1. Introduction to Direct and Iterative Methods

1.1 Introduction

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. where a's are constants ($a_0 \neq 0$) and n is a positive integer, is called a polynomial in x of degree n. The polynomial $f(x) = 0$ is called an algebraic equation of degree n. If $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential etc., then $f(x) = 0$ is called a transcendental equation.

Definition 1.1.1 — Root. The value α of x which satisfies

$$f(x) = 0 \tag{1.1}$$

is called a root of $f(x) = 0$. Geometrically, a root of (1.1) is that value of x where the graph of $y = f(x)$ crosses the x-axis.

The process of finding the roots of an equation is known as the solution of that equation. This is a problem of basic importance in applied mathematics.

If $f(x)$ is a quadratic, cubic or a biquadratic expression, algebraic solutions of equations are available. But the need often arises to solve higher degree or transcendental equations for which no direct methods exist. Such equations can best be solved by approximate methods. In this chapter, we shall discuss some numerical methods for the solution of algebraic and transcendental equations.

1.1.1 Basic Properties of Equations

1. If $f(x)$ is exactly divisible by $x - \alpha$, then α is a root of $f(x) = 0$.
2. Every equation of the nth degree has only n roots (real or imaginary).

Conversely if $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the ‘ nth degree equation $f(x) = 0$, then where A is a constant.

$$f(x) = A(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n) \tag{1.2}$$

where A is a constant.

Note 1.1.1 If a polynomial of degree n vanishes for more than n values of x, it must be identically zero.

Note 1.1.2 Every equation of the odd degree has atleast one real root.

Note 1.1.3 If an equation of the nth degree has at the most p positive roots and at the most q negative roots, then it follows that the equation has at least $n-(p+q)$ imaginary roots.

1.2 Iterative Methods

The limitations of analytical methods for the solution of equations have necessitated the use of iterative methods. An iterative method begins with an approximate value of the root which is generally obtained with the help of Intermediate value property of the equation. This initial approximation is then successively improved iteration by iteration and this process stops when the desired level of accuracy is achieved. The various iterative methods begin their process with one or more initial approximations. Based on the number of initial approximations used, these iterative methods are divided into two categories: Bracketing Methods and Open-end Methods.

Bracketing methods begin with two initial approximations which bracket the root. Then the width of this bracket is systematically reduced until the root is reached to desired accuracy. The commonly used methods in this category are :

1. Graphical method
2. Bisection method
3. Method of False position.

Open-end methods are used on formulae which require a single starting value or two starting values which do not necessarily bracket the root. The following methods fall under this category :

1. Secant method
2. Iteration method
3. Newton-Raphson method

1.2.1 Rate of Convergence

Let $x_0, x_1, x_2, \cdots$ be the values of a root (α) of an equation at the 0th, 1st, 2nd, $\cdots$, iterations while its actual value is 3.5567. The values of this root calculated by three different methods, are as given below :

Root	1st method	2nd method	3rd method
x_0	5	5	5
x_1	5.6	3.8527	3.8327
x_2	6.4	3.5693	3.56834
x_3	8.3	3.55798	3.55743
x_4	9.7	3.55687	3.55672
x_5	10.6	3.55676	
x_6	11.9	3.55671	

The values in the 1st method do not converge towards the root 3.5567. In the 2nd and 3rd methods, the values converge to the root after 6th and 4th iterations respectively. Clearly 3rd met hod converges faster than the 2nd method. This fastness of convergence in any method is represented by its rate of convergence.

If e be the error then $e_i = \alpha - x_i = x_{i+1} - x_i$.

If $\frac{e_{i+1}}{e_i}$ is almost constant, convergence is said to be linear i.e. slow.
If $\frac{e_{i+1}}{e_i^p}$ is nearly constant, convergence is said to be of order p i.e. faster.

1.3 Iteration Method

To find the roots of the equation

$$f(x) = 0 \tag{1.3}$$

by successive approximations, we rewrite (1.3) in the form

$$x = \phi(x) \tag{1.4}$$

The roots of (1.3) are the same as the points of intersection of the straight line $y = x$ and the curve representing $y = \phi(x)$ Fig. 2.6 illustrates the working of the iteration method which provides a spiral solution.

Let $x = x_0$ be an initial approximation of the desired root α. Then the first approximation xi is given by $x_1 = \phi(x_0)$.

Now treating x_1 as the initial value, the second approximation is $x_2 = \phi(x_1)$.

Proceeding in this way, the nth approximation is given $x_n = \phi(x_{n-1})$.

1.4 Sufficient condition for convergence of iterations

Now it is not sure whether the sequence of approximations $x_1, x_2, \dots, x_n$ always converges to the same number which is a root of (1) or not. As such, we have to choose the initial approximation x_0 suitably so that the successive approximations $x_1, x_2, \dots, x_n$ converge to the root α. The following theorem helps in making the right choice of x_0.

