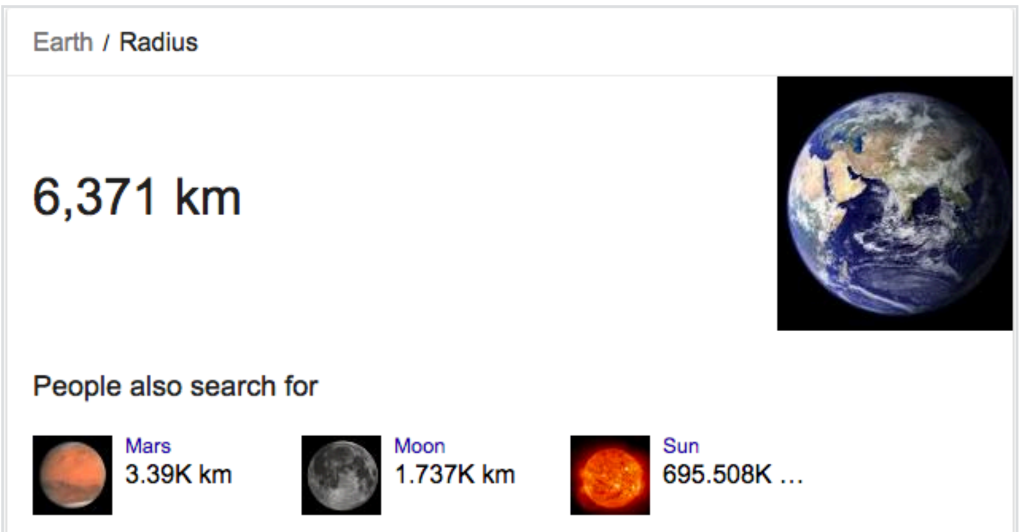
Workshop 3
Sources of Errors

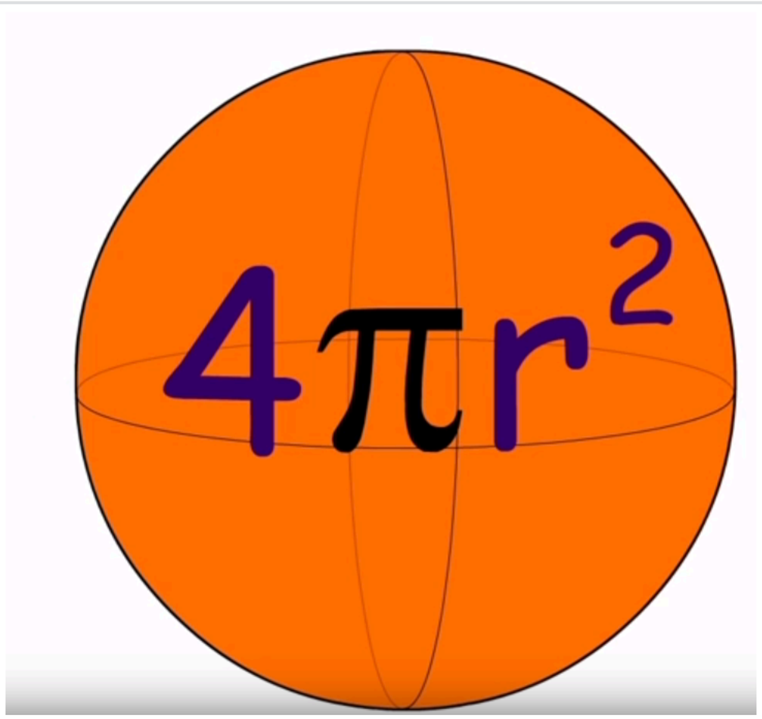
FIT 3139

Computational Modelling and Simulation

MONASH University
Information Technology

What's the surface area of the earth?





https://www.youtube.com/watch?v=6EzQEdBX_30&t=306s

≈ 510064472 km^2

- **Modelling error**: the earth is not a sphere.
- **Data errors**: Measuring earth radius is error prone.
- **Computational errors**: the number π has to be truncated in a computer

Sources of errors…

- **Modelling error**:
  Arise from simplifying assumptions inherent to models.

- **Data errors**:

  Arising from:

  - **Empirical measurements**, which have physical limitations.
  - **Previous computations**, which "propagate" an error

- **Computational Errors**:

  Arising from:

  - **Truncation errors**, e.g, approximate an infinite sum with a finite sum.
  - **Rounding error**, because computers can only cope with finite precision.

Measuring error…

Measure	9999 cm	7 cm
True value	10000 cm	10 cm
Error	**1 cm**	**3 cm**

Errors often need to be put in **context**, which we do by normalising.

