If $\frac{e_{i+1}}{e_i}$ is almost constant, convergence is said to be linear i.e. slow.
If $\frac{e_{i+1}}{e_i^p}$ is nearly constant, convergence is said to be of order p i.e. faster.

2.3 Secant Method

This method is an improvement over the method of false position as it does not require the condition $f(x_0)f(x_1) < 0$ of that method (Fig. 2.5). Here also the graph of the function $y = f(x)$ is approximated by a secant line but at each iteration, two most recent approximations to the root are used to find the next approximation. Also it is not necessary that the interval must contain the root.

Taking x_0, x_1 as the initial limits of the interval, we write the equation of the chord joining these as

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1)$$

Then the abscissa of the point where it crosses the x-axis $(y = 0)$ is given by

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

which is an approximation to the root. The general formula for successive approximations is, therefore, given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), n \geq 1$$

2.3.1 Rate of Convergence

If at any iteration $f(x_n) = f(x_{n-1})$, this method fails and shows that it does not converge necessarily. This is a drawback of secant method over the method of false position which always converges. But if the secant method once converges, its rate of convergence is 1.6 which is faster than that of the method of false position.

■ **Example 2.1** Find a root of the equation $x^3 - 2x - 5 = 0$ using secant method correct to three decimal places. ■

Let $f(x) = x^3 - 2x - 5$ so that

$$f(0) = (0)^3 - 2(0) - 5 = -5 = -ve$$
$$f(1) = (1)^3 - 2(1) - 5 = -6 = -ve$$
$$f(2) = (2)^3 - 2(2) - 5 = -1 = -ve$$
$$f(3) = (3)^3 - 2(3) - 5 = 16 = +ve$$

$\therefore$ Taking initial approximations $x_0 = 2$ and $x_1 = 3$, by secant method, we have

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

So we get,

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 3 - \frac{3-2}{16+1} 16 = 2.058823$$

Now $f(x_2) = -0.390799$

$$\therefore \quad x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 2.081263$$

and

$$f(x_3) = -0.147204$$

Therefore,

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.094824$$

and

$$f(x_4) = 0.003042$$

Therefore,

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) = 2.094549$$

Hence the root is 2.094 correct to 3 decimal places.

■ **Example 2.2** Find the root of the equation $xe^x = \cos x$ using the secant method correct to four decimal places. ■

Let $f(x) = \cos x - xe^x = 0$. So that

$$f(0) = \cos 0 - (0)e^{(0)} = 1 = +ve$$
$$f(1) = \cos(1) - (1)e^{(1)} == 0.5403 - 2.7183 = -2.17798 = -ve$$

Taking the initial approximations $x_0 = 0, x_1 = 1$.

Then by secant method, we have

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 1 + \frac{1}{3.17798}(-2.17798) = 0.31467$$

Now $f(x_2) = 0.51987$

Therefore

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 0.44673$$

and

$$f(x_3) = 0.20354$$

Therefore

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 0.53171.$$

Repeating this process, the successive approximations are $x_5 = 0.51690, x_6 = 0.51775$, $x_7 = 0.51776$ etc.

Hence the root is 0.5177 correct to 4 decimal places.

4. Using Regula falsi method, compute the real root of the following equations correct to three decimal places:
   (a) $xe^x = 2$
   (b) $\cos x = 3x - 1$
   (c) $xe^x = \sin x$
   (d) $x \tan x = -1$
   (e) $2x - \log x = 7$
   (f) $3x + \sin x = e^x$.
5. Find the fourth root of 12 correct to three decimal places by interpolation method.
6. Locate the root of $f(x) = x^{10} - 1 = 0$, between 0 and 1.3 using bisection method and method of false position. Comment on which method is preferable.
7. Find a root of the following equations correct to three decimal places by the method :
   (a) $x^3 + x^2 + x + 7 = 0$
   (b) $x - e^{-x} = 0$
   (c) $x \log_{10} x = 1.9$.
8. Use the iteration method to find a root of the equations to four decimal places:
   (a) $x^3 + x^2 - 100 = 0$
   (b) $x^3 - 9x + 1 = 0$
   (c) $x = \frac{1}{2} + \sin x$
   (d) $\tan x = x$
   (e) $e^x - 3x = 0$
   (f) $2^x - x - 3 = 0$ which lies between $(-3, -2)$
9. Evaluate $\sqrt{30}$ by (i) secant method (ii) iteration method correct to four decimal places.
10. Find the root of the equation $2x = \cos x + 3$ correct to three decimal places using Iteration method.
11. Find the real root of the equation $x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \ldots = 0.443$, correct to three decimal places using iteration method.

