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Workshop 13
Runge Kutta

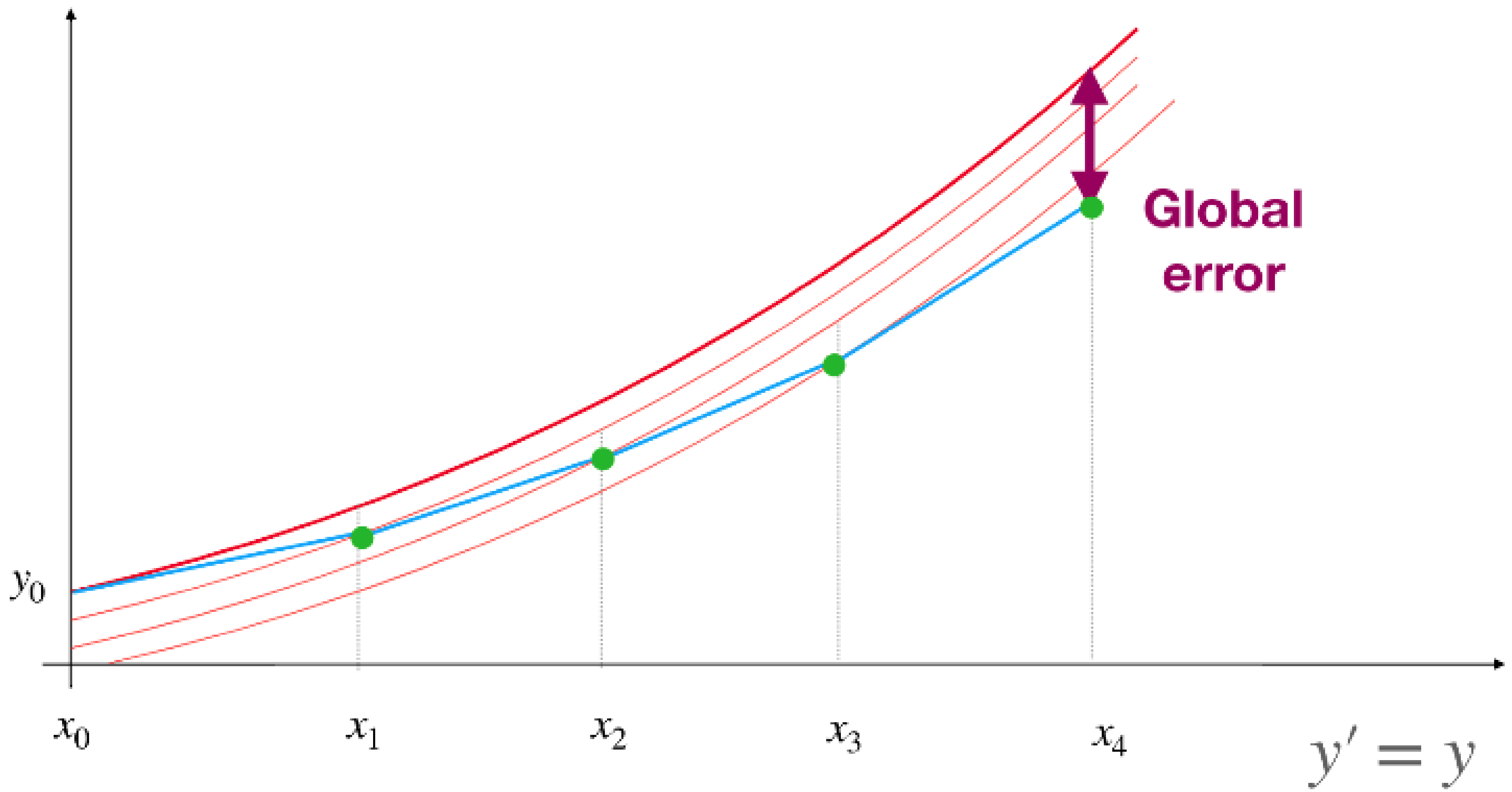
FIT 3139
Computational Science

MONASH University
Information Technology

$$y_{i+1} = y_i + h\phi$$

Error in Euler method

$$y_{i+1} = y_i + hf(x_i, y_i)$$



