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Operators

Finite difference operators :-

Let the tabular points $x_0, x_1, x_2, x_3 \dots x_n$ be Equispaced with step length h, that

$$x_i = x_0 + ih \quad i = 1, 2, \dots n$$

or

$$x_{i+1} - x_i = h \quad \text{for all } i$$

For Equispace data we have to define the following data -

$$\xleftarrow{\quad h \quad}$$
$x_0 \qquad x_1$

Shift operators (E) -

$$E f(x_i) = f(x_i + h) = f(x_{i+1}) = f_{i+1}$$

that

$$E f(x_0) = f(x_0 + h) = f_1$$
$$E f(x_1) = f(x_1 + h) = f(x_2) \dots\dots$$

$$E^2 f(x_i) = E[E f(x_i)] = E \cancel{f(x_i + h)}$$

$$E[f(x_{i+1})] = f(x_{i+2}) = \underline{f_{i+2}} \text{ Ans}$$

$$\therefore \quad E^3 f(x_i) = f_{i+3}$$

In general,

$$\boxed{E^K f(x_i) = f(x_i + Kh) = f(x_{i+K}) = f_{i+K}}$$

where K is any real number

Ex -

$$E^{1/2} f(x_i) = f(x_i + \tfrac{1}{2} h) = f_{i+\frac{1}{2}}$$

Forward difference operator (Δ):-

$$\Delta f(x_i) = f(x_i + h) - f(x_i)$$
$$= f(x_{i+1}) - f(x_i)$$
$$= f_{i+1} - f_i \quad — (2)$$

We define for any positive integer n

$$\left\{ \Delta^n f(x_i) = \Delta^{n-1}[\Delta f(x_i)] = \Delta^{n-1}[f_{i+1} - f_i] \right.$$
$$\left. = \Delta^{n-1}[f_{i+1}] - \Delta^{n-1} f_i \right\}$$

In particular-

$$\Delta^2 f(x_i) = \Delta^{2-1}[\Delta f(x_i)] = \Delta^1[f_{i+1} - f_i]$$

$\{\Delta f_{i+1} = f_{i+2} - f_{i+1}\}$

$$= \Delta f_{i+1} - \Delta f_i$$
$$= f_{i+2} - f_{i+1} - f_{i+1} + f_i$$
$$= f_{i+2} - 2f_{i+1} + f_i$$

(*) $$\boxed{\Delta^3 f(x_i) = f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i}$$

Now, from Eqⁿ (1) and (2) then-

$\{\because E f(x_i) = f(x_{i+1})\}$

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$$
$$= f(x_i + h) - f(x_i) = f_{i+1} - f_i \Rightarrow f(x_{i+1}) - f_i$$

comparing, we get $\Rightarrow E f(x_i) - f(x_i)$
$$\Rightarrow (E-1) f(x_i)$$

$\boxed{\Delta = E - 1}$ or $\boxed{E = \Delta + 1}$,

$$\boxed{\Delta^n f(x_i) = (E-1)^n f(x_i) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i+n-k}}$$

The forward differences can be written in a tabular form as

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	$f(x_0)$	$\Delta f_0 = f_1 - f_0$	$\Delta^2 f_0 = f_2 - 2f_1 + f_0$	$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$
x_1	$f(x_1)$	$\Delta f_1 = f_2 - f_1$	$\Delta^2 f_1 = f_3 - 2f_2 + f_1$	
x_2	$f(x_2)$	$\Delta f_2 = f_3 - f_2$		
x_3	$f(x_3)$			

Backward difference operator ∇ -

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = f_i - f_{i-1} \quad \text{— (3)}$$

we define

$$\nabla^n f(x_i) = \nabla^{n-1}[\nabla f(x_i)] = \nabla^{n-1}[f_i - f_{i-1}]$$

In particular -

$$\Rightarrow \nabla^{n-1} f_i - \nabla^{n-1} f_{i-1}$$

$$\nabla^2 f_i = \nabla[\nabla f(x_i)] = \nabla[f_i - f_{i-1}] = \nabla f_i - \nabla f_{i-1}$$
$$= f_i - 2f_{i-1} + f_{i-2}$$

