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Workshop 11
Solving ODEs

- FIT3139: Computational Modelling and Simulation

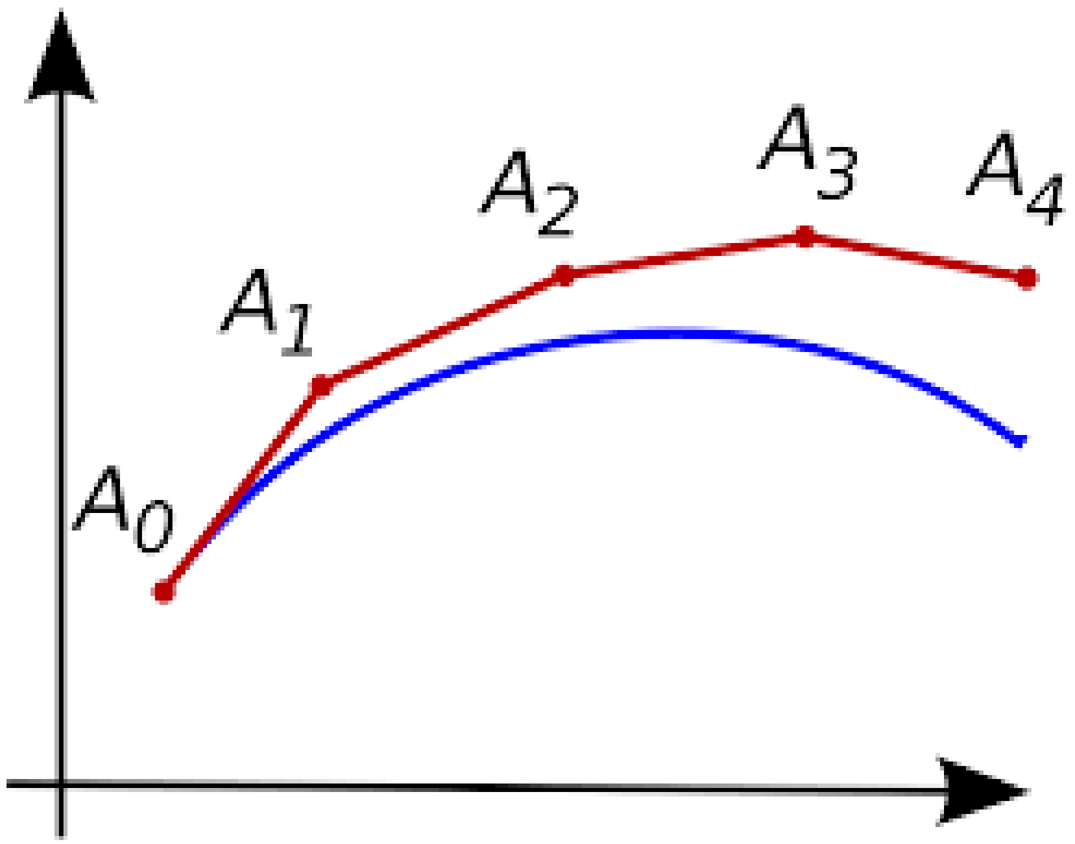
MONASH University
Information Technology

Odeint: Numerical integration

$$\frac{\mathbf{dx}}{\mathbf{dt}} = \mathbf{f}(\mathbf{x}, \mathbf{t})$$

$$x(t) = \int f(x)$$

$$x(0) = x_0$$



Terminology

degree 3

order 1

$$2x^2\left(\frac{dy}{dx}\right)^3 - 3y = 0$$

The **order** of an ordinary differential equation is determined by the order of the highest appearing derivative

The **degree** of an ODE is the power to which the highest-order derivative is raised.

This **ODE**, where the O is for **ordinary**; i.e., no partial derivatives.

