Prepared by Julian García, based on material by
David Albrecht and María García de la Banda

Workshop 10
Dynamical Systems
with Differential Equations

- **FIT 3139** Computational Modelling and Simulation

Outline

- Introduction to dynamical systems.
- **Continuous** logistic model
- Coupled **continuous** models
- Phase-plane analysis.

Calculus is integral and dynamical systems

- **Calculus** is the mathematics of change.
- **Derivative** gives the concept of instantaneous rate of change.
- **Differential** calculus deals with problems involving the derivative.
- **Systems dynamics** involves rates of change of various variables, hence derivatives are fundamental to it.

A **differential equation** is simply an equation containing derivatives.

A **solution** to a differential equation implies finding a *function* that *satisfies* the differential equation.

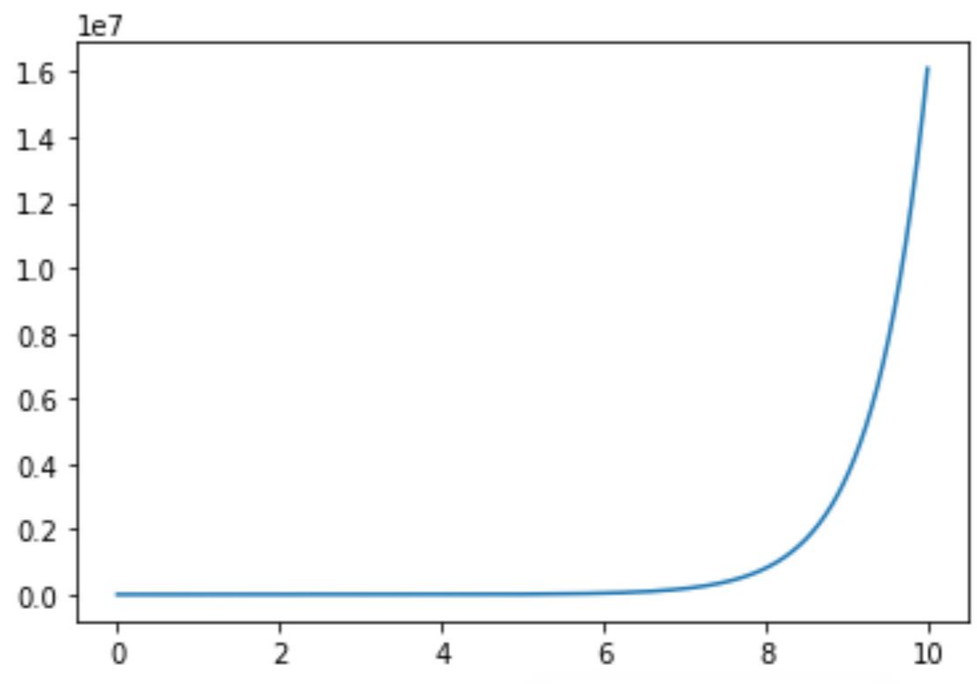
Unlike looking for a value, in an algebraic equation; in a differential equation you are looking **for a function**.

