Prepared by Julian García, based on material by
Arun Konagurthu

Workshop 22
Nash Equilibria

FIT 3139 Computational Modelling and Simulation

MONASH University
Information Technology

	L	C	R
U	0,2	3,1	2,3
M	1,4	2,1	4,1
D	2,1	4,4	3,2

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

Row player

Is U dominated? (need to check against M and D)

Does M dominate U?

$$U_{row}(M, \square) > U_{row}(U, \square)$$

$\square = L \rightarrow$ 1 > 0 ✓

$\square = C \rightarrow$ 2 > 3 ✗

	L	C	R
U	0,2	3,1	2,3
M	1,4	2,1	4,1
D	2,1	4,4	3,2

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Does M dominate U?

$$U_{row}(M, \square) > U_{row}(U, \square)$$

$\square = L \rightarrow$ 1 > 0 ✓

$\square = C \rightarrow$ 2 > 3 ✗

No

Does D dominate U?

$$U_{row}(D, \square) > U_{row}(U, \square)$$

$\square = L \rightarrow$ 2 > 0 ✓

$\square = C \rightarrow$ 4 > 3 ✓

$\square = R \rightarrow$ 3 > 2 ✓

Yes

	L	C	R
U	~~0,2~~	~~3,1~~	~~2,3~~
M	1,4	2,1	4,1
D	2,1	4,4	3,2

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

Row player

Is U dominated? **Yes**

Does M dominate U?

$$U_{row}(M, \square) > U_{row}(U, \square)$$

$\square = L \rightarrow$ 1 > 0 ✓

$\square = C \rightarrow$ 2 > 3 ✗

No

Does D dominate U?

$$U_{row}(D, \square) > U_{row}(U, \square)$$

$\square = L \rightarrow$ 2 > 0 ✓

$\square = C \rightarrow$ 4 > 3 ✓

$\square = R \rightarrow$ 3 > 2 ✓

Yes

	L	C	R
M	1,4	2,1	4,1
D	2,1	4,4	3,2

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

Row player

Is M dominated?

Does D dominate M?

$$U_{row}(D, \square) > U_{row}(M, \square)$$

$\square = L$ 2 > 1 ✓

$\square = C$ 4 > 2 ✓

$\square = R$ 3 > 4 ✗

No

Is D dominated?

DoesM dominate D?

$$U_{row}(M, \square) > U_{row}(D, \square)$$

$\square = L$ 1 > 2 ✗

$\square = C$ 2 > 4 ✗

$\square = R$ 4 > 3 ✓

No

	L	C	R
M	1,4	2,1	4,1
D	2,1	4,4	3,2

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

<u>Column player</u>

<u>Is L dominated?</u> (need to check against C and R)

Does C dominate L?

$$U_{column}(\square, C) > U_{column}(\square, L)$$

$$\square = M \rightarrow \quad 1 > 4$$

No

	L	C	R
M	1,4	2,1	4,1
D	2,1	4,4	3,2

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

Column player

Is L dominated?

Does C dominate L?

$$U_{column}(\square, C) > U_{column}(\square, L)$$

$$\square = M \rightarrow \quad 1 > 4 \quad \times$$

No

Does R dominate L?

$$U_{column}(\square, R) > U_{column}(\square, L)$$

$$\square = M \rightarrow \quad 1 > 4 \quad \times$$

No

	L	C	R
M	1,4	2,1	4,1
D	2,1	4,4	3,2

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

Column player

Is L dominated?

What about a combination of C and R?

$$U_{column}(\square, (0,\alpha,1-\alpha)) > U_{column}(\square, L)$$

$$\square = M \rightarrow \quad \alpha + (1-\alpha) = 1 > 4 \quad \times$$

No

L is not dominated

	L	C	R
M	1,4	2,1	4,1
D	2,1	4,4	3,2

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

Column player

<u>Is C dominated?</u> (need to check against L and R)

Does L dominate C?

$$U_{column}(\square, L) > U_{column}(\square, C)$$

$\square = M \rightarrow$ 4 > 1 ✓

$\square = D \rightarrow$ 1 > 4 ✗

No

Does R dominate C?

$$U_{column}(\square, R) > U_{column}(\square, C)$$

$\square = M \rightarrow$ 1 > 1 ✗

No

	L	C	R
M	1,4	2,1	4,1
D	2,1	4,4	3,2

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

Column player

Is C dominated?