Terminology

order 4 *degree* 1

$$5x\frac{d^{\boxed{4}}y^{\boxed{}}}{dx^4} + 2x^2\left(\frac{dy}{dx}\right)^3 - 3y =$$

notational shorthand $y' = \frac{dy}{dx}$ $y'' = \frac{d^2y}{dx^2}$ $y^{(n)} = \frac{d^ny}{dx^n}$

$$5xy'''' + 2x^2(y')^3 - 3y$$

Linear differential equation

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + a_3(x)y''' + \cdots + a_n y^{(n)} + b(x) = 0$$

- $a_1(x)$, $a_2(x)$ ... $a_n(x)$, and $b(x)$ are arbitrary <u>differentiable</u> functions that do not need to be linear.
- Note that derivatives are <u>degree</u> 1.
- A solution, is a function $y(x)$ that satisfies the equation.
- Anything else is non-linear.

An **initial value problem** involves an ODE together with a specified value, called the **initial condition**, of an unknown function at a given point in the domain of the solution.

We'll focus on **first order** initial value problems.

$$y' = \frac{dy}{dx} = f(x, y)$$

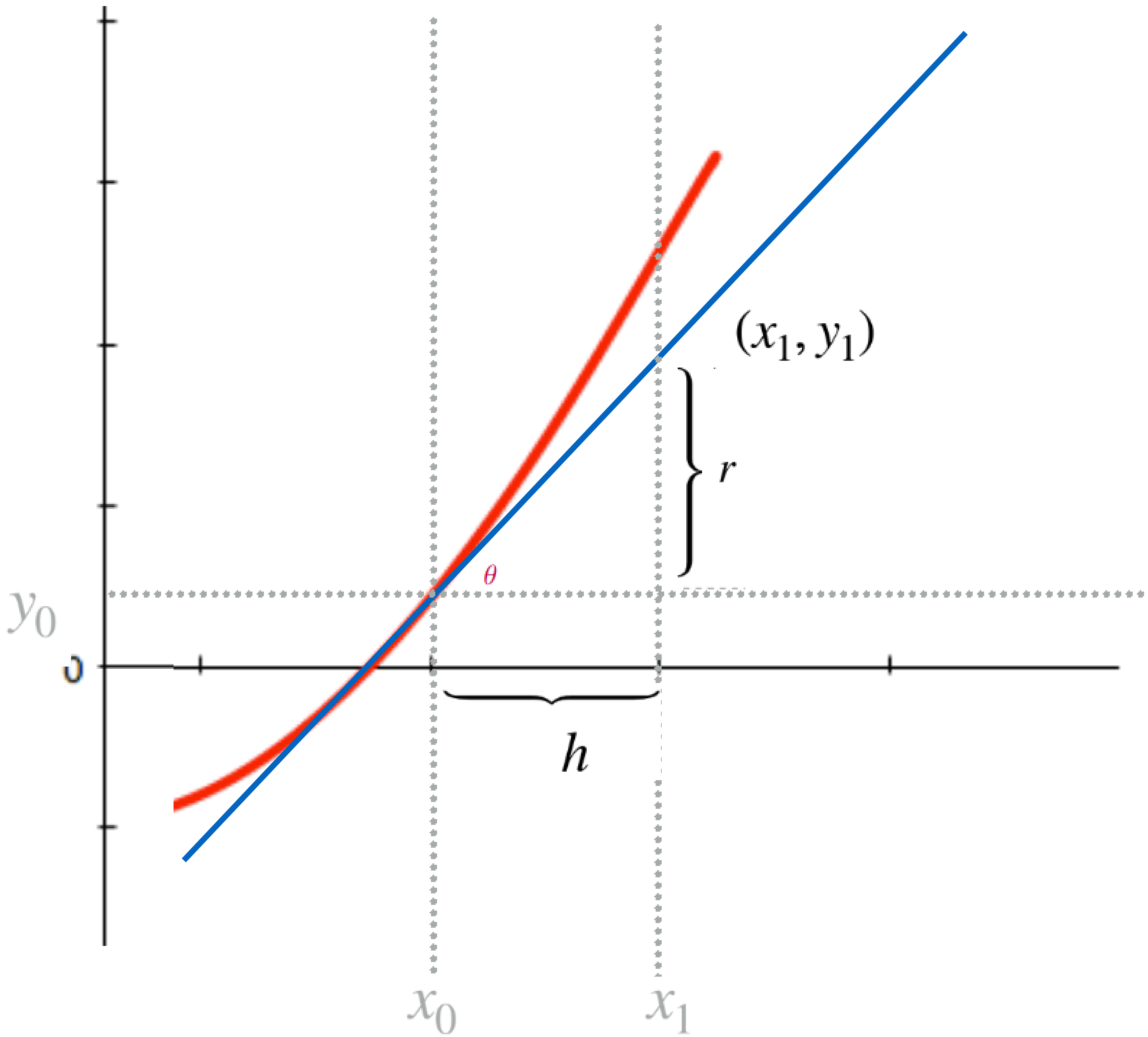
(x_0, y_0) i.e., $y_0 = y(x_0)$

$y(x$ is unknown.

Euler’s method

known…

$y' = f(x, y), (x_0, y_0)$



$y(x)$ is unknown.

