Let $\frac{dy}{dx} = f(x,y)$ is a differential equation whose solution is $y = f(x)$ and $y(x_0) = y_0$ be initial conditions.
Taylor's series method:

$$y = f(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

Problem 5.1.1

Find by Taylor's series method the value of y at $x = 0.1$ and $x = 0.2$ to five places of decimal places from

$$\frac{dy}{dx} = x^2 y - 1, \; y(0) = 1$$

Solution:
Given

$$y' = \frac{dy}{dx} = x^2 y - 1$$
$$y(0) = 1$$

Also know, $y(x_0) = y_0$
$\therefore x_0 = 0, y_0 = 1$
As we know that, Taylor's method to find $y(x)$ is given by

$$y = f(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots \quad (58)$$

From the initial condition, $x_0 = 0$, $y_0 = 1$.
Given that

$$y' = x^2 y - 1 \tag{59}$$

At initial,

$$\begin{aligned} y_0' &= x_0^2\, y_0 - 1 \\ &= 0^2(1) - 1 = -1 \\ y_0' &= -1 \end{aligned}$$

Find the derivative of equation (59),

$$y'' = x^2 y' + y.2x - 0 \tag{60}$$

At initial,

$$\begin{aligned} y_0'' &= x_0^2\, y_0' + y_0\, 2\, x_0 - 0 \\ &= 0^2(-1) + 1(2)(0) - 0 \\ y_0'' &= 0 \end{aligned}$$

Find the derivative of equation (60),

$$y''' = x^2y'' + y'.2x + 2y + 2xy' \tag{61}$$

At initial,

$$\begin{aligned} y_0''' &= x_0^2y_0'' + y_0'.2x_0 + 2y_0 + 2x_0y_0' \\ &= (0)^2(0) + (-1).2(0) + 2(1) + 2(0)(-1) \\ &= 0 + 0 + 2(1) + 0 = 2 \\ y_0''' &= 2 \end{aligned}$$

Find the derivative of equation (61),

$$\begin{aligned} y'''' &= \left(x^2y''' + 2xy''\right) + \left(2y'(1) + 2xy''\right) + 2y' + \left(2(1)y' + 2xy''\right) \\ y'''' &= x^2y''' + 6xy'' + 6y' \end{aligned}$$

At initial,

$$\begin{aligned} y_0'''' &= x_0^2y_0''' + 6x_0y_0'' + 6y_0' \\ &= (0)^2(2) + 6(0)(0) + 6(-1) = 0 + 0 + 6(-1) = -6 \\ y_0'''' &= -6 \end{aligned}$$

Substitute these values in the formula at equation (58), we get

$$
\begin{aligned}
y(x) &= y_0 + (x-0)(y_0') + \frac{(x-0)^2}{2!}(y_0'') + \frac{(x-0)^3}{3!}y_0''' + \frac{(x-0)^4}{4!}y_0'''' \\
&= 1 + (x-0)(-1) + \frac{(x-0)^2}{2!}(0) + \frac{(x-0)^3}{3!}(2) + \frac{(x-0)^4}{4!}(-6) \quad = 1 \\
y(x) &= 1 - x + \frac{x^3}{3} - \frac{x^4}{4} \\
y(0.1) &= 1 - (0.1) + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} \\
&\approx 0.90030 \\
y(0.2) &= 1 - (0.2) + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} \\
&\approx 0.80226
\end{aligned}
$$

Problem 5.1.2

Solve the given equation for $y(1.1)$ using Taylor's series method.

$$\frac{dy}{dx} = 2y + 3e^x, \; y(0) = 0,$$

Given,

$$\begin{aligned} y' = \frac{dy}{dx} &= 2y + 3e^x \\ y(0) &= 0 \\ \Rightarrow \; y(x_0) &= y_0 \\ \Rightarrow x_0 &= 0, \; y_0 = 0. \end{aligned}$$