What about a combination of L and R?

$$U_{column}(\square, (\alpha, 0, 1-\alpha)) > U_{column}(\square, C)$$

$$\square = M \rightarrow \quad 4\alpha + (1-\alpha) > 1$$

$$\square = D \rightarrow \quad \alpha + 2(1-\alpha) > 4$$

✕ **No**

C is not dominated

	L	C	R
M	1,4	2,1	4,1
D	2,1	4,4	3,2

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

Column player

Is R dominated? (need to check against L and C)

Does L dominate R?

$$U_{column}(\square, L) > U_{column}(\square, R)$$

$\square = M \rightarrow$ 4 > 1 ✓

$\square = D \rightarrow$ 1 > 2 ✗

No

Does C dominate R?

$$U_{column}(\square, C) > U_{column}(\square, R)$$

$\square = M \rightarrow$ 1 > 1 ✗

No

	L	C	R
M	1,4	2,1	4,1
D	2,1	4,4	3,2

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

Column player

Is R dominated?

What about a combination of L and C?

$$U_{column}(\square, (\alpha, 1-\alpha, 0)) > U_{column}(\square, R)$$

$$\square = M \rightarrow \quad 4\alpha + (1-\alpha) > 1$$

$$\square = D \rightarrow \quad \alpha + 4(1-\alpha) > 2$$

✓

$$\frac{2}{3} > \alpha > 0$$

R is dominated

	L	C
M	~~1,4~~	~~2,1~~
D	2,1	4,4

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

Row player

Is M dominated?

Does D dominate M?

$$U_{row}(D, \square) > U_{row}(M, \square)$$

$\square = L \rightarrow 2 > 1$ ✓

$\square = C \rightarrow 4 > 2$ ✓

Yes

	L	C
D	2,1	4,4

A strategy y_i strictly dominates x_i if

$$u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$$

for all $z \in \Theta$

x_i is dominated if there is a y_i that dominates it.

Column player

Is L dominated?

Does C dominate L?

$$U_{column}(\square, C) > U_{column}(\square, L)$$

$\square = D \rightarrow$ 4 > 1

Yes

(D, C)

(D, C)

	L	C	R
U	0,2	3,1	2,3
M	1,4	2,1	4,1
D	2,1	4,4	3,2

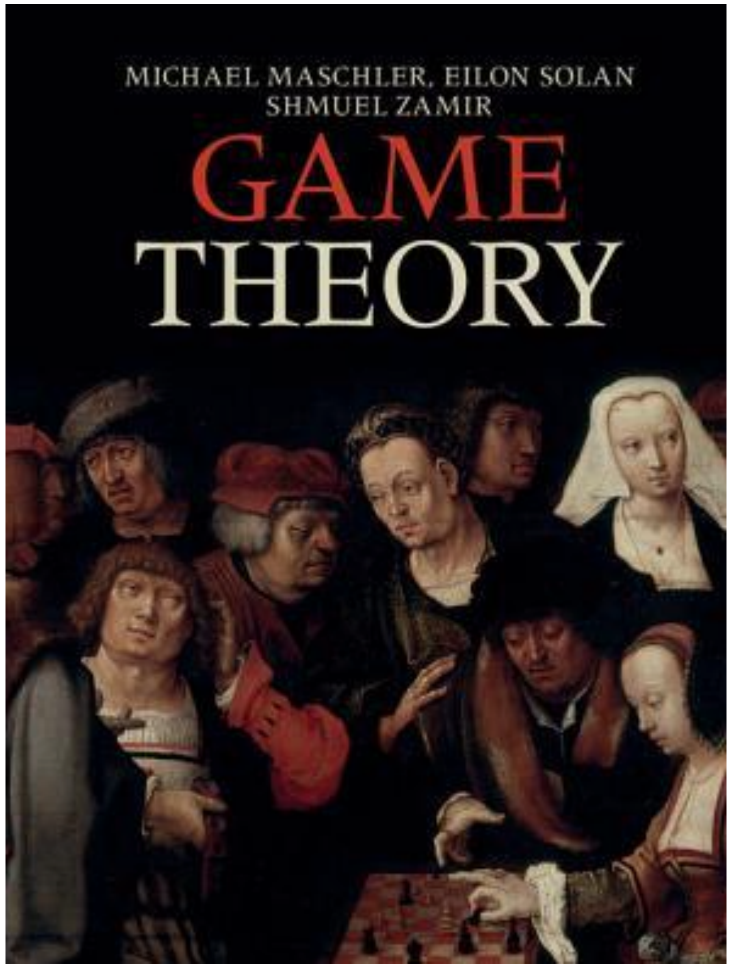
s=((0,0,1), (0,1,0))

On Dominance

- There are (many) games that cannot be solved using this method.

- Straightforwardly combinatorial .