2.6 Newton-Raphson Method

Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root then $f(x_1) = 0$.

$\therefore$ Expanding $f(x_0 + h)$ by Taylor's series $f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots = 0$.
Since h is small, neglecting h^2 and higher powers of h, we get $f(x_0) + hf'(x_0) = 0$
or

$$h = -\frac{f(x_0)}{f'(x_0)} \tag{2.10}$$

$\therefore$ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,

$$\mathbf{x_{n+1}} = \mathbf{x_n} - \frac{\mathbf{f}(\mathbf{x_n})}{\mathbf{f}'(\mathbf{x_n})} \quad (n = 0, 1, 2 \ldots) \tag{2.11}$$

which is known as the Newton-Raphson formula or Newton's iteration formula.

Note 2.6.1 Newton's method is useful in cases of large values of $f'(x)$ i.e., when the graph of $f(x)$ while crossing the x-axis is nearly vertical.

For if $f'(x)$ is small in the vicinity of the root, then by (2.10) h will be large and the computation of the root is slow or may not be possible. Thus this method is not suitable in those cases where the graph of $f(x)$ is nearly horizontal while crossing the x-axis.

Note 2.6.2 **Geometrical interpretation.** Let x_0 be a point near the root α of the equation $f(x)=0$ (Fig. 2.7). Then the equation of the tangent at $A_0\left[x_0, f\left(x_0\right)\right]$ is

$$y-f\left(x_0\right)=f'\left(x_0\right)\left(x-x_0\right).$$

It cuts the x-axis at $x_1=x_0=\frac{f(x_0)}{f'(x_0)}$. which is a first approximation to the root α. If A_1 is the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x-axis at x_2 which is nearer to α and is, therefore, a second approximation to the root. Repeating this process, we approach the root α quite rapidly. Hence the method consists in replacing the part of the curve between the point A_0 and the x-axis by means of the tangent to the curve at A_0.

Note 2.6.3 Newton's method is generally used to improve the result obtained by other methods. It is applicable to the solution of both algebraic and transcendental equations.

2.6.1 Convergence of Newton-Raphson Method

Newton's formula converges provided the initial approximation x_0 is chosen sufficiently close to the root.

If it is not near the root, the procedure may lead to an endless cycle. A bad initial choice will lead one astray. Thus a proper choice of the initial guess is very important for the success of the Newton's method.

Comparing (2.11) with the relation $x_{n+1}=\phi\left(x_n\right)$ of the iteration method, we get

$$\phi\left(x_n\right)=x_{n+1}=x_n-\frac{f\left(x_n\right)}{f'\left(x_n\right)}$$

In general, $\quad \phi(x)=x-\frac{f(x)}{f'(x)} \quad$ which gives $\quad \phi'(x)=\frac{f(x)f''(x)}{[f'(x)]^2}$.

Since the iteration method (\$2.10) converges if $|\phi'(x)|<1$

$\therefore$ Newton's formula will converge if $|f(x)f''(x)|<|f'(x)|^2$ in the interval considered.

Assuming $f(x), f'(x)$ and $f''(x)$ to be continuous, we can select a small interval in the vicinity of the root α, in which the above condition is satisfied. Hence the result.

