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Lecture 8
Intro to Dynamical Systems: Difference Equations

FIT 3139

Computational Modelling and Simulation

MONASH University
Information Technology

Outline

- Introduction to Dynamical Systems
- Population Growth:
  - Differential equations
  - Difference equations

Introduction to dynamical systems

In many situations a **quantity** that is being measured, which **changes with time**, can be modelled using a smooth function (of time):

- Population growth
- Interest repayment
- Radioactive decay
- Quantities involved in classical mechanics.

A **dynamical system** is any system evolving with time. i.e each **state** is <u>time dependent</u>.

A **rule** (or set of rules) guides the <u>state changes</u>, describing what the state of the system will be in a short time-interval into the future from any given state.

$$P(t+\Delta t) = P(t) + \{\text{ births }\} - \{\text{ deaths }\}$$

$$P(t+\Delta t) = P(t) + r_b P(t)\Delta t - \{\text{ deaths }\}$$

$$P(t+\Delta t) = P(t) + r_b P(t)\Delta t - r_d P(t)\Delta t$$

$$P(t+\Delta t) = P(t) + \underbrace{(r_b - r_d)}_{r} P(t)\Delta t$$

$$P(t+\Delta t) = P(t) + rP(t)\Delta t$$

We know an explicit solution

$$P(t+\Delta t) = P(t)(1 + r\Delta t)$$

first order

$$P_{t+1} = P_t(1+r)$$

A **difference equation** is a rule that expresses a sequence, in terms of previous members of the sequence (starting from some initial values)

A **steady state solution** is a solution in which the measured values <u>do not change</u> with time.

$$x_0, x_1, x_2, x_3, \dots, x_k, x_{k+1}, x_{k+2}, \dots$$

A **steady state solution** is achieved when x_k does not change with k, thus

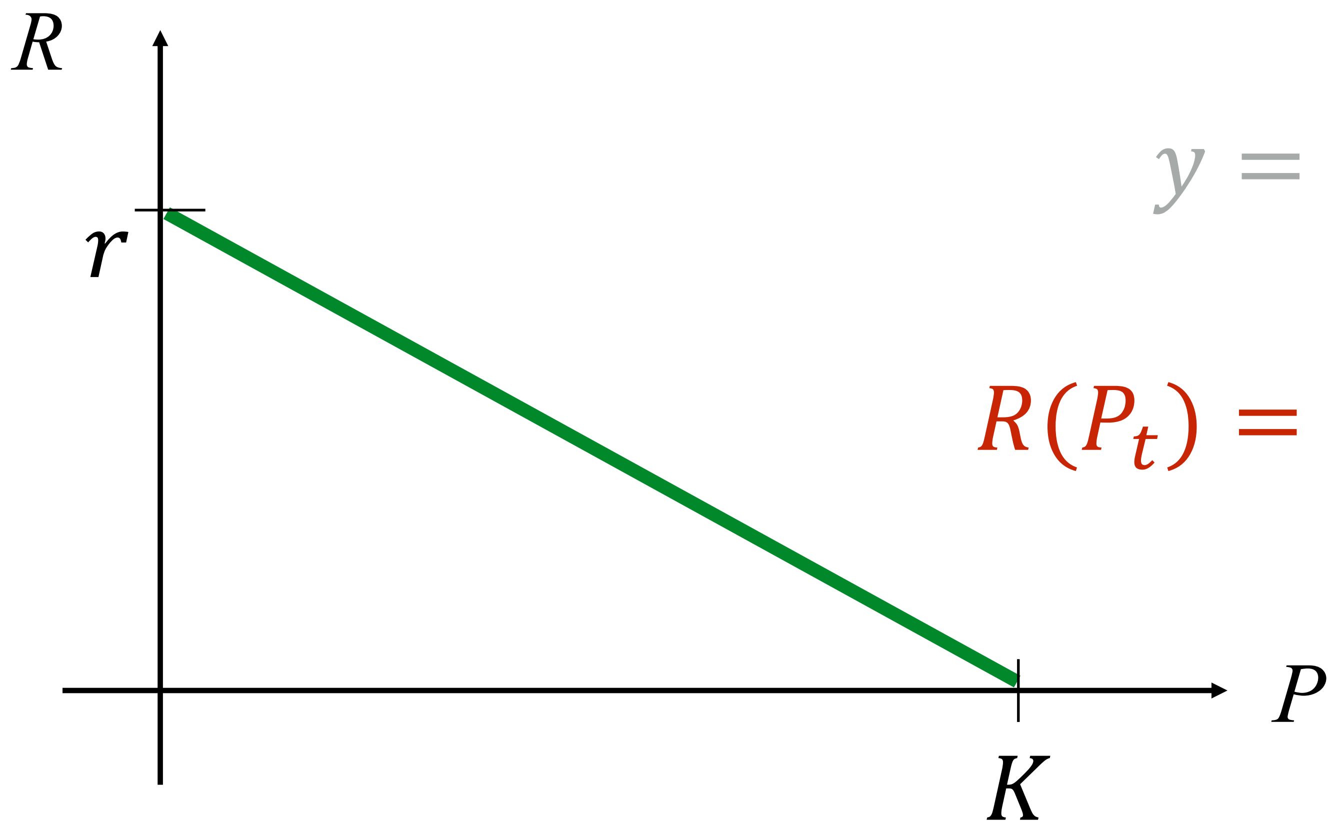
$$x_k = x_{k+1} = x_{k+2} = x_{k+3} \dots$$

More realistic population growth

- So far we have **unbounded** growth
- Does this happen in nature?
- Resources may limit growth, so populations grow up to a certain limit
- This limit is known as the carrying capacity of the system

Let the rate of growth depend on the population size itself….

$$P_{t+1} = P_t(1 + r) \longrightarrow P_{t+1} = P_t(1 + R(P_t))$$



$$y = mx + b$$

$$R(P_t) = -\frac{r}{K}P_t + r$$

- R should decrease with population size, due to overcrowding.
- When does the rate of growth become zero?at carrying capacity K
- As $P \rightarrow 0$ the rate of growth approaches..the <u>unconstrained</u> rate of growth r.