$$r = b - d$$

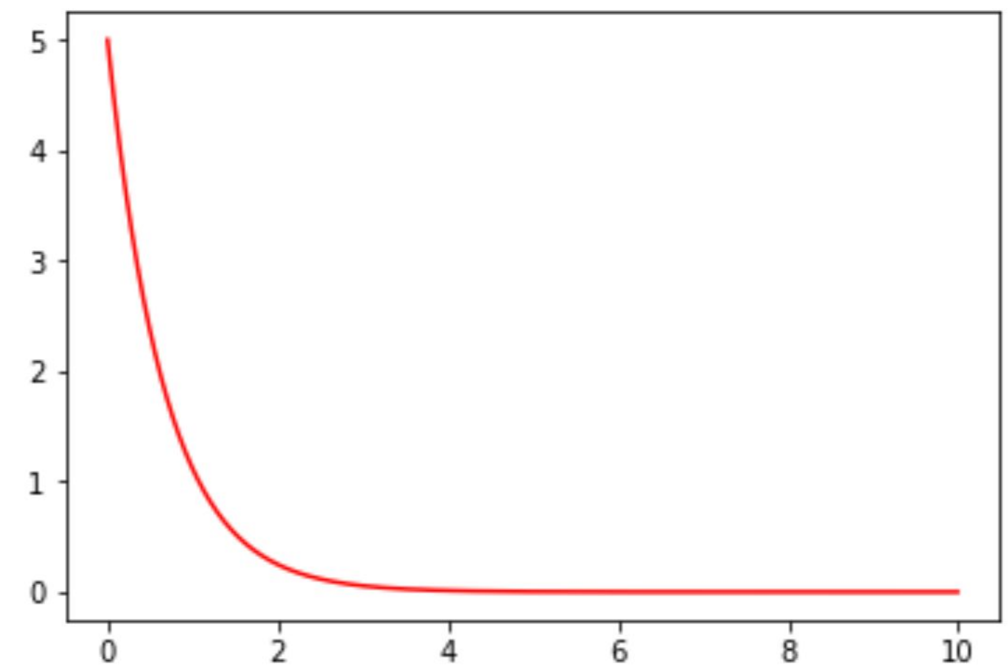
$$\frac{dP}{P} = r\,dt$$

$$\boxed{\frac{dP}{dt} = rP}$$

$$\boxed{P(t) = P_0 e}$$



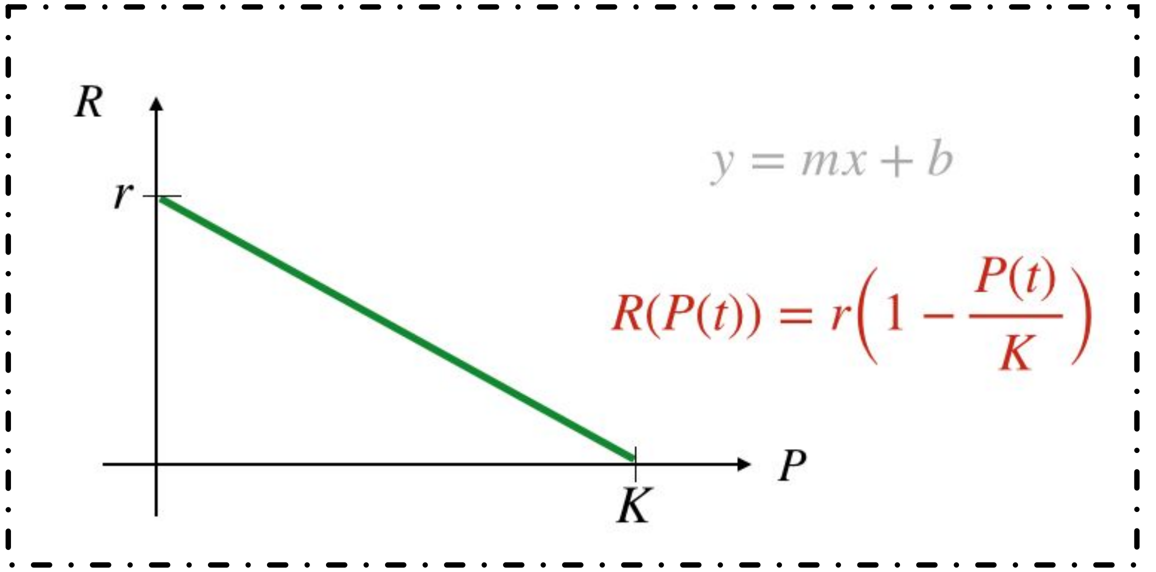
$r > 0$



$r < 0$

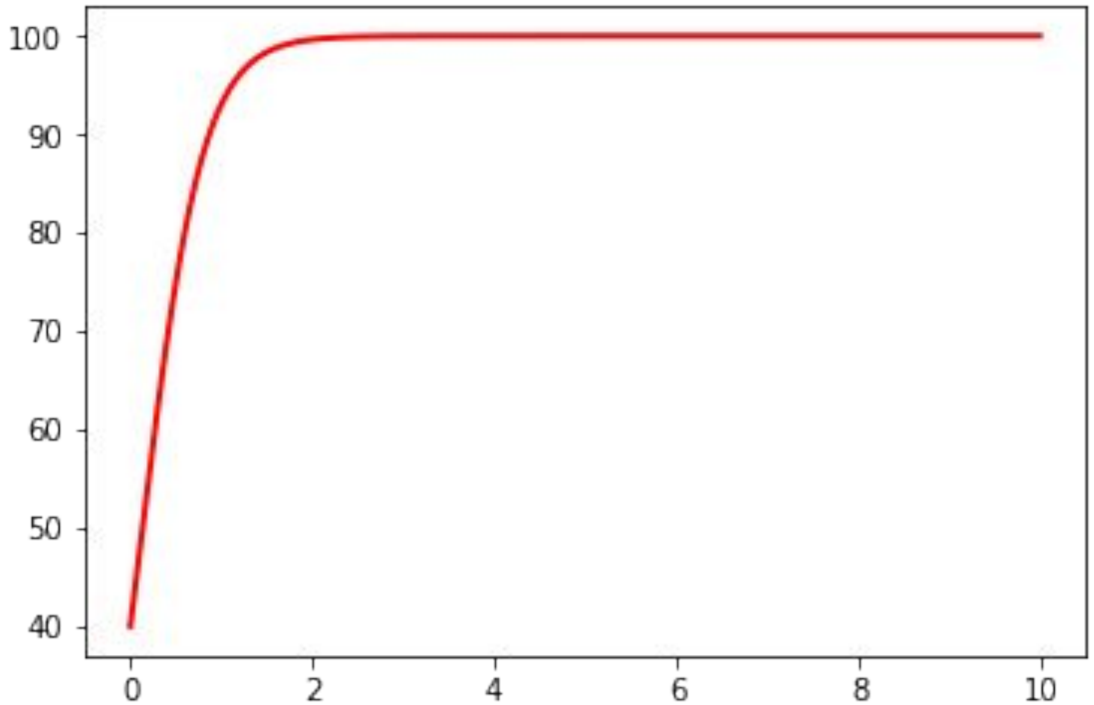
$$\frac{dP}{dt} = rP(t)$$

$$\frac{dP}{dt} = R(P(t))P(t)$$

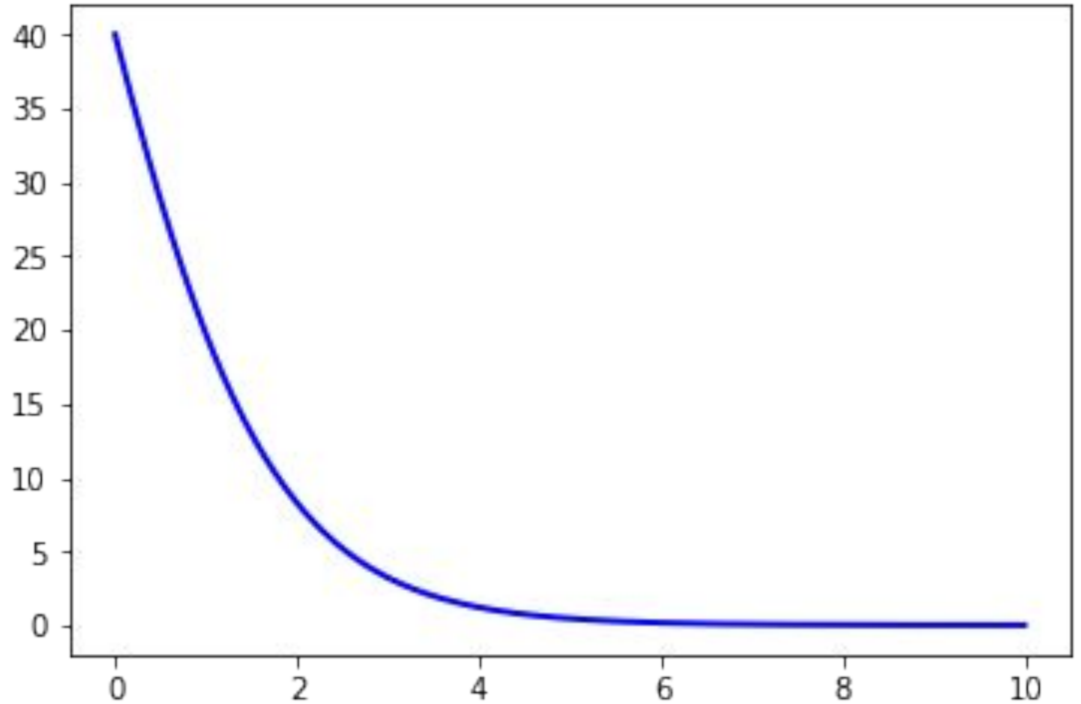


$$\frac{dP}{dt} = r(1 - \frac{P(t)}{K})P(t)$$

$$\boxed{P(t) = \frac{K}{(\frac{K}{P_0} - 1)e^{-rt} + 1}}$$

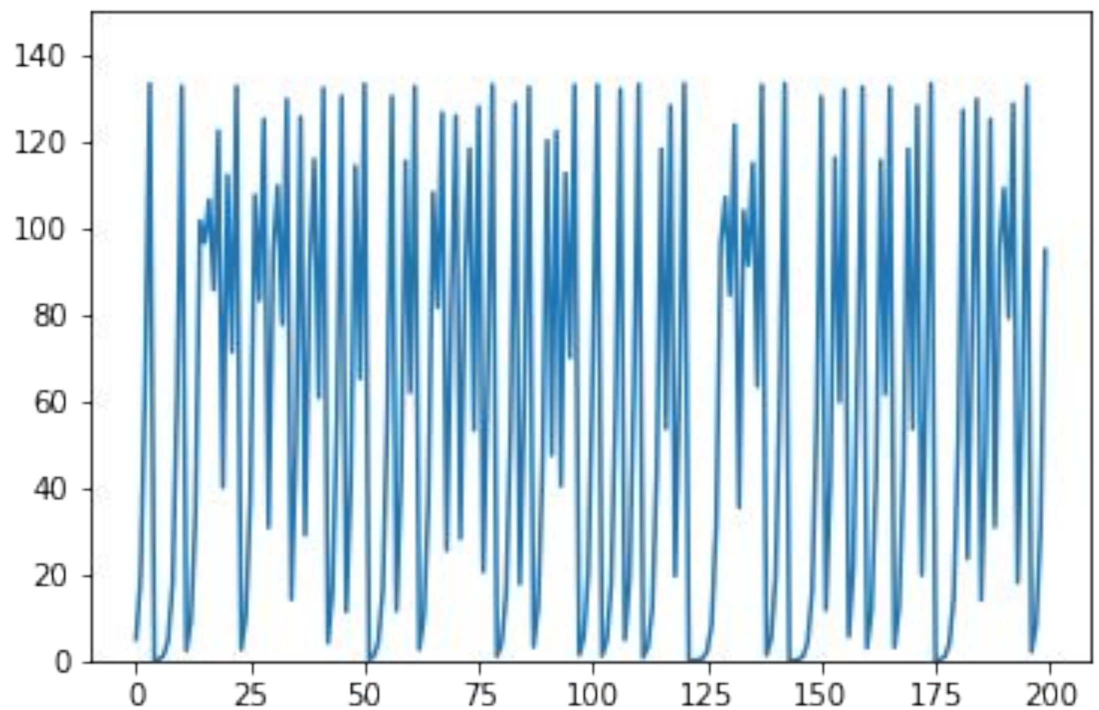


$r = 3$

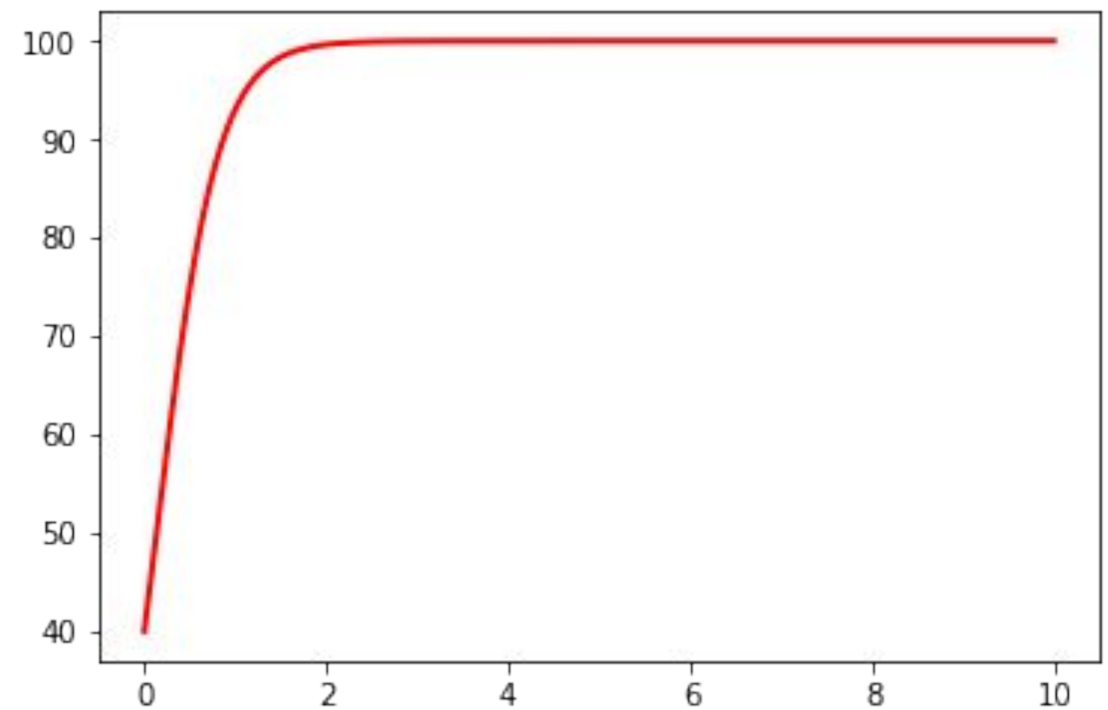


$r = -1$

Discrete vs Continuous



$r = 3$

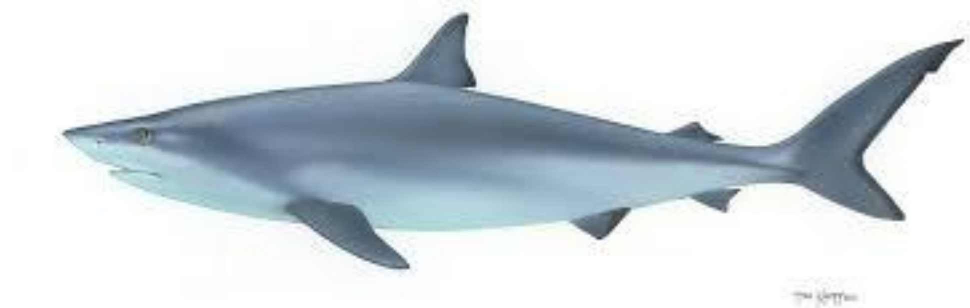


$r = 3$

$$\frac{dP}{dt} = r(1 - \frac{P(t)}{K})P(t)$$

$$\frac{dP}{P} = r\,dt$$

Coupled models:
Prey-Predator interactions



- Differential equations in a Lotka-Volterra model describe interactions between predators and their prey.

- This model arose when they (independently) applied this model to the study of fish and shark populations during world war I.

Question: What will happen to the fish population when there is a decrease in fishing?

- Extensive records were kept of the yearly catches of fish and sharks at an Italian sea port (Fiume 1914-1923).
