Prepared by Julian García, based on material by
Arun Konagurthu

Workshop 23
Support Enumeration

FIT 3139 Computational Modelling and Simulation

MONASH University
Information Technology

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So, what should Nick do?

		Receiver	
		Left	Right
Server	Left	58, 42	79, 21
	Right	73, 27	49, 51

There is no dominant strategy. What about Nash?

		Receiver	
		Left	Right
Server	Left	58, 42	79, 21
	Right	73, 27	49, 51

$B_{Nick}(\text{Left}) = \text{Right}$ $B_{Rafa}(\text{Left}) = \text{Left}$

$B_{Nick}(\text{Right}) = \text{Left}$ $B_{Rafa}(\text{Right}) = \text{Right}$

No Nash. In "pure strategies".

EQUILIBRIUM POINTS IN N-PERSON GAMES

By John F. Nash, Jr.*

Princeton University

Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an *n*-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the *n* players corresponds to each *n*-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability

Every game with a finite number of strategies has at least a Nash Equilibrium in (possibly) mixed strategies.

		Receiver	
		Left	Right
Server	Left	58, 42	79, 21
	Right	73, 27	49, 51

There must be a mixed strategy Nash equilibrium.

How do we find it?

- In a bi-matrix game, a profile (**x**,**y**) is Nash iff **x** is a best response to **y** and **y** is a best response to **x**.
- The **support** of a vector is the set of components that are non-zero.

Best response condition

If a Nash equilibrium contains a mixed strategy, each pure strategy in the support of it, is a best response to the equilibrium mixture.

Some useful notation

$x = (x_1, x_2, ..., x_n)$ is a mixed strategy profile

$$(x_{-i}, \tau_i) = (\tau_i, x_{-i}) = \begin{cases} (\tau_1, x_2, ..., x_n) & \text{if } i = 1 \\ (x_1, ...x_{i-1}, \tau_i, x_{i+1}, ..., x_n) & \text{if } 1 < i < n \\ (x_1, ...x_{n-1}, \tau_n) & \text{if } i = n \end{cases}$$

replace a strategy in an otherwise identical profile

$$p = (p_i, p_{-i})$$

Best response condition

If a player i's mixed strategy p_i is a best response to the (mixed) strategies of the other players, p_{-i}, then for each pure strategy s_i , such that $p_i(s_i) > 0$ it must be the case that s_i is itself a best response to p_{-i} .

In particular $u_i(s_i, p_{-i})$ must be the same for all s_i such that $p_i(s_i) > 0$

Lemma: A mixed strategy ***x*** is a best response to a mixed strategy **y** if and only if all pure strategies in its support are best responses to **y**

Lemma: A mixed strategy ***x*** is a best response to a mixed strategy **y** if and only if all pure strategies in its support are best responses to **y**

Proof:

$(Ay)_i$ is the expected payoff of player 1 when playing row *i*. Let $u = \max_i[(Ay)_i]$

$$
\begin{aligned}
xAy &= \sum_i x_i (Ay)_i \\
&= \sum_i x_i (u - u + (Ay)_i) \\
&= \sum_i x_i (u - (u - (Ay)_i)) \\
&= \sum_i x_i u - \sum_i x_i (u - (Ay)_i) \\
&= u - \sum_i x_i (u - (Ay)_i)
\end{aligned}
$$

Lemma: A mixed strategy ***x*** is a best response to a mixed strategy **y** if and only if all pure strategies in its support are best responses to **y**

Proof:

$(Ay)_i$ is the expected payoff of player 1 when playing row *i*. Let $u = \max_i [(Ay)_i]$

$$xAy = \sum_i x_i (Ay)_i$$

$$xAy = u - \underbrace{\sum_i x_i \overbrace{(u - (Ay)_i)}^{\geq 0}}_{\geq 0}$$

when does row player maximise her payoff?

xAy achieves the maximum iff the sum is zero.

$$\boxed{x_i > 0 \rightarrow (Ay)_i = u}$$

x is best response to y iff

$$x_i > 0 \implies (Ay)_i = u = \max\{(Ay)_k \mid k \in 1 \ldots n\}$$

		Receiver	
		Left	Right
Server	Left	58, 42	79, 21
	Right	73, 27	49, 51

Find Nash using the best response condition.

