Prepared by Julian García, based on material by
Arun Konagurthu

# Workshop 21
# Fundamentals of Game Theory

**FIT 3139** Computational Modelling and Simulation

MONASH University
Information Technology

# Outline

- Strategic interaction.
- Games:
  - Normal form.
  - Bimatrix games
- Mixed strategies.
- Dominance (if we have time).

# Strategic interaction

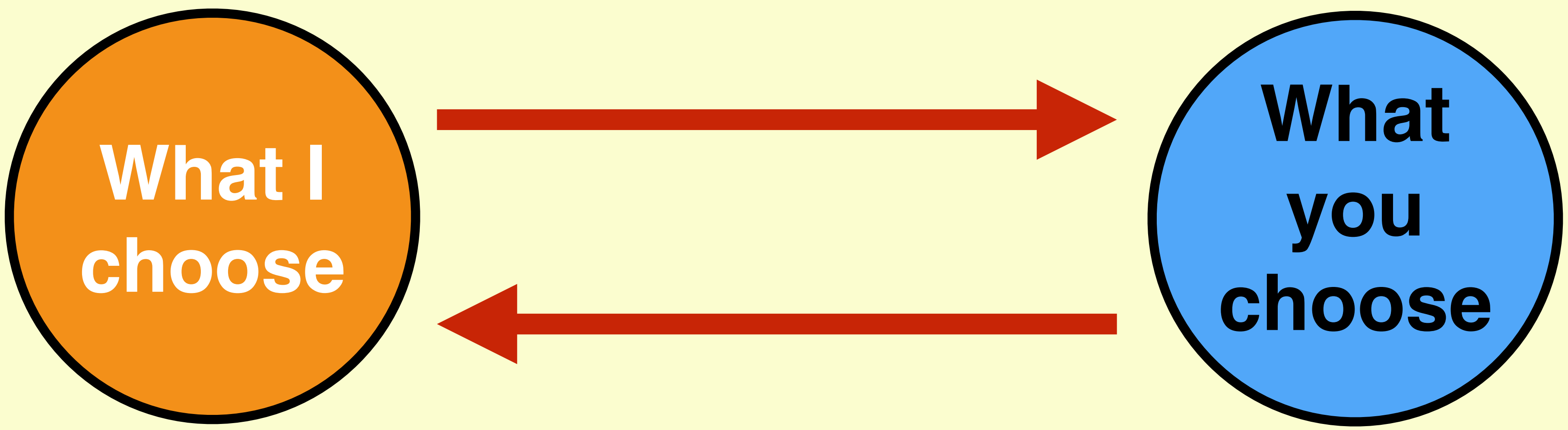
**Choose an integer number between 1 and 100**

**The winner is the person whose number is closest to 2/3 of the average choice**

(Ties broken randomly)


https://www.analyticsvidhya.com/blog/2019/12/game-theory-101-decision-making-normal-form-games/

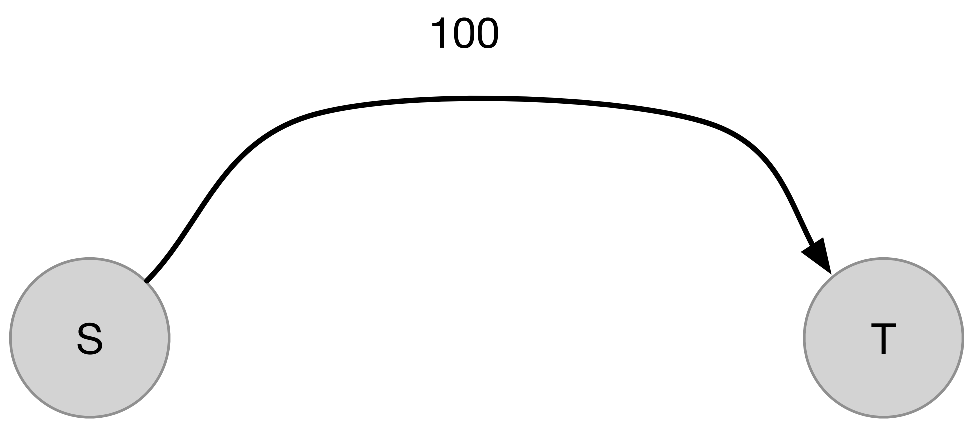
the weather forecast
does not influence the weather….



