Multi EIASC: An Efficient Type-reduction Algorithm for Multi-dimensional Fuzzy Sets

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***Abstract*—In this paper, we propose an efficient algorithm for estimating the centroid bound of multi-dimensional interval type-2 fuzzy sets. We further prove the convergence and optimality of the proposed algorithm.**

***Index Terms*—Multi EIASC, Multi-dimensional centroid, Centroid boundary, Switch line, Switch plane, Switch hyperplane.**

I. Introduction

Uncertainty modeling of type-2 fuzzy sets (T2-FSs) have been widely used in many applications such as pattern recognition, industrial control systems, and image processing, to name a few. For example, the use of fuzzy set operations, which have been extended from traditional crisp set operations, T2 fuzzy logic controllers (T2-FLCs) give way to an improved and relatively finer control for many control applications [1]. Due to this advantage over crisp sets, it has been also used in many theoretical facets and mathematical tools, like clustering, classification, pattern recognition [2] [3]. An important step in all such application is the type-reduction of the final resultant T2-FSs into traditional crisp sets for binary real life actuator control. Out of all type reduction mechanisms, the centroid type-reduction is the most commonly and widely used mechanism [1]. The Karnik-Mendel (KM) algorithm has proven to be a super exponentially fast type-reduction method for interval type-2 fuzzy sets (IT2-FSs). Many developments have been extended leading to more efficient algorithms such as the enhanced KM [4], iterative algorithm plus stopping condition (IASC) [5], and enhanced IASC (EIASC) [6]. A major drawback of all these algorithms is that they only work for fuzzy sets where domain elements must have a cardinal ordering defined. As a result, this necessity restricts the domain to be of only one dimension (1-D) for the type reduction technique to be applicable. For many applications in pattern recognition that involve multidimensional data there is great need of efficient multidimensional type-reduction techniques due to possible importance in joint dependencies among the features. Typically, a forced cardinal ordering is imposed on elements of higher dimensional fuzzy sets, such as treating the order of features in each dimension independently, and then KM or related type-reduction is performed [3]. This leads to an obvious loss of correlation among the features as each dimension is considered independently and thus valuable correlation information governing the dataset maybe lost in type reduction.

An intitutive way to extend type-reduction to multiple dimensions would be to consider switch hyper-surfaces instead of switch points as in KM. The major resason we suspect for the lack of progress for an efficient multi dimensional type reduction technique is the formidable computation requirements needed to deal with such hyper-surfaces. Furthermore, without any proofs of correctness and convergence of any such hypothetical type reduction method, it would only be an unreliable heuristic. In our previous work [7] we have extended KM for two-dimensions by defining a centroid bound and proposed two heuristics for such a reduction. The first contribution of this paper is using the extended centroid definition and proving the existence of a separating hyper-surface for which a special embedded set, that can be constructed by using only the upper membership function (UMF) on one side and only the lower membership function (LMF) on the other, gives a point on the centroid boundary. The second contribution is proving the linearity of such a hyper-surface, making it a hyper-plane. Using the proofs, we develop a type-reduction algorithm that gives the correct bounds for multidimensional centroids efficiently.

The remaining of the paper is organized as follows. In Section II we define the centroid bound of a multi dimensional fuzzy set and how it can be used in type reduction and defuzzfication. Section III derives a very important equation governing various properties of such a centroid. We propose a special embedded set with only higher and lower membership as secondary membership values and argue that for every point on the centroid boundary, there exists such a special embedded set which we shall compute. We further prove that the switch-curve (hyper curves for more than 2D) used in construction of such an embedded set should necessarily be a switch-plane (hyper planes for more than 2D). In Section IV we show how the 1-D case of the proposed algorithm is similar to the optimzations involved in K-M algorithm. In Section V we finally propose the multi dimensional EIASC algorithm following which we prove the correctness and convergence of our proposed algorithm by stating and proving the *shape property* and *location property* in Section VI. Then we show the working of our method and illustrations of various properties for a 2-D IT2 fuzzy set. We finally conclude our paper in Section VIII.