We want to find a **mixed** equilibrium.

This means both strategies have full support.

$$s^* = (x, y) \qquad x = (\theta, 1-\theta)$$

$$0 < \theta, \phi < 1 \qquad y = (\phi, 1-\phi)$$

		Receiver: Left (ϕ)	Receiver: Right ($1-\phi$)
Server	Left	58, 42	79, 21
Server	Right	73, 27	49, 51

Best response condition

If x is a best response to y, then the pure strategies in the support of x are also a best response to y

$$[58]\phi + [79](1-\phi) = [73]\phi + [49](1-\phi)$$

We want to find a **mixed** equilibrium.

This means both strategies have full support.

$$s^* = (x, y)$$

$0 < \theta, \phi < 1$

$$x = (\theta, 1 - \theta)$$

$$y = (\phi, 1 - \phi)$$

			Receiver Left	Receiver Right
Server	θ	Left	58, 42	79, 21
	$1 - \theta$	Right	73, 27	49, 51

Best response condition

If x is a best response to y, then the pure strategies in the support of x are also a best response to y

$$[58]\phi + [79](1 - \phi) = [73]\phi + [49](1 - \phi)$$

Best response condition

If y is a best response to x, then the pure strategies in the support of y are also a best response to x

$$[42]\theta + [27](1 - \theta) = [21]\theta + [51](1 - \theta)$$

We want to find a **mixed** equilibrium.

This means both strategies have full support.

$$A = \begin{pmatrix} 58 & 79 \\ 73 & 49 \end{pmatrix}$$

$$s^* = (x, y) \qquad x = (\theta, 1-\theta)$$

$$0 < \theta, \phi < 1 \qquad y = (\phi, 1-\phi)$$

$$B = \begin{pmatrix} 42 & 21 \\ 27 & 51 \end{pmatrix}$$

$$[58]\phi + [79](1-\phi) = [73]\phi + [49](1-\phi) \longrightarrow \phi = \frac{2}{3}$$

$$[42]\theta + [27](1-\theta) = [21]\theta + [51](1-\theta) \longrightarrow \theta = \frac{8}{15}$$

Verify it is indeed a best response…

Payoff for row, in eq.

