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David Albrecht and María García de la Banda

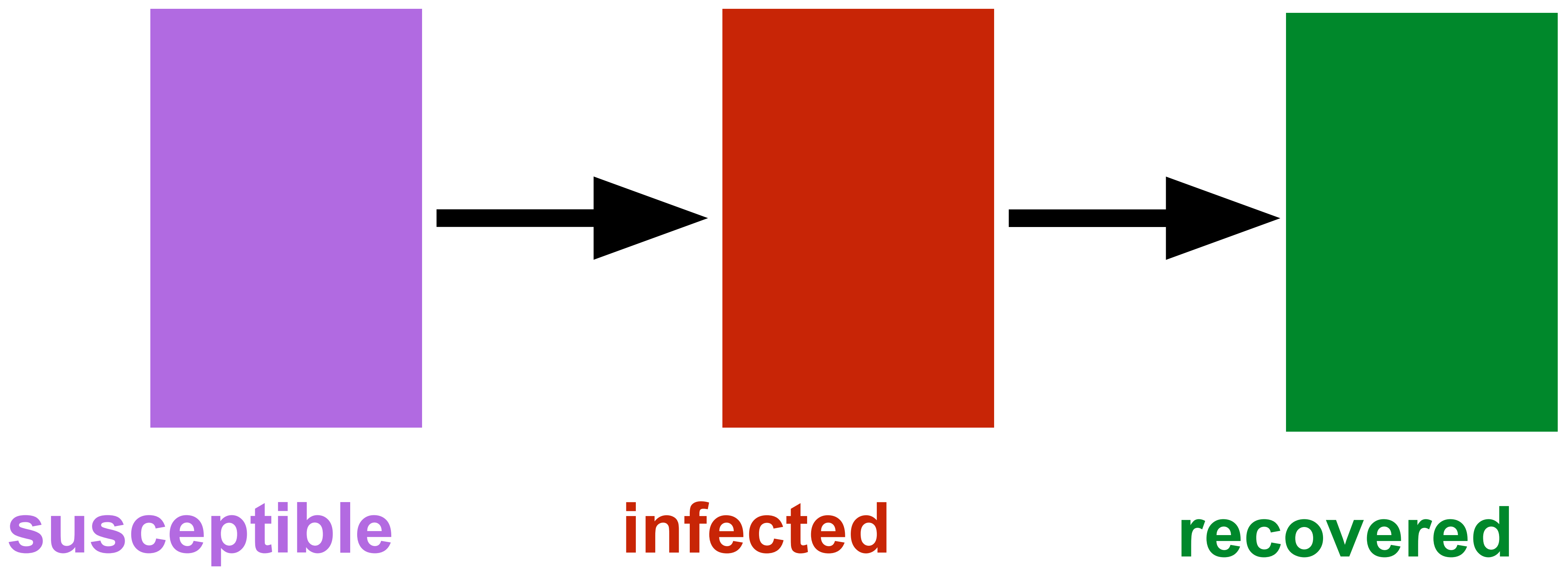
Workshop 9
Intro to Dynamical Systems: Coupled Models

- FIT3139: Computational Modelling and Simulation

MONASH University
Information Technology

Outline

- Coupled models
- SIR Models
- Measles


susceptible
infected
recovered

A map *is not* the territory it represents, but, if correct, it has a *similar structure* to the territory, which accounts for its usefulness.

