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Workshop 13
Intro to Stochastic Models: Montecarlo

FIT 3139 Computational Modelling and Simulation

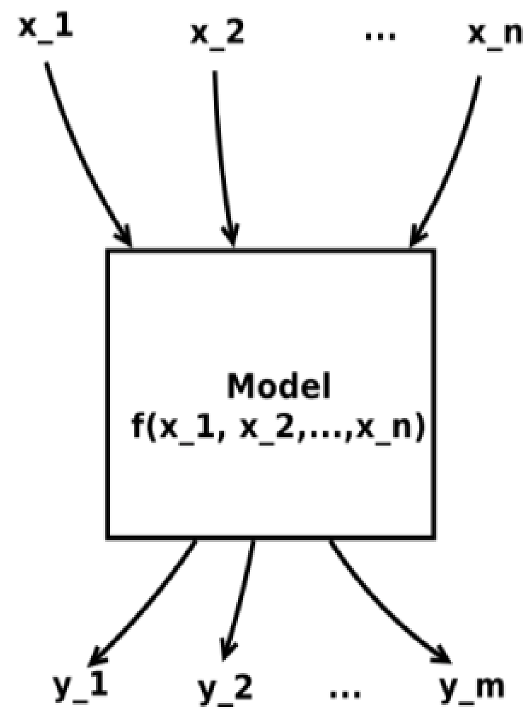
MONASH University
Information Technology

Outline

- Basic elements of probability
- Buffon needle problem:
  - Axiomatic
  - Montecarlo
- (Pseudo) random numbers

Deterministic models

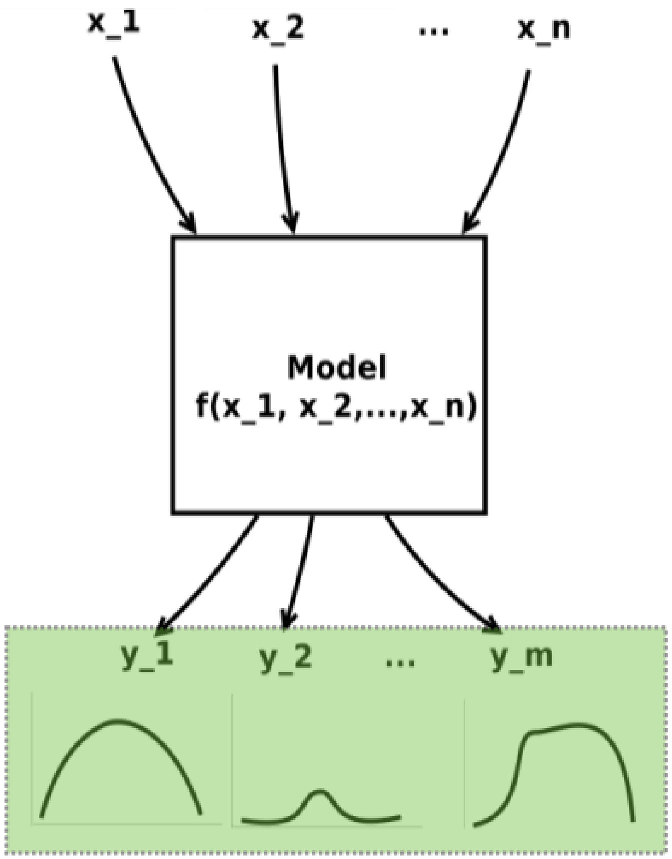
- No stochastic (i.e., random) components.
- Every state of the variables is *uniquely* determined by previous states of the variables.
- A set of input **always** leads to same outputs.
- Same initial conditions, same output



Stochastic models

- Inherent randomness
- State of variables are described by probability distributions, not unique values.
- Same initial conditions lead to different outcomes.

