PHYSICS

Module - I: Relativistic Mechanics

Frames of Reference:

The coordinate system is known as frame of reference. The simplest frame of reference is the familiar Cartesian system of coordinates in which the position of the particle is specified by its three coordinates x, y, z along the three perpendicular axes and time coordinate t.

There are two types of frame of reference:

(1) Inertial or un-accelerated frames.

(2) Non-inertial or accelerated frames.

Inertial Frames:

A frame of reference is said to be inertial when bodies in this frame obey Newton's law of inertia and other laws of Newtonian mechanics. In this frame, a body not acted upon by external force, is at rest or moves with a constant velocity.

Thus un-accelerated reference frames in uniform motion of translation relative to one another are called Galilean Frames or Inertial frames.

Non-inertial frames: Accelerated reference frames are called Non-inertial frames.

Galilean transformation equations:

Let us consider two inertial frames S and S' having observer at the origin O and O' respectively. Let frame S' is moving with uniform velocity v ($v \ll c$) along the positive X direction as shown in Fig 1. Let the origins of the two frames coincide at t = 0. Suppose some event occurs at point P. The observer O in frame S determines the position of the event by the coordinates x, y, z. The observer O' in the frame S' determines the position of the same event by the coordinates x', y', z'.

Since, there is no relative motion between S and S' along the Y and Z axes.

So, y = y' and z = z'.

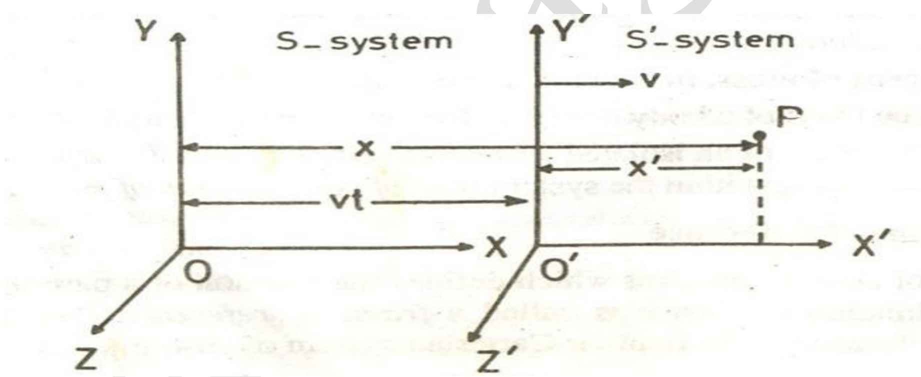

Figure (1)

The distance moved by S' in the positive X direction in time t = v t.

Therefore, x' = x – v t.

Thus the transformation equations from System S to System S' are given by,

$x' = x - v\,t$

$y' = y$

$z' = z$

$t' = t$ (Classically time is assumed to be absolute)

The transformation of velocities from one to another system is obtained by taking time derivatives i.e.

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v \qquad \text{Or} \qquad u' = u - v \qquad \text{as } t' = t \text{ so } dt' = dt$$

$$\frac{dy'}{dt'} = \frac{dy}{dt}$$

$$\frac{dz'}{dt'} = \frac{dz}{dt}$$

Now let a and a' be the accelerations of the particle as measured by observers O and O' respectively, we have

$$a = \frac{du}{dt} \text{ and } a' = \frac{du'}{dt}$$

Since $u' = u - v$

Therefore, $\frac{du'}{dt} = \frac{du}{dt}$ (as v is constant)

Or a = a', the accelerations as measured by the two observers in the two frames are the same. *Hence acceleration is invariant under Galilean transformation.*

Michelson-Morley Experiment: A. A. Michelson in 1881 and Michelson and E. W. Morley in 1887 carried out an optical experiment to detect the presence of ether.