$$xAy = 65 \qquad Ay = \begin{pmatrix} 65 \\ 65 \end{pmatrix}$$

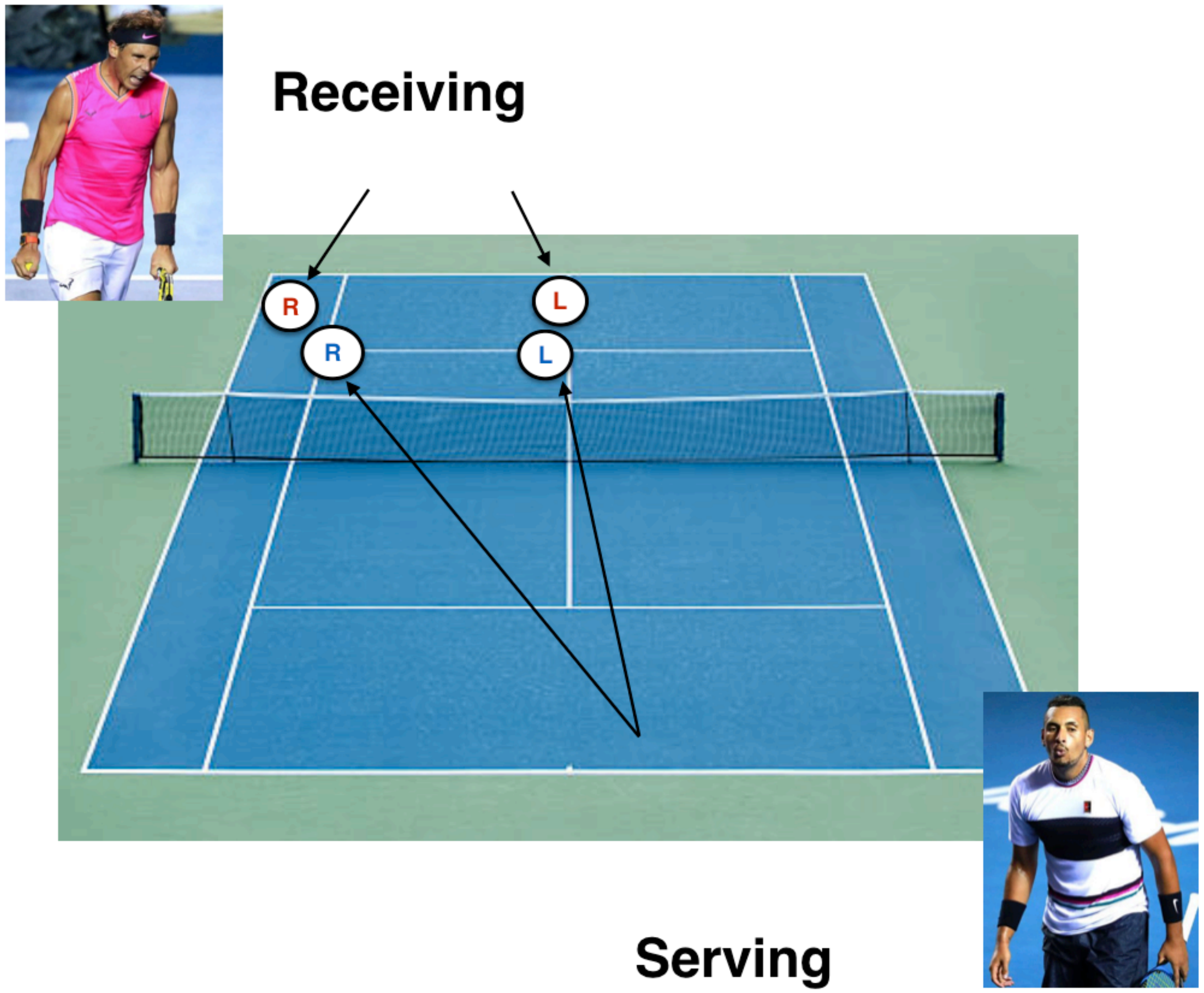
Payoff for column, in eq.

$$xBy = 35 \qquad xB = \begin{pmatrix} 35 \\ 35 \end{pmatrix}$$

Can row/column improve playing something outside the support?

No

$$s^* = \left[\left(\frac{8}{15}, \frac{7}{15}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right] \longrightarrow \text{Nash equilibrium}$$



$$s^* = \left[\left(\frac{8}{15}, \frac{7}{15}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right]$$

The American Ec... / Vol. 91, No. 5,... / Minimax Play at...

JOURNAL ARTICLE

Minimax Play at Wimbledon

Mark Walker and John Wooders

The American Economic Review

Vol. 91, No. 5 (Dec., 2001), pp. 1521-1538

Published by: American Economic Association

https://www.jstor.org/stable/2677937

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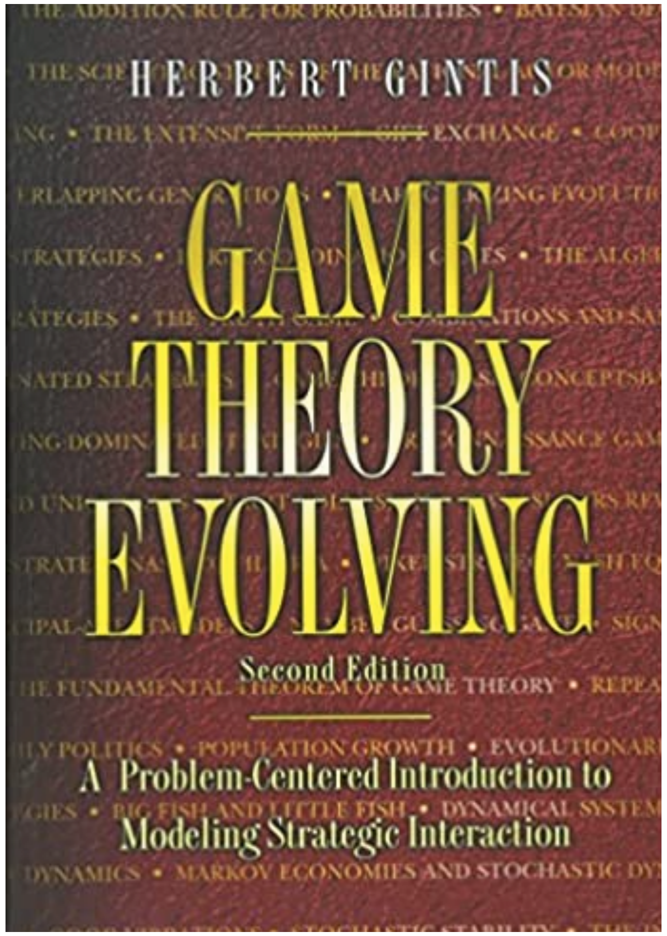