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II. Multi-dimensional centroid bound

The centroid of an *n* dimensional T1 FS is an *n* dimensional vector given by

$$c(\mathbf{A}_e) = \frac{\int_{\mathbf{x}\in\mathbf{X}} \mu_{\mathbf{A}_e}(\mathbf{x}) \cdot \mathbf{x}\,\mathrm{d}x}{\int_{\mathbf{x}\in\mathbf{X}} \mu_{\mathbf{A}_e}(\mathbf{x})\,\mathrm{d}x} \tag{1}$$

and the centroid of an IT2 FS is a subset of $\mathbb{R}^n$ comprising the centroids of constituting embedded fuzzy sets and can be expressed in terms of centroids of its embedded sets as

$$C(\widetilde{\mathbf{A}}) = 1\Big/\bigcup_{\forall \mathbf{A}_e \in \widetilde{\mathbf{A}}} c(\mathbf{A}_e), \tag{2}$$

which was introduced in [7]

The centroid given in (2) represents a subset of the feature space $\mathbf{X} \subset \mathbb{R}^n$ with each $\mathbf{x} \in \mathbf{X}$ having membership of 1, which is basically a crisp subset of $\mathbf{X}$. For a 1-D IT2 FS ($\mathbf{X} \subset \mathbb{R}$), the centroid is a range on $\mathbb{R}$ bounded by the centroid bounds

$$c_l(L) = \frac{\int_{x\leq L} x\overline{\mu}_{\widetilde{\mathbf{A}}}(x)\,\mathrm{d}x + \int_{x>L} x\underline{\mu}_{\widetilde{\mathbf{A}}}(x)\,\mathrm{d}x}{\int_{x\leq L} \overline{\mu}_{\widetilde{\mathbf{A}}}(x)\,\mathrm{d}x + \int_{x>L} \underline{\mu}_{\widetilde{\mathbf{A}}}(x)\,\mathrm{d}x} \tag{3}$$

and

$$c_r(R) = \frac{\int_{x\leq R} x\underline{\mu}_{\widetilde{\mathbf{A}}}(x)\,\mathrm{d}x + \int_{x>R} x\overline{\mu}_{\widetilde{\mathbf{A}}}(x)\,\mathrm{d}x}{\int_{x\leq R} \underline{\mu}_{\widetilde{\mathbf{A}}}(x)\,\mathrm{d}x + \int_{x>R} \overline{\mu}_{\widetilde{\mathbf{A}}}(x)\,\mathrm{d}x} \tag{4}$$

where L and R are the switch-points [8].

Similarly, for a 2-D IT2 FS, the centroid is a bounded region in $\mathbb{R}^2$, the boundary being any arbitrary contour. Extrapolating further, for a n-D IT2 FS the centroid is a bounded region in $\mathbb{R}^n$.

The $c_l(L)$ and $c_r(R)$ are the leftmost and rightmost centroids of the 1-D IT2 FS, which can also be viewed as the nearest and farthest centroids from the "reference point" origin. Generalizing this idea, for a 2-D IT2 FS we need to find the nearest and farthest centroids from a "reference line." One reference line will give us two points on the contour forming the boundary. Thus, the added dimensionality requires us to consider multiple reference lines covering all directions which will give us the complete contour. Similarly for n-D IT2 FS, we need to consider "reference hyperplanes" and find centroids nearest and farthest from them to find the boundary of the centroid region.

For a 1-D IT2 FS, it has been proved that the nearest and farthest centroids from the origin were obtained from embedded sets governed by switch-points where the embedded sets contained the maximum membership on one side of the switch-point and minimum on the other [8]. In the following section, we show that the nearest and farthest centroid from a reference line, in case of 2-D IT2 FS, is also obtained by such embedded set containing maximum membership on one side and minimum membership on the other side of a switch-curve. We further prove that the switch-curve is actually a "switch-line" not any arbitrary curve, which for a n-D IT2 FS becomes a "switch-hyperplane".

