.e. power set of $A = \{a, b, c\}$, is a partial ordering.	
H	3	Show that $(P(A), \subseteq)$ is a poset.	

C	4	Let $A = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ if x divides y. Draw the Hasse diagram of $(A, \leq)$. **Answer :**  	
H	5	Let $\subseteq$ be a relation on a set P(A), i. e. power set of A. Then, draw Hasse diagram for (1) $A = \{a\}$, (2) $A = \{a, b\}$ and (3) $A = \{a, b, c\}$. **Answer :**  (1) (2) (3)	
C	6	Give an example of a set A such that $(P(A), \subseteq)$ is a totally ordered set. **Answer :** $\mathbf{A = \{a\}}$	
C	7	Draw the Hasse diagram of the following sets under the partial order relation "divides" and indicate those which are totally ordered sets. (a) $\{2, 6, 24\}$, (b) $\{3, 5, 15\}$, (c) $\{1, 2, 3, 6, 12\}$ and (d) $\{3, 9, 27, 54\}$ **Answer : sets (a) and (d) are totally ordered sets.**  (a) (b) (c) (d)	

H **8** Give a relation which is both a partial order relation and an equivalence relation on a set.

Answer : $\mathbf{R = \{(x, y) : x = y\}}$

C **9** Hasse diagram of a poset (P, R), where $P = \{x_1, x_2, x_3, x_4, x_5\}$, is given below. Find out which of the followings are true?



a) $x_1\ R\ x_2$
b) $x_3\ R\ x_5$
c) $x_1\ R\ x_1$
d) $x_4\ R\ x_5$
e) $x_4\ R\ x_1$
f) $x_2\ R\ x_5$
g) $x_2\ R\ x_3$

Answer : **false, false, true, false, true, false, false**

C **10** For above poset given in example 9. Find least and greatest member in P if exists. Also, find minimal and maximal elements. Find upper and lower bounds. Find LUB and GLB if exists.

Answer : **least member does not exist, greatest member is x_1**
minimal elements are x_4 & x_5, maximal element is x_1
lower bound does not exist, upper bound & LUB is x_1

C **11** Let $P = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ if x divides y. Then, find least and greatest member in P if exists. Also, find minimal and maximal elements. Find upper bounds, lower bounds, LUB and GLB if exists for sets (a) $\{2, 3, 6\}$, (b) $\{2, 3\}$, (c) $\{12, 6\}$, (d) $\{24, 36\}$ and (e) $\{3, 12, 24\}$.

Answer : **least and greatest member does not exists**
minimal elements are 2 & 3, maximal elements are 24 and 36

	Upper bounds	Lower bounds	LUB	GLB
(a)	6,12, 24,36	Not exist	6	Not exist
(b)	6,12, 24,36	Not exist	6	Not exist
(c)	12,24, 36	2,3, 6	12	6
(d)	Not exist	2,3, 6,12	Not exist	12
(e)	24	3	24	3

❖ LATTICE

- ✓ A lattice is a poset $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound and a least upper bound.
- ✓ The greatest lower bound of a subset $\{a, b\} \subseteq L$ is called meet and denoted by GLB $\{a, b\}$ or $a * b$ or $a \wedge b$ or $a \cdot b$. The least upper bound is called join and denoted by LUB $\{a, b\}$ or $a \oplus b$ or $a \vee b$ or $a + b$.
- ✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively.

❖ PROPERTIES OF LATTICES

- ✓ Let $(L, \leq)$ be a lattice and $*$ and $\oplus$ be two binary operations meet and join. Then, for

$$a, b, c \in L$$

  - ➢ $a * a = a$ and $a \oplus a = a$ (Idempotent)
  - ➢ $a * b = b * a$ and $a \oplus b = b \oplus a$ (Commutative)
  - ➢ $(a * b) * c = a * (b * c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (Associative)
  - ➢ $a * (a \oplus b) = a$ and $a \oplus (a * b) = a$ (Absorption)

❖ COMPLETE LATTICE

- ✓ A lattice is called complete if each of its non-empty subsets has a least upper bound and a greatest lower bound.
- ✓ Note: Every finite lattice must be complete.
- ✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively.

❖ DISTRIBUTIVE LATTICE

- ✓ A lattice $(L, *, \oplus)$ is called a distributive lattice if for any $a, b, c \in L$,

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

- ✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively.

❖ MODULAR LATTICE

- ✓ A lattice $(L,*,\oplus)$ is said to be modular if $a \leq c \Rightarrow a\oplus(b * c) = (a\oplus b) * c$.
- ✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations ∩ and ∪ respectively.

❖ BOUNDED LATTICE

- ✓ A lattice is said to be bounded if it has both least and greatest elements, i.e. 0 and 1.
- ✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations ∩ and ∪ respectively, and least element is ϕ and greatest element is S.

❖ COMPLEMENT

- ✓ In a bounded Lattice $(L,*,\oplus,0,1)$ an element $b \in L$ is called a complement of an element $a \in L$ if $a * b = 0$ and $a\oplus b = 1$.

❖ COMPLEMENTED LATTICE

- ✓ A lattice $(L,*,\oplus,0,1)$ is said to be a complemented lattice if every element of L has at least one complement.
- ✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations ∩ and ∪ respectively, and least element is ϕ and greatest element is S.

❖ BOOLEAN LATTICE

- ✓ A Boolean lattice (Boolean algebra) is a complemented, distributive lattice.
- ✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations ∩ and ∪ respectively, and least element is ϕ and greatest element is S.

❖ PSEUDO BOOLEAN LATTICE

- ✓ A bounded lattice $(L, \leq)$ is called pseudo Boolean lattice if for all $a, b \in L$, there exists $c \in L$ such that $a * x \leq b \Leftrightarrow x \leq c, \forall x \in L$.
- ✓ If such element c exists, then it is unique and will be denoted by $b : a$.

✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively, and least element is ϕ and greatest element is S.

METHOD-5: BASIC EXAMPLES ON LATTICE

H	**1**	Define With example: Lattice, Complete Lattice, Distributive lattice, Modular lattice, Bounded lattice, Complement element, Complemented lattice, Boolean lattice(algebra), Pseudo Boolean lattice(algebra).	
C	**2**	Let $A = \{a, b, c\}$. Then, show that the poset $(p(A), \subseteq)$ is a lattice.	
H	**3**	Let $A = \{1,2,3\}$. Check whether the poset $(p(A), \subseteq)$ is a lattice. **Answer : yes, $(p(A), \subseteq)$ is a lattice**	
C	**4**	Let $A = \{2,3,4,6,8,12,24,36\}$. Check whether the poset $(A, \mid)$ is a lattice. **Answer : no, $(A, \mid)$ is not a lattice**	
H	**5**	Let S_n be the set of factors(divisors) of positive integer n. Draw Hasse diagram for lattice $(S_n, \mid)$ for $n = 6,24,30,45$.	
C	**6**	Check whether $([0,1], \leq)$ is a lattice. If yes, what are meet and join? **Answer : yes, $([0,1], \leq)$ is a lattice with meet $\{a, b\} = \min\{a, b\}$ and join $\{a, b\} = \max\{a, b\}$**	
H	**7**	Define lattice. Determine whether poset $(\{1,2,3,4,5\}, \mid)$ is a lattice. **Answer : no, $(\{1,2,3,4,5\}, \mid)$ is not a lattice**	

PART-III RECURSION

❖ RECURSION

- ✓ Suppose that n is a natural number. We often defines n! as $n! = n \times (n-1) \times \ldots \times 2 \times 1$. Sometimes, it is difficult to define a computation explicitly and it is easy to define it in terms of itself, that is, recursively.
- ✓ Recursion is an elegant and powerful problem solving technique, used extensively in both discrete mathematics and computer science. We can use recursion to define sequence, functions, sets, algorithms and many more.

❖ RECURRENCE RELATION

- ✓ A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more preceding terms of the sequence like $a_0, a_1, \ldots, a_{n-1}$.
- ✓ Example: Consider the following instructions for generating a sequence.
  - ➢ Start with 1.
  - ➢ Given a term of sequence, get the next term by adding two to it.
  - ➢ If we generate terms of sequence with these two rules, we obtain

$$1, 3, 5, 7, \ldots$$

  - ➢ If we denote the sequence as $a_1, a_2, a_3, a_4, \ldots$
  - ➢ Then, we can rephrase instruction as, $a_n = a_{n-1} + 2$; $n \geq 2$ and $a_1 = 1$.
  - ➢ The equation $a_n = a_{n-1} + 2$; $n \geq 2$ is an recurrence relation and $a_1 = 1$ is an initial condition.

❖ SOLVING RECURRENCE RELATION

- ✓ Solving recurrence relation means finding an explicit formula for a_n $\big(\text{as } f(n)\big)$.
- ✓ The following are methods for this.
  - ➢ Generating function
  - ➢ Undetermined coefficients

❖ GENERATING FUNCTION METHOD

✓ Example: Let $a_n - 3a_{n-1} = 2$; $n \geq 1$ with $a_0 = 1$.

- Given recurrence relation is $a_n - 3a_{n-1} = 2$.
- Multiplying both sides by z^n, we obtain

$$a_n z^n - 3a_{n-1} z^n = 2z^n$$

- Since $n \geq 1$, summing for all n. we get,

$$\sum_{n=1}^{\infty} a_n z^n - 3\sum_{n=1}^{\infty} a_{n-1} z^n = 2\sum_{n=1}^{\infty} z^n \quad \dots\dots \quad (1)$$

- Let $A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$

$$\sum_{n=1}^{\infty} a_n z^n = a_1 z + a_2 z^2 + a_3 z^3 + \cdots = A(z) - a_0$$

$$\sum_{n=1}^{\infty} a_{n-1} z^n = z\sum_{n=1}^{\infty} a_{n-1} z^{n-1} = z(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots) = zA(z)$$

$$\sum_{n=1}^{\infty} z^n = z + z^2 + z^3 + \cdots = \frac{z}{1-z}$$

- Substituting values in equation (1). We get,

$$[A(z) - a_0] - 3zA(z) = 2\frac{z}{1-z}$$

$$\Longrightarrow (1-3z)A(z) = \frac{2z}{1-z} + a_0$$

$$\Longrightarrow (1-3z)A(z) = \frac{2z}{1-z} + 1 \qquad (\because \text{given that } a_0 = 1)$$

$$\Longrightarrow A(z) = \frac{2z}{(1-z)(1-3z)} + \frac{1}{(1-3z)}$$

$$\Longrightarrow A(z) = \frac{1+z}{(1-z)(1-3z)}$$

$$\Longrightarrow A(z) = \frac{2}{(1-3z)} - \frac{1}{(1-z)}$$

$$\Longrightarrow A(z) = 2[1 + (3z) + (3z)^2 + (3z)^3 + \cdots] - [1 + z + z^2 + z^3 + \cdots]$$

$$\Longrightarrow A(z) = 2\sum_{n=0}^{\infty}(3)^n z^n - \sum_{n=0}^{\infty} z^n$$

$$\Longrightarrow a_n = 2(3)^n - 1\,;n \geq 0$$

- Which is required solution of the given recurrence relation.

❖ UNDETERMINED COEFFICIENTS METHOD

✓ Example: Let $a_n - 3a_{n-1} = 2\,; n \geq 1$ with $a_0 = 1$.

- Here, the total solution a_n is given by $a_n = a_n^{(h)} + a_n^{(P)}$.

Part-I $\left(\text{how to find } a_n^{(h)}\right)$

- The characteristic equation is $(\lambda - 3) = 0$.

$$\Longrightarrow \lambda = 3$$

$$\Longrightarrow a_n^{(h)} = C_1(3)^n$$

$a_n^{(h)}$ according to λ	
λ	$a_n^{(h)}$
$\lambda_1 \neq \lambda_2 \neq \lambda_3$	$C_1(\lambda_1)^n + C_2(\lambda_2)^n + C_3(\lambda_3)^n$
$\lambda_1 = \lambda_2 = \lambda_3$	$(C_1 + C_2 n + C_3 n^2)(\lambda_1)^n$

Part-II $\left(\text{how to find } a_n^{(P)}\right)$

- Here, $f_n = 2$.
- Therefore, particular solution is of the type $f_n =$ constant.
- Hence, we will consider $a_n^{(P)} = P_0$.

$$\Longrightarrow a_{n-1} = P_0$$

- Substituting these values in given recurrence relation. We get,

$$P_0 - 3P_0 = 2$$

$$\Longrightarrow -2P_0 = 2$$

$$\Longrightarrow P_0 = -1$$

- So, $a_n^{(P)} = P_0 = -1$.
- Hence, required solution is $a_n = a_n^{(h)} + a_n^{(P)}$

$$a_n = C_1(3)^n - 1$$

$a_n^{(P)}$ according to f_n	
f_n	$a_n^{(P)}$
constant	P_0
$a + bn + cn^2 + dn^3 + \cdots$	$P_0 + P_1 n + P_2 n^2 + P_3 n^3 + \cdots$
ab^n $(b \neq \lambda)$	$P_0 b^n$
ab^n ($b = \lambda$ with multiplicity m)	$P_0 n^m b^n$

Part-III (how to find constants using given conditions)

- We are given that $a_0 = 1$.
- So, $a_n = C_1(3)^n - 1$

  $\Longrightarrow a_0 = C_1(3)^0 - 1$

  $\Longrightarrow 1 = C_1 - 1$

  $\Longrightarrow C_1 = 2$
- Hence, required solution is $a_n = 2(3)^n - 1$.

METHOD-6: EXAMPLES ON RECURRENCE RELATION

H	1	Solve the recurrence relation using the method of generating function $$a_n = 3a_{n-1}\,;n \geq 1.$$ **Answer :** $\mathbf{a_n = (3)^n a_0\,; n \geq 0}$	
H	2	Solve the recurrence relation using the method of generating function $$a_n + 2a_{n-1} - 15a_{n-2} = 0\,; n \geq 2, a_0 = 0, a_1 = 1.$$ **Answer :** $\mathbf{a_n = \frac{1}{8}(3)^n - \frac{1}{8}(-5)^n\,; n \geq 0}$	
C	3	Solve the recurrence relation using the method of generating function $$a_n - 3a_{n-1} = 2\,; n \geq 1, a_0 = 1.$$ **Answer :** $\mathbf{a_n = 2(3)^n - 1\,; n \geq 0}$	
C	4	Solve the recurrence relation using the method of generating function $$a_n - 5a_{n-1} + 6a_{n-2} = 3^n\,; n \geq 2, a_0 = 0, a_1 = 2.$$ **Answer :** $\mathbf{a_n = (3)^n\left(\frac{1}{2} - (2)^n + \frac{1}{2}(3)^n\right) + 2((3)^n - (2)^n)\,; n \geq 0}$	
C	5	Solve the following recurrence relation using the method of undetermined coefficients. a) $a_n = 2a_{n-1} - a_{n-2}\,; a_1 = 1.5, a_2 = 3$ b) $a_n = 3a_{n-1} - 2a_{n-2}\,; a_1 = -2, a_2 = 4$ c) $a_n - 7a_{n-1} + 10a_{n-2} = 0\,; a_0 = 0, a_1 = 3$ d) $a_n - 4a_{n-1} + 4a_{n-2} = 0\,; a_0 = 1, a_1 = 6$ e) $a_n + 2a_{n-1} - 15a_{n-2} = 0\,; a_0 = 0, a_1 = 1$ **Answer :** $\mathbf{a_n = (1.5n)(1)^n}$, $\mathbf{a_n = (-8)(1)^n + 3(2)^n}$, $\mathbf{a_n = (5)^n - (2)^n}$, $\mathbf{a_n = (1 + 2n)(2)^n}$, $\mathbf{a_n = \frac{1}{8}(3)^n - \frac{1}{8}(-5)^n}$	
C	6	Solve the recurrence relation $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 2^n\,; n \geq 3, a_0 = a_1 = 0, a_2 = 2$ using the method of undetermined coefficients. **Answer :** $\mathbf{a_n = \left(-\frac{1}{8} + \frac{1}{4}n\right)(-2)^n + (2)^{(n-3)}}$	

H	7	Solve the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2} + 5^n\ ; n \geq 2, a_0 = -2, a_1 = 1$ using the method of undetermined coefficients. **Answer :** $\mathbf{a_n = -\frac{17}{24}(-1)^n - \frac{27}{8}(3)^n + \frac{25}{12}(5)^n}$	
C	8	Solve the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$ using the method of undetermined coefficients. **Answer :** $\mathbf{a_n = C_1(-2)^n + C_2(-3)^n + \frac{115}{228} + \frac{17}{24}n + \frac{1}{4}n^2}$	
C	9	Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = n + 2^n\ ; n \geq 2, a_0 = 1, a_1 =$ 1 using the method of undetermined coefficients. **Answer :** $\mathbf{a_n = \frac{17}{4}(3)^n - 5(2)^n + \frac{7}{4} + \frac{1}{2}n - 2n(2)^n}$	
H	10	Solve the recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = n + 3^n$ using undetermined coefficients method. **Answer :** $\mathbf{a_n = (C_1 + C_2 n)(2)^n + 4 + n + 9(3)^n}$	

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UNIT-4 » ALGEBRAIC STRUCTURES

PART-I ALGEBRAIC STRUCTURES WITH ONE BINARY OPERATION

❖ IMPORTANT SETS

- ✓ $\mathbb{N}$(The set of natural numbers) $= \{1, 2, 3, \dots, \}$.
- ✓ $\mathbb{I} = \mathbb{Z}$(The set of integers) $= \{, \dots, -3, -2, -1, 0, 1, 2, 3, \dots, \}$.
- ✓ $\mathbb{Q}$(The set of rational numbers) $= \left\{\frac{p}{q} / \ p, q \in \mathbb{Z} \ \& \ q \neq 0\right\}$.
- ✓ $\mathbb{Q}^c$(The set of irrational numbers).
- ✓ $\mathbb{R}$(The set of real numbers) $= \mathbb{Q} \cup \mathbb{Q}^c$.
- ✓ $\mathbb{C}$(The set of complex numbers) $= \{z = x + iy / x, y \in \mathbb{R}\}$.
- ✓ $\mathbb{R}^+$(The set of positive real numbers).
- ✓ $\mathbb{Z}^*$(The set of invertible elements of $\mathbb{Z}$ with respect to multiplication) $= \{1, -1\}$.
- ✓ $\mathbb{Z}_n = \{[0], [1], [2], [3], \dots, [n-1]\}$.
  (The set of equivalance class for the relation congruence modulo n).
- ✓ Let A be a non-empty set then $A \times A = \{(a, b) / a, b \in A\}$. (Cartesian product).

❖ BINARY OPERATION

- ✓ A function $*: A \times A \to A$ is said to be a binary operation on A.
- ✓ i.e., $*(a, b) = a * b \in A, \forall a, b \in A$ then $*$ is said to be a binary operation on A.
- ✓ Examples
  - ➢ '$+$' is a binary operation on $\mathbb{N}$.
  - ➢ '$\times$' is a binary operation on $\mathbb{R}$.
  - ➢ '$-$' is not binary operation on $\mathbb{N}$.
  - ➢ '/' is not binary operation on $\mathbb{Z}$.

❖ ALGEBRAIC STRUCTURE

- ✓ A nonempty set G equipped with one or more binary operation is said to be an algebraic structure.

- ✓ Examples: $(\mathbb{N},+)$, $(\mathbb{Z},+)$ & $(\mathbb{R},+,\ \cdot)$.