Measuring error…

Measure	9999 cm	7 cm
True value	10000 cm	10 cm
Absolute Error	**1 cm**	**3 cm**
Relative Error	(10000-9999) / 10000 **0.01%**	(10-7) / 10 **30%**

Absolute Error

Absolute error = | Approximate - True value |

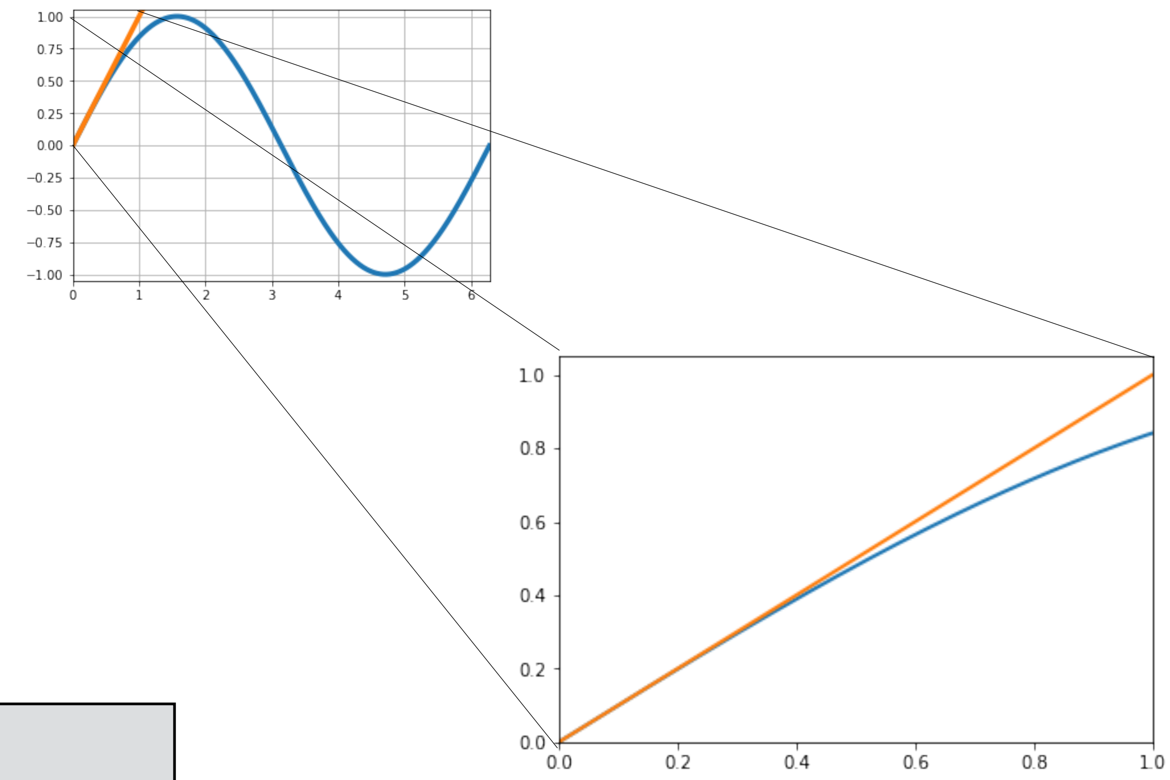
Relative Error

Relative error = (Absolute error / True value) x 100%

Data error vs Computational Error

$$\boxed{\sin(\frac{\pi}{8})}$$

$$f(x) = \sin(x)$$



$$f(x) = x - \frac{x}{3\,!} + \frac{x}{5\,!} - \frac{x}{7\,!} + \ldots$$

$$\tilde{f}(x) \approx x \qquad \pi \approx 3$$

$$\sin(\frac{\pi}{8}) \approx \sin(\frac{3}{8}) \approx \frac{3}{8} = 0.37500$$

Data error: $f(\tilde{x}) - f(x) = \sin(\frac{3}{8}) - \sin(\frac{\pi}{8}) = -0.0164$

$+$

Computation Error: $\tilde{f}(\tilde{x}) - f(\tilde{x}) = \frac{3}{8} - \sin(\frac{3}{8}) = 0.0087$

Total Error: $\tilde{f}(\tilde{x}) - f(x) = \frac{3}{8} - \sin(\frac{\pi}{8})$

$= 0.37500 - 0.38267 = -0.0077$

Total error = Data error + Computational Error

- Let f be a true function, and x a true value.
- Let $\tilde{f}$ be an approximation of f and $\tilde{x}$ be an inexact measurement of x

$$\text{Total error} = \tilde{f}(\tilde{x}) - f(x)$$

$$= \underbrace{\tilde{f}(\tilde{x}) - f(\tilde{x})}_{\text{Computational Error}} + \underbrace{f(\tilde{x}) - f(x)}_{\text{Data Error}}$$

Forward and Backward Error

Forward error: Difference between **computed** and **true value**

$$f(x) = \sqrt{x} \qquad \tilde{y} = 1.4 \qquad y = \sqrt{2} = 1.41421...$$

$$|\Delta y| = |\tilde{y} - y| = |1.4 - 1.41421...| \approx 0.0142$$

Backward error:
Discrepancy in the input that would lead to the observed discrepancy in output

$$|\Delta x| = |\tilde{x} - x| \quad = |1.96 - 2| = 0.04$$

$\tilde{x} = f^{-1}(\tilde{y})$ ← Input $\tilde{x}$ for which f produces $\tilde{y}$