Newtons method converges conditionally while Regula-falsi method always converges. However when once Newton-Raphson method converges, it converges faster and is preferred.

2.6.2 Newton's method has a quadratic convergence

Suppose x_n differs from the root α by a small quantity ε_n so that

$$x_0=\alpha+\varepsilon_n \text{ and } x_{n+1}=\alpha+\varepsilon_{n+1}$$

Then (2.11) becomes $\alpha+\varepsilon_{n+1}=\alpha+\varepsilon_n-\frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)}$

i.e.

$$\begin{aligned}
\varepsilon_{n+1} &= \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)} \\
&= \varepsilon_n - \frac{f(\alpha)+\varepsilon_n f'(\alpha)+\frac{1}{2!}\varepsilon_n^2 f''(\alpha)+\ldots\ldots}{f'(\alpha)+\varepsilon_n f''(\alpha)+\ldots\ldots} \text{ by Taylor's expansion.} \\
&= \varepsilon_n - \frac{\varepsilon_n f'(\alpha)+\frac{1}{2}{\varepsilon_n}^2 f''(\alpha)+\ldots\ldots}{f'(\alpha)+\varepsilon_n f''(\alpha)+\ldots\ldots} = \frac{{\varepsilon_n}^2}{2}\frac{f''(\alpha)}{f'(\alpha)}. \quad [\because f(\alpha)=0]
\end{aligned}$$

This shows that the subsequent error at each step, is proportional to the square of the previous error and as such the convergence is quadratic. Thus Newton-Raphson method has second order convergence.

■ **Example 2.8** Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method. ■

Let $f(x) = x^4 - x - 10$.
so that

$$\begin{aligned}
f(0) &= 0^4 - 0 - 10 = -10 = -ve \\
f(1) &= 1^4 - 1 - 10 = -10 = -ve, \\
f(2) &= 2^4 - 2 - 10 = 16 - 2 - 10 = 4 = +ve.
\end{aligned}$$

$\therefore$ A root of $f(x) = 0$ lies between 1 and 2.
Let us take $x_0 = 2$.
Also $f'(x) = 4x^3 - 1$.
Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.12}$$

Putting $n = 0$, the first approximation x_1 is given by

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} \\
&= 2 - \frac{4}{4\times 2^3 - 1} = 2 - \frac{4}{31} = 1.871
\end{aligned}$$

Putting $n = 1$ in (2.12), the second approximation is

$$\begin{aligned}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)} \\
&= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1} \\
&= 1.871 - \frac{0.3835}{25.199} = 1.856
\end{aligned}$$

Putting $n = 2$ in (2.12), the third approximation is

$$\begin{aligned}
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1} \\
&= 1.856 - \frac{0.010}{24.574} = 1.856
\end{aligned}$$

Here $x_2 = x_3$. Hence the desired root is 1.856 correct to three decimal places.

■ **Example 2.9** Find by Newton's method, the real root of the equation $3x = \cos x + 1$, correct to four decimal places. ■

Let

$$
\begin{aligned}
f(x) &= 3x - \cos x - 1 \\
f(0) &= -2 = -ve \\
f(1) &= 3 - 0.5403 - 1 = 1.4597 = +ve.
\end{aligned}
$$

So a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1 . Let us take $x_0 = 0.6$.

Also

$$f'(x) = 3 + \sin x$$

$\therefore$ Newton's iteration formula gives

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} \\
&= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}
\end{aligned}
\tag{2.13}
$$

Putting $n = 0$, the first approximation x_1 is given by

$$
\begin{aligned}
x_1 &= \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)} \\
&= \frac{0.6 \times 0.5729 + 0.82533 + 1}{3 + 0.5729} = 0.6071
\end{aligned}
$$

Putting $n = 1$ in (2.13), the second approximation is

$$
\begin{aligned}
x_2 &= \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)} \\
&= \frac{0.6071 \times 0.57049 + 0.8213 + 1}{3 + 0.57049} = 0.6071
\end{aligned}
$$

Here $x_1 = x_2$. Hence the desired root is 0.6071 correct to four decimal places.

