FIT3139: Applied exercises for Week 7

Question 1

Consider the guerrilla warfare model discussed in the lectures

$$\frac{dx}{dt} = -ay$$
$$\frac{dy}{dt} = -bxy$$

for positive constants, a and b, where x is the number of home soldiers and y is the number of enemy soldiers.

- Use phase plane analysis to determine if (and when), it is possible to send a specific number of enemy soldiers y_0, that will guarantee a target level of casualties in the enemy army.
- What factors determine the number of casualties in the winning army.

Question 2

Implement a function that solves a differential equation using a Runge-Kutta 2 method. The function should take as input the differential equation, initial conditions, step size, number of iterations, and a variable b which will be used to determine what kind of RK2 method is used.

Use this function to solve the problem:

$$\frac{dy}{dx} = xy$$

with $y(0) = 1$ and a step size of $h = 1$, over the domain $x \in [0, 10]$.

Question 3

Adapt your code from question 2 so that it can be used to solve coupled dynamical systems and use this function to solve the problem:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

Use the parameters $\sigma = 10, \beta = 8/3$, and $\rho = 28$ and the initial condition $x(0) = y(0) = z(0) = 1$.

Plot the solution of each plot over time or generate a 3D line plot with the axes x, y, and z. In MATLAB, this can be done using the plot3 function and in Python this can be done using matplotlib.

Question 4

Write the schema for approximating the dynamics of the guerrilla warfare model discussed in Question 1 using Euler's method and Heun's method.

Question 5: Optional Extension

As already discussed, Euler's method take the following iterative form:

$$\begin{aligned} y_{i+1} &= y_i + hf(x_i, y_i) \\ x_{i+1} &= x_i + h. \end{aligned}$$

This method is sometimes referred to as Euler's Forward method and uses the gradient at the point (x_i, y_i) to approximate the value at y_{i+1}. An alternative method is to use the gradient at the point (x_{i+1}, y_{i+1}) to approximate the value y_{i+1}. Doing so gives the following iterative form:

$$\begin{aligned} y_{i+1} &= y_i + hf(x_{i+1}, y_{i+1}) \\ x_{i+1} &= x_i + h \end{aligned}$$

This is known as Euler's Backward Method.
For the differential equation in question 3, find an expression for y_{i+1} in terms of y_i, x_i, and h.
Repeat for the differential equation

$$\frac{dy}{dx} = \sin(y).$$

What issue arises? How can this be overcome? Roughly compare the computational time of this method to Euler's Forward method. What possible reasons may lead us to using this method over Euler's Forward method?