- Italian mathematician **Vito Volterra** (1860-1940) developed a model of **predator-prey interactions** in response to some unusual data.

Year	Shark (% total catch)
1914	11.9
1915	21.4
1916	22.1
1917	21.2
1918	36.4
1919	27.3
1920	16.0
1921	15.9
1922	14.8
1923	10.7

The model:

things Volterra considered

Natural births and deaths of sharks and fish in isolation from each other.

Decline of fish population due to the fish being the food for sharks.

Increase in the shark population due to the presence of more fish.

Fishing of both sharks and fish

X $\longrightarrow$ number of Fish at time t

Y $\longrightarrow$ number of Sharks at time t

$$\frac{dx}{dt} = \quad - \quad - \quad -$$

fish born in isolation | fish deaths in isolation | fish eaten by sharks | fish caught by fishermen

$$\frac{dy}{dt} = \quad - \quad + \quad -$$

sharks born in isolation | shark deaths in isolation | sharks feed from fish | sharks caught by fishermen

Assumptions

- The change in the shark and fish populations, in isolation, is **proportional** to the present population of sharks and fish, respectively.
- The number of sharks and fish caught by fishermen is directly **proportional** to the present population of the shark and fish population.
- The number of fish eaten by sharks is directly **proportional** to the product of the number of fishes present and the number of sharks present.
- The additional number of sharks surviving is directly **proportional** to the number of fish eaten.

X → number of Fish at time t

Y → number of Sharks at time t

$$\frac{dx}{dt} = \qquad - \qquad - \qquad -$$

fish born in isolation	fish deaths in isolation	fish eaten by sharks	fish caught by fishermen

$$\frac{dy}{dt} = \qquad - \qquad + \qquad -$$

sharks born in isolation	shark deaths in isolation	sharks feed from fish	sharks caught by fishermen

$$\frac{dx}{dt} = bx - dx - \alpha xy - fx$$

fish born in isolation

fish deaths in isolation

fish eaten by sharks

fish caught by fishermen

$$\frac{dx}{dt} = (b - d - f)x - \alpha xy$$

r is the fish rate of growth

f is the fraction of fish caught

α is the prop. constant of fish being eaten by shark

$$\frac{dx}{dt} = (r - f)x - \alpha xy$$

$$\frac{dy}{dt} = \qquad - \qquad + \qquad -$$

sharks born in isolation

shark deaths in isolation

sharks feed from fish

sharks caught by fishermen

$$\frac{dy}{dt} = b'y - d'y + \beta xy - fy$$

sharks born in isolation | **shark deaths in isolation** | **sharks feed from fish** | **sharks caught by fishermen**

