

Differential Calculus II

Unit: 3



ENGINEERING MATHEMATICS – I

**B Tech 1st Sem (CSE/ CS/IT/IOT/
ME/M.TECH (INTEGRATED)/EC)**

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Content

- Taylor's and Maclaurin's Theorems for a function of one variables
- Taylor's and Maclaurin's Theorems for a function of two variables
- Jacobians
- Approximation of errors
- Maxima and Minima of functions of several variables
- Lagrange's method of multipliers

Course Objective

- The objective of this course is to familiarize the engineering students with techniques of solving matrices, differential calculus, multivariable calculus problems. It aims to equip the students with adequate knowledge of mathematics that will enable them in formulating problems and solving problems analytically.

Unit Objective

- The use of differential calculus to solve physics, geometry, and optimization problems.
- Apply calculus concepts to solve real-world problems such as optimization and related rates problems. For example, students will be expected to construct related rate equations to compute the value of a variable and interpret the resulting value in the context of the given problem.
- The students will learn the essential concepts of series expansion as an Taylor's & Maclaurin's series, Jacobian, maxima-minima and approximation of errors in a comprehensive manner.

Course Outcomes 2020-21 (B. Tech. – 1st Sem)

Course Name: Engineering Mathematics-I (AAS0103)

CO1	Apply the concept of matrices to solve linear simultaneous equations
CO2	Apply the concept of successive differentiation and partial differentiation to solve problems of Leibnitz theorems and total derivatives .
CO3	Apply partial differentiation for evaluating maxima, minima, Taylor's series and Jacobians.
CO4	Illustrate multiple integral to find area, volume, centre of mass and centre of gravity.
CO5	Demonstrate the basic concept of Profit, Loss, Number & Series, Coding & decoding.

Program Outcomes (PO)

- 1. Engineering knowledge**
- 2. Problem analysis**
- 3. Design/development of solutions**
- 4. Conduct investigations of complex problems**
- 5. Modern tool usage**
- 6. The engineer and society**

7. Environment and sustainability

8. Ethics

9. Individual and team work

10. Communication

11. Project management and finance

12. Life-long learning

CO-PO Mapping 2021-22 (B. Tech. – 1st Sem)

Course Name: Engineering Mathematics-I (AAS0103)												
CO	PO-1	PO-2	PO-3	PO-4	PO-5	PO-6	PO-7	PO-8	PO-9	PO-10	PO-11	PO-12
CO1	3	2	1	1	3	2	-	-	-	2	2	3
CO2	3	3	2	3	3	-	-	-	-	2	3	3
CO3	3	2	3	3	3	2	-	-	-	2	3	3
CO4	3	2	3	3	2	2	-	-	-	2	2	3
CO5	1	1	1	1	1	-	-	-	-	2	-	3
Mean	2.6	2	2	2.2	2.4	2.0	-	-	-	2	2.5	3

Prerequisite and Recap

- Basic knowledge of function of one and two variables
- Basic knowledge of differentiation

Taylor's and Maclaurin's Theorem (CO 3)

- Students will be able to understand the concepts of expansion of a function of several variables.
- Taylor's Theorem is used in physics when it's necessary to write the value of a function at one point in terms of the value of that function at a nearby point.

Taylor's Theorem

If the $f(x + h)$ be a function of h (x being independent of h) which can be expanded in powers of h and the expansion be differentiable any number of times, then

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^n}{n!} f^n(x) + \dots \quad \text{-----(1)}$$

Case 1: When we put $x = a$ in equation (1) then

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^n(a) + \dots \quad \text{-----(2)}$$

Expansion of function of one variable (CO3)

Case 2: If we put $h = x - a$ in equation (2) then

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^n}{n!} f^n(a) + \dots$$

Case 3: If we Put $a = 0$ and $h = x$ in equation (2) then

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} f^n(0) \end{aligned}$$

which is known as **Maclaurin's theorem**.

Example 1

Question: Expand $\log(1+x)$ in powers of x . Then find the series for $\log\left(\frac{1+x}{1-x}\right)$ and hence determine the value of $\log\left(\frac{11}{9}\right)$ upto five places of decimal. (MTU 2012)

Solution: Let $f(x) = \log(1+x)$ $f'(x) = \frac{1}{1+x}$ $f''(x) = -\frac{1}{(1+x)^2}$
 $f(0) = 0$ $f'(0) = 1$ $f''(0) = -1$

We know by Maclaurin's theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

Then we get

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Example 1 Contd....

Replacing x by $-x$ in it then

$$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

Then we can find

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right]$$

by putting $x = 1/10$, we get

$$\log\left(\frac{11}{9}\right) = 0.20067$$

Example 2

Question: Use Taylors theorem to Express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x - 2)$.

Solution:

Let $f(x) = 2x^3 + 7x^2 + x - 6$	$f(2) = 40$
$f'(x) = 6x^2 + 14x + 1$	$f'(2) = 53$
$f''(x) = 12x + 14$	$f''(2) = 38$
$f'''(x) = 12$	$f'''(2) = 12$

By Taylors theorem ,we know

$$\begin{aligned}
 f(x) &= f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!} f''(a) + \dots + \frac{(x - a)^n}{n!} f^n(a) + \dots \\
 &= 40 + 53(x - 2) + 19(x - 2)^2 + 2(x - 2)^3
 \end{aligned}$$

Example 3

Question: Expand e^x in powers of x by Maclaurin's theorem

Solution:

$$\text{Let } f(x) = e^x \qquad f'(x) = e^x \qquad f''(x) = e^x \qquad f^n(x) = e^x$$

$$f(0) = 1 \qquad f'(0) = 1 \qquad f''(0) = 1 \qquad f^n(0) = 1$$

$$f(x) = f(0) + \frac{f'(0)}{1!}(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)}{3!}(x - 0)^3 + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \dots \dots$$

FAQ

1. Expand $e^{2x} \sin x$ in ascending powers of x up to x^4 .

Ans: $x + 2x^2 + \frac{11}{3!}x^3 + \dots + \frac{5^{n/2}}{n!}x^n \sin\left(n \tan^{-1} \frac{1}{2}\right)$ (UPTU 2018)

2. Show that

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right)$$

(UPTU2017)

3. Calculate the approximate value of $\sqrt{10}$ to four decimal places by taking the first four terms of an appropriate Taylor's expansion.

Ans: 3.16227

Expansion of function of several variable (CO3)

If $f(x + h, y + k)$ be a given function which can be expanded into a series of positive ascending powers of h and k then

$$f(x + h, y + k) = f(x, y) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(x, y) + \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f(x, y) + \frac{1}{3!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^3 f(x, y) + \dots$$

----- (1)

This is known as Taylor's series expansion of $f(x + h, y + k)$ in powers of h and k .

Expansion of function of several variable (CO3)

Case 1: Putting $h = x - a$, $x = a$, $k = y - b$, $y = b$ in equation (1) then

$$\begin{aligned} f(x, y) &= f(a, b) + [(x - a)f_x(a, b) + (y - b)f_y(a, b)] \\ &+ \frac{1}{2!} [(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] \\ &+ \dots \end{aligned}$$

Case 2: Putting $h = x$, $x = 0$, $k = y$, $y = 0$ in equation (1) then

$$\begin{aligned} f(x, y) &= f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] \\ &+ \frac{1}{2!} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \dots \end{aligned}$$

Which is known as Maclaurin's series.

Example 1

Question: Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ up to second degree terms.

Solution: We have $f(x, y) = x^2y + 3y - 2$, where $a = 1, b = -2$

		$x = 1, y = -2$
$f(x, y)$	$x^2y + 3y - 2$	-10
$f_x(x, y)$	$2xy$	-4
$f_y(x, y)$	$x^2 + 3$	4
$f_{xx}(x, y)$	$2y$	-4
$f_{xy}(x, y)$	$2x$	2
$f_{yy}(x, y)$	0	0
$f_{xxx}(x, y)$	0	0
$f_{xxy}(x, y)$	2	2
$f_{xyy}(x, y)$	0	0
$f_{yyy}(x, y)$	0	0

Example 1 Contd...

By Taylor's expansion

$$\begin{aligned}
 f(x, y) &= f(a, b) + [(x - a)f_x(a, b) + (y - b)f_y(a, b)] \\
 &+ \frac{1}{2!} [(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] \\
 &+ \dots
 \end{aligned}$$

$$f(x, y) = -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2) + \dots$$

Example 2

Question: Find the Taylor's series expansion of $f(x, y) = x^3 + xy^2$ about the point (2,1)

Solution: we have

		$x = 2, y = 1$
$f(x, y)$	$x^3 + xy^2$	10
$f_x(x, y)$	$3x^2 + y^2$	13
$f_y(x, y)$	$2xy$	4
$f_{xx}(x, y)$	$6x$	12
$f_{xy}(x, y)$	$2y$	2
$f_{yy}(x, y)$	$2x$	4
$f_{xxx}(x, y)$	6	6
$f_{xxy}(x, y)$	0	0
$f_{xyy}(x, y)$	2	2
$f_{yyy}(x, y)$	0	0

Example 2 Contd...