■ **Example 2.10** Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places. ■

Let

$$
\begin{aligned}
& f(x) = x \log_{10} x - 1.2 \\
& f(0) = 0 - 1.2 = -\text{ ve}, \\
& f(1) = 0 - 1.2 = -\text{ ve}, \\
& f(2) = 2 \log_{10} 2 - 1.2 = 0.59794 = -\text{ ve} \\
\text{and} \quad & f(3) = 3 \log_{10} 3 - 1.2 = 1.4314 - 1.2 = 0.23136 = +\text{ ve}.
\end{aligned}
$$

So a root of $f(x) = 0$ lies between 2 and 3 . Let us take $x_0 = 2$.

Also

$$f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e = \log_{10} x + 0.43429 \qquad [\log_{10} e = 0.43429]$$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + 0.43429} = \frac{0.43429 x_n + 1.2}{\log_{10} x_n + 0.43429} \tag{2.14}$$

Putting $n = 0$, the first approximation is

$$\begin{aligned} x_1 &= \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 1.2}{\log_{10} 2 + 0.43429} \\ &= \frac{0.86858 + 1.2}{0.30103 + 0.43429} = 2.81 \end{aligned}$$

Similarly putting $n = 1, 2, 3, 4$ in (1), we get

$$\begin{aligned} x_2 &= \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741 \\ x_3 &= \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.741 + 0.43429} = 2.74064 \\ x_4 &= \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.74064 + 0.43429} = 2.74065 \\ x_5 &= \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429} = 2.74065 \end{aligned}$$

Here $x_4 = x_5$. Hence the required root is 2.74065 correct to five decimal places.

2.6.3 Some Deductions from Newton-Raphson Formula

Theorem 2.6.4 We can derive the following useful results from the Newton's iteration formula :

1. Iterative formula to find $1/N$ is $\mathbf{x_{n+1} = x_n (2 - N x_n)}$.
2. Iterative formula to find $\sqrt{N}$ is $\mathbf{x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)}$.
3. Iterative formula to find $1/\sqrt{N}$ is $\mathbf{x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{N x_n}\right)}$.
4. Iterative formula to find $\sqrt[k]{N}$ is $\mathbf{x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n{}^{k-1}}\right)}\Big]$.

(1) Let $x = 1/N$ or $1/x - N = 0$.
Taking $f(x) = 1/x - N$, we have $f'(x) = -x^{-2}$.
Then Newton's formula gives

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-x_n^{-2}} = x_n + \left(\frac{1}{x_n} - N\right) x_n^2 \\ &= x_n + x_n - N x_n^2 = x_n (2 - N x_n) \end{aligned}$$

(2) Let $x = \sqrt{N}$ or $x^2 - N = 0$.
Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$.
Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n{}^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

(3) Let $x = \frac{1}{\sqrt{N}}$ or $x^2 - \frac{1}{N} = 0$.
Taking $f(x) = x^2 - \frac{1}{N}$, we have $f'(x) = 2x$.
Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n{}^2 - \frac{1}{N}}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{N x_n}\right)$$

(4) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$.
Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$.
Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \frac{{x_n}^k - N}{k{x_n}^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right].$$

■ **Example 2.11** Evaluate the following (correct to four decimal places) by Newton's iteration method:

1. $\frac{1}{31}$
2. $\sqrt{5}$
3. $\frac{1}{\sqrt{14}}$
4. $\sqrt[3]{24}$
5. $(30)^{\frac{-1}{5}}$

■

(1). Since, Iterative formula to find $1/N$ is $\mathbf{x_{n+1} = x_n\,(2 - Nx_n)}$.
Taking $N = 31$ in the above formula, we get

$$x_{n+1} = x_n\,(2 - 31x_n)$$

Since an approximate value of $1/31 = 0.03$, we take $x_0 = 0.03$. Then

$$\begin{aligned}
x_1 &= x_0\,(2 - 31x_0) = 0.03(2 - 31 \times 0.03) = 0.0321\\
x_2 &= x_1\,(2 - 31x_1) = 0.0321(2 - 31 \times 0.0321) = 0.032257\\
x_3 &= x_2\,(2 - 31x_2) = 0.032257(2 - 31 \times 0.032257) = 0.03226
\end{aligned}$$

Since $x_2 = x_3$ upto 4 decimal places, we have $\frac{1}{31} = 0.0323$.