- **Stochastic Simulations**



An **experiment**

Let’s thing of something that has a “random” outcome…

(and think of some useful definitions on the way) https://commons.wikimedia.org/wiki/File:Military_cyclists_in_pace_line.jpg

An **experiment**

$S = \{$ all possible orderings of $(1,2,3,4,5,6,7)\}$

sample space

$(4,1,2,5,3,7,6)$ → **outcome**

of the random experiment

$A = \{$all outcomes starting with 4 $\}$ → **event**

Any subset of the sample space is an event.

For any two events A and B, the event $A \cup B$ consists of all events in either A or B. It's called the **Union**

$\bigcup_{i=1}^{n} A_i$ it's the union of all outcomes in any of $A_1, A_2, \ldots A_n$

Similarly for **intersections** of events.

Axioms of probability

Consider an **experiment** with **sample space** S, for each event A, there is a number $P(A)$, called **the probability** of the event A, which is in accord with the following axioms:

$0 \le P(A) \le 1$	the probability that the outcome of the experiment lies in A is some number between 0 and 1
$P(S) = 1$	with probability 1 the outcome of the experiment is in the sample space
$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$ $i = 1,2\ldots,\infty$	for any sequence of **mutually exclusive** events $\{A_i\}$ the probability that at least one of these events occurs is equal to the sum of their respective probabilities

These axioms, can be used to prove a variety of results….

Example:

$$S = A \cup A^c$$

$$1 \underset{\text{axiom 2}}{=} P(S) = P(A \cup A^c) \underset{\text{axiom 3}}{=} P(A) + P(A^c)$$

$$P(A^c) = 1 - P(A)$$

"The probability that an event does not occur is 1 minus the probability that it does"

Empirical probability

If a **random experiment** is repeated $\underline{n \text{ times}}$ and n_a is the number of times an event E occurs, then the relative frequency of occurrence of event E is $\frac{n_a}{n}$.

This relative frequency converges to the probability of E as $n \to \infty$:

$$Pr(E) = \lim_{n \to \infty} \frac{n_a}{n}$$

This is the basis of the Montecarlo technique, to be discussed..

Experiment

Throw two dice.

$$S = \{(d_1, d_2), 0 \leq d_i \leq \ 6, d_i \in \mathbb{Z}, i = 1,2\}$$

Say I am interested, in the sum of the dice X

X is a random variable

Random variable

A **quantity** of interest determined by the result of an experiment.

We say a random variable is **discrete**, if it can take a finite (or at most countable) number of values.

$$p(x) = P\{X = x\} \longrightarrow \text{Probability mass function}$$

[for a discrete RV]

We say a random variable is **continuous**, if the set of possible values is an interval.

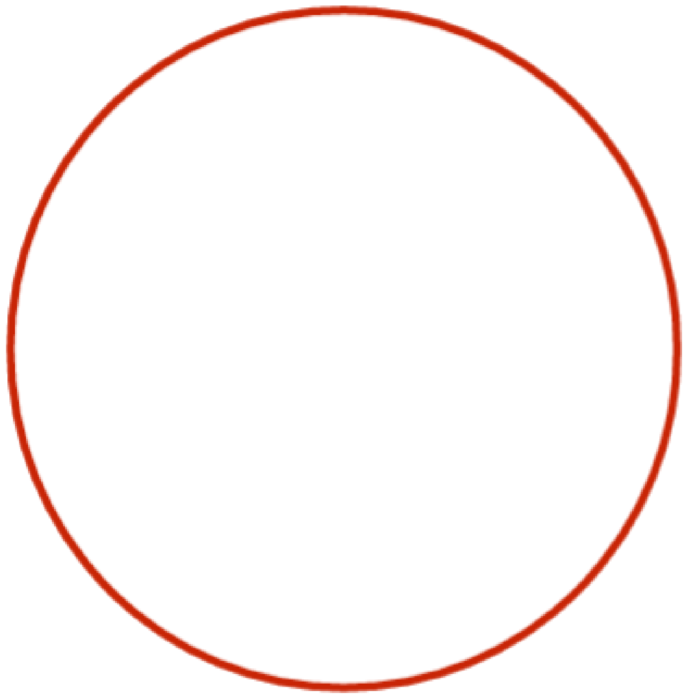
$$f(x) \longrightarrow \text{Probability density function}$$

$$P\{X \in C\} = \int_C f(x)dx$$

[for a continuous RV]