From the Taylor's series, we have

$$y = f(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots \quad (62)$$

From the initial condition, $x_0 = 0,\ y_0 = 0$.
Given that,

$$y' = 2y + 3e^x \tag{63}$$

At initial,

$$y_0' = 2y_0 + 3e^{x_0} = 2(0) + 3e^0 = 0 + 3(1) = 3$$
$$y_0' = 3$$

Find the derivative of equation (63),

$$y'' = 2y' + 3e^x \tag{64}$$

At initial,

$$\begin{aligned} y_0'' &= 2y_0' + 3e^{x_0} \\ &= 2(3) + 3e^0 = 6 + 3(1) \\ y_0'' &= 9 \end{aligned}$$

Find the derivative of equation (64),

$$y''' = 2y'' + 3e^x \tag{65}$$

At initial,

$$\begin{aligned} y_0''' &= 2y_0'' + 3e^{x_0} = 2(9) + 3e^0 = 21 \\ y_0''' &= 21 \end{aligned}$$

Find the derivative of equation (65),

$$y'''' = 2y''' + 3e^x$$

At initial,

$$\begin{aligned} y_0'''' &= 2y_0''' + 3e^{x_0} = 2(21) + 3e^0 \\ y_0'''' &= 45 \end{aligned}$$

Substituting these values in equation (62), we get

$$y = f(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0'''' + \cdots$$

$$y(x) = 0 + \frac{(x-0)}{1!}(3) + \frac{(x-0)^2}{2!}(9) + \frac{(x-0)^3}{3!}(21) + \frac{(x-0)^4}{4!}(45)$$

$$y(x) = 3x + \frac{x^2}{2}(9) + \frac{x^3}{6}(21) + \frac{x^4}{24}(45)$$

$$y(1.1) = 3(1.1) + \frac{(1.1)^2}{2}(9) + \frac{(1.1)^3}{6}(21) + \frac{(1.1)^4}{24}(45)$$

$$= 3.3 + 5.445 + 4.6585 + 2.7452 = 16.1457$$

Problem 5.1.3

From Taylor's series method, find $y(0.1)$ considering up to fourth degree term if $y(x)$ satisfies the equation $\frac{dy}{dx} = x - y^2, y(0) = 1$.

Taylor's series expansion is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \frac{(x - x_0)^3}{3!}y_0''' + \frac{(x - x_0)^4}{4!}y_0'''' + \cdots \tag{66}$$

From initial condition $y(0) = 1$, we can conclude $x_0 = 0 \Rightarrow y_0 = 1$. Given equation can be written as $y' = x - y^2$, Therefore, we substitute the values in (66), we have

$$y(x) = 1 + (x - 0)^2 y'(0) + \frac{(x - 0)^2}{2!}y''(0) \tag{67}$$

$$+ \frac{(x - 0)^3}{3!}y'''(0) + \frac{(x - 0)^4}{4!}y''''(0) \tag{68}$$

Given,

$$y' = x - y^2 \tag{69}$$

At initial,

$$y_0' = 0 - 1^2 = -1$$

Find the derivative of equation (69)

$$y'' = 1 - 2yy' \tag{70}$$

At initial,

$$y_0'' = 1 - 2(1)(-1) = 3$$

Find the derivative of equation (70)

$$y''' = 0 - 2\left[yy'' + (y')^2\right] \tag{71}$$

At initial,

$$y_0''' = -2\left[(1)(3) + (-1)^2\right] = -8$$

Find the derivative of equation (71)

$$y'''' = -2\left[yy''' + y'y'' + 2y'y''\right] = -2\left[yy'' + 3y'y''\right] \tag{72}$$

At initial,

$$y_0'''' = -2\left[(1)(-8) + 3(-1)(3)\right] = 34$$

$y(x) \Rightarrow y(0.1)$ Substitute y', y'', y''', y''''' in equation (66). Therefore

$$y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34)$$
$$y(0.1) = 0.9138$$

Practice Problem

1. Consider the first order differential equation $\frac{dy}{dx} = x + y$, with initial condition $y(1) = 0$. Find y value at $x = 1.2$.
2. Use Taylor's series method to approximate y when $x = 0.1$, convert to 4 decimal places given that $\frac{dy}{dx} = 3x + y^2$ and $y = 1$ when $x = 0$ by taking the first five terms of Taylor's series expansions.