- **Gintis, H**. "Game Theory Evolving: A Problem-Centered Introduction to Modelling Strategic Interaction" Princeton University Press, Cambridge, (2009).

[Chapters 5 and 6]

Available online, Monash Library

What about an Algorithm?

Prepared by Julian García, based on material by
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Nash Equilibrium

A profile $x^* = (x_1^*, \ldots, x_{x_n}^*)$ is a Nash Equilibrium

if and only if x_i^* is a **best response** to $x_{-i}^* = (x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_{x_n}^*)$

for each i

Players are "best-responding" to each other

How to check if a mixture is a best response ?

x is best response to y iff

$$x_i > 0 \Longrightarrow (Ay)_i = u = \max\{(Ay)_k \,|\, k \in 1 \ldots n\}$$

We want to find a **mixed** equilibrium.

This means both strategies have full support.

$$A = \begin{pmatrix} 58 & 79 \\ 73 & 49 \end{pmatrix}$$

$$s^* = (x, y) \qquad x = (\theta, 1-\theta)$$

$$0 < \theta, \phi < 1 \qquad y = (\phi, 1-\phi)$$

$$B = \begin{pmatrix} 42 & 21 \\ 27 & 51 \end{pmatrix}$$

$$[58]\phi + [79](1-\phi) = [73]\phi + [49](1-\phi) \longrightarrow \phi = \frac{2}{3}$$

$$[42]\theta + [27](1-\theta) = [21]\theta + [51](1-\theta) \longrightarrow \theta = \frac{8}{15}$$

Verify it is indeed a best response…

Payoff for row, in eq.

$$xAy = 65 \qquad Ay = \begin{pmatrix} 65 \\ 65 \end{pmatrix}$$

Payoff for column, in eq.

$$xBy = 35 \qquad xB = \begin{pmatrix} 35 \\ 35 \end{pmatrix}$$

Can row/column improve playing something outside the support?

No

$$s^* = \left[\left(\frac{8}{15}, \frac{7}{15}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right] \longrightarrow \text{Nash equilibrium}$$

We want to find a **mixed** equilibrium.

$$A = \begin{pmatrix} 58 & 79 \\ 73 & 49 \end{pmatrix}$$

This means both strategies have full support.

$$s^* = (x, y) \qquad x = (\theta, 1-\theta)$$

$$B = \begin{pmatrix} 42 & 21 \\ 27 & 51 \end{pmatrix}$$

$$0 < \theta, \phi < 1 \qquad y = (\phi, 1-\phi)$$

We could do this because we "guessed" the support.

$$[58]\phi + [79](1-\phi) = [73]\phi + [49](1-\phi) \longrightarrow \phi = \frac{2}{3}$$

$$[42]\theta + [27](1-\theta) = [21]\theta + [51](1-\theta) \longrightarrow \theta = \frac{8}{15}$$

(1, 1), (1, 1)

Verify it is indeed a best response…

Payoff for row, in eq.

$$xAy = 65 \qquad Ay = \begin{pmatrix} 65 \\ 65 \end{pmatrix}$$

Payoff for column, in eq.

$$xBy = 35 \qquad xB = \begin{pmatrix} 35 \\ 35 \end{pmatrix}$$

Can row/column improve playing something outside the support?