III. Switch planes for embedded sets

Let us consider an n-dimensional type 2 fuzzy set $\widetilde{\mathbf{F}}$, i.e the *primary variable* is an n-dimensional vector. $\widetilde{\mathbf{F}}$ is a bivariate function on the space generated by $\mu_{\widetilde{\mathbf{F}}} : \mathbf{X} \times [0,1] \rightarrow [0,1]$, where $\mathbf{X} \subset \mathbb{R}^n$ where $\mathbf{x} \in \mathbf{X}$ and $u \in U \subset [0,1]$ [1].

In set builder notation, a T2 FS is expressed as

$$\widetilde{\mathbf{A}} = \big\{\big((\mathbf{x},u), \mu_{\widetilde{\mathbf{A}}}(\mathbf{x},u)\big) | \forall \mathbf{x} \in \mathbf{X}, \forall u \in U\big\}, \tag{5}$$

where $0 \leq \mu_{\widetilde{\mathbf{A}}}(\mathbf{x},u) \leq 1$. In the rest of the paper, we consider Interval type-2 fuzzy sets unless otherwise mentioned. so we restrict the secondary member ship value to be 1, i.e $\mu_{\widetilde{\mathbf{A}}}(\mathbf{x},u) = 1 \ \ \forall (\mathbf{x},u) \in \mathbf{X} \times U$.

We calculate each point of the boundary one by one, by computing the centroid which is farthest from a line passing through the origin and making an angle $\pi/2 + \theta$ from the x-axis. In the following we have fixed a θ and so we perform the entire method given below for multiple discrete $\theta \in [0, 2\pi]$.

The coordinates of the centroid (x^*, y^*) are given by

$$x^* = \sum\sum\nolimits_{\forall \text{i,j in range}} \mathbf{X}_{ij}^{(1)} \Theta_{ij} \Big/ \sum\sum \Theta_{ij} \tag{6}$$

$$y^* = \sum\sum\nolimits_{\forall \text{i,j in range}} \mathbf{X}_{ij}^{(2)} \Theta_{ij} \Big/ \sum\sum \Theta_{ij} \tag{7}$$

where $\mathbf{X}_{ij}^{(1)}$ represents the first feature of the 2-D point $\mathbf{X}_{ij}$ and $\mathbf{X}_{ij}^{(1)}$ represents the first feature of the 2-D point $\mathbf{X}_{ij}$

Let g be the distance of this centroid from a line (hyperplane passing through origin for higher dimensions and the point at the origin for a single dimension case) passing through the origin and making an angle $\pi/2 + \theta$ from the x-axis.

$$x\cos\theta + y\sin\theta = 0 \tag{8}$$

g can be given as

$$g_\theta = \begin{bmatrix} x^* & y^* \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \tag{9}$$

Our aim is to find membership values for the 2-D points in domain such that this g_θ is maximized. Hence, let's evaluate the following differential for inference.

$$\frac{dg_\theta}{d\Theta_{ij}} = \cos\theta \frac{dx^*}{d\Theta_{ij}} + \sin\theta \frac{dy^*}{d\Theta_{ij}} \tag{10}$$

$$\begin{aligned} \implies \frac{dg_\theta}{d\Theta_{ij}} = \cos\theta &\frac{\mathbf{X}_{ij}^{(1)} - x^*}{\sum\sum_{\forall \text{l,m in range}} \Theta_{lm}} \\ + \sin\theta &\frac{\mathbf{X}_{ij}^{(2)} - y^*}{\sum\sum_{\forall \text{l,m in range}} \Theta_{lm}} \end{aligned} \tag{11}$$

For this derivative to be $\geq$ 0, we have the following condition,

$$\begin{bmatrix} \mathbf{X}_{ij}^{(1)} & \mathbf{X}_{ij}^{(2)} \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} - \begin{bmatrix} x^* & y^* \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \geq 0 \tag{12}$$

The above equation means that for every point $\mathbf{X}$ at a greater distance from the reference line than that of the centroid, more is the membership (Θ) value the farther the centroid is.