<u>Predictor</u>: Slope 1

$$f(x_i, y_i)$$

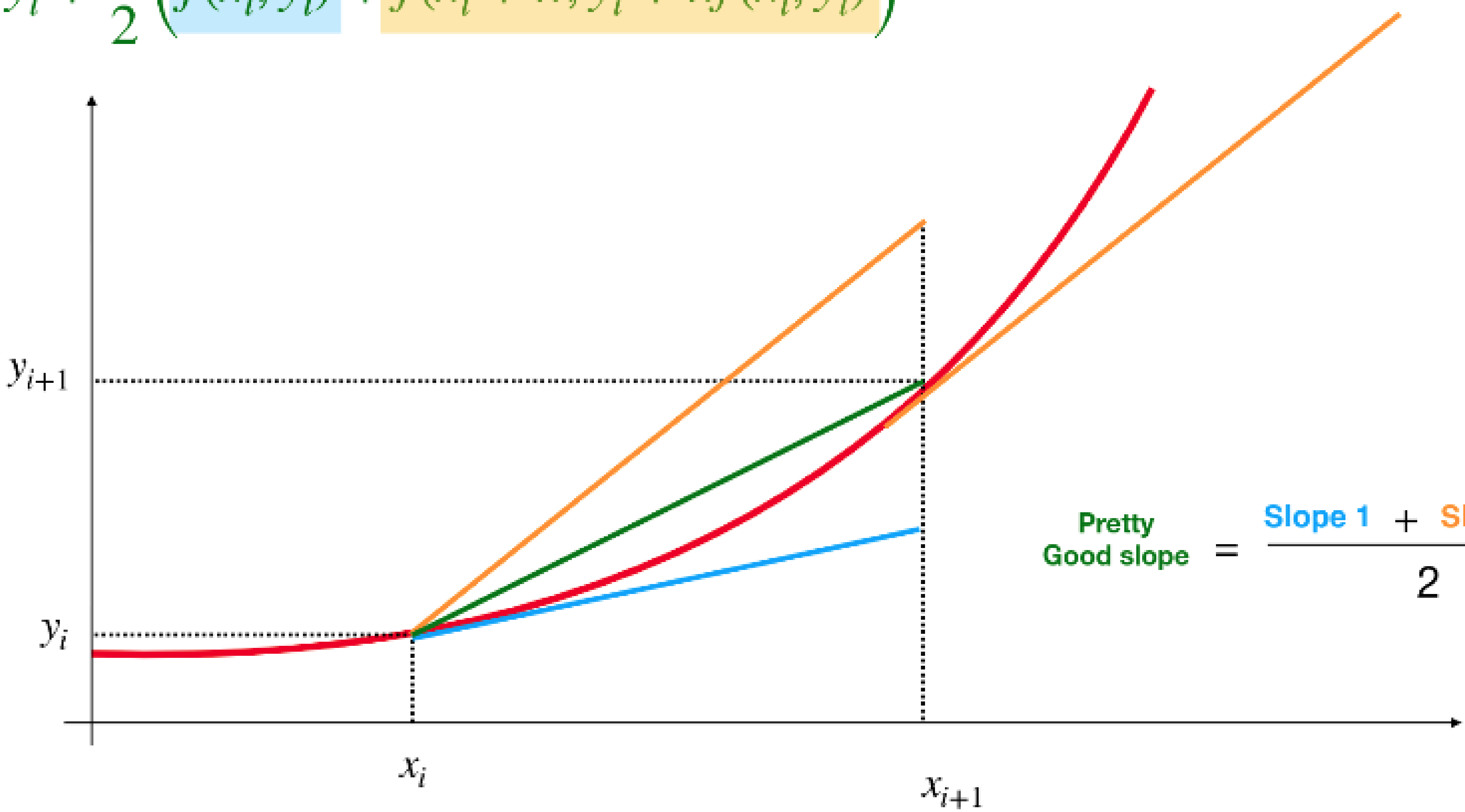
<u>Corrector</u>: Slope 2

$$f(x_{i+1}, y_{i+1})$$

$$f(x_i + h, \underline{\quad\quad})$$

$$f(x_i + h, y_i + hf(x_i, y_i))$$

$$y_{i+1} = y_i + \frac{h}{2}\Big(f(x_i, y_i) + f(x_i + h, y_i + hf(x_i, y_i)\Big)$$