Reading more…

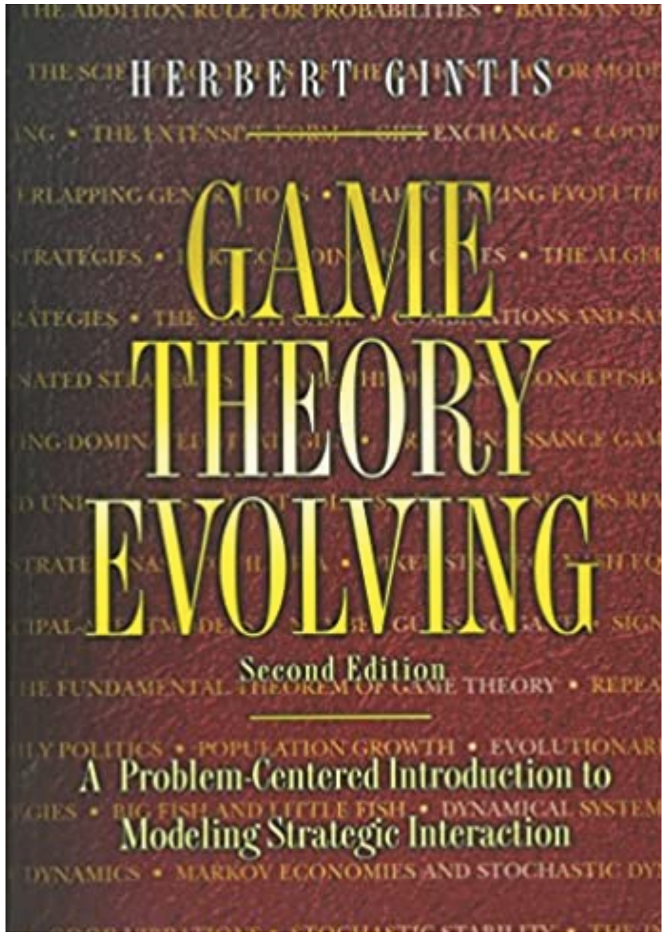


- **Maschler, Solan, Zamir**. "Game Theory" Cambridge University Press, Cambridge, (2013).

[Chapters 4 and 5]

Available online, Monash Library

Reading more…



- **Gintis, H**. "Game Theory Evolving: A Problem-Centered Introduction to Modelling Strategic Interaction" Princeton University Press, Cambridge, (2009).

[Chapters 3 and 4]

Available online, Monash Library

So, what should Nick do?

		Receiver	
		Left	Right
Server	Left	58, 42	79, 21
	Right	73, 27	49, 51

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Solution concept

A way to justify a prediction in a game

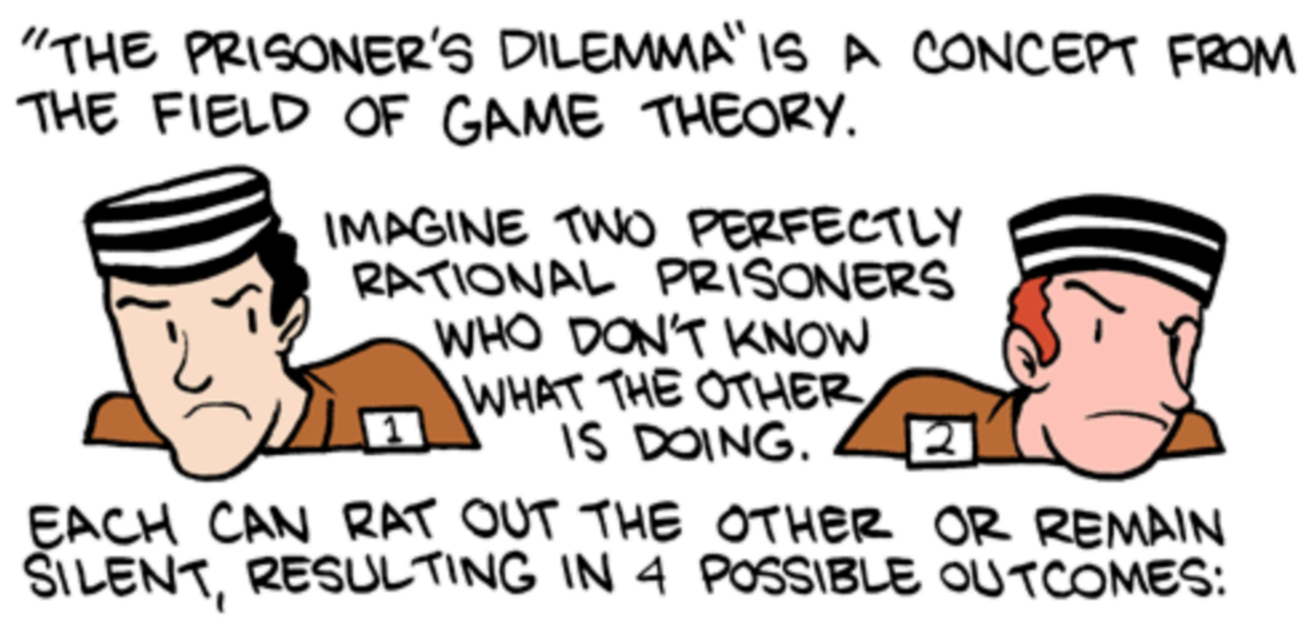
- ✓ Iterated removal of dominated strategies
- Nash equilibrium