Theorem 1.4.1 If
1. α be a root of $f(x) = 0$ which is equivalent to $x = \phi(x)$,
2. I, be any interval containing the point $x = \alpha$,
3. $|\phi'(x)| < 1$ for all x in I,

then the sequence of approximations $x_0, x_1, x_2, \dots, x_n$ will converge to the root α provided the initial approximation x_0 is chosen in I.

Since α is a root of $x = \phi(x)$, we have $\alpha = \phi(\alpha)$.

If x_{n-1} and x_n be 2 successive approximations to α, we have $x_n = \phi(x_{n-1})$. Therefore,

$$x_n - \alpha = \phi(x_{n-1}) - \phi(\alpha) \tag{1.5}$$

By mean value theorem,

$$\frac{\phi(x_{n-1}) - \phi(\alpha)}{x_{n-1} - \alpha} = \phi'(\xi) \text{ where } x_{n-1} < \xi < \alpha$$

Hence (1.5) becomes $x_n - \alpha = (x_{n-1} - \alpha)\,\phi'(\xi)$.

If $|\phi'(x_i)| \le k < 1$ for all i, then

$$|x_n - \alpha| \le k\,|x_{n-1} - \alpha| \tag{1.6}$$

i.e., Similarly

$$|x_{n-1} - \alpha| \le k\,|x_{n-2} - \alpha|$$

That is

$$|x_n - \alpha| \le k^2 |x_{n-2} - \alpha|$$

Proceeding in this way,

$$|x_n - \alpha| \le k^n |x_0 - \alpha|$$

As $n \to \infty$, the R.H.S. tends to zero, therefore, the sequence of approximations converges to the root α.

Note 1.4.2 The smaller the value of $\phi'(x)$, the more rapid will be the convergence.

Note 1.4.3 This method of iteration is particularly useful for finding the real roots of an equation given in the form of an infinite series.

1.4.1 Acceleration of convergence

From (1.6), we have

$$|x_n - \alpha| \le k |x_{n-1} - \alpha|, k < 1.$$

It is clear from this relation that the iteration method is linearly convergent. This slow rate of convergence can be improved by using the following method :

1.4.2 Aitken's Δ^2 method

Let x_{i-1}, x_i, x_{i+1} be three successive approximations to the desired root α of the equation $x = \phi(x)$. Then we know that

$$\alpha - x_i = k(\alpha - x_{i-1}), \alpha - x_{i+1} = k(\alpha - x_i)$$

Dividing, we get $\frac{\alpha - x_i}{\alpha - x_{i+1}} = \frac{\alpha - x_{i-1}}{\alpha - x_i}$
whence

$$\alpha = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}} \tag{1.7}$$

But in the sequence of successive approximations, we have

$$\begin{aligned} \Delta x_i &= x_{i+1} - x_i \\ \Delta^2 x_i &= \Delta(\Delta x_1) = \Delta(x_{i+1} - x_i) = \Delta x_{i+1} - \Delta x_i \\ &= x_{i+2} - x_{i+1} - (x_{i+1} - x_i) = x_{i+2} - 2x_{i+1} + x_i \end{aligned}$$

Therefore,

$$\Delta^2 x_{i-1} = x_{i+1} - 2x_i + x_{i-1}$$

Hence (1.7) can be writtern as

$$\alpha = x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}} \tag{1.8}$$

which yields successive approximations to the root α.

■ **Example 1.1** Find a real root of the equation $\cos x = 3x - 1$ correct to three decimal aces using Iteration method. ■

We have $f(x) = \cos x - 3x + 1 = 0$,

$$f(0) = \cos 0 - 3(0) + 1 = 1 - 0 + 1 = 2 = +\text{ ve}$$
$$\text{and } f\left(\frac{\pi}{2}\right) = -3\left(\frac{\pi}{2}\right) + 1 = -3\left(\frac{3.14}{2}\right) = -3(1.57) + 1 = -4.71 + 3 = -3.71 = -\text{ ve}$$

$\therefore$ A root lies between 0 and $\frac{\pi}{2}$. Rewriting the given equation as $x = \frac{1}{3}(\cos x + 1) = \phi(x)$, we have

$$\phi'(x) = \frac{\sin x}{3} \text{ and } \left|\phi'(x)\right| = \frac{1}{3}|\sin x| < 1 \text{ in } \left(0, \frac{\pi}{2}\right).$$

Hence the iteration method can be applied.