Given $y' = \frac{dy}{dx} = f(x,y)$ and initial condition $y(x_0) = y_0$.
Euler's method formula

$$y_{i+1} = y_i + hf(x_i, y_i)$$

Problem 5.2.1

Using Euler's method solve for y *at* $x = 0.1$ *from* $\frac{dy}{dx} = x + y + xy$, $y(0) = 1$ *taking* $h = 0.025$

Given $h = 0.025$

$$
\begin{aligned}
x_0 &= 0 \\
x_1 &= x_0 + h = 0 + 0.025 = 0.025 \\
x_2 &= x_0 + 2h = 0 + 2(0.025) = 0.05 \\
x_3 &= x_0 + 3h = 0 + 3(0.025) = 0.75 \\
x_4 &= x_0 + 4h = 0 + 4(0.025) = 0.1
\end{aligned}
$$

Given

$$
\begin{aligned}
y' = \frac{dy}{dx} &= x + y + xy \\
y(0) = 1 &\Rightarrow y(x_0) = y_0 \\
\therefore x_0 &= 0, y_0 = 1 \\
\text{Also given, } h &= 0.025
\end{aligned}
$$

Now, we can calculate y_1 using Euler formula

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ y_1 &= 1 + 0.025\,(x_0 + y_0 + x_0 y_0) \\ y_1 &= 1 + 0.025\,(0 + 1 + 0) \\ y_1 = y(x_1) &= 1 + 0.025 \\ i.e)\ y(0.025) &= 1.025 \end{aligned}$$

$$x_1 = 0.025\ y_1 = 1.025$$

$$\begin{aligned} y_2 &= y_1 + hf(x_1, y_1) = y_1 + h(x_1 + y_1 + x_1 y_1) \\ y_2 &= 1.025 + 0.025\,(0.025 + 1.025 + 0.025(1.025)) \\ y_2 &= 1.025 + 0.025\,(1.0756) \\ \therefore y(0.05) = y(x_2) &= 1.05189 \end{aligned}$$

$$y_3 = y_2 + hf(x_2, y_2)$$
$$y_3 = y_2 + h(x_2 + y_2 + x_2 y_2)$$
$$y_3 = y(x_3) = 1.05189 + 0.025\,(0.05 + 1.05189 + 0.05(1.05189))$$
$$\therefore\; y(0.075) = 1.08075$$

$$x_3 = 0.075,\; y_3 = 1.08075$$

$$y_4 = y_3 + hf(x_3, y_3)$$
$$y_4 = y_3 + h(x_3 + y_3 + x_3 y_3)$$
$$y_4 = 1.08075 + 0.025\,(0.075 + 1.08075 + 0.075(1.08075))$$
$$y_4 = y(x_4) = 1.11167$$

$$\therefore y(0.1) = 1.11167$$

Example 5.2.2

Using Euler's method solve for y at $x = 1$ from $\frac{dy}{dx} = 2e^x + y^2$, $y(0) = \frac{1}{2}$ taking $h = 0.25$.

Given $h = 0.25$

$$
\begin{aligned}
x_0 &= 0 \\
x_1 &= x_0 + h = 0 + 0.25 = 0.25 \\
x_2 &= x_0 + 2h = 0 + 2(0.50) = 0.50 \\
x_3 &= x_0 + 4h = 0 + 4(0.25) = 1
\end{aligned}
$$

Given

$$
\begin{aligned}
y' &= \frac{dy}{dx} = 2e^x + y^2 \\
y(0) &= \frac{1}{2} \Rightarrow y(x_0) = y_0 \\
\therefore x_0 &= 0, y_0 = \frac{1}{2}
\end{aligned}
$$