(2). Since, Iterative formula to find $\sqrt{N}$ is $\mathbf{x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)}$.
Taking $N = 5$ in the above formula, we get

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right).$$

Since an approximate value of $\sqrt{5} = 2$, we take $x_0 = 2$. Then

$$\begin{aligned}
x_1 &= \frac{1}{2}\left(x_0 + \frac{5}{x_0}\right) = \frac{1}{2}\left(2 + \frac{5}{2}\right) = 2.25\\
x_2 &= \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right) = 2.2361\\
x_3 &= \frac{1}{2}\left(x_2 + \frac{5}{x_2}\right) = 2.2361
\end{aligned}$$

Since $x_2 = x_3$ upto 4 decimal places, we have $\sqrt{5} = 2.2361$.

(3). Since Iterative formula to find $\frac{1}{\sqrt{N}}$ is $\mathbf{x_{n+1}} = \frac{1}{2}\left(\mathbf{x_n} + \frac{1}{\mathbf{Nx_n}}\right)$.
Taking $N = 14$ in the above formula, we get

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{1}{14x_n}\right].$$

Since an approximate value of $\frac{1}{\sqrt{14}} = \frac{1}{\sqrt{16}} = \frac{1}{4} = 0.25$, we take $x_0 = 0.25$, Then

$$x_1 = \frac{1}{2}\left[x_0 + \frac{1}{14x_0}\right] = \frac{1}{2}\left[0.25 + \frac{1}{14\times 0.25}\right] = 0.26785$$
$$x_2 = \frac{1}{2}\left[x_1 + \frac{1}{14x_1}\right] = \frac{1}{2}\left[0.26785 + \frac{1}{14\times 0.26785}\right] = 0.2672618$$
$$x_3 = \frac{1}{2}\left[x_2 + \frac{1}{14x_2}\right] = \frac{1}{2}\left[0.2672618 + \frac{1}{14\times 0.2672618}\right] = 0.2672612$$

Since $x_2 = x_3$ upto 4 decimal places, we take $\frac{1}{\sqrt{14}} = 0.2673$.

(4). Since Iterative formula to find $\sqrt[k]{N}$ is $\mathbf{x_{n+1}} = \frac{1}{\mathbf{k}}\left[(\mathbf{k}-1)\mathbf{x_n} + \frac{\mathbf{N}}{\mathbf{x_n}^{\mathbf{k}-1}}\right]$.
Taking $N = 24$ and $k = 3$, in the above formula, we get

$$x_{n+1} = \frac{1}{3}\left[2x_n + \frac{24}{x_n}\right].$$

Since an approximate value of $(24)^{\frac{1}{3}} = (27)^{\frac{1}{3}} = 3$, we take $x_0 = 3$. Then

$$x_1 = \frac{1}{3}\left(2x_0 + \frac{24}{x_0^2}\right) = \frac{1}{3}\left(6 + \frac{24}{9}\right) = 2.88889$$
$$x_2 = \frac{1}{3}\left(2x_1 + \frac{24}{x_1^2}\right) = \frac{1}{3}\left[(2\times 2.88889) + \frac{24}{2.88889^2}\right] = 2.88451$$
$$x_3 = \frac{1}{3}\left(2x_2 + \frac{24}{x_2^2}\right) = \frac{1}{3}\left[2\times 2.88451 + \frac{24}{2.88451^2}\right] = 2.8845$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(24)^{\frac{1}{3}} = 2.8845$.