$$\tan(\theta) = \frac{r}{h}$$

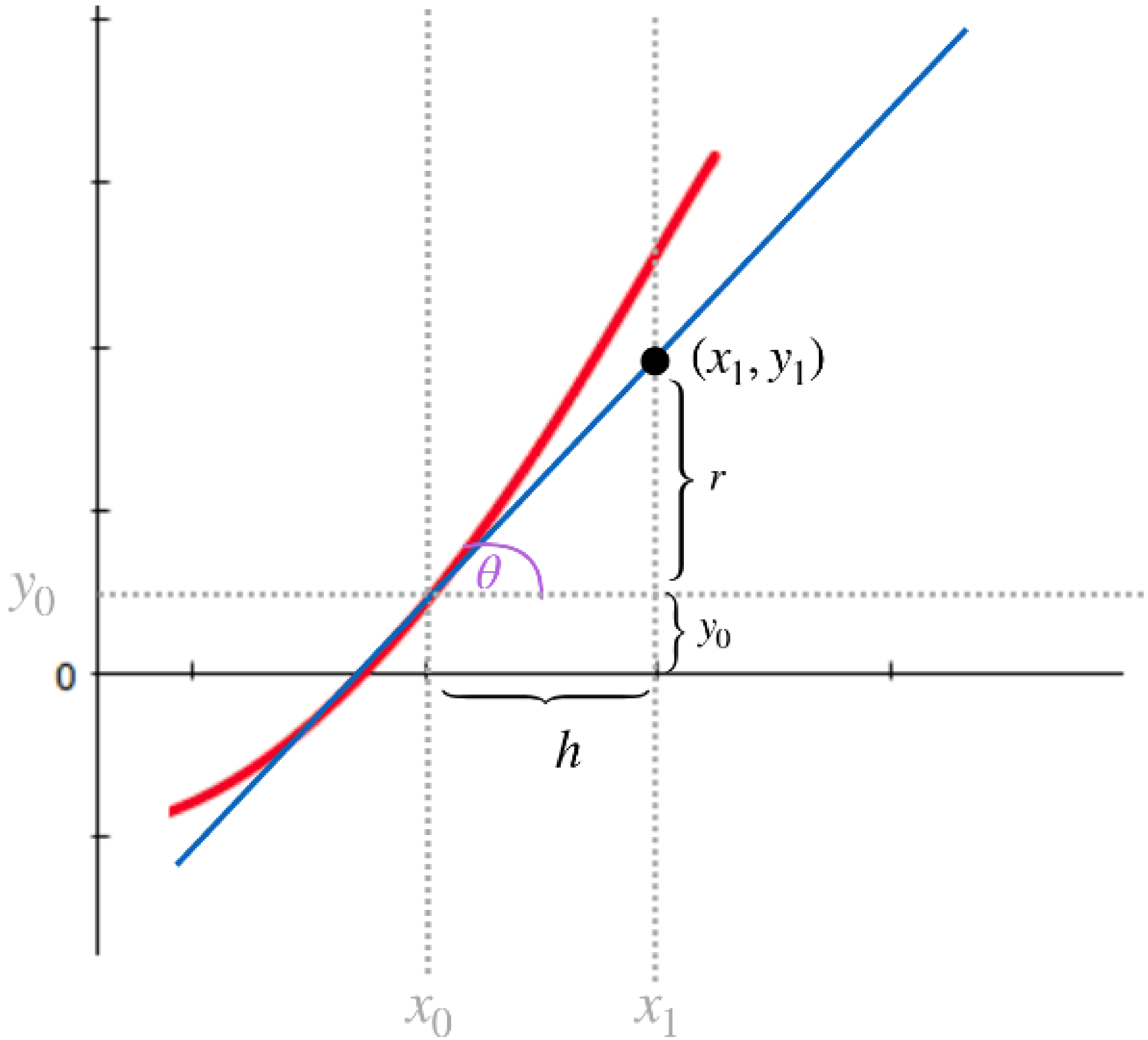
$$f(x_0, y_0) = \frac{r}{h}$$

$$hf(x_0, y_0) = r$$

$$x_1 = x_0 + h$$

$$y_1 = y_0 + r$$