Similarly,

$Ef(x_i) = f(x_{i+1})$

$$\nabla^3 f(x_i) = \nabla(\nabla^2 f(x_i)] = \nabla[f_i - 2f_{i-1} + f_{i-2}]$$
$$= f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}$$

Now, from (1) & (3)

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = f(x_i) - E^{-1} f(x_i)$$
$$\nabla f(x_i) = (1 - E^{-1}) f(x_i)$$

compare

$$\boxed{\nabla = 1 - E^{-1}} \quad \text{or} \quad \boxed{E^{-1} = 1 - \nabla}$$

$f_i - f_{i-1}$

$$\text{or} \quad \boxed{E = (1-\nabla)^{-1}}$$

$-2(f_{i-1} - f_{i-2})$

$f_{i-2} - f_{i-3}$

$E^k f(x_i) = f(x_i + kh)$

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x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
x_0	$f(x_0)$	$f_0 - f_0$		
x_1	$f(x_1)$	$\nabla f_1 = f_1 - f_0$		
x_2	$f(x_2)$	$\nabla f_2 = f_2 - f_1$	$\nabla^2 f_2 = f_2 - 2f_1 + f_0$	
x_3	$f(x_3)$	$\nabla f_3 = f_3 - f_2$	$\nabla^2 f_3 = f_3 - 2f_2 - f_1$	$\nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2$

Central Difference operator δ -

$$\delta f(x_i) = f(x_i + h/2) - f(x_i - h/2)$$
$$= f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \quad - (4)$$

$$\delta f(x_i + h/2) = \delta f_{i+\frac{1}{2}}$$
$$= f_{i+1} - f_i \quad - (5)$$

$\to f(x_i - \frac{h}{2} + \frac{h}{2})$
$\to f(x_i)$
$\to f_i$

$$f\left(x_i + \frac{h}{2} + \frac{h}{2}\right) = f(x_i + h) = f(x_{i+1})$$
$$= f_{i+1}$$

$$\delta^2 f(x_i) = \delta[\delta f(x_i)] = \delta\left[f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)\right]$$
$$= \delta\left[f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right]$$
$$= \delta[f_{i+\frac{1}{2}}] - \delta(f_{i-\frac{1}{2}})$$
$$= f_{i+1} - f_i + f_{i-1} - f_i$$
$$= f_{i+1} - 2f_i + f_{i-1}$$

$$\delta^3 f(x_i) = \delta(\delta^2 f(x_i)) = f_{i+2} - 3f_{i+1} + 3f_i - f_{i-1}$$

f_0

Now, from ① and ④, we get -

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

Average (Mean) operator 'μ'

$$\mu f(x_i) = \frac{1}{2}\left[f\left(x_i + \frac{h}{2}\right) + f\left(x_i - \frac{h}{2}\right)\right] =$$

$$\frac{1}{2}\left[f_{i+\frac{1}{2}} + f_{(i-\frac{1}{2})}\right]$$

$$= \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x_i)$$

comparing -

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$$

Q. Show that $\Delta^n f_i = \nabla^n f_{i+n} = \delta^n f_{i+\frac{n}{2}}$.

We have $\nabla = 1 - E^{-1} = (E-1)E^{-1} = \Delta E^{-1}$

$$\nabla^n f_{i+n} = \nabla^n E^{-n} f_{i+n} = \Delta^n f_i$$

$$\delta = (E^{1/2} - E^{-1/2}) = (E-1)E^{-1/2} = \Delta E^{-1/2}$$

$$\delta^n f_{i+\frac{n}{2}} = \Delta^n E^{-n/2} f_{i+\frac{n}{2}} = \Delta^n f_i$$