Let the rate of growth depend on the population size itself….

$$P_{t+1} = P_t(1+r) \qquad R(P_t) = -\frac{r}{K}P_t + r$$

$$P_{t+1} = P_t(1 + (\qquad\qquad))$$

$$P_{t+1} = P_t(1 + (-\frac{r}{K}P_t + r))$$

$$\boxed{P_{t+1} = P_t + rP_t\left(1 - \frac{P_t}{K}\right)}$$

Discrete Logistic Equation

non-linear

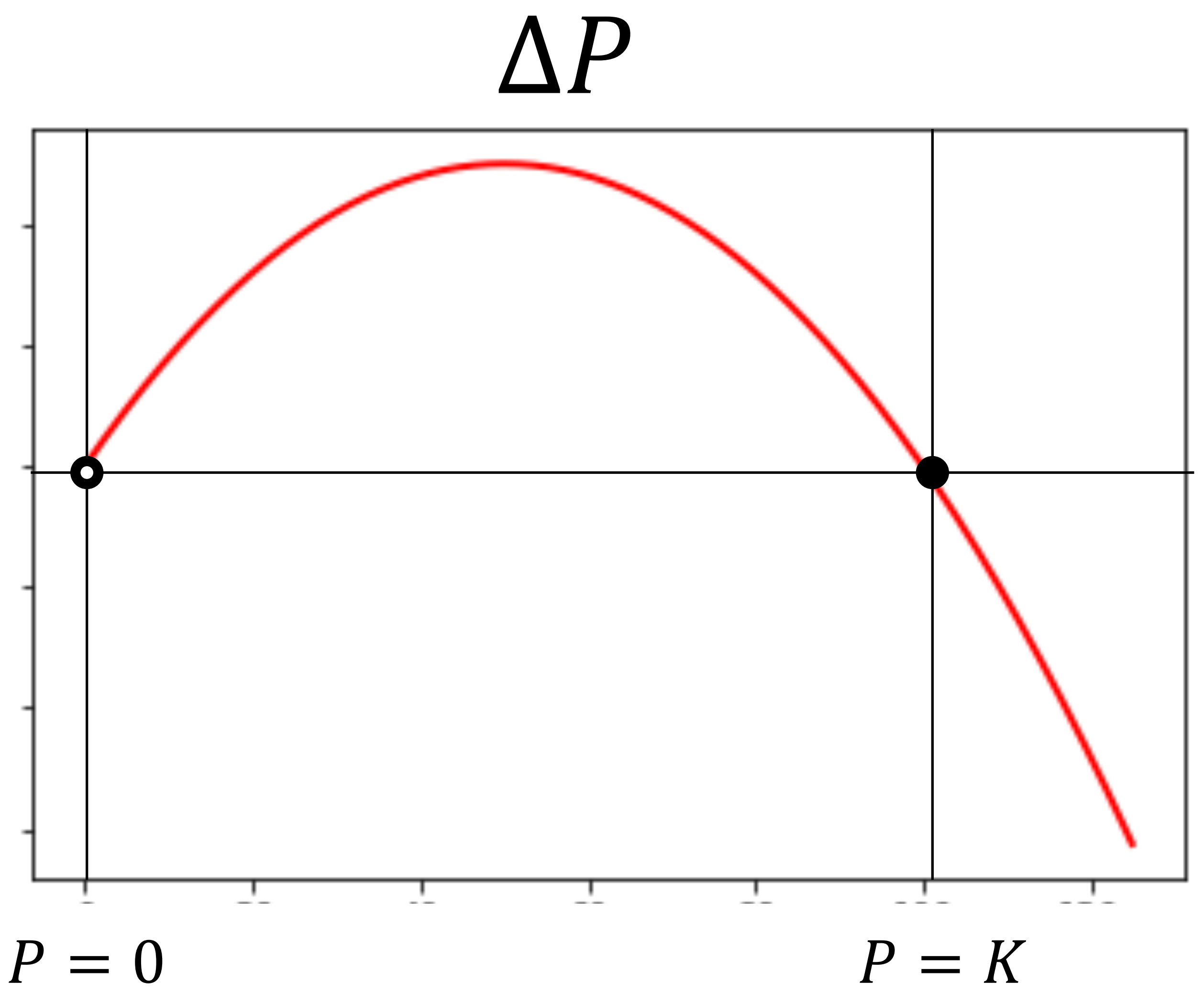
$$P_{t+1} = P_t + rP_t\left(1 - \frac{P_t}{K}\right)$$

$$P_{t+1} - P_t = rP_t\left(1 - \frac{P_t}{K}\right)$$

$$\Delta P = rP_t\left(1 - \frac{P_t}{K}\right)$$

A **steady state solution** is a solution in which the measured values do not change with time.

$$0 = rP_t\left(1 - \frac{P_t}{K}\right)$$



- Steady state solutions of 0 and k
- If $Pt < K$, the total population will increase in the next time interval.
- If $Pt > K$, the total population will decrease in the next time interval.
- Change in P is proportional to r:

small r, steady growth towards K

large r, oscillations around K

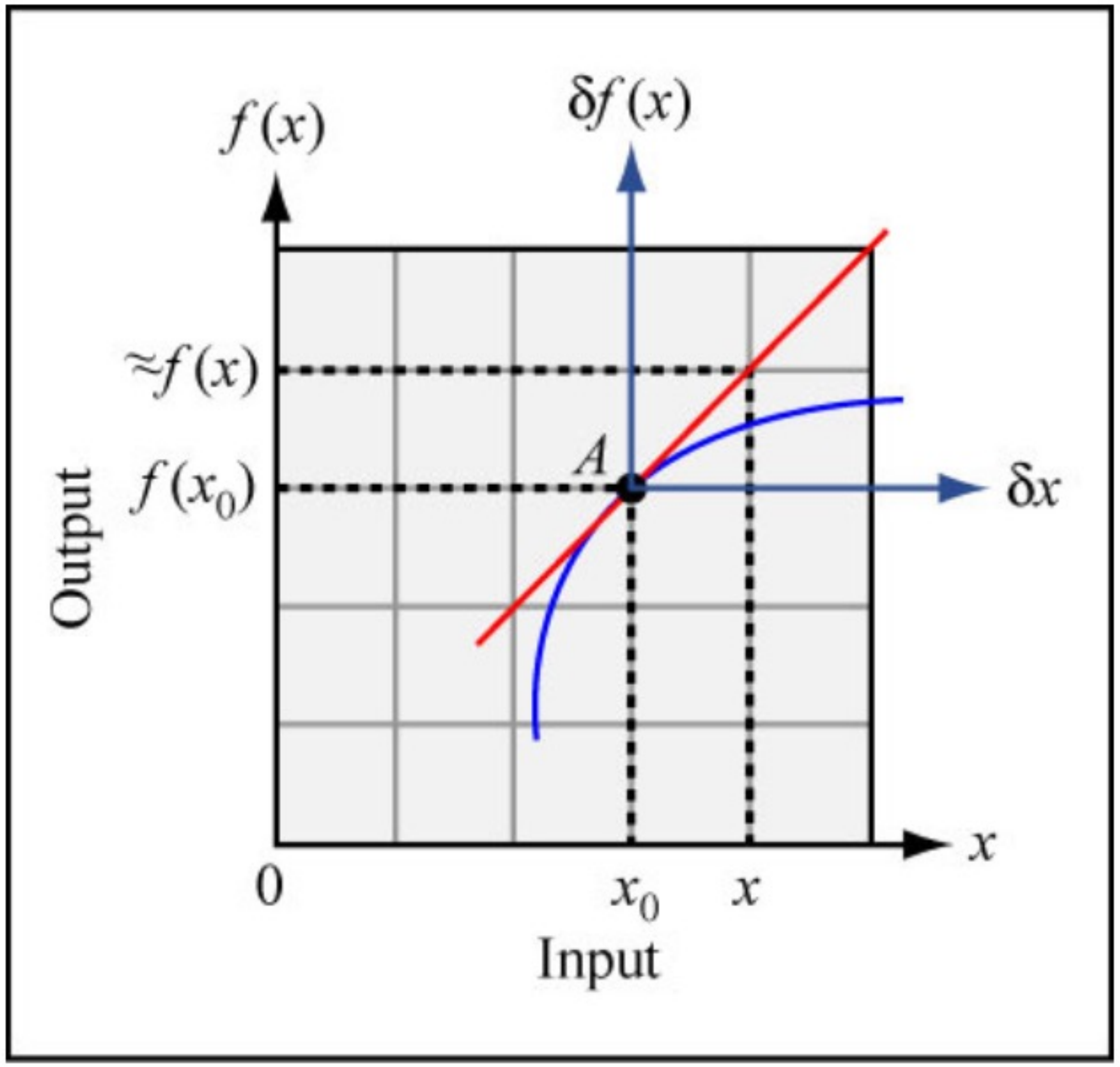
from numerical sims

$$\Delta P = rP_t\left(1 - \frac{P_t}{K}\right)$$

- We studied the discrete logistic model via numerical iterations.
- **In general**, no closed-form solutions are possible in non-linear systems.
- Sometimes you "pretend" that things are linear, around steady states… to try and gain insight.

Linearisation

Conceptually…



Linearisation

Linearisation

$$P_{t+1} = P_t + rP_t(1 - \frac{P_t}{K})$$

$$P^* = K$$

$$\frac{P_{t+1}}{K} = \frac{P_t}{K} + \frac{rP_t}{K}(1 - \frac{P_t}{K})$$

$$X_t = \frac{P_t}{K}$$

look at "fractions"

$$X_{t+1} = X_t + rX_t(1 - X_t)$$

$$X^* = 1$$

$$\delta_t = X_t - 1$$

$$\delta_{t+1} = X_{t+1} - 1$$

$$\delta_{t+1} = (1 - r)\delta_t - r\delta_t^2$$

K is hidden away (for now)

when $X_t \to 1$ then $\delta_t = X_t - 1 \to 0$

$$\delta_{t+1} \approx (1 - r)\delta_t$$

$$\delta_{t+1} \approx (1-r)\delta_t$$

This has a closed-form solution we know…

$$\delta_{t+1} \approx (1-r)^{t+1}\delta_0$$

$\delta_t = X_t - 1$

$$X_{t+1} - 1 \approx (1-r)^{t+1}(X_0 - 1)$$

$$X_{t+1} \approx (1-r)^{t+1}(X_0 - 1) + 1$$

$X_t = \frac{P_t}{K}$

$$\frac{P_{t+1}}{K} \approx (1-r)^{t+1}(\frac{P_0}{K} - 1) + 1$$

$$\boxed{P_{t+1} \approx (1-r)^{t+1}(P_0 - K) + K}$$

$$\boxed{P_{t+1} \approx (1-r)^{t+1}(P_0 - K) + K}$$

r=0 $\quad P_{t+1} \approx (\ 1\ )^{t+1}(P_0 - K) + K \longrightarrow P_{t+1} \approx P_0$

r=1 $\quad P_{t+1} \approx (\ 0\ )^{t+1}(P_0 - K) + K \longrightarrow P_{t+1} \approx K$