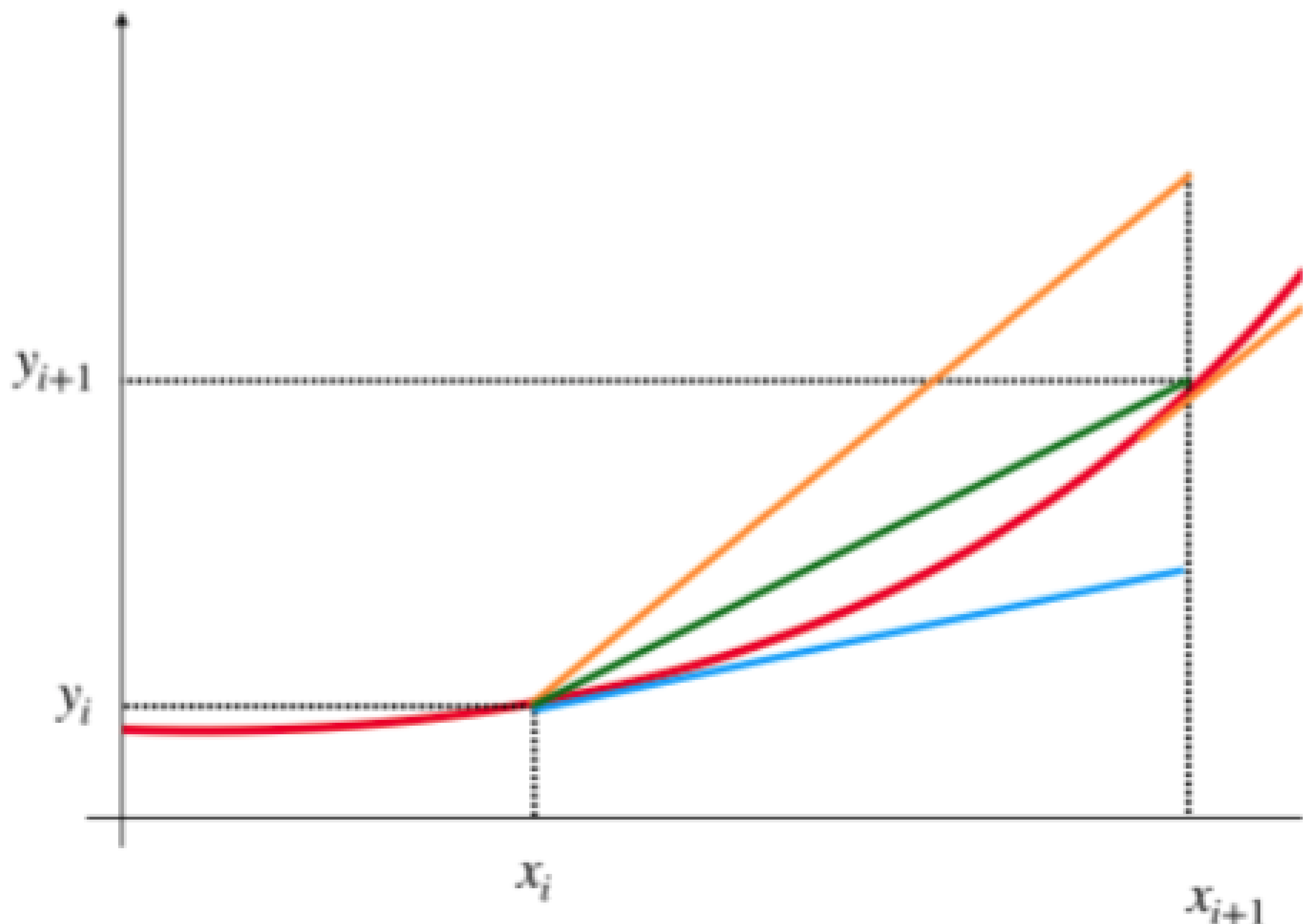


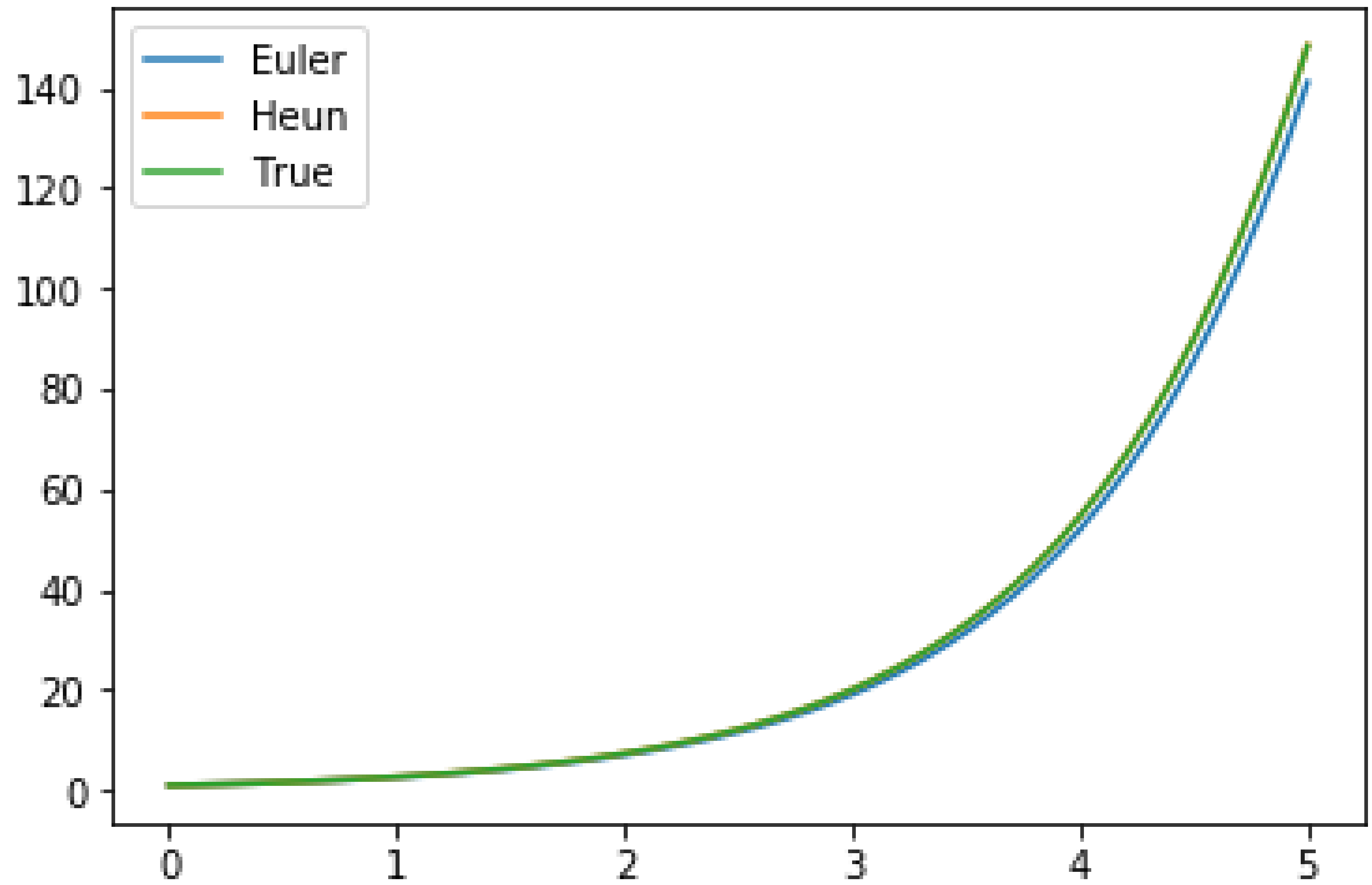
Pretty Good slope $= \dfrac{\text{Slope 1} + \text{Slope 2}}{2}$