Relation between differences and derivatives -

using Taylor Series Expansion, we get

$$\Delta f(x) = f(x+h) - f(x) = hf'(x) + \frac{h^2}{2!} f''(x) + \dots$$

Neglect the higher order -

$$\boxed{\Delta f(x) \approx hf'(x) \text{ or } f'(x) \approx \frac{1}{h} \Delta f(x)}$$

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$$E f(x) = f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots\dots$$

$$= \left(1 + hD + \frac{h^2}{2!} D^2 + \dots\dots\right) f(x),$$

$$E f(x) = e^{hD} f(x)$$

$$\Rightarrow \boxed{E = e^{hD}} \quad \text{or} \quad hD = \log(E) = \log(1+\Delta)$$

$\ln(E)$

$$hD = \log(1-\nabla)^{-1} = -\log(1-\nabla)$$

$$\left[\delta = E^{1/2} - E^{-1/2} = e^{hD/2} - e^{-hD/2}\right]$$
$$= 2\sinh\left(\frac{hD}{2}\right)$$

To prove that following relations -

(i) $\Delta\left(\frac{1}{f_i}\right) = \frac{-\Delta f_i}{f_i f_{i+1}}$ (ii) $\Delta\left(\frac{f_i}{g_i}\right) = \frac{g_i \Delta f_i - f_i \Delta g_i}{g_i g_{i+1}}$

(iii) $\Delta(f_i^2) = (f_i + f_{i+1}) \Delta f_i$ (iv) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

(1) $\Delta\left(\frac{1}{f_i}\right) = \frac{1}{f_{i+1}} - \frac{1}{f_i} = \frac{f_i - f_{i+1}}{f_i f_{i+1}} = \frac{-\Delta f_i}{f_i f_{i+1}}$

(3) $\Delta(f_i^2) = f_{i+1}^2 - f_i^2 = (f_{i+1} + f_i)(f_{i+1} - f_i)$
$= \Delta f_i (f_{i+1} + f_i)$

(4) LHS - $\nabla + \Delta = (E-1) + (1-E^{-1}) = E - E^{-1}$

RHS = $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} =$

$$\frac{E-1}{(1-E^{-1})} - \frac{(1-E)^{-1}}{E-1}$$

$$= \frac{(1-E^{-1})\cdot E}{(1-E^{-1})} - \frac{E^{-1}(E-1)}{(E-1)}$$

RHS $= E - E^{-1}$

$$\boxed{\therefore LHS = RHS}$$

Q. Prove that the following ~~vector~~ relations-

① $\delta = \nabla(1-\nabla)^{-1/2}$ ② $\mu = \left(1+\frac{\delta^2}{4}\right)^{1/2}$

① $(1-E^{-1})(1-(1-E^{-1}))^{-1/2} = (1-E^{-1})(E^{1/2})$

$= E^{1/2} - E^{-1/2} = \delta$

$\nabla\Delta\nabla = \nabla\Delta$

$\{ \star \ \Delta\nabla = \nabla\Delta \ ?? \}$

Q. Find (a) Δe^{ax} (b) $\Delta^2 e^x$ (c) $\Delta \sin x$

(d) $\Delta \log x$ (e) $\Delta \tan^{-1} x$

Q. Prove that $E\Delta = \Delta E$

Q. Evalute $\left(\frac{\Delta^2}{E}\right) x^3$

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Interpolation and approximation

Let $f(x)$ be a continuous function in some interval $[a,b]$, and defined at $n+1$ distinct point $x_0, x_1, x_2, \dots x_n$, such that $a = x_0 < x_1 < x_2 < \dots\dots < x_n = b$. These points may be Equispaced. that is $x_{i+1} - x_i = h$, $i = 0, 1, 2, \dots n-1$.

Or non-Equispaced

The problem of poly. interpolation is to find a poly. $P(x)$ of degree $\le n$, which fits the given distinct data, Exactly. that $P_n(x_i) = f(x_i)$,

$$i = 0, 1, 2 \dots n \quad \text{— (1)}$$

Such a poly. is called the interpolating poly. The condition (1) are called interpolating conditions.