r=2 $\quad P_{t+1} \approx (\ -1\ )^{t+1}\underbrace{(P_0 - K) + K}_{+\ -}$

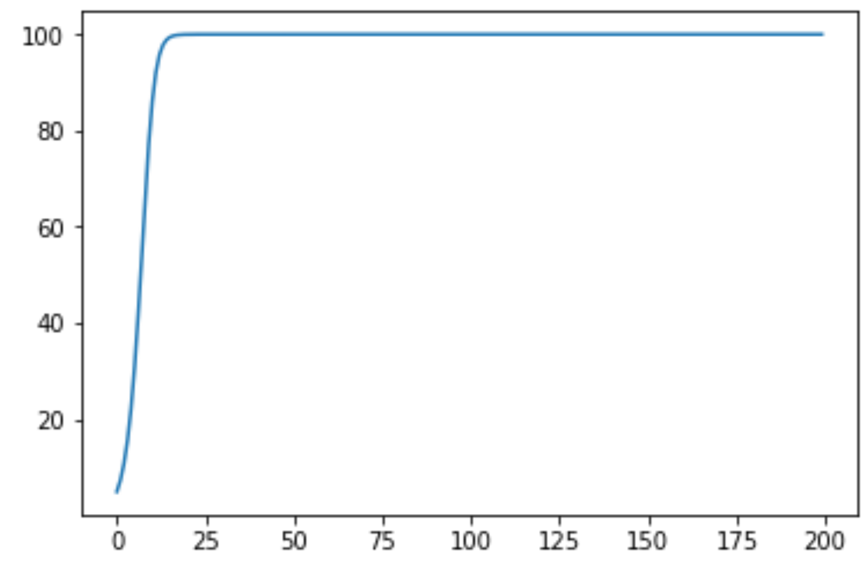
oscillations

r>2 $\quad P_{t+1} \approx (\quad\ )^{t+1}\underbrace{(P_0 - K) + K}_{+\ -}$

amplified

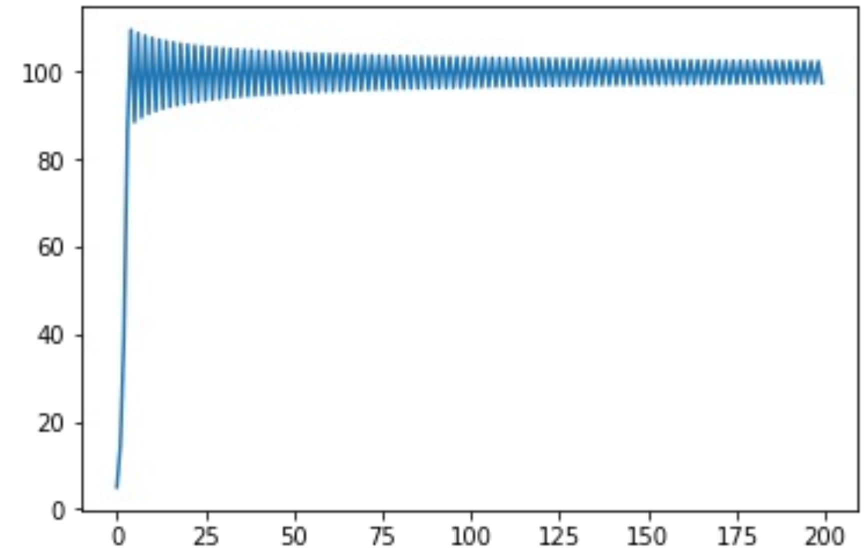
divergent oscillations

$$0 \leq r \leq 1$$

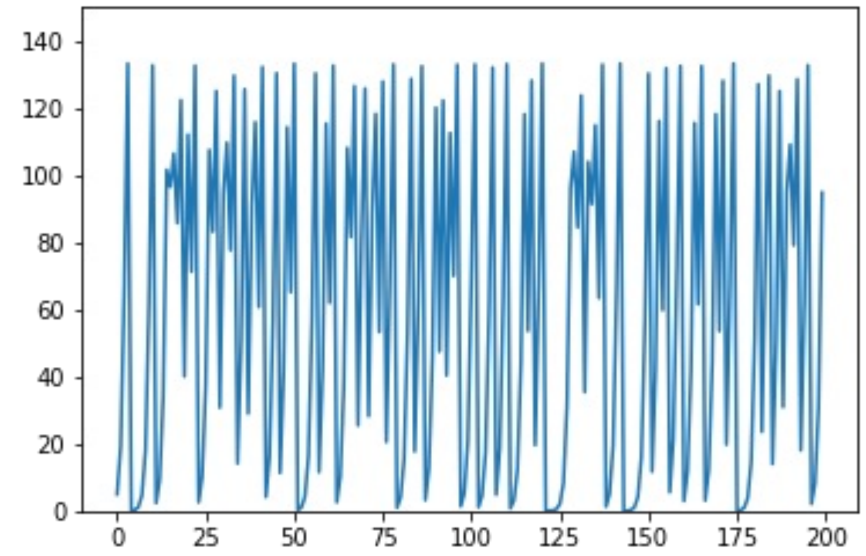


oscillations

$$1 < r \leq 2$$



divergent oscillations



Even when things are non-linear, around points of interest…things can be approximated by something linear….

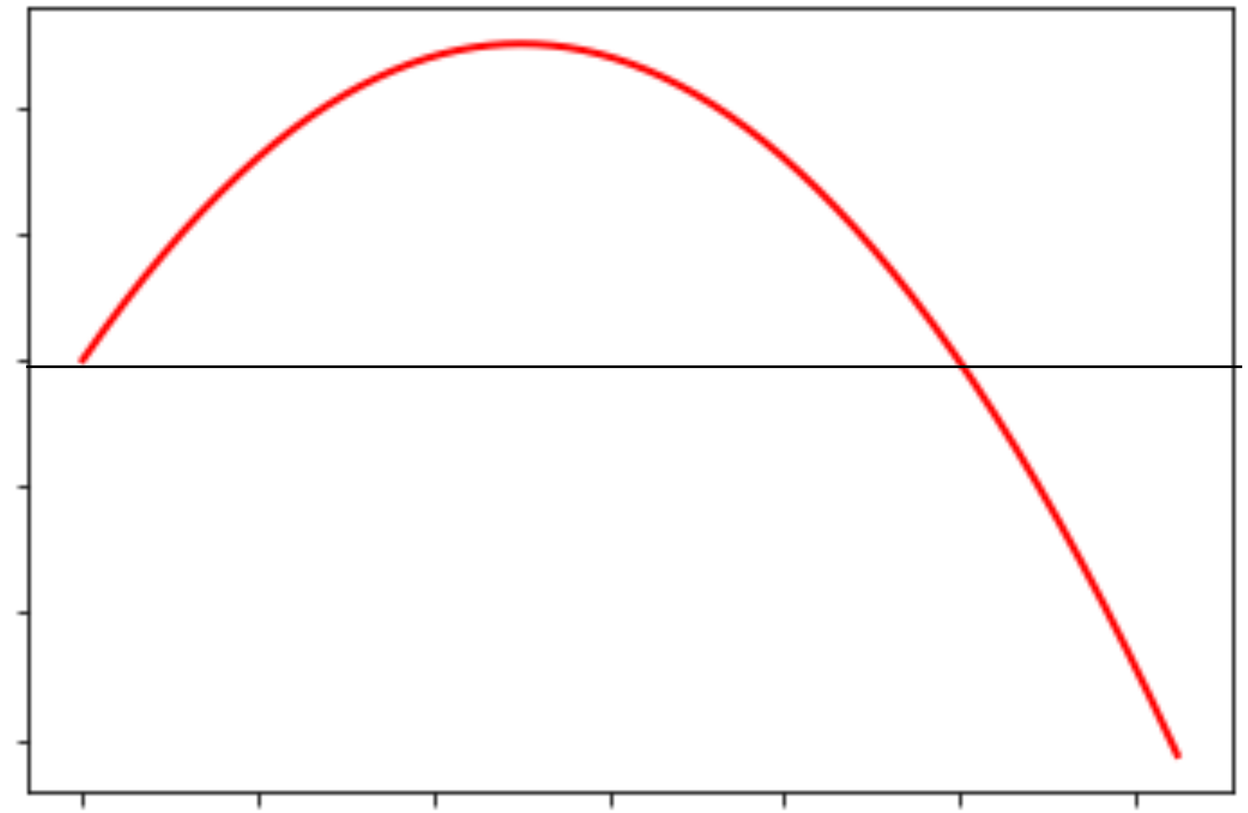
Graphical analysis is also a powerful tool….