❖ SEMIGROUP

- ✓ Let S be a nonempty set and '$*$' be a binary operation on S. Then $(S,\ *)$ is said to be a semigroup if the binary operation '$*$' is associative in S.
- ✓ i.e., $\forall\ a, b, c \in S$
  - ➢ $a * b \in S$ (Binary operations/Closure property).
  - ➢ $a * (b * c) = (a * b) * c$ (Associative property on $*$).
- ✓ Example: $(\mathbb{N},+)$, $(\mathbb{N},\ \times)$, $(\mathbb{Z},+)$ & $(\mathbb{R},\ \times)$.

❖ MONOID

- ✓ Let M be a nonempty set and ' $*$ ' be any operation on M. Then, $(M,\ *)$ is said to be a Monoid if the following conditions are satisfied.
  - ➢ Closure property: M is closed under ' $*$ '. i.e. $\forall\ a, b \in M \Rightarrow a * b \in M$.
  - ➢ Associative property: ' $*$ ' is associative in M. i.e.

$$\forall\ a, b, c \in M \Rightarrow (a * b) * c = a * (b * c).$$

  - ➢ Existence of an identity element: There exists an element $e \in M$ such that

$$a * e = e * a = a;\ \forall a \in M$$

- ✓ Examples
  - ➢ $(\mathbb{N},\ \times)$, $(\mathbb{Z},\ \times)$ & $(\mathbb{R},\ \times)$.
  - ➢ Let S be a non-empty set and $P(S)$ be its power set. Then
    1) $(P(S),\cup)$ is monoid with the identity ϕ.
    2) $(P(S),\cap)$ is monoid with the identity S.

❖ GROUP

- ✓ Let G be a nonempty set and ' $*$ ' be an operation on G. Then $(G,\ *)$ is said to be a Group if the following conditions are satisfied.
  - ➢ Closure property: G is closed under ' $*$ ' . i.e. $\forall\ a, b \in G \Rightarrow a * b \in G$.
  - ➢ Associative property: ' $*$ ' is associative in G. i.e. $\forall\ a, b, c \in G \Rightarrow (a * b) * c = a * (b * c)$.
  - ➢ Existence of an identity element: There exists an element $e \in G$ such that

$$a * e = e * a = a;\ \forall a \in G.$$

- Existence of an inverse for each element of G: For every $a \in G$, there exists $b \in G$ such that $a * b = e = b * a$. Here $b = a^{-1}$ is said to be an inverse of a.

❖ ABELIAN GROUP / COMMUTATIVE GROUP

- ✓ A group $(G, *)$ is said to be an Abelian group/commutative group if $a * b = b * a;\ \forall a, b \in G$.
- ✓ Example: $(\mathbb{Z}, +), (\mathbb{R}, +), (\mathbb{R} - \{0\}, \times)$ & $(\mathbb{Q} - \{0\}, \times)$.

❖ COMPOSITION TABLE

- ✓ Let $G = \{a_1, a_2, \dots, a_n\}$ is a finite set having n elements then there is a binary operation defined on G multiplicatively. All possible binary composition elements of G can be arranged in a tabular form as:
- ✓ Write elements of S in a horizontal row, say it column header and in a vertical column, say it row header.
- ✓ The element $a_i * a_j$ associated with the ordered pair (a_i, a_j) is placed at the intersection of the row headed by a_i and column a_j.
- ✓ Notes
  - Let 'e' be an identity element in group $(G, *)$ then 'e' is unique.
  - Inverse of each element of a group $(G, *)$ is unique.
  - If a^{-1} is the inverse of 'a' of group $(G, *)$ then $(a^{-1})^{-1} = a$.
  - If $(G, *)$ be a group then $\forall a, b \in G \Rightarrow (a * b)^{-1} = b^{-1} * a^{-1}$.
  - Let $(G, *)$ be a group and $a, b, c \in G$ then $a * b = a * c \Rightarrow b = c$ (Left cancellation law).
  - Let $(G, *)$ be a group and $a, b, c \in G$ then $b * a = c * a \Rightarrow b = c$ (Right cancellation law).
  - If each element of group G is self-invertible then G is Abelian.
  - Let $(G, *)$ be a group and if $\forall a, b \in G \Rightarrow (a * b)^2 = a^2 * b^2$ then $(G, *)$ must be Abelian.

METHOD-1: EXAMPLES ON ALGEBRAIC STRICTURES WITH ONE BINARY OPERATIONS

C	1	Let $*$ be a binary operation on $\mathbb{R}$ defined by $a * b = a + b + 2ab$. a) Find $2 * 3, 3 * (-5)$ & $7 * (1/2)$. b) Is $(\mathbb{R}, *)$ semigroup? Is it commutative? c) Find the identity element in $\mathbb{R}$ with respect to $*$. d) Which elements has inverses and what are they?	
H	2	Show that the set of square roots of unity forms a group under multiplication.	
C	3	Show that the set of cube roots of unity forms a group under multiplication.	
T	4	Show that the set of fourth roots of unity forms a group under multiplication.	
H	5	Show that the set of rational numbers excluding zero is an Abelian group under multiplication. i.e., $(\mathbb{Q}^*, \times)$ is an Abelian group.	
C	6	Show that the set of all positive rational numbers forms an Abelian group under the composition defined by $a * b = \frac{ab}{2}$.	
T	7	Show that the set $G = \{a + b\sqrt{2} \,/ a, b \in \mathbb{Q}\}$ is a group with respect to addition.	
C	8	Show that $G = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} / ad - bc \neq 0 \;\&\; a, b, c, d \in \mathbb{R}\right\}$ is a group under matrix multiplication.	
T	9	Show that the set of matrices $\left\{A\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} / \alpha \in \mathbb{R}\right\}$ is form a group under matrix multiplication.	
T	10	Show that $G = \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} / a, b \in \mathbb{R}\right\}$ is a commutative group under matrix addition.	
H	11	Show that $G = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right\}$ is an Abelian group under matrix multiplication.	
C	12	Show that $(\mathbb{Z}_6, +_6)$ is an Abelian group.	
C	13	Show that $(\mathbb{Z}_5, \times_5)$ is a monoid but not group.	
T	14	Show that $(\mathbb{Z}_5 - \{0\}, \times_5)$ is group.	
H	15	Write down the composition tables for $(\mathbb{Z}_7, +_7)$ and $(\mathbb{Z}_7^*, \times_7)$.	

C	16	Show that the set of real polynomials in variable x is a group under the operation of addition.	

❖ ORDER OF A GROUP

- ✓ Let $(G, \ *)$ be a finite group then the number of distinct elements in G is called the order of a group G. It is denoted by $O(G)$ **or** $|G|$.
- ✓ Examples
  - ➢ Let $G = \{1, i, -1, -i\}$ then $O(G) = 4$.
  - ➢ $O(\mathbb{Z}_n) = n$.

❖ PERMUTATIONS

- ✓ Let S be a finite set having n-distinct elements. Then a one-one mapping from S onto S is called a permutation of S of degree n.
- ✓ The number of elements in the finite set S is known as the degree of permutation.
- ✓ Notes
  - ➢ Let $S = \{a_1, a_2, \dots, a_n\}$ then permutation $f = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$. Where, $f : S \to S$ define as $f(a_i) = b_i, \forall\, b_i \in S$.
  - ➢ If $S = \{a, b\}$ then set of all permutation on S_2 is $\left\{p_1 = I = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, p_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\right\}$.
  - ➢ If S_n be the set consisting of all permutation of degree n, then S_n will have n! distinct elements. i.e., $o(S_n) = n!$.
  - ➢ $S_n = \{f :\ f \text{ is a permutation of degree } n\}$ is called symmetric group.

❖ IDENTITY PERMUTATION

- ✓ If I is a permutation of degree ‘n’ such that $I(a) = a\ ; \forall a \in S$, then I is called the identity permutation of degree ‘n’.
- ✓ Thus if $S = \{1, 2, 3, \dots, n\}$ then $I = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$ is an identity permutation on S of degree ‘n’.

❖ EQUALITY OF TWO PERMUTATION

- ✓ Let S be any finite set. Then, two permutation f and g of degree 'n' on S are said to be equal if we have $f(a) = g(a) : \forall a \in S$.
- ✓ Example: If $S = \{1, 2, 3, 4\}$ and $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1 \ \ 2 \ \ 3 \ \ 4)$ & $g = \begin{pmatrix} 2 & 4 & 3 & 1 \\ 3 & 1 & 4 & 2 \end{pmatrix} = (2 \ \ 3 \ \ 4 \ \ 1)$ are two permutation of degree 4 then we have $f = g$.
- ✓ Note: If $S = \{1, 2, 3\}$ then the set S_3 of all permutations of degree 3 will have $3! = 6$ distinct elements. Then,

$$S_3 = \left\{ \begin{matrix} p_1 = I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \\ p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, p_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \end{matrix} \right\}$$

❖ PRODUCT OR COMPOSITION OF TWO PERMUTATIONS

- ✓ The product(composition)of two permutations f and g of degree 'n' denoted by $f \cdot g$ is obtained by first carrying out the operation defined by f and then by g.
- ✓ Suppose, S_n is the set of all permutation of degree n. Let $f = \begin{pmatrix} b_1 & b_2 & \dots & b_n \\ c_1 & c_2 & \dots & c_n \end{pmatrix}$ and $g = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$ be any two elements of S_n.
- ✓ Then $f \cdot g = \begin{pmatrix} b_1 & b_2 & \dots & b_n \\ c_1 & c_2 & \dots & c_n \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ c_1 & c_2 & \dots & c_n \end{pmatrix} \in S_n$.

METHOD-2: EXAMPLES ON PERMUTATION GROUP

T	1	Show that S_2(the set of all permutations on two symbols 1 & 2) is a group of order 2 with respect to composition of permutation.	
C	2	Show that S_3(the set of all permutations on three symbols 1, 2 & 3) is a finite non-Abelian group of order 6 with respect to composition of permutation.	
H	3	Let the permutations of the elements of $\{1, 2, 3, 4, 5\}$ be given by $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 1 & 2 \end{pmatrix}$ & $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$. Find $\alpha\beta, \beta\alpha, \alpha^2, \gamma\beta, \delta^{-1}, \alpha\beta\gamma$ and solve the equation $\alpha x = \beta$.	

C	4	Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 4 & 6 & 2 \end{pmatrix}$ & $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{pmatrix}$. Find $\tau\sigma, \sigma\tau, \sigma^2, \sigma^{-1}$.	
C	5	Let $f = (1\ \ 2\ \ 3)(1\ \ 4\ \ 5) \in S_5$ then find f^{-1} & f^{99}.	

❖ CYCLIC GROUP

- ✓ A group $(G, *)$ is said to be cyclic group if there exists an element $a \in G$ such that every element of G can be written as some power of 'a'. i.e., a^n for some integer n. In such a case cyclic group is said to be generated by 'a' or 'a' is a generator of the group G.
- ✓ Notes
  - ➢ $< a >= \{a^n / n \in \mathbb{Z}\}$ if binary operation is multiplication and $< a >= \{na / n \in \mathbb{Z}\}$ if binary operation is addition.
  - ➢ A cyclic group is Abelian. Because for any $p, q \in G \Rightarrow p = a^r$ & $q = a^s$ for some $r, s \in I$ and

$$p * q = a^r * a^s = a^{r+s} = a^{s+r} = a^s * a^r = q * p$$

  - ➢ The generators of the cyclic group $(\mathbb{Z}_n, +_n)$ are those elements from $\mathbb{Z}_n$ which are relatively prime to 'n'. i.e., for $a \in \mathbb{Z}_n$ such that $(a, n) = 1$ then 'a' is a generator of $\mathbb{Z}_n$.

❖ SUBGROUP

- ✓ Let G be a group under the binary operation '$*$'. A nonempty subset H of a group G is said to be a subgroup of G if H itself a group under the same binary operation '$*$'.
- ✓ Examples
  - ➢ $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Q}, +)$.
  - ➢ $(\mathbb{Q}, +)$ is a subgroup of $(\mathbb{R}, +)$.
- ✓ Note: For any group $(G, *)$ we always have two subgroups which are $(\{e\}, *)$ and $(G, *)$. Where 'e' is the identity element of G. These two subgroups are called improper/trivial subgroups of $(G, *)$ and other subgroups are called proper/non-trivial subgroups of $(G, *)$.
- ✓ Theorem 1: A non-empty subset H of a group G is a subgroup of $(G, *)$ if and only if

$$a \in H, b \in H \Rightarrow a * b \in H$$

$$a \in H \Rightarrow a^{-1} \in H$$

- ✓ Theorem 2: A non-empty subset H of a group G is a subgroup of $(G, *)$ if and only if

$$a \in H, b \in H \Rightarrow a * b^{-1} \in H$$

- ✓ Theorem 3: If H_1 and H_2 are two subgroups of a group G, then $H_1 \cap H_2$ is also a subgroup of G.
- ✓ Notes
  - ➢ The union of two subgroups is not necessarily a subgroup.
  - ➢ The number of subgroups of the group $(\mathbb{Z}_n, +_n)$ is the number of positive divisors of 'n'.

❖ LANGRAGE'S THEOREM

- ✓ The order of a subgroup of a finite group divides the order of the group.

METHOD-3: EXAMPLES ON CYCLIC GROUP AND SUBGROUP

C	1	Show that $\mathbb{Z}$ is a cyclic group.	
C	2	Find generators of cyclic groups (a) $(\mathbb{Z}_5, +_5)$ and (b) $(\mathbb{Z}_6, +_6)$.	
T	3	Determine all the proper and improper subgroups of symmetric group $(S_3, \circ)$.	
C	4	Consider the group $G = \{1, 2, 3, 4, 5, 6\}$ under the binary operation multiplication modulo 7. i.e., $(\mathbb{Z}_7^*, \times_7)$. a) Find the multiplication table of G. b) Find $2^{-1}, 3^{-1}$ and 6^{-1}. c) Find the subgroup generated by 2 and 3. Also find its order. d) Is G cyclic?	
C	5	Find all subgroups of $(\mathbb{Z}_7^*, \times_7)$.	
H	6	Find all subgroups of $(\mathbb{Z}_{10}^*, \times_{10})$.	
C	7	Find all subgroups of $(\mathbb{Z}_{12}, +_{12})$.	
H	8	Find all subgroups of $(\mathbb{Z}_{30}, +_{30})$.	
T	9	Show that $(\{1, 4, 13, 16\}, \times_{17})$ is a subgroup of $(\mathbb{Z}_{17}^*, \times_{17})$.	

❖ COSET

- ✓ Let $(H, *)$ be a subgroup of $(G, *)$. For any $a \in G$, the set aH defined by $aH = \{a * h / h \in H\}$ is called the left coset of H in G determined by the element $a \in G$. The element 'a' is called the representative element of the left coset aH.
- ✓ Similarly, $Ha = \{h * a / h \in H\}$ is called right coset of H in G determined by a.
- ✓ Notes
  - ➢ Ha and aH both are subsets of G.
  - ➢ No left or right coset can be empty.
  - ➢ $a \in H$ iff $aH = H$.
  - ➢ If the group G is Abelian then we have $aH = Ha, \forall a \in H$.
  - ➢ If G is additive group then its coset are generated by 'a' defined as

    $H + a = \{h + a : h \in H\}$(Right Coset) and $a + H = \{a + h : h \in H\}$(Left Coset).
  - ➢ The number of left cosets of H in G is called the index of H in G.

❖ NORMAL SUBGROUP

- ✓ A subgroup H of a group G is said to be normal subgroup of G if

$$\forall x \in G \quad \& \quad \forall h \in H \text{ such that } xhx^{-1} \in H$$

- ✓ i.e., $xHx^{-1} = H, \forall x \in H$

  $\Longrightarrow xH = Hx, \forall x \in H$

  $\Longrightarrow$ Every left coset of H in G is right coset of H in G.
- ✓ Notes
  - ➢ If a group G is Abelian group then every subgroup of G is normal subgroup of G.
  - ➢ G and $\{e\}$ are improper normal subgroups of G.
  - ➢ A group having no proper normal subgroup is called simple group.
  - ➢ A subgroup H of a group G is normal subgroup of G if and only if the product of two right coset of H in G is again a right coset.
  - ➢ Intersection of two normal subgroups is a normal subgroup.

❖ GROUP HOMOMORPHISM

- ✓ Let $(G, *)$ and (H, Δ) be two groups. A mapping $g: G \to H$ is called a group homomorphism from $(G, *)$ to (H, Δ) if for any $a, b \in G \Rightarrow g(a * b) = g(a)\, \Delta\, g(b)$.

❖ CONGRUENCE RELATION AND QUOTIENT STRUCTURES

- ✓ Consider the set of integers $\mathbb{Z}$ and a positive integer $m > 1$. Then we say that a is congruent to b modulo m written as $a \equiv b \pmod{m}$ if m divides the difference $a - b$. This relation is an equivalence relation on $\mathbb{Z}$.
- ✓ Let N be a normal subgroup of group G, then the set of all distinct right(left) cosets of N in G is called quotient group under multiplication/addition of cosets of N in G.
- ✓ Let N be a normal subgroup of group G then $G/N = \{Na / a \in G\}$.
- ✓ For $N_a, N_b \in G/N$, the multiplication of cosets is defined as $N_a . N_b = N_{ab}$.
- ✓ If G is an Abelian group then G/N is also an Abelian group.
- ✓ If $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$ then $a + b \equiv c + d \pmod{m}$ &
- ✓ If $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$ then $ab \equiv cd \pmod{m}$.

METHOD-4: EXAMPLES ON COSET, HOMOMORPHISM, LANGRAGE'S THEOREM, CONGRUENCE RELATION AND QUOTIENT STRUCTURES

H	1	Find the left cosets of $\{[0], [3]\}$ in the group $(\mathbb{Z}_6, +_6)$.	
C	2	Find the left cosets of $H = \{p_1, p_5, p_6\}$ in the group $G = (S_3, \circ)$.	
C	3	Find the left cosets of $\{[0], [2]\}$ in the group $(\mathbb{Z}_4, +_4)$.	
T	4	Let $H = \{p_1, p_2\}$. Find the left cosets of H in the group $(S_3, \circ)$.	
T	5	Let $G = (\mathbb{Z}, +)$ be a group and $H = 5\mathbb{Z}$. Then, find G/H and show that $(G/H, +)$ is group.	

PART-II ALGEBRAIC STRUCTURES WITH TWO BINARY OPERATIONS

❖ RING

- ✓ An algebraic structure $(R, +, \cdot)$ is called a ring if the binary operations '+' and '·' on S satisfy the following three properties.
  - ➢ $(S, +)$ is an Abelian group.
  - ➢ $(S, \cdot)$ is a semigroup.
  - ➢ The operation '·' is distributive over '+'. i.e., for any $a, b, c \in S$,

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (b + c) \cdot a = b \cdot a + c \cdot a$$

- ✓ Example: $(\mathbb{Z}, +, \cdot), (\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot), (\mathbb{C}, +, \cdot)$ & $(2\mathbb{Z}, +, \cdot)$.
- ✓ Notes
  - ➢ The ring $(S, +, \cdot)$ is called a ring with identity if $(S, \cdot)$ is monoid.
  - ➢ The ring $(S, +, \cdot)$ is called a commutative ring if $(S, \cdot)$ is commutative.