Heun's method

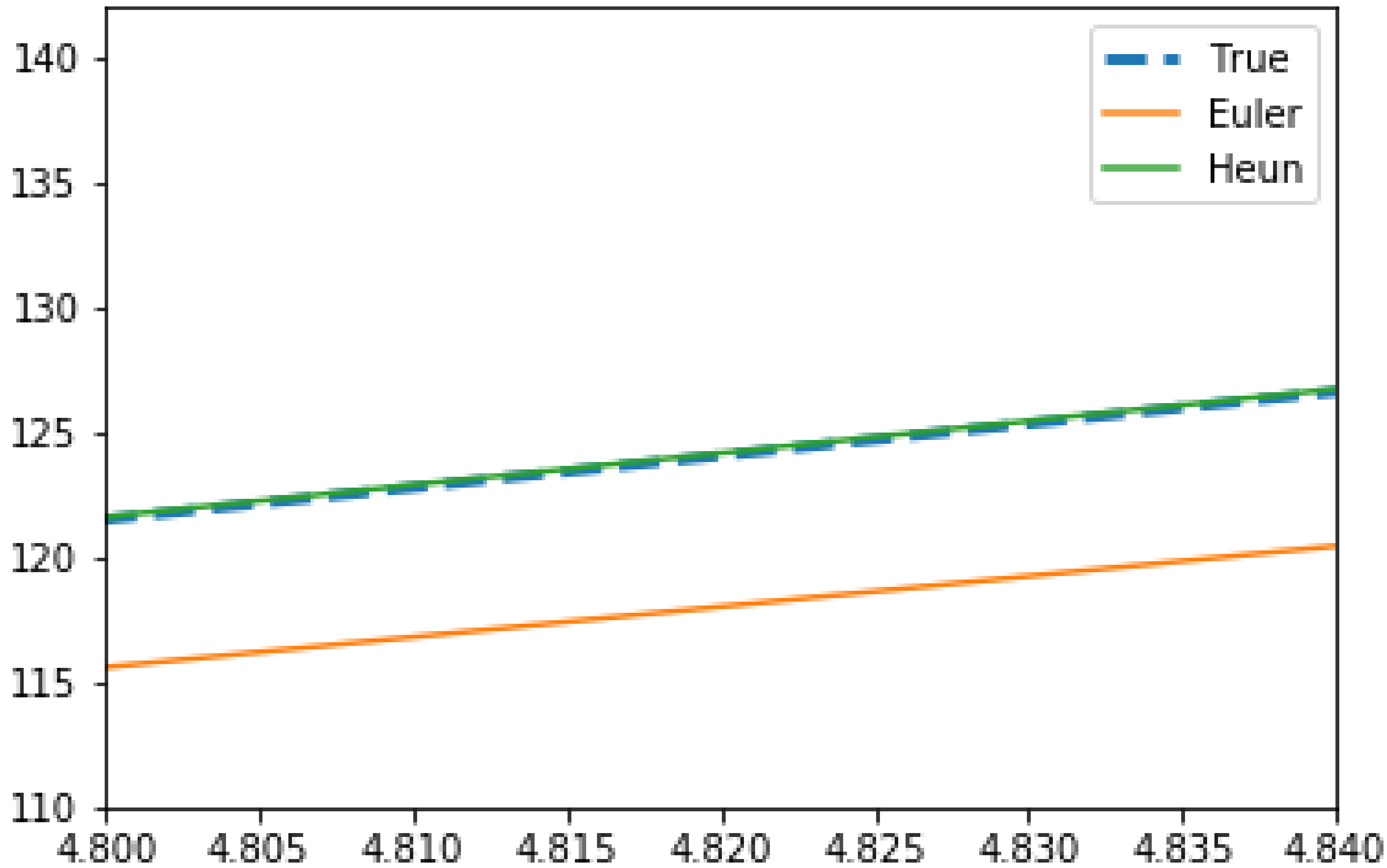
$$y_{i+1} = y_i + \frac{h}{2}\Big(f(x_i, y_i) + f(x_i + h, y_i + hf(x_i, y_i)\Big)$$