As shown in the figure 2, S is the source of monochromatic light, a parallel beam from which it falls upon a thin glass plate P thinly silvered on the back surface. The incident beam is thus split up into two parts at P. The reflected portion travels in a direction at right angles to the incident beam, falls normally at B on the plane mirror M_1 and is reflected back to P. It gets refracted through P and enters the telescope T. The transmitted portion travels along the direction of the initial beam, falls normally at A on the plane mirror M_2 and is reflected back to P. After reflection from the back surface of P, it enters the telescope T. The two reflected beams interfere and the interference fringes are viewed with the help of telescope T. The beam reflected upwards to M_1 traverses the thickness of the plate P thrice whereas the beam reflected onto mirror M_2 traverses plate P only once. The effective distance of the mirrors M_1 and M_2 from the plate P is made to be same by use of compensating plate not shown in figure2.

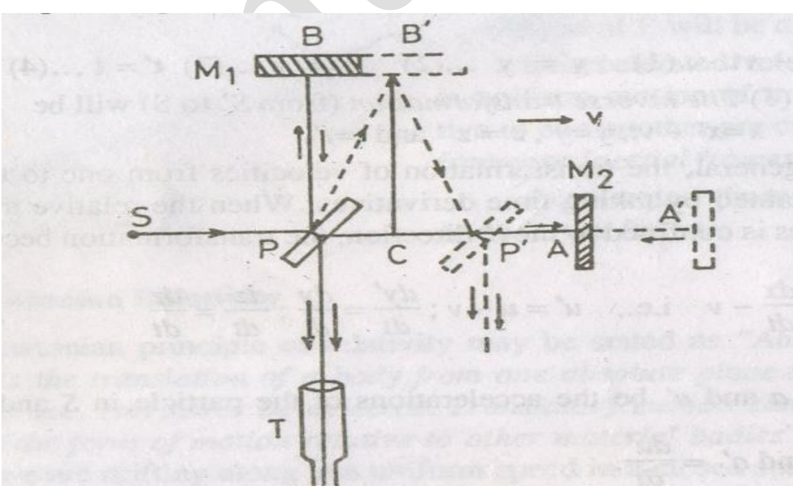

Figure (2)

The whole apparatus was floating on mercury. The one arm PA was pointed in the direction of earth motion round the sun and the other PB was pointed perpendicular to this motion. The paths of the two beams and the positions of their reflections from M_1 and M_2 will be as shown by the dotted lines.

Assume that the velocity of the apparatus (or earth) relative to fixed ether be v in the direction PA. The relative velocity of light ray traveling along PA is (c – v) while its value would be (c + v) for the returning ray.

Time difference $\Delta t = t - t' = \frac{Lv^2}{c^3}$

Now the distance traveled by light in this time Δt is $c \times \Delta t = \frac{Lv^2}{c^2}$

This is the path difference between the two parts of incident beam reaching to the telescope T. If the apparatus is turned through 90º, the role of the perpendicular arm and the horizontal arm interchanges. This further induces a path difference of $\frac{Lv^2}{c^2}$ between the two beams so that the net path difference becomes $\frac{2Lv^2}{c^2}$. Therefore, the rotation should cause a shift in the fringe pattern. Let ΔN represent the number of fringes moving past the crosshairs as the pattern shifts. Then, if light of wavelength λ is used, then $\Delta N = \frac{2Lv^2}{\lambda c^2}$. In this case, L=11 m and $\lambda = 5.5 \text{ X } 10^{-7}$ m and $v/c = 10^{-4}$, so $\Delta N = \frac{2Lv^2}{\lambda c^2} = 0.4$. Thus, Michelson and Morley expected a fringe shift of about 0.4 in their apparatus when it was rotated by 90º. Observations were made day and night (as the earth spins about its axis) and during all seasons of the year (as the earth rotates about the sun), but the expected fringe shift was not observed. Indeed, the experimental conclusion was that there was no fringe shift at all. This negative result suggests that it is impossible to measure the speed of earth relative to the ether. This null result (ΔN = 0) was such a blow to the ether hypothesis that the experiment was repeated by many workers over a 50-year period. The null result was amply confirmed. Thus all attempts to establish ether as a fixed frame of reference failed.