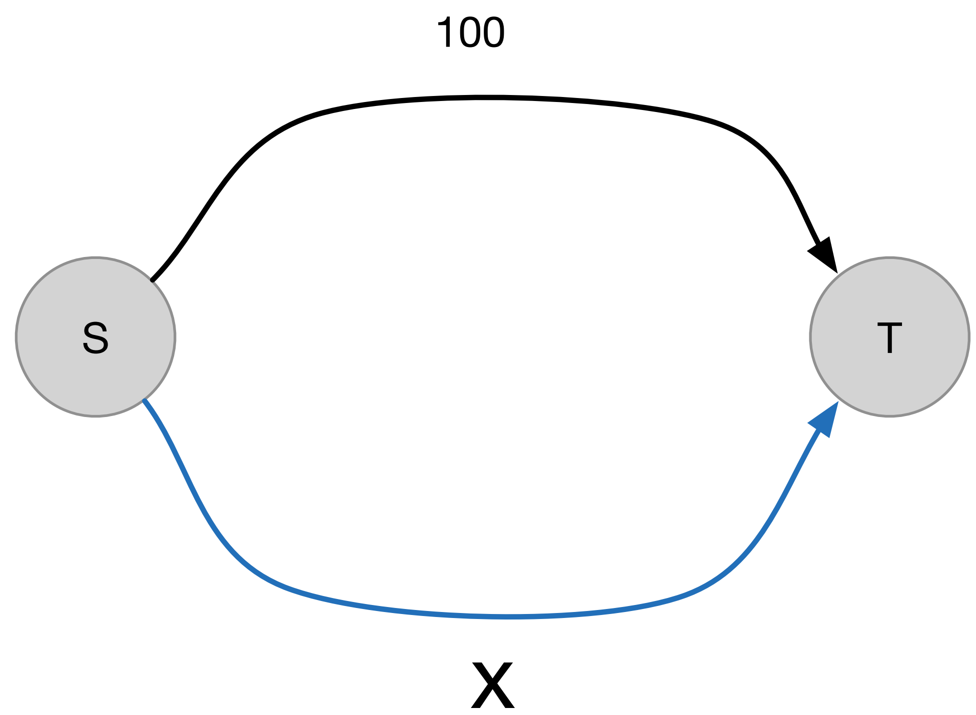

Strategic interaction

**GAME THEORY**

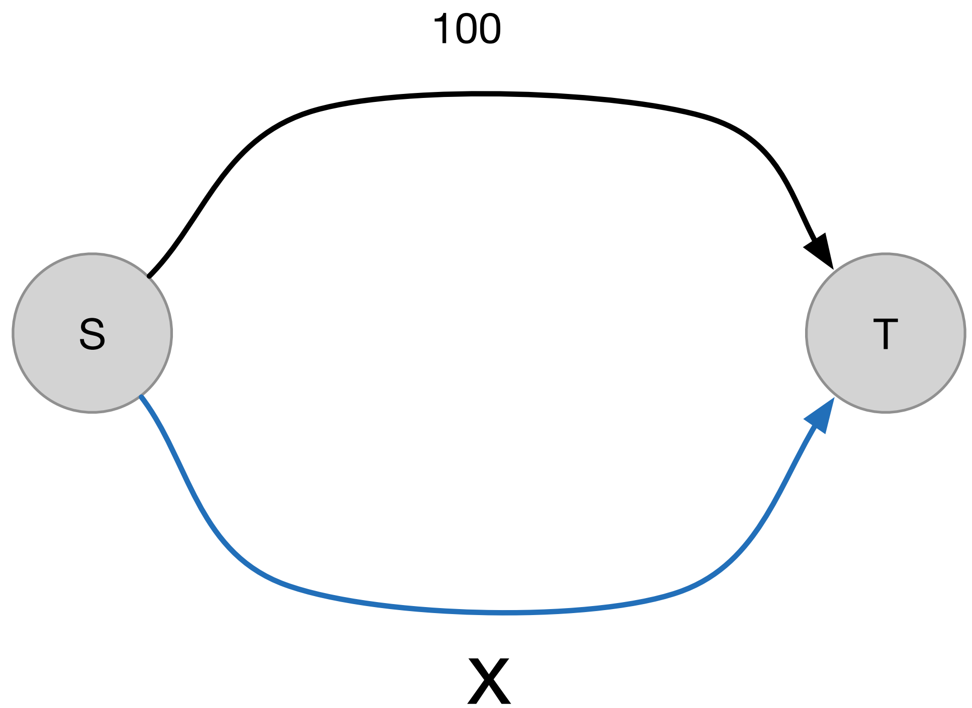

DagenH




- 100 agents commute from S to T.




- 100 agents commute from S to T.
  - ➡ Option 1: Long no congestion.
  - ➡ Option 2: Congestion potentially shorter.
- How <u>do</u> agents use the road?
- How <u>should</u> agents use the road?




Road user

$$\Pi_1 = 100 \qquad \Pi_2 = x$$

No matter what  others are doing, I am <u>always</u> better off choosing option 2