$$\frac{dy}{dt} = (b' - d' - f)y + \beta xy$$

s is the shark rate of growth

f is the fraction of shark caught

β is the prop. constant of shark surviving by eating fish

$$\frac{dy}{dt} = (s - f)y + \beta xy$$

$$\frac{dx}{dt} = (r - f)x - \alpha xy$$

$$\frac{dy}{dt} = (s - f)y + \beta xy$$

r is the fish rate of growth

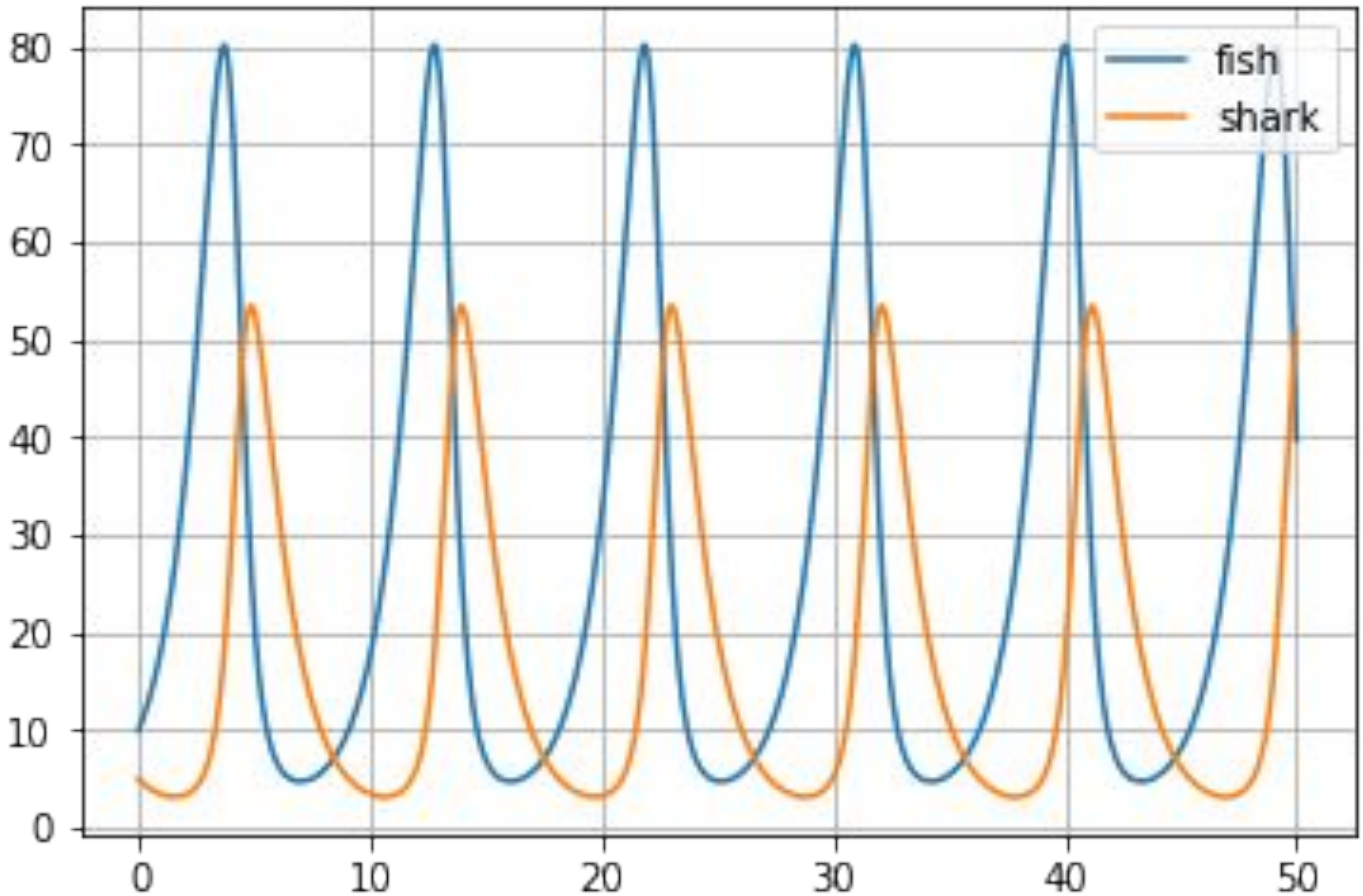
s is the shark rate of growth

f is the fraction of shark (fish) caught

α is the prop. constant of fish being eaten by shark

β is the prop. constant of shark surviving by eating fish

For numerical exploration:
https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.odeint.html