Explanation of negative results:

The proper explanation of negative result of Michelson-Morley experiment was given by Einstein. He proposed that the speed of light is an invariant, thus light would take equal times to travel the two paths in Michelson-Morley apparatus. i.e. $t = t' = \frac{2L}{c}$. And hence no fringe shift is to be expected.

Postulates of special theory of relativity:

(1) The laws of physics are the same in all inertial systems. No preferred inertial system exists. (The Principle of Relativity.)

(2) The speed of light in free space has the same value c in all inertial systems. (ThePrinciple of the Constancy of the Speed of Light.)

Lorentz Transformation Equations: **T**he Galilean transformation equations must be replaced by new ones consistent with experiment, using the postulates of special relativity theory.

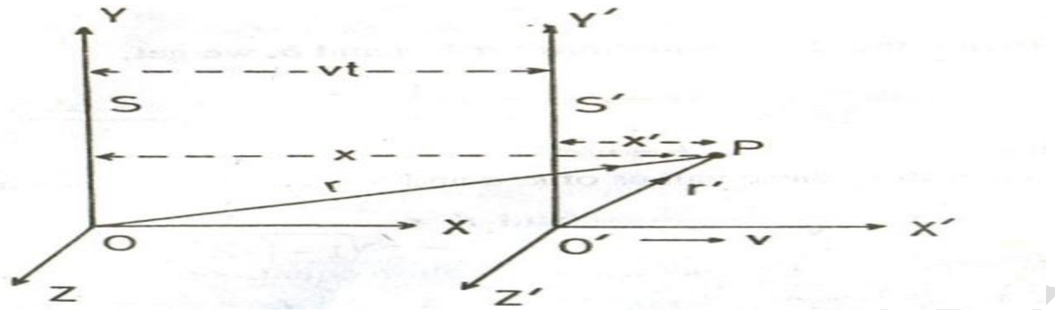

Figure 3

Lorentz transformation equations from system S to S' are:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}; \; y' = y; \; z' = z; \; t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

The inverse transformation equations can be obtained by replacing x by x', y by y', z by z', t by t' and v by (–v).

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y = y', \; z = z', \quad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Consequences of the Lorentz Transformation Equations

<u>Length Contraction (Lorentz - Fitzerald Contraction)</u> : *A body's length is measured to be greatest when it is at rest relative to the observer. When it moves with a velocity v relative to the observer its measured length is contracted in the direction of its motion by the factor* $\sqrt{1 - \frac{v^2}{c^2}}$ *whereas its dimensions perpendicular to the direction of motion are unaffected.*

Let us consider two inertial frames S and S' having observers O and O' respectively. System S' is moving with constant velocity v $(v \approx c)$ along the positive direction of axis of X with respect to system S as shown in Figure 4. Let a rod AB is at rest relative to frame of reference S' and the coordinates of ends A and B as observed by observer O' are x_1' and x_2' respectively. It is common in relativity to speak of the frame in which the observed body is at rest as the proper frame. The length of a rod in such a frame is then called the proper length. Therefore the proper length, $l_0 = x'_2 - x'_1$.

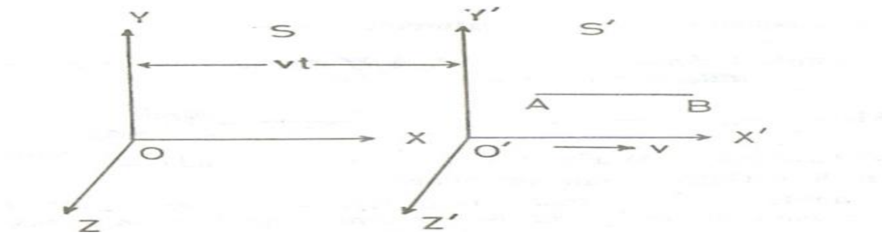

Figure 4

Similarly let x_1 and x_2 be the coordinates of the ends of the rod at the same instant of time in S. Then $l = x_2 - x_1$ is the length of rod as measured by the observer O in frame of reference S. According to Lorentz transformation,