Newton's forward difference interpolation formula-

Let h be the step length in the given Equispaced data we have-

$$f(x) = f(x_0) + (x-x_0)\frac{\Delta f(x_0)}{1!\,h} + (x-x_0)(x-x_1)\frac{\Delta^2 f(x_0)}{2!\,h^2} + (x-x_0)(x-x_1)(x-x_2)\frac{\Delta^3 f(x)}{3!\,h^3}$$

$$+ \dots\dots + (x-x_0)(x-x_1)\dots\dots(x-x_{n-1})\frac{\Delta^n f(x_0)}{n!\,h^n}$$

This is called the Newton forward difference interpolation ~~form~~ formula (Poly).

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Newton's backward difference interpolation :-

$$f(x) = f(x_n) + (x-x_n)\frac{\nabla f_n}{1!\,h} + (x-x_n)(x-x_{n-1})\frac{\nabla^2 f_n}{2!\,h^2} + \cdots\cdots + (x-x_n)(x-x_{n-1})\cdots\cdots(x-x_1)\frac{\nabla^n f_n}{n!\,h^n}.$$

This is called the Newton's backward difference interpolation.

Ex- For the data

x-	-4	-2	0	2	4	6
$f(x)$-	-139	-21	1	23	141	451

construct the forward and backward difference tables using the corresponding interpolation, show that the interpolation poly. is same.

The step length is $h=2$

	x	$f(x)$	$\Delta f / \Delta\nabla f$	$\Delta^2 f / \nabla^2 f$	$\Delta^3 f / \nabla^3 f$
x_0	-4	-139			
x_1	-2	-21	118		
x_2	0	1	22	-96	
x_3	2	23	22	0	96
x_4	4	141	118	96	96
x_5	6	451	310	192	96

$\Delta^3 f / \nabla^3 f$

0

0

0

① using the Newton forward difference interpolating poly. we get

$$f(x) \approx f(x) \simeq f(x_0) + (x-x_0)\frac{\Delta f(x_0)}{1!\,h} + (x-x_0)(x-x_1)$$

$$\frac{\Delta^2 f(x_0)}{2!\,h^2} + (x-x_0)(x-x_1)(x-x_2)\frac{\Delta^3 f(x_0)}{3!\,h^3}$$

$$= -139 + (x+4)\frac{110}{1!\,2} + \frac{(x+4)(x+2)(-96)}{2\cdot 2^2}$$

$$+ \frac{(x+4)(x+2)(x-0)(96)}{6\times 8}$$

$$f(x) = 2x^3 + 3x + 1$$

② using Newton Backward difference interpolating poly - we get -

$$f(x) \simeq f(x) \simeq f(x_n) + (x-x_n)\frac{\nabla f(x_n)}{1!\,h} +$$

$$(x-x_n)(x-x_{n-1})\frac{\nabla^2 f_n}{2!\,h^2} + \cdots$$

$$f(x) = f(x_5) + (x-x_5)\frac{\nabla f_5}{1!\,h} + (x-x_5)(x-x_4)\frac{\nabla^2 f_5}{2!\,h^2}$$

$$+ (x-x_5)(x-x_4)(x-x_3)\frac{\nabla^3 f_5}{3!\,h^3} + \cdots$$

$$f(x) = 451 + (x-6)\frac{\overset{155}{\cancel{310}}}{1!\,\cancel{2}} + (x-6)(x-4)\frac{\overset{24}{+92}}{2!\,2^2}$$