Also given, $h = 0.25$

Now, we can calculate y_1 using Euler formula

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = \frac{1}{2} + 0.25\left(2e^{x_0} + y_0^2\right)$$

$$y_1 = \frac{1}{2} + 0.25\,(2 + 1)$$

$$y_1 = y(x_1) = 1.0625$$

$$i.e)\ y(0.25) = 1.0625$$

$$x_1 = 0.25\ y_1 = 1.0625$$

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + h(2e^{x_1} + y_0^1)$$

$$y_2 = +0.25\left(2e^{0.25} + y_0^2\right)$$

$$y_2 = 1.025 + 0.025\,(1.0756)$$

$$
\begin{aligned}
y_3 &= y_2 + hf(x_2, y_2) \\
y_3 &= y_2 + h(x_2 + y_2 + x_2 y_2) \\
y_3 = y(x_3) &= 1.05189 + 0.025\,(0.05 + 1.05189 + 0.05(1.05189)) \\
\therefore\ y(0.075) &= 1.08075
\end{aligned}
$$

$$
x_3 = 0.075,\ y_3 = 1.08075
$$

$$
\begin{aligned}
y_4 &= y_3 + hf(x_3, y_3) \\
y_4 &= y_3 + h(x_3 + y_3 + x_3 y_3) \\
y_4 &= 1.08075 + 0.025\,(0.075 + 1.08075 + 0.075(1.08075)) \\
y_4 = y(x_4) &= 1.11167
\end{aligned}
$$

$\therefore y(0.1) = 1.11167$

Given

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0$$

Modified Euler's formula is given by

$$y_{n+1}^{(r+1)} = y_n + \frac{h}{2}\left[f(x_n,y_n) + f\left(x_{n+1}, y_{n+1}^{(r)}\right)\right]$$

where, $r = 0, 1, 2, \cdots$ where

$$y_{n+1}^{(0)} = y_n + hf(x_n,y_n), n = 0, 1, 2, \cdots,$$

using Euler's formula.

Example 5.3.1

Using Modified Euler's method find $y(0.2)$ given that $\frac{dy}{dx} = x + y$, $y(0) = 1$. Correct to 4 decimal places.

Given

$$
\begin{aligned}
f(x, y) &= x + y, \\
x_0 &= 0, \; y_0 = 1, \\
x_1 &= 0.2, \\
h &= x_1 - x_0 = 0.2 - 0 \\
h &= 0.2
\end{aligned}
$$

Initial approximation

$$y_{n+1}^{(0)} = y_n + hf(x_n, y_n)$$

Put $n = 0$

$$\begin{aligned} y_1^{(0)} &= y_0 + hf(x_0, y_0) \\ &= 1 + 0.2(0+1) \\ y_1^{(0)} &= 1.2 \end{aligned}$$

Modified Euler formula is

$$y_{n+1}^{(r+1)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(r)})\right] \tag{73}$$

First approximation:
Put $r = 0$ and $n = 0$ in equation (73)

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 1 + \frac{0.2}{2}\left[(x_0 + y_0) + (x_1 + y_1^{(0)})\right]$$
$$= 1 + \frac{0.2}{2}\left[(0+1) + (0.2 + 1.2)\right]$$
$$y_1^{(1)} = 1.24$$

Second Approximation
Put $r = 1$, $n = 0$ in equation (73)

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 1 + \frac{0.2}{2}\left[(0+1) + (0.2 + 1.24)\right]$$
$$y_1^{(2)} = 1.244$$

Third approximation:

Put $r = 2, n = 0$ in equation (73)

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$
$$= 1 + \frac{0.2}{2}\left[(0+1) + (0.2 + 1.244)\right]$$
$$y_1^{(3)} = 1.2444$$

Fourth Approximation

Put $r = 3, x = 0$ in equation (73)

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$
$$= 1 + \frac{0.2}{2}\left[(0+1) + (0.2 + 1.2444)\right]$$
$$y_1^{(4)} = 1.24444$$