$$x'_2 = \frac{x_2 - vt}{\sqrt{1-\frac{v^2}{c^2}}} \text{ and } x'_1 = \frac{x_1 - vt}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{Thus,} \quad x'_2 - x'_1 = \frac{x_2 - x_1}{\sqrt{1-\frac{v^2}{c^2}}} \text{ or } \quad l_0 = \frac{l}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\boxed{l = l_0\sqrt{1-\frac{v^2}{c^2}}}$$ It is clear that $l < l_0$.

Therefore to the observer in S it would appear that the length of rod has contracted by the factor $\sqrt{1-\frac{v^2}{c^2}}$.

Time Dilation: A clock is measured to go at its fastest rate when it is at rest relative to the observer. When it moves with a velocity v relative to the observer, its rate is measured to have slowed down by a factor $\sqrt{1-\frac{v^2}{c^2}}$.

Let us consider two coordinate systems S and S' having observers O and O' respectively. S' is moving with velocity v $(v \approx c)$ along the positive direction of axis of X with respect to S. Suppose a gun is placed at the position (x', y', z') in system S'. Let it fires two shots at times t'_1 and t'_2 measured with respect to observer in system S'. In S' the clock is at rest relative to the observer O'. The time interval measured by a clock at rest relative to observer O' is the proper time interval $t_0 = t'_2 - t'_1$. (The proper time interval is the time interval recorded by a clock attached to the observed body.)

Let $t = t_2 - t_1$ represent the time interval between two shots as measured by observer O in S.

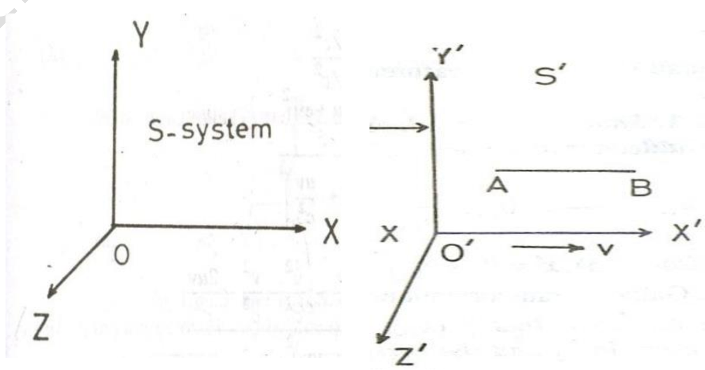

Figure 5

From inverse Lorentz transformations we have,

$t_1 = \frac{t'_1 + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$, $t_2 = \frac{t'_2 + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$ Thus, $t_2 - t_1 = \frac{t'_2 - t'_1}{\sqrt{1-\frac{v^2}{c^2}}}$, $t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$ i.e.

$t > t_0$ which shows that an interval of time observed in a moving frame of reference will be less than the same interval of time observed in a stationary frame of reference. This effect is called time dilation.

Addition of Velocities:

Let us consider two inertial frames S and S' having observers O and O' respectively. S' is moving with a constant velocity v $(v \approx c)$ along the positive axis of X with respect to S

Let a particle **P** is moving in the positive direction of X and X' axes.

Suppose the velocity of the particle **P** as measured by an observer O in system S be u, then $u = \frac{dx}{dt}$ and $u' = \frac{dx'}{dt'}$.

Where u' is the velocity of the particle as measured by an observer O' in system S'.

Using inverse Lorentz transformation equations we have,

$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}} \quad , \quad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Differentiating above equations we get,

$$dx = \frac{dx' + vdt'}{\sqrt{1-\frac{v^2}{c^2}}} \quad , \quad dt = \frac{dt' + \frac{vdx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{Therefore} \quad \frac{dx}{dt} = \frac{dx' + vdt'}{dt' + \frac{vdx'}{c^2}} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}.\frac{dx'}{dt'}}$$

Or $$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

This is the relativistic, or Einstein velocity addition theorem.