$$\Delta P = rP_t\left(1 - \frac{P_t}{K}\right)$$

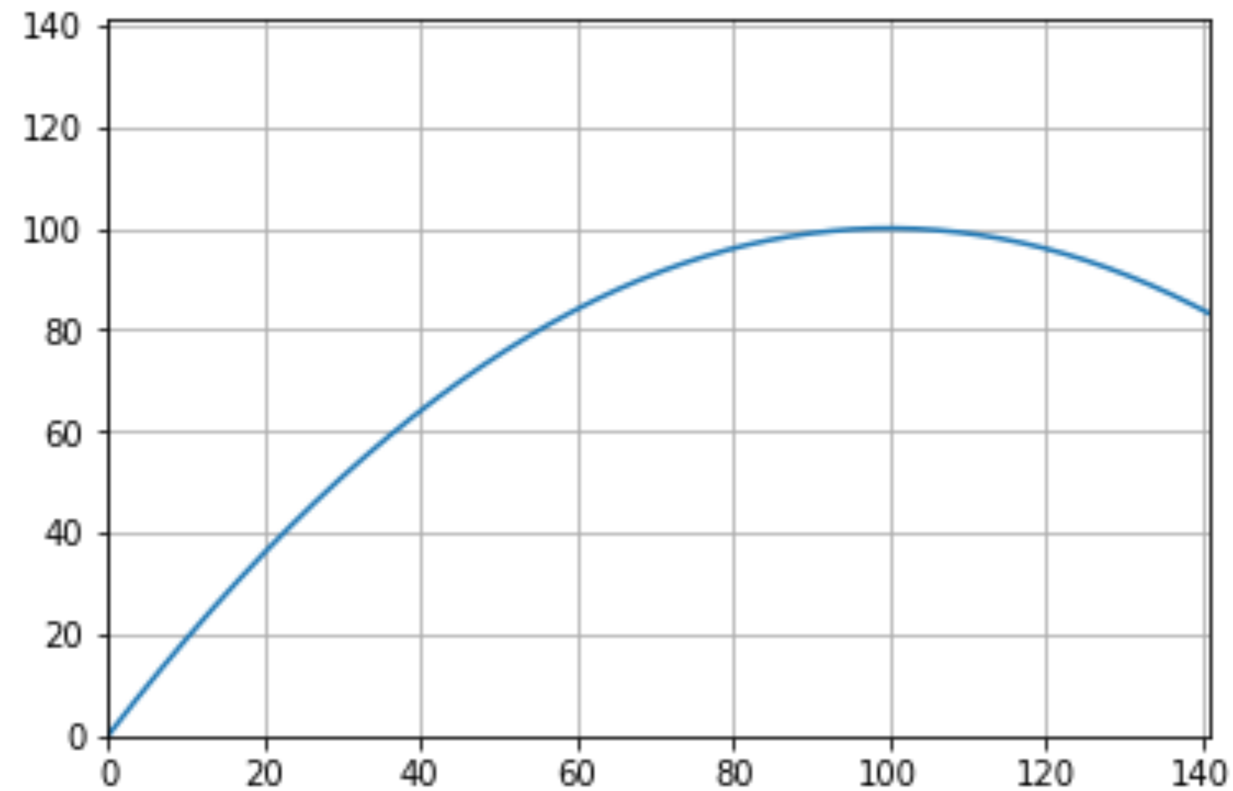
$$P_{t+1} - P_t = rP_t\left(1 - \frac{P_t}{K}\right)$$