$$y_1 = y_0 + hf(x_0, y_0)$$



$$x_1 = x_0 + h$$
$$y_1 = y_0 + hf(x_0, y_0)$$

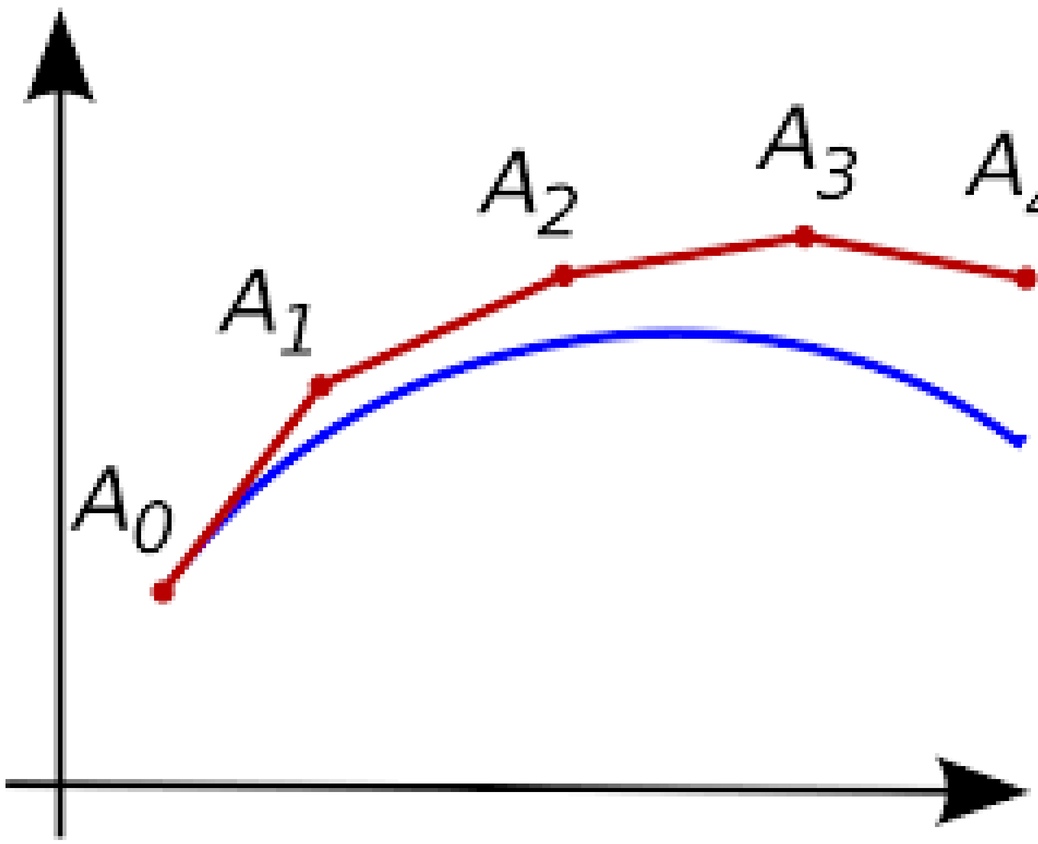
$$x_2 = x_1 + h$$
$$y_2 = y_1 + hf(x_1, y_1)$$