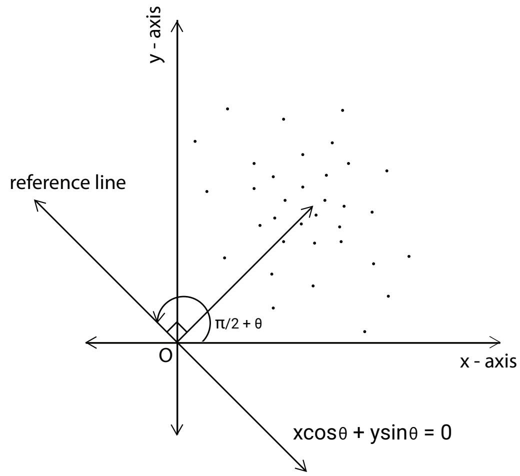

Fig. 1. Reference line construction

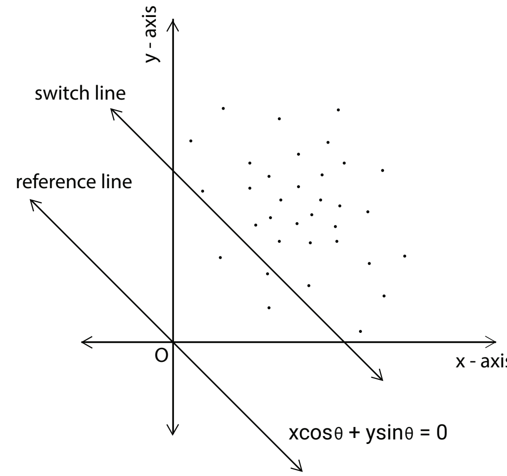

Fig. 2. Switch line corresponding to the farthest centroid from the reference line

Likewise for the points nearer to the reference line than that of the centroid, lesser is the Θ value the farther the centroid moves.

Argument that the embedded set which gives farthest centroid has either maximum or minimum memberships only:
Assume that the embedded set which gives the farthest centroid is formed by a switch curve whose embedded set has some memberships less than the maximum membership or greater than the lowest memberships i.e. not binary. Now consider one such 2-D point P at which the membership value assigned is neither the maximum nor the minimum membership. If the centroid (formed with this embedded set) is closer to line 8 than the current point P , then from equation 12 we get that if we increase its membership a bit, then the new centroid (obtained from the modified embedded set) is farther to line 8 than the previous centroid. But this contradicts the hypothesis that the initially considered embedded set gives the farthest centroid possible - In the similar way, we can arrive at contradiction even in the other complementary case of centroid with the initial embedded set be far from the point P in consideration - Hence all the memberships for the embedded set that gives the farthest centroid should have only either Upper membership or lower membership (No middle values)

Argument that the embedded set which gives farthest centroid is given by a switch-line and not a random curve:
Assume that the embedded set which gives the farthest centroid is formed by a switch curve which is not a straight line, then again from equation 12, we get that, **every point** having larger (smaller) perpendicular distance from line 8 than that of previous centroid should have the highest (lowest) membership possible. Such a property is guaranteed only when the switch curve coincides with the **line** parallel to the reference line and passing through the centroid. Hence, switch curve that gives farthest centroid must be "Line".

IV. Reducing the 2-D case to 1-D

To reduce the above scenario to 1-D, we put $\theta = 0$ in the above equations. We will have $g = x^*$, which is the distance of the centroid from the origin ($x = 0$). So calculating leftmost and the right most centorids incase of KM or EIASC algorithm is similar to calculating the centroid nearest and farthest from the reference hyper-plane passing the origin, which in 1-D case is the origin itself, in our newly proposed framework.

Basically, we consider SWITCH LINES here, in the same way as SWITCH POINTS that are for in the 1-D case. Recall, in the 1-D case, all the points to the left of this switch point are given the lower membership function and all the points to the right are given the upper membership value given by the 2-D FOU at that point. The use of switch lines, is clearly explained in the next section.

The equation of such a switch line is given by

$$x^* \cos\theta + y^* \sin\theta = 0 \tag{13}$$

where $[x^* \;\; y^*]$ is the existing centroid value given by the equations 6 and 7

We use a similar version of EIASC algorithm to reach the switch point .We shall prove the 'location property' and 'shape property', 'monotonicity property'.