By Taylor theorem for a function of two variables,

$$f(x, y) = f(2, 1) + (x - 2) f_x(2, 1) + (y - 1) f_y(2, 1) + \frac{1}{2!} \{ (x - 2)^2 f_{xx}(2, 1) + 2(x - 2)(y - 1) f_{xy}(2, 1) + (y - 1)^2 f_{yy}(2, 1) \} \dots$$

$$\begin{aligned} f(x, y) &= 10 + (x - 2)13 + (y - 1)4 \\ &+ \frac{1}{2} \{ (x - 2)^2 12 + 2(x - 2)(y - 1)2 + (y - 1)^2 4 \} + \dots \end{aligned}$$

Example 3

Question: Find the Taylor's series expansion of $f(x, y) = e^x \sin y$ in powers of x and y as far as terms of third degree.

Solution: we have

		$x = 0, y = 0$
$f(x, y)$	$e^x \sin y$	0
$f_x(x, y)$	$e^x \sin y$	1
$f_y(x, y)$	$e^x \cos y$	0
$f_{xx}(x, y)$	$e^x \sin y$	0
$f_{xy}(x, y)$	$e^x \cos y$	1
$f_{yy}(x, y)$	$-e^x \sin y$	0
$f_{xxx}(x, y)$	$e^x \sin y$	0
$f_{xxy}(x, y)$	$e^x \cos y$	1
$f_{xyy}(x, y)$	$-e^x \sin y$	0
$f_{yyy}(x, y)$	$-e^x \cos y$	-1

Example 3 Contd...

By Taylor theorem for a function of two variables,

$$f(x, y) = f(0,0) + (x - 0) f_x(2,1) + (y - 0) f_y(0,0) \\ + \frac{1}{2!} \{ (x - 0)^2 f_{xx}(0,0) + 2(x - 0)(y - 0) f_{xy}(0,0) + (y - 0)^2 f_{yy}(0,0) \} \dots$$

$$f(x, y) = y + xy + \frac{1}{2} x^2 y - \frac{1}{6} y^3 + \dots$$

1. Expand $e^x \log(1 + y)$ in Taylor's series about the origin up to the terms of degree three. **[UPTU 2014]**

Ans: $y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \dots \dots \dots$

2. Expand $e^x \sin y$ in the powers of x and y in the neighborhood of $(0, \frac{\pi}{4})$ up to the third degree terms. **[U.P.T.U.2008]**

Ans:

$$\frac{1}{\sqrt{2}} \left[1 + x + \left(y - \frac{\pi}{4}\right) + \frac{x^2}{2!} + x \left(y - \frac{\pi}{4}\right) - \frac{1}{2!} \left(y - \frac{\pi}{4}\right)^2 + \frac{x^3}{3!} - \frac{1}{3!} \left(y - \frac{\pi}{4}\right)^3 + 3 \frac{x^2}{3!} \left(y - \frac{\pi}{4}\right) - \frac{3x}{3!} \left(y - \frac{\pi}{4}\right)^2 \right]$$

Quiz

- Expansion of $e^x = \dots\dots\dots$
- Expansion of $f(x, y) = f(a, b) + \dots\dots\dots$
- Maclaurian's expansion of $\sin x = \dots\dots\dots$

- **Taylor's theorem**
- NPTEL-Lecture 17: Taylor's Theorem
<https://www.youtube.com/watch?v=jiEaKYlOATY>
- NPTEL-Lecture 16: Taylor's Theorem for function of two variable
<https://www.youtube.com/watch?v=r6lDwJZmfGA>
- **Maclaurin's Theorem**
- Lecture 27- Taylor's Theorem and Maclaurin's Series
<https://www.youtube.com/watch?v=Jk9xMY4mPH8>

Weekly Assignment (CO3)

Taylor and Maclaurin Series

- Expand $e^{2x} \sin x$ in ascending power of x up to x^5 . **Ans:** $x + 2x^2 + \frac{11}{6}x^3 + x^4 + \frac{41}{120}x^5$
- Expand $\log(1+x)$ in powers of x . Then find series for $\log_e \left[\frac{1+x}{1-x} \right]$ and hence determine the value of $\log_e \left(\frac{11}{9} \right)$ upto five places of decimal. **Ans:** $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$,
 $\log_e \left[\frac{1+x}{1-x} \right] = 2 \left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \right]$, 0.20067
- Expand x^y in the powers of $(x-1)$ and $(y-1)$ up to the third degree terms. **Ans:** $1 + (x-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1) + \dots$
- Expand $e^x \sin y$ in the powers of x and y in the neighborhood of $\left(0, \frac{\pi}{4}\right)$ up to the third degree terms. **Ans:** $\frac{1}{\sqrt{2}} \left[1 + x + \left(y - \frac{\pi}{4}\right) + \frac{x^2}{2!} + x \left(y - \frac{\pi}{4}\right) - \frac{1}{2!} \left(y - \frac{\pi}{4}\right)^2 + \frac{x^3}{3!} - \frac{1}{3!} \left(y - \frac{\pi}{4}\right)^3 + 3 \frac{x^2}{3!} \left(y - \frac{\pi}{4}\right) - \frac{3x}{3!} \left(y - \frac{\pi}{4}\right)^2 \right] + \dots$
- Expand $e^x \log(1+y)$ in Taylor's series about the origin up to the terms of degree three. **Ans:** $y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \dots \dots \dots$
- Expand $\tan^{-1} \frac{y}{x}$ in the neighbourhood of $(1, 1)$ upto and inclusive of second degree terms. Hence compute $f(1.1, 0.9)$ approximately. **Ans:** $\frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$; 0.6857 approx.

Weekly Assignment (CO3)

➤ MCQ:

Q.1 Expansion of e^{x+y} in power of $(x - 1)$ and $(y + 1)$ up to first-degree is

(a) $f(x) = 1 + (x - 1) + (y + 1)$

(b) $f(x) = 1 + (x + 1) + (y + 1)$

(c) $f(x) = 1 + (x - 1) + (y - 1)$

(d) $f(x) = 1 + xy$

Q.2 Maclaurin's Expansion of $\sin x$ is

(a) $x - \frac{x^3}{\sqrt{3}} + \frac{x^5}{\sqrt{5}} - \dots$

(b) $x - \frac{x^3}{\sqrt{3}} - \frac{x^5}{\sqrt{5}} - \dots$

(c) $x + \frac{x^3}{\sqrt{3}} + \frac{x^5}{\sqrt{5}} \dots$

(d) $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

Q.3 Expansion of $\sin x$ in ascending powers of $(x - a)$ is

(a) $\sin a + (x - a)\cos a - \frac{(x-a)^2}{\sqrt{2}}\sin a - \dots$

(b) $\sin a + (x - a)\cos a + \frac{(x-a)^2}{\sqrt{2}}\sin a + \dots$

(c) $\sin a + (x + a)\cos a - \frac{(x+a)^2}{\sqrt{2}}\sin a - \dots$

(d) $\sin a + (x + a)\cos a + \frac{(x+a)^2}{\sqrt{2}}\sin a + \dots$

Q.4 Expansion of x^y near the point $(1,1)$ up to the first degree terms is

(a) $1 + (x - 1) + (y - 1) \dots$

(b) $1 + (x - 1)(y - 1) \dots$

(c) $1 + (x - 1) + \dots$

(d) $1 + (x + 1) + (y + 1) \dots$

Ans. 1- (a), 2- (a), 3- (a), 4- (c)

Weekly Assignment (CO3)

➤ Very Short Question:

(a) Find the Maclaurin's series of $\cos x$.

$$\text{Ans: } \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \dots \dots \dots \right]$$

(b) Expand a^x about $x = 0$.

$$\text{Ans} \left[1 + x \log a + \frac{x^2}{2!} + \frac{x^3}{3!} (\log a)^3 \right] + \dots$$

➤ Short Questions:

(c) Find the Taylor's series expansion of $f(x, y) = x^3 + xy^2$ about $(2, 1)$.

[UPTU 2018]

$$\text{Ans: } 10 + 13(x - 2) + 4(y - 1) + 6(x - 2)^2 + 2(y - 1)^2 + 2(x - 2)(y - 1) + \dots$$

(d) Expand $e^{2x} \sin x$ in ascending power of x up to x^5 .

$$\text{Ans: } x + 2x^2 + \frac{11}{6}x^3 + x^4 + \frac{41}{120}x^5$$

➤ Short Questions:

(e) Find the Taylor's series expansion of $f(x, y) = x^3 + xy^2$ about $(2, 1)$.

[UPTU 2017]

$$\text{Ans: } 10 + 13(x - 2) + 4(y - 1) + 6(x - 2)^2 + 2(y - 1)^2 + 2(x - 2)(y - 1) + \dots$$

(f) Expand $e^{2x} \sin x$ in ascending power of x up to x^5 .