Monte Carlo method

- An (approximate) method of simulation based on **random sampling**.
- Typically used to attack some **intractable models**, integrals, or sums.
- Initially used in 1930's by Enrico Fermi (*neutron difussion*), popularised by Stanislaw Ulam in the 1940's at Los Alamos Nat Lab.
- **Computers** make this method extremely relevant.
- **Used extensively** in mathematical, computational, statistical modelling



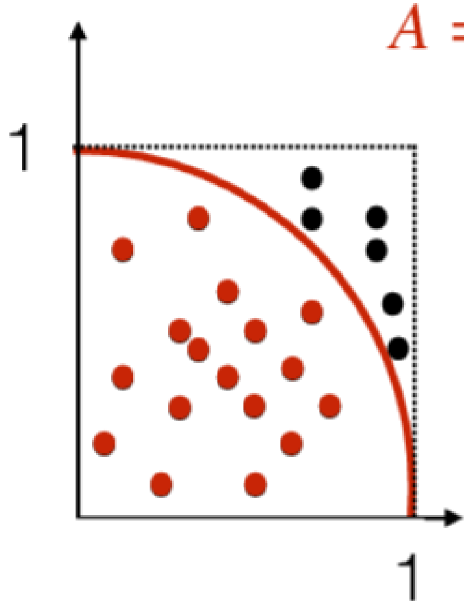