— *Alfred Korzybski, Science and Sanity*

So let’s keep the model **simple.**

Assumptions

- Latent period is exactly **1 week.**
- **Infective** period is exactly **1 week.**
- All contact between **infectives** and **susceptibles** occurs during weekend.
  - *number of infectives and susceptibles remain constant throughout the week.*
- Infection rate: a single **infective** infects a constant fraction (f) of **susceptibles**
- Population changes through births and deaths:
  - *Constant number of births per week (B).*
  - *No death.*

$S_k \longrightarrow$ *number of susceptibles in week k*

$I_k \longrightarrow$ *number of infective in week k*

$$S_{k+1} = F(S_k, I_k)$$

$$I_{k+1} = G(S_k, I_k)$$

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$S_k \longrightarrow$ *number of susceptibles in week k*

$I_k \longrightarrow$ *number of infective in week k*

$$I_{k+1} = \underbrace{\qquad}_{\substack{\text{number of susceptibles who} \\ \text{caught measles at} \\ \text{the beginning of week k}}}$$

$$S_{k+1} = \underbrace{\qquad}_{\substack{\text{number of susceptibles} \\ \text{in week k}}} - I_{k+1} + \underbrace{\qquad}_{\substack{\text{number of births} \\ \text{during week k}}}$$

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$$I_{k+1} = \underbrace{\qquad}$$

number of **susceptibles** who
caught measles at
the beginning of week k

$$S_{k+1} = \underbrace{S_k}\qquad - I_{k+1} + \underbrace{\qquad}$$

number of **susceptibles**
in week k

number of **births**
during week k

Assumptions

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$$I_{k+1} = f \underbrace{S_k}\, I_k$$

number of **susceptibles** who caught measles at the beginning of week k

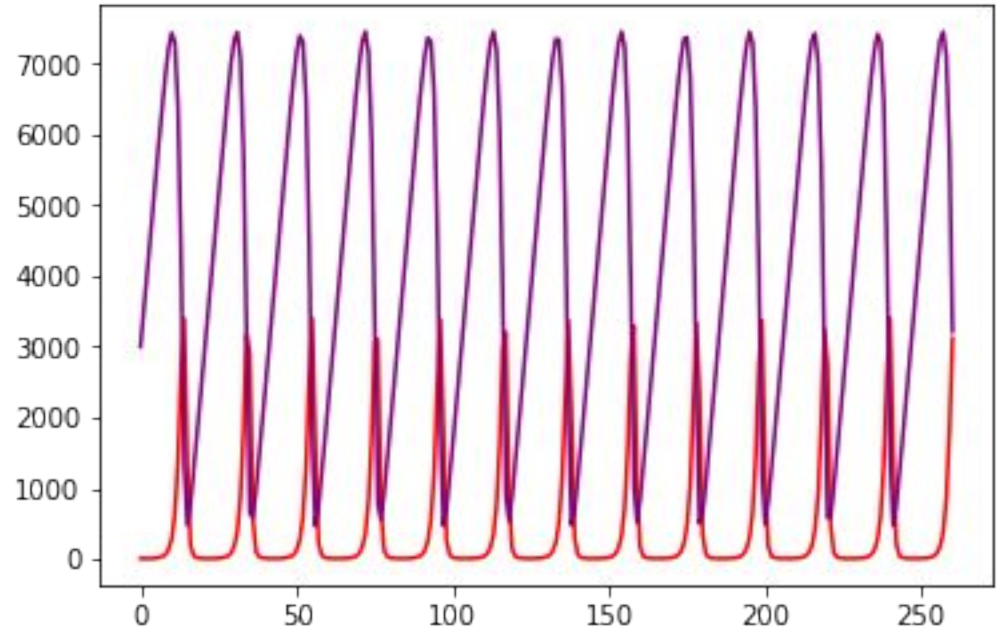
$$S_{k+1} = \underbrace{S_k} - f S_k I_k + \underbrace{B}$$