[GBTU 2017]

$$\text{Ans: } x + 2x^2 + \frac{11}{6}x^3 + x^4 + \frac{41}{120}x^5$$

Weekly Assignment (CO3)

➤ Long Questions:

(a) Expand $\log(1+x)$ in powers of x . Then find series for $\log_e \left[\frac{1+x}{1-x} \right]$ and hence determine the value of $\log_e \left(\frac{11}{9} \right)$ upto five places of decimal.

[MTU 2012]

Ans: $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$, $\log_e \left[\frac{1+x}{1-x} \right] = 2 \left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \right]$, 0.20067

(b) Expand x^y in the powers of $(x-1)$ and $(y-1)$ up to the third degree terms. [UKTU 2018]

Ans: $1 + (x-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1) + \dots$

(c) Expand $e^x \sin y$ in the powers of x and y in the neighborhood of $(0, \frac{\pi}{4})$ up to the third degree terms.

[U.P.T.U.2018 **Ans:** $\frac{1}{\sqrt{2}} \left[1 + x + \left(y - \frac{\pi}{4} \right) + \frac{x^2}{2!} + x \left(y - \frac{\pi}{4} \right) - \frac{1}{2!} \left(y - \frac{\pi}{4} \right)^2 + \frac{x^3}{3!} - \frac{1}{3!} \left(y - \frac{\pi}{4} \right)^3 + 3 \frac{x^2}{3!} \left(y - \frac{\pi}{4} \right) - \frac{3x}{3!} \left(y - \frac{\pi}{4} \right)^2 \right] + \dots$

(d) Compute $\tan^{-1} \left(\frac{0.9}{1.1} \right)$ approximately.

[U.P.T.U.2018, 2017(SUM)]

Ans:0.6904

(e) Expand $e^x \log(1+y)$ in Taylor's series about the origin up to the terms of degree three.

[UPTU 2018]

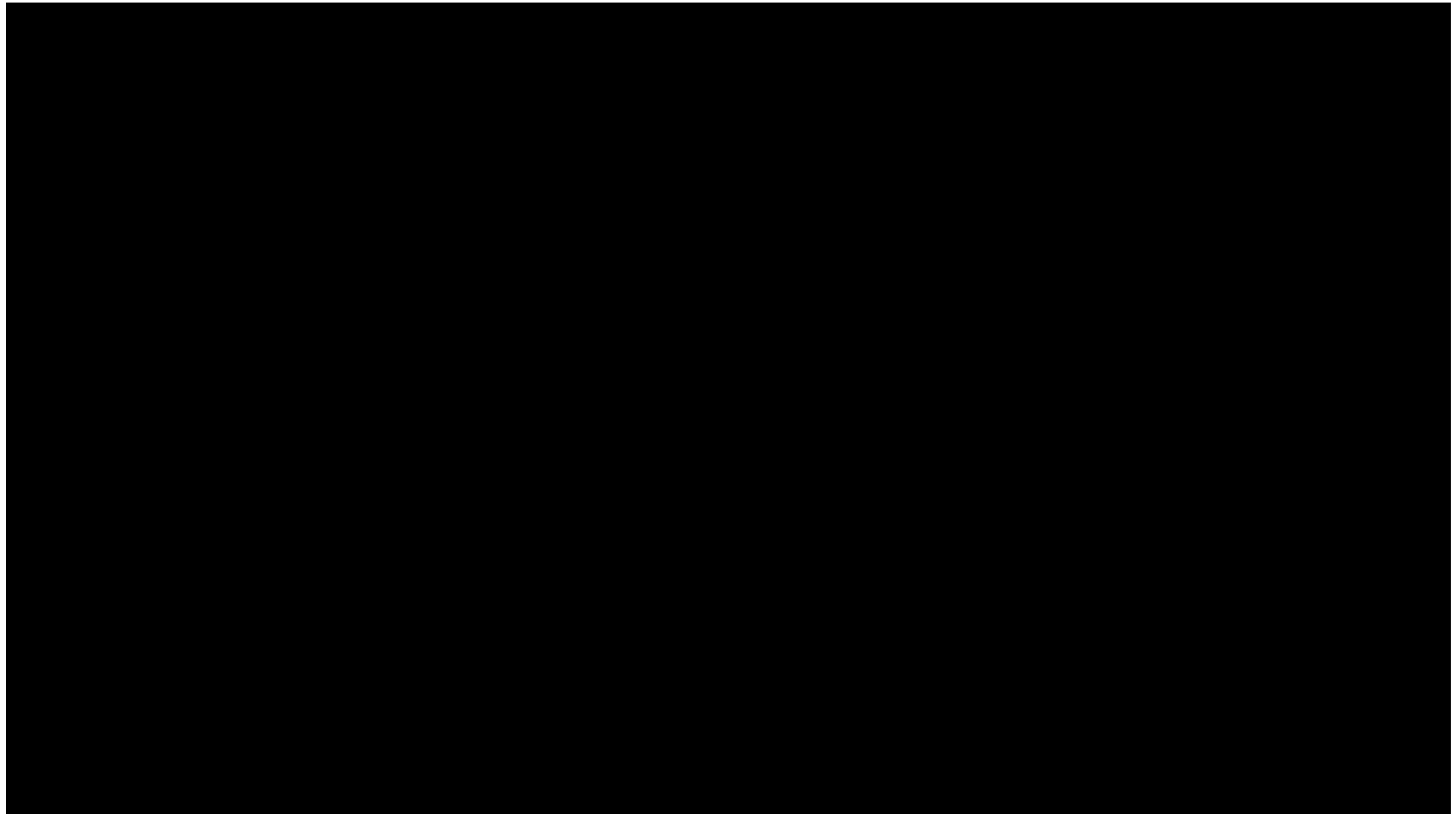
Ans: $y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \dots$

(f) Expand $\tan^{-1} \left(\frac{y}{x} \right)$ in the neighbourhood of $(1, 1)$ upto and inclusive of second degree terms. Hence compute $f(1.1, 0.9)$ approximately.

[UPTU 2016, 2018; GBTU 2010;UKTU 2011,2012]

Ans: $\frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$; 0.6857 approx.

Taylor's Theorem



Taylor and Maclaurin's theorems

1. Find a Taylor series for the function $f(x) = \ln(x)$ centered at $c = 1$.

Recap

- ✓ Taylor's & Maclaurin's Theorems for one variable.
- ✓ Expansion of function of several variables
- ✓ Discussion on assignments.

Jacobians (CO3)

- Students will be able to evaluate of Jacobians of two and more variables.
- In our physical world many things can be represented as a vector field etc.
- The Jacobian matrix of such a field is just how the field changes with the input (position, time, etc.), to the first order.

Jacobians (CO3)

If u and v continuous and differentiable function of two variables x and y , i.e.,

$u = f_1(x, y)$ and $v = f_2(x, y)$ then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the jacobian of u and v with respect to x, y and is denoted as

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Property-1

If u and v are functions of x and y then $J \cdot J^* = 1$ where

$$J = \frac{\partial(u,v)}{\partial(x,y)} \quad \text{and} \quad J^* = \frac{\partial(x,y)}{\partial(u,v)}.$$

Similarly if u , v and w are functions of x , y and z then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \times \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

Property-2

If u, v are functions of r, s and r, s are function of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}.$$

Similarly, If u, v, w are functions of r, s, t and r, s, t are functions of x, y and z then

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{\partial(u, v, w)}{\partial(r, s, t)} \times \frac{\partial(r, s, t)}{\partial(x, y, z)}$$

Property-3

If functions u, v of two independent variables x, y are dependent, then $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

Jacobian of Implicit Functions (CO3)

If u, v are implicit functions of x, y connected by f_1, f_2 such that $f_1(u, v, x, y) = 0$, and $f_2(u, v, x, y) = 0$

$$\text{then } \frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\frac{\partial(f_1, f_2)}{\partial(x, y)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}}$$

If u, v and w are implicit functions of x, y and z connected by f_1, f_2 and f_3 such that

$f_1(u, v, w, x, y, z) = 0, f_2(u, v, w, x, y, z) = 0$ and $f_3(u, v, w, x, y, z) = 0$

$$\text{then } \frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}$$

Example 1

Question: If $x = u(1 + v)$ and $y = v(1 + u)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.