r = 0.8
f = 0.0001
s = -0.8
alpha = 0.045
beta = 0.03

Steady state analysis

$$\frac{dx}{dt} = (r - f)x - \alpha xy$$

$$\frac{dy}{dt} = (s - f)y + \beta xy$$

Steady state analysis

$$0 = (r - f)x - \alpha xy$$

$$0 = (s - f)y + \beta xy$$

$$(x, y) = (0, 0)$$

$$(x, y) = (\frac{f - s}{\beta}, \frac{r - f}{\alpha})$$

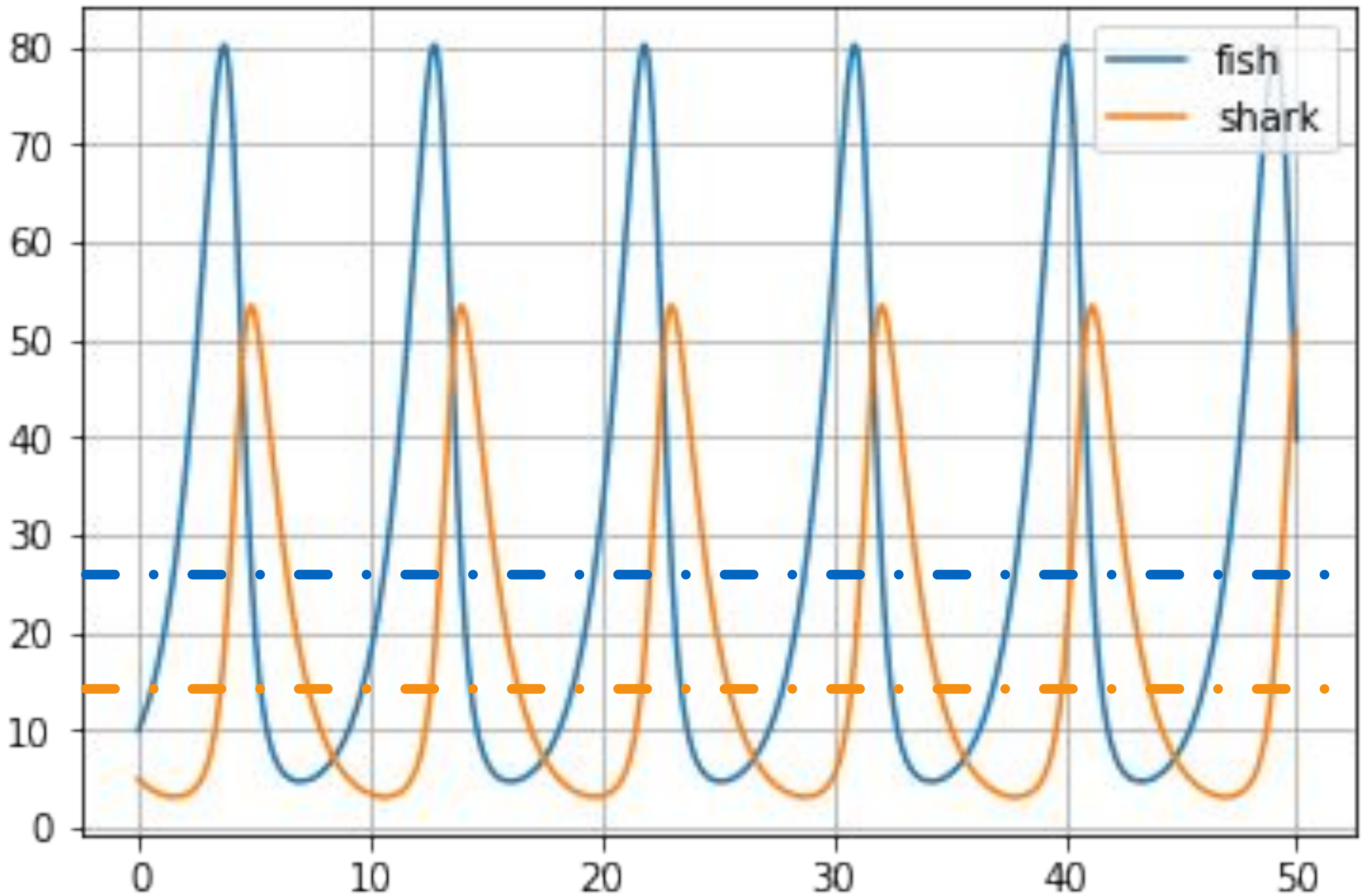
r is the fish rate of growth

s is the shark rate of growth

f is the fraction of shark (fish) caught

α is the prop. constant of fish being eaten by shark

β is the prop. constant of shark surviving by eating fish



r = 0.8
f = 0.0001
s = -0.8
alpha = 0.045
beta = 0.03

x* = 26.67
y*= 17.77

Steady state analysis

$$(x, y) = (\frac{f - s}{\beta}, \frac{r - f}{\alpha})$$

Steady state *(0, 0)* corresponds to the extinction of both species. Uninteresting

and unlikely…. for the most part.

Non-trivial steady state: x* does not depend on r and y*

does not depend on s.

We will use **phase-plane analysis** to verify that oscillations are expected around

the non-trivial steady state.

Phase-plane analysis

$$\frac{dx}{dt} = f(x, y) \qquad\qquad \frac{dy}{dt} = g(x, y)$$

- Lotka-Volterra is **first-order**, **autonomous** system.
- Instead of solving for $x(t)$ and $y(t)$, see how x reacts to changes in y.
- Resulting curves in (x, y) are called **phase-plane trajectories.**

Starting point:

$$\frac{dx}{dt} = f(x, y) \qquad \frac{dy}{dt} = g(x, y)$$

Goal:

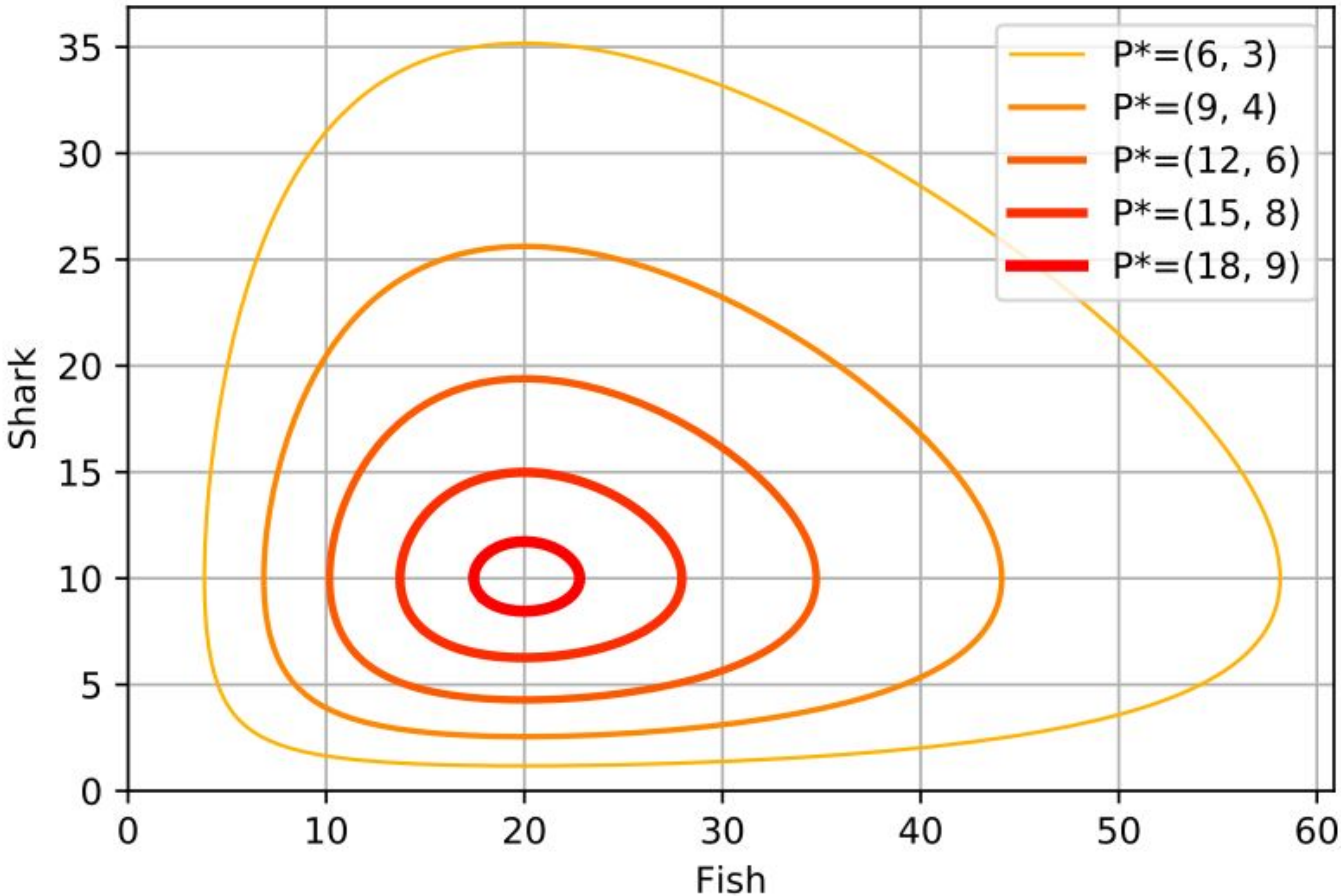
chain rule

$$\frac{dy}{dx} = ? \longrightarrow \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dt}$$

