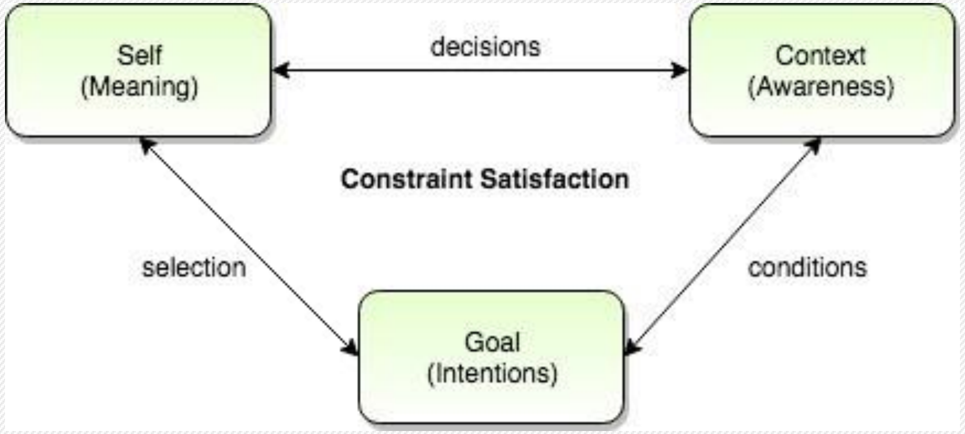
Lab Exercise-3 Constraint Satisfaction Problems

- states and goal test conform to a standard, structured and simple representation
- general-purpose heuristic functions
- Solve: design the variable, domain and **constraints** set.
- Then, look for an optimal solution. The optimal solution should satisfy all **constraints**



Constraint Satisfaction Problems (CSP)

- CSP is defined by 3 components (X, D, C):
  - state: a set of variables X, each X_i , with values from domain D_i
  - goal test: a set of constraints C, each C_i involves some subset of the variables and specifies the allowable combinations of values for that subset
  - Each constraint C_i consists of a pair <scope, rel>, where scope is a tuple of variables and rel is the relation, either represented explicitly or abstractly
- X1 and X2 both have the domain {A, B}
  - Constraints:
    - <(X1, X2), [(A, B), (B, A)]>, or
    - <(X1, X2), X1 ≠ X2>

Solution

- Each state in a CSP is defined by an assignment of values to some or all of the variables
- An assignment that does not violate any constraints is called a consistent or legal assignment
- A complete assignment is one in which every variable is assigned
- A solution to a CSP is consistent and complete assignment
- Allows useful general-purpose algorithms with more power than standard search algorithms

CSP as a Search Problem

- Initial state:
  - {} – all variables are unassigned
- Successor function:
  - a value is assigned to one of the unassigned variables with no conflict
- Goal test:
  - a complete assignment
- Path cost:
  - a constant cost for each step
- Solution appears at depth *n* if there are *n* variables
- Depth-first or local search methods work well

CSP Solvers Can be Faster

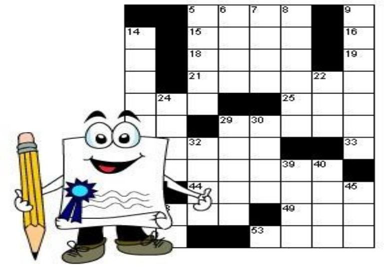
- CSP solver can quickly eliminate large part of search space
- In a CSP, if a partial assignment is not a solution, we can immediately discard further refinements of it
- Reduce Complexity

Types of Variables

- **Discrete variables**
  - finite domains:
    - n variables, domain size d^n, $O(d^n)$ complete assignments
    - e.g., Boolean CSPs, such as 3-SAT (NP-complete)
    - Worst case, can't solve finite-domain CSPs in less than exponential time
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
- **Continuous variables**
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

Examples

- Consider the problem of crossword puzzle: fitting words into a rectangular grid. Assume that a list of words is provided and that the task is to fill in the blank squares using any subset of the list. Formulate this problem precisely in two ways:
  - As a general search problem. Choose an appropriate search algorithm.
  - As a constraint satisfaction problem.
  - Word vs. letters
  - Some of the popular CSP **problems** include Sudoku, Crypt-arithmetic, crosswords, n-Queen, etc.