Nash Equilibrium

- A profile is a Nash Equilibrium if each player plays an optimal strategy against the strategies of other players.
- A profile is a Nash Equilibrium if no player has an incentive to deviate unilaterally.
- Everyone is doing **the best they can** against what everyone else is doing.

build a "best response *function*"

	Silent	Incriminate
Silent	-1, -1	-4, 0
Incriminate	0, -4	-3, -3



build a "best response *function*"

column's player action

row's player action

B_{row}(**Silent**) = **Incriminate**

B_{column}(**Silent**) = **Incriminate**

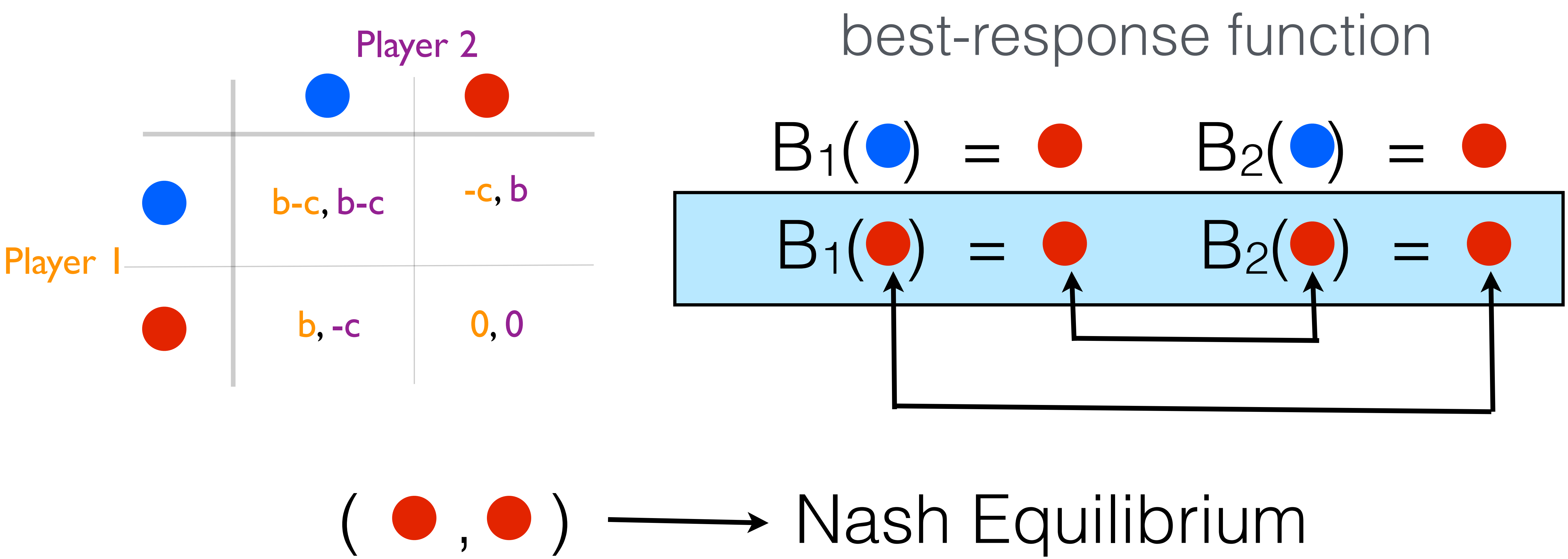
B_{row}(**Incriminate**) = **Incriminate**

B_{column}(**Incriminate**) = **Incriminate**

Is there an outcome where everyone is "best-responding" to everyone else?

(Incriminate, Incriminate) is a Nash equilibrium

Helping game



Why be nice?

Player 2

Player 1	b-c, b-c	-c, b
	b, -c	0, 0

Why be nice?