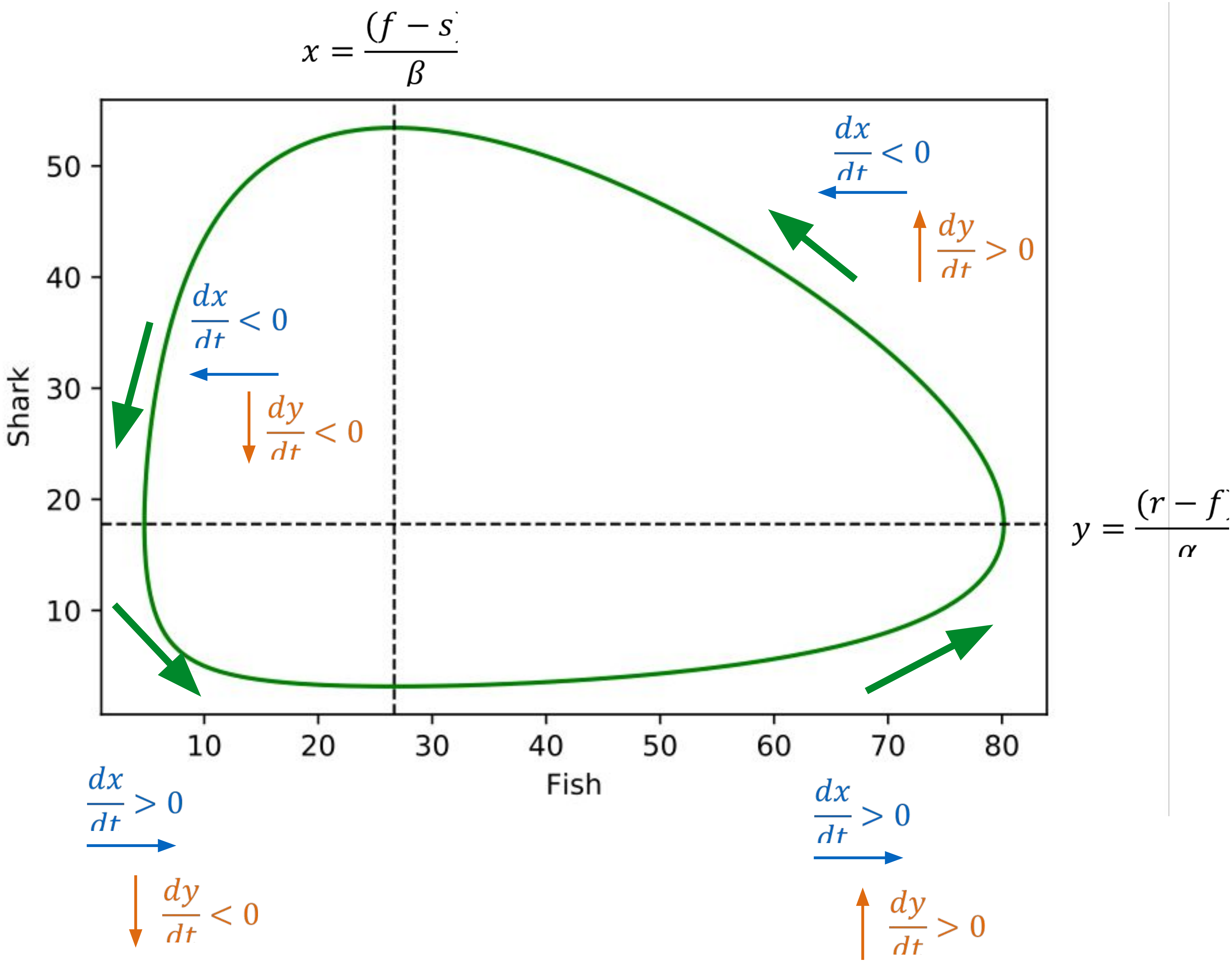
$$\frac{dy}{dx} f(x, y) = g(x, y)$$

$$\frac{dy}{dx} = h(x, y)$$

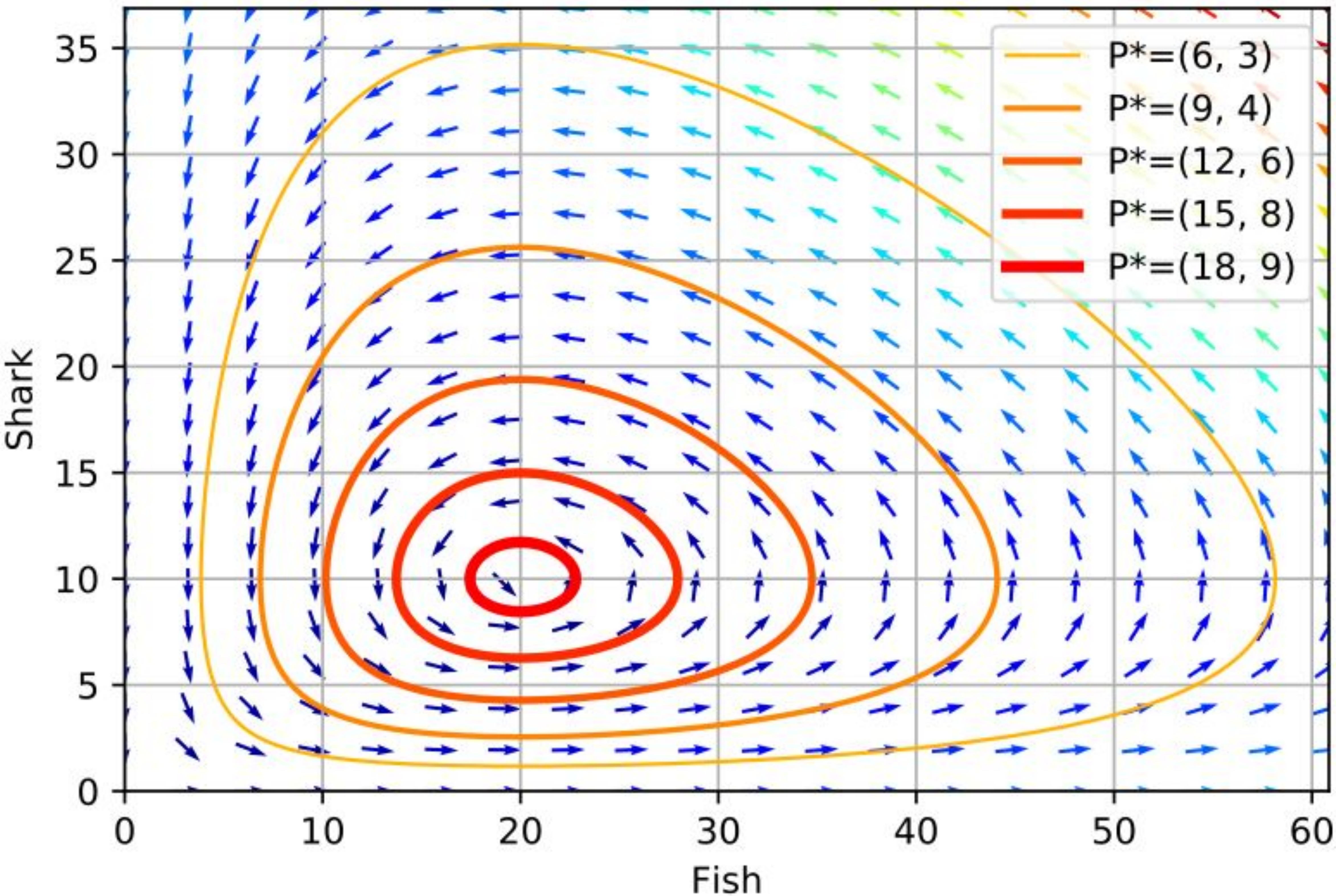
Solve for x and y, and **plot in the (x, y) space** using the given initial values.