$A = \pi r^2$

Use Montecarlo to estimate $\boldsymbol{\pi}$

Use Montecarlo to estimate π

$$A = \pi r^2$$

$$A = \frac{\pi}{4}$$



throw *many* random points

ratio of inside/total samples
must equal $\pi/4$

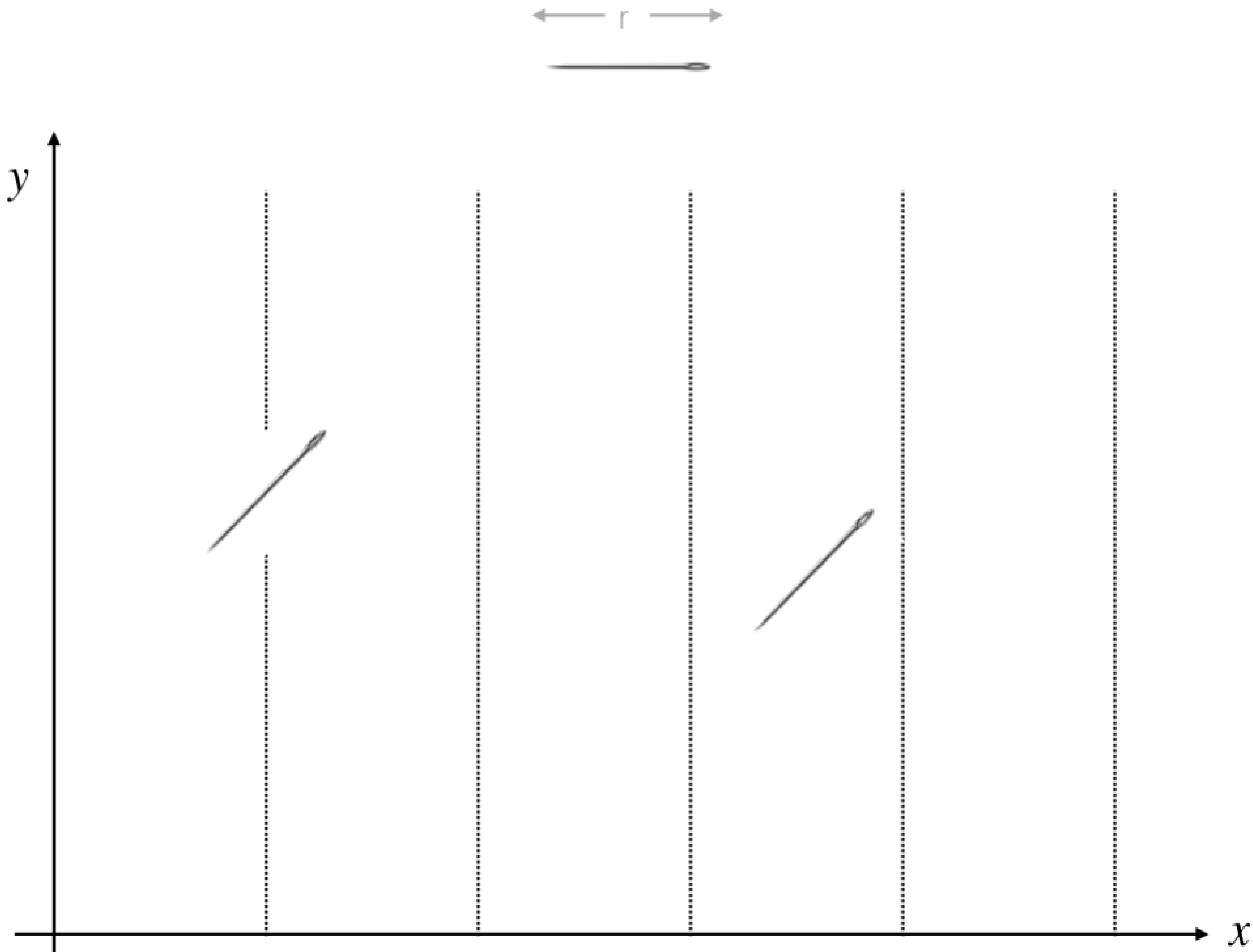
Montecarlo method:
1. Define a domain for the samples.
2. Generate samples (using a Random Number Generator).
3. Apply a deterministic computation on the samples.
4. Aggregate the results.