$$P_{t+1} = \underbrace{rP_t\left(1 - \frac{P_t}{K}\right) + P_t}_{f(P_t)}$$

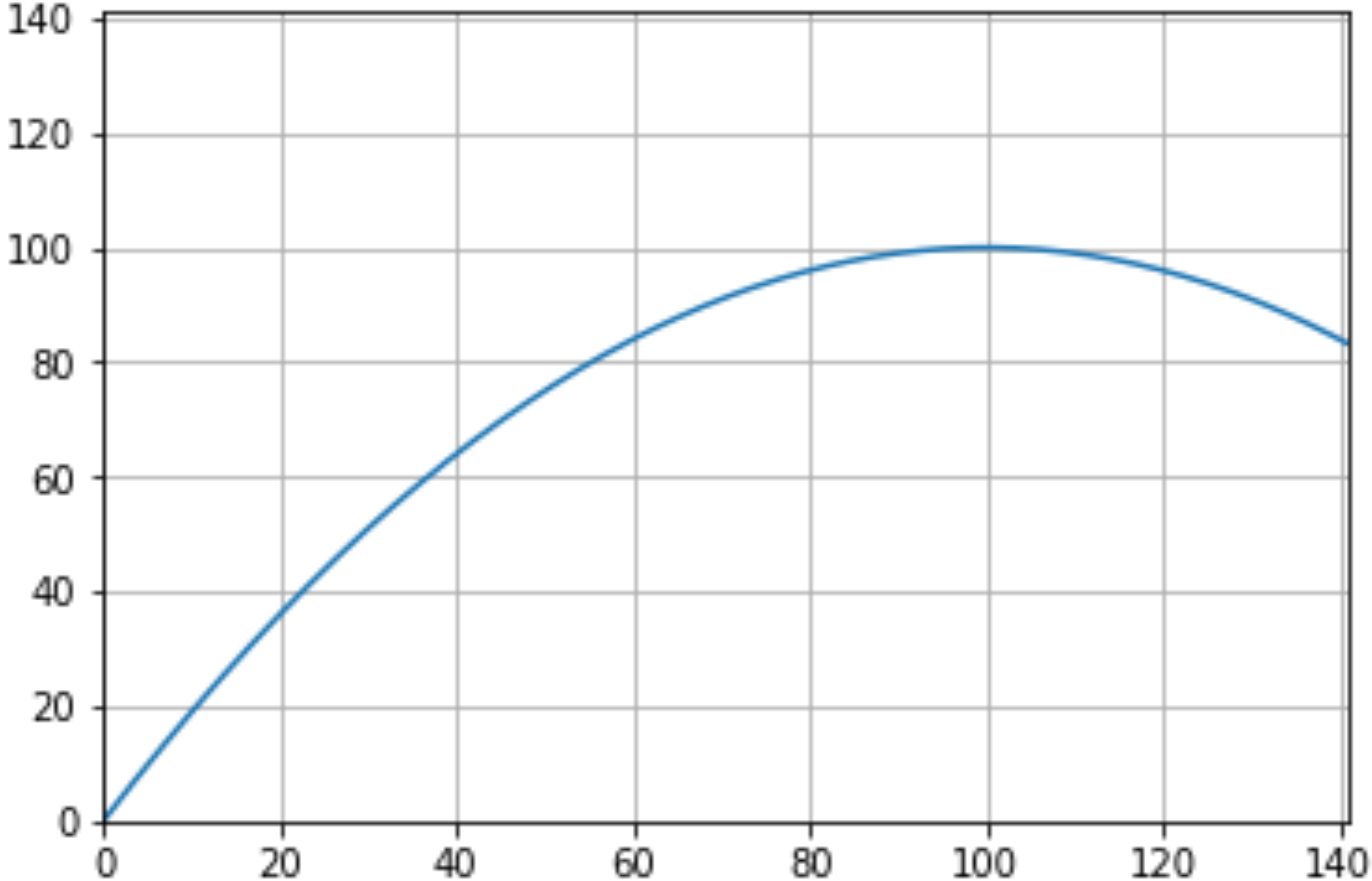
ΔP



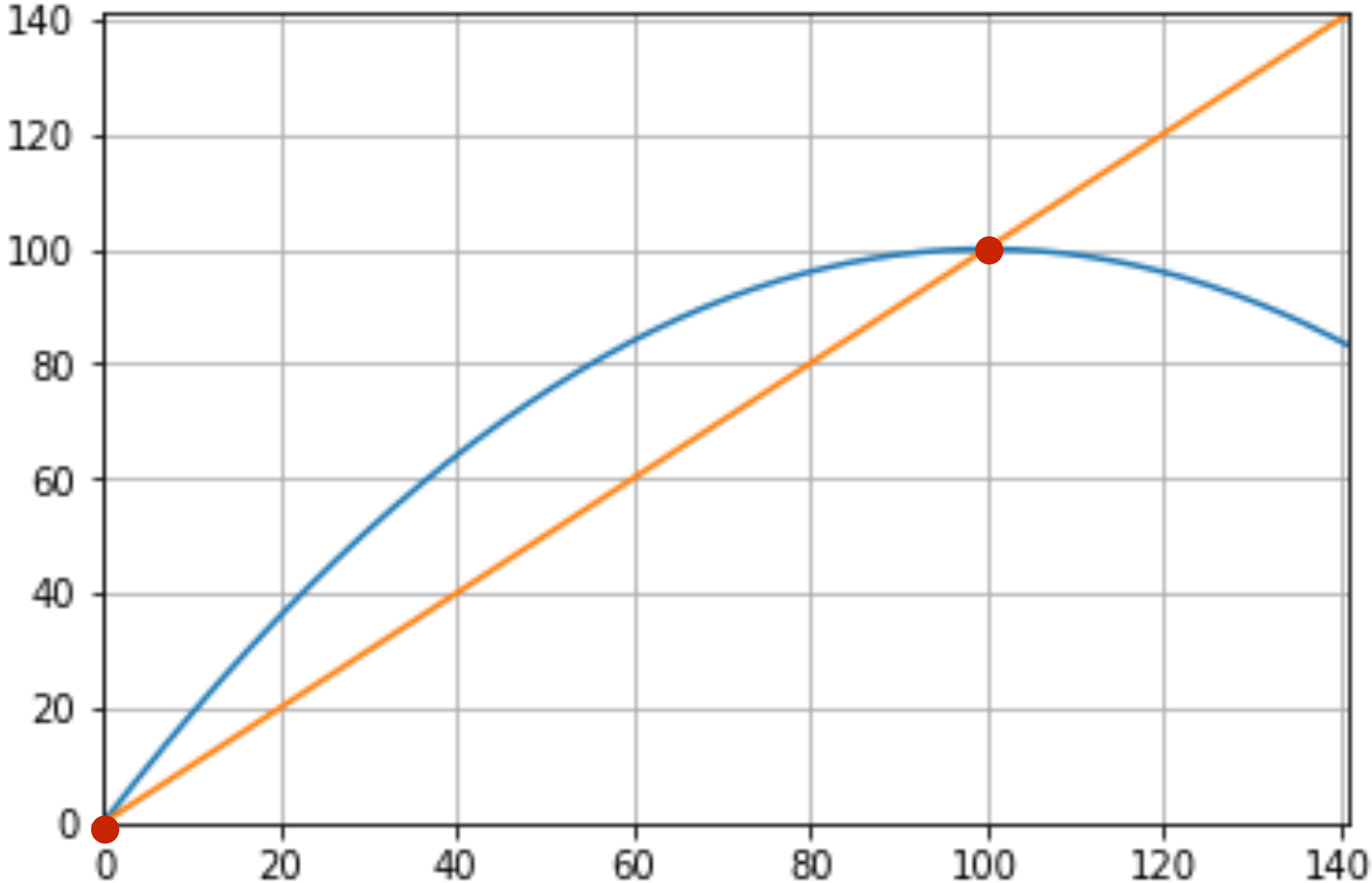
$P_{t+1} = f(P_t)$



$$P_{t+1} = f(P_t)$$



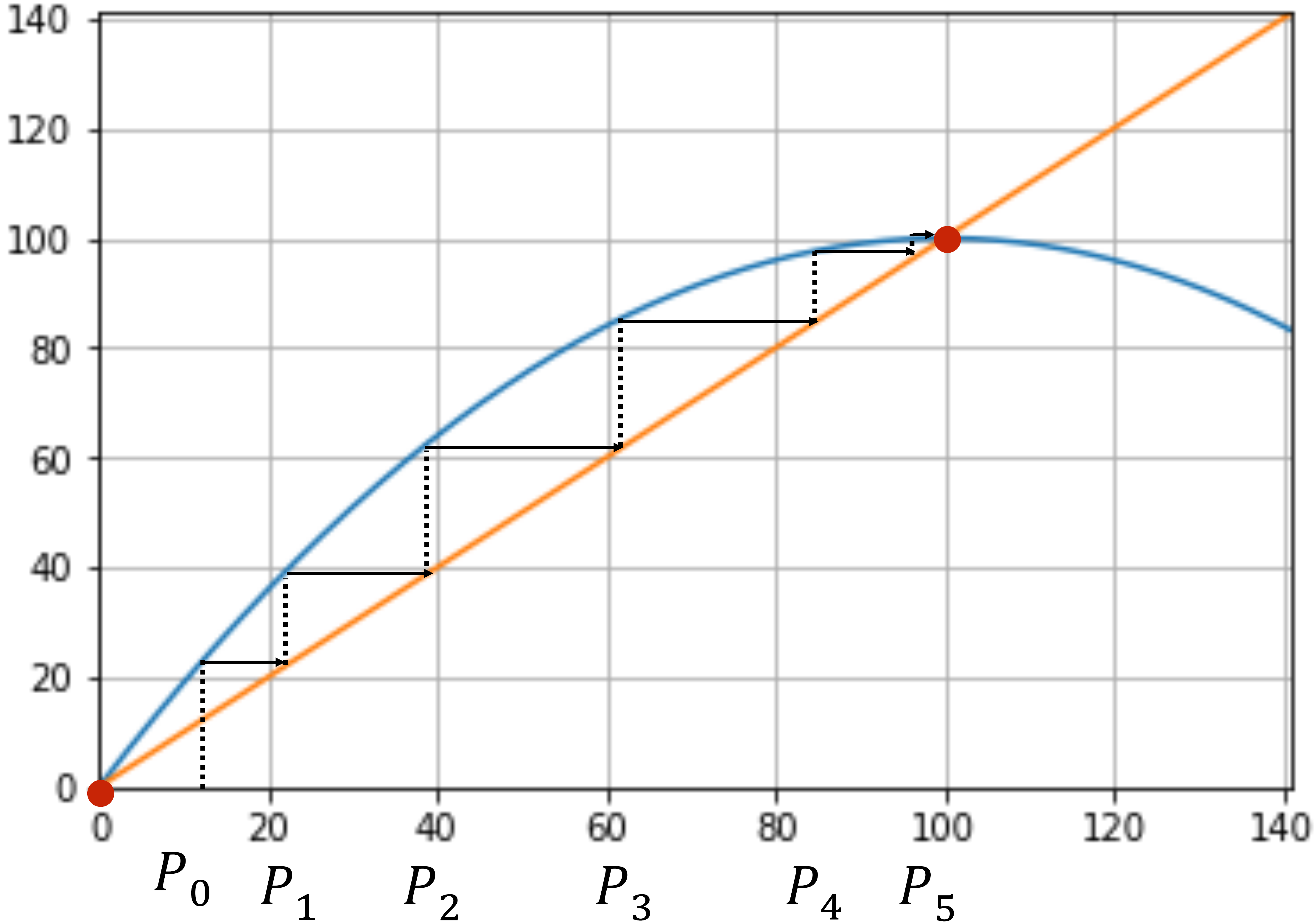
$$P_{t+1} = f(P_t) \qquad P_{t+1} = P_t$$