Solution: we have $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} 1 + v & u \\ v & 1 + u \end{vmatrix}$$

$$= (1 + v)(1 + u) - uv \\ = 1 + u + v$$

Example 2

Question: If $u = xyz$, $v = x^2 + y^2 + z^2$ and $w = x + y + z$ then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

Solution:

$$\begin{aligned} \text{We have } \frac{\partial(u,v,w)}{\partial(x,y,z)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} \\ &= -2(x - y)(y - z)(z - x). \end{aligned}$$

Also, we know that $\frac{\partial(u,v,w)}{\partial(x,y,z)} \times \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{-1}{2(x - y)(y - z)(z - x)}.$$

Example 3

Question: If $x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$ and $u = r \sin \theta \cos \varphi$,
 $v = r \sin \theta \sin \varphi$, $w = r \cos \theta$. Find $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$. (UPTU2008)

Solution:

$$\text{We have } \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & \frac{1}{2} \sqrt{\frac{w}{v}} & \frac{1}{2} \sqrt{\frac{v}{w}} \\ \frac{1}{2} \sqrt{\frac{w}{u}} & 0 & \frac{1}{2} \sqrt{\frac{u}{w}} \\ \frac{1}{2} \sqrt{\frac{v}{u}} & \frac{1}{2} \sqrt{\frac{u}{v}} & 0 \end{vmatrix} = \frac{1}{4}$$

$$\text{and } \frac{\partial(u,v,w)}{\partial(r,\theta,\varphi)} = r^2 \sin \theta$$

$$\text{Thus } \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \frac{1}{4} r^2 \sin \theta$$

Example 4

Question: If u, v, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ , find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Solution: We have $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$

$$\lambda^3 - 3(x + y + z)\lambda^2 + 3(x^2 + y^2 + z^2)\lambda - (x^3 + y^3 + z^3) = 0$$

If u, v, w are the roots of the equation

$$u + v + w = x + y + z$$

$$uv + vw + wu = x^2 + y^2 + z^2$$

$$uvw = \frac{1}{3}(x^3 + y^3 + z^3)$$

Example 4 Contd...

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

After solving, we get $\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = -2(x - y)(y - z)(z - x)$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ v + w & w + u & u + v \\ vw & wu & uv \end{vmatrix}$$

Example 4 Contd...

We get

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = (w - u)(w - v)(u - v)$$

then

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}$$

Thus

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = - \frac{-2(x - y)(y - z)(z - x)}{(w - u)(w - v)(u - v)}$$

Example 5

Question: Determine whether the following functions are functionally dependent or not. If functionally dependent, find the relation between them.

$$u = \sin^{-1}x + \sin^{-1}y \text{ and } v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

Solution:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix} = 0$$

i.e., u and v are functionally dependent.

Example 5 Contd...

Relation between u and v

Let $\sin^{-1}x = \alpha$ and $\sin^{-1}y = \beta$

$$\begin{aligned}\text{then } v &= x\sqrt{1-y^2} + y\sqrt{1-x^2} \\ &= \sin\alpha \cos\beta + \sin\beta \cos\alpha \\ &= \sin u.\end{aligned}$$

Example 6

Question: If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution:

Let $f_1 = u^3 + v^3 - x - y = 0$

$f_2 = u^2 + v^2 - x^3 - y^3 = 0.$

$$\begin{aligned} \frac{\partial(u,v)}{\partial(x,y)} &= (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x,y)} \div \frac{\partial(f_1, f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_1}{\partial y} \\ \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_2}{\partial y} \end{vmatrix} \div \begin{vmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_1}{\partial v} \\ \frac{\partial \phi_2}{\partial u} & \frac{\partial \phi_2}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} \div \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix} \\ &= \frac{3(y^2 - x^2)}{6uv(u - v)} = \frac{1}{2} \frac{y^2 - x^2}{uv(u - v)} \end{aligned}$$

1. If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$ and $y_3 = \frac{x_1 x_2}{x_3}$ then show that $\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = 4$.

2. If $x + y + z = u$, $y + z = uv$ and $z = uvw$ then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$. **(GBTU2018)**

3. If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$, show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin \theta$. and find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$. **(GBTU2019)**

4. If $u_1 = x_1 + x_2 + x_3 + x_4$, $u_1 u_2 = x_2 + x_3 + x_4$, $u_1 u_2 u_3 = x_3 + x_4$, $u_1 u_2 u_3 u_4 = x_4$ then show that $\frac{\partial(x_1 x_2 x_3 x_4)}{\partial(u_1 u_2 u_3 u_4)} = u_1^3 u_2^2 u_3$. **(MTU2017)**

5. Verify the chain rule for the Jacobians if $x = u$, $y = u \tan v$, $z = w$. **(UPTU 2019)**

6. If $u = x + 2y + z$, $v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between u , v and w . **(UPTU2017, GBTU2018)**

7. If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy$, prove that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{x^2 - y^2}{u^2 + v^2}$

8. If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$ then show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2 - x^2}{2uv(u-v)}$. **(UPTU2017)**

9. If $u = x(1 - r^2)^{-\frac{1}{2}}$, $v = y(1 - r^2)^{-\frac{1}{2}}$, $w = z(1 - r^2)^{-\frac{1}{2}}$ where $r^2 = x^2 + y^2 + z^2$, then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = (1 - r^2)^{-\frac{5}{2}}$. **(UPTU2019)**

Quiz

- Calculate $\frac{\partial(u, v)}{\partial(x, y)}$ for $x = e^u \cos v$ and $y = e^u \sin v$.
- What is the condition for two functions $u(x, y)$ and $v(x, y)$ to be functionally dependent?
- The Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ for the function
 $u = e^x \sin y, v = x + \log \sin y$ is.....
- If $x = u(1 + v), y = v(1 + u)$, the Jacobian $\frac{\partial(u, v)}{\partial(x, y)} = \dots\dots\dots$

Weekly Assignment (CO3)

➤ MCQ

Q.1 If $x = u, y = u \tan v, z = w$ then value of $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ is

- (a) $\frac{x}{x^2+y^2}$ (b) $u \sec^2 v$ (c) 1 (d) 0

Q.2 Two functions are functionally dependent. If their jacobian with respect to independent variable is

- (a) 1 (b) 0 (c) $1/2$ (d) None of these

Q.3 If $u = x + y - z, v = x - y + z$ and $w = x^2 + y^2 + z^2 - 2xz$ are functionally dependent then the value of $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ is

- (a) 1 (b) 2 (c) 3 (d) 0

Q.4 Evaluate the value of $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)}$

- (a) 0 (b) 1 (c) 100 (d) None of these

Ans. 1 - (b), 2 - (b), 3 - (d), 4 - (b)

Weekly Assignment (CO3)

➤ Very Short Question:

(a) If $x = r \cos \theta$, $y = r \sin \theta$ then find the value of $\frac{\partial(x,y)}{\partial(r,\theta)}$.

Ans: r

(b) If $x = u(1 + v)$, $y = v(1 + u)$, then find the Jacobian of u, v with respect to x, y .

[UPTU 2019]

Ans: $1 + u + v$

➤ Short Questions:

(c) If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$. **[UPTU2017]**

Ans:

$$-\frac{1}{2(x-y)(y-z)(z-x)}$$

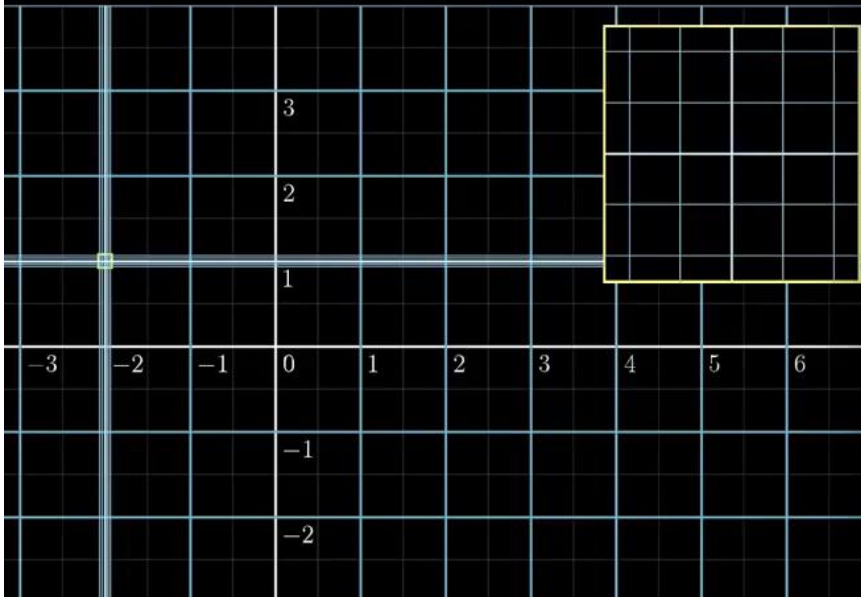
(d) Calculate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $u = \frac{2yz}{x}$, $v = \frac{3zx}{y}$, $w = \frac{4xy}{z}$. **[UPTU 2011, 2017. AKTU. 2018]**

Ans: 96

Jacobian

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + \sin(y) \\ y + \sin(x) \end{bmatrix}$$

+



- **Jacobian**
- Khan Acad- The Jacobian determinant
- <https://www.youtube.com/watch?v=p46QWyHQE6M>
- NPTEL-Lecture
- https://www.youtube.com/watch?v=fqq_UR4zhfl

Recap

- ✓ Jacobian and their properties.
- ✓ Jacobian of Implicit Function.
- ✓ Discussion on assignments

Errors and Approximations (CO3)

- Student will be able to understand the concepts of error and evaluate the approximations.

Errors and Approximations (CO3)

- An approximation error can occur because:
- The measurement of the data is not precise due to the instruments.
- Approximations are used instead of the real data.
- The relative error is often used to compare approximations of numbers of widely differing size.