V. EIASC Algorithm for centroid bound estimation

As we deal with discrete datasets here, we will discretize the domain at first in the following way. To find the g_θ, we consider lines making a slope of $\pi/2 + \theta$ at equal intervals of ϵ as shown in the figure 3.
Recapitulating the above discussion, a switch line for g_θ always has a fixed slope of $\pi/2 + \theta$. So, given any point on that line, we can construct the switch line because the slope is fixed and known. Hence **all the points on the switch line**

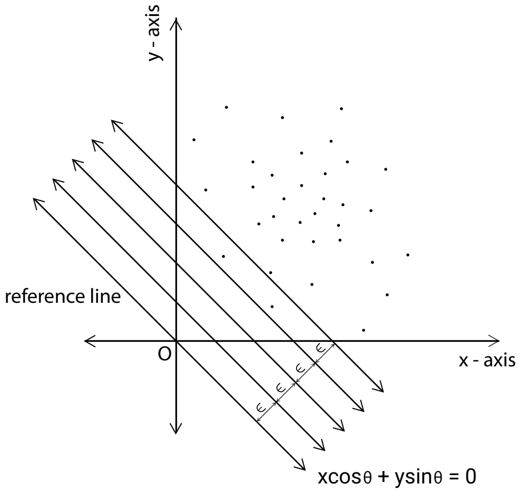

Fig. 3. Movement of switch lines for multi-EIASC

can be referred as switch point (this way of referring is used in the proofs).

In the rest of the paper we use certain notations as explained below

- Let $d_\theta(X)$ be the perpendicular distance of the point X from the line 8. For a simpler understanding and to be consistent with the notations used, while calculating the nearest most centroid, we represent this distance as $d_\theta(X_r)$. While calculating the nearest centroid, this distance is represented as $d_\theta(X_l)$. Here after the terms *nearer* and *farther* are used w.r.t the perpendicular distance from the reference line.
- $c(X_r)$ represents the multi dimensional centroid vector of the embedded set S obtained by constructing the switch line as given in equation 13, where x^* and y^* are x and y co-ordinates of the point X_r. The elements of the embedded set S which are nearer than the switch line are given lowest membership possible (from its corresponding LMF) and the elements which are farther away than the switch line are assigned highest primary membership values (from its corresponding UMF).
- $c(X_l)$ represents the multi dimensional centroid vector of the embedded set S obtained by constructing the switch line as given in equation 13, where x^* and y^* are x and y co-ordinates of the point X_l. The elements of the embedded set S which are nearer than the switch line are assigned the highest membership possible and the elements which are farther away than the switch line are assigned the lowest primary membership values possible.
- $f_\theta(X_r)$ as the function that gives perpendicular distance of the $c(X_r)$ from the line 8
  i.e $f_\theta(X_r) \quad = \quad d_\theta(c(X_r))$

  Similarly $f_\theta(X_l)$ is the function that gives perpendicular distance of $c(X_l)$ from the rerence line 8.

EIASC for g_θ

Step1: Initialize

Take the line farthest from the line $x\cos\theta + y\sin\theta = 0$ and has atleast one point in the domain as the initial switch line. Lets call the set of points on that side of the switch line which are closer to $x\cos\theta + y\sin\theta = 0$ as L and the set of points on the other side of switch line be represented as R

$a^{(1)} = \sum\sum_{\forall \mathrm{i,j} \in \mathrm{L}} \mathbf{X}_{ij}^{(1)} \underline{\theta_{ij}}$

$a^{(2)} = \sum\sum_{\forall \mathrm{i,j} \in \mathrm{L}} \mathbf{X}_{ij}^{(2)} \underline{\theta_{ij}}$

$b = \sum\sum_{\forall \mathrm{i,j} \in \mathrm{L} \cup \mathrm{R}} \underline{\theta_{ij}}$

Step2: Compute

Now, move the switch line towards the L side by ϵ parallely, and compute again let BREG represent the set of points in the region between the previous switch line and the current switch line