Steady states

Cobwebbing

$P_{t+1} = f(P_t)$ $\quad$ $P_{t+1} = P_t$

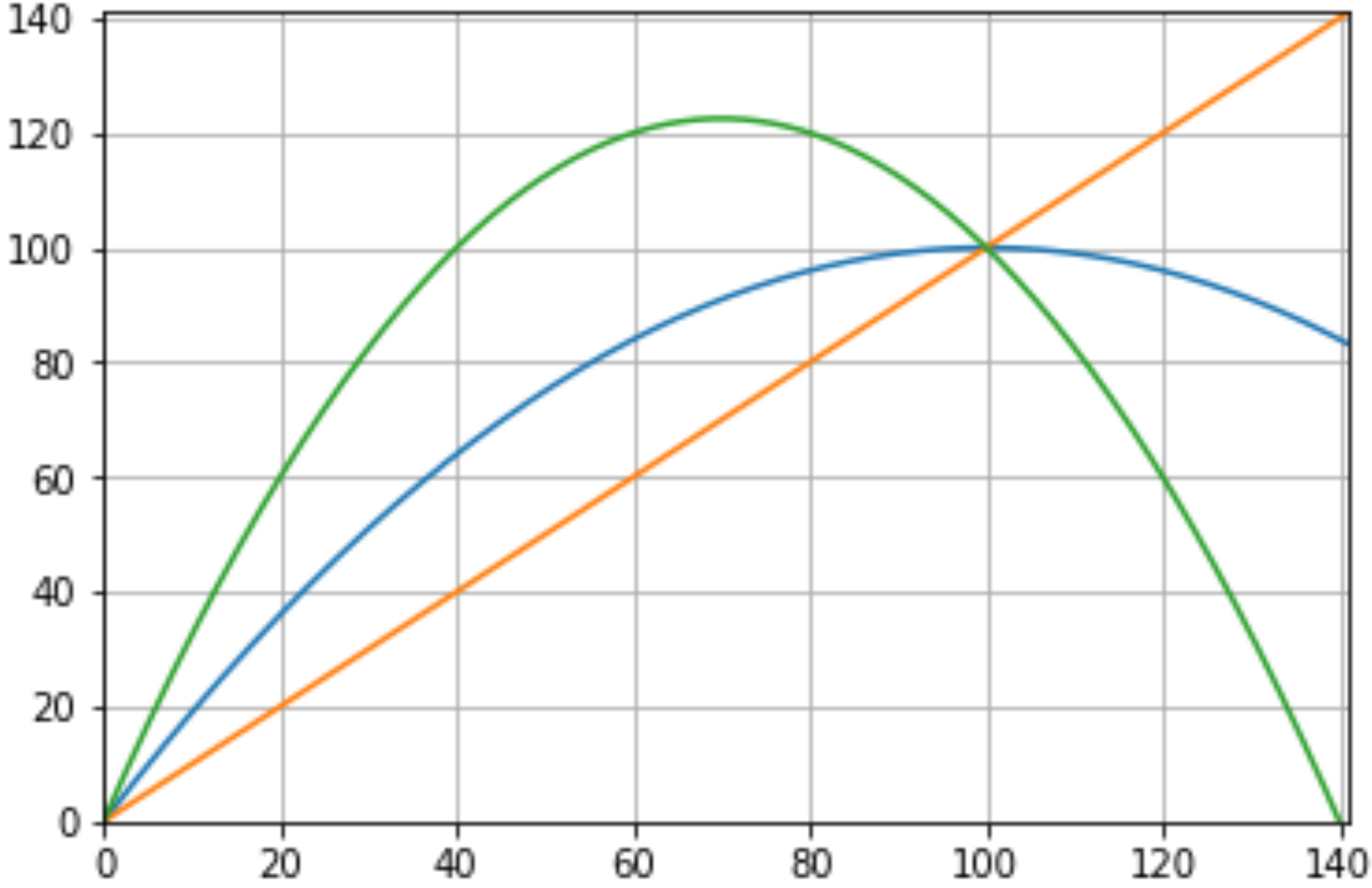


$r = 2.5$

$r = 1$

$P_{t+1} = f(P_t)$

$P_{t+1} = P_t$



$r = 2.5$

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$P_{t+1} = f(P_t)$

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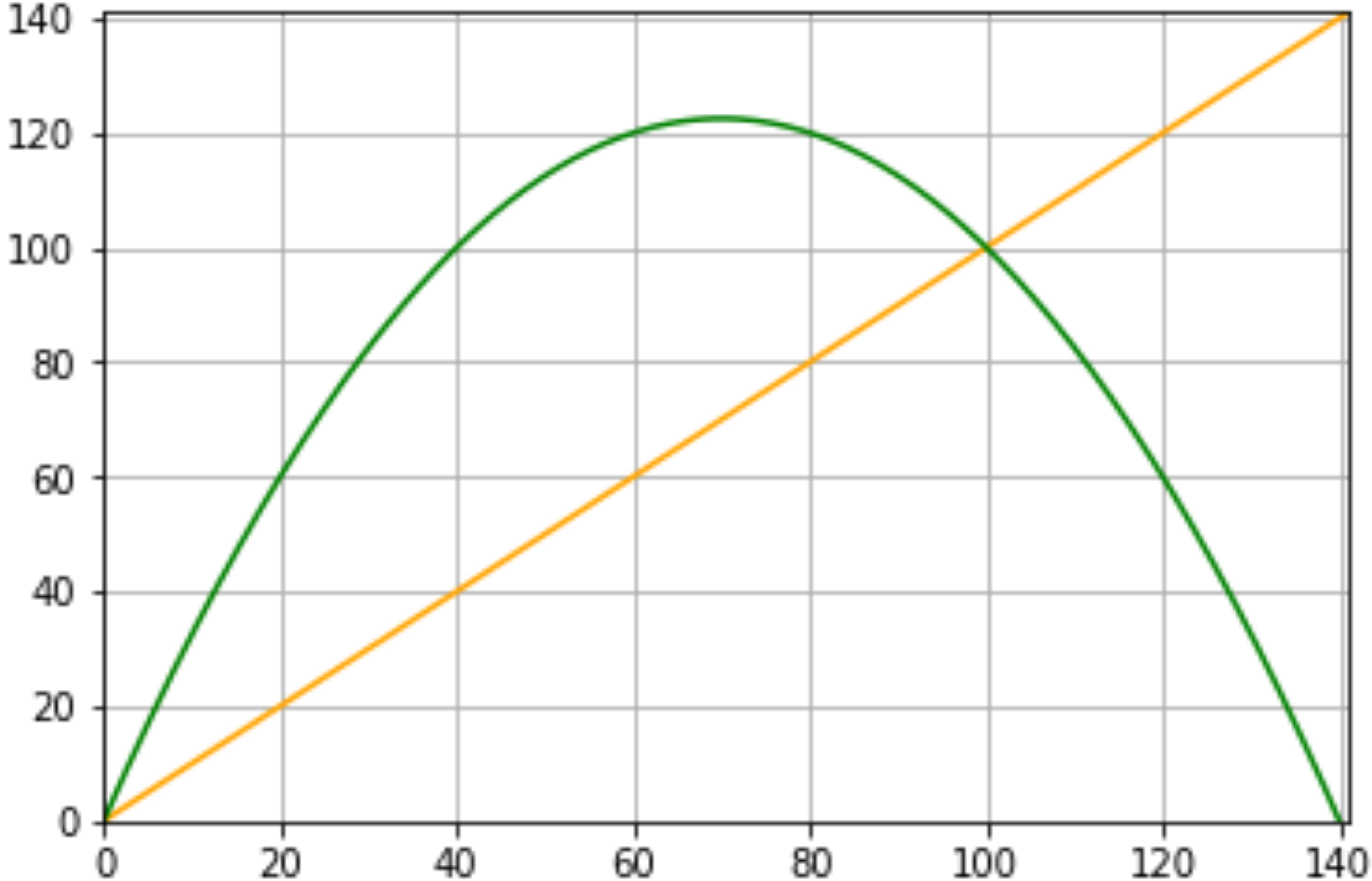
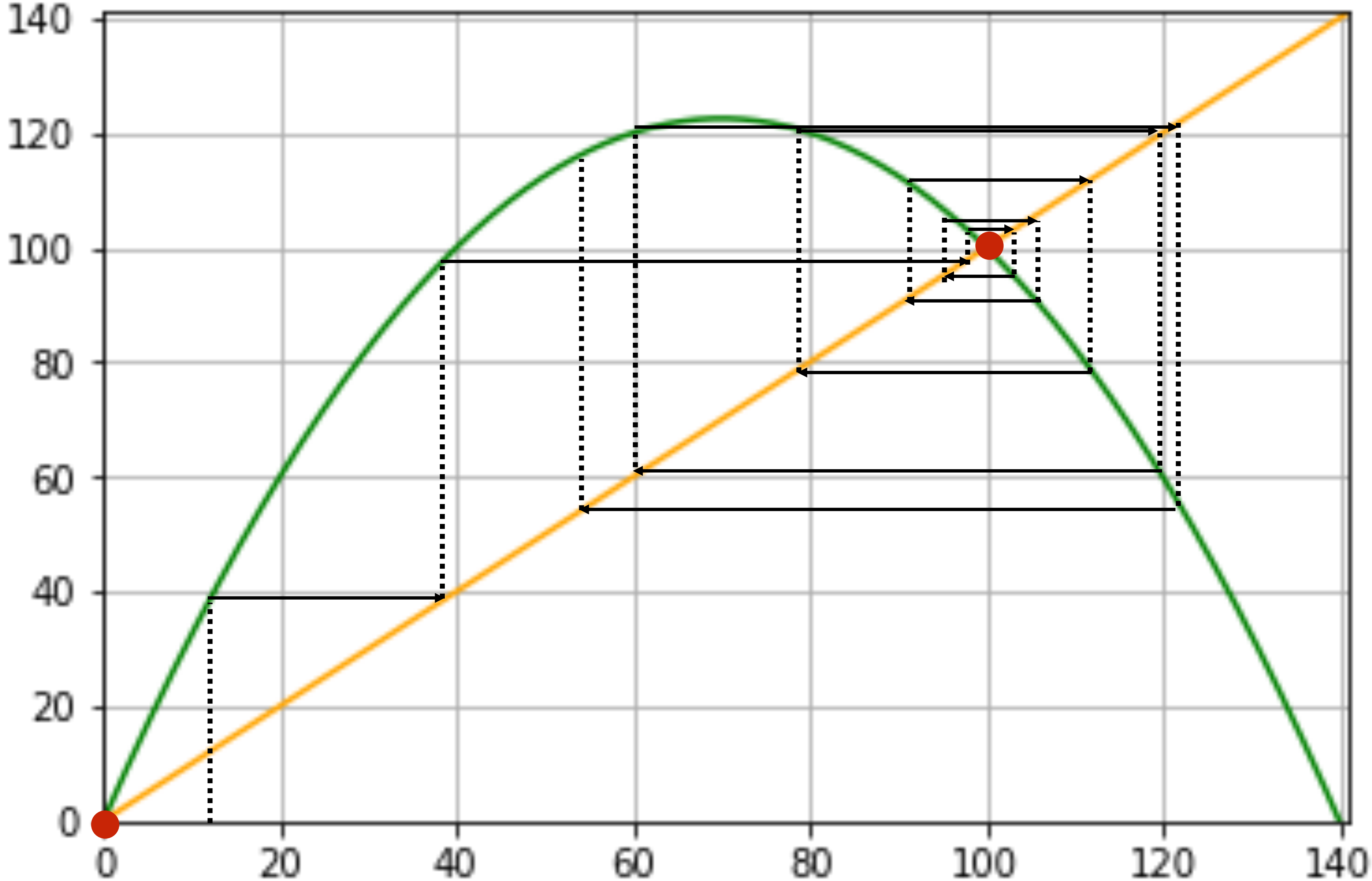


$r = 2.5$ $P_{t+1} = f(P_t)$ $P_{t+1} = P_t$



- For $r > 3$ the equation can yield negative numbers, so it is unsuitable to model population growth
- For $3.75 < r < 4.0$ the system is chaotic. “Very sensitive” to initial conditions.
- Things that *look like* “noise” need not be noise.
- Simple models do not necessarily lead to simple dynamics.

Nature **261** 459–67 (1976)

Simple mathematical models with very complicated dynamics

Robert M. May*

First-order difference equations arise in many contexts in the biological, economic and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behaviour, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random fluctuations. There are consequently many fascinating problems, some concerned with delicate mathematical aspects of the fine structure of the trajectories, and some concerned with the practical implications and applications. This is an interpretive review of them.

Extra reading…

http://abel.harvard.edu/archive/118r_spring_05/docs/may.pdf

Next up: Coupled models