https://en.wikipedia.org/wiki/Georges-Louis_Leclerc,_Comte_de_Buffon

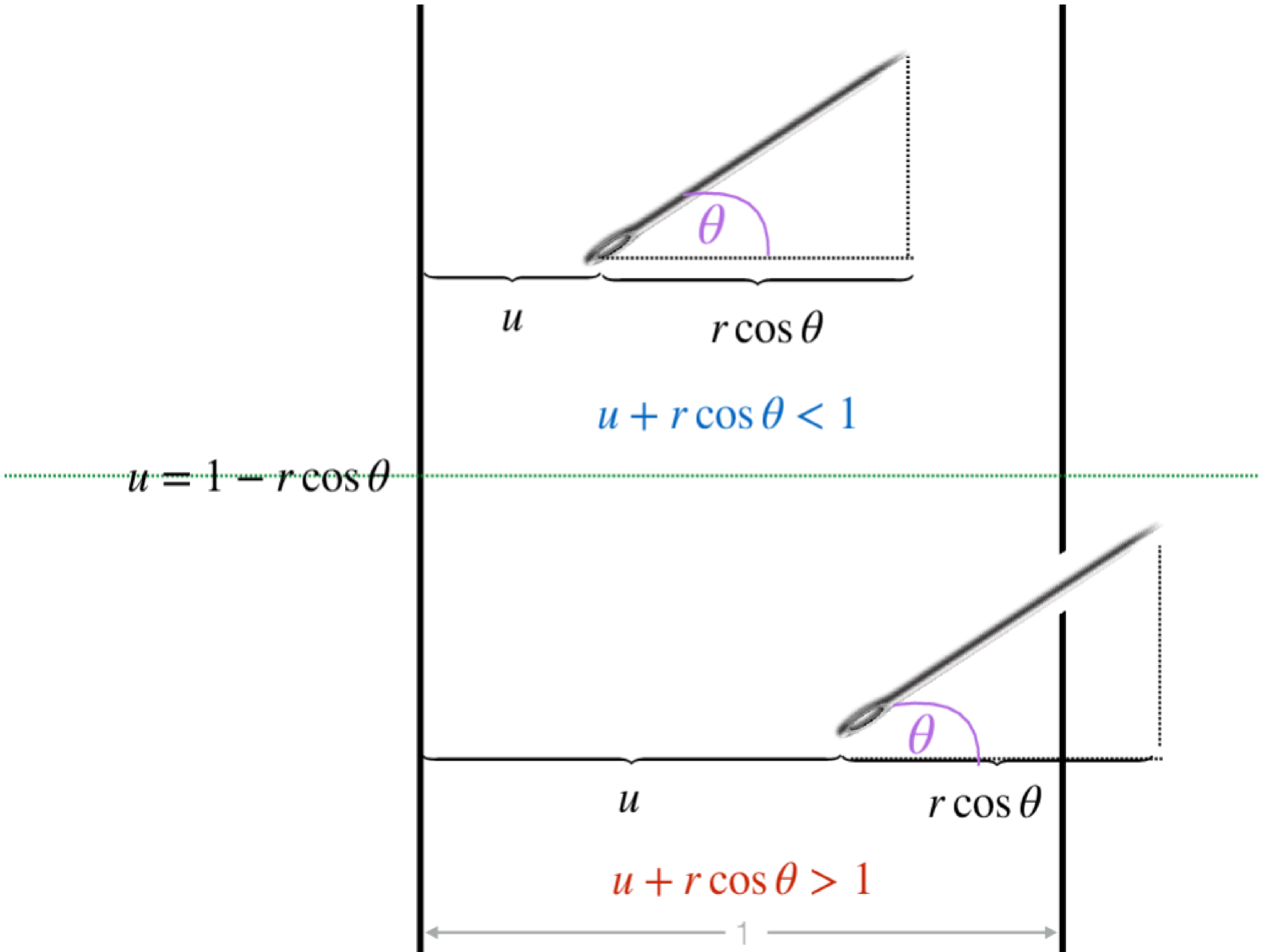
Georges-Louis Leclerc, Comte de Buffon

Buffon's needle

An infinite family of long vertical lines are spaced one unit apart in the (x,y) plane. If a needle of length $r > 0$ (and negligible width) is dropped at random onto the plane, what is the probability that it will land crossing at least one line?