Since, $y_1^{(3)}$ and $y_1^{(4)}$ are the same at corrected to four decimal places.

$$y_1 = y(x_1) = y(0.2) = 1.2444$$

Example 5.3.2

Use Modified Euler's Method to find the approximate value of $y(1.1)$ for the solution of the initial value problem $\frac{dy}{dx} = 2xy$, $y(1) = 1$ correct to 3 decimal places, perform 2 iterations.

$$f(x,y) = 2xy$$

$$x_0 =, y_0 = 1, x_1 = 1.1$$

$$h = x_1 - x_0 = 1.1 - 1 = 0.1$$

$$y(1.1) = 1.2355$$

Example 5.3.3

Find $y(1.2)$ and $y(1.4)$ by modified Euler's method given that $\frac{dy}{dx} = \frac{2y}{x} + x^3$, $y(1) = 0.5$ correct to 3 decimal places.

Given $f(x, y) = \frac{2y}{x} + x^3$

$$x_0 = 1, \qquad y_0 = 0.5,$$
$$\text{and } x_1 = 1.2, \qquad x_2 = 1.4$$

$x_0 = 1$	$x_1 = x_0 + h = 1.2$	$x_2 = x_0 + 2h = 1.4$
$y_0 = 0.5$	?	?

(i). To find $y(1.2)$

$$h = x_1 - x_0 = 1.2 - 1 = 0.2$$

Modified Euler's formula is given by

$$y_{n+1}^{(r+1)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y_{n+1}^{(r)}\right)\right] \tag{74}$$

where,

$$y_{n+1}^{(0)} = y_n + hf(x_n, y_n) \tag{75}$$

Step:1

To find $y_1 = y(x_1) = y(1.2)$

Put $n = 0$ in equation (74) and (75)

$$y_1^{(r+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(r)})\right] \tag{76}$$

$$\text{where } y_1^{(0)} = y_0 + hf(x_0, y_0) \tag{77}$$

Initial approximation from equation (77)

$$\begin{aligned} y_1^{(0)} &= 0.5 + 0.2\left[\frac{2y_0}{x_0} + (x_0)^3\right] \\ &= 0.5 + 0.2\left[\frac{(2)(0.5)}{1} + (1)^3\right] \\ y_1^{(0)} &= 0.9 \end{aligned}$$

First Approximation, put $r = 0$ in equation (76)

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 0.5 + \frac{0.2}{2}\left[\left(\frac{2y_0}{x_0} + (x_0)^3\right) + \left(\frac{2y_1^{(0)}}{x_1} + (x_1)^3\right)\right]$$

$$y_1^{(1)} = 1.0227$$

Second Approximations, put $r = 1$ in equation (76)

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 0.5 + \frac{0.2}{2}\left[\left(\frac{2y_0}{x_0} + (x_0)^3\right) + \left(\frac{2y_1^{(1)}}{x_1} + (x_1)^3\right)\right]$$

$$y_1^{(2)} = 1.043$$

Similarly, $y_1^{(3)} = 1.046$ and $y_1^{(4)} = 1.046$.
Since $y_1^{(3)}$ and $y_1^{(4)}$ are the same correct to 4 decimal places

$$y(1.2) = 1.046 \tag{78}$$

Step: To find $y_2 = y(x_2) = y(1.4)$
Put $n = 1$ in equations (74) and (75)

$$y_2^{(r+1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(r)})\right] \tag{79}$$

$$y_2^{(0)} = y_1 + h(x_1, y_1) \tag{80}$$

Initial Approximation from equation 80

$$y_2^{(0)} = 1.046 + 0.2\left[\frac{2y_1}{x_1} + (x_1)^3\right]$$
$$= 1.046 + 0.2\left[\frac{2(1.046)}{1.2} + (1.2)^3\right]$$
$$y_2^{(0)} = 1.740$$

First Approximation

Put $r = 0$ in equation (79)