$$x_3 = x_2 + h$$
$$y_3 = y_2 + hf(x_2, y_2)$$

$$\vdots$$

$$x_n = x_{n-1} + h$$
$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

$(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$ approximates $y(x)$


A_0
A_1
A_2
A_3
A_4

Another view…

Remember Taylor…

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + \ldots$$

known…

$$y' = f(x, y)$$

$y(x)$ is unknown.

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + \dots$$

$$y_{i+1} = y(x_{i+1}) = y(x_i + h)$$

Taylor

$$= y(x_i + h) = y(x_i) + hy'(x_i) + \boxed{\cancel{\frac{h^2}{2}y''(x_i) + \dots}}$$

small h

$$y_{i+1} \approx y(x_i) + hy'(x_i)$$

$$y_{i+1} \approx y_i + hy'(x_i)$$

$$\boxed{\begin{aligned} y_{i+1} &\approx y_i + hf(x_i, y_i) \\ x_{i+1} &\approx x_i + h \end{aligned}}$$

$\rightarrow$ Start at initial value and iterate

known…

$$y' = f(x, y)$$

$$(x_0, y_0)$$

$y(x)$ is unknown.

$$\boxed{\begin{aligned} y_{i+1} &\approx y_i + hf(x_i, y_i) \\ x_{i+1} &\approx x_i + h \end{aligned}} \longrightarrow$$

Start at initial value and iterate

Example

$$\frac{dP}{dt} = r\left(1 - \frac{P(t)}{K}\right)P(t)$$

Carl **Runge**

Martin Kutta

Runge-Kutta methods circa 1901

https://en.wikipedia.org/wiki/Martin_Wilhelm_Kutta

https://en.wikipedia.org/wiki/Carl_David_Tolm%C3%A9_Runge

Runge-Kutta methods circa 1901

$$y_{i+1} = y(x_{i+1}) = y(x_i + h) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \dots$$

$$\approx y(x_i) + h\left(\boxed{f(x_i, y_i) + \frac{h}{2}y''(x_i)}\right)$$

how is this computed

Runge-Kutta methods circa 1901

$$y_{i+1} = y_i + \phi h$$

slope

Methods of this family
will differ in **how the slope is computed**.

$$\phi = \frac{dy}{dx} = f(x, y)$$

RK1

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$y_i = f(x_i, y_i)(x_{i+1} - x_i)$$