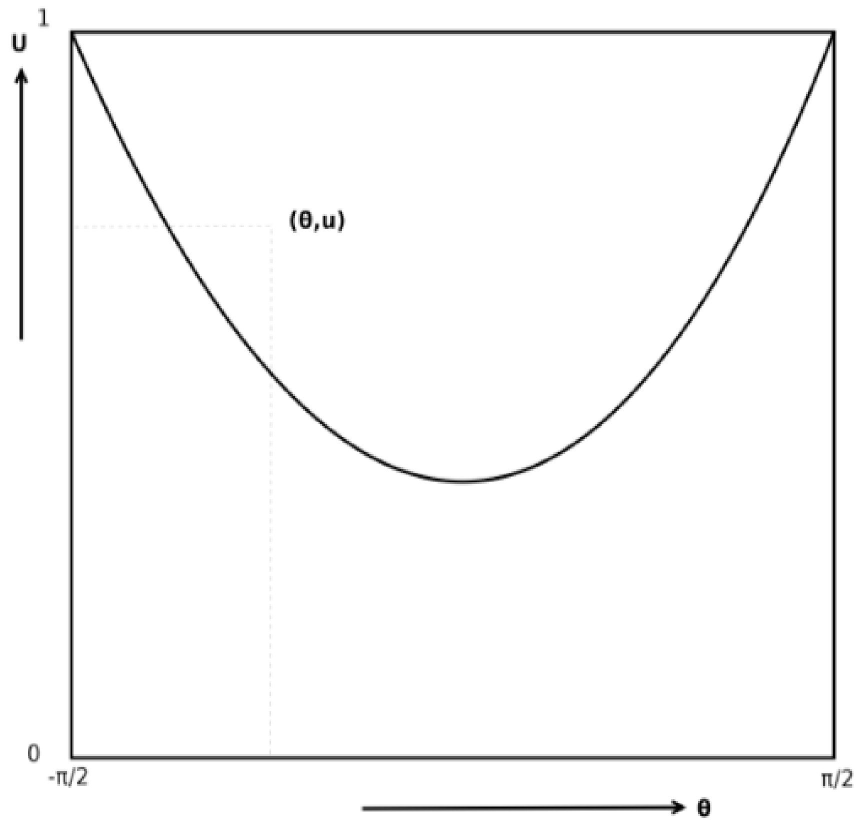


r
y
x

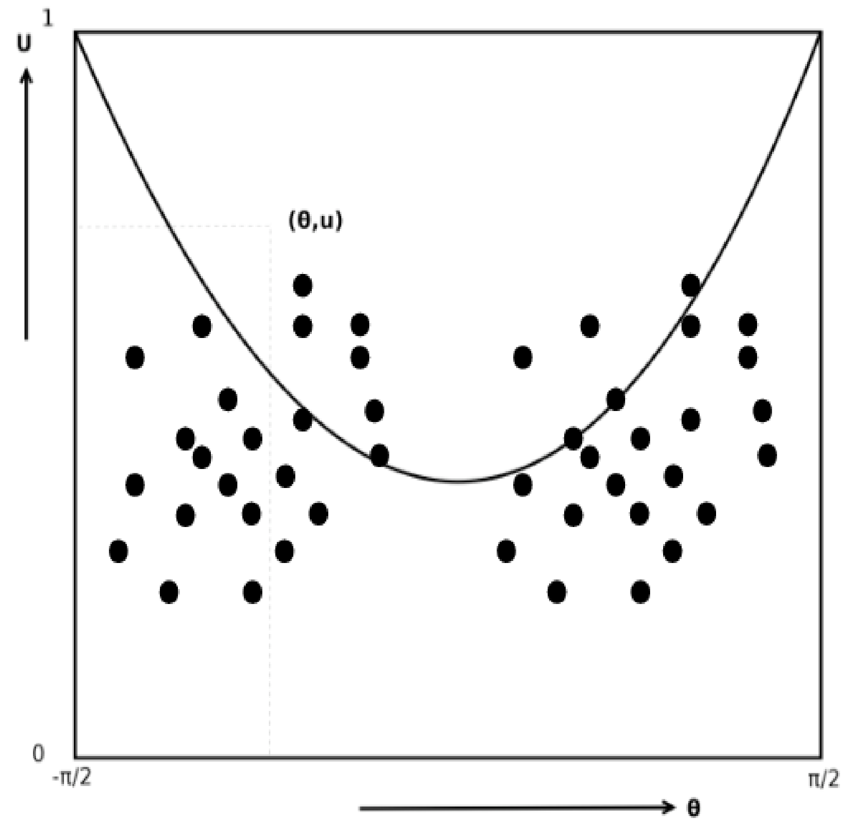

θ
u
$r\cos\theta$
$u + r\cos\theta < 1$
$u = 1 - r\cos\theta$
θ
u
$r\cos\theta$
$u + r\cos\theta > 1$
1