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$
$$= 1.046 + \frac{0.2}{2}\left[\frac{2(1.046)}{1.2} + (1.2)^3 + \frac{2(1.74)}{1.4} + (1.4)^3\right]$$
$$y_2^{(1)} = 1.916.$$

Similarly,

$$y_2^{(2)} = 1.941, \qquad y_2^{(3)} = 1.944$$
$$y_2^{(4)} = 1.945, \qquad y_2^{(5)} = 1.945$$

Since, $y_2^{(4)}$ and $y_2^{(5)}$ are the same correct to the decimal places

$$y(1.4) = 1.945$$

Given

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0$$

The Runge-kutta method of 4^{th} order is given by

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where

$$\begin{aligned}
k_1 &= hf(x_n, y_n) \\
k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\
k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\
k_4 &= hf(x_n + h, y_n + k_3)
\end{aligned}$$

Example 5.4.1

Apply Runge Kutta Method of fourth order to find an approximate value of $y(0.1)$ and $y(0.2)$ of $\frac{dy}{dx} = x + y^2$, $y(0) = 1$, correct to three decimal places.

Given $f(x,y) = x + y^2$

$$x_0 = 0.1, \qquad y_0 = 1$$
$$x_1 = 0.1, \qquad x_2 = 0.2$$

$$h = x_1 - x_0 = 0.1 - 0 = 0.1$$

We have to calculate y_1 and y_2.
The Runge-Kutta method of 4^{th} order is given by

$$y_{n+1} = y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right],$$

where,

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$
$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$
$$k_4 = hf\left(x_n + h, y_n + k_3\right) \tag{81}$$

Step 1: Finding $y(0.1)$ i.e. y_1
Put $n = 0$ in equation (81)

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \tag{82}$$

where

$$\begin{aligned}
k_1 &= hf(x_0, y_0) = h\left[x_0 + (y_0)^2\right] \\
&= 0.1\left[0 + (1)^2\right] \\
k_1 &= 0.1 \\
k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
k_2 &= h\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)^2\right] \\
&= 0.1\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)^2\right] \\
k_2 &= 0.1152
\end{aligned}$$

$$
\begin{aligned}
k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
&= h\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_2}{2}\right)^2\right] \\
&= 0.1\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1152}{2}\right)^2\right] \\
k_3 &= 0.1168
\end{aligned}
$$

$$
\begin{aligned}
k_4 &= hf\left(x_0 + h, y_0 + k_3\right) \\
&= h\left[(x_0 + h) + (y_0 + k_3)^2\right] \\
&= 0.1\left[(0 + 0.1) + (1 + 0.1168)^2\right] \\
k_4 &= 0.1347
\end{aligned}
$$

Substitute y_0, k_1, k_2, k_3 and k_4 in equation (82), we get

$$\begin{aligned} y_1 &= 1 + \frac{1}{6}\left[0.1 + 2(0.1152) + 2(0.1168) + 0.1347\right] \\ y_1 &= 1.1164 \\ \text{i.e., } y(0.1) &= 1.1164 \end{aligned}$$

Step 2: Finding $y(0.2)$ i.e., y_2
Put $x = 1$ in equation 81, we get

$$y_2 = y_1 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

where,

$$\begin{aligned} k_1 &= hf(x_1, y_1) \\ k_1 &= 0.1\left[x_1 + y_1^2\right] \\ &= 0.1\left[0.1 + (1.1164)^2\right] \\ k_1 &= 0.1346 \end{aligned}$$

$$
\begin{aligned}
k_2 &= hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) \\
&= h\left[\left(x_1 + \frac{h}{2}\right) + \left(y_1 + \frac{k_1}{2}\right)^2\right] \\
&= 0.1\left[\left(0.1 + \frac{0.1}{2}\right) + \left(1.1164 + \frac{0.1346}{2}\right)^2\right] \\
k_2 &= 0.1551
\end{aligned}
$$

$$
\begin{aligned}
k_3 &= hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) \\
&= h\left[\left(x_1 + \frac{h}{2}\right) + \left(y_1 + \frac{k_2}{2}\right)^2\right] \\
&= 0.1\left[\left(0.1 + \frac{0.1}{2}\right) + \left(1.1164 + \frac{0.1551}{2}\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
k_4 &= hf\left[x_1 + h, y_1 + k_3\right] \\
&= h\left[(x_1 + h) + (y_1 + k_3)^2\right] \\
&= 0.1\left[(0.1 + 0.1) + (1.1164 + 0.1575)^2\right] \\
k_4 &= 0.1822
\end{aligned}
$$