$a^{(1)} = a^{(1)} + \sum\sum_{\forall \mathrm{i,j} \in \mathrm{BREG}} \mathbf{X}_{ij}^{(1)} (\overline{\theta_{ij}} - \underline{\theta_{ij}})$

$a^{(2)} = a^{(2)} + \sum\sum_{\forall \mathrm{i,j} \in \mathrm{BREG}} \mathbf{X}_{ij}^{(2)} (\overline{\theta_{ij}} - \underline{\theta_{ij}})$

$b = b + \sum\sum_{\forall \mathrm{i,j} \in \mathrm{BREG}} (\overline{\theta_{ij}} - \underline{\theta_{ij}})$

$c^{(1)} = a^{(1)}/b$

$c^{(2)} = a^{(2)}/b$

$c = [c^{(1)} \quad c^{(2)}]$

Step3: Stopping condition

if $d(c) \geq$ (distance of switch line in prev iteration) stop, otherwise goto step-2

VI. Proof of convergence

Inorder to show that the above proposed algorithm will necessarily reach the stopping condition in finite time, we first prove two essential properties.

A. Location property

If X_R (observe capital R in the notation here) is the switch point that gives us the centroid farthest from the line 8,then centroid of its corresponding embedded set should be in the epsilon discretized range of X_R

i.e,

$$d_\theta(X_R) \leq f_\theta(X_R) \leq d_\theta(X_R) + \epsilon \tag{14}$$

1) proof: We use proof by contradiction technique. Suppose the relation 14 is not valid.

so, let $|f_\theta(X_R) - d_\theta(X_R)| > \epsilon$,i.e violating relation 14

Now if we construct a switch line passing through $c(X_R)$, and slightly decrease the membership value of points just towards left of the line (nearer to the line 8). From the equation 12 we can see that we will thus get a better centroid (farther than $c(X_R)$). But the hypothesis of the theorem that X_R is the farthest centroid gets contradicted.

The technique used above is that, all the centroids in the any given ϵ discretized domain are treated same and all those centroids not falling in that ϵ neighborhood are treated different .Hence the theorem, thus the location property.

B. Shape property

Let X_R be the switch point,whose embedded set gives the centroid farthest from the line 8

$$f_\theta(X_r) = \begin{cases} < d_\theta(X_r) \text{ ,for } X_r \text{ such that } d_\theta(X_r) > d_\theta(X_R) + \epsilon \\ \geq d_\theta(X_r) \text{ ,for } X_r \text{ such that} d_\theta(X_r) < d_\theta(X_R) + \epsilon \end{cases} \tag{15}$$

i.e. The centroid resulting due to an embdedded set thus constructed from a point P nearer to the reference line (compared to the distance of the actual switch point leading to farthest centroid) will be to farther away from the reference line (compared to the distance of the point P itself.)

1) proof:
Let us prove the first part of the above relation.

Let's consider only those X such that $d_\theta(X_r) > d_\theta(X_R) + \epsilon$ from relation 14,we already have

$d_\theta(X_R) \leq f_\theta(X_R) \leq d_\theta(X_R) + \epsilon$

and from the hypothesis of the theorem, X_R gives the best centroid, i.e

$f_\theta(X_r) \leq f_\theta(X_R)$

by combining all the above three equations, we get

$f_\theta(X_r) \ \leq \ f_\theta(X_R) \ \leq \ d_\theta(X_R) + \epsilon \ < \ d_\theta(X_r)$

consider the first and last terms of the above equation

$f_\theta(X_r) \ < \ d_\theta(X_r)$

This proves the first part of above theorem. The second part can be proved in the same way, it has been proved for 1D case. [8]

2) Explanation:
The above theorem states that, the centroids of the embedded sets obtained by switch points that are to the right of the best switch point (X_R) (farther side away from the line 8), are nearer than the switch point itself. (perpendicular distances from the line 8).

This is the similar 'shape' property that has been proved for the 1D case.