In <u>equilibrium</u>, everyone chooses path B. $x^\star = 100$

Benevolent dictator

Total time spent on the road: $F(x) = x^2 + 100(100 - x)$

$$F'(x) = 2x - 100 = 0 \qquad \bar{x} = 50$$

If we could dictate to agents what to do.  Average commuting time is 75

Road user

$\Pi_1 = 100$ $\Pi_2 = x$

No matter what others are doing, I am always better off choosing option 2

In equilibrium, everyone chooses path B. $x^\star = 100$

Benevolent dictator

Average time spent on the road: $F(x) = x^2 + 100(100 - x)$

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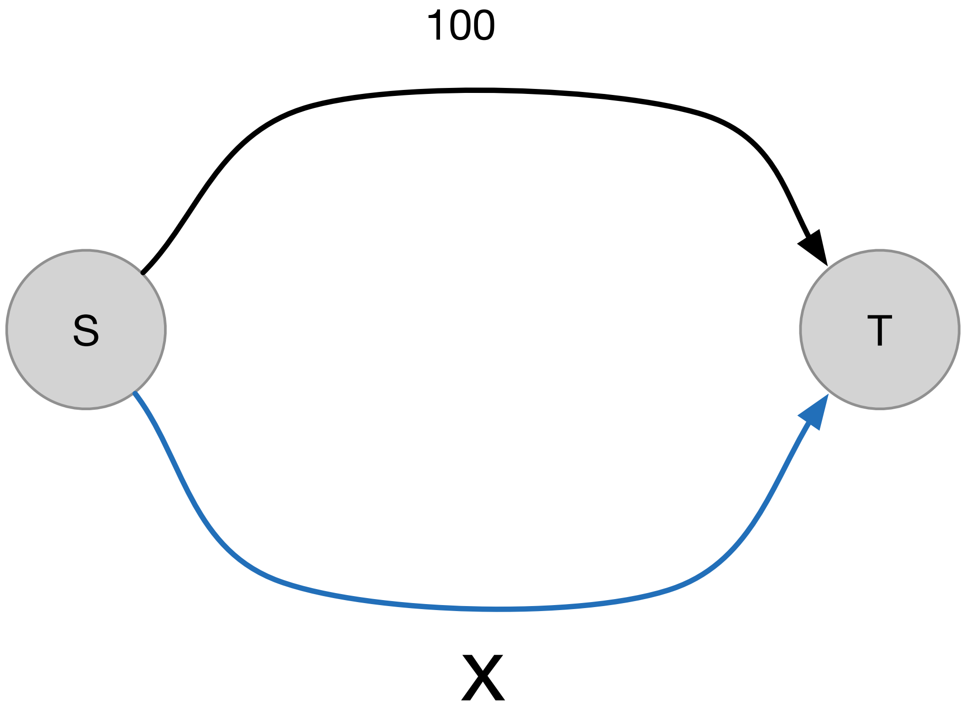
If we could dictate to agents what to do. Average commuting time is 75

$$\frac{100}{75} \longrightarrow \text{Price of anarchy}$$

the Price of Anarchy is the system cost of the worst-case Nash Equilibrium over the optimal system cost, that would be achieved if the players were forced to coordinate.

measures how the efficiency of a system degrades due to selfish behaviour of its agents




In equilibrium, everyone chooses path B. $x^\star = 100$

# Type of analysis.

- Specify assumptions (a.k.a *a model*).
- Equilibrium:
  Situation in which there is no incentive for anyone to try to “switch” unilaterally.
- Static: I don’t care how agents *reach* equilibrium.

# Games and Normal Form

# A game

List of players in the game.

Set of strategies available to each player.

Payoffs associated with any  strategy combination.

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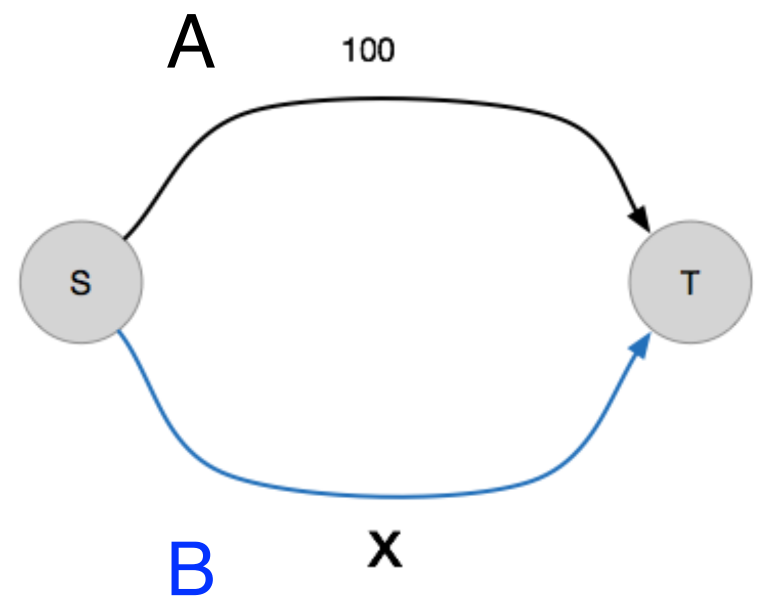



$$P = \{1, 2, \cdots, i, \cdots 100\}$$

$$S_i = \{A, B\}$$

$$\Pi : S_1 \times S_2 \times \cdots \times S_{100} \rightarrow \mathbb{R}^{100}$$

# A game

(in normal form)

| | |
|---|---|
| $I = \{1, ..., n\}$ | set of players |
| $S_i$ for $i = 1, ..., n$ | for each player, a set of strategies |
| $s = (s_1, ..., s_n)$ with $s_i \in S_i$ | (pure) **profile** |
| $\pi_i : S \to \mathbb{R}$ for $i = 1, ..., n$<br>$S = \times_i S_i$ | for each player, a payoff function |

# Acapulco 2019

| | | | | |
|---|---|---|---|---|
|  | **Rafa:** | 6 | 6 | 6 |
|  | **Nick:** | 3 | 7 | 7 |