No

$$s^* = \left[\left(\frac{8}{15}, \frac{7}{15}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right] \longrightarrow \text{Nash equilibrium}$$

Finding Mixed Nash

- If the support of the mixed Nash Equilibria are known, the problem reduces to a system of linear equations.
- The set of supports for a finite game is a combinatorial space.
- Finding mixed Nash Equilibria is a combinatorial problem.

“Algorithmic Game Theory”, Nissan et al (2007).

Column player

	U	M	D
L	1, 1/2	1, -1	-1, -1/2
R	2, - 1	-1, 3	0, 2

Row player

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ -1 & 3 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ -1 & 3 & 2 \end{pmatrix}$$

A = {1, 2} B = {1, 2, 3}

Possible supports of size 2

Case 1: (1, 2) (1, 2) $x = (\alpha, \beta)$ $y = (\delta, \gamma, 0)$

Case 2: (1, 2) (1, 3) $x = (\alpha, \beta)$ $y = (\delta, 0, \gamma)$

Case 3: (1, 2) (2, 3) $x = (\alpha, \beta)$ $y = (0, \delta, \gamma)$

Apply best response condition in each case to find mixture, and check if Nash

Case 1: (1, 2) (1, 2) $x = (\alpha, \beta)$ $y = (\delta, \gamma, 0)$

row player

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

column player

$$B = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ -1 & 3 & 2 \end{pmatrix}$$

Case 1: (1, 2) (1, 2) $x = (\alpha, \beta)$ $y = (\delta, \gamma, 0)$

row player

$$\hat{A} = \begin{array}{cc} \delta & \gamma \\ \end{array}$$
$$\hat{A} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$\delta + \gamma = u$$
$$2\delta - \gamma = u$$
$$\delta + \gamma = 1$$

$$Mx = b \quad M = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

row's payoff

$$\delta = \frac{2}{3} \quad \gamma = \frac{1}{3} \quad u = 1$$

column player

$$\hat{B} = \begin{array}{c} \alpha \\ \beta \end{array} \begin{pmatrix} \frac{1}{2} & -1 \\ -1 & 3 \end{pmatrix}$$

$$0.5\alpha - \beta = v$$
$$-\alpha + 3\beta = v$$
$$\alpha + \beta = 1$$

$$Mx = b \quad M = \begin{bmatrix} 0.5 & -1 & -1 \\ -1 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

column's payoff

$$\alpha = \frac{8}{11} \quad \beta = \frac{3}{11} \quad v = \frac{1}{11}$$

Candidate Nash

$$[x, y] = \left[\left(\frac{8}{11}, \frac{3}{11}\right), \left(\frac{2}{3}, \frac{1}{3}, 0\right)\right]$$

Case 1: (1, 2) (1, 2) $x = (\alpha, \beta)$ $y = (\delta, \gamma, 0)$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ -1 & 3 & 2 \end{pmatrix}$$

Candidate Nash

$$[x, y] = \left[\left(\frac{8}{11}, \frac{3}{11}\right), \left(\frac{2}{3}, \frac{1}{3}, 0\right)\right]$$

row's payoff

$$u = 1$$

column's payoff

$$v = \boxed{\frac{1}{11}}$$

Verify

$$Ay = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$xB = \left(\frac{1}{11}, \frac{1}{11}, \boxed{\frac{2}{11}}\right)$$

Column player is better off, playing something else.

Not Nash

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ -1 & 3 & 2 \end{pmatrix}$$

A = {1, 2}

B = {1, 2, 3}

Possible supports of size 2

✕	Case 1:	(1, 2)	(1, 2)	$x = (\alpha, \beta)$	$y = (\delta, \gamma, 0)$
	Case 2:	(1, 2)	(1, 3)	$x = (\alpha, \beta)$	$y = (\delta, 0, \gamma)$
	Case 3:	(1, 2)	(2, 3)	$x = (\alpha, \beta)$	$y = (0, \delta, \gamma)$

Apply best response condition in each case to find mixture, and check if Nash

Case 2: (1, 2) (1, 3) $x = (\alpha, \beta)$ $y = (\delta, 0, \gamma)$

row player

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

column player

$$B = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ -1 & 3 & 2 \end{pmatrix}$$