C. Convergence of EIASC

While calculating the farthest centroid as explained in V, we first start with a farthest point in the domain as switch point. From the Shape property, the centroid thus obtained will be nearer to the reference line. As we keep looping on Step-2, the switch point we consider in each iteration gets nearer and nearer to the reference line. At one point, when we reach the switch point corresponding to the farthest centroid, the centroid due to its corresponding embedded set falls in the same ϵ region because of the Location property. Thus the stopping condition is necessarily reached as the domain is finite.

VII. Visualization of multi EIASC in two dimensions

In this section we illustrate the working of multi EIASC in a special 2-D IT2 fuzzy set case. For simplicity the domain to have uniformly spaced points in the xy plane. Lower membership function (LMF) was constructed over the domain as a gaussian and the same gaussian was scaled to form the upper membership function (UMF) as shown in the figure 4

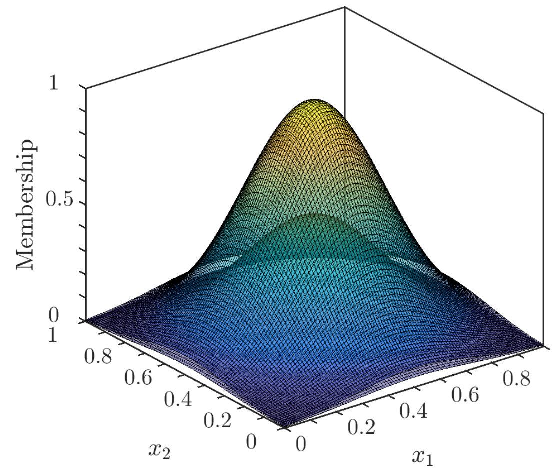

Fig. 4. 2-D IT2 fuzzy set with gaussian MFs

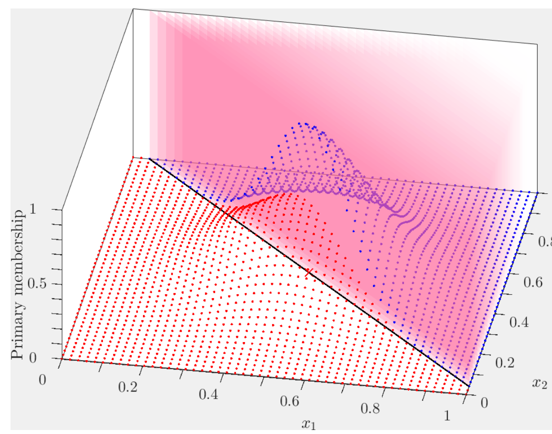

Fig. 5. Movement of switch planes till convergence

To calculate a centroid farthest c away from a considered reference line passing through origin, as explained in the section V, we consider switch planes parallel to the reference plane starting with the farthest such plane possible and iterate till the centroid for the switch plane lies on the switch plane (in discrete domain, the centroid lies in an ϵ neighbourhood). The figure 5 illustrates this process. Switch planes considered in the earlier iterations are lighter in color and those considered in the later iterations are relatively brighter in intensity. The dark black line on the xy axis is the final switch line that gives the farthest centroid.

Figure 6 and figure 7 illustrate how the shape property holds holds in 2D case (and in general for any n-D case). The x-axis represent the distance of the switch line w.r.t reference line and y-axis represent the distance of the centroid of the embedded set constructed from the switch planes.

Similarly figure 8 and figure 9 illustrate how the location property holds. The x-axis represent the distance of the switch line from the reference line and the values on the y axis represent the absolute perpendicular distance of the centroids and the corresponding switch line with which the centroid was calculated.

In all of the above figures, the y axis is cropped and zoomed in for better visualization. All the notations used in the labels