Variation Of Mass With Velocity:

In general if m denotes the mass of the body when it is moving with velocity v then,

$$\boxed{m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}}$$

This is relativistic formula for variation of mass with velocity.

Thus the relativistic momentum is $\boxed{p = mu = \dfrac{m_0 u}{\sqrt{1-\dfrac{u^2}{c^2}}}}$

Mass-Energy Equivalence:

Force is defined as rate of change of momentum,

i.e. $F = \dfrac{d}{dt}.p$

$= \dfrac{d}{dt}(mu)$ ------------------- (1)

According theory of relativity both mass and velocity are variable, therefore

$F = m.\dfrac{du}{dt} + u.\dfrac{dm}{dt}$ ------------------- (2)

Let the force F displace the body through a distance dx. Then the increase in the kinetic energy (dE_k) of the body will be given by –

$dE_k = F.dx$

Or $dE_k = m.\dfrac{du}{dt}.dx + u.\dfrac{dm}{dt}.dx$

Or $dE_k = m.u.du + u^2 dm$ ------------------ (3)

Now according to law of variation of mass with velocity,

$$m = \frac{m_0}{\sqrt{1-\dfrac{u^2}{c^2}}}$$

Squaring we get,

$$m^2 = \frac{{m_0}^2}{1-\dfrac{u^2}{c^2}}$$

$m^2(c^2 - u^2) = {m_0}^2 c^2$

$m^2 c^2 = {m_0}^2 c^2 + m^2 u^2$

Differentiating, $c^2 2m.dm = 0 + u^2 2m.dm + m^2 2u.dv$

$c^2 dm = u^2 dm + mu.du$ ------------------- (4)

From equation (3) and (4) we have,

$dE_k = c^2 dm$ ------------------- (5)

Integrating we get,

$E_k = \int_0^{E_k} dE_k = c^2 \int_{m_0}^{m} dm$ ------------------ (6)

Or $E_k = c^2(m - m_0)$

Or $E_k = mc^2 - m_0 c^2$ ------------------ (7)

When body is at rest, the internal energy stored in the body is m_0c^2. It is also known as rest mass energy. The total energy of the body is given by –

$$\text{Total energy} = E_k + m_0c^2$$

$$= \left(mc^2 - m_0c^2\right) + m_0c^2$$

$$\boxed{E = mc^2}$$

This is Einstein's mass energy relation.

Relation between Total energy, Rest energy and Momentum:

The expression for total energy is given by,

$$E = mc^2$$

$$E = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \times c^2$$

Squaring both sides, we get,

$$E^2 = \frac{m_0{}^2c^4}{1-\frac{u^2}{c^2}}$$

Or $$\left(\frac{E}{c}\right)^2 = \frac{m_0{}^2c^2}{1-\frac{u^2}{c^2}}$$

Or $$m_0{}^2c^2 = \left(\frac{E}{c}\right)^2\left(1-\frac{u^2}{c^2}\right)$$

$$m_0{}^2c^2 = \left(\frac{E}{c}\right)^2 - \left(\frac{E}{c}\right)^2 . \frac{u^2}{c^2}$$

$$m_0{}^2c^2 = \left(\frac{E}{c}\right)^2 - \left(\frac{mc^2}{c}\right)^2 . \frac{u^2}{c^2}$$

$$m_0{}^2c^2 = \left(\frac{E}{c}\right)^2 - m^2c^2 . \frac{u^2}{c^2}$$

$$m_0{}^2c^2 = \frac{E^2}{c^2} - p^2$$

Or $$m_0{}^2c^4 = E^2 - p^2c^2$$

$$\therefore \boxed{E^2 = m_0{}^2c^4 + p^2c^2}$$