❖ ZERO DIVISORS

- ✓ If there exist nonzero elements $a, b \in (R, +, \cdot)$ such that $a \cdot b = 0$ then a and b are called zero divisors.
- ✓ Example: In a ring $(\mathbb{Z}_6, +_6, \cdot_6)$, $[2], [3] \in \mathbb{Z}_6$ such that $[2].[3] = [6] = [0]$. Hence [2] and [3] are zero divisors in $\mathbb{Z}_6$.

❖ INTEGRAL DOMAIN

- ✓ A commutative ring $(S, +, \cdot)$ with identity and without zero divisors is called an integral domain.
- ✓ Example: $(\mathbb{Z}, +, \cdot), (\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot)$.

❖ FIELD

- ✓ A commutative ring $(S, +, \cdot)$ with identity element such that every nonzero element of S has a multiplicative inverse in S is called a field.
- ✓ Example: $(\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot)$ & $(\mathbb{C}, +, \cdot)$.

METHOD-5: EXAMPLES ON ALGEBRAIC STRUCTURE WITH TWO BINARY OPERATIONS

H	1	Show that $(\mathbb{Z}_6, +_6, \times_6)$ is not an integral domain.	
C	2	Show that $(\mathbb{Z}_7, +_7, \times_7)$ is an integral domain.	
C	3	Show that $(\mathbb{Z}_n, +_n, \times_n)$ is ring.	
T	4	Show that $(\mathbb{Z}_n, +_n, \times_n)$ is field if and only if n is prime integer.	
C	5	Prove that $F = \{a + b\sqrt{2} \,/ a, b \in \mathbb{Q}\}$ is a field.	
H	6	Prove that $F = \{a + b\sqrt{2} \,/ a, b \in \mathbb{Z}\}$ is an integral domain but not a field.	
C	7	Prove that $R = \{a + ib / a, b \in \mathbb{Z}\}$ is an integral domain. Also find it's units.	
C	8	Consider the ring $(\mathbb{Z}_{10}, +_{10}, \times_{10})$. a) Find the units of $\mathbb{Z}_{10}$. b) Find -3, -8 and 3^{-1}. c) Let $f(x) = 2x^2 + 4x + 4$. Find the roots of f(x) over $\mathbb{Z}_{10}$.hb	

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UNIT-5 » GRAPH THEORY

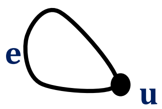
❖ INTRODUCTION

- ✓ Graph theory plays an important role in several areas of computer science such as switching theory and logical design, artificial intelligence, formal languages, computer graphics, operating systems, compiler writing, information organization and retrieval.

❖ GRAPH

- ✓ A graph $G = < V, E, \emptyset >$ consists of a nonempty set V called the set of nodes (points, vertices) of the graph, E is said to be the set of edges of the graph and $\emptyset$ is a mapping from the set of edges E to a set of ordered or unordered pairs of elements of V.
- ✓ $V = V(G) = \{v_1, v_2, v_3, \dots\}$ = The set of nodes (vertices/points/dots/junctions).
- ✓ $E = E(G) = \{e_1, e_2, e_3, \dots\}$ = The set of edges (branch/line/arc).
- ✓ The elements of V are called nodes/vertices of a graph G and the elements of E are called edges of a graph G.

- ✓ Notes
  - ➢ Throughout, we shall assume that the sets V and E of a graph G are finite.
  - ➢ Any edge e can be made by one OR two nodes (u, u) OR (u, v) respectively.



❖ ADJACENT NODES

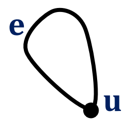
- ✓ If two nodes u and v are joined by an edge e then u and v are said to be adjacent nodes.

❖ INCIDENT EDGE

- ✓ An edge $e \in E$ (directed/undirected) which joins the nodes u and v is said to be incident to the nodes u and v.

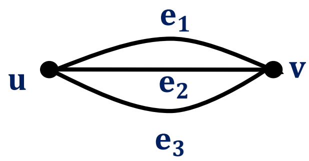
❖ LOOP(SLING)

✓ An edge e of a graph G that joins a node u to itself is called a loop. A loop is an edge $e = (u, u)$.



❖ PARALLEL EDGES

✓ If two nodes of a graph are joined by more than one edge then these edges are called parallel edges/multiple edges.

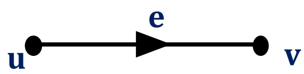


❖ DIRECTED EDGES

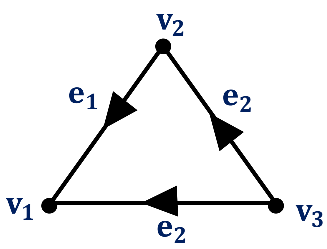
✓ In a graph G an edge 'e' which is associated with an ordered pair of nodes 'u' to 'v' is called directed edge of graph G.

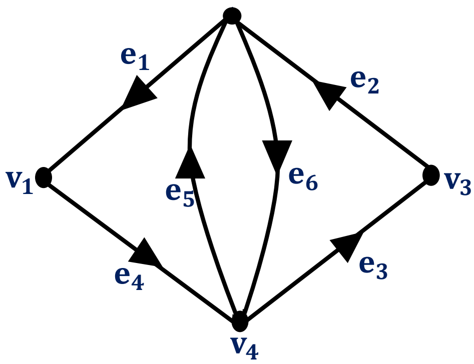
❖ INITIATING NODE AND TERMINATING NODE

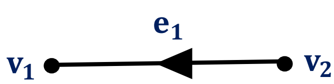
✓ Let $G = (V, E)$ be a graph and let $e \in E$ be a directed edge associated with the order pair $< u, v >$ of the nodes 'u' and 'v' then the edge 'e' is said to be initiating(originating) in the node 'u' and terminating(ending) in the node 'v'. The nodes 'u' and 'v' are called initial node and terminal node of the edge 'e' respectively.



❖ DIRECTED GRAPH/DIGRAPH







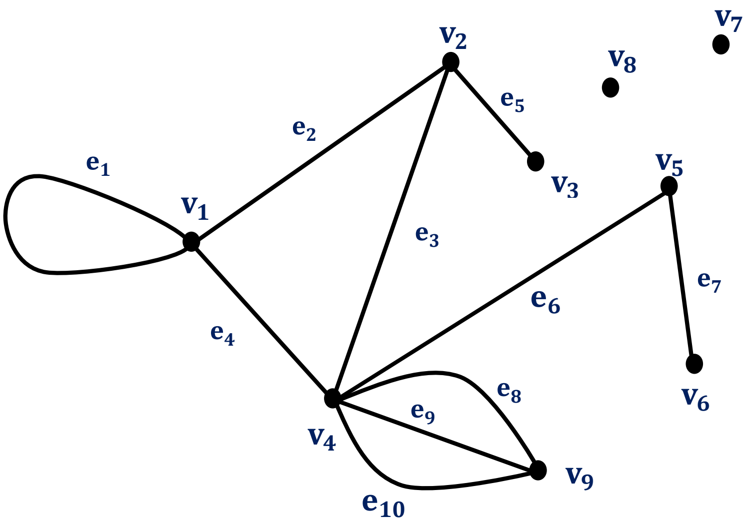
✓ A graph in which every edge is directed is called a directed graph (digraph).

❖ UNDIRECTED EDGE

- ✓ In a graph G an edge 'e' which is associated with an unordered pair (u, v) of nodes 'u' and 'v' is called undirected edge of graph G.

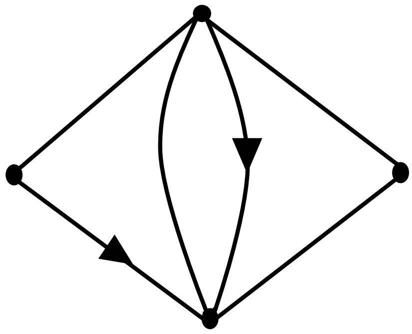
❖ UNDIRECTED GRAPH

- ✓ A graph in which every edge is undirected is called an undirected graph.



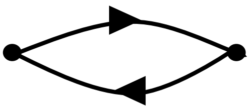
❖ MIXED GRAPH

- ✓ If some edges of a graph G are directed and some are undirected then G is said to be a mixed graph.



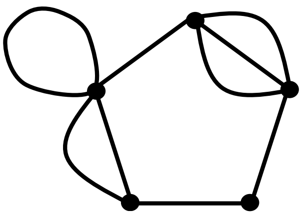
❖ DISTINCT EDGES

- ✓ The two possible edges between a pair of nodes which are opposite in direction which are known as distinct edges.



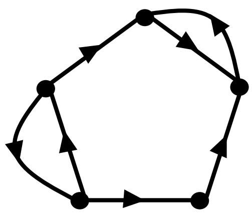
❖ MULTI GRAPH

- ✓ Any graph which contains some parallel edges is called a multigraph.



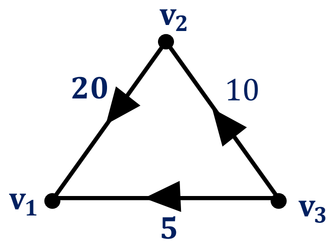
❖ SIMPLE GRAPH

- ✓ A graph which has neither loop nor parallel edges is called a simple graph.



❖ WEIGHTED GRAPH

- ✓ A graph in which weighted are assigned to every edge is called a weighted graph.

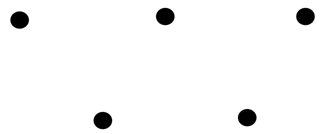


❖ ISOLATED NODE

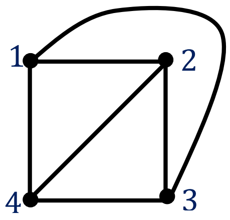
- ✓ In a graph a node which is not adjacent to any other node is called an isolated node.

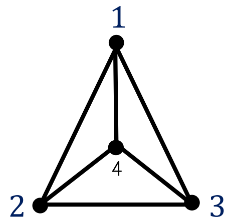
❖ NULL GRAPH

- ✓ A graph containing only isolated nodes is called a null graph.
- ✓ The set of edges in the null graph is empty.



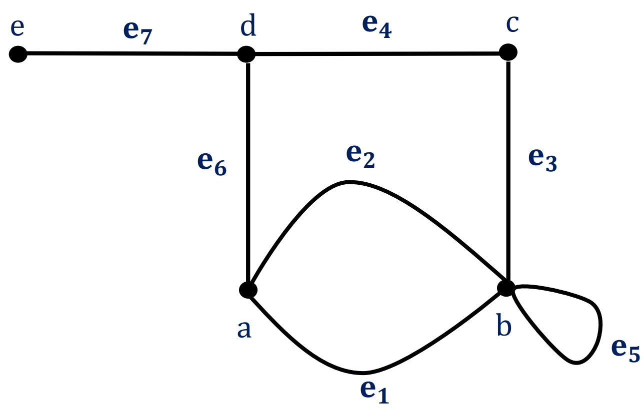
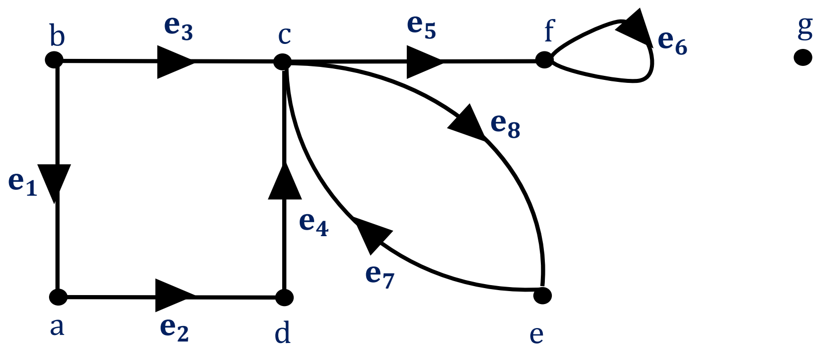
- ✓ Note
  - ➢ It can happen that two diagrams which look entirely different but both may represent the same graph.





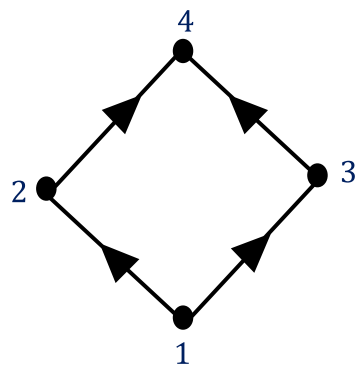
METHOD-1: BASIC DEFINITIONS AND RELATED EXAMPLES

H	**1**	Define with example: graph, nodes and edges.	
T	**2**	Define with example: adjacent nodes, initiating node, terminating node and isolated node.	
T	**3**	Define with example: incident edges, loop, parallel edges, directed edges, undirected edges and distinct edges.	
H	**4**	Define with example: directed graph, undirected graph, mixed graph, multi graph, simple graph, weighted graph and null graph.	

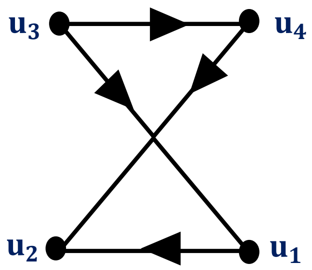
C	5	Draw the undirected graph $G = (V, E)$ where, $V = \{a, b, c, d, e\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and its incidence relation given as: $e_1 = (a, b), e_2 = (a, b), e_3 = (b, c), e_4 = (c, d), e_5 = (b, b), e_6 = (a, d)$ & $e_7 = (e, d)$. Discuss the terms defines in example 1 to 4 for G. **Answer :**  	
C	6	Draw the directed graph $G =< V, E >$ where, $V = \{a, b, c, d, e, f, g\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ and its incidence relation given as: $e_1 =< b, a >, e_2 =< d, a >, e_3 =< b, c >, e_4 =< d, c >, e_5 =< c, f >, e_6 =< f, f >, e_7 =< e, c >$ & $e_8 =< c, e >$. Discuss the terms defines in example 1 to 4 for G. **Answer :**  	

❖ ISOMORPHIC GRAPH

- ✓ A graph $G_1 = (V_1, E_1)$ is said to be isomorphic to the graph $G_2 = (V_2, E_2)$ if there exists a bijection between the set of nodes V_1 and V_2 and a bijection between the set of edges E_1 and E_2 such that if e is an edge with end nodes u and v in G_1 then the corresponding edge e' has its end nodes u' and v' in G_2 which correspond to u and v respectively. If such pair of bijections exist then it is called a graph isomorphism and it is denoted by $G_1 \cong G_2$.
- ✓ According to the definition of isomorphism we note that any two nodes in one graph which are joined by an edge must have the corresponding nodes in the other graph also joined by an edge and hence a one to one correspondence exists between the edges as well.
- ✓ Example



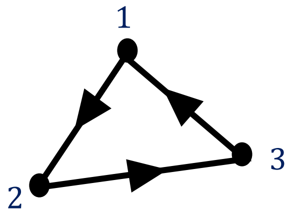
$\mathbf{G_1}$



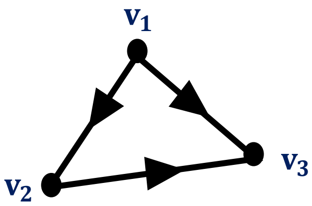
$\mathbf{G_2}$

  - ➢ Here, G_1 and G_2 are isomorphic because of the existence of a mapping $1 \rightarrow u_3, 2 \rightarrow u_1, 3 \rightarrow u_4$ & $4 \rightarrow u_2$.
  - ➢ Under this mapping the edges $< 1, 3 >, < 1, 2 >, < 2, 4 >$ & $< 3, 4 >$ are mapped into $< u_3, u_4 >, < u_3, u_1 >, < u_1, u_2 >$ & $< u_4, u_2 >$ which are the only edges of the graph in G_2.
- ✓ Note
  - ➢ The two graphs which are isomorphic have the same number of nodes and edges but converse need not be true. i.e., If the two graphs which has same number of nodes and edges implies both graphs need not be isomorphic.

- Example



G_1

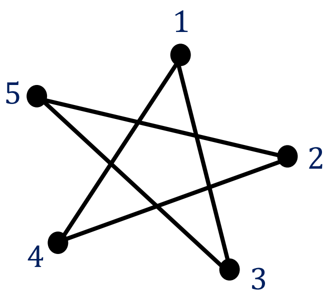


G_2

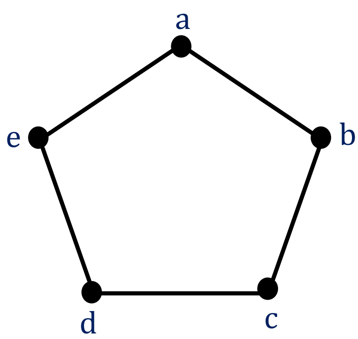
- Here G_1 and G_2 both has same number of nodes and edges but G_1 is not isomorphic to G_2. i.e., $G_1 \ncong G_2$ because the edge $< 3, 1 > \nrightarrow < v_1, v_3 >$.

- Note
  - The concept of isomorphism also defines in undirected graphs with the same definition given for directed graphs.

- Example



G_1



G_2

- Here G_1 and G_2 are isomorphic because of the existence of a mapping $1 \to a, 2 \to d, 3 \to b, 4 \to e$ & $5 \to c$.
- Under this mapping, the edges $(1, 3), (3, 5), (5, 2), (2, 4)$ & $(4, 1)$ are mapped into $(a, b), (b, c), (c, d), (d, e),$ & (e, a) which are the only edges of the graph in G_2.

❖ ORDER OF A GRAPH

- The number of nodes in a graph G is called order of the graph G.

❖ SIZE OF A GRAPH

- The number of edges in a graph G is called size of the graph G.

❖ DEGREE OF A NODE

- ✓ Let G be an undirected graph then the degree of a node v in G is defined as the number of edges incident on v. It is denoted by $d(v)$ or $d_G(v)$ or $deg(v)$.
- ✓ Note
  - ➢ Self-loop will be counted twice in the degree of corresponding node.

❖ ODD NODE

- ✓ A node with odd degree is called an odd node.

❖ EVEN NODE

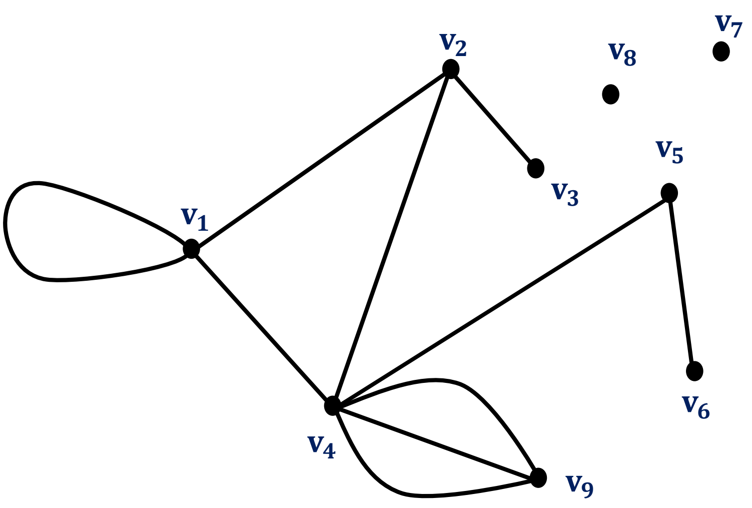
- ✓ A node with even degree is called an even node.

❖ ISOLATED NODE

- ✓ A node with degree zero is called isolated node.

❖ PENDANT NODE

- ✓ A node with degree one is called a pendent node.
- ✓ Example



  - ➢ Here, $d(v_1) = 4, d(v_2) = 3, d(v_3) = 1, d(v_4) = 6, d(v_5) = 2, d(v_6) = 1, d(v_7) = 0, d(v_8) = 0$ & $d(v_9) = 3$.