- **Problem formulation as CSP:**
  - Class scheduling: There is a fixed number of professors and classrooms, a list of classes to be offered, and a list of possible time slots for classes. Each professor has a set of classes that he or she can teach.

Real-World CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

Illustration

Crypt-Arithmetic puzzle

Problem:

```
  SEND
+ MORE
......
 MONEY
```

Initial State:

No two letters have the same value.
The sums of the digits must be as shown in the problem.

A Cryptarithmetic Problem

- **Problem Statement:**
  - Solve the following puzzle by assigning numeral (0-9) in such a way that each letter is assigned unique digit which satisfy the following addition.
  - Constraints : No two letters have the same value. (The constraints of arithmetic).

	S	E	N	D
+	M	O	R	E
M	O	N	E	Y

- Initial Problem State
  - S = ? ; E = ? ;N = ? ; D = ? ; M = ? ;O = ? ; R = ? ;Y = ?

Carries :

$C_4 = ?$; $C_3 = ?$; $C_2 = ?$; $C_1 = ?$

C_4	C_3	C_2	C_1	$\leftarrow$	Carry
	S	E	N	D	
+	M	O	R	E	
M	O	N	E	Y	

Constraint equations:

$$Y = D + E \longrightarrow C_1$$
$$E = N + R + C_1 \longrightarrow C_2$$
$$N = E + O + C_2 \longrightarrow C_3$$
$$O = S + M + C_3 \longrightarrow C_4$$
$$M = C_4$$

- We can easily see that M has to be non zero digit, so the value of C4 =1

1. M = C4 $\Rightarrow$ **M = 1**
2. O = S + M + C3 → C4

   For C4 =1, S + M + C3 > 9 $\Rightarrow$

   S + 1 + C3 > 9 $\Rightarrow$ S+C3 > 8.

   If C3 = 0, then S = 9 else if C3 = 1,

   then S = 8 or 9.

- We see that for S = 9
  - C3 = 0 or 1
  - It can be easily seen that C3 = 1 is not possible as O = S + M + C3 $\Rightarrow$ O = 11 $\Rightarrow$ O has to be assigned digit 1 but 1 is already assigned to M, so not possible.
  - Therefore, only choice for C3 = 0, and thus O = 10. This implies that O is assigned 0 (zero) digit.
- **Therefore, O = 0 : M = 1, O = 0**

C_4	C_3	C_2	C_1	← Carry
	S	E	N	D
+	M	O	R	E
M	O	N	E	Y

Y = D + E → C1

E = N + R + C1 → C2

N = E + O + C2 → C3

O = S + M + C3 → C4

M = C4

3. Since C3 = 0; N = E + O + C2 produces no carry.

- As O = 0, N = E + C2 .
- Since N ≠ E, therefore, C2 = 1.

Hence N = E + 1

- Now E can take value from 2 to 8 {0,1,9 already assigned so far }
  - If E = 2, then N = 3.
  - Since C2 = 1, from E = N + R + C1 , we get 12 = N + R + C1
    - If C1 = 0 then R = 9, which is not possible as we are on the path with S = 9
    - If C1 = 1 then R = 8, then
      - From Y = D + E , we get 10 + Y= D + 2 .
      - For no value of D, we can get Y.
  - Try similarly for E = 3, 4. We fail in each case.

C_4	C_3	C_2	C_1	←	Carry
	S	E	N	D	
+	M	O	R	E	
M	O	N	E	Y	

Y = D + E → C1
E = N + R + C1 → C2
N = E + O + C2 → C3
O = S + M + C3 → C4
M = C4

- If E = 5, then N = 6
  - Since C2 = 1, from E = N + R + C1 , we get 15 = N + R + C1 ,
  - If C1 = 0 then R = 9, which is not possible as we are on the path with S = 9.
  - If C1 = 1 then R = 8, then
    - From Y = D + E , we get 10 + Y= D + 5 i.e., 5 + Y = D.
    - If Y = 2 then D = 7. These values are possible.