$$+ \frac{(x-6)(x-4)(x-2)\,\overset{2}{96}}{\cancel{6}\times 8}$$

$$= 451 + \frac{(x-6)\,155}{1} + (x-6)(x-4)\,24 + (x-6)(x-4)(x-2)\,2$$

$$= 451 + 155x - 930 + (x^2 - 10x + 24)\,24 + (x^2 - 10x + 24)(x-2)\{2\}$$

$$= 155x - 479 + 24x^2 - 240x + 576 + (x^3 - 10x^2 + 24x - 2x^2 + 20x - 48)\,2$$

$$= 24x^2 - 85x + 97 + 2x^3 - 20x^2 + 48x - 4x^2 + 40x - 96$$

$$\boxed{f(x) = 2x^3 + 3x + 1}$$ Ans

Q. For the following data, calculate the differences and obtain the Newton's forward and backward difference interpolation polys. Are these polys different? Interpolate at $x = 0.25$ and $x = 0.35$.

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.00	2.28

x	$f(x)$	$\Delta f / \nabla f$	$\Delta^2 f / \nabla^2 f$	$\Delta^3 f / \nabla^3 f$
0.1	1.40	~~0.16~~		
0.2	1.56	0.16		0
0.3	1.76	0.20	0.04	0
0.4	2.00	0.24	0.04	0
0.5	2.28	0.28	0.04	

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Using Newton forward method -

$$f(x) = f(x_0) + (x-x_0)\frac{\Delta f(x_0)}{1!h} + (x-x_0)(x-x_1)\frac{\Delta^2 f}{2!h^2}$$

$$= 1.40 + (x-0.1)\frac{0.16}{1!\,0.1} + (x-0.1)(x-0.2)\frac{\cancel{0.04}\ 0.2}{2!(0.1)^2}$$

$$= 1.40 + \frac{(x-0.1)(0.16)}{0.1} + (x-0.1)(x-0.2)\frac{0.2}{0.1}$$

$$= \cancel{1.40 + 0.16x}$$

$$= 1.40 + (x-0.1)(1.6) + (x-0.1)(x-0.2)2$$

$$= 1.40 + 1.6x - 0.16 + (x^2 - 0.3x + 0.02)2$$

$$= 1.40 + 1.6x - 0.16 + 2x^2 - 0.6x + 0.04$$

$$\boxed{f(x) = 2x^2 + x + 1.28} \text{ Ans}$$

Using Newton backward's method

$$f(x) = f(x_n) + (x-x_n)\frac{\nabla f_n}{1!h} + (x-x_n)(x-x_{n-1})\frac{\nabla^2 f_n}{2!h^2}$$

$$= f(x_5) + (x-x_5)\frac{\nabla f_5}{1!h} + (x-x_5)(x-x_4)\frac{\nabla^2 f_5}{2!h^2}$$

$$= 2.28 + (x-0.5)\frac{0.28}{1!\,0.1} + (x-0.5)(x-0.4)\frac{\cancel{0.04}\ 0.02}{2\times 0.01}$$

$$= 2.28 + (x-0.5)2.8 + (x-0.5)(x-0.4)2$$

$= 2.28 + 2.8x - 1.4 + (x^2 - 0.9x + 0.2)2$

$\boxed{f(x) = 2x^2 + x + 1.28}$ Ans

Lagrange Interpolation-

Consider the following data value-

x	x_0	x_1	x_2	x_n
$f(x)$	f_0	f_1	f_2	f_n

where $f_i = f(x_i)$

For this data we can fit a unique poly. of degree $\le n$

This poly. must we all the ordinates $f_0, f_1 \dots f_n$

Hence, the poly. can be written as a linear combination of $f_0, f_1, f_2, f_3 \dots f_n$ as-

$$f_n(x) = l_0(x) f_0 + l_1(x) f_1 + \dots l_n(x) f_n \quad -(1)$$

where $l_i(x)$ & $i = 0, 1, 2, \dots n$ are poly. degree n

Since $f_n(x)$ fits the data Exactly we get at $x = x_0$.

The only possibility is $l_0(x_0) = 1$ and $l_i(x_0) = 0$, $i \ne 0$.

$$f(x_i) = f_n(x_i) = l_0(x_i) f_0 + \dots l_i(x_i) f_i + \dots l_n(x_i) f_n.$$

This Eq^n is satisified only when $l_i(x_i) = 1$ and $l_i(x_j) = 0$, $i \ne j$.