Since $|f(0)| < |f(1)|$ the root is near to 0, we can start with $x_0 = 0$. Then the successive approximations are,

$$x_1 = \phi\,(x_0) = \frac{1}{3}(\cos 0 + 1) = 0.6667$$
$$x_2 = \phi\,(x_1) = \frac{1}{3}(\cos 0.6667 + 1) = 0.5953$$
$$x_3 = \phi\,(x_2) = \frac{1}{3}(\cos 0.5953 + 1) = 0.6093$$
$$x_4 = \phi\,(x_3) = \frac{1}{3}(\cos 0.6093 + 1) = 0.6067$$
$$x_5 = \phi\,(x_4) = \frac{1}{3}(\cos 0.6067 + 1) = 0.6072$$
$$x_6 = \phi\,(x_5) = \frac{1}{3}(\cos 0.6072 + 1) = 0.6071$$

Hence x_5 and x_6 being almost the same, the root is 0.607 correct to 3 decimal places.

■ **Example 1.2** Using iteration method, find a root of the equation $x^3 + x^2 - 1 = 0$ correct to four decimal places. ■

We have $f(x) = x^3 + x^2 - 1 = 0$. Since,

$$f(0) = 0^3 + 0^2 - 1 = -1$$
$$\text{and } f(1) = 1^3 + 1^2 - 1 = 1,$$

a root lies between 0 and 1.

Rewriting the given equation as

$$x^3 + x^2 - 1 = 0 \quad \Rightarrow \quad x^2(x+1) - 1 = 0$$
$$x^2 = \frac{1}{(x+1)} \quad \Rightarrow \quad x = (x+1)^{\frac{-1}{2}} = \phi(x)$$

We have $\phi'(x) = -\frac{1}{2}(x+1)^{-3/2}$ and $|\phi'(x)| < 1$ for $x \in (0,1)$. Hence the iteration method can be applied.

Since $|f(1)| < |f(0)|$ the root is near to 1, we can starting with $x_0 = 0.75$, the successive approximations are

$$x_1 = \phi\,(x_0) = \frac{1}{\sqrt{(x_0+1)}} = 0.7559$$
$$x_2 = \phi\,(x_1) = \frac{1}{\sqrt{(0.7559+1)}} = 0.75466$$
$$x_3 = 0.75492,$$
$$x_4 = 0.75487,$$
$$x_5 = 0.75488$$

Hence x_4 and x_5 being almost the same, the root is 0.7548 correct to 4 decimal places.

■ **Example 1.3** Apply iteration method to find the negative root of the equation $x^3-2x+5=0$ correct to four decimal places. ■

If α,β,γ are the roots of the given equation, then $-\alpha,-\beta,-\gamma$ are the roots of

$$(-x)^3-2(-x)+5=0 \quad \Rightarrow \quad -(x^3-2x-5=0)$$

$\therefore$ The negative root of the given equation is the negative of the positive root of

$$f(x)=x^3-2x-5=0. \tag{1.9}$$

Since,

$$f(0)=0^3-2(0)-5=-5=-ve$$
$$f(1)=1^3-2(1)-5=-6=-ve$$
$$f(2)=2^3-2(2)-5=-1=-ve$$
$$f(3)=3^3-2(3)-5=16=+ve$$

a root lies between 2 and 3 .

Rewriting Eq. (1.9)

$$x^3-2x-5=0 \Rightarrow x^3=2x+5 \Rightarrow x=(2x+5)^{\frac{1}{3}}=\phi(x)$$

We have $\phi'(x)=\frac{1}{3}(2x+5)^{\frac{-2}{3}}.2$ and $|\phi'(x)|<1$ for $x\in(2,3)$. $\therefore$ The iteration method can be applied.

Since $|f(2)|<|f(3)|$ the root is near to 2, we can starting with $x_0=2$, the successive approximations are

$$x_1=\phi\,(x_0)=(2x_0+5)^{\frac{1}{3}}=2.08008$$
$$x_2=\phi\,(x_1)=2.09235,$$
$$x_3=2.09422$$
$$x_4=2.09450,$$
$$x_5=2.09454$$

Since x_4 and x_5 being almost the same, the root of (1.9) is 2.0945 correct to 4 decimal places.

Hence the negative root of the given equation is -2.0945.

■ **Example 1.4** Find a real root of $2x-\log_{10}x=7$ correct to four decimal places using iteration method. ■

We have

$$f(x)=2x-\log_{10}x-7$$
$$f(0)=0-\log_{10}0-7=0-\infty-7=-\infty=-ve$$
$$f(1)=2-\log_{10}1-7=2-0-7=-5=-ve$$
$$f(2)=4-\log_{10}2-7=4-0.0310-7=-2.6989=-ve$$
$$f(3)=6-\log_{10}3-7=6-0.4771-7=-1.4471=-ve$$
$$f(4)=8-\log_{10}4-7=8-0.602-7=0.398=+ve$$

$\therefore$ A root lies between 3 and 4.