Errors and Approximations (CO3)

Let $z = f(x, y)$. If $\delta x, \delta y$ are small increments in x, y and δz , the corresponding increment in z then

$$z + \delta z = f(x + \delta x, y + \delta y)$$

$$\delta z = f(x + \delta x, y + \delta y) - f(x, y)$$

$$\delta z = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}$$

If δx and δy are small changes (or errors) in x and y respectively, then an approximate change (or error) in z is δz . Replacing $\delta x, \delta y, \delta z$ by dx, dy, dz respectively, we have

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Errors and Approximations (CO3)

If δx is the error in x then

- δx is known as absolute error in x
- $\frac{\delta x}{x}$ is known as relative error in x .
- $\frac{\delta x}{x} \times 100$ is known as percentage error in x .

Example 1

Question: Find the percentage error in the area of an ellipse when errors of 2% and 3% are made in measuring its major and minor axes respectively. **(GBTU2019)**

Solution: We have, Area of an ellipse, $A = \pi ab$

$$\log A = \log \pi + \log a + \log b$$

$$\frac{1}{A} \delta A = 0 + \frac{1}{a} \delta a + \frac{1}{b} \delta b$$

$$\frac{1}{A} \delta A \times 100 = 2\% + 3\% = 5\%$$

Example 2

Question:

Find approximate value of $[(0.98)^2 + (2.01)^2 + (1.94)^2]^{1/2}$ (MTU2018)

Solution: We have $u(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$

by taking $x = 1.0, y = 2, \text{ and } z = 2$

Therefore we get, $x + \delta x = 0.98$

$$\delta x = -0.02$$

Similarly $\delta y = 0.01 \text{ and } \delta z = -0.06$

When $x = 1.0, y = 2, z = 2$ then $u = 3$.

Corresponding error δu in u is given by

$$\delta u = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial z} \delta z = -0.04$$

Required approximate value $= u + \delta u = 2.96$

1. Find the possible percentage error in computing the parallel resistance r of two resistances r_1 and r_2 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ where r_1, r are each in error by 2%.
(UKTU2017)
2. In estimating the number of bricks in a pile which is measured to be $(5m \times 10m \times 5m)$, the count of bricks is taken as 100 bricks per m^3 . Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of the bricks is Rs.2000 per thousand bricks.**(UKTU2018)**
3. A balloon is in the form of a right cylinder of radius 1.5m and length 4.0m surmounted by hemispherical ends. If the radius is increased by 0.01m and length by 0.05m, find the percentage .
(UPTU2018)
4. Evaluate $(1.99)^2 (3.01)^3 (0.98)^{1/10}$ using approximation.**(UPTU2018)**

Quiz

- If an error of 1% is made in measuring its length and breadth. what is the percentage error in the area of a rectanle?
- The percentage error in computing the area of an ellipse when an error of 1% is made in measuring the major and minor axes is.....

Weekly Assignment (CO3)

➤ MCQ:

Q.1 If $rt - s^2 > 0$ and $r < 0$ at the point (a,b) then the value of function has

- (a) Minimum
- (b) Neither maximum nor minimum
- (c) Zero
- (d) Maximum

Q.2 If $rt - s^2 < 0$ at the point (a,b) then the value of function has

- (a) No extreme value
- (b) Maximum
- (c) Minimum
- (d) Saddle point

Q.3 If $f(x, y) = 1 - x^2y^2$ then stationary point is

- (a) (0,0)
- (b) (1,1)
- (c) None of these
- (d) 1

Weekly Assignment (CO3)

➤ MCQ:

Q.4 The minimum value of $f(x, y) = x^2 + y^2$ is

- (a) 1
- (b) 2
- (c) 5
- (d) 0

Ans. 1- (d), 2- (a), 3- (c), 4 - (d)

➤ Very Short Question:

Q.1 Find the stationary points of $f(x, y) = 5x^2 + 10y^2 + 12xy - 4x - 6y + 1$.

[G.B.T.U.2013] Ans: (1 7 , 3 14)

Q.2 Find the maximum value of the function $f(xyz) = (z - 2x^2 - 2y^2)$ where $3xy - z + 7 = 0$.

[AKTU. 2016]

Q.3 Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

[M.T.U.2013] Ans: (0,0), ($\sqrt{2}$, $-\sqrt{2}$), ($-\sqrt{2}$, $\sqrt{2}$)

Weekly Assignment (CO3)

➤ Short Question:

Q.4 Examine for extreme values: $x^3 + y^3 - 3xy$.

[M.T.U.(SUM) 2011, G.B.T.U.(C.O) 2010]

Ans: Minimum. Value = - 1 at (1, 1)

Q.5 What is the maximum value of the function $f(x, y) = 1 - x^2 - y^2$.

[M.T.U.(SUM)2011]

Ans: 1

➤ Long Question:

Q.6 What is the shortest and longest distance from the point (1,2,-1) to the sphere $2x^2 + 2y^2 + 2z^2 = 24$.

[G.B.T.U.2010, 2013]

Ans: $\sqrt{6}$, $3\sqrt{6}$

Q.7 A rectangular box open at the top is of given volume, what must be the dimensions so that the surface is minimum.

[UPTU 2012, GBTU 2010]

Ans: $x = y = (2V)^{1/3}$

Weekly Assignment (CO3)

Q.8 Show that the rectangular parallelepiped of maximum volume that can be inscribed within the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is of the volume $8abc/3\sqrt{3}$.

[UKTU 2010]

Q.9 Find the maximum and minimum distances from the origin to the curve $x^2 + 4xy + 6y^2 = 140$.

[MTU 2012]


Ans: 21.6589(max), 4.5706(min)

Q.10 Find the minimum value of $x^2 + y^2 + z^2$ given that $ax + by + cz = p$.

[UKTU 2012, UPTU 2009]

Ans: $u = p^2 / (a^2 + b^2 + c^2)$

Approximation of errors

 If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

+

- **Approximation of Error**
- NPTEL-Lecture 18: Error Approximation
https://www.youtube.com/watch?v=G0V_yp0jz5c

Recap

✓ Errors

✓ Approximation

Maxima and minima of Functions of Several Variables (CO 3)

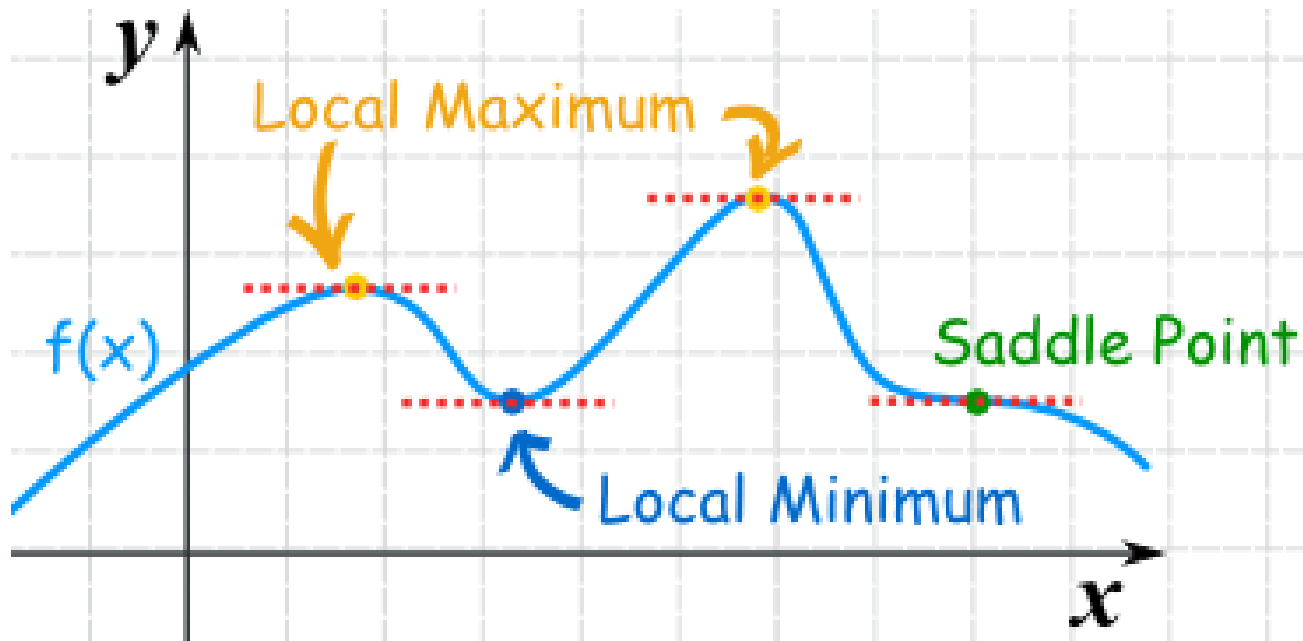
- Students will be able to evaluate maxima and minima of a function of two and several variables.