Therefore, $l_i(x)$, the polys of degree n satisfy the conditions

$$l_i(x_j) = 0, \quad i \ne j \qquad -(2)$$
$$= 1 \quad i = j$$

Therefore, $l_i(x) = A(x-x_0)(x-x_1) \cdots (x-x_{i-1})(x-x_{i+1}) \cdots (x-x_n)$ — ③

$\boxed{i \neq 0}$

where A = constant.

Now, since $l_i(x_i) = 1$, we get

$$A = \frac{1}{[(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)]}$$

Hence,

$$l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

The poly. ① where $l_i(x)$ are defined by ④ is called the Lagrange interpolating poly. and $l_i(x)$ are called Lagrange fundamental polys.

<u>Lagrange Linear polynomials</u> - For $n=1$, we have

x	x_0	x_1
$f(x)$	$f(x_0)$	$f(x_1)$

The Lagrange fundamental polys. are given by -

$$l_0(x) = \frac{(x-x_1)}{(x_0-x_1)}, \quad l_1(x) = \left(\frac{x-x_0}{x_1-x_0}\right)$$

Hence $P_1(x) = l_0(x) f_0 + l_1(x) f_1$

is the linear Lagrange interpolation poly. which ~~satisfied~~ fits in the given data.

Lagrange Quadratic Interpolation

For $i=2$, we have

x	x_0	x_1	x_2
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$

The Lagrange fundamental polys. are given by -

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

Hence, $f_2(x) = l_0(x) f_0 + l_1(x) f_1 + l_2(x) f_2$
is the quadratic langrang interpolation poly. which fits in given data.

Q. use langrage formula, to find quadratic poly. that takes the value

	x_0	x_1	x_2
x	0	1	3
$f(x)$	0	1	0

The Langrange formula - for Quadratic Equation

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{(x-1)(x-3)}{(0-1)(1-3)} \Rightarrow \frac{(x^2-4x+3)}{3}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$= \frac{(x-0)(x-3)}{(1-0)(1-3)} = \frac{x^2-3x}{-2}$$

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$$l_2(x) = \frac{(x-x_1)(x-x_0)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{(x-1)(x-0)}{(3-0)(3-1)} = \frac{x^2-x}{6}$$

$$f_2(x) = l_0(x) f_0 + l_1(x) f_1 + l_2(x) f_2$$

$$= l_0(x)\cdot 0 + \frac{(x^2-3x)}{-2}\cdot 1 + l_2(x)\cdot 0$$

$$= \frac{(3x-x^2)}{2} \text{ Ans}$$

Q. Given that $f(0) = 1$, $f(1) = 3$, $f(3) = 55$, find the unique poly. of degree 2 or less, which fits the given data.

	x_0	x_1	x_2
x	0	1	3
$f(x)$	1	3	55

Langrange formula -

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)}$$

$$= \frac{x^2-4x+3}{3}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)}$$

$$= \frac{x^2-3x}{-2}$$

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$$l_2(x) = \frac{(x-x_1)(x-x_0)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-0)}{(3-0)(3-1)} =$$

$$= \frac{x^2-x}{6}$$

$$f_2(x) = l_0(x) f_0 + l_2(x) f_2 + l_3(x) f_3$$

$$= \frac{x^2-4x+3}{3}(1) + \frac{(x^2-3x)}{-2} \times 3 + \left(\frac{x^2-x}{6}\right) 55$$

$$= \frac{x^2-4x+3}{3} + \frac{-3x^2+9x}{2} + \frac{55x^2-55x}{6}$$

$$= \cancel{\frac{3x^2-12x+9-9x^2+27x+55x^2-55x}{6}}$$

$$= \frac{2x^2-8x+6-9x^2+27x+55x^2-55x}{6}$$

$$= \frac{48x^2-36x+6}{6} = 8x^2-6x+1 \text{ Ans}$$

Q. Find the Ly Interpolynation that find the following data values.

(i) Interpolate at $x = 3$

x	2.5	3.5
$f(x)$	6	8

(ii) Determine approximation value of $f(3)$

x	1	2	4
$f(x)$	1	7	61

(iii) Interpolate at $x = 1.5$.