Substituting y_1, k_1, k_2, k_3 and k_4 in equation (81), we get

$$
\begin{aligned}
y_2 &= 1.1164 + \frac{1}{6}\left[0.1346 + 2(0.1575) + 0.1822\right] \\
y_2 &= 1.2734 \\
\text{i.e., } y(0.2) &= 1.2734.
\end{aligned}
$$

Example 5.4.2

Use Runge-kutta method of fourth order to approximate y when $x = 0.1$, given that $y = 1$ when $x = 0$, $\frac{dy}{dx} = x + y$, correct to 4 decimal places.

Given,

$$
\begin{aligned}
f(x, y) &= x + y \\
x_0 = 0, y_0 &= 1, x_1 = 0.1 \\
h = x_1 - x_0 &= 0.1 - 0 = 0.1 \\
y_1 = ? &\Rightarrow y(0.1) = ?
\end{aligned}
$$

The Runga-Kutta method of 4^{th} order is given by

$$y_{n+1} = y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \tag{83}$$

$$\text{where } k_1 = hf(x_n, y_n) \tag{84}$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \tag{85}$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_n + h, y_n + k_3\right) \tag{86}$$

Put $n = 0$ in equation (83)

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

where,

$$\begin{aligned}
k_1 &= hf(x_0, y_0) \\
k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
k_4 &= hf\left(x_0 + h, y_0 + k_3\right)
\end{aligned} \tag{87}$$

$$\therefore\ k_1 = 0.1\left[x_0 + y_0\right]$$
$$= 0.1\left[0 + 1\right] = 0.1$$
$$k_2 = 0.1\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)\right]$$
$$= 0.1\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)\right]$$
$$k_2 = 0.11$$
$$k_3 = 0.1\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_2}{2}\right)\right]$$
$$= 0.1\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.11}{2}\right)\right]$$
$$k_3 = 0.1105$$
$$k_4 = 0.1\left[(x_0 + h) + (y_0 + k_3)\right]$$
$$= 0.1\left[(0 + 0.1) + (1 + 0.1105)\right]$$
$$k_4 = 0.1210$$

$$k_1 = 0.1,\ k_2 = 0.11,\ k_3 = 0.1105,\ k_4 = 0.1210,\ y_0 = 1.$$

Substitute all the values in equation (87), we get

$$\begin{aligned} y_1 &= 1 + \frac{1}{6}\left[0.1 + 2(0.11) + 2(0.1105) + 0.1210\right] \\ y_1 &= 1.1103 \\ i.e.\ y(0.1) &= 1.1103 \end{aligned}$$

Example 5.4.3

Use Runge-kutta method of fourth order to obtain an approximation to $y(1.5)$ for the solution of $\frac{dy}{dx} = 2xy$, $y(1) = 1$, correct to 4 decimal places.

Hint:

$$
\begin{aligned}
f(x,y) &= 2xy, x_0 = 1, y_0 = 1 \\
x_1 &= 1.5 \\
h &= x_1 - x_0 = 1.5 - 1 = 0.5 \\
y_1 &= 3.4543
\end{aligned}
$$

Given

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0$$

Milnet's predictor and corrector formula is given by

$$y_{4,p} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \qquad (\textit{Predictor Formula}) \qquad (88)$$

$$y_{4,c}^{(r+1)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^{(r)}\right) \qquad (\textit{Corrector Formula}) \qquad (89)$$

where

$$f_1 = f(x_1, y_1), \qquad f_2 = f(x_2, y_2), \qquad f_3 = f(x_3, y_3)$$

$$f_4^{(r)} = f(x_4, y_4^{(r)}), \qquad f_4^{(0)} = f(x_4, y_4^{(0)})$$

where

$$y_4^{(0)} = y_{4,p}, \; for \; r = 0, \qquad y_4^{(r)} = y_{4,c}^{(r)}, \; for \; r \neq 0.$$