- From degree of nodes we conclude that nodes v_1, v_4 & v_5 are even nodes, nodes v_2, v_3, v_6 & v_9 are odd nodes, nodes v_7 & v_8 are isolated nodes and nodes v_3 & v_6 are pendent nodes.

❖ HANDSHAKING THEOREM/DEGREE SUM FORMULA/FIRST THEOREM OF GRAPH THEORY

✓ STATEMENT:

Any undirected graph G with n nodes $v_1, v_2, \dots, v_n$ and e edges, $\sum_{i=1}^{n} d(v_i) = 2e$.

PROOF:

- Let $G = (V, E)$ be a graph with n nodes $v_1, v_2, v_3, \dots, v_n$ and e edges.
- Let $e_k \in E$, then e_k have two terminated nodes say v_i & v_j.
- By definition of degree of node e_k is count in degree of v_i and degree of v_j.

- Therefore, every edge of G is count twice and if e_k is loop then it is also count twice.
- Hence, the sum of degree of all nodes of G is twice to the number of edges.

$$\text{i.e.,} \sum_{i=1}^{n} d(v_i) = 2e.$$

✓ Example

- In above graph,

$$\sum_{i=1}^{9} d(v_i) = d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) + d(v_7) + d(v_8) + d(v_9)$$

$$= 4 + 3 + 1 + 6 + 2 + 1 + 0 + 0 + 3 = 20 = 2(10) = 2e.$$

❖ SECOND THEOREM OF GRAPH THEORY

✓ STATEMENT:

In any undirected graph G, number of odd nodes must be even.

PROOF:

- Let $G = (V, E)$ be a graph with n nodes $v_1, v_2, v_3, \dots, v_n$ and e edges.

- By first theorem of graph theory $\sum_{i=1}^{n} d(v_i) = 2e$.
- We divide V into two sets say U and W. where, U is the set of even nodes and W is the set of odd nodes.
- Therefore, $\sum_{v \in U} d(v) + \sum_{v \in W} d(v) = 2e$

$$\Rightarrow \sum_{v \in W} d(v) = 2e - \sum_{v \in U} d(v) = \text{even} - \text{even} = \text{even}$$

$$\Rightarrow \sum_{v \in W} d(v) \text{ is even number}$$

- Hence, number of odd vertices in G are even.

✓ Example

- In above graph there are 4(even) number of odd nodes.

❖ INDEGREE

✓ Let G be a directed graph then for any node v in G, the number of edges which have v as their terminal node is called the indegree of the node v.

OR

✓ In a directed graph G, the number of edges directed towards node v is called indegree of a node v. It is denoted by $d^-(v)$.

❖ OUTDEGREE

✓ Let G be a directed graph then for any node u in G, the number of edges which have u as their initial node is called the outdegree of the node u.

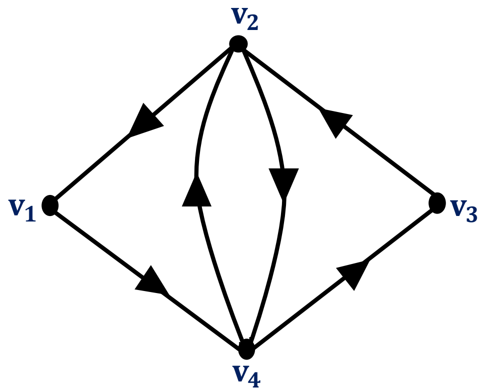
OR

✓ In a directed graph G, the number of edges directed outwards node u is called outdegree of a node u. It is denoted by $d^+(u)$.

❖ TOTAL DEGREE OF A NODE

✓ Sum of the indegree and the outdegree of a node v is called total degree of a node v. It is denoted by d(v). i.e., $d(v) = d^-(v) + d^+(v)$.

- Example
  - $d^+(v_1) = 1, d^-(v_1) = 1 \Rightarrow d(v_1) = 2$
  - $d^+(v_2) = 2, d^-(v_2) = 2 \Rightarrow d(v_2) = 4$
  - $d^+(v_3) = 1, d^-(v_3) = 1 \Rightarrow d(v_3) = 2$
  - $d^+(v_4) = 2, d^-(v_4) = 2 \Rightarrow d(v_4) = 4$



- Notes
  - The total degree of an isolated node is 0.
  - The node with degree 1 is known as pendent node.

DEGREE SUM FORMULA FOR DIRECTED GRAPH

- STATEMENT:

In directed graph G with n nodes $v_1, v_2, \dots, v_n$ and e edges

$$\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i) = e \text{ and } \sum_{i=1}^{n} d(v_i) = 2e.$$

PROOF:

- Let $G = \langle V, E \rangle$ be a directed graph with n nodes $v_1, v_2, \dots, v_n$ and e edges.
- A node has indegree corresponding to simple edge is exactly one. Once an edge is count for indegree it will not count for outdegree for the same node.
- When we are adding the indegree of nodes, each edge is counted exactly once.
- Hence, total of indegrees of all nodes is same as the number of edges.

$$\text{i.e.,} \sum_{i=1}^{n} d^-(v_i) = e$$

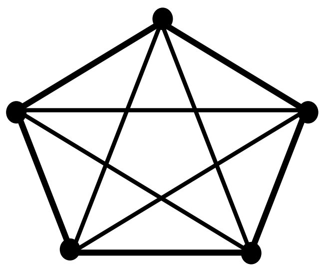
- Similarly, we can prove $\sum_{i=1}^{n} d^+(v_i) = e$.

- Example
  - In above example,

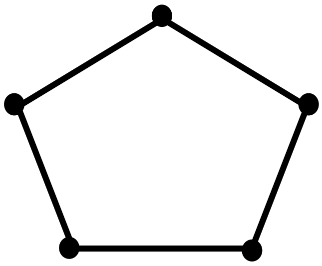
$$\sum_{i=1}^{4} d^+(v_i) = \sum_{i=1}^{4} d^-(v_i) = 6 \text{ and } \sum_{i=1}^{4} d(v_i) = 12 = 2 \text{ times no. of edges .}$$

❖ SUBGRAPHS

- ✓ Let G and H be two graphs. Then H is said to be a subgraph of G if $V(H) \subseteq V(G)$ & $E(H) \subseteq E(G)$. Here G is called super graph of H.
- ✓ Example



G



H

  - ➢ Here H is subgraph of G.
- ✓ Note
  - ➢ The graph G as well as the null graph obtained from G by deleting all the edges of G are subgraphs of G. Other subgraphs of G can be obtained by deleting certain nodes and edges of G.

❖ NODE DELETED SUBGRAPH

- ✓ The graph obtained by deletion of a node v from a given graph G is called node deleted subgraph of G. It is denoted by $G - \{v\}$.

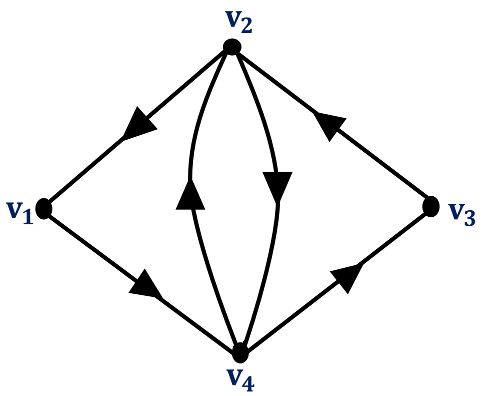
❖ EDGE DELETED SUBGRAPH

- ✓ The graph obtained by deletion of an edge e from a given graph G is called edge deleted subgraph of G. It is denoted by $G - \{e\}$.

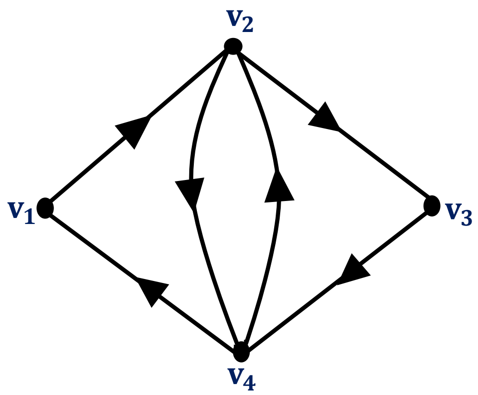
❖ CONVERSE (REVERSAL/DIRECTIONAL DUAL) OF A DIGRAPH

- ✓ The converse of a digraph $G = < V, E >$ to be a digraph $\tilde{G} = < V, \tilde{E} >$ in which the relation $\tilde{E}$ is the converse of the relation E. The diagram $\tilde{G}$ is obtained from G by simply reversing the directions of the edges in G. The converse $\tilde{G}$ is also called the reversal or directional dual of a digraph G.

✓ Example



G



$\tilde{G}$

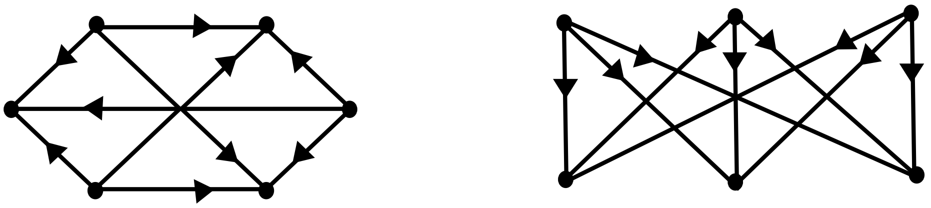
➢ In the above diagram G & $\tilde{G}$ are converse of each other.

METHOD-2: PROPERTIES OF GRAPHS

H	**1**	Define with example: isomorphism of graphs.	
C	**2**	Draw all possible different simple digraphs having three nodes up to isomorphism. Show that there is only one digraph with no edges, one with one edge, four with two edges, four with three edges, four with four edges, one with five edges and one with six edges. Assume that there are no loops.	
T	**3**	Define with example: degree of a node, odd node, even node, pendant node and isolated node for undirected graph.	
T	**4**	Define with example: indegree, outdegree and total degree for directed graph.	
C	**5**	Show that the sum of indegrees of all the nodes of a simple digraph is equal to the sum of outdegrees of all its nodes and that this sum is equal to the number of edges of the graph.	
H	**6**	Define with example: subgraph of a graph and converse of a digraph.	

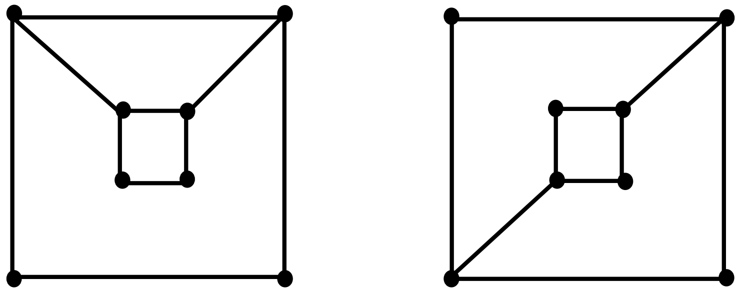
C	7	Check whether the following pair of graphs G & H are isomorphic or not with description.

(A).



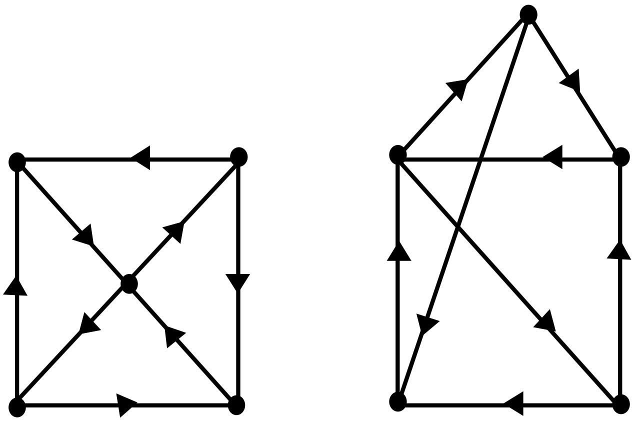
G H

(B).



G H

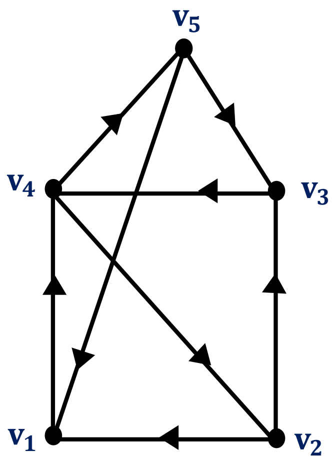
(C).



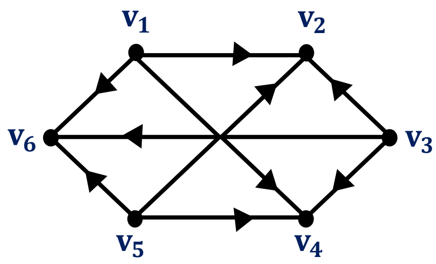
G H

Answer : (**a**) Isomorphic, (**b**) Not Isomorphic, (**c**) Isomorphic

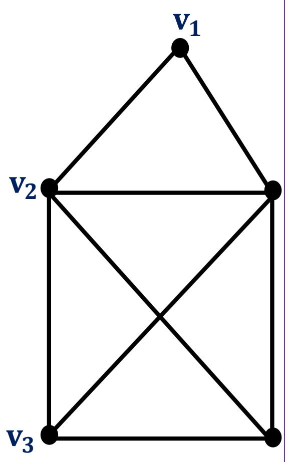
T **8** Consider the following graphs: Determine the degree of each node and verify Handshaking theorem.



I



II



III

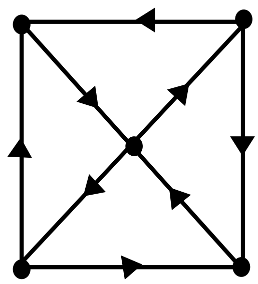
Answer :

(I) $\mathbf{d(v_1) = 3, d(v_2) = 3, d(v_3) = 3, d(v_4) = 4, d(v_5) = 3}$

(II) $\mathbf{d(v_1) = 3, d(v_2) = 3, d(v_3) = 3, d(v_4) = 3, d(v_5) = 3, d(v_6) = 3}$

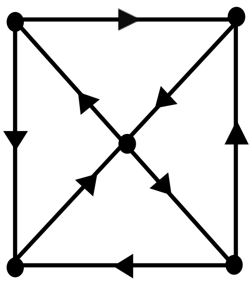
(III) $\mathbf{d(v_1) = 2, d(v_2) = 4, d(v_3) = 3, d(v_4) = 3, d(v_5) = 4}$

H **9** Define converse of a digraph and find it for given graph G.



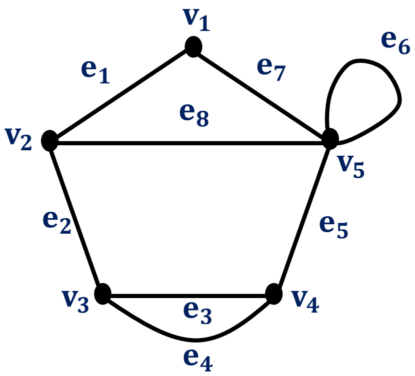
G

Answer :



G

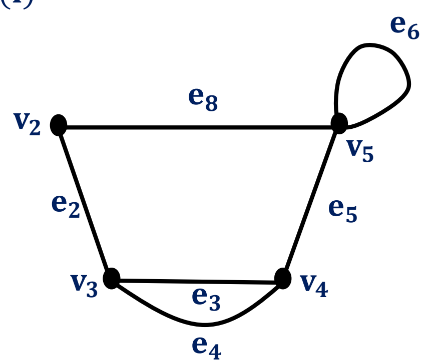
C **10** Define node deleted subgraph and edge deleted subgraph. Also find subgraphs from the given graph G by deleting
(I) node $v_1 (G - \{v_1\})$
(II) edge $e_4 (G - \{e_4\})$.



G

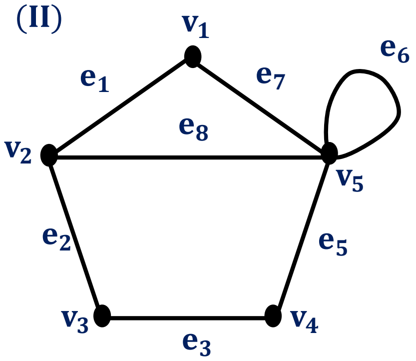
Answer :

(I)



$G - v_1$

(II)



$G - e_4$

✓ Note

➢ In this method we introduce some additional terminology associated with a simple digraph.

❖ PATH OF A GIVEN GRAPH

✓ Let $G = (V, E)$ be a simple digraph. Consider a sequence of edges of G such that the terminal node of any edge in the sequence is the initial node of next edge. Such a sequence is called a path of the graph G.

✓ A path is said to traverse through the nodes appearing in the sequence originating in the initial node of the first edge and ending in the terminal node of the last edge in the sequence.

❖ LENGTH OF PATH

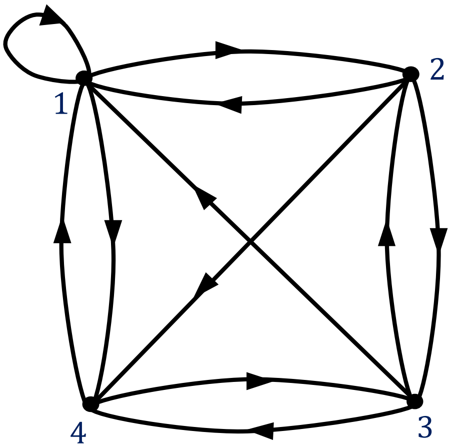
- ✓ The number of edges appearing in the sequence of a path is called the length of the path.

❖ SIMPLE PATH (EDGE SIMPLE)

- ✓ A path in a diagraph in which all the edges are distinct is called a simple path (edge simple).

❖ ELEMENTARY PATH (NODE SIMPLE)

- ✓ A path in a digraph in which all the nodes through which it traverses are distinct is called an elementary path (node simple).
- ✓ Note
  - ➢ Naturally every elementary path of a digraph is also simple path.
- ✓ Example
  - ➢ Paths originating in the node 1 and ending in node 3 are



$P_1 = (<1,2>, <2,3>)$

$P_2 = (<1,4>, <4,3>)$

$P_3 = (<1,2>, <2,4>, <4,3>)$

$P_4 = (<1,2>, <2,4>, <4,1>, <1,2>, <2,3>)$

$P_5 = (<1,2>, <2,4>, <4,1>, <1,4>, <4,3>)$

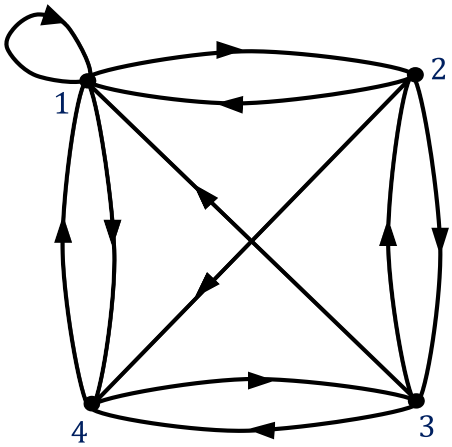
$P_6 = (<1,1>, <1,1>, \dots, <1,2>, <2,3>)$

  - ➢ The paths P_1, P_2 & P_3 of the digraph in above figure are elementary. The path P_5 is simple but not elementary.
  - ➢ If there exist a path from u to v then there must be an elementary path from u to v.