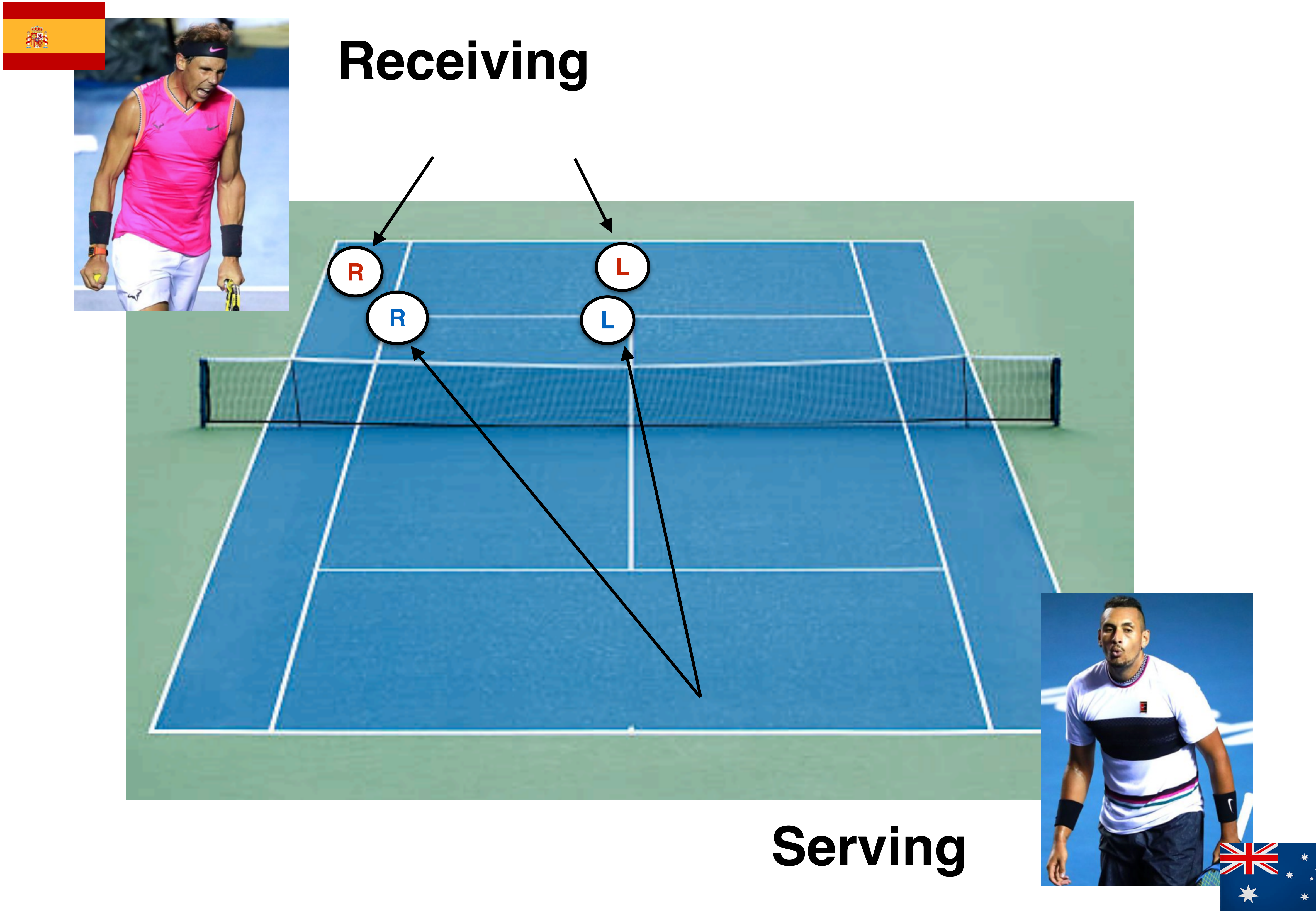
https://www.youtube.com/watch?v=qMN298irRLw

# Serving….

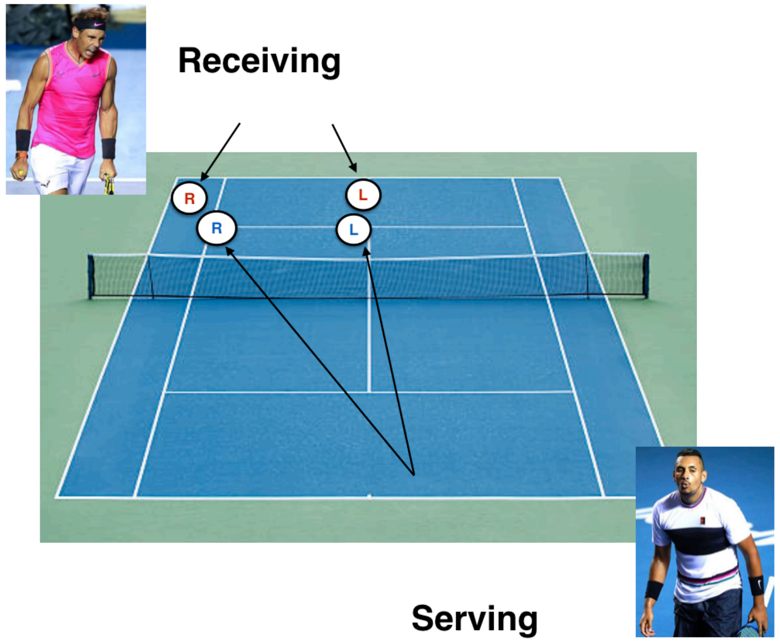
- speed and spin on the serve are important, but virtually every **first serve** is delivered as far toward **one side or the other** of the service court as the server feels is prudent

- receiver often “overplays” one side

- serve is for many players an extremely important factor in **determining the winner of the point**




We will define payoff to be the probability that the point is ultimately won by either player




List of players in the game.

Set of strategies available to each player.

Payoffs associated with any strategy combination.