Introduction

- A maximum is a high point and a minimum is a low point.
- The terms maxima and minima refer to extreme values of a function, that is, the maximum and minimum values that the function attains. Maximum means upper bound or largest possible quantity. The absolute maximum of a function is the largest number contained in the range of the function.
- Example: A ball is thrown in the air. Its height at any time t is given by, $h = 3 + 14t - 5t^2$. What is its maximum height?

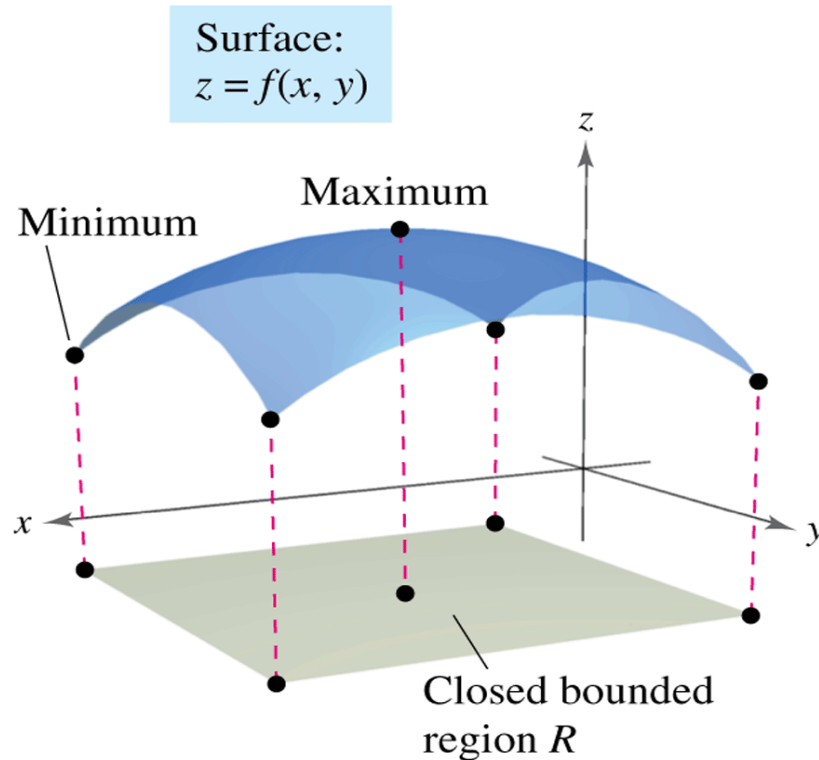
Extreme of Functions of Several Variables (CO 3)

For single variable



Extreme of Functions of Several Variables (CO 3)

For several variable variable



Extreme of Functions of Several Variables (CO 3)

Working rule to find extreme value

Step 1: Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y^2}$

Step 2:

Find stationary point by solving $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously.

Let (a,b), (c,d)are the solutions.

Step 3: For each solution in step 2, find r, s and t as follows-

$$r = \frac{\partial^2 f}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x \partial y}, \quad t = \frac{\partial^2 f}{\partial y^2}$$

Working rule to find extreme value

Step 4: At the point (a,b)

(i) If $rt - s^2 > 0$ and $r < 0$ at (a,b)

Then $f(x,y)$ is maximum at (a,b)

(ii) If $rt - s^2 > 0$ and $r > 0$ at (a,b)

Then $f(x,y)$ is minimum at (a,b)

(iii) If $rt - s^2 < 0$ and $r > 0$ at (a,b)

Then $f(x,y)$ is neither maximum nor minimum at (a,b) . Such points are known as saddle point.

(iv) If $rt - s^2 = 0$ at (a,b)

Then no conclusion can be made about extreme values of $f(x,y)$
And further investigation is required.

Example 1

Question:

Find the stationary value of $x^3 + y^3 - 3axy$, $a > 0$ (UPTU2009,GBTU2012)

Solution:

Step (i):- We have, stationary points are $(0,0)$ and (a,a)

Step (ii):- $r = 6x$, $s = -3a$ and $t = 6y$

Step(iii):- At $(0,0)$, $r = 0$, $s = -3a$, $t = 0$, then $rt - s^2 = -9a^2 < 0$.

Hence $f(x,y)$ is neither maximum nor minimum at $(0,0)$

At (a,a) , $r = 6a$, $s = -3a$, $t = 6a$

then $rt - s^2 = 27a^2 > 0$ also $r = 6a > 0$

Hence $f(x,y)$ is minimum at (a,a)

Therefore $f_{\min} = -a^3$

Example 2

Question: Find the extreme values of the function $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$.

Solution: Calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and equating it with zero

$$f_x = y - 2x - 2 = 0, \quad f_y = x - 2y - 2 = 0 \Rightarrow x = y = -2$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \quad f_{yy} = -2, \quad f_{xy} = 1$$

$$rt - s^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

$r < 0$ and $rt - s^2 > 0$ tells us that f has a maximum at $(-2, -2)$.

The value of f at this point is $f(-2, -2) = 8$.

Lagrange's Method of Undetermined Multipliers (CO3)

This method is useful to find the extreme values (i.e., maximum and minimum) for the given function, whenever the variables are three or more. To find the Maxima and Minima for the given function using Lagrange's Method, the following procedure must be followed:

Step 1: Let us consider given function to be $f(x, y, z)$ subject to the condition $\Phi(x, y, z) = 0$.

Step 2: Let us define a Lagrange's function $F = f + \lambda\Phi$, where λ is called the Lagrange multiplier.

Lagrange's Method of Undetermined Multipliers (CO3)

Step 3: Find first order partial derivatives and equate to zero

$$i.e. \frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial \Phi}{\partial x} = 0 \dots (1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \lambda \frac{\partial \Phi}{\partial y} = 0 \dots (2)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \lambda \frac{\partial \Phi}{\partial z} = 0 \dots (3)$$

Let the given condition be $\Phi(x, y, z) = 0 \dots (4)$

Step 4: Solve (1), (2), (3) & (4) , eliminate λ to get the values of x, y, z

Lagrange's Method of Undetermined Multipliers (CO3)

Step 5: The values so obtained will give the stationary point of $f(x, y, z)$

Step 6: The minimum/maximum value will be obtained by substituting the values of x, y, z in the given function.

Note: Lagrange's method does not enable us to find whether there is a maximum or minimum.

Example 3

Question:

Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$

Solution: Let us consider given function to be $f = x^2 + y^2 + z^2$ and $\Phi = xyz - a^3$. Let us define Lagrange's function $F = f + \lambda\Phi$, where λ is called the Lagrange multiplier.

$$\Rightarrow F = (x^2 + y^2 + z^2) + \lambda(xyz - a^3)$$

$$\text{Now, } \frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda yz = 0 \Rightarrow \frac{\lambda}{2} = -\frac{x}{yz} \dots\dots (1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda xz = 0 \Rightarrow \frac{\lambda}{2} = -\frac{y}{xz} \dots\dots (2)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda xy = 0 \Rightarrow \frac{\lambda}{2} = -\frac{z}{xy} \dots\dots (3)$$

$$\text{Solving (1), (2) \& (3)} \Rightarrow \frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy}$$

Example 3 Contd...

Now, consider $\frac{x}{yz} = \frac{y}{xz} \Rightarrow x^2 = y^2 \dots\dots (4)$

Again, consider $\frac{y}{xz} = \frac{z}{xy} \Rightarrow z^2 = y^2 \dots\dots (5)$

Again solving (4) & (5) $\Rightarrow x^2 = y^2 = z^2 \Rightarrow x = y = z$

Given $g = xyz - a^3 = 0$ At $x = y = z \Rightarrow x^3 = a^3 \Rightarrow x = a$

Similarly, we get $y = a, z = a$

Hence, the minimum value of the function is given by

$$(f)_{(a,a,a)} = a^2 + a^2 + a^2 = 3a^2$$

Example 4

Question: Find the extreme value of $x^2 + y^2 + z^2$, subject to the condition $xy + yz + zx = p$ (UPTU2018)

Solution: Lagrange's equation ,

$$2x + \lambda(y + z) = 0$$

$$2y + \lambda(x + z) = 0$$

$$2z + \lambda(x + y) = 0$$

Solving these equation, we get $x = y = z$

$$x^2 = \frac{p}{3}, y^2 = \frac{p}{3}, z^2 = \frac{p}{3}$$

Extreme value of $f(x, y, z) = \frac{p}{3} + \frac{p}{3} + \frac{p}{3} = p$

1. Find the maximum value of the function $f(xyz) = (z - 2x^2 - 2y^2)$ where $3xy - z + 7 = 0$. **[AKTU. 2017]**

2. A rectangular box open at the top is of given volume, what must be the dimensions so that the surface is minimum.

[UPTU 2019, GBTU 2017] **Ans:** $x = y = (2V)^{\frac{1}{3}}$

3. Find the maximum and minimum distances from the origin to the curve $x^2 + 4xy + 6y^2 = 140$. **[MTU 2018]**

Ans: 21.6589(max), 4.5706(min)

Quiz

- What is maximum value of function $f(x, y) = 1 - x^2 - y^2$?
- Determine the point(s) where the function $u = x^2 + y^2 + 6x + 12$ has a maximum or minimum.
- Find the stationary points of $f(x, y) = 5x^2 + 10y^2 + 12xy - 4x - 6y + 1$.