❖ CYCLE (CIRCUIT)

- ✓ A path which originates and ends in the same node is called a cycle (circuit).

- Example
  - The following are some of the cycles in the given graph



$C_1 = (<1,1>)$

$C_2 = (<1,2>, <2,1>)$

$C_3 = (<1,2>, <2,3>, <3,1>)$

$C_4 = (<1,4>, <4,3>, <3,1>)$

$C_5 = (<1,4>, <4,3>, <3,2>, <2,1>)$

❖ SIMPLE CYCLE

- A cycle is called simple if its path is simple path. i.e., no edge in the cycle appears more than once in the path.

❖ ELEMENTARY CYCLE

- A cycle is called elementary if it does not traverse through any node more than once.
- Notes
  - In a cycle the initial node appears at least twice even if it is an elementary cycle.
  - Observe that any path which is not elementary contains cycle traversing through those nodes which appear more than once in the path.
  - In above graph all the cycles C_1, C_2, C_3, C_4 & C_5 are simple as well as elementary.

❖ ACYCLIC GRAPH

- A simple digraph which does not have any cycles is called acyclic. An acyclic graph does not have any loop.

❖ REACHABILITY

- A node v of a simple digraph is said to be reachable (accessible) from the node u of the same digraph if there exist a (at least one) path from u to v.
- It is clear from the definition that reachability is a binary relation on the set of nodes of a simple digraph. Reachability is reflexive and transitive relation. Reachability is not necessarily symmetric nor it is antisymmetric.

❖ GEODESIC

- ✓ If a node v is a reachable from the node u then a path of minimum length from u to v is called a geodesic.

❖ DISTANCE

- ✓ The length of a geodesic from the node u to the node v is called the distance and it is denoted by $d < u, v >$.

❖ DIAMETER

- ✓ The diameter of a simple digraph $G =< V, E >$ is given by δ, where $\delta = \max_{u,v \in V} d(u, v)$.
- ✓ Note
  - ➢ It is assumed that $d < u, u > = 0$ for any node u.

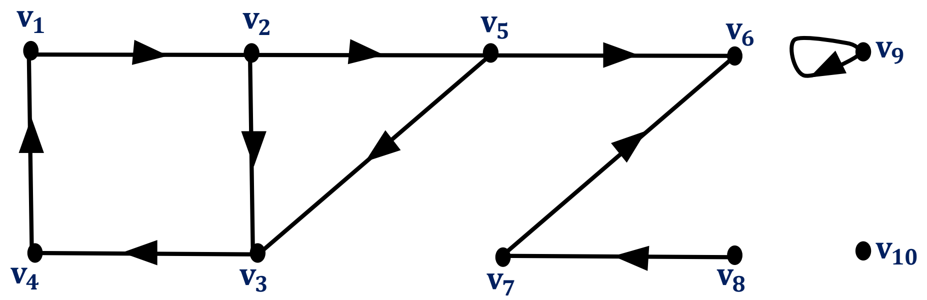
❖ PROPERTIES OF REACHABILITY

- ✓ If v is reachable from u then $d < u, v >$ satisfies the following properties:
  - ➢ $d < u, v > \geq 0$.
  - ➢ $d < u, u >= 0$.
  - ➢ $d < u, v > +d < v, w > \geq d < u, w >$. (Triangle inequality)
- ✓ Notes
  - ➢ If v is not reachable from u then it is customary to write $d < u, v >= \infty$(not defined).
  - ➢ If v is reachable from u and u is reachable from v then $d < u, v >$ is not necessarily equal to $d < v, u >$.

❖ REACHABLE SET OF A GIVEN NODE

- ✓ The set of nodes which are reachable from a given node v is said to be the reachable set of v. The reachable set of v is written as $R(v)$. For any subset $S \subseteq V$, the reachable set of S is the set of nodes which are reachable from any node of S. This set is denoted by $R(S)$.

✓ Example



➢ In the above graph all the reachable sets are given as below

$R(v_1) = R(v_2) = R(v_3) = R(v_4) = R(v_5) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$R(v_6) = \{v_6\}, \quad R(v_7) = \{v_6, v_7\}, \quad R(v_8) = \{v_6, v_7, v_8\}, \quad R(v_9) = \{v_9\}$

$R(v_{10}) = \{v_{10}\}, \quad R(v_5, v_8, v_9, v_{10}) = V = R(v_1, v_8, v_9, v_{10})$

❖ NODE BASE

✓ In a digraph $G = (V, E)$, a subset $X \subseteq V$ is called a node base if its reachable set is V and no proper subset of X has this property.

✓ In the above graph the set $\{v_1, v_8, v_9, v_{10}\}$ is a node base and similarly the set $\{v_5, v_8, v_9, v_{10}\}$ is a node base.

❖ CONNECTEDNESS

✓ An undirected graph is said to be connected if for any pair of nodes of the graph the two nodes are reachable from one another.

❖ WEAKLY CONNECTED

✓ A digraph is said to be weakly connected(connected) if it is connected as an undirected graph in which the direction of the edge is neglected. i.e., if the graph when treated as an undirected graph is connected.

❖ STRONGLY CONNECTED

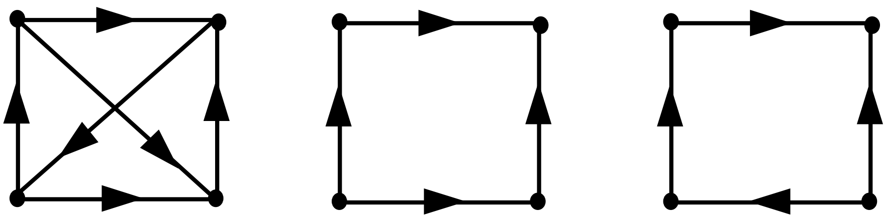
✓ If for any pair of nodes of the graph both the nodes of the pair are reachable from one another then the graph is called strongly connected.

❖ UNILATERALLY CONNECTED

✓ A simple digraph is said to be unilaterally connected if for any pair of nodes of the graph at least one of the node of the pair is reachable from the other node.

✓ Notes

- Observe that a unilaterally connected digraph is weakly connected but a weakly connected digraph is not necessarily unilaterally connected.
- A strongly connected digraph is both unilaterally and weakly connected.



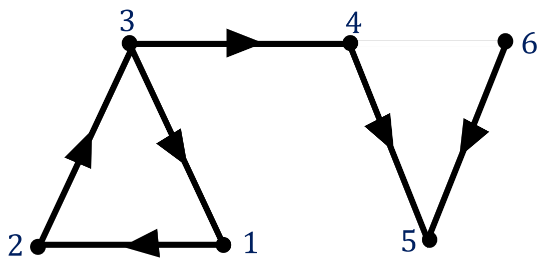
A B C

- The digraph in figure (A) is strongly connected, (B) is weakly connected but not unilaterally connected while (C) is unilaterally connected but not strongly connected.

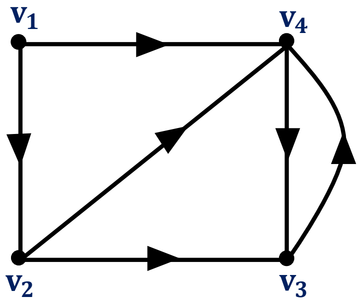
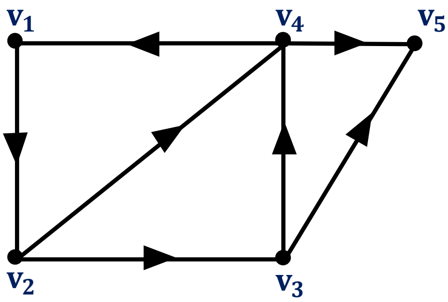
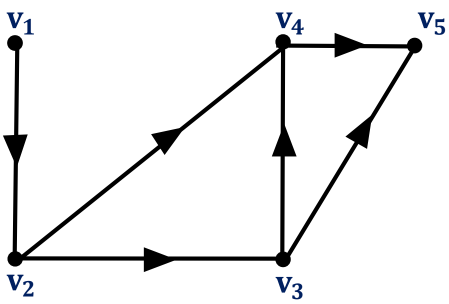
❖ STRONG, WEAK AND UNILATERAL COMPONENTS OF A GRAPH

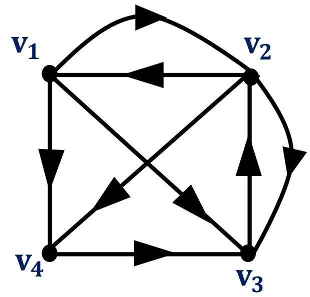
✓ For a simple digraph, a maximal strongly connected subgraph is called a strong component. Similarly, a maximal unilaterally connected subgraph is called a unilateral component and maximal weakly connected subgraph is called a weak component.

✓ For the digraph given in above figure {1,2,3}, {4}, {5}, {6} are the strong components. {1,2,3,4,5}, {6} are the unilateral components. {1,2,3,4,5,6} is the weak component because the graph is weakly connected.



METHOD-3: PATH, REACHABILITY AND CONNECTEDNESS

H	1	Give three elementary paths from $\mathbf{v_1}$ to $\mathbf{v_3}$ for the digraph given in following figure. Is there any cycle in the graph? What is the shortest distance between $\mathbf{v_1}$ and $\mathbf{v_3}$.  **Answer :** $\mathbf{P_1 = (< v_1, v_2 >, < v_2, v_3 >), P_2 = (< v_1, v_4 >, < v_4, v_3 >)}$ $\mathbf{P_3 = (< v_1, v_2 >, < v_2, v_4 >, < v_4, v_3 >), c_1 = (< v_4, v_3 >, < v_3, v_4 >)}$ $\mathbf{P_1}$ **and** $\mathbf{P_2}$ **are the shortest distance.**	
C	2	Give all elementary cycles for the following graph. Obtain an acyclic digraph by deleting one edge of the given digraph.  **Answer :** $\mathbf{c_1 = (< v_1, v_2 >, < v_2, v_3 >, < v_3, v_4 >, < v_4, v_1 >)}$ $\mathbf{c_2 = (< v_1, v_2 >, < v_2, v_4 >, < v_4, v_1 >)}$  	
C	3	Prove that in a simple digraph, the length of any elementary path is less than or equal to $n - 1$, where n is the number of nodes in the graph. Similarly, the length of any elementary cycle does not exceed n.	
T	4	Find the diameter of the digraphs given in the examples 1 and 2. **Answer :** $\mathbf{(1)\ \delta = \max_{v_1, v_4 \in V} d(v_1, v_4), (2)\ \delta = \max_{v_1, v_5 \in V} d(v_1, v\ \ )}$	

H	**5**	Find the reachable set of $\{v_1, v_4\}, \{v_4, v_5\}$ and $\{v_3\}$ for the digraph given in example-2. **Answer** : $R(v_1, v_4) = \{v_1, v_2, v_3, v_4\}, R(v_4, v_5) = \{v_1, v_2 v_3, v_4, v_5\},$ $R(v_3) = \{v_1, v_2, v_3, v_4, v_5\}$	
C	**6**	Find the reachable set for all the nodes in the following digraph.  **Answer** : $R(v_1) = \{v_1, v_2, v_3, v_4\}, R(v_2) = \{v_1, v_2 v_3, v_4\}, R(v_3) = \{v_1, v_2, v_3, v_4\}$ $R(v_4) = \{v_1, v_2, v_3, v_4\}$	
C	**7**	Find a node base for each of the digraphs given in the examples 1 and 2. **Answer** : (1) $\{v_1, v_2, v_4\}$, (2) $\{v_1, v_2, v_5\}$	
C	**8**	Determine whether the digraphs in example 1 and 2 are strongly, weakly or unilaterally connected. **Answer** : (1) **Unileterally Connected ⇒ Weakly connected** (2) **Unileterally Connected ⇒ Weakly connected**	
H	**9**	Find the strong, weak and unilateral component for the digraph given in the example 2. **Answer** : **Strong Component** : $\{v_1, v_2 v_3, v_4\}, \{v_5\}$ **Unilateral & weak Component** : $\{v_1, v_2, v_3, v_4, v_5\}$	

❖ APPLICATIONS TO REPRESENT RESOURCE ALLOCATION STATUS OF AN OPERATING SYSTEM AND DETECTION AND CORRECTION OF DEADLOCKS:

- ✓ We shall now show how a simple digraph can be used to represent the resource allocation status of an operating system.
- ✓ In a multi programmed computer system it appears that several programs are executed at one time. In reality, the programs are sharing the resources of the computer system such as tape units, disk devices, the central processor, main memory and compilers. A special set of programs called an operating system controls the allocation of these resources to the

programs. When a program requires the use of a certain resource and the operating system must ensure that the request is satisfied.

- ✓ It may happen that requests for resources are in conflict. For example, program A may have control of resource r_1 and require resource r_2 but program B has control of resource r_2 and requires resource r_1. In such a case the computer system is said to be in a state known as deadlock and the conflicting requests must be resolved. A directed graph can be used to model resource requests and assist in the detection and correction of deadlocks.
- ✓ It is assumed that all resource requests of a program must be satisfied before that program can complete execution. If any requested resources are unavailable at the time of the request the program will assume control of the resources which are available but must wait for the unavailable resources.
- ✓ Let $P_t = \{p_1, p_2, \dots, p_m\}$ represent the set of programs in the computer system at time t. Let $A_t \subseteq P_t$ be the set of active programs or programs that have been allocated at least a portion of their resource requests at time t. Let $R_t = \{r_1, r_2, \dots, r_n\}$ represent the set of resources in the system at time t. An allocation graph G_t is a directed graph representing the resource allocation status of the system at time t and consisting of a set of nodes $V = R_t$ and a set of edges E. Each resource is represented by a node of the graph. There is a directed edge from node r_i to r_j if and only if there is a program p_k in A_t that has been allocated resource r_i but is waiting for r_j.
- ✓ Example
  - ➢ let $R_t = \{r_1, r_2, r_3, r_4\}$ and $A_t = \{p_1, p_2, p_3, p_4\}$ and the resource allocation status be

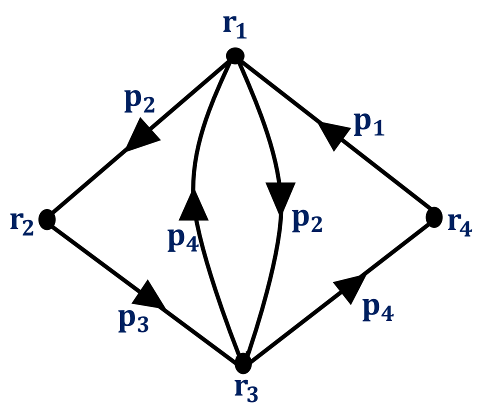
    p_1 has resource r_4 and requires r_1

    p_2 has resource r_1 and requires r_2 and r_3

    p_3 has resource r_2 and requires r_3

    p_4 has resource r_3 and requires r_1 and r_4.

    Then the allocation graph at time t is given in figure.



  - ➢ It can be shown that the state of deadlock exists in a computer system at time t if and only if the allocation graph G_t contains strongly connected components. In the case of our example the graph G_t is strongly connected.

❖ MATRIX REPRESENTATION OF A GRAPH

- ✓ A diagrammatic representation of a graph has limited usefulness. Furthermore, such a representation is only possible when the number of nodes and edges is reasonably small. An alternating method of representing graphs using matrices has several advantages. It is easy to store and manipulate matrices and the graphs represented by them in a computer. Well known operations of matrix algebra can be used to calculate paths, cycles and other characteristics of a graph.
- ✓ Given a simple digraph $G = (V, E)$, it is necessary to assume some kind of ordering of the nodes of the graph in the sense that a particular node is called a first node, another a second node and so on. Our matrix representation of G depends upon the ordering of the nodes.

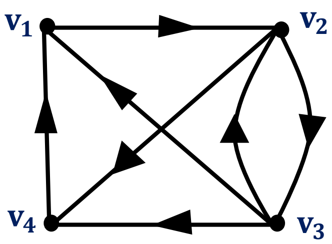
❖ ADJACENCY MATRIX

- ✓ Let $G = (V, E)$ be a simple digraph in which $V = \{v_1, v_2, \dots, v_n\}$ and the nodes are assumed to be ordered from v_1 to v_n. An $n \times n$ matrix A whose elements a_{ij} are given by

$$a_{ij} = \begin{cases} 1 \ ; & \text{if } < v_i, v_j > \in E(\text{ If there is an edge from } v_i \text{ to } v_j) \\ 0 \ ; & \text{otherwise} \end{cases}$$

is called the adjacency matrix of the graph G.

- ✓ Recall that the adjacency matrix is the same as the relation matrix or the incidence matrix of the relation E in V. Any element of the adjacency matrix is either 0 or 1.
- ✓ Example
  - ➢ The adjacency matrix of the above graph is



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- ✓ Notes
  - ➢ The sum of all 1's in a row indicates the outdegree of the corresponding node.
  - ➢ The sum of all 1's in a column indicates the indegree of the corresponding node.

- ✓ Example
  - ➢ From the adjacency matrix(A) of the above digraph we can calculate outdegree, indegree and total degree of each nodes which as follow.

NODE	OUTDEGREE	INDEGREE	TOTAL DEGREE
v_1	1	2	3
v_2	2	2	4
v_3	3	1	4
v_4	1	2	3

  - ➢ An adjacency matrix completely defines a simple digraph.

❖ BOOLEAN (BIT) MATRIX

- ✓ Any matrix whose elements are either 0 or 1 is called a Boolean matrix or bit matrix.
- ✓ Notes
  - ➢ For a given digraph $G = < V, E >$, an adjacency matrix depends upon the ordering of the elements of V. For different ordering of the elements of V we get different adjacency matrices of the same graph G.
  - ➢ However, any one of the adjacency matrices of G can be obtained from another adjacency matrix of the same graph by interchanging some of the rows and the corresponding column of the matrix. But the digraphs of both the matrix are isomorphic.
  - ➢ If a digraph is **reflexive** then the diagonal elements of the adjacency matrix are 1s.
  - ➢ The adjacency matrix for symmetric digraph is also **symmetric**. i.e., $a_{ij} = a_{ji}$ for all i and j.
  - ➢ If a digraph is **antisymmetric**, then $a_{ij} = 1$ implies $a_{ji} = 0$ and $a_{ij} = 0$ implies that $a_{ji} = 1$ for all i and j.

- For a null graph which consists of only n nodes but no edges, the adjacency matrix is a null matrix.
- If there are loops at each node but no other edges in the graph then the adjacency matrix is the identity matrix.
- If $G = (V, E)$ is a simple digraph whose adjacency matrix is A then the adjacency matrix of $\widetilde{G}$, the converse of G, is the transpose of A, that is A^T.

❖ PATH (REACHABILITY) MATRIX OF A GRAPH

✓ Let $G =< V, E >$ be a simple digraph in which $|V| = n$ and the nodes of G are assumed to be ordered. An $n \times n$ matrix P whose elements are given by

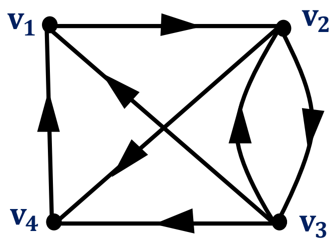
$$p_{ij} = \begin{cases} 1\ ; & \text{if there exists a path from } v_i \text{ to } v_j \\ 0\ ; & \text{otherwise} \end{cases}$$

is called the path matrix (reachability matrix) of the graph G.