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	b-c, b-c	-c, b
	Defect	b, -c	0, 0

$b = 3,\ c = 1$ $\quad +1$ for everybody

Why be nice?

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	3, 3	0, 4
	Defect	4, 0	1, 1

Why be nice?

Repetition opens the door for reciprocation.

		Player 2 Cooperate	Player 2 Defect
Player 1	Cooperate	3, 3	0, 4
Player 1	Defect	4, 0	1, 1

- Players face this game, repeatedly
- A strategy now takes into account previous moves (i.e., history)
- Payoff: sum (or average) of all encounters

Example Strategies

- Cooperate no matter what the other does ALLC
- Defect no matter what the other does ALLD
- **TFT** cooperate on the first move and then do what the other did in the previous round

TFT	C	C	D	D	C	C .... C ...... C ....
Other	C	D	D	C	C	C .... C ...... C ....
		*		*		

Repetition opens the door for reciprocation.

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	3, 3	0, 4
	Defect	4, 0	1, 1

- Players face this game, repeatedly
- A strategy now takes into account previous moves (i.e., history)
- Payoff: sum (or average) of all encounters

Payoff structure in a repeated game

- To give cooperation a chance we need the number of rounds to be uncertain.
- Let δ be the continuation probability. Thus, δ^i is the chance that agents play round *i*
- The payoff of a player using strategy *P* against strategy *Q* is:

$$\Pi(P,Q) = (1-\delta)\sum_{i=0}^{\infty}\left(\delta^i \cdot \boxed{\pi(a_i^P, a_i^Q)}\right)$$

payoff in round i given actions induced by P and Q

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	3, 3	0, 4
	Defect	4, 0	1, 1

$$\Pi(P,Q) = (1-\delta)\sum_{i=0}^{\infty}\left(\delta^i \cdot \pi(a_i^P, a_i^Q)\right)$$

TFT vs ALLD

TFT	C	C	C ... C
TFT	C	C	C ... C

$$(1-\delta)\sum_{i=0}^{\infty} 3\delta^i = (1-\delta)\frac{3}{1-\delta} = 3$$

	0	1	1 ... 1
TFT	C	D	D ... D
ALLD	D	D	D ... D
	4	1	1 ... 1

$$(1-\delta)\left(0 + \sum_{i=1}^{\infty}\delta^i\right) = \delta$$

$$(1-\delta)\left(4 + \sum_{i=1}^{\infty}\delta^i\right) = 4 - 3\delta$$

ALLD	D	D	D ... D
ALLD	D	D	D ... D

$$(1-\delta)\sum_{i=0}^{\infty}\delta^i = (1-\delta)\frac{1}{1-\delta} = 1$$

	TFT	ALLD
TFT	3, 3	2/3, 2
ALLD	2, 2/3	1, 1

$\delta = 2/3$

Repeated Prisoner's Dilemma

	TFT	ALLC	ALLD
TFT	3, 3	3, 3	2/3, 2
ALLC	3, 3	3, 3	0, 4
ALLD	2, 2/3	4, 0	1, 1

Nash equilibria?

	TFT	ALLC	ALLD
TFT	3, 3	3, 3	2/3, 2
ALLC	3, 3	3, 3	0, 4
ALLD	2, 2/3	4, 0	1, 1

best-response ***correspondence***

$B_1(\text{ALLC}) = \text{ALLD}$

$B_1(\text{ALLD}) = \text{ALLD}$

$B_1(\text{TFT}) = \{\text{TFT, ALLC}\}$

$B_2(\text{ALLC}) = \text{ALLD}$

$B_2(\text{ALLD}) = \text{ALLD}$

$B_2(\text{TFT}) = \{\text{TFT, ALLC}\}$

Nash set: (ALLD, ALLD)
(TFT, TFT)

A game can have multiple Nash equilibria!

frontiers in Robotics and AI

ORIGINAL RESEARCH
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No Strategy Can Win in the Repeated Prisoner's Dilemma: Linking Game Theory and Computer Simulations

Julián García[1]* and *Matthijs van Veelen*[2]

[1] *Faculty of Information Technology, Monash University, Melbourne, VIC, Australia,* [2] *Department of Economics, Universiteit van Amsterdam, Amsterdam, Netherlands*

Computer simulations are regularly used for studying the evolution of strategies in repeated games. These simulations rarely pay attention to game theoretical results that can illuminate the data analysis or the questions being asked. Results from evolutionary game theory imply that for every Nash equilibrium, there are sequences of mutants that would destabilize them. If strategies are not limited to a finite set, populations move between a variety of Nash equilibria with different levels of cooperation. This instability is inescapable, regardless of how strategies are represented. We present algorithms that show that simulations do agree with the theory. This implies that cognition itself may only have limited impact on the cycling dynamics. We argue that the role of mutations or exploration is more important in determining levels of cooperation.