Rewriting the given equation as $x = \frac{1}{2}(\log_{10} x + 7) = \phi(x)$, we have

$$\phi'(x) = \frac{1}{2}\left(\frac{1}{x}\log_{10} e\right) \quad \left[\because \frac{d}{dx}\log_a xx = \frac{1}{x\log a}\right]$$

$$\therefore \quad \left|\phi'(x)\right| < 1 \text{ when } 3 < x < 4 \quad [\because \log_{10} e = 0.4343]$$

Since $|f(4)| < |f(3)|$, the root is near to 4.

Hence the iteration method can be applied. Taking $x_0 = 3.6$, the successive approximations are

$$x_1 = \phi(x_0) = \frac{1}{2}(\log_{10} 3.6 + 7) = 3.77815$$
$$x_2 = \phi(x_1) = \frac{1}{2}(\log_{10} 3.77815 + 7) = 3.78863$$
$$x_3 = \phi(x_2) = \frac{1}{2}(\log 3.78863 + 7) = 3.78924$$
$$x_4 = \phi(x_3) = \frac{1}{2}(\log 3.78924 + 7) = 3.78927$$

Hence x_3 and x_4 being almost equal, the root is 3.7892 correct to 4 decimal places.

■ **Example 1.5** Find the smallest root of the equation

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = 0$$

. ■

Writing the given equation as

$$x = 1 + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = \phi(x)$$

Omitting x^2 and higher powers of x, we get $x = 1$ approximately.

Taking $x_0 = 1$, we obtain

$$x_1 = \phi(x_0) = 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} - \frac{1}{(5!)^2} + \ldots = 1.2239$$
$$x_2 = \phi(x_1) = 1 + \frac{(1.2239)^2}{(2!)^2} - \frac{(1.2239)^3}{(3!)^2} + \frac{(1.2239)^4}{(4!)^2} - \frac{(1.2239)^5}{(5!)^2} + \ldots = 1.3263$$

Similarly $x_3 = 1.38, x_4 = 1.409, x_5 = 1.425, x_6 = 1.434, x_7 = 1.439, x_8 = 1.442$.

The values of x_7 and x_8 indicate that the root is 1.44 correct to 2 decimal places.

1.5 Practice Problem

1. Find a root of the following equations, using the bisection method correct to three decimal places:
   (a) $x^3 - x - 1 = 0$
   (b) $x^3 - x^2 - 1 = 0$
   (c) $2x^3 + x^2 - 20x + 12 = 0$
   (d) $x^4 - x - 10 = 0$.
2. Evaluate a real root of the following equations by bisection method:
   (a) $x - \cos x = 0$

(b) $e^{-x}-x=0$
(c) $e^x=4\sin x$.

3. Find a real root of the following equations correct to three decimal places, by the method of false position:
(a) $x^{-}5x+1=0$
(b) $x^3-4x-9=0$
(c) $x^6-x^4-x^3-1=0$
4. Using Regula falsi method, compute the real root of the following equations correct to three decimal places:
(a) $xe^x=2$
(b) $\cos x=3x-1$
(c) $xe^x=\sin x$
(d) $x\tan x=-1$
(e) $2x-\log x=7$
(f) $3x+\sin x=e^x$.
5. Find the fourth root of 12 correct to three decimal places by interpolation method.
6. Locate the root of $f(x)=x^{10}-1=0$, between 0 and 1.3 using bisection method and method of false position. Comment on which method is preferable.
7. Find a root of the following equations correct to three decimal places by the method :
(a) $x^3+x^2+x+7=0$
(b) $x-e^{-x}=0$
(c) $x\log_{10}x=1.9$.
8. Use the iteration method to find a root of the equations to four decimal places:
(a) $x^3+x^2-100=0$
(b) $x^3-9x+1=0$
(c) $x=\frac{1}{2}+\sin x$
(d) $\tan x=x$
(e) $e^x-3x=0$
(f) $2^x-x-3=0$ which lies between $(-3,-2)$
9. Evaluate $\sqrt{30}$ by (i) secant method (ii) iteration method correct to four decimal places.
10. Find the root of the equation $2x=\cos x+3$ correct to three decimal places using Iteration method.
11. Find the real root of the equation $x-\frac{x^3}{3}+\frac{x^5}{10}-\frac{x^7}{42}+\frac{x^9}{216}-\frac{x^{11}}{1320}+\ldots=0.443$, correct to three decimal places using iteration method.

1.6 Newton-Raphson Method

Let x_0 be an approximate root of the equation $f(x)=0$. If $x_1=x_0+h$ be the exact root then $f(x_1)=0$.

$\therefore$ Expanding $f(x_0+h)$ by Taylor's series $f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\ldots\ldots=0$.
Since h is small, neglecting h^2 and higher powers of h, we get $f(x_0)+hf'(x_0)=0$
or

$$h=-\frac{f(x_0)}{f'(x_0)} \tag{1.10}$$

$\therefore$ A closer approximation to the root is given by

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}.$$