number of **susceptibles** in week k

number of **births** during week k

$$I_{k+1} = f S_k I_k$$

$$S_{k+1} = S_k - f S_k I_k + B$$

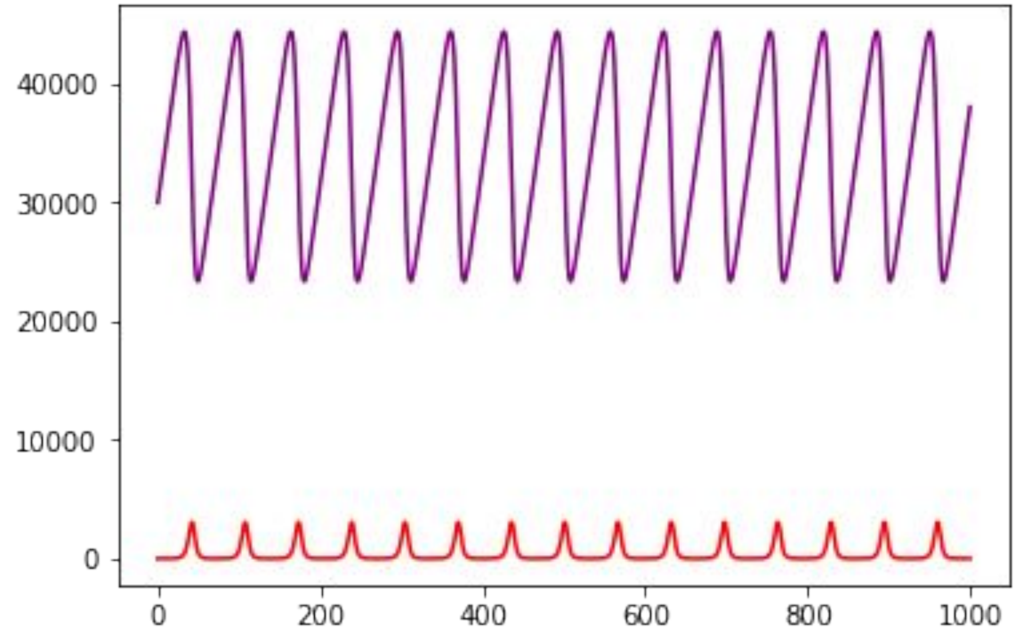


$B = 15$

$f = 3 \times 10^{-4}$

$I_0 = 5$

$$I_{k+1} = f S_k I_k$$



$B = 120$

$f = 3 \times 10^{-5}$

$I_0 = 10$

$S_0 = 30000$

Steady state solution

$$\boxed{\begin{aligned} I_{k+1} &= f S_k I_k \\ S_{k+1} &= S_k - f S_k I_k + B \end{aligned}}$$