x	-1	2	3	4
$f(x)$	-1	11	31	69

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(i)

	x_0	x_1
x	2.5	3.5
$f(x)$	6	8

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-3.5)}{(2.5-3.5)}$$

$$= \frac{(x-3.5)}{-1}$$

$$l_1(x) = \frac{(x-x_0)}{(x_1-x_0)} = \frac{(x-2.5)}{(3.5-2.5)} = \frac{x-2.5}{1}$$

$$P_1(x) = l_0(x) f_0 + l_1(x) f_1$$

$$= \frac{(x-3.5)\,6}{-1} + \frac{(x-2.5)}{1} \times 8$$

$$= +21 - 6x + 8x - 20$$

$$P_1(x) = 2x + 1$$

$$P_1(3) = 2(3)+1 = 7$$

$$\boxed{P_1(3) = 7}$$

(ii)

	x_0	x_1	x_2
x	1	2	4
$f(x)$	1	7	61

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2)(x-4)}{(1-2)(1-4)}$$

$$= \frac{(x-2)(x-4)}{(-1)(-3)} = \frac{(x-2)(x-4)}{3}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-4)}{(2-1)(2-4)}$$

$$= \frac{(x-1)(x-4)}{-2}$$

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$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-2)}{(4-1)(4-2)}$$

$$= \frac{(x-1)(x-2)}{(+3)(+2)} \rightarrow \frac{(x-1)(x-2)}{+6}$$

$$P_2(x) = l_0(x)f_0 + l_1(x)f_1 + l_2(x)f_2$$

$$= \frac{(x-2)(x-4)}{3}1 + \frac{(x-1)(x-4)}{-2}7 + \frac{(x-1)(x-2)}{-6} \times 61$$

$$= \frac{x^2-6x+8}{3} + \frac{7x^2-35x+28}{-2} + \frac{61x^2-183x+122}{-6}$$

$$= \frac{2x^2-12x+16-21x^2+105x-84+61x^2+183x+122}{6}$$

$$\cancel{P_2(x) = \frac{-80x^2+276x-190}{6}}$$

$$\cancel{P_2(3) = \frac{-80(3)^2+276(3)-190}{6}}$$

$$\cancel{= \frac{-720+828-190}{6} = \frac{-82}{6}}$$

$$P_2(3) = -13.667$$

$$P_2(3) = \frac{42x^2-90x+54}{6} = \frac{378-270+54}{6}$$

$$\boxed{P_2(3) = 27}$$

Newton's divided difference Interpolation -

Divided differences -

Let the data (x_i, f_i), $i = 0, 1, 2, \dots n$ be given then we define the divided differences as follow -

First divided difference -

The 1^{st} divided difference of any two consecutive data values is defined as -

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \Rightarrow i = 0, 1, 2, \dots n.$$

Second divided difference -

The second divided difference using three consecutive data values is defined as -

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

On simplifying, we get -

$$f[x_0, x_1, x_2] = \frac{1}{x_2 - x_0}\left[\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_2}{x_1 - x_0}\right]$$

$$= \frac{f_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{f_2}{(x_2 - x_0)(x_2 - x_1)}$$

In general,

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

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Table for divided difference -

x	$f(x)$	first did.	Second did.	Third did.
x_0	f_0	$f[x_0, x_1]$		
x_1	f_1	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
x_2	f_2	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$
x_3	f_3			

→ Using divided differences, show that the data -

x	-3	-2	-1	1	2	3
$f(x)$	18	12	8	6	8	12

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representation a second degree poly. Hence the interpolating poly.