Recap

- ✓ Extrema of functions of several variables
- ✓ Stationary points
- ✓ Rule to find maxima and minima of function of two variable
- ✓ Lagrange's method of multipliers
- ✓ Discussion on assignment

Weekly Assignment (CO3)

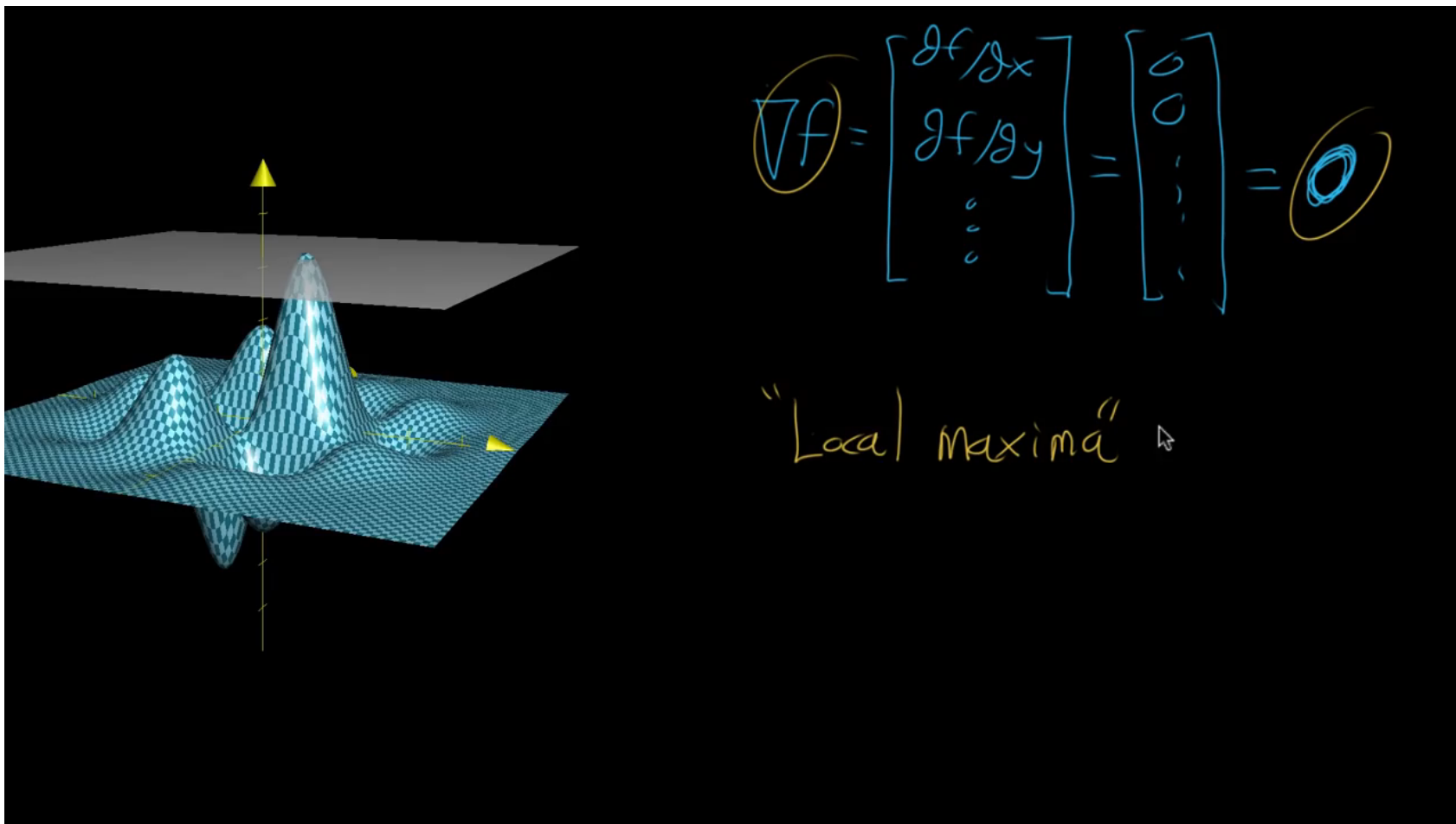
Jacobian

1. If $u_1 = \frac{x_2 x_3}{x_1}$, $u_2 = \frac{x_3 x_1}{x_2}$ and $u_3 = \frac{x_1 x_2}{x_3}$, find the value of $\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}$. **Ans. 4**
2. If $x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$ and $u = r \sin \theta \cos \phi$ and $v = r \sin \theta \sin \phi$, $w = r \cos \theta$ find the value of $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$. **Ans. $\frac{1}{4} r^2 \sin \theta$**
3. If $u = \frac{x+y}{z}$, $v = \frac{z+y}{x}$, $w = \frac{y(x+y+z)}{xz}$, then show that u, v and w are not independent and find relation between them. **Ans. Relation is $uv - w = 1$**
4. If $x + y + z = u$, $y + z = uv$, $z = uvw$. Then show $\frac{\partial(u, v, w)}{\partial(x, y, z)} = u^2 v$.
5. Prove that the functions $u = 3x + 2y - z$, $v = x - 2y + z$, $w = x^2 - 2xy - xz$ are not independent, and find the relation between them
6. If u, v and w are the roots of the equation $(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$, then find $\frac{\partial(u, v, w)}{\partial(a, b, c)}$.
Ans. $\frac{-2(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}$

Maxima and Minima

1. A rectangular box open at the top is to have a volume of 32 cubic feet. Determine the dimensions of the box requiring least material for its construction. **Ans.** $l = 4, b = 4, h = 2$
2. Find the dimension of the rectangular box of maximum capacity whose surface area is given. When (i) box is open at the top (ii) box is closed. **Ans.** (i) $l = b = 2h$ (ii) $l = b = h$
3. Divide 24 into three parts such that the continued product of the first ,square of the second and the cube of the third may be maximum. **Ans.** 4,8,12.
4. Find the shortest and maximum distance from the point (1, 2, -1) to the sphere $x^2 + y^2 + z^2 = 24$. **Ans.** $\sqrt{6}, \sqrt{54}$
5. Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values.

Saddle points



Maxima and Minima

Maxima and Minima



Extrema of functions of several variables

Extrema of Functions of Two Variable

Calculate all the critical points of the function. Then determine if each point represents a relative maximum, relative minimum, or a saddle point.

$$f(x, y) = x^2 - xy + 2y^2$$

1. Locate the critical numbers.
 - $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
 - $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ do not exist
2. Perform the second partials test.

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$
 - If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a **relative minimum** at (a, b) .
 - If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a **relative maximum** at (a, b) .
 - If $d < 0$ then $(a, b, f(a, b))$ is a **saddle point**.
 - The test is **inconclusive** if $d = 0$.

Lagrange Method of Multipliers



MCQ s

1. If $x = r \cos\theta$, $y = r \sin\theta$ then the value of $\frac{\partial(x,y)}{\partial(r,\theta)}$ is

- (i) r (ii) $1/r$ (iii) 1 (iv) 2 **Ans. (i)**

2. If $x = u$, $y = u \tan v$, $z = w$ then value of $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ is

- (i) $\frac{x}{x^2+y^2}$ (ii) $u \sec^2 v$ (iii) 1 (iv) 0 **Ans. (ii)**

3. Two functions are functionally dependent. If their jacobian with respect to independent variable is

- (i) 1 (ii) 0 (iii) $\frac{1}{2}$ (iv) None of these

Ans. (ii)

4. If $rt - s^2 > 0$ and $r < 0$ at the point (a,b) then the value of function has

- (i) Minimum (ii) Neither maximum nor minimum
(iii) Zero (iv) Maximum **Ans. (iv)**

MCQ s

5. If $f(x, y) = 1 - x^2y^2$ then stationary point is

- (i) (0,0) (ii) (1,1) (iii) None of these (iv) 1

Ans.(iii)

6. The minimum value of $f(x, y) = x^2 + y^2$ is

- (i) 1 (ii) 2 (iii) 5 (iv) 0 **Ans. (iv)**

7. Expand e^{x+y} in power of $(x - 1)$ and $(y + 1)$ up to first-degree is

- (i) $f(x) = 1 + (x - 1) + (y + 1)$ (ii) $f(x) = 1 + (x + 1) + (y + 1)$
(iii) $f(x) = 1 + (x - 1) + (y - 1)$ (iv) $f(x) = 1 + xy$ **Ans.(i)**

8. Maclaurin's Expansion of $\sin x$ is

- (i) $x - \frac{x^3}{\sqrt{3}} + \frac{x^5}{\sqrt{5}} - \dots$ (ii) $x - \frac{x^3}{\sqrt{3}} - \frac{x^5}{\sqrt{5}} - \dots$ (iii) $x + \frac{x^3}{\sqrt{3}} + \frac{x^5}{\sqrt{5}} \dots$
(iv) $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ **Ans.(iv)**