✓ Example

- The path matrix of given graph is

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



✓ Note

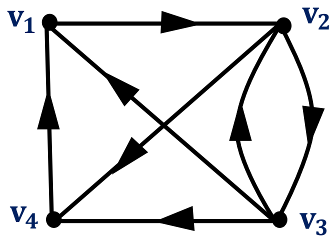
- The path matrix can be calculated from the matrix B_n by choosing $p_{ij} = 1$ if the element $b_{n_{ij}}$ of B_n is nonzero and $p_{ij} = 0$ if the element $b_{n_{ij}}$ of B_n is zero. Where, $B_n = A + A^2 + A^3 + \cdots + A^n$. A is adjacency matrix of given graph and n is number of nodes in given graph.

❖ DETERMINE NUMBER OF PATHS OF LENGTH N THROUGH ADJACENCY MATRIX:

✓ Result

- Let A be the adjacency matrix of a digraph G. The element in the i^{th} row and j^{th} column of A^n (n is a nonnegative integer) is equal to the number of paths of length n from the i^{th} node to the j^{th} node.

✓ Example



- Find the path matrix of given graph using adjacency matrix. Also find number of paths of length 4 between v_2 & v_4 from the adjacency matrix and mention it from the given graph.
- The adjacency matrix of the given graph is $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.
- Here no. of nodes in the given graph is $|V| = 4$.
- Hence, $B_n = B_4 = A + A^2 + A^3 + A^4$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 2 & 3 \\ 2 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 & 3 \\ 5 & 5 & 4 & 6 \\ 7 & 7 & 4 & 7 \\ 3 & 2 & 1 & 2 \end{bmatrix}.$$

- From B_4 we conclude that all entries are nonzero implies all entries of path matrix P are 1(one).
- Hence, path matrix P for the given graph is $P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.
- The number of paths of length 4 between v_2 & v_4 is the element of 2^{nd} row and 4^{th} column of A^4, which is 3.
- The paths of length 4 from v_2 to v_4 are

  $P_1 = (<v_2, v_3>, <v_3, v_1>, <v_1, v_2>, <v_2, v_4>)$

  $P_2 = (<v_2, v_4>, <v_4, v_1>, <v_1, v_2>, <v_2, v_4>)$

  $P_3 = (<v_2, v_3>, <v_3, v_2>, <v_2, v_3>, <v_3, v_4>)$.

✓ Notes

- The path matrix only shows the presence or absence of at least one path between a pair of points and also the presence or absence of a cycle at any node. It does not show all the paths that may exist. In this sense a path matrix does not complete information about a graph as does the adjacency matrix. The path matrix is important in its own right.

- It may be remarked that if we are interested in knowing the reachability of one node from another, it is sufficient to calculate $B_{n-1} = A + A^2 + \cdots + A^{n-1}$, because a path of length n cannot be elementary for graph with n nodes.
- For the purpose of reachability, every node is assumed to be reachable from itself.

❖ WARSHALL'S ALGORITHM TO PRODUCE PATH MATRIX

- ✓ Let G be a directed graph with m vertices $v_1, v_2, \dots, v_m$. Suppose we want to find the path matrix P of the graph G. Warshall gave an algorithm which is much more efficient than calculating the powers of the adjacency matrix A.
- ✓ First we define m-square Boolean matrices $P_0, P_1, \dots, P_m$ as follows. Let $P_k[i,j]$ denote the ij^{th} entry of the matrix P_k. Then we define:

$$P_k[i,j] = \begin{cases} 1 \;;\; & \text{if there is a simple path from } v_i \text{ to } v_j \text{ which does not use} \\ & \quad \text{any others vertices except possibly } v_1, \dots, v_k. \\ 0 \;;\; & \text{otherwise} \end{cases}$$

- ✓ That is $P_0[i,j] = 1$ if there is an edge from v_i to v_j.
- ✓ $P_1[i,j] = 1$ if there is a simple path from v_i to v_j which does not use any other vertex except possibly v_1.
- ✓ $P_2[i,j] = 1$ if there is a simple path from v_i to v_j which does not use any other vertices except possibly v_1 and v_2. And so on.
- ✓ Observe that the first matrix $P_0 = A$ is the adjacency matrix of G. Furthermore, since G has only m vertices, the last matrix $P_m = P$ is the path matrix of G.
- ✓ Warshall observed that $P_k[i,j] = 1$ can occur only if one of the following two cases occurs:
  - There is a simple path from v_i to v_j which does not use any other vertices except possibly $v_1, v_2, \dots, v_{k-1}$; hence $P_{k-1}[i,j] = 1$.
  - There is a simple path from v_i to v_k and a simple path from v_k to v_j where each simple path does not use any other vertices except possibly $v_1, v_2, \dots, v_{k-1}$. Hence $P_{k-1}[i,k] = 1$ & $P_{k-1}[k,j] = 1$.
- ✓ These two cases are pictured as follows:

  (1) $v_i \to \cdots \to v_j$ and (2) $v_i \to \cdots \to v_k \to \cdots \to v_j$

- Here $\rightarrow \cdots \rightarrow$ denotes part of a simple path which does not use any other vertices except possibly $v_1, v_2, \ldots, v_{k-1}$. Accordingly the elements of P_k can be obtained by

  $P_k[i,j] = P_{k-1}[i,j] \vee (P_{k-1}[i,k] \wedge P_{k-1}[k,j])$.

- where we use the logical operations of $\wedge$(AND) and $\vee$(OR). In other words we can obtain each entry in the matrix P_k by looking at only three entries in the matrix P_{k-1}.

ALGORITHM

- A directed graph G with M vertices is maintained in memory by its adjacency matrix A. This algorithm finds the (Boolean) path matrix P of the graph G.
- Step 1

  Let I, J, K

- Step 2

```
for I = 1 to M
    for J = 1 to M
    if (A[ I ] [ J ]=0), then P[ I ] [ J ] = 0
else
    P[ I ] [ J ] = 1
    [End of if loop]
[End of J loop]
[End of I loop]
```

- Step 3

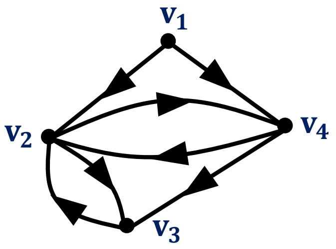
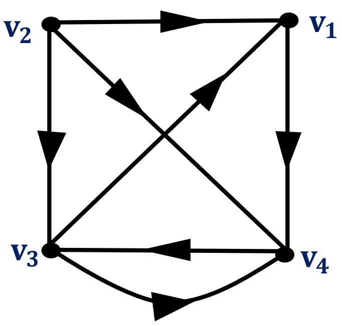
```
for K = 1 to M
    for I = 1 to M
        for J = 1 to M
        P[ I ] [ J ] = P[ I ] [ J ](OR)
        (P[ I ] [ K ] AND [ K ] [ J ])
        [End of loop]
  [End of loop]
```

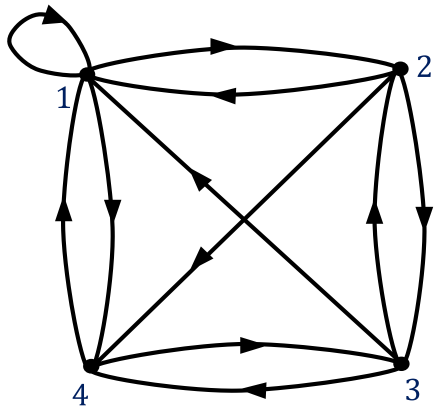
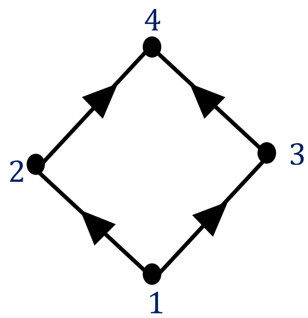
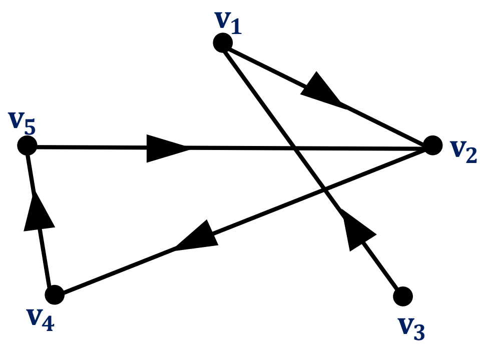
[End of loop]

- ✓ Step 4

Exit.

METHOD-4: MATRIX REPRESENTATION OF A GRAPH

T	1	Obtain the adjacency matrix A of the digraph given in the figure. Find the elementary paths of lengths 1 and 2 from v_1 to v_4. Show that there is also a simple path of length 4 from v_1 to v_4. Verify the results by calculating A^2, A^3 & A^4.  **Answer** : **Path of length 1 : $P_1 = (<v_1, v_4>)$** **Path of length 2 : $P_2 = (<v_1, v_2>, <v_2, v_4>)$** **Simple Path of length 4 : $P_3 = (<v_1, v_2>, <v_2, v_3>, <v_3, v_2>, <v_2, v_4>)$**	
C	2	Find A and A^2 without matrix multiplication for the given digraph. Where, A is an adjacent matrix.  **Answer** : $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathbf{A^2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$	

T	3	From the adjacency matrix of given graph calculate outdegree, indegree and total degree of each nodes and also verify it from the graph.  **Answer :** $\mathbf{d^+(1) = 4, d^-(v_1) = 3 \Rightarrow d(v_1) = 7, d^+(2) = 2, d^-(v_2) = 3 \Rightarrow d(v_2) = 5}$ $\mathbf{d^+(3) = 2, d^-(v_1) = 3 \Rightarrow\ d(v_1) = 5, d^+(4) = 3, d^-(v_1) = 2 \Rightarrow d(v_1) = 5}$	
H	4	Find the adjacency matrix A and path matrix from $B_4 = A + A^2 + A^3 + A^4$ for the given graph. Also verify path matrix from given graph.  **Answer :** $\mathbf{B_4 = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$	
C	5	Apply Warshall's algorithm to produce a path matrix for given graph.  **Answer :** $\mathbf{P = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & \end{bmatrix}}$	

❖ INTRODUCTION OF TREE

- ✓ Trees are useful in describing any structure which involves hierarchy. Familiar examples of such structures are family trees, the decimal classifications of books in a library, the hierarchy of positions in an organization, an algebraic expression involving operations for which certain rules precedence are prescribed, etc.

❖ ACYCLIC GRAPH

- ✓ A digraph which does not have any cycle is called acyclic graph.

❖ TREE

- ✓ A tree is a connected acyclic graph.

❖ DIRECTED TREE

- ✓ A directed tree is an acyclic graph which has one node called root with indegree 0, while all other nodes have indegree 1.
- ✓ Note
  - ➢ Every directed tree must have at least one node. An isolated node is also a directed tree.

❖ FOREST

- ✓ A set of disjoint trees is called a forest.

❖ ROOT

- ✓ A directed tree which has a node with indegree 0 is called roots of tree.

❖ LEAF (TERMINAL) NODE

- ✓ In a directed tree, any node which has outdegree 0 is called a terminal node or a leaf.

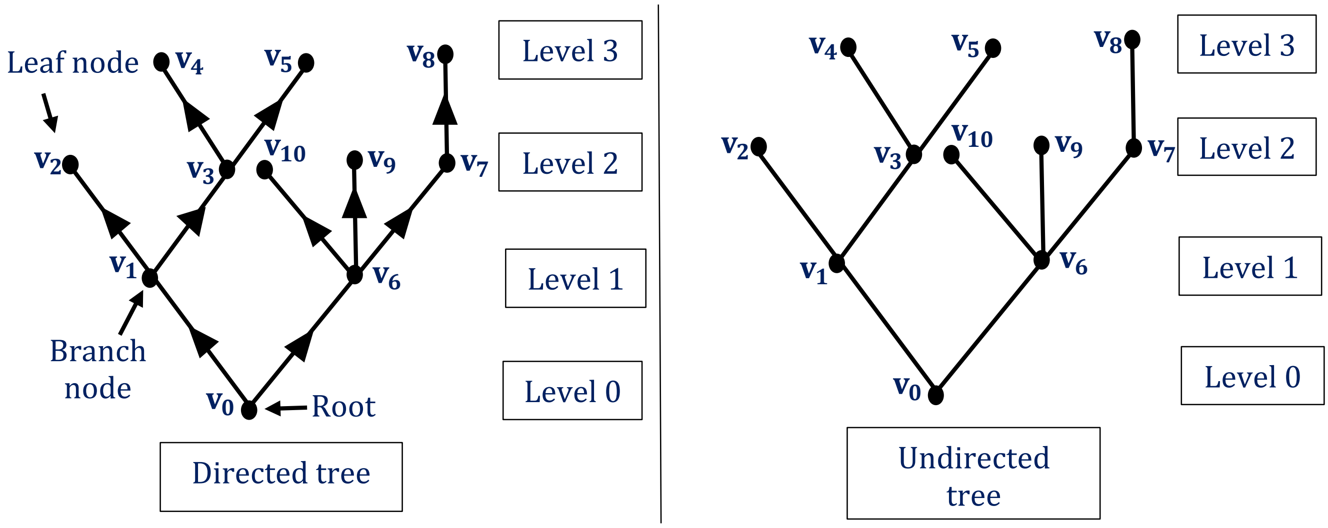
❖ BRANCH NODE

- ✓ The nodes which are not terminal nodes are known as branch nodes.

❖ LEVEL OF A NODE

- ✓ The level of any node is the length of its path from the root.
- ✓ The level of the root of a directed tree is 0, while the level of any node is equal to its distance from the root.

- ✓ Observe that all the paths in a directed tree are elementary and the length of a path from any node to another node if such a path exists is the distance between the nodes because a directed tree is acyclic.
- ✓ Example



❖ DIFFERENT REPRESENTATIONS OF A TREE

- ✓ There are several other ways in which a directed tree can be represented graphically.
- ✓ These methods of representation for the directed tree of figure 1 are given in figures A, B and C. The method(A) uses the familiar technique of Venn diagrams to show subtrees, the method(B) uses the convention of nesting parentheses and the method(C) method is the one used in the list of contents of books.

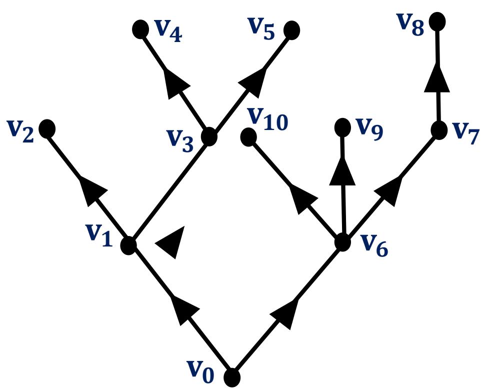
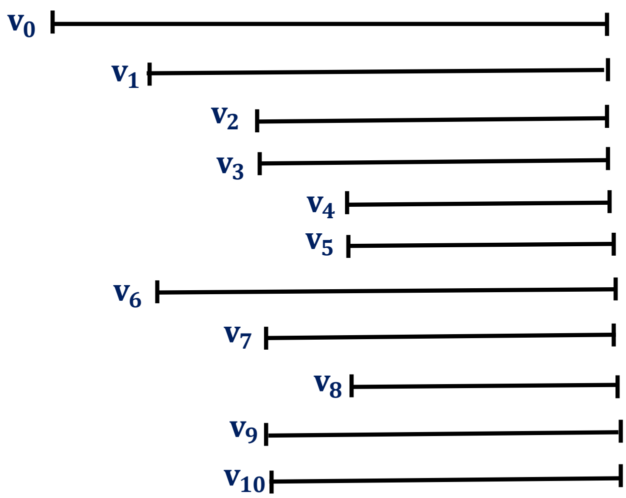


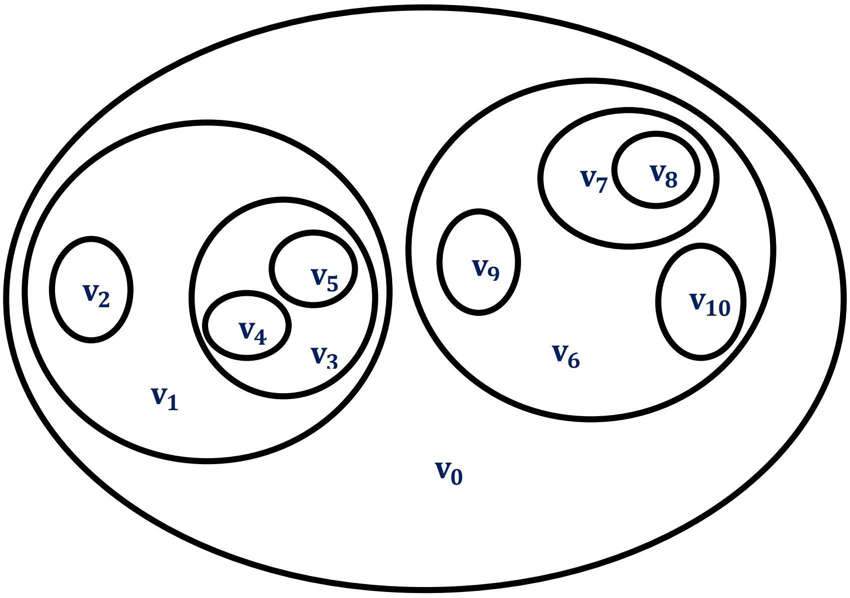
Figure 1



Method C

$$\left(v_0 \left(v_1 (v_2) \left(v_3 (v_4)(v_5)\right)\right)\left(v_6 \left(v_7 (v_8)\right)(v_9)(v_{10})\right)\right)$$

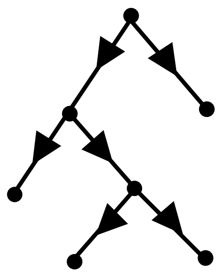
Method B



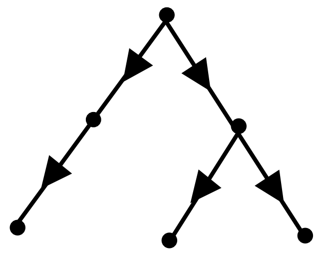
Method A

❖ BINARY TREE

- ✓ In a directed tree the outdegree of every node is less than or equal to 2 then the tree is called binary tree.
- ✓ Example: Figure A and B shows binary tree.



A



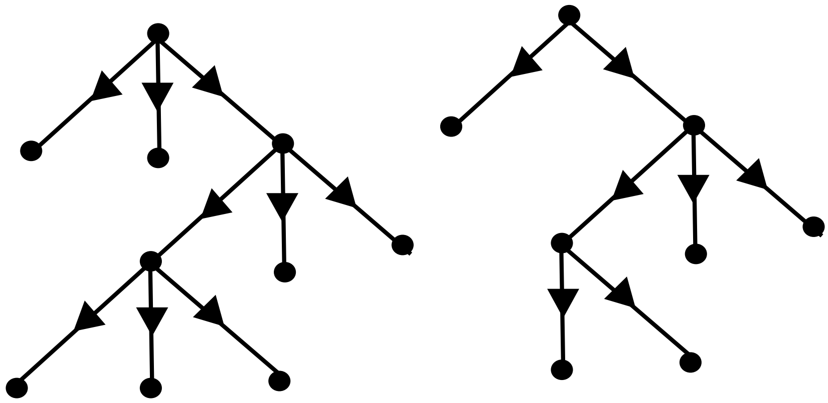
B

❖ FULL (COMPLETE)BINARY TREE

- ✓ If the outdegree of every node is exactly equal to 2 or 0 then the tree is called a full or complete binary tree.
- ✓ Example: Figure A shows full binary tree.