$$\hat{I} = I_{k+1} = I_k$$

$$\hat{S} = S_{k+1} = S_k$$

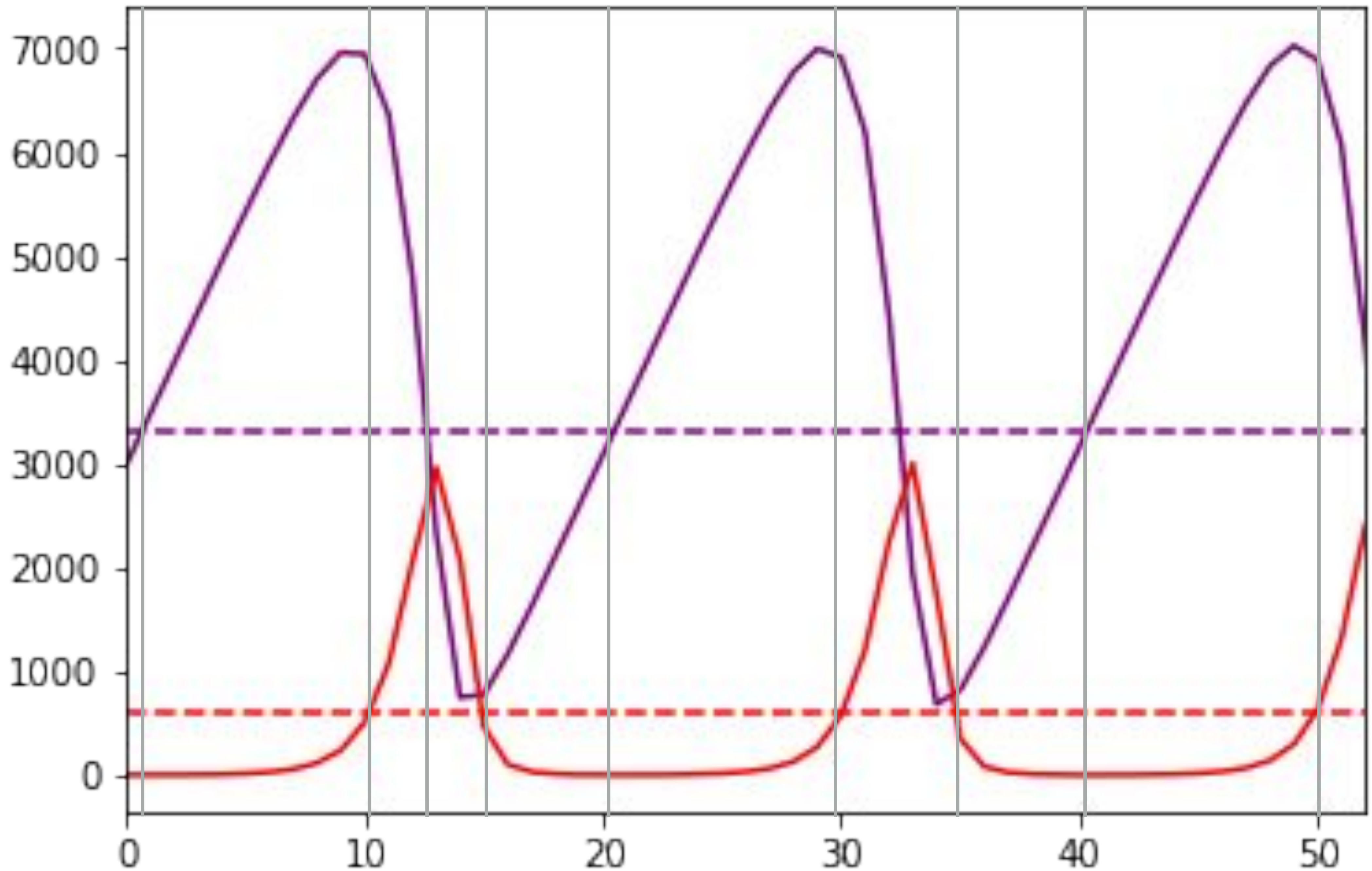
$$\longrightarrow$$

$$\hat{I} = f \hat{S} \hat{I}$$

$$\hat{S} = \hat{S} - f \hat{S} \hat{I} + B$$

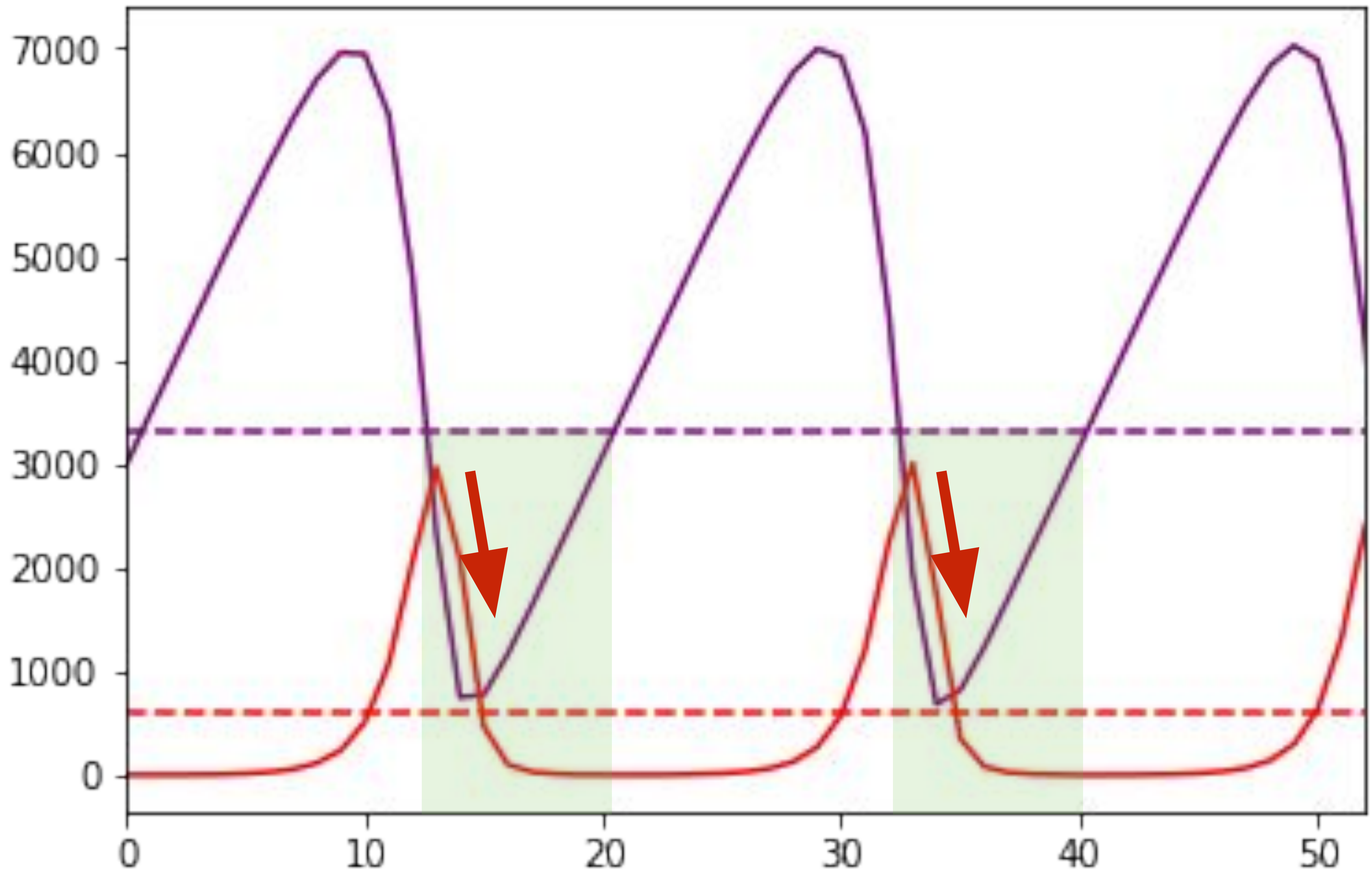
$$\longrightarrow$$

$$\boxed{\begin{aligned} \hat{S} &= 1/f \\ \hat{I} &= B \end{aligned}}$$