$$P = \{N, R\}$$

$$S = \{S_N, S_R\} \qquad S_R = \{L, R\} \qquad S_N = \{L, R\}$$

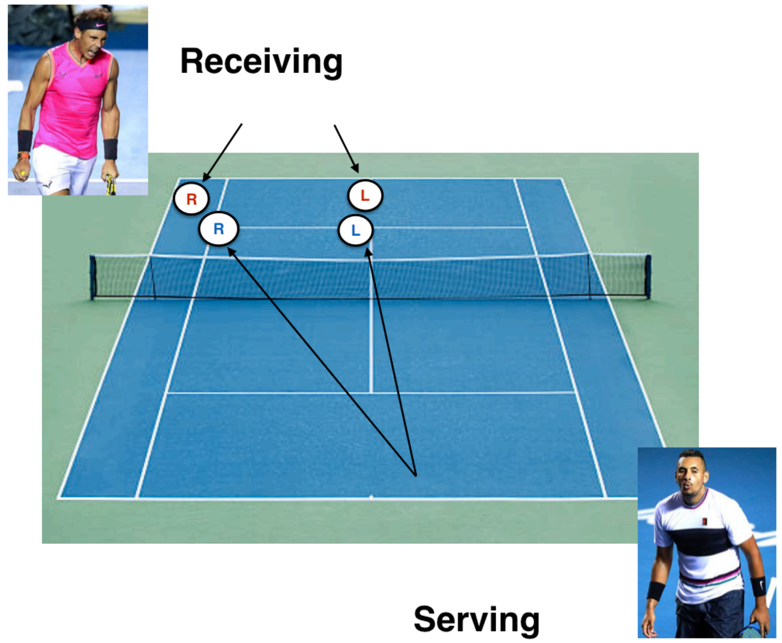
$$\Pi_N(L, L) = 0.58 \qquad \Pi_R(L, L) = 0.42$$

$$\Pi_N(L, R) = 0.79 \qquad \Pi_R(L, R) = 0.21$$

$$\Pi_N(R, L) = 0.73 \qquad \Pi_R(R, L) = 0.27$$

$$\Pi_N(R, R) = 0.49 \qquad \Pi_R(R, R) = 0.51$$

For 2-player games with a finite number of strategies it is customary to encapsulate all this info in <u>matrix form</u>




$$\Pi_N(L, L) = 0.58$$
$$\Pi_N(L, R) = 0.79$$
$$\Pi_N(R, L) = 0.73$$
$$\Pi_N(R, R) = 0.49$$

$$\Pi_R(L, L) = 0.42$$
$$\Pi_R(L, R) = 0.21$$
$$\Pi_R(R, L) = 0.27$$
$$\Pi_R(R, R) = 0.51$$

| | | Receiver | |
|---|---|---|---|
| | | Left | Right |
| Server | Left | 58, 42 | 79, 21 |
| | Right | 73, 27 | 49, 51 |

$$A = \begin{pmatrix} 58 & 79 \\ 73 & 49 \end{pmatrix} \qquad B = \begin{pmatrix} 42 & 21 \\ 27 & 51 \end{pmatrix}$$

# Bi-matrix Game

(special case, n=2)

$$A = \begin{pmatrix} 58 & 79 \\ 73 & 49 \end{pmatrix} \quad B = \begin{pmatrix} 42 & 21 \\ 27 & 51 \end{pmatrix}$$

$\pi_1(s_1, s_2)$ and $\pi_2(s_1, s_2)$ are now given by matrices

More generally, if we have $m_1$ strategies for player 1 and $m_2$ for player 2

$A_{(m_1 \times m_2)}$ and $B_{(m_1 \times m_2)}$

$a_{kh}$ payoff for player 1, when she chooses k and player 2 chooses h

$b_{kh}$ payoff for player 2, when she chooses h and player 1 chooses k

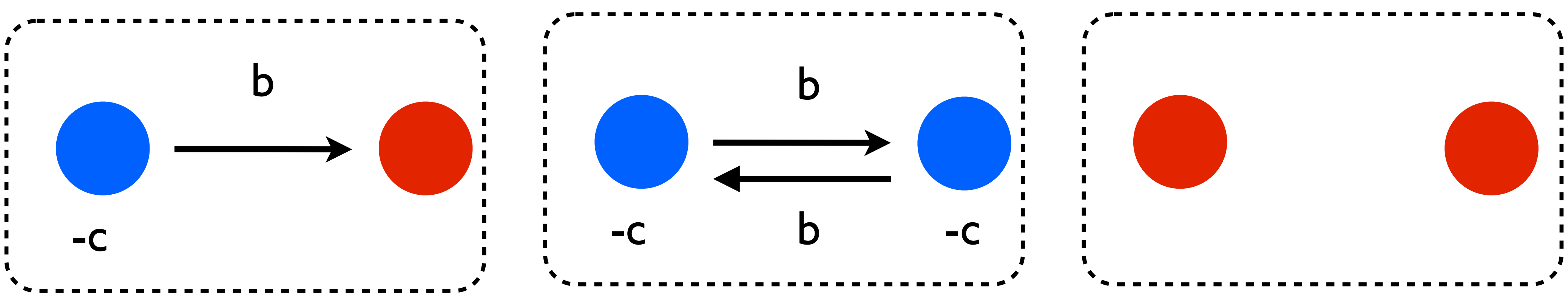
# Helping game

| | | Player 2 | |
|---|---|---|---|
| | | ● | ● |
| Player 1 | ● | b-c, b-c | -c, b |
| | ● | b, -c | 0, 0 |

$$A = \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \qquad B = \begin{pmatrix} b-c & b \\ -c & 0 \end{pmatrix}$$

*b= benefit, c = cost*

b > c > 0



# Symmetric vs Asymmetric

Player 2

Player 1

| | | |
|---|---|---|
| | b-c, b-c | -c, b |
| | b, -c | 0, 0 |

$$A = \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \qquad B = \begin{pmatrix} b-c & b \\ -c & 0 \end{pmatrix}$$

$$A = B^T$$

Symmetric

$$A = \begin{pmatrix} 58 & 79 \\ 73 & 49 \end{pmatrix} \qquad B = \begin{pmatrix} 42 & 21 \\ 27 & 51 \end{pmatrix}$$

$$A \neq B^T$$

Asymmetric

# Mixed Strategies

Receiver

| Server | | Left | Right |
|---|---|---|---|
| | Left | **58, 42** | **79, 21** |
| | Right | **73, 27** | **49, 51** |

[if you face this situation repeatedly]

What’s the best strategy?