Keywords: evolution, game theory, simulations, cooperation, evolutionary computation

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So, what should Nick do?

		Receiver	
		Left	Right
Server	Left	58, 42	79, 21
	Right	73, 27	49, 51

There is no dominant strategy. What about Nash?

		Receiver	
		Left	Right
Server	Left	58, 42	79, 21
	Right	73, 27	49, 51

$B_{Nick}(\text{Left}) = \text{Right}$

$B_{Rafa}(\text{Left}) = \text{Left}$

$B_{Nick}(\text{Right}) = \text{Left}$

$B_{Rafa}(\text{Right}) = \text{Right}$

No Nash. In "pure strategies".

EQUILIBRIUM POINTS IN N-PERSON GAMES

BY JOHN F. NASH, JR.*

PRINCETON UNIVERSITY

Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an *n*-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the *n* players corresponds to each *n*-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability

Every game with a finite number of strategies has at least a Nash Equilibrium in (possibly) mixed strategies.

		Receiver	
		Left	Right
Server	Left	58, 42	79, 21
	Right	73, 27	49, 51

There must be a mixed strategy Nash equilibrium.

How do we find it?

- In a bi-matrix game, a profile (**x**,**y**) is Nash iff **x** is a best response to **y** and **y** is a best response to **x**.
- The **support** of a vector is the set of components that are non-zero.

Best response condition

If a Nash equilibrium contains a mixed strategy, each pure strategy in the support of it, is a best response to the equilibrium mixture.

Some useful notation

$$x = (x_1, x_2, ..., x_n)$$ is a mixed strategy profile

$$(x_{-i}, \tau_i) = (\tau_i, x_{-i}) = \begin{cases} (\tau_1, x_2, ..., x_n) & \text{if } i = 1 \\ (x_1, ...x_{i-1}, \tau_i, x_{i+1}, ..., x_n) & \text{if } 1 < i < n \\ (x_1, ...x_{n-1}, \tau_n) & \text{if } i = n \end{cases}$$

replace a strategy in an otherwise identical profile

$$p = (p_i, p_{-i})$$

Best response condition

If a player i's mixed strategy p_i is a best response to the (mixed) strategies of the other players, p_{-i}, then for each pure strategy s_i , such that $p_i(s_i) > 0$ it must be the case that s_i is itself a best response to p_{-i} .

In particular $u_i(s_i, p_{-i})$ must be the same for all s_i such that $p_i(s_i) > 0$

Lemma: A mixed strategy ***x*** is a best response to a mixed strategy **y** if and only if all pure strategies in its support are best responses to **y**

Lemma: A mixed strategy ***x*** is a best response to a mixed strategy **y** if and only if all pure strategies in its support are best responses to **y**

Proof:

$(Ay)_i$ is the expected payoff of player 1 when playing row *i*. Let $u = \max_i [(Ay)_i]$

$$
\begin{aligned}
xAy &= \sum_i x_i (Ay)_i \\
&= \sum_i x_i (u - u + (Ay)_i) \\
&= \sum_i x_i (u - (u - (Ay)_i)) \\
&= \sum_i x_i u - \sum_i x_i (u - (Ay)_i) \\
&= u - \sum_i x_i (u - (Ay)_i)
\end{aligned}
$$

Lemma: A mixed strategy ***x*** is a best response to a mixed strategy **y** if and only if all pure strategies in its support are best responses to **y**

Proof:

$(Ay)_i$ is the expected payoff of player 1 when playing row *i*. Let $u = \max_i [(Ay)_i]$

$$xAy = \sum_i x_i (Ay)_i$$

$$xAy = u - \underbrace{\sum_i x_i \overbrace{(u - (Ay)_i)}^{\geq 0}}_{\geq 0}$$

when does row player maximise her payoff?

xAy achieves the maximum iff the sum is zero.

$$\boxed{x_i > 0 \rightarrow (Ay)_i = u}$$

x is best response to y iff

$$x_i > 0 \Longrightarrow (Ay)_i = u = \max\{(Ay)_k \mid k \in 1 \ldots n\}$$

x is best response to y iff

$$x_i > 0 \implies (Ay)_i = u = \max\{(Ay)_k \mid k \in 1 \ldots n\}$$

		Receiver	
		Left	Right
Server	Left	58, 42	79, 21
	Right	73, 27	49, 51

Find Nash using the best response condition.

We want to find a **mixed** equilibrium.