(5). Since Iterative formula to find $\sqrt[k]{N}$ is $\mathbf{x_{n+1}} = \frac{1}{\mathbf{k}}\left[(\mathbf{k}-1)\mathbf{x_n} + \frac{\mathbf{N}}{\mathbf{x_n}^{\mathbf{k}-1}}\right]$.
Taking $N = 30$ and $k = -5$, in the above formula, we get

$$x_{n+1} = \frac{1}{-5}\left(6x_n + \frac{30}{x_n^{-6}}\right) = \frac{x_n}{5}\left(6 - 30x_n^5\right)$$

Since an approximate value of $(30)^{\frac{-1}{5}} = (32)^{\frac{-1}{5}} = \frac{1}{2}$, we take $x_0 = \frac{1}{2}$, Then

$$x_1 = \frac{x_0}{5}\left(6 - 30x_0^5\right) = \frac{1}{10}\left(6 - \frac{30}{2^5}\right) = 0.50625$$
$$x_2 = \frac{x_1}{5}\left(6 - 30x_1^5\right) = \frac{0.50625}{5}\left[6 - 30(0.50625)^5\right] = 0.506495$$
$$x_3 = \frac{x_2}{5}\left(6 - 30x_2^5\right) = \frac{0.506495}{5}\left[6 - 30(0.506495)^5\right] = 0.506496$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(30)^{\frac{-1}{5}} = 0.5065$.

2.7 Practice Problem

1. Find by Newton-Raphson method, a root of the following equations correct to 3 decimal places :
   (a) $x^3 - 3x + 1 = 0$
   (b) $x^3 - 2x - 5 = 0$
   (c) $x^3 - 5x + 3 = 0$

(d) $3x^3 - 9x^2 + 8 = 0$.

2. Using Newton's iterative method, find a root of the following equations correct to 4 decimal places :
   (a) $x^4 + x^3 - 7x^2 - x + 5 = 0$ which lies between 2 and 3.
   (b) $x^5 - 5x^2 + 3 = 0$
3. Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to 2 decimal places by Newton's method.
4. Using Newton-Raphson method, find a root of the following equations correct to 3 decimal places :
   (a) $x^2 + 4\sin x = 0$
   (b) $x\sin x + \cos x = 0$ or $x\tan x + 1 = 0$
   (c) $e^x = x^3 + \cos 25x$ which is near 4.5 .
   (d) $x\log_{10} x = 12.34$, start with $x_0 = 10$.
   (e) $\cos x = xe^x$
   (f) $10^x + x - 4 = 0$
5. The equation $2e^{-x} = \frac{1}{x+2} + \frac{1}{x+1}$ has two roots greater than -1 . Calculate these roots correct to five decimal places.
6. The bacteria concentration in a reservoir varies as $C = 4e^{-2t} + e^{-0.1t}$. Using Newton Raphson method, calculate the time required for the bacteria concentration to be 0.5 .
7. Use Newton's method to find the smallest root of the equation $e^x \sin x = 1$ to four places of decimal.
8. The current i in an electric circuit is given by $i = 10e^{-t}\sin 2\pi t$ where t is in seconds. Using Newton's method, find the value of t correct to 3 decimal places for $i = 2$amp.
9. Find the iterative formulae for finding $\sqrt{N}, \sqrt[3]{N}$ where N is a real number, using Newton-Raphson formula. Hence evaluate : (a) $\sqrt{15}$. (b) $\sqrt{21}$ (c) the cube-root of 17 to three places of decimal.
10. Develop an algorithm using N.R. method, to find the fourth root of a positive number N. Hence find $\sqrt[4]{32}$.
11. Evaluate the following (correct to 3 decimal places) by using the Newton-Raphson method.
    (a) $1/18$
    (b) $1/\sqrt{15}$
    (c) $(28)^{-1/4}$.