Euler

$$\frac{dy}{dx} = f(x, y)$$

$$\int dy = \int f(x, y)dx$$

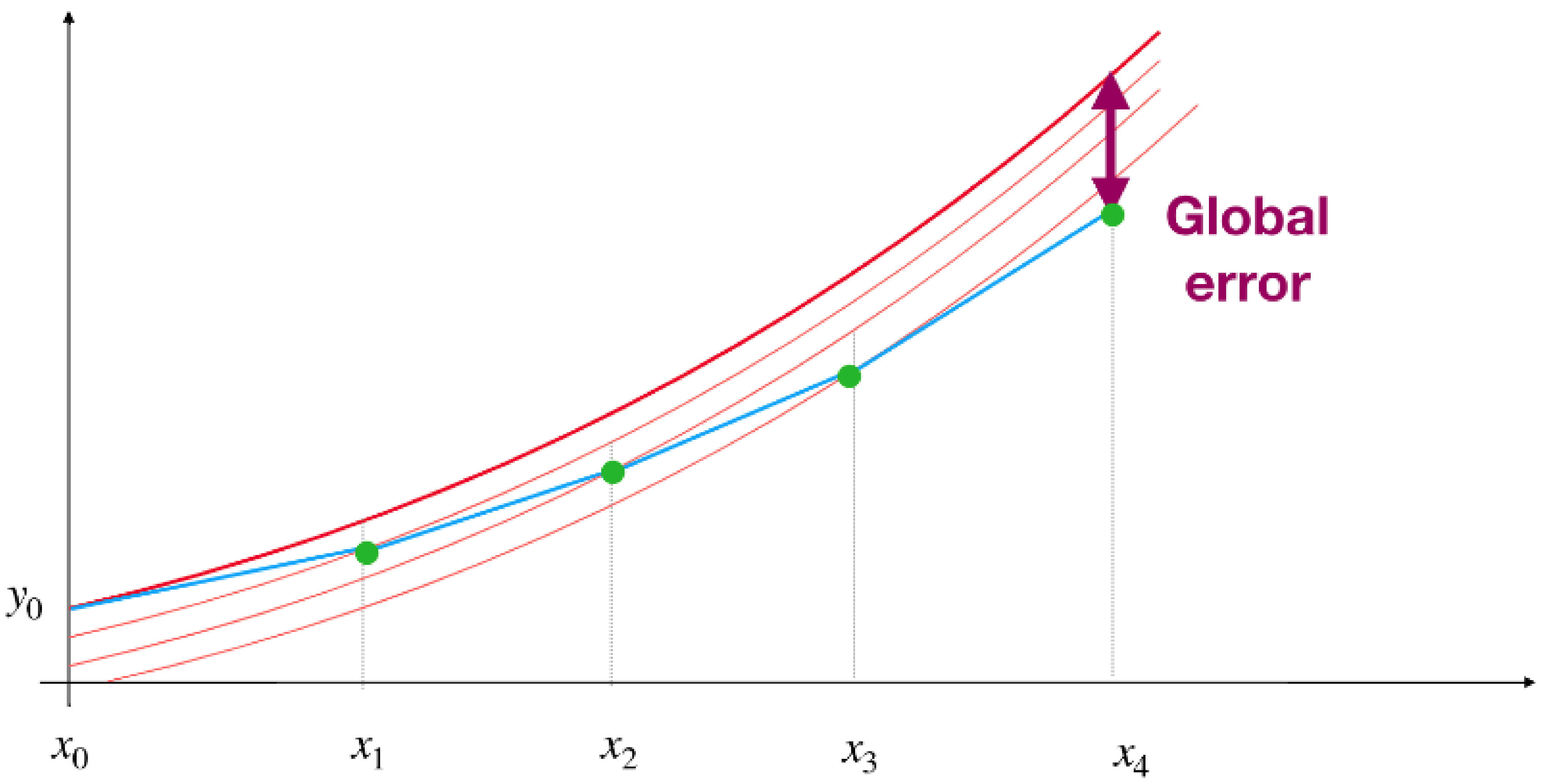
$$\int_{y_i}^{y_{i+1}} dy = \int_{x_i}^{x_{i+1}} f(x, y)dx$$

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} f(x, y)dx$$

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x, y)dx$$

Error in Euler method

$$y' = y$$



How to improve our prediction?