This means both strategies have full support.

$$s^* = (x, y) \qquad x = (\theta, 1-\theta)$$

$$0 < \theta, \phi < 1 \qquad y = (\phi, 1-\phi)$$

		Receiver: Left (ϕ)	Receiver: Right ($1-\phi$)
Server	Left	**58, 42**	**79, 21**
Server	Right	**73, 27**	**49, 51**

Best response condition

If x is a best response to y, then the pure strategies in the support of x are also a best response to y

$$[58]\phi + [79](1-\phi) = [73]\phi + [49](1-\phi)$$

We want to find a **mixed** equilibrium.

This means both strategies have full support.

$$s^* = (x, y)$$

$$0 < \theta, \phi < 1$$

$$x = (\theta, 1 - \theta)$$

$$y = (\phi, 1 - \phi)$$

			Receiver: Left	Receiver: Right
Server	θ	Left	58, 42	79, 21
	$1 - \theta$	Right	73, 27	49, 51

Best response condition

If x is a best response to y, then the pure strategies in the support of x are also a best response to y

$$[58]\phi + [79](1 - \phi) = [73]\phi + [49](1 - \phi)$$

Best response condition

If y is a best response to x, then the pure strategies in the support of y are also a best response to x

$$[42]\theta + [27](1 - \theta) = [21]\theta + [51](1 - \theta)$$

We want to find a **mixed** equilibrium.

$$A = \begin{pmatrix} 58 & 79 \\ 73 & 49 \end{pmatrix}$$

This means both strategies have full support.

$$s^* = (x, y)$$

$0 < \theta, \phi < 1$

$x = (\theta, 1 - \theta)$

$y = (\phi, 1 - \phi)$

$$B = \begin{pmatrix} 42 & 21 \\ 27 & 51 \end{pmatrix}$$

$$[58]\phi + [79](1 - \phi) = [73]\phi + [49](1 - \phi) \longrightarrow \phi = \frac{2}{3}$$

$$[42]\theta + [27](1 - \theta) = [21]\theta + [51](1 - \theta) \longrightarrow \theta = \frac{8}{15}$$

Verify it is indeed a best response…

Payoff for row, in eq.

$$xAy = 65$$

Can row improve, playing something else?

No

$$Ay = \begin{pmatrix} 65 \\ 65 \end{pmatrix}$$

Payoff for column, in eq.