P*=(6, 3)
P*=(9, 4)
P*=(12, 6)
P*=(15, 8)
P*=(18, 9)
Shark
Fish
0
10
20
30
40
50
60
0
5
10
15
20
25
30
35



$x = \frac{(f - s}{\beta}$
$y = \frac{(r - f}{\alpha}$
Shark
Fish
10
20
30
40
50
60
70
80
$\frac{dx}{dt} < 0$
$\frac{dy}{dt} > 0$
$\frac{dx}{dt} < 0$
$\frac{dy}{dt} < 0$
$\frac{dx}{dt} > 0$
$\frac{dy}{dt} < 0$
$\frac{dx}{dt} > 0$
$\frac{dy}{dt} > 0$



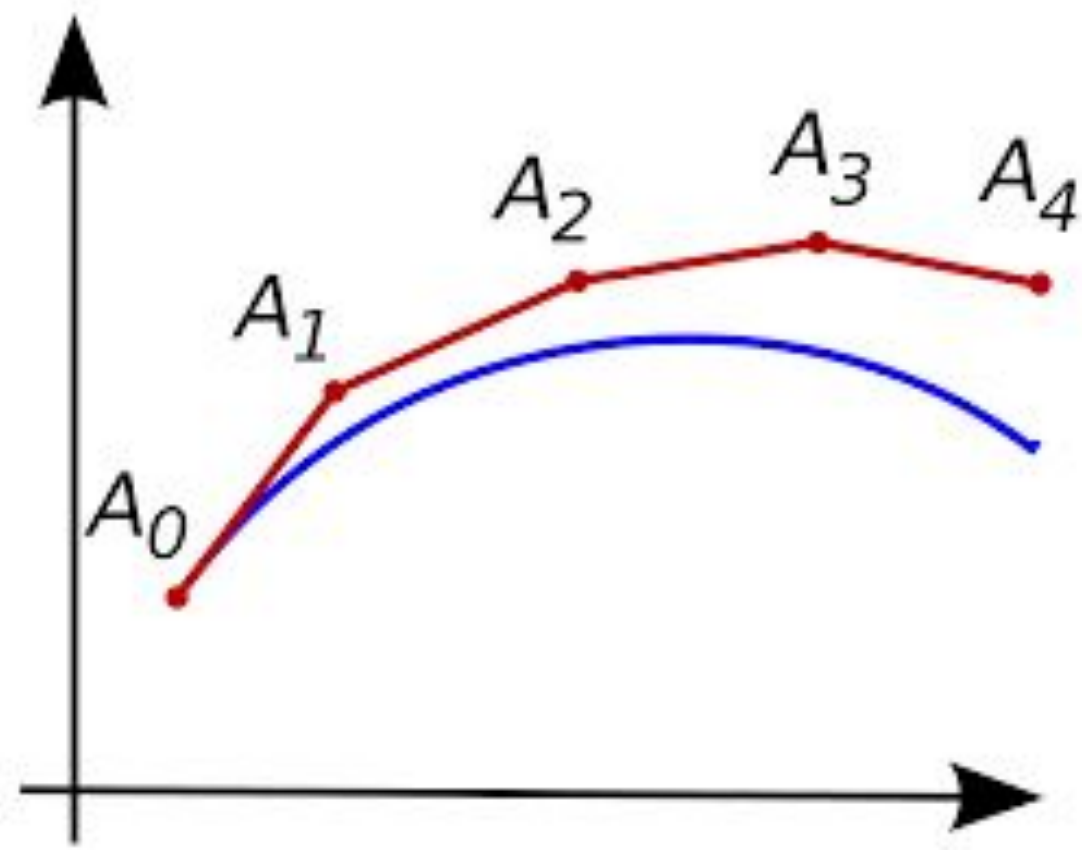
P*=(6, 3)
P*=(9, 4)
P*=(12, 6)
P*=(15, 8)
P*=(18, 9)
Shark
Fish
0
5
10
15
20
25
30
35
0
10
20
30
40
50
60

Odeint: Numerical integration

$$\frac{dx}{dt} = f(x, t$$

$$x(t) = \int f(x$$

$$x(0) = x$$



(we will discuss this next week)

Visualising a dynamical system…

https://vimeo.com/23839605

Kill Math, by Bret Victor:

http://worrydream.com/KillMath/