MCQ s

9. In calculating the volume of a right circular cylinder, errors of 2% and 1% are found in measuring height and base radius respectively. Then the percentage error in volume of the cylinder is

- (i) 2% (ii) 1% (iii) 4% (iv) 10% **Ans.(iii)**

10. If δx is the error in x then $\frac{\delta x}{x} \times 100$ is called

- (i) Percentage error (ii) relative error (iii) absolute error
(iv) none of these **Ans(i)**

11. The errors in measuring the radius of the base of a cone and its volume are found to be 1% and 4%, then the error in the height of the cone is

- (i) 1% (ii) 2% (iii) 3% (iv) 4% **Ans.(ii)**

12. Approximate value of $(1.04)^{3.01}$ is

- (i) 1.00 (ii) 1.11 (iii) 1.12 (iv) 1.13 **Ans.(iii)**

Old Year Paper 2019 (AAS-0103)

Differential Calculus 2 (CO3)

Section A

- h. If $u = \cos^{-1} \left(\frac{x+y}{\sqrt{x}+\sqrt{y}} \right)$, then find the value of $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$. 2
- i. Find $\frac{du}{dt}$ if $u = x^3 + y^3, x = a \cos t, y = b \sin t$. 2

Section B

If $u^3 + v^3 + w^3 = x + y + z, u^2 + v^2 + w^2 = x^3 + y^3 + z^3$ and $u + v + w = x^2 + y^2 + z^2$, then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$. 10

Section C

- a. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2} \frac{\partial u}{\partial x} + \frac{1}{3} \frac{\partial u}{\partial y} + \frac{1}{4} \frac{\partial u}{\partial z} = 0$. 10
- b. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. 10

Old Year Paper 2020 (AAS-0103)

Differential Calculus 2 (CO3)

Section A

- a. The function $z=y^2+x^2y+x^4$ has a minimum at $(0,0)$. (T/F) 2
- b. If $u = x(1-y)$, $v = xy$, then the value of the Jacobian $\partial(u,v)/\partial(x,y)$ is..... 2

Section B

- c. Prove that $1(1-x)=1^3+(x+2)^3+2^3+3^3+(x+2)^3+3^3+4^3+\dots$ 10

Section C

- d. Expand xy in powers of $(x-1)$ and $(y-1)$ upto the third degree terms. 10
- e. Expand $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's theorem for several variables. 10

Old Year Paper 2020 (AAS-0103)

Differential Calculus 2 (CO3)

Section A

- a. The stationary points of $f(x,y)=5x^2+10y^2+12xy-4x-6y+1$ is 2

Section C

- b. If $u=x+2y+z, v=x-2y+3z$ and $w=2xy-xz+4yz-2z^2$, show that $\partial(u,v,w)/\partial(x,y,z)=0$ and find a relation between u, v , and w . 10
- c. In estimating the number of bricks in a pile which is measured to be (5m x 10m x 5m) count of bricks is taken as 100 bricks per metre cube. Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks is Rs. 2,000 per thousand bricks. 10

Old Year Paper 2019 (AAS-0103)

Printed Page 1 of 2

Sub Code:KAS103

Paper Id: **199103**

Roll No:

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B. TECH.
(SEM I) THEORY EXAMINATION 2019-20
MATHEMATICS-I

Time: 3 Hours

Total Marks: 100

Note: 1. Attempt all Sections. If require any missing data; then choose suitably.

SECTION A

1. Attempt all questions.

Q. No.	Question	Marks	CO
a.	Show that vectors (1, 6, 4), (0, 2, 3) and (0, 1, 2) are linearly independent.	2	1
b.	Define Lagrange's mean value theorem.	2	2
c.	If $u = x(1 - y)$, $v = xy$, find $\frac{\partial(u,v)}{\partial(x,y)}$.	2	3
d.	Show that vector $\vec{V} = (x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k}$ is solenoidal.	2	5
e.	Find the value of 'b' so that rank of $A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & b \end{bmatrix}$ is 2.	2	1
f.	Evaluate $\int_0^2 \int_0^1 (x^2 + 3y^2) dy dx$.	2	4
g.	Find grad ϕ at the point (2, 1, 3) where $\phi = x^2 + yz$	2	5
h.	If $u = \cos^{-1} \left(\frac{x+y}{\sqrt{x}+\sqrt{y}} \right)$, then find the value of $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$.	2	3
i.	Find $\frac{du}{dt}$ if $u = x^3 + y^3$, $x = a \cos t$, $y = b \sin t$.	2	3
j.	Find the area lying between the parabola $y = 4x - x^2$ and above the line $y = x$.	2	4

Old Year Paper 2019 (AAS-0103)

SECTION B

2. Attempt any *three* of the following:

Q. No.	Question	Marks	CO
a.	Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and hence find A^{-1} .	10	1
b.	If $y = e^{m \cos^{-1} x}$, prove that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0$. Hence find y_n when $x = 0$.	10	2
c.	If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$ and $u + v + w = x^2 + y^2 + z^2$, then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$.	10	3
d.	Evaluate the integral by changing the order of integration: $I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$.	10	4
e.	Verify Stoke's theorem for the vector field $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ integrated round the rectangle in the plane $z = 0$ and bounded by the lines $x = 0, y = 0, x = a, y = b$.	10	5

Old Year Paper 2019 (AAS-0103)

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Sub Code:KAS103

Paper Id: **199103**

Roll No:

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3. Attempt any one part of the following:

Q. No.	Question	Marks	CO
a.	For what values of λ and μ the system of linear equations: $\begin{aligned} x + y + z &= 6 \\ x + 2y + 5z &= 10 \\ 2x + 3y + \lambda z &= \mu \end{aligned}$ has (i) a unique solution (ii) no solution (iii) infinite solution Also find the solution for $\lambda = 2$ and $\mu = 8$.	10	1
b.	Find the rank of the matrix $A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$ by reducing it to normal form.	10	1

4. Attempt any one part of the following:

Q. No.	Question	Marks	CO
a.	Verify the Cauchy's mean value theorem for the function e^x and e^{-x} in the interval $[a, b]$. Also show that 'c' is the arithmetic mean between a and b.	10	2
b.	Trace the curve $r^2 = a^2 \cos 2\theta$.	10	2

5. Attempt any one part of the following:

Q. No.	Question	Marks	CO
a.	If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2} \frac{\partial u}{\partial x} + \frac{1}{3} \frac{\partial u}{\partial y} + \frac{1}{4} \frac{\partial u}{\partial z} = 0$.	10	3
b.	Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.	10	3

Old Year Paper 2019 (KAS-103)

6. Attempt any *one* part of the following:

Q. No.	Question	Marks	CO
a.	Evaluate $\iint (x + y)^2 dx dy$, where R is the parallelogram in the xy -plane with vertices $(1, 0), (3, 1), (2, 2), (0, 1)$ using the transformation $u = x + y, v = x - 2y$.	10	4
b.	Find the volume of the region bounded by the surface $y = x^2, x = y^2$ and the planes $z = 0, z = 3$.	10	4

7. Attempt any *one* part of the following:

Q. No.	Question	Marks	CO
a.	Verify the divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over the rectangular parallelepiped $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.	10	5
b.	Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at $(1, -2, 1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. Find also the greatest rate of increase of ϕ .	10	5

Links for old year papers

Mathematics I, 2019-20 (KAS-103)

<http://www.aktuonline.com/papers/btech-1-sem-mathematics-1-kas103-2020.html>

Mathematics I, 2018-19 (NAS-103)

<http://www.aktuonline.com/papers/btech-1-sem-engineering-mathematics-1-nas-103-2018-19.html>

Mathematics I, 2017-18 (RAS-103)

<http://www.aktuonline.com/papers/btech-1-sem-engineering-mathematics-ras-103-2017-18.html>

Expected Questions for University Exam

- If $u = e^{xyz}$ then prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2 y^2 z^2)u$
- If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, show that $xu_x + yu_y = \sin 2u$
- If $v = f(2x - 3y, 3y - 4z, 4z - 2x)$ prove that $6v_x + 4v_y + 3v_z = 0$.
- Expand $e^x \log(1 + y)$ in Taylor's series about the origin up to the terms of degree three.
- If u, v and w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$, then find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.
- A rectangular box open at the top is to have a volume of 32 cubic feet. Determine the dimensions of the box requiring least material for its construction.
- Find the dimension of the rectangular box of maximum capacity whose surface area is given. When (i) box is open at the top (ii) box is closed.

➤ **Taylor's and Maclaurin's Theorems**

Expansion of a function of one variables

Expansion of a function of two variables

➤ **Jacobian**

Jacobian and their properties

Jacobian of implicit function

➤ **Approximation of errors**

Errors

Approximations

➤ **Maxima and Minima of functions of several variables**

Stationary points

Rules to find maxima and minima for function of two variable

Lagrange's method of multiplier

References

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- P. Sivaramakrishna Das and C. Vijayakumari, Engineering Mathematics, 1st Edition, Pearson India Education Services Pvt. Ltd.



Thank You