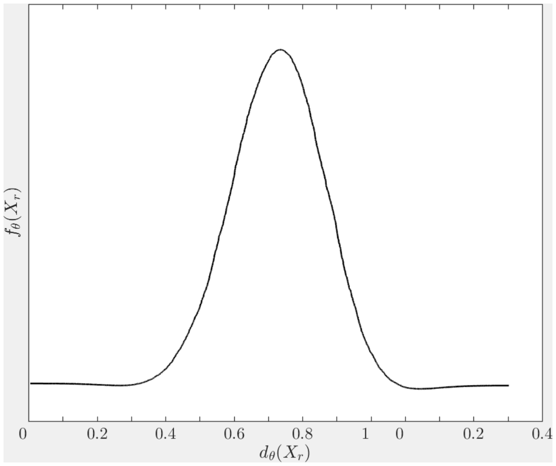

Fig. 6. Illustration of shape property for calculating farthest centroid

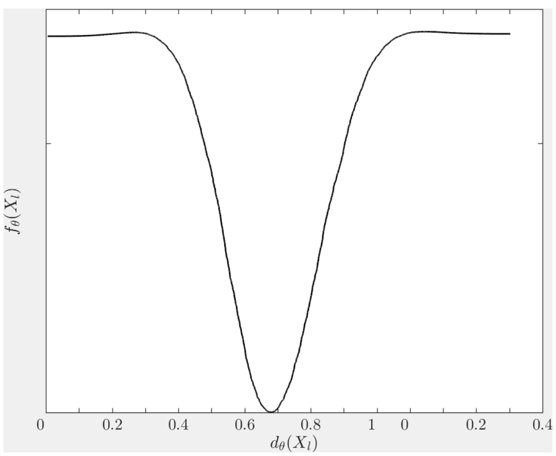

Fig. 7. Illustration of shape property for calculating nearest centroid

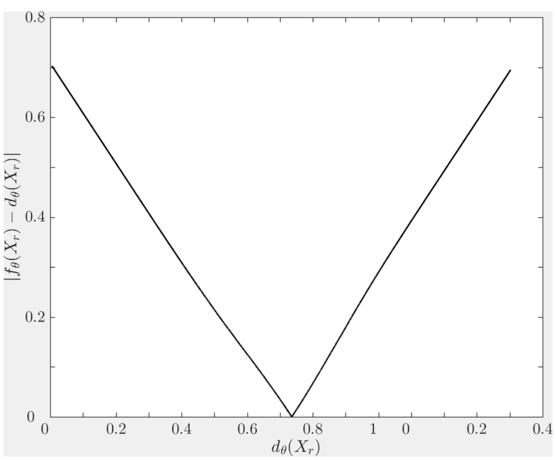

Fig. 8. Illustration of location property for calculating nearest centroid

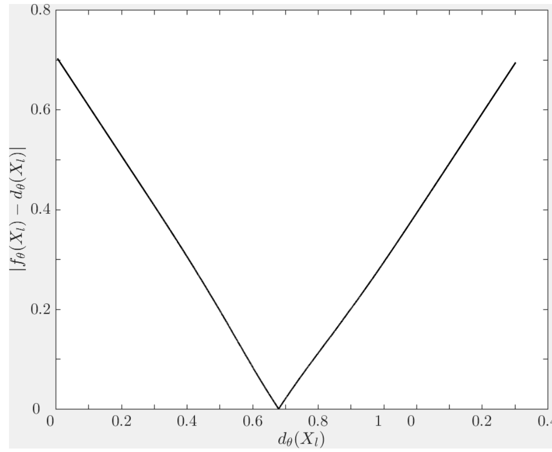

Fig. 9. Illustration of location property for calculating farthest centroid

of the figures are in accordance with that of section V.

The time complexity of the proposed algorithm for two dimension case depends on the number of different reference planes being considered. Assuming a uniform K level discretization of angles in the range $[0, 2\pi]$ for slopes of reference lines passing through origin, the time complexity of computing the bounding centroids for N points belonging to an IT2-FS would be $O(K \times N \log N)$. The $N \log N$ portion comes from the need to sort the transformed N points for every reference line for computing the points under BREG as descrribed in section V efficiently, which overshadows the linear $O(N)$ time complexity for 1-D EIASC itself.

VIII. Conclusion

There exists a switch line whose embedded set switches from lower membership values for primary varibles at the line, and gives the centroid that is farthest from the line 8 as shown in figure 2 and our aim to find such a switch line. We propose a multi EIASC algorithm in section V and prove its optimality and convergence guarentee in section VI. We also show how the 1-D special case of our newly proposed algorithm is similar to the K-M algorithm.

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