❖ M-ARY TREE

- ✓ In a directed tree the outdegree of every node is less than or equal to m then the tree is called m-ary tree.
- ✓ Example: Figure C and D shows 3-ary tree.



C D

❖ FULL (COMPLETE) M-ARY TREE

- ✓ If the outdegree of every node is exactly equal to m or 0 then the tree is called a full or complete m-ary tree.
- ✓ Example: Figure C shows full 3-ary tree.

❖ DESCENDENT OF NODE U

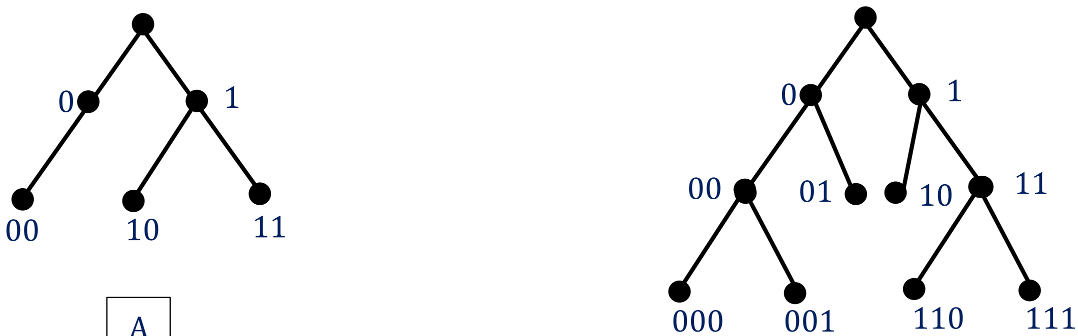
- ✓ The node which is reachable from u is called descendent of u.

❖ SON OF NODE U

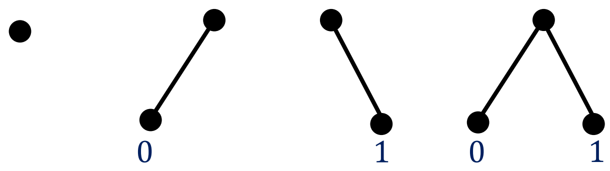
- ✓ The node which is reachable from u through a single edge is called son of u.

❖ POSITIONAL M-ARY TREE

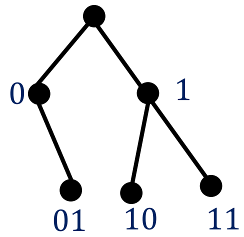
- ✓ If we consider m-ary trees in which the m sons of any node are assumed to have m distinct positions. If such positions are taken into account then the tree is called a positional m-ary tree.
- ✓ Example: Representation of nodes of a binary tree.



A B



C



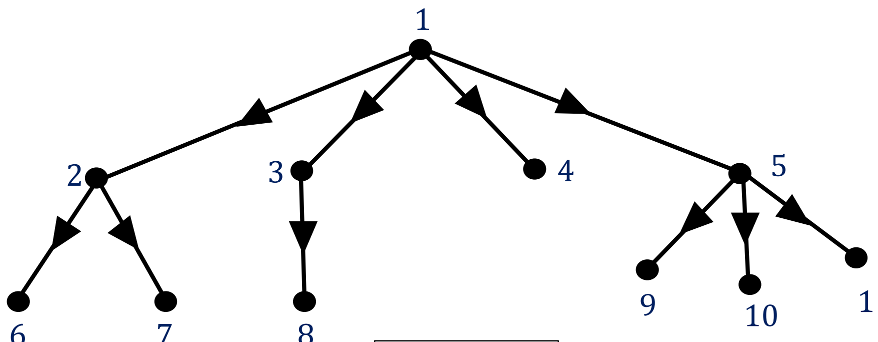
D

- ✓ Figure A shows a binary tree, B shows a full binary tree and C shows all four possible arrangements of sons of a node in a binary tree. The binary trees shown in figure A and D are distinct positional trees although they are not distinct ordered trees. In a positional binary tree, every node is uniquely represented by a string over the alphabet $\{0, 1\}$, the root being represented by an empty string. Any son of a node u has a string which is prefixed by the string of u. The string of any terminal node is not prefixed to the string of any other node. The set of strings which correspond to terminal node from a prefix code. Thus, the prefix code of the binary tree in B is $\{000, 001, 01, 10, 110, 111\}$. A similar representation of nodes of a positional m-ary tree by means of string over an alphabet {0, 1, …, m-1} is possible.

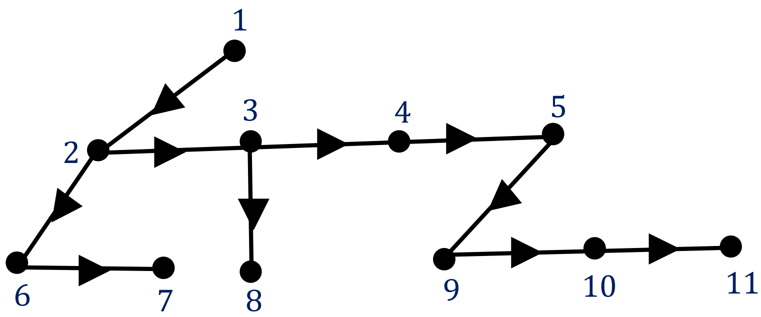
❖ CONVERTING ANY M-ARY TREE TO A BINARY TREE

- ✓ Delete all the branches originating in every node except the left most branch.
- ✓ Draw edges from a node to the node on the right, if any, which is situated at the same level.
- ✓ Choose its left and right sons as below.
  - ➢ (a). The left son is the node which is immediately below the given node.
  - ➢ (b). The right son is the node to the immediate right of the given node on some horizontal line.

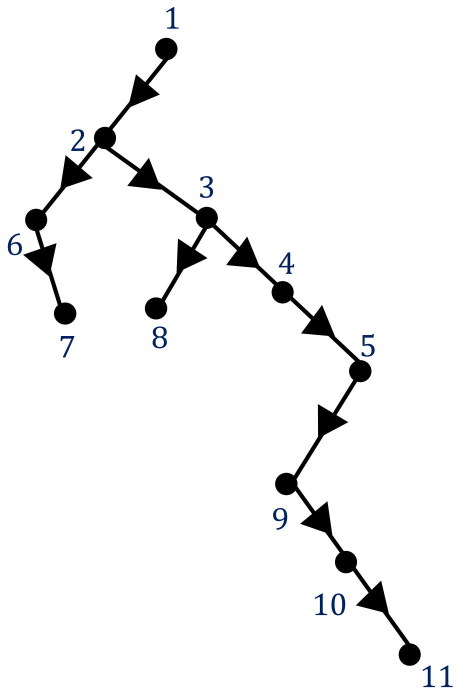
✓ Example:



Given tree



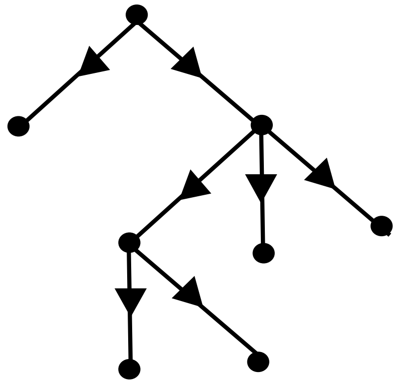
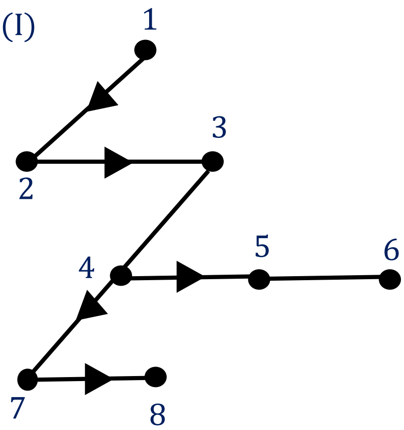
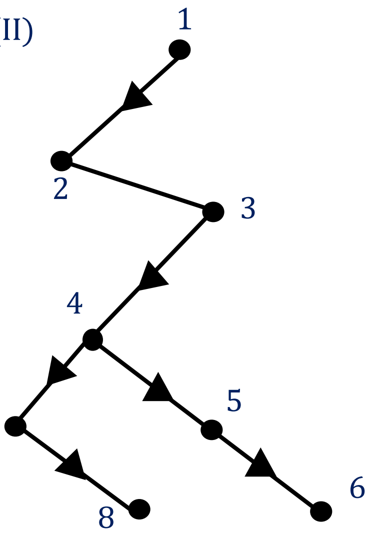
Stage 1



Binary tree representation

METHOD-5: PROPERTIES OF TREE AND ITS REPRESENTATIONS

H	1	Define with example: acyclic graph, tree, directed tree, forest, root, leaf node, branch node and level of a node.	
T	2	Explain different representation of a tree with example.	

H	3	Define with example: binary tree, complete binary tree, m-ary tree and complete m-ary tree.	
C	4	Write the steps of converting any m-ary tree to a binary tree and convert the following tree into a binary tree.  **Answer :** (I)  (II) 	

❖ REPRESENTATIONS OF A BINARY TREE

✓ The representation of a binary tree is simple compared to those for general tree.

❖ LINKED-LIST

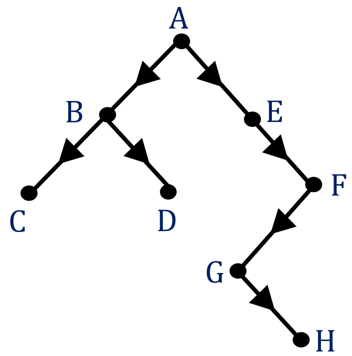
✓ Computer representation of trees based on linked allocation seems to be more popular because of the ease with which nodes can be inserted in and deleted from a tree and because tree structures can grow to an arbitrary size, a size which is often unpredictable.

✓ Linked allocation techniques will be used to represent binary trees. A number of possible traversals which can be performed on binary trees are described. The subsections end with a symbol table algorithm based on a tree structure.

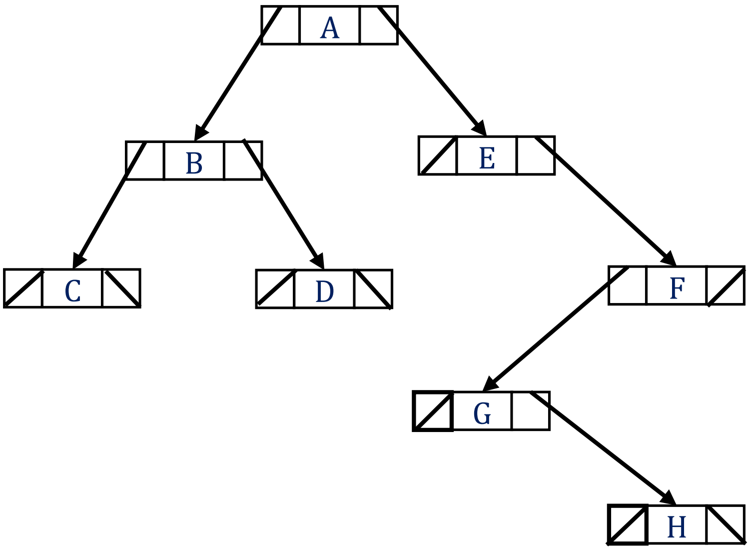
- A binary tree has one root node with no descendants or else a left, or a right, or a left and right subtree descendent(s). Each subtree descendant is also a binary tree and we do make the distinction between its left and right branches. A convenient way of representing binary trees is to used linked allocation techniques involving nodes with structure where LLINK or RLINK contain a pointer to the left subtree respectively of the node in question. Data contains the information which is to be associated with this particular node. Each pointer can have a value NULL.

LLINK	DATA	RLINK

- An example of a binary tree as a graph and its corresponding linked representation in memory are given in figure A and B respectively. Observe the very close similarity between the figures as drawn. Such a similarity illustrates that the linked storage representation of a tree is closer to the logical structuring of the data involved. This property can be useful in designing algorithms which process tree structures.



A



B

❖ TREE TRAVERSAL

- Tree traversal is a procedure by which each node is processed exactly once in some systematic manner.

❖ PRE-ORDER TRAVERSAL

- ✓ Process the root node.

  Traverse the left sub tree in pre-order.

  Traverse the right subtree in pre-order.

❖ IN-ORDER TRAVERSAL

- ✓ Traverse the left subtree in in-order.

  Process the root node.

  Traverse the right subtree in in-order.

❖ POST ORDER TRAVERSAL

- ✓ Traverse the left subtree in post-order.

  Traverse the right subtree in post-order.

  Process the root node.

- ✓ Example

  - ➢ The pre-order, in-order and post-order traversals of the tree A given in following table which process the nodes in the following order:

    ABCDEFGH (pre-order)

    CBDAEGHF (in-order)

    CDBHGFEA (post-order)

❖ ALGORITHM PREORDER

- ✓ Given a binary tree whose root node address is given by a variable T and whose node structure is the same as previously described, this algorithm traverses the tree in preorder. An auxiliary stack S is used and TOP is the index of the top element of S. P is a temporary variable which denotes when we are in the tree.

  (1). [Initialize] If T = NULL, then Exit (the tree has no root and therefore is not a proper binary tree); otherwise set P ← T and TOP ← 0.

  (2). [Visit node, stack right branch address and go left] Process node P. If RLINK(P) ≠ NULL, then set TOP ← TOP + 1 and S[TOP] ← RLINK(P). Set P ← LLINK(P).

  (3). [End of chain?] If P ≠ NULL, then go to step 2.

(4). [Unstack a right branch address] If TOP=0, then Exit; otherwise set $P \leftarrow S[TOP], TOP \leftarrow TOP - 1$, and go to step 2.

- ✓ In the second and third steps of the algorithm, we visit and process a node. The address of the right branch of such a node, if it exists, is stacked and a chain of left branches is followed until this chain ends. At this point we enter step 4 and delete from the stack the address of the root node of the most recently encountered right subtree and process it according to steps 2 and 3. A trace of the algorithm for the binary tree given in the above graph appears in above table, where the rightmost element in the stack is considered to be its top element and the notation "NE," for example, denotes the address of node E. The visit of a node in this case merely involves the output of the label for that node.

❖ ALGORITHM POSTORDER

- ✓ The same node structure described previously is assumed and T is again a variable which contains address of the root of the tree. A stack S with its top element pointer is also required, but in this case each node will be stacked twice namely once when its left subtree is traversed and once when its right subtree is traversed. On completion of these two traversals, the particular node being considered is proceed. Hence, we must be able to distinguish two types of stack entries. The first type of entry indicates that a left subtree is being traversed, while the second indicates the traversal of a right subtree. For convenience we will use negative pointer values for the second type of entry. This of course assumes that valid pointer data is always nonzero and positive.

(1). [Initialize] If $T = NULL$, then Exit (the tree has no root and therefore is not a proper binary tree); otherwise set $P \leftarrow T$ and $TOP \leftarrow 0$.

(2). [Stack node address and go left] Set $TOP \leftarrow TOP + 1, S[TOP] \leftarrow P$ and $P \leftarrow LLINK(P)$.

(3). [End of chain?] If $P \neq NULL$, then go to step 2.

(4). [Unstack a node address] If TOP=0, then Exit; otherwise set $P \leftarrow S[TOP], TOP \leftarrow TOP - 1$, and go to step 2.

(5). [Restack address if right sub tree is not traversed] If P<0, then go to step 6; otherwise set $TOP \leftarrow TOP + 1, S[TOP] \leftarrow -P, P \leftarrow RLINK(P)$, and go to step 3.

(6). [Visit node] Set $P \leftarrow -P$, process node P, and go to step 4.

- ✓ In the second and third steps, a chain of left branches is followed and the address of each node which is encountered is stacked. At the end of such a chain, the stack entry for the last node encountered is checked against zero. If it is positive, the negative address of that node is restacked and the right branch of this node is taken and processed according to steps 2 and 3. If the stack value is negative however we have finished traversing the right subtree of that node. The node is then processed and the next stack entry is subsequently checked.

❖ APPLICATIONS OF LIST STRUCTURES AND GRAPHS

- ✓ Representation of a structure which is more general than a tree such a structure is called a list structure and several programming languages have been developed to allow easy programming languages have been developed to allow easy processing of structures similar to those that will be described. The need for list processing arose from the high cost of rapid computer storage and the unpredictable nature of the storage requirements of computer programs and data. There are many symbol manipulation applications in which this unpredictability is particularly acute. It will be shown that a list structure can be used to represent a directed graph. The representations of a general graph structures are based not only nature of the data but also on the operations which are to be performed on the data.

METHOD-6: TYPES OF TREE TRAVERSAL AND ITS ALGORITHMS

T	1	Explain representation of a binary tree by linked allocation technique with example.	
H	2	Define: tree traversal, pre-order traversal, in-order traversal and post-order traversal.	
C	3	Write algorithm on pre-order traversal.	
C	4	Write algorithm on post-order traversal.	
H	5	Discuss application of list structures and graphs.	

★ ★ ★ ★ ★ ★ ★ ★ ★ ★

GUJARAT TECHNOLOGICAL UNIVERSITY

Bachelor of Engineering
Subject Code: 3140708
Semester – IV
Subject Name: Discrete Mathematics

Type of course: Undergraduate

**Prerequisite : ** Algebra, Logic

**Rationale : ** This course introduces the basic concepts of discrete mathematics in the field of computer science. It covers sets, logic, functions, relations, graph theory and algebraic structures. These basic concepts of sets, logic functions and graph theory are applied to Boolean Algebra and logic networks, while the advanced concepts of functions and algebraic structures are applied to finite state machines and coding theory.

