Numerical Study of Scalar Field Dynamics in Slow-Roll Inflation

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1 Introduction

Cosmological inflation provides a framework for explaining both the large-scale homogeneity of the universe and the origin of primordial fluctuations. To study inflation quantitatively, one must understand the background scalar field dynamics, the theory of cosmological perturbations, and their quantization.

This project followed a systematic approach: first consolidating the theoretical background, then extending to perturbations and their quantum treatment, and finally implementing numerical solutions of the governing equations.

2 Theoretical Foundations

2.1 Review of Inflationary Cosmology

The project began with a careful study of the dynamics of a scalar inflaton field in an expanding FLRW background. This included:

- Friedmann equations and the role of scalar fields in driving accelerated expansion.
- The slow-roll approximation and its key parameters:

$$\epsilon = \frac{M_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2, \quad \eta = M_{\rm Pl}^2 \frac{V''}{V}.$$

- The connection between inflationary dynamics and observables such as the spectral index and tensor-to-scalar ratio.

2.2 Cosmological Perturbation Theory

The framework of linear perturbations in FLRW spacetime was then studied. Key aspects included:

- Scalar–vector–tensor decomposition of perturbations.
- Gauge choices and construction of gauge-invariant quantities.
- The Bardeen potentials Φ, Ψ, and conditions under which they coincide.
- The Mukhanov–Sasaki variable v, combining field and metric perturbations into a single gauge-invariant variable.

The resulting equation for each Fourier mode is:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0, \quad z = \frac{a\phi'}{\mathcal{H}}.$$

This clarified why individual Fourier modes evolve independently and how horizon crossing determines their behavior.

2.3 Quantum Field Theory Aspects

The quantization of perturbations was then explored:

- Expansion of perturbations into Fourier modes and quantization in terms of creation/annihilation operators.
- Sub-horizon initial conditions given by the Bunch–Davies vacuum:

$$v_k(\eta) \sim \frac{1}{\sqrt{2k}} e^{-ik\eta}, \quad k \gg aH.$$

- Understanding the "freezing" of modes on super-horizon scales, which underpins the nearly scale-invariant primordial power spectrum.

This part established the connection between quantum vacuum fluctuations and the seeds of cosmic structure.

3 Transition to Numerical Work

After building the theoretical foundation, the project moved to numerical analysis. The focus was on translating the key dynamical equations into solvable systems of ODEs.

3.1 Background Equations

- Klein–Gordon equation for the homogeneous inflaton field.
- Friedmann equation for the scale factor $a(\eta)$.

3.2 Perturbations

- Numerical solution of the Mukhanov–Sasaki equation for selected Fourier modes.
- Implementation of sub-horizon vacuum initial conditions and evolution across horizon crossing.

4 Results So Far

- **Background evolution:** Verified the slow-roll behavior of the inflaton field and accelerated expansion of the scale factor.
- **Mode dynamics:** Confirmed that perturbation modes oscillate inside the horizon and freeze upon horizon exit.
- **Independent evolution:** Observed numerically that each k-mode evolves separately, consistent with theoretical predictions.

5 Planned Extensions

1. Compute the numerical power spectrum:

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} \left| \frac{v_k}{z} \right|^2,$$

and compare with analytic predictions.

2. Investigate different inflaton potentials (quadratic, quartic, plateau-type).

3. Extend simulations to link numerical results with observable quantities such as spectral index and tensor-to-scalar ratio.

6 Conclusion

The project has developed a rigorous understanding of inflationary dynamics, from background evolution to quantum perturbations, and successfully transitioned this framework into numerical implementation. Early results reproduce key theoretical expectations, establishing a base for extracting observable predictions in the next stages.