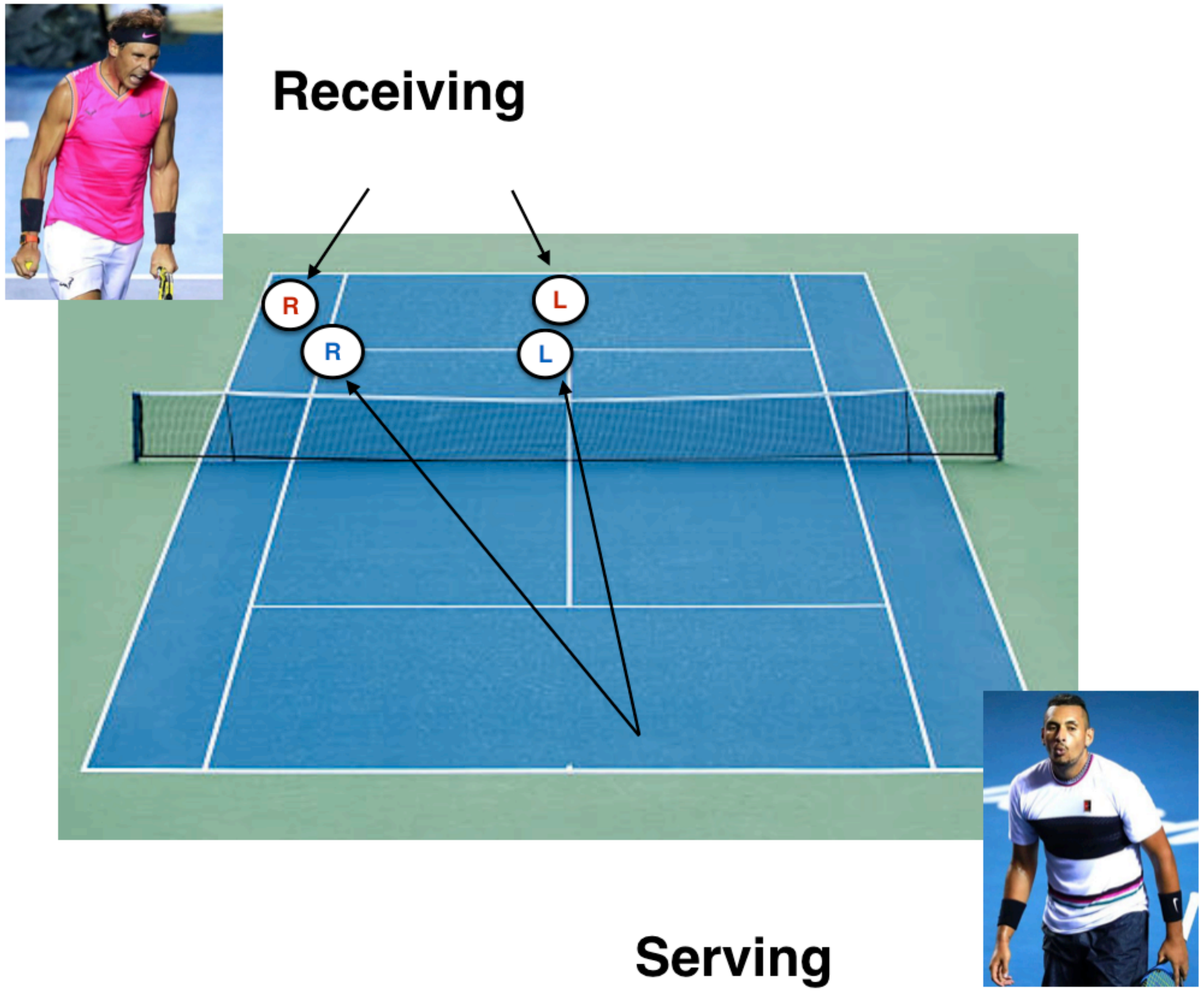
$$xBy = 35$$

Can column improve, playing something else?

No

$$xB = \begin{pmatrix} 35 \\ 35 \end{pmatrix}$$

$$s^* = \left[\left(\frac{8}{15}, \frac{7}{15}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right] \longrightarrow \text{Nash equilibrium}$$



$$s^* = \left[\left(\frac{8}{15}, \frac{7}{15}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right]$$

The American Ec... / Vol. 91, No. 5,... / Minimax Play at...

JOURNAL ARTICLE

Minimax Play at Wimbledon

Mark Walker and John Wooders
The American Economic Review
Vol. 91, No. 5 (Dec., 2001), pp. 1521-1538

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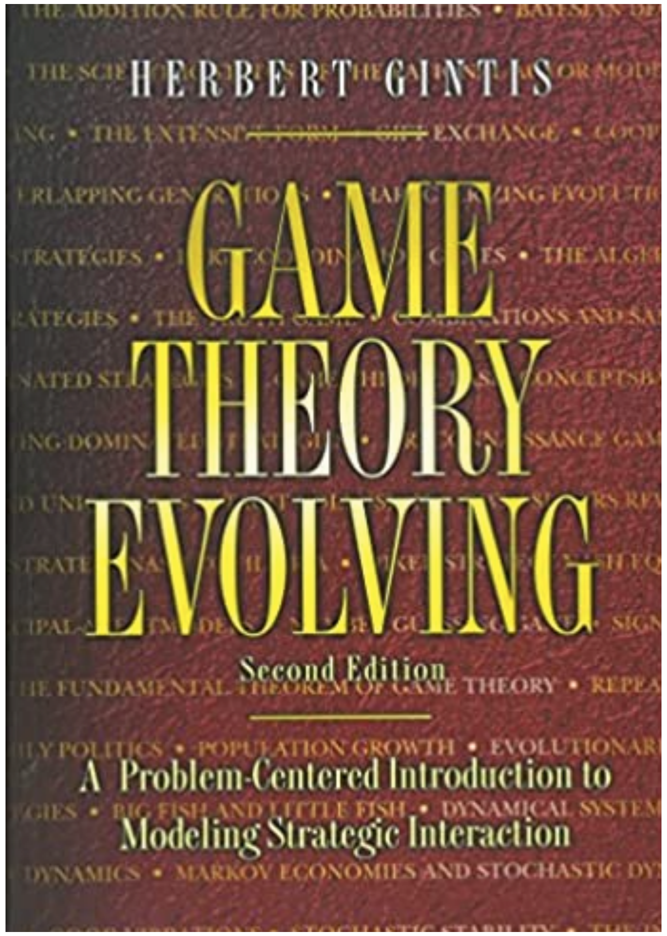
Page Count: 18

What about an Algorithm?

Reading more…



- **Gintis, H**. "Game Theory Evolving: A Problem-Centered Introduction to Modelling Strategic Interaction" Princeton University Press, Cambridge, (2009).

[Chapters 5 and 6]

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