Teaching and Examination Scheme:

Teaching Scheme			Credits	Examination Marks				Total
L	T	P	C	Theory Marks		Practical Marks		
				ESE(E)	PA (M)	ESE(V)	PA(I)	
3	2	0	5	70	30	0	0	100

Contents:

Sr. No.	Content	Total Hrs.	% weightage
01	**Set Theory**: Basic Concepts of Set Theory: Definitions, Inclusion, Equality of Sets, Cartesian product, The Power Set, Some operations on Sets, Venn Diagrams, Some Basic Set Identities **Functions**: Introduction & definition, Co-domain, range, image, value of a function; Examples, surjective, injective, bijective; examples; Composition of functions, examples; Inverse function, Identity map, condition of a function to be invertible, examples; Inverse of composite functions, Properties of Composition of functions; **Counting:** The Basics of Counting, The Pigeonhole Principle, Permutations and Combinations, Binomial Coefficients, Generalized Permutations and Combinations, Generating Permutations and Combinations	**06**	**12%**
02	**Propositional Logic**: Definition, Statements & Notation, Truth Values, Connectives, Statement Formulas & Truth Tables, Well-formed Formulas, Tautologies, Equivalence of Formulas, Duality Law, Tautological Implications, Examples **Predicate Logic:** Definition of Predicates; Statement functions, Variables, Quantifiers, Predicate Formulas, Free & Bound Variables; The Universe of Discourse, Examples, Valid Formulas & Equivalences, Examples	**06**	**13%**
03	**Relations**: Definition, Binary Relation, Representation, Domain, Range, Universal Relation, Void Relation, Union, Intersection, and Complement Operations on Relations, Properties of Binary Relations in a Set: Reflexive, Symmetric, Transitive, Anti-symmetric Relations, Relation Matrix and Graph of a Relation; Partition and Covering of a Set, Equivalence Relation, Equivalence Classes, Compatibility Relation, Maximum Compatibility Block, Composite Relation, Converse of a Relation, Transitive Closure of a Relation R in Set X **Partial Ordering**: Definition, Examples, Simple or Linear Ordering, Totally Ordered Set (Chain), Frequently Used Partially Ordered Relations, Representation of Partially Ordered Sets, Hesse Diagrams, Least & Greatest Members, Minimal & Maximal Members, Least Upper Bound (Supremum), Greatest Lower Bound (infimum), Well-ordered Partially Ordered Sets (Posets). Lattice as Posets, complete, distributive	**10**	**25%**

	modular and complemented lattices Boolean and pseudo Boolean lattices. (Definitions and simple examples only) **Recurrence Relation:** Introduction, Recursion, Recurrence Relation, Solving, Recurrence Relation		
04	**Algebraic Structures:** Algebraic structures with one binary operation- Semigroup, Monoid, Group, Subgroup, normal subgroup, group Permutations, Coset, homomorphic subgroups, Lagrange's theorem, Congruence relation and quotient structures. Algebraic structures (Definitions and simple examples only) with two binary operation- Ring, Integral domain and field.	**10**	**25%**
05	**Graphs:** Introduction, definition, examples; Nodes, edges, adjacent nodes, directed and undirected edge, Directed graph, undirected graph, examples; Initiating and terminating nodes, Loop (sling), Distinct edges, Parallel edges, Multi-graph, simple graph, weighted graphs, examples, Isolated nodes, Null graph; Isomorphic graphs, examples; Degree, Indegree, out-degree, total degree of a node, examples; Subgraphs: definition, examples; Converse (reversal or directional dual) of a digraph, examples; Path: Definition, Paths of a given graph, length of path, examples; Simple path (edge simple), elementary path (node simple), examples; Cycle (circuit), elementary cycle, examples; **Reachability:** Definition, geodesic, distance, examples; Properties of reachability, the triangle inequality; Reachable set of a given node, examples, Node base, examples; **Connectedness:** Definition, weakly connected, strongly connected, unilaterally connected, examples; Strong, weak, and unilateral components of a graph, examples, Applications to represent Resource allocation status of an operating system, and detection and correction of deadlocks; Matrix representation of graph: Definition, Adjacency matrix, boolean (or bit) matrix, examples; Determine number of paths of length n through Adjacency matrix, examples; Path (Reachability) matrix of a graph, examples; Warshall's algorithm to produce Path matrix, Flowchart. **Trees:** Definition, branch nodes, leaf (terminal) nodes, root, examples; Different representations of a tree, examples; Binary tree, m-ary tree, Full (or complete) binary tree, examples; Converting any m-ary tree to a binary tree, examples; Representation of a binary tree: Linked-list; Tree traversal: Pre-order, in-order, post-order traversal, examples, algorithms; Applications of List structures and graphs	**10**	**25%**

Reference Books:

1. J. P. Tremblay and R. Manohar, Discrete Mathematical Structures with Applications to Computer Science, Tata McGraw-Hill,1997.
2. S. Lipschutz and M. L. Lipson, Schaum's Outline of Theory and Problems of Discrete Mathematics, 2nd Ed., Tata McGraw-Hill,1999.
3. K. H. Rosen, Discrete Mathematics and its applications, Tata McGraw-Hill, 6th Ed., 2007.
4. David Liben-Nowell, Discrete Mathematics for Computer Science, Wiley publication, July 2017.
5. Eric Gossett, Discrete Mathematics with Proof, 2nd Edition,Wiley publication, July 2009.

Suggested Specification table with Marks (Theory):					
R Level	U Level	A Level	N Level	E Level	C Level
10	20	20	10	10	

Legends: R: Remembrance; U: Understanding; A: Application, N: Analyze and E: Evaluate C: Create and above Levels (Revised Bloom's Taxonomy).

Course Outcomes:

After Completion of this course students will be able

Sr. No.	Course Outcomes	Weightage in %
1	Understand the basic principles of sets and operations in sets and apply counting principles to determine probabilities, domain and range of a function, identify one-to- one functions, perform the composition of functions and apply the properties of functions to application problems.	12%
2	Write an argument using logical notation and determine if the argument is or is not valid. To simplify and evaluate basic logic statements including compound statements, implications, inverses, converses, and contra positives using truth tables and the properties of logic. To express a logic sentence in terms of predicates, quantifiers, and logical connectives.	13%
3	Apply relations and to determine their properties. Be familiar with recurrence relations	25%
4	Use the properties of algebraic structures.	25%
5	Interpret different traversal methods for trees and graphs. Model problems in Computer Science using graphs and trees.	25%

List of Open Source Software/learning website: NPTEL Discrete Mathematics lectures

Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER–IV (NEW) EXAMINATION – WINTER 2021

Subject Code:3140708 Date:24/12/2021

Subject Name:Discrete Mathematics

Time:10:30 AM TO 01:00 PM Total Marks: 70

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**
4. **Simple and non-programmable scientific calculators are allowed.**

Q.1 **(a)** Show that for any two sets A and B, $A-(A\cap B)=A-B$. **03**

(b) If $S=\{a,\ b,\ c\}$, find nonempty disjoint sets A_1 and A_2 such that $A_1\cup A_2=S$. Find the other solutions to this problem. **04**

(c) Using truth table state whether each of the following implication is tautology. **07**

a) $(p\wedge r)\to p$
b) $(p\wedge q)\to(p\to q)$
c) $p\to(p\vee q)$

Q-2 **(a)** Given $S=\{1,2,3,----,10\}$ and a relation R on S. Where, $R=\{\langle x,y\rangle|x+y=10\}$. What are the properties of relation R? **03**

(b) Let L denotes the relation "less than or equal to" and D denotes the relation "divides". Where xDy means "x divides y". Both L and D are defined on the set $\{1,2,3,\ 6\}$. Write L and D as sets, find $L\cap D$. **04**

(c) Let $X=\{1,2,3,4,5,6,7\}$ and $R=\{\langle x,y\rangle|x-y\ is\ divisible\ by\ 3\}$. Show that R is an equivalence relation on. Draw the graph of R. **07**

OR

(b) Define equivalence class generated by an element $x\in X$. Let Z be the set of integers and let R be the relation called "congruence modulo 3" defined by **07**

$$R=\{\langle x,y\rangle|x\in Z\wedge y\in Z\wedge(x-y)\ is\ divisible\ by\ 3\}$$

Determine the equivalences classes generated by the element of Z.

Q.3 **(a)** Let $f(x)$ be any real valued function. Show that $g(x)=\frac{f(x)+f(-x)}{2}$ is always an even function where as $h(x)=\frac{f(x)-f(-x)}{2}$ is always an odd function. **03**

(b) The Indian cricket team consist of 16 players. It includes 2 wicket keepers and 5 bowlers. In how many ways can cricket eleven be selected if we have select 1 wicket keeper and at least 4 bowlers? **04**

(c) Let A be the set of factors of particular positive integer m and $\leq$ be the relation divides, that is **07**

$\leq=\{\langle x,y\rangle|x\in A\wedge y\in A\wedge(x\ divides\ y)\}$ Draw the Hasse diagrams for

a) $m=45$
b) $m=210$.

OR

Q-3 **(a)** Find the composition of two functions $f(x)=e^x$ and $g(x)=x^3$, $(f\circ g)(x)$ and $(g\circ f)(x)$. Hence, show that $(f\circ g)(x)\neq(g\circ f)(x)$. **07**

(b) In a box, there are 5 black pens, 3 white pens and 4 red pens. In how many ways can 2 black pens, 2 white pens and 2 red pens can be chosen? **04**

(c) Let A be a given finite set and $\rho(A)$ its power set. Let $\subseteq$ be the inclusion relation on the elements of $\rho(A)$. Draw Hass diagram for $\langle\rho(A),\subseteq\rangle$ for **07**

a) $A=\{a,b,c\}$
b) $A=\{a,b,c,d\}$

Q.4 **(a)** Let $\langle L, \leq \rangle$ be a lattice. Show that for $a, b, c \in L$, following inequalities holds. **07**

$$a \oplus (b \star c) = (a \oplus b) \star (a \oplus c)$$
$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

(b) Let $G = \{(a, b) | a, b \in R\}$. Define binary operation $(*)$ on G as $(a, b), (c, d) \in G, (a, b) * (c, d) = (ac, bc + d)$. Show that an algebraic structure $(G, *)$ is a group. **07**

OR

Q-4 **(a)** Let G be the set of non-zero real numbers. Define binary operation $(*)$ on G as $a * b = \frac{ab}{2}$. Show that an algebraic structure $(G, *)$ is an abelian group. **07**

(b) Explain the following terms with proper illustrations. **07**

a) Directed graphs
b) Simple and elementary path
c) Reachability of a vertex
d) Connected graph.

Q-5 **(a)** Show that sum of in-degrees of all the nodes of simple digraph is equal to the sum of out-degrees of all the nodes and this sum equal to the number of edges in it. **07**

(b) Let $= \{1, 2, 3, 4\}$. For the relation R whose matrix is given, find the matrix of the transitive closure by using Warshall's algorithm. **07**

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

OR

Q-5 **(a)** Define tree and root. Also prove that tree with n vertices has $n - 1$ edges. **07**

(b) Define in-degree and out-degree of a vertex and matrix of a relation. Let $A = \{a, b, c, d\}$ and let R be the relation on A that has the matrix **07**

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

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Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER–IV (NEW) EXAMINATION – SUMMER 2021

Subject Code:3140708 Date:07/09/2021

Subject Name:Discrete Mathematics

Time:02:30 PM TO 05:00 PM Total Marks: 70

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**
4. **Simple and non-programmable scientific calculators are allowed.**

			MARKS
Q.1	**(a)**	Among 100 people at least how many of them were born in the same month?	**03**
	(b)	Prove that: $(A \cup B)' \equiv A' \cap B'$.	**04**
	(c)	Define the following: 1) Composition of functions 2) Monoid 3) Existential Quantifier 4) Partially Ordered Set 5) Boolean Algebra 6) Tree 7) Complete Graph	**07**
Q.2	**(a)**	Explain types of a Relation with a suitable example.	**03**
	(b)	Rewrite the following statements using quantifier variables and predicate symbols: 1) All birds can fly. 2) Some women are genius. 3) There is a student who likes Discrete Mathematics but not Probability and Statistics. 4) Each integer is either even or odd.	**04**
	(c)	Determine the validity of the argument given: If I study, then I will not fail in Discrete Mathematics. If I do not play cricket, then I will study. But I failed in Discrete Mathematics. -- Therefore I must have played cricket.	**07**
		OR	
	(c)	Find if the following is a tautology, contradiction or contingency. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$	**07**
Q.3	**(a)**	Define: Bounded, Distributive and Complemented Lattices.	**03**
	(b)	Find the transitive closure of $R = \{(1,2), (3,4), (4,5), (4,1), (1,1)\}$. Where, $A = \{1,2,3,4,5\}$.	**04**
	(c)	Let A be a set of factors of positive integer m and relation is divisibility on A. For $m = 45$, show that POSET $(A, \leq)$ is a Lattice.	**07**
		OR	
Q.3	**(a)**	Draw the Hasse diagram of the set $\{1,3,9,18\}$ under partial order relation ‘divides’ and indicate those which are chains.	**03**
	(b)	Let $X = \{1,2,3, \dots ,7\}$ and $R = \{(x, y): x - y\ is\ divisble\ by\ 3\}$. Show that R is an equivalence relation. Draw the graph of R.	**04**

(c) Solve the recurrence relation: **07**

$$a_{n+2} - 5a_{n+1} + 6a_n = 2.$$

Q.4 **(a)** Define group with example. Give an example of a non-abelian group. **03**

(b) Let $H = \{[0], [3]\}$ in Z_6 under addition. Find left and right cosets in $< Z_6, +_6 >$. **04**

(c) Prove that $G = \{1,2,3,4,5,6\}$ is a finite abelian group of order 6 with respect to multiplication modulo 7. **07**

OR

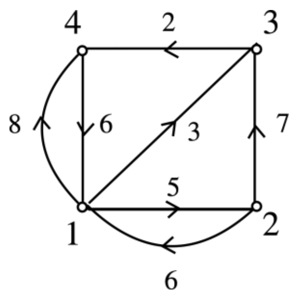
Q.4 **(a)** Define subgroup and group Homomorphism. **03**

(b) Is addition a binary operation on {-1,0,1}? Justify. **04**

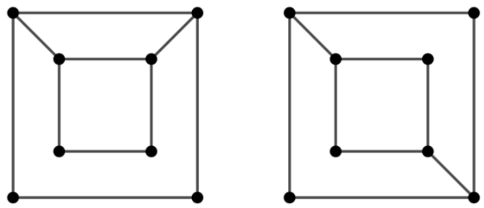
(c) Explain Cosets and Lagrange's theorem. **07**

Q.5 **(a)** How many nodes are necessary to construct a graph with exactly 8 edges in which each node is of degree 2. **03**

(b) Find the shortest path between each pair of vertices for a simple digraph using Warshall's Algorithm. **04**



(c) Define Isomorphic Graphs. Verify the following graphs are Isomorphic or not (Justify). **07**



OR

Q.5 **(a)** Define Cyclic graph, Null graph and Strongly connected graph. **03**

(b) Draw a graph which is regular but not bipartite. **04**

(c) For the following set of weights construct an optimal binary prefix code. For each weight in the set, give corresponding code word. **07**

5, 7, 8, 15, 35, 40

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Seat No.: ________ Enrolment No.___________

GUJARAT TECHNOLOGICAL UNIVERSITY

BE - SEMESTER– IV EXAMINATION – SUMMER 2020

Subject Code: 3140708 **Date:29/10/2020**

Subject Name: Discrete Mathematics

Time: 10:30 AM TO 01:00 PM **Total Marks: 70**

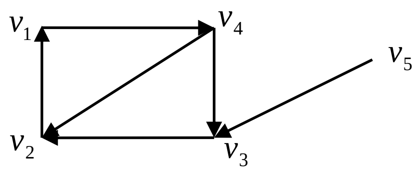
Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

			Marks
Q.1	**(a)**	If $A = \{a, b\}$ and $B = \{c, d\}$ and $C = \{e, f\}$ then find *(i)* $(A\times B)U(B\times C)$ $(ii) A\times(BUC)$.	**03**
	(b)	Define even and odd functions. Determine whether the function $f: I \to R^+$ defined by $f(x0 = 2x+7$ is one-to-one or bijective.	**04**
	(c)	**(i)** Show that the relation $x \equiv y(\text{mod}\, m)$ defined on the set of integers I is an equivalence relation.	**03**
		(ii) Draw the Hasse diagram for the partial ordering $\{(A,B)/A\subseteq B\}$ on the power set $P(S)$, where $S=\{a,b,c\}$.	**04**
Q.2	**(a)**	Define equivalence class. Let R be the relation on the set of integers I defined by $(x-y)$ is an even integer, find the disjoint equivalence classes	**03**
	(b)	A committee of *5* persons is to be formed from *6* men and *4* women. In how many ways can this be done when (i) at least *2* women are included (ii) at most *2* women are included ?	**04**
	(c)	Solve the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$ using the method of undetermined coefficients.	**07**
		OR	
	(c)	Solve the recurrence relation using the method of generating function $a_n - 5a_{n-1} + 6a_{n-2} = 3^n, n \ge 2; a_0 = 0, a_1 = 2.$	**07**
Q.3	**(a)**	Define simple graph, degree of a vertex and complete graph.	**03**
	(b)	Define tree. Prove that there is one and only one path between every pair of vertices in a tree *T*.	**04**
	(c)	**(i)** A graph *G* has *15* edges, *3* vertices of degree *4* and other vertices of degree *3*. Find the number of vertices in *G*.	**03**
		(ii) Define vertex disjoint and edge disjoint subgraphs by drawing the relevant graphs.	**04**
		OR	
Q.3	**(a)**	Show that $(G,+_5)$ is a cyclic group, where *G={ 0, 1, 2, 3, 4 }.*	**03**
	(b)	Define the following by drawing graphs (i) weak component (ii) unilateral component (iii) strong component.	**04**
	(c)	**(i)** Construct the composite tables for *(i)* addition modulo *4* and *(ii)* multiplication modulo *4* for $Z_4 = \{0,1,2,3\}$. Check whether they have identity and inverse element.	**03**
		(ii) Define ring. Show that the set $M = \left\{\begin{bmatrix} a & 0 \\ b & 0\end{bmatrix} / a,b \in R\right\}$ is not a ring under the operations of matrix addition and multiplication.	**04**

Q.4 **(a)** Define algebraic structure, semi group and monoid. Also give related examples. **03**

(b) Use Warshall's algorithm to obtain path matrix from the adjacency matrix of **04**



(c) **(i)** Is the algebraic system *(Q, *)* a group? Where Q is the set of rational numbers and * is a binary operation defined by $a*b = a+b-ab,\ \forall a,b \in Q$. **03**

(ii) Let *(Z, +)* be a group, where Z is the set of integers and + is an operation of addition. Let H be a subgroup of Z consisting of elements multiple of 5. Find the left cosets of H in Z. **04**

OR

Q.4 **(a)** Prove that there are always an even number of vertices of odd degree in a graph. **03**

(b) Prove that every subgroup H of an abelian group is normal. **04**

(c) **(i)** Find the number of edges in a r – regular graph with n – vertices. **03**

(ii) A tree T has 4 vertices of degree 2, 4 vertices of degree 3, 2 vertices of degree 4. Find the number of pendant vertices in T. **04**

Q.5 **(a)** Show that the operation * defined by $x*y = x^y$ on the set N of natural numbers is neither commutative nor associative. **03**

(b) Prove that an algebraic structure *(G, *)* is an abelian group, where G is the set of non-zero real numbers and * is a binary operation defined by $a*b = \dfrac{ab}{2}$. **04**

(c) **(i)** Find out using truth table, whether $(p \wedge r) \rightarrow p$ is a tautology. **03**

(ii) Obtain the dnf of the form $\neg(p \rightarrow (q \wedge r))$. **04**

OR

Q.5 **(a)** If $f : x \rightarrow 2x$, $g : x \rightarrow x^2$ and $h : x \rightarrow (x+1)$, then show that $(fog)oh = fo(goh)$. **03**

(b) Define lattice. Determine whether the POSET $(\{1,2,3,4,5\};1)$ is lattice. **04**

(c) **(i)** If *p:* product is good, *q:* service is good, *r:* company is public limited, write the following in symbolic notations *(i)* either product is good or the service is good or both, *(ii)* either the product is good or service is good but not both, *(iii)* it is not the case that both product is good and company is public limited. **03**

(ii) For the universe of integers, let *p(x), q(x), r(x), s(x)* and *t(x)* be the following open statements*: p(x): x > 0; q(x): x* is even*; r(x): x* is a perfect square; *s(x): x* is divisible by *4*; *t(x): x* is divisible by *5*. Write the following statements in symbolic form: *(i)* At least one integer is even, *(ii)* There exists a positive integer that is even, *(iii)* If *x* is even then *x* is not divisible by *5, (iv)* No even integer is divisible by *5, (v)* There exists an even integer divisible by *5*. **04**