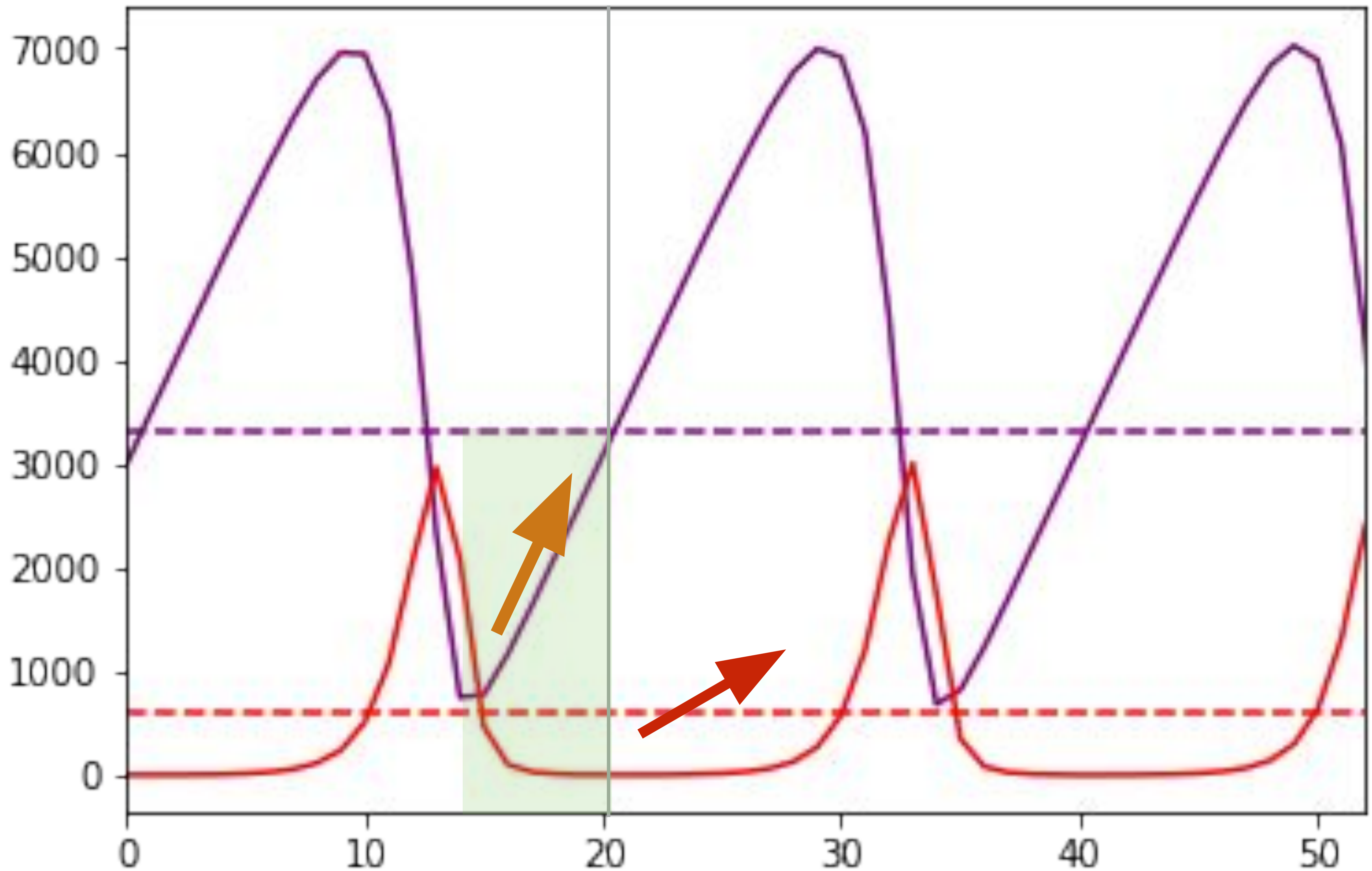


$$\hat{S} = 1/f$$

$$\hat{I} = B$$



From $I_{k+1} = f S_k I_k$, if $S_k < 1/f$



From $S_{k+1} = S_k - fS_kI_k + B$ for small I_k

increases until $S_k > \frac{1}{f}$ then I_k grows again

$$S_k < 1/f$$

Observations

- The difference equation $I_{k+1} = fS_kI_k$ tell its that if $\frac{I_{k+1}}{I_k} < 1$ then $S_k < 1/f$.
- From the coupled difference equation $S_{k+1} = S_k - fS_kI_k + B$ for small I_k , S_k increases due to birth until S_k eventually becomes greater than $1/f$. Then I_k grows again.
- This shows the importance of B.
- Keeping susceptibles below $1/f$ is important:
- this can be achieved through **vaccination**.

A simple model can explain seasonal epidemics….