Case 2: (1, 2) (1, 3) $x = (\alpha, \beta)$ $y = (\delta, 0, \gamma)$

row player

$$\hat{A} = \begin{array}{cc} \delta & \gamma \\ \end{array}$$

$$\hat{A} = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$$

$$\delta - \gamma = u$$

$$2\delta = u$$

$$\delta + \gamma = 1$$

$$Mx = b \qquad M = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

No solution!

column player

$$\hat{B} = \begin{array}{c} \alpha \\ \beta \end{array} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix}$$

$$0.5\alpha - \beta = v$$

$$-0.5\alpha + 2\beta = v$$

$$\alpha + \beta = 1$$

$$Mx = b \qquad M = \begin{bmatrix} 0.5 & -1 & -1 \\ -0.5 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

column's payoff

$$\alpha = \frac{3}{4} \qquad \beta = \frac{1}{4} \qquad v = \frac{1}{8}$$

No Nash

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ -1 & 3 & 2 \end{pmatrix}$$

A = {1, 2}

B = {1, 2, 3}

Possible supports of size 2

✕ Case 1: (1, 2) (1, 2) $x = (\alpha, \beta)$ $y = (\delta, \gamma, 0)$

✕ Case 2: (1, 2) (1, 3) $x = (\alpha, \beta)$ $y = (\delta, 0, \gamma)$

Case 3: (1, 2) (2, 3) $x = (\alpha, \beta)$ $y = (0, \delta, \gamma)$

Apply best response condition in each case to find mixture, and check if Nash

Case 3: (1, 2) (2, 3) $x = (\alpha, \beta)$ $y = (0, \delta, \gamma)$

row player

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

column player

$$B = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ -1 & 3 & 2 \end{pmatrix}$$

Case 3: (1, 2) (2, 3) $x = (\alpha, \beta)$ $y = (0, \delta, \gamma)$

row player

$$\hat{A} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \delta & \gamma \\ \left(\begin{array}{c} 1 \\ -1 \end{array}\right. & \left.\begin{array}{c} -1 \\ 0 \end{array}\right) \end{array}$$

$$\delta - \gamma = u$$
$$-\delta = u$$
$$\delta + \gamma = 1$$

$$Mx = b \qquad M = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

row's payoff

$$\delta = \frac{1}{3} \qquad \gamma = \frac{2}{3} \qquad u = -\frac{1}{3}$$

column player

$$\hat{B} = \begin{array}{c} \alpha \\ \beta \end{array}\begin{pmatrix} -1 & -\frac{1}{2} \\ 3 & 2 \end{pmatrix}$$

$$-\alpha + 3\beta = v$$
$$-0.5\alpha + 2\beta = v$$
$$\alpha + \beta = 1$$

$$Mx = b \qquad M = \begin{bmatrix} -1 & 3 & -1 \\ -0.5 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

column's payoff

$$\alpha = \frac{2}{3} \qquad \beta = \frac{1}{3} \qquad v = \frac{1}{3}$$

Candidate Nash

$$[x, y] = \left[\left(\frac{2}{3}, \frac{1}{3}\right), \left(0, \frac{1}{3}, \frac{2}{3}\right)\right]$$

Case 3: (1, 2) (2, 3) $x = (\alpha, \beta)$ $y = (0, \delta, \gamma)$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ -1 & 3 & 2 \end{pmatrix}$$

Candidate Nash

$$[x, y] = \left[\left(\frac{2}{3}, \frac{1}{3}\right), \left(0, \frac{1}{3}, \frac{2}{3}\right)\right]$$

$$\underset{\text{row's payoff}}{u} = -\frac{1}{3} \qquad \underset{\text{column's payoff}}{v} = \frac{1}{3}$$

Verify

$$Ay = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} \qquad xB = \left(0, \frac{1}{3}, \frac{1}{3}\right)$$

Mixtures are best responses to each other…

It is Nash!