Sample space

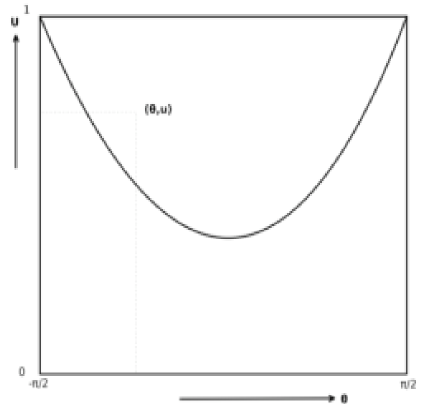
$$u = 1 - r\cos\theta$$



Montecarlo simulation



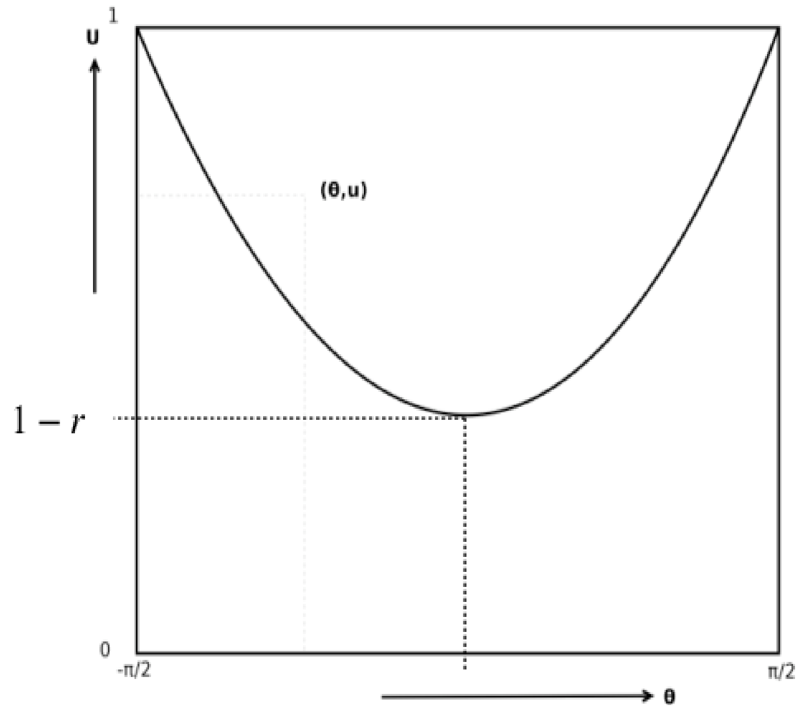
Montecarlo simulation

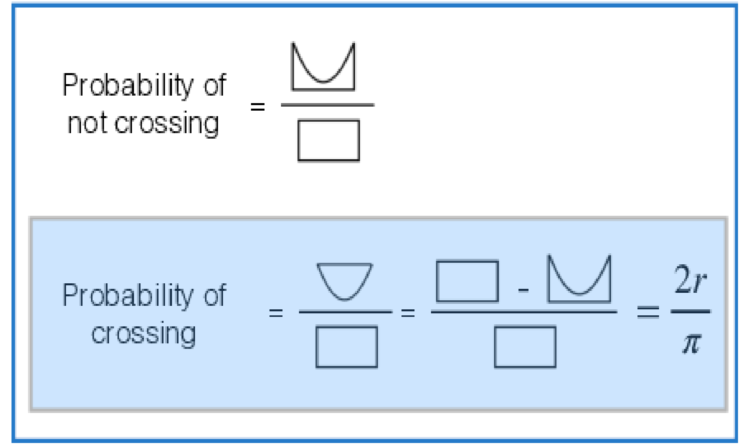


- Randomly sample: $(\theta_1, u_1), (\theta_2, u_2), \ldots, (\theta_n, u_n)$

- If some (θ_i, u_i) falls above the curve (i.e., $u_i > 1 - r\cos\theta$) call it a **hit**, otherwise a **miss**.

- Probability of the needle <u>crossing</u> can be approximated as number of **hits** over number of samples (n)





$$= \pi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 - r\cos\theta d\theta$$

$$= \theta - r\sin\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \pi - 2r$$

How do we generate random numbers?

https://www.random.org/