* Newton's divided difference -
Second degree difference -

	x	$f(x)$	first did	Second did
x_0	-3	18	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$
x_1	-2	12	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$
x_2	-1	8	$f[x_2, x_3]$	$f[x_2, x_3, x_4]$
x_3	1	6	$f[x_3, x_4]$	$f[x_3, x_4, x_5]$
x_4	2	8	$f[x_4, x_5]$	
x_5	3	12		

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{12 - 18}{-2 + 3} \rightarrow -6$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{8 - 12}{-1 + 2} \Rightarrow -4$$

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{6 - 8}{1 + 1} \Rightarrow -1$$

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$$f[x_3, x_4] = \frac{f(x_4) - f(x_3)}{x_4 - x_3} = \frac{8-6}{2-1}$$
$$= 2$$
$$f[x_4, x_5) = \frac{f(x_5) - f(x_4)}{x_5 - x_4} = \frac{12-8}{3-2}$$
$$= 4$$

$$f[x_0, x_1, x_2] = \frac{f_0}{(x_0-x_1)(x_0-x_2)} + \frac{f_1}{(x_1-x_0)(x_1-x_2)}$$
$$+ \frac{f_2}{(x_2-x_0)(x_2-x_1)}$$
$$= \frac{18}{(-3+2)(-3+1)} + \frac{12}{(-2+3)(-2+1)}$$
$$+ \frac{8}{(-1+3)(-1+2)}$$
$$= \frac{18}{(-1)(-2)} + \frac{12}{(1)(-1)} + \frac{8}{(2)(1)}$$
$$= +9 - 12 + 4$$
$$f[x_0, x_1, x_2] = 1$$
$$f[x_1, x_2, x_3] = \frac{12}{(-2+1)(-2-1)} + \frac{8}{(-1+2)(-1-1)}$$
$$+ \frac{6}{(1+2)(1+1)}$$
$$= \frac{12}{(-1)(-3)} + \frac{8}{(1)(-2)} + \frac{6}{(3)(2)}$$
$$= 4 + -4 + 1 \Rightarrow 1$$

Similarly,

$$f[x_2, x_3, x_4] = 1 \qquad f[x_3, x_4, x_5) = 1$$

$$P_2(x_2) = f_2 = a_0 + (x_2-x_0)a_1 + (x_2-x_0)(x_2-x_1)a_2$$
$$= 18 + (x+3)\cancel{(-6)} + (x+3)(x+2)1$$
$$= 18 - 6x - 18 + x^2 + 5x + 6$$
$$\boxed{P_2(x) = x^2 - x + 6 \quad \text{Ans}}$$

Newton's divided difference Interpolation

We write the interpolating poly' as

$$f(x) = P_n(x) = a_0 + (x-x_0)a_1 + (x-x_0)(x-x_1)a_2 + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})a_n$$

Substituting $x = x_0, x = x_1, \ldots\ldots, x = x_n$, and using the interpolating conditions -

$$P_n(x_i) = f_i, \text{ we get -}$$

$$P_n(x_0) = f_0 = a_0$$

$$P_n(x_1) = f_1 = a_0 + (x_1 - x_0)a_1$$

$$P_n(x_2) = f_2 = a_0 + (x_2 - x_0)a_1 + (x_2 - x_0)(x_2 - x_1)a_2$$

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0} = f[x_0, x_1]$$

$$a_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)}\left[f_2 - f_0 - (x_2 - x_0)\cdot\frac{(f_1 - f_0)}{x_1 - x_0}\right]$$

$$= \frac{f_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{f_2}{(x_2 - x_0)(x_2 - x_1)}$$

$$= f[x_0, x_1, x_2]$$

using by induction, we can prove that

$$a_n = f[x_0, x_1, \ldots x_n]$$

Hence, we obtain -

$$P_n(x) = f_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_0, x_1, \ldots, x_n]$$

This poly. is called the Newton's divided difference interpolating poly.