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ -1 & 3 & 2 \end{pmatrix}$$

A = {1, 2} B = {1, 2, 3}

Possible supports of size 2

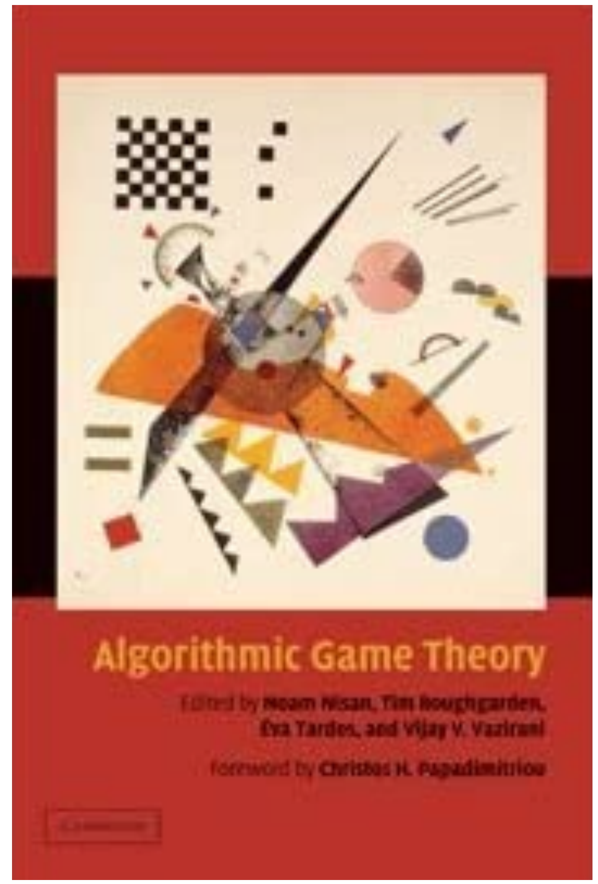
✗ Case 1: (1, 2) (1, 2) $x = (\alpha, \beta)$ $y = (\delta, \gamma, 0)$

✗ Case 2: (1, 2) (1, 3) $x = (\alpha, \beta)$ $y = (\delta, 0, \gamma)$

✓ Case 3: (1, 2) (2, 3) $x = (\alpha, \beta)$ $y = (0, \delta, \gamma)$

$$[x, y] = \left[\left(\frac{2}{3}, \frac{1}{3}\right), \left(0, \frac{1}{3}, \frac{2}{3}\right)\right]$$

Reading more…



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[Chapter 3]

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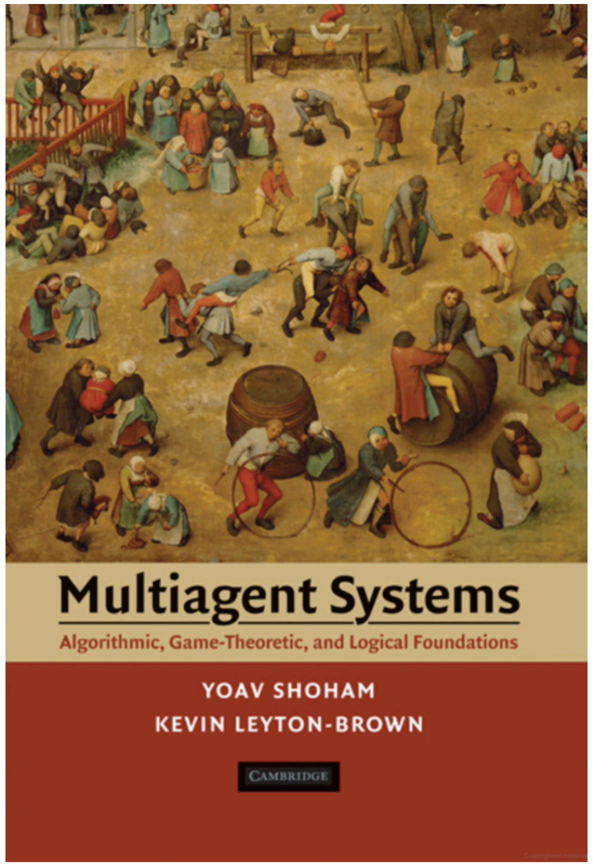

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[Chapter 4]

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