- Hence we get the final solution as given below and on backtracking, we may find more solutions.

S = 9 ; E = 5 ; N = 6 ; D = 7 ;

M = 1 ; O = 0 ; R = 8 ;Y = 2

C_4	C_3	C_2	C_1	←	Carry
	S	E	N	D	
+	M	O	R	E	
M	O	N	E	Y	

$$
\begin{aligned}
Y &= D + E && \rightarrow C1 \\
E &= N + R + C1 && \rightarrow C2 \\
N &= E + O + C2 && \rightarrow C3 \\
O &= S + M + C3 && \rightarrow C4 \\
M &= C4
\end{aligned}
$$

Constraints:

$Y = D + E \longrightarrow C_1$

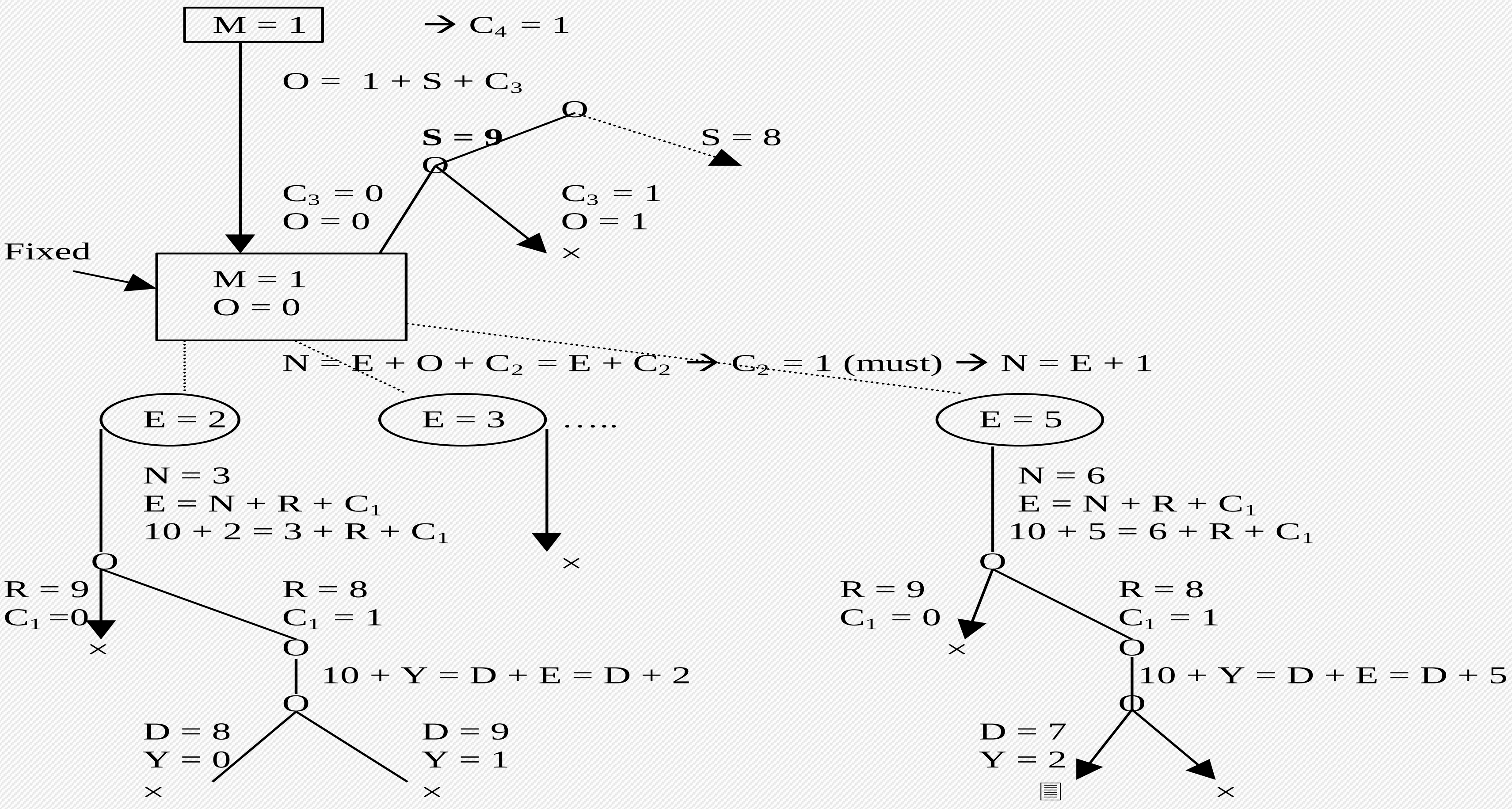
$E = N + R + C_1 \longrightarrow C_2$

$N = E + O + C_2 \longrightarrow C_3$

$O = S + M + C_3 \longrightarrow C_4$

$M = C_4$

Initial State



The first solution obtained is:

M = 1, O = 0, S = 9, E = 5, N = 6, R = 8, D = 7, Y = 2

C4	C3	C2	C1		← Carries
	B	A	S	E	
+	B	A	L	L	
G	A	M	E	S	

Constraints equations are:

$E + L = S \rightarrow C1$

$S + L + C1 = E \rightarrow C2$

$2A + C2 = M \rightarrow C3$

$2B + C3 = A \rightarrow C4$

$G = C4$

Initial Problem State

G = ?; A = ?;M = ?; E = ?; S = ?; B = ?; L = ?

1. $G = C_4 \Rightarrow \mathbf{G = 1}$
2. $2B + C_3 = A \quad \rightarrow C_4$

   2.1 Since $C_4 = 1$, therefore, $2B + C_3 > 9 \Rightarrow$ B can take values from 5 to 9.

   2.2 Try the following steps for each value of B from 5 to 9 till we get a possible value of B.

- If B = 5