$$x_{i+1} = x_i + h$$



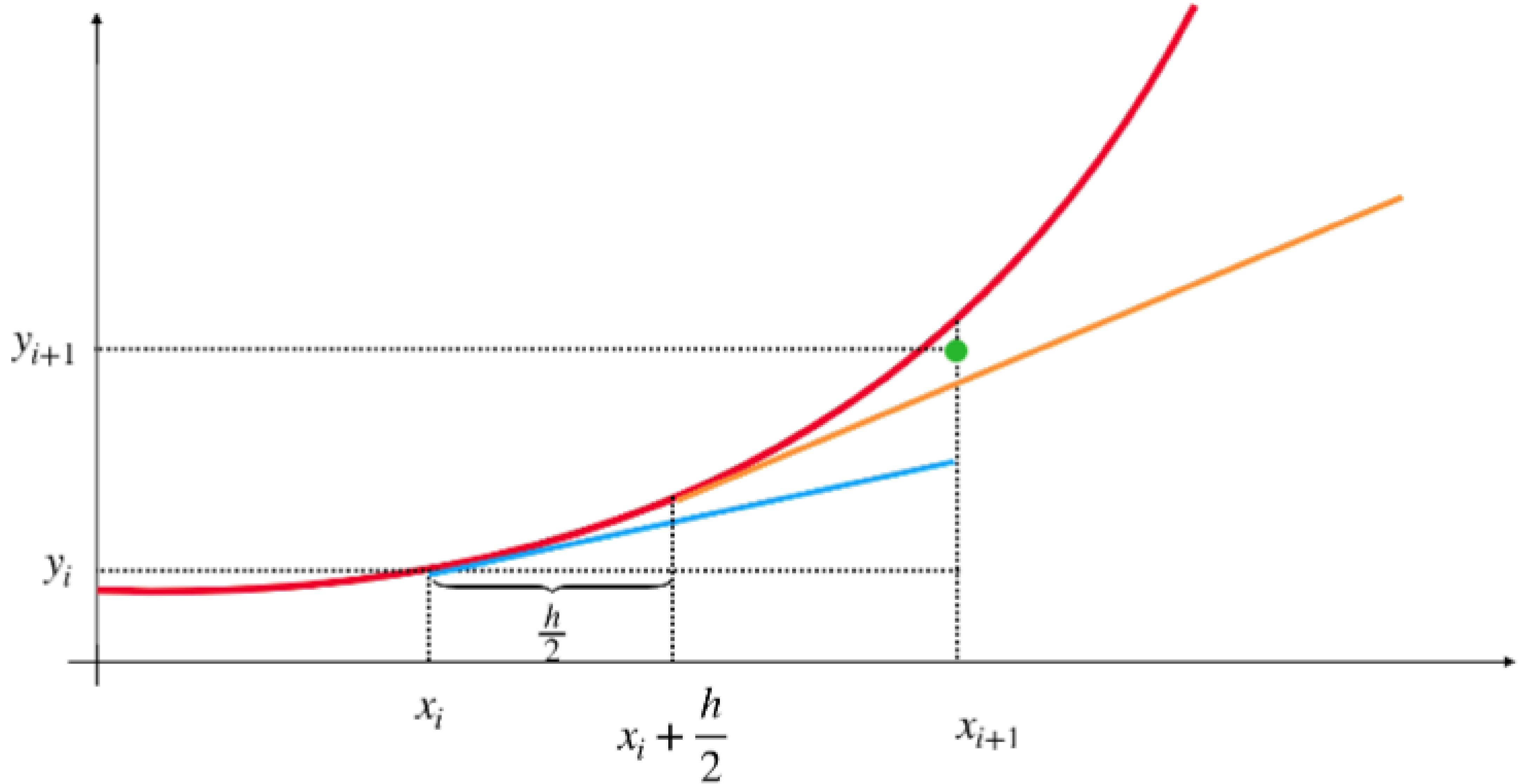


$$y' = y$$


140
135
130
125
120
115
110
4.800
4.805
4.810
4.815
4.820
4.825
4.830
4.835
4.840
True
Euler
Heun

$y' = y$

Midpoint method



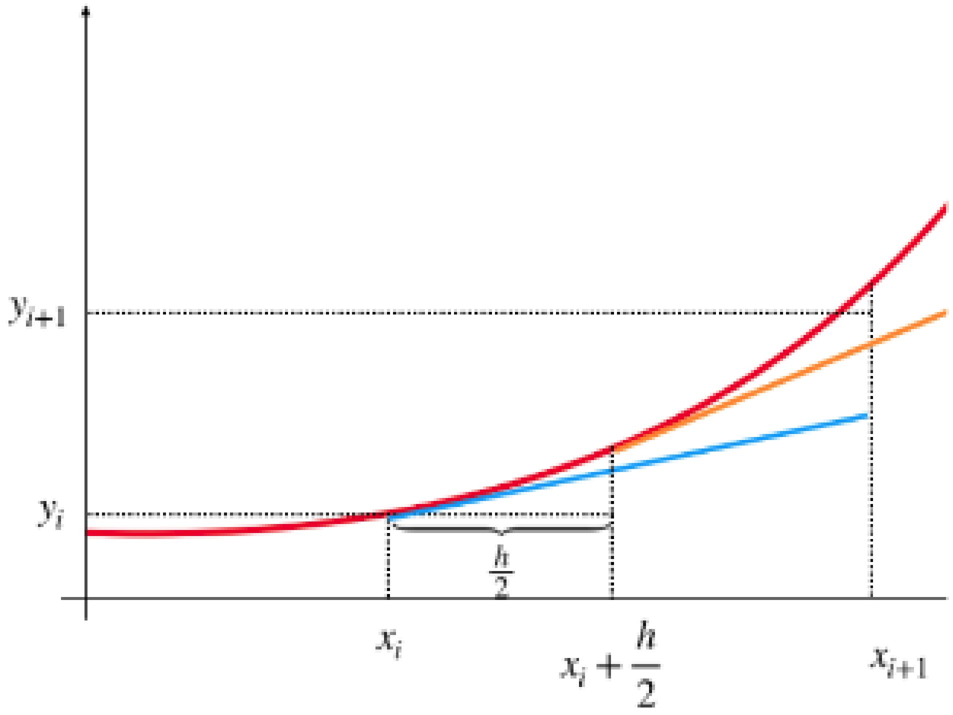
$f(x_i, y_i)$

$y_{i+1} = y_i + hf(x_i, y_i)$

$f\left(x_i + \frac{h}{2}, \qquad\qquad\qquad \right)$

$f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i)\right)$

Midpoint method



$$f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i)\right)$$

$$y_{i+1} = y_i + h\phi$$

$$\boxed{\begin{aligned} y_{i+1} &= y_i + hf\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i)\right) \\ x_{i+1} &= x_i + h \end{aligned}}$$

So far…

Method	
Euler	RK1
Heun's	RK2
Midpoint	RK2

$$y_{i+1} = \underbrace{\underbrace{y_i + hy'(x_i)}_{RK1} + \frac{h^2}{2}y''(x_i) + \ldots}_{RK2}$$

$$y_{i+1} \approx y_i + hy'(x_i) + \frac{h^2}{2}y''(x_i)$$

Runge-Kutta 2

General formula

$$\frac{dy}{dx} = f(x, y)$$

$$y_{i+1} \approx y_i + hy'(x_i) + \frac{h^2}{2} y''(x_i)$$

$$y_{i+1} \approx y_i + h\left(y'(x_i) + \frac{h}{2} y''(x_i) \right)$$

$$y_{i+1} \approx y_i + h\left(f(x_i, y_i) + \frac{h}{2} \boxed{f'(x_i, y_i)} \right) \longrightarrow \textbf{unknown}$$