**Randomise**.

The American Ec... / Vol. 91, No. 5,... / Minimax Play at...


JOURNAL ARTICLE

**Minimax Play at Wimbledon**

Mark Walker and John Wooders

*The American Economic Review*

Vol. 91, No. 5 (Dec., 2001), pp. 1521-1538

Published by: American Economic Association

https://www.jstor.org/stable/2677937

Page Count: 18

# Mixed strategies

- A mixed strategy is a probability distribution over the set of actions.
- A collection of $n$ mixed strategies forms a mixed strategy profile.
- It is assumed that randomisations are independent. Thus, we can compute expected payoffs for a given strategy profile.

| | | Receiver | |
|---|---|---|---|
| | | Left | Right |
| Server | Left | 58, 42 | 79, 21 |
| | Right | 73, 27 | 49, 51 |

| | |
|---|---|
| **Nick Kyrgios** | **(0.4, 0.6)** |
| **Rafa Nadal** | **(0.7, 0.3)** |

→ Profile

$S = \{(L, L), (L, R), (R, L), (R, R)\}$

| | | | |
|---|---|---|---|
| 0.28 | 0.12 | 0.42 | 0.18 |
| 58 | 79 | 73 | 49 |
| 42 | 21 | 27 | 51 |

**Expected Payoff**

**Kyrgios = 65.2**

**Rafa = 34.8**

$$A = \begin{pmatrix} 58 & 79 \\ 73 & 49 \end{pmatrix} \qquad B = \begin{pmatrix} 42 & 21 \\ 27 & 51 \end{pmatrix}$$

$$x = (x_1, x_2) \longrightarrow \text{Profile}$$

$$x_1 = (0.4, 0.6) \qquad x_2 = (0.7, 0.3)$$

$$(0.4,0.6) \cdot \begin{pmatrix} 58 & 79 \\ 73 & 49 \end{pmatrix} \cdot \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} = 65.2$$

$$(0.4,0.6) \cdot \begin{pmatrix} 42 & 21 \\ 27 & 51 \end{pmatrix} \cdot \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} = 34.8$$

For a bimatrix game:

$$u_1(x) = x_1 \cdot A x_2$$

$$u_2(x) = x_1 \cdot B x_2$$

$S_i$ set of pure actions for player i

$S = \times_i S_i$ set of all pure profiles

if $|S_i| = m_i$ a mixed strategy for player i $x_i \in \mathbb{R}^{m_i}$

mixed strategies live in a **unit simplex**

$x = (x_1, x_2, ..., x_n)$ is a **mixed strategy profile**

$s = (s_1, s_2, ..., s_n)$ is a profile in pure actions.

$x(s) = \Pi_i x_{is_i}$ is the probability that a pure profile s is played, given mixed strategy profile x

$u_i(x) = \sum_{s \in S} x(s) \pi_i(s)$ **payoff** that player i gets, given a mixed strategy profile x