Predictor: Slope 1

$f(x_i, y_i)$

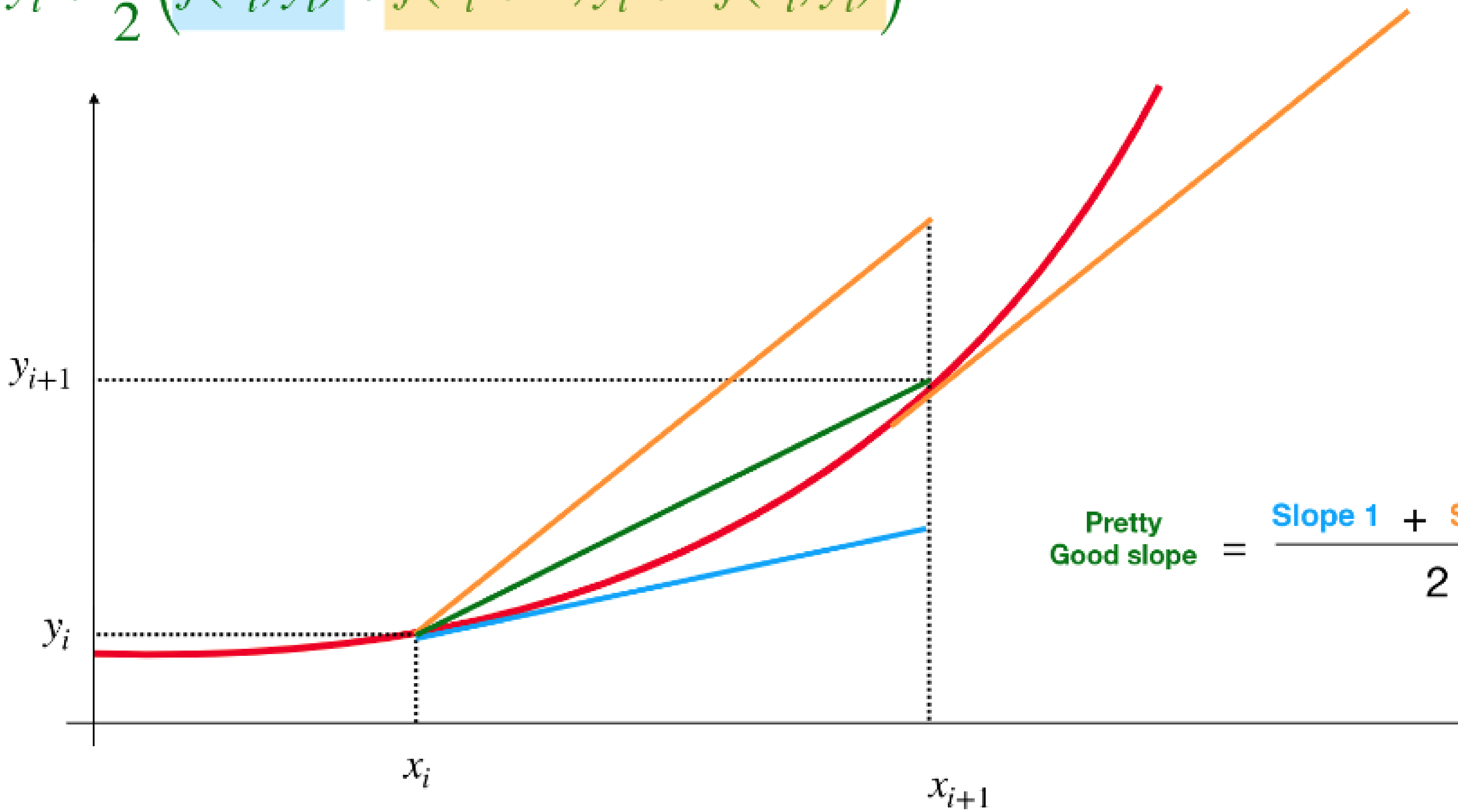
Corrector: Slope 2

$f(x_{i+1}, y_{i+1})$

$f(x_i + h, \boxed{}\,)$

$f(x_i + h, y_i + hf(x_i, y_i))$

$$y_{i+1} = y_i + \frac{h}{2}\Big(f(x_i, y_i) + f(x_i + h, y_i + hf(x_i, y_i))\Big)$$