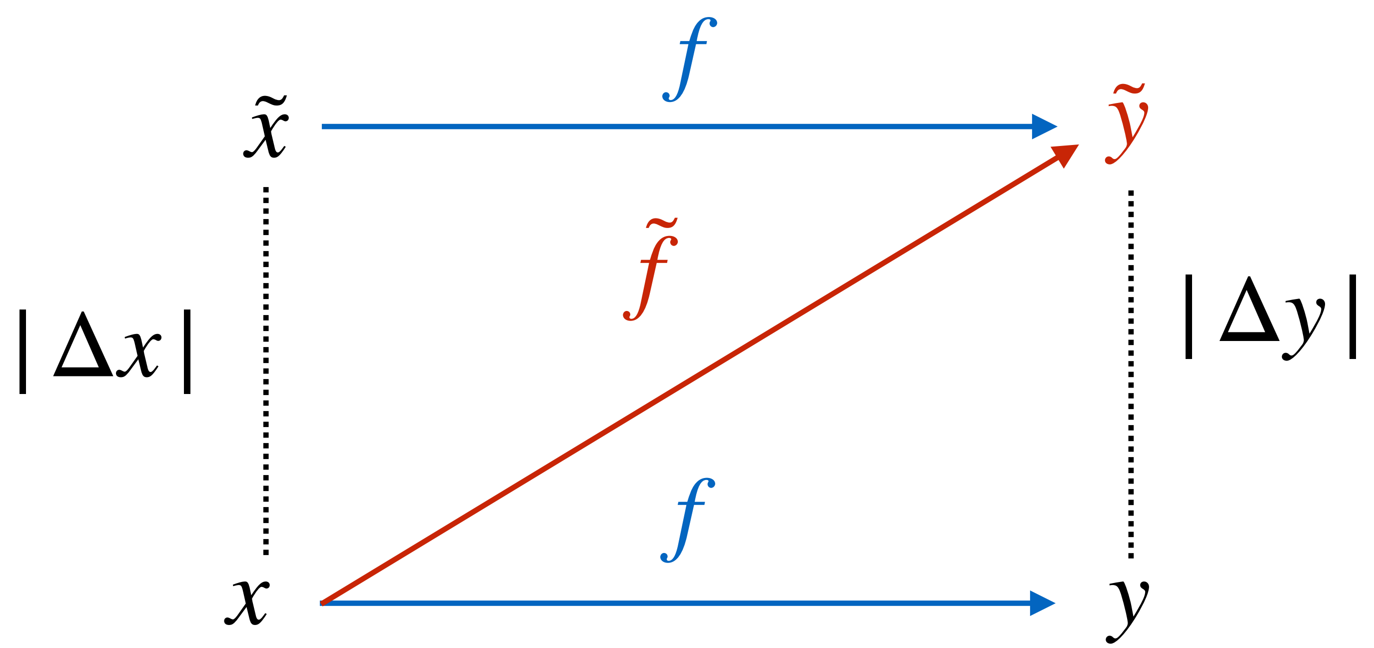
$$\sqrt{\tilde{x}} = 1.4$$

$$\tilde{x} = 1.4^2 = 1.96$$

Forward and Backward Error

Forward error: Difference between **computed** and **true value**

Backward error:
Discrepancy in the input that would lead to the observed discrepancy in output



Backward **Forward**

Some functions deal better than others…

with propagation of error

Sensitivity: A qualitative statement of propagated data error.

Conditioning: Quantitive measure of sensitivity.

$$\overset{\text{changes}}{\textbf{INPUT}} \longrightarrow \overset{\text{changes}}{\textbf{OUTPUT}}$$

A problem is **insensitive** or **well-conditioned** if a given relative change in the input data causes a reasonably commensurate **change in the solution.**

$$\text{Condition number} = \left|\frac{\Delta y}{y}\right| \Big/ \left|\frac{\Delta x}{x}\right| = \frac{\text{Relative forward error}}{\text{Relative backward error}}$$

For an ill-conditioned problem, condition number >> 1

Approximating the condition number

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\left|\frac{\Delta y}{y}\right| = \left|\frac{\tilde{y} - y}{y}\right| \approx \frac{f'(x)\Delta x}{f(x)} \longrightarrow$$ relative forward error

$$\left|\frac{\Delta x}{x}\right| \longrightarrow$$ relative backward error

$$\frac{\left|\dfrac{f'(x)\Delta x}{f(x)}\right|}{\left|\dfrac{\Delta x}{x}\right|} \approx \left|\frac{x f'(x)\Delta x}{f(x)\Delta x}\right| = \left|\frac{x f'(x)}{f(x)}\right|$$

Stability is to an algorithm
what **conditioning** is to a problem…..