Can be approximated as follows...

$$y_{i+1} = y_i + h(ak_1 + bk_2)$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \alpha h, y_i + \beta k_1 h)$$

$$a + b = 1$$

$$b\alpha = \frac{1}{2} \qquad b\beta = \frac{1}{2}$$

Heun's method

General RK2

$$y_{i+1} = y_i + h(ak_1 + bk_2)$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + \alpha h, y_i + \beta k_1 h)$$

$$a + b = 1$$
$$b\alpha = \frac{1}{2} \qquad b\beta = \frac{1}{2}$$

$$y_{i+1} = y_i + \frac{h}{2}\Big(f(x_i, y_i) + f(x_i + h, y_i + hf(x_i, y_i)\Big)$$

$$a = \frac{1}{2} \qquad b = \frac{1}{2} \qquad \alpha = 1 \qquad \beta = 1$$

$$y_{i+1} = y_i + h\left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h, y_i + k_1 h)$$

Midpoint method

General RK2

$$y_{i+1} = y_i + h(ak_1 + bk_2)$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + \alpha h, y_i + \beta k_1 h)$$

$$a + b = 1$$
$$b\alpha = \frac{1}{2} \qquad b\beta = \frac{1}{2}$$

$$y_{i+1} = y_i + hf\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i)\right)$$

$$a = 0 \qquad b = 1 \qquad \alpha = \frac{1}{2} \qquad \beta = \frac{1}{2}$$

$$y_{i+1} = y_i + hk_2$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{h}{2}, y_i + k_1 \frac{h}{2}\right)$$

Ralston's method

General RK2

$$y_{i+1} = y_i + h(ak_1 + bk_2)$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + \alpha h, y_i + \beta k_1 h)$$

$$a + b = 1$$
$$b\alpha = \frac{1}{2} \qquad b\beta = \frac{1}{2}$$

$$a = \frac{1}{3} \qquad b = \frac{2}{3} \qquad \alpha = \frac{3}{4} \qquad \beta = \frac{3}{4}$$

$$y_{i+1} = y_i + h\left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{3h}{4}, y_i + k_1\frac{3h}{4}\right)$$

Homework: graphic interpretation

Runge-Kutta 2

$$y_{i+1} = y_i + h(ak_1 + bk_2)$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \alpha h, y_i + \beta k_1 h)$$

$$a + b = 1$$

$$b\alpha = \frac{1}{2} \qquad b\beta = \frac{1}{2}$$

Please refer to lecture recording for a full derivation.

Named methods

Method	
Heun's	RK2
Midpoint	RK2
Ralston's	RK2

You can choose any b…..

Not examinable

Runge Kutta 4

$$y_{i+1} = y_i + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y''''(x_i)\ldots$$

Not examinable

Runge Kutta 4

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 +$$

$$k_1 = f(x_i,$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k$$

$$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k$$

$$k_4 = f(x_i + h, y_i + k$$

Not examinable

Runge Kutta 4

$$y_{i+1} = y_i + \frac{1}{8}(k_1 + 3k_2 + 3k_3 +$$

$$k_1 = f(x_i,$$

$$k_2 = f(x_i + \frac{1}{3}h, y_i + \frac{1}{3}k$$

$$k_3 = f(x_i + \frac{2}{3}h, y_i - \frac{1}{3}k_1 h +$$

$$k_4 = f(x_i + h, y_i + k_1 h - k_2 h +$$

Not examinable

Runge Kutta 4

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

Notes…

- The RK methods can be generalised for systems of several ODE.

- State of the art algorithms common in computational science libraries: RK4, RK5 and variations.