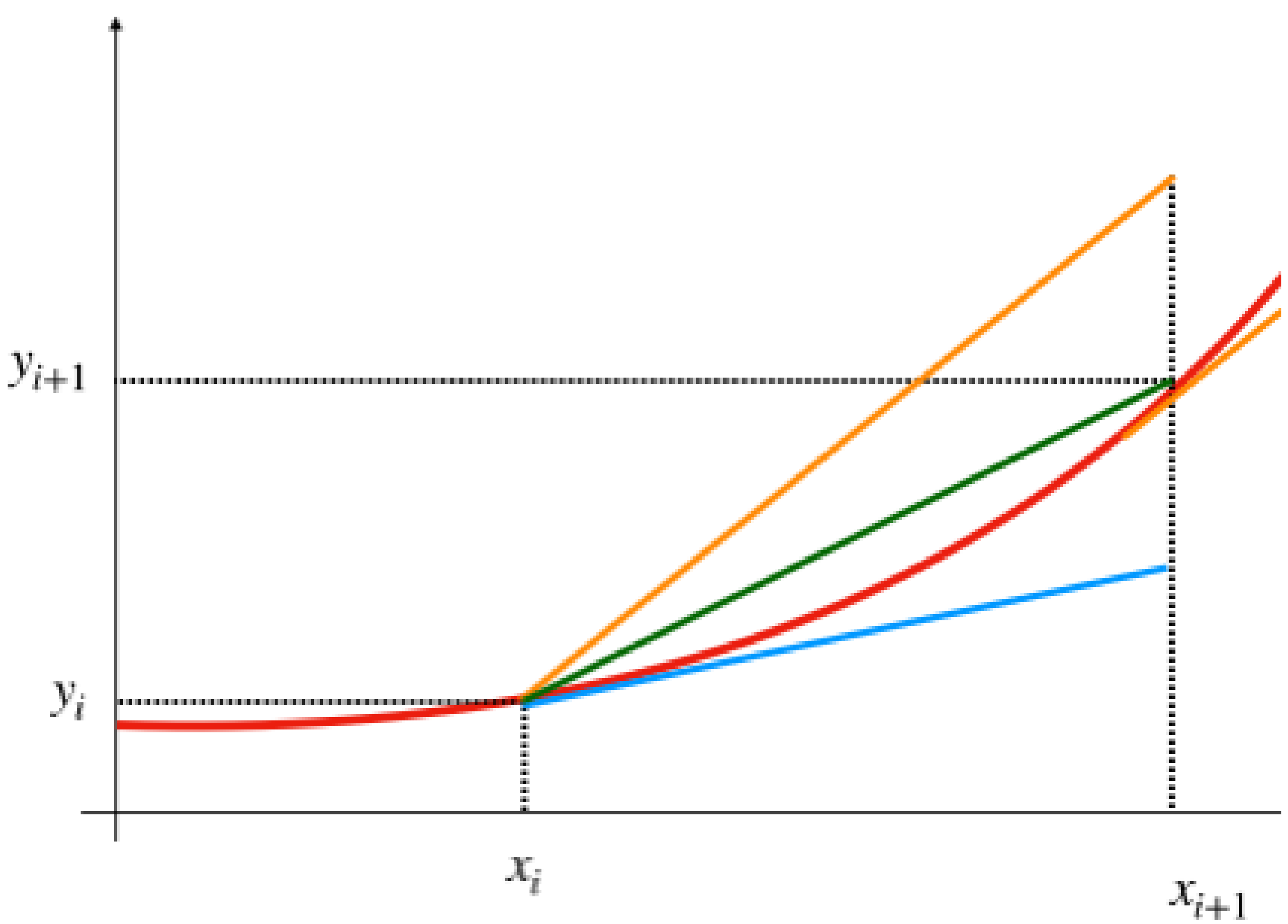


$$\text{Pretty Good slope} = \frac{\text{Slope 1} + \text{Slope 2}}{2}$$

Heun's method

$$y_{i+1} = y_i + \frac{h}{2}(f(x_i, y_i) + f(x_i + h, y_i + hf$$

$$x_{i+1} = x_i +$$



What about Dynamical Systems?

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dI}{dt} = +\beta SI - \alpha I$$
$$\frac{dR}{dt} = +\alpha I$$

We know the starting values S_0, I_0, and R_0…

Can we use them to find $S(t_1)$, $I(t_1)$, and $R(t_1)$?

Recall

$$y_{i+1} \approx y_i + hf(x_i, y_i)$$
$$x_{i+1} \approx x_i + h$$

Let's apply this schema to each Differential equation

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dI}{dt} = +\beta SI - \alpha I$$
$$\frac{dR}{dt} = +\alpha I$$

$$S_{i+1} = S_i - h\beta S_i I_i$$
$$I_{i+1} = I_i + h(\beta S_i I_i - \alpha I_i)$$
$$\mathrm{R}_{i+1} = R_i + h\alpha I_i$$
$$t_{i+1} = t_i + h$$

Try it out….