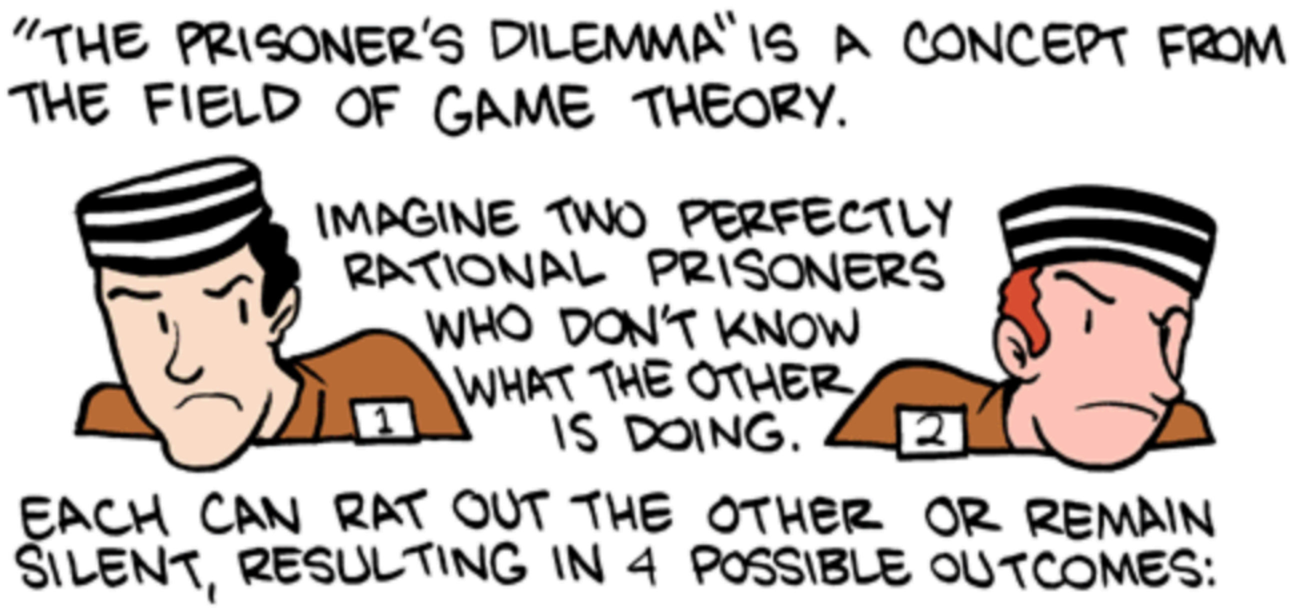
# Solving games

**Solution concept**

A way to justify a prediction in a game

# Dominance

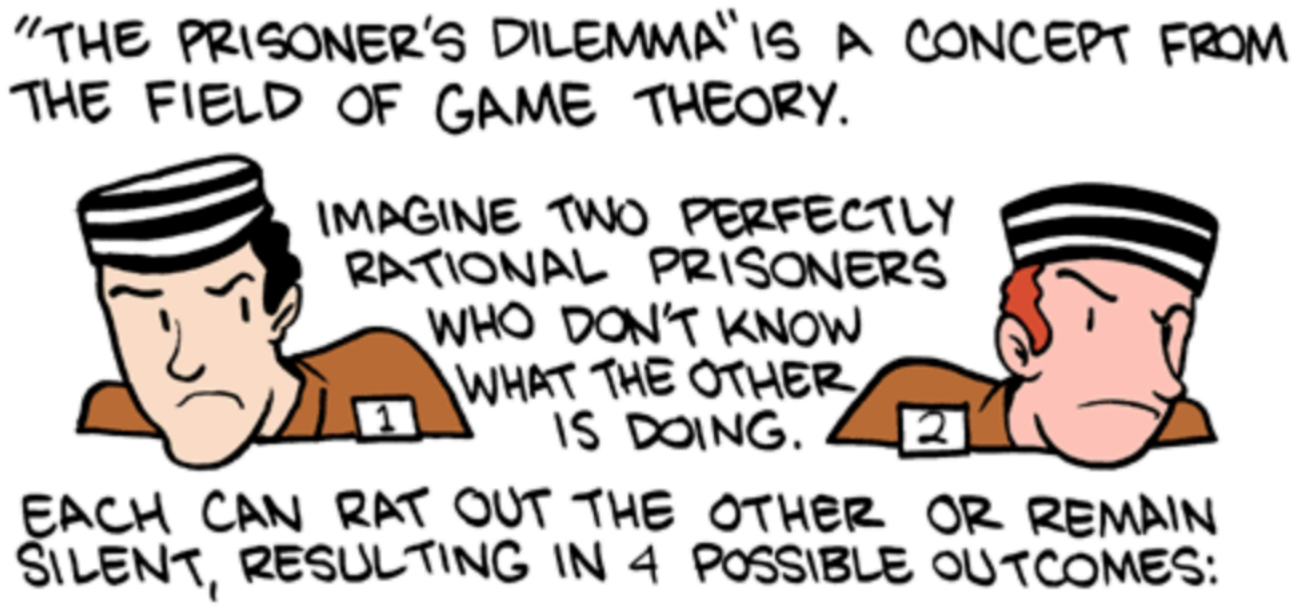
# Prisoner's dilemma




https://www.smbc-comics.com/comic/2010-06-05

Silent or Incriminate

# Prisoner's dilemma




| | Silent | Incriminate |
|---|---|---|
| Silent | -1, -1 | -4, 0 |
| Incriminate | 0, -4 | -3, -3 |

# Dominance

- A strategy **x** strictly dominates a strategy **y** if **x** is always better than **y**.
- A strategy is dominated if some strategy dominates it.
- Rational players: do not play dominated strategies.
- Solution concept: remove dominated strategies.

**Outcome:** (Incriminate, Incriminate)

| | Silent | Incriminate |
|---|---|---|
| Silent | **-1, -1** | **-4, 0** |
| Incriminate | **0, -4** | **-3, -3** |

Are there any dominated strategies?

Incriminate <u>dominates</u> Silent

[always better to incriminate regardless of my opponent's strategy]

… more formally

# Remember….

- A **profile** $z$ is a collection of strategies, one for each player.

- The **solution** to a game is a subset of the set of all profiles $\Theta$

- Remember the payoff for player i is given by a function:

$$u_i : \Theta \rightarrow \mathbb{R}$$

- A profile $(x_i, z_{-i})$ replaces with $x_i$ the strategy of player $i$ in profile $z$

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Approach:** Reduce the game by removing dominated strategies

| | L | C | R |
|---|---|---|---|
| U | 0,2 | 3,1 | 2,3 |
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Row player**

Is U dominated? (need to check against M and D)

*Does M dominate U?*

$$U_{row}(M, \square) > U_{row}(U, \square)$$

$\square = L \rightarrow$ 1 > 0 ✓

$\square = C \rightarrow$ 2 > 3 ✗

| | L | C | R |
|---|---|---|---|
| U | 0,2 | 3,1 | 2,3 |
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Row player**

Is U dominated? (need to check against M and D)

*Does M dominate U?*

$$U_{row}(M, \square) > U_{row}(U, \square)$$

$\square = L \rightarrow$ 1 > 0 ✓

$\square = C \rightarrow$ 2 > 3 ✗

**No**

*Does D dominate U?*

$$U_{row}(D, \square) > U_{row}(U, \square)$$

$\square = L \rightarrow$ 2 > 0 ✓

$\square = C \rightarrow$ 4 > 3 ✓

$\square = R \rightarrow$ 3 > 2 ✓

**Yes**

| | L | C | R |
|---|---|---|---|
| U | ~~0,2~~ | ~~3,1~~ | ~~2,3~~ |
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Row player**