  - if $C_3 = 0 \Rightarrow A = 0 \Rightarrow M = 0$ for $C_2 = 0$ or $M = 1$ for $C_2 = 1$ $\times$
  - if $C_3 = 1 \Rightarrow A = 1$ $\times$ (as G = 1 already)
- For B = 6 we get similar contradiction while generating the search tree.
- If $\boxed{\mathbf{B = 7}}$, then for $C_3 = 0$, we get $\boxed{\mathbf{A = 4}}$ $\Rightarrow M = 8$ if $C_2 = 0$ that leads to contradiction, so this path is pruned. If $C_2 = 1$, then $\boxed{\mathbf{M = 9}}$.

3. Let us solve $S + L + C_1 = E$ and $E + L = S$
   - Using both equations, we get $2L + C_1 = 0 \Rightarrow \boxed{\mathbf{L = 5}}$ and $C_1 = 0$
   - Using L = 5, we get $S + 5 = E$ that should generate carry $C_2 = 1$ as shown above
   - So $S+5 > 9 \Rightarrow$ Possible values for E are {2, 3, 6, 8} (with carry bit $C_2 = 1$)
   - If E = 2 then $S + 5 = 12 \Rightarrow S = 7$ $\times$ (as B = 7 already)
   - If E = 3 then $S + 5 = 13 \Rightarrow S = 8$.
   - Therefore $\boxed{\mathbf{E = 3}}$ and $\boxed{\mathbf{S = 8}}$ are fixed up.
4. Hence we get the final solution as given below and on backtracking, we may find more solutions. In this case we get only one solution.

   G = 1; A = 4; M = 9; E = 3; S = 8;B = 7; L = 5

CROSS+ROADS=DANGER

- 3 solutions:

  59488 + 64218 = 123706

  90733 + 67513 = 158246

  92344 + 83714 = 176058

- **TWO+TWO=FOUR**

setting F = 1, O = 4, R = 8, T = 7, W = 3, U = 6 gives 734+734=1468