Example 5.5.1

Solve the initial value problem $\frac{dy}{dx} = 1 + xy^2$, $y(0) = 1$ for $x = 0.4$ by Milne's predictor and corrector method correct to three decimal places, given that

x	0.1	0.2	0.3
y	1.105	1.223	1.355

Given $f(x,y) = 1 + xy^2, h = x_2 - x_1 = 0.1$

$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$	$x_4 = 0.4$
$y_0 = 1$	$y_1 = 1.105$	$y_2 = 1.223$	$y_3 = 1.355$	$y_4 = ?$

Milne's predictor formula is given by

$$y_{4,p} = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right) \tag{90}$$

x_i	y_i	$f_i = f(x_i, y_i) = 1 + x_i y_i^2$
x_1=0.1	y_1=1.105	$f_1 = 1 + x_1 y_1^2$ =1+(0.1)(1.105)^2 =1.122
x_2=0.2	y_2=1.223	$f_2 = 1 + x_2 y_2^2$ =1+(0.2)(1.223)^2 =1.299
x_3=0.3	y_3=1.355	$f_3 = 1 + x_3 y_3^2$ =1+(0.3)(1.355)^2 =1.550

Substituting all the values in equation (90) we get,

$$y_{4,p} = 1 + \left(\frac{4(0.1)}{3}\right)[2(1.122) - 1.299 + 2(1.550)]$$
$$y_{4,p} = 1.526$$

Milne's corrector formula is given by

$$y_{4,c}^{(r+1)} = y_2 + \left(\frac{h}{3}\right)\left(f_2 + 4f_3 + f_4^{(r)}\right) \tag{91}$$

where

$$f_4^{(r)} = f(x_4, y_4^{(r)})$$
$$y_4^{(0)} = y_{4,p}, \; r = 0$$
$$y_4^{(r)} = y_{4,c}^{(r)}, \; r \neq 0.$$

First improvement:

Put $r = 0$ in equation (91)

$$y_{4,c}^{(1)} = y_2 + \left(\frac{h}{3}\right)\left(f_2 + 4f_3 + f_4^{(0)}\right)$$

where

$$\begin{aligned} f_4^{(0)} &= f(x_4, y_4^{(0)}) = f(x_4, y_{4,p}) \\ &= 1 + x_4\,(y_{4,p})^2 \\ f_4^{(0)} &= 1 + (0.4)(1.526)^2 = 1.931 \\ \therefore\ y_{4,c}^{(1)} &= 1.223 + \left(\frac{0.1}{3}\right)(1.299 + 4(1.550) + 1.931) \\ y_{4,c}^{(1)} &= 1.537 \end{aligned}$$

Second improvement:

Put $r = 1$ in equation (91)

$$y_{4,c}^{(2)} = y_2 + \left(\frac{h}{3}\right)\left(f_2 + 4f_3 + f_4^{(1)}\right)$$

where

$$\begin{aligned} f_4^{(1)} &= f\left(x_4, y_4^{(1)}\right) \\ &= 1 + x_4\left(y_{4,c}^{(1)}\right)^2 \\ &= 1 + (0.4)(1.537)^2 \\ f_4^{(1)} &= 1.944 \\ \therefore\ y_{4,c}^{(2)} &= 1.223 + \left(\frac{0.1}{3}\right)(1.299 + 4(1.550 + 1.944)) \\ y_{4,c}^{(2)} &= 1.537 \end{aligned}$$

Since, $y_{4,c}^{(1)}$ and $y_{4,c}^{(2)}$ are the same up to three decimal places

$$y(0.4) = 1.537$$