Is U dominated? **Yes**

*Does M dominate U?*

$$U_{row}(M, \square) > U_{row}(U, \square)$$

$\square = L \rightarrow$ $1 > 0$ ✓

$\square = C \rightarrow$ $2 > 3$ ✗

**No**

*Does D dominate U?*

$$U_{row}(D, \square) > U_{row}(U, \square)$$

$\square = L \rightarrow$ $2 > 0$ ✓

$\square = C \rightarrow$ $4 > 3$ ✓

$\square = R \rightarrow$ $3 > 2$ ✓

**Yes**

| | L | C | R |
|---|---|---|---|
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Row player**

## Is M dominated?

### Does D dominate M?

$$U_{row}(D, \square) > U_{row}(M, \square)$$

$\square = L$ 2 > 1 ✓

$\square = C$ 4 > 2 ✓

$\square = R$ 3 > 4 ✗

**No**

## Is D dominated?

### DoesM dominate D?

$$U_{row}(M, \square) > U_{row}(D, \square)$$

$\square = L$ 1 > 2 ✗

$\square = C$ 2 > 4 ✗

$\square = R$ 4 > 3 ✓

**No**

| | L | C | R |
|---|---|---|---|
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Column player**

Is L dominated? (need to check against C and R)

*Does C dominate L?*

$$U_{column}(\square, C) > U_{column}(\square, L)$$

$$\square = M \rightarrow \quad 1 > 4$$

**No**

| | L | C | R |
|---|---|---|---|
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**<u>Column player</u>**

<u>Is L dominated?</u>

*Does C dominate L?*

$$U_{column}(\square, C) > U_{column}(\square, L)$$

$\square = M \rightarrow$ 1 > 4 ✕

**No**

*Does R dominate L?*

$$U_{column}(\square, R) > U_{column}(\square, L)$$

$\square = M \rightarrow$ 1 > 4 ✕

**No**

|   | L | C | R |
|---|---|---|---|
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Column player**

Is L dominated?

What about a combination of C and R?

$$U_{column}(\square, (0,\alpha,1-\alpha)) > U_{column}(\square, L)$$

$$\square = M \rightarrow \quad \alpha + (1-\alpha) = 1 > 4 \; \times$$

**No**

**L is not dominated**

| | L | C | R |
|---|---|---|---|
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Column player**

Is C dominated? (need to check against L and R)

*Does L dominate C?*

$$U_{column}(\square, L) > U_{column}(\square, C)$$

$\square = M \rightarrow$ 4 > 1 ✓

$\square = D \rightarrow$ 1 > 4 ✗

**No**

*Does R dominate C?*

$$U_{column}(\square, R) > U_{column}(\square, C)$$

$\square = M \rightarrow$ 1 > 1 ✗

**No**

|   | L | C | R |
|---|---|---|---|
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Column player**

Is C dominated?

What about a combination of L and R?

$$U_{column}(\square, (\alpha, 0, 1-\alpha)) > U_{column}(\square, C)$$

$$\square = M \rightarrow \quad 4\alpha + (1-\alpha) > 1$$

$$\square = D \rightarrow \quad \alpha + 2(1-\alpha) > 4$$

✕ **No**

**C is not dominated**

| | L | C | R |
|---|---|---|---|
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Column player**

Is R dominated? (need to check against L and C)

*Does L dominate R?*

$$U_{column}(\square, L) > U_{column}(\square, R)$$

$\square = M \rightarrow$ 4 > 1 ✓

$\square = D \rightarrow$ 1 > 2 ✗

**No**

*Does C dominate R?*

$$U_{column}(\square, C) > U_{column}(\square, R)$$

$\square = M \rightarrow$ 1 > 1 ✗

**No**

| | L | C | R |
|---|---|---|---|
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Column player**

Is R dominated?

What about a combination of L and C?

$$U_{column}(\square, (\alpha, 1-\alpha, 0)) > U_{column}(\square, R)$$

$$\square = M \rightarrow \quad 4\alpha + (1-\alpha) > 1$$

$$\square = D \rightarrow \quad \alpha + 4(1-\alpha) > 2$$

✓

$$\frac{2}{3} > \alpha > 0$$

**R is dominated**

| | L | C |
|---|---|---|
| M | ~~1,4~~ | ~~2,1~~ |
| D | 2,1 | 4,4 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Row player**

Is M dominated?

Does D dominate M?

$$U_{row}(D, \square) > U_{row}(M, \square)$$

$\square = L \rightarrow$ 2 > 1 ✓

$\square = C \rightarrow$ 4 > 2 ✓

**Yes**

| | L | C |
| --- | --- | --- |
| D | 2,1 | 4,4 |

A strategy $y_i$ strictly dominates $x_i$ if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

$x_i$ is dominated if there is a $y_i$ that dominates it.

**Column player**

Is L dominated?

*Does C dominate L?*

$$U_{column}(\square, C) > U_{column}(\square, L)$$

$$\square = D \rightarrow \quad 4 > 1$$

**Yes**

(D, C)

# (D, C)

| | L | C | R |
|---|---|---|---|
| U | 0,2 | 3,1 | 2,3 |
| M | 1,4 | 2,1 | 4,1 |
| D | 2,1 | 4,4 | 3,2 |

$s=((0,0,1),\ (0,1,0))$

# On Dominance

- There are (many) games that cannot be solved using this method.

- Straightforwardly combinatorial .

# So, what should Nick do?

| | | Receiver | |
|---|---|---|---|
| | | Left | Right |
| Server | Left | 58, 42 | 79, 21 |
| | Right | 73, 27 | 49, 51 |

# Reading more…

- Maschler, Solan, Zamir. "Game Theory" Cambridge University Press, Cambridge, (2013).

[Chapters 4 and 5]

Available online, Monash Library