Bender-Schmidt method is used to solve the heat equation and similar partial equations.
The general equation of heat equation is

$$\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0; \text{ where } \begin{cases} x & \rightarrow \text{ indicate distance} \\ y & \rightarrow \text{ indicate time} \end{cases}$$

General formula of Bender-Schmidt explicit method is

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

1. $\lambda = \frac{k}{ah^2}$ assume $\left[\lambda = \frac{1}{2}\right]$
2. $u_{xx} = au_t$
3. $h = \Delta x$ and $k = \Delta t$

Example 5.6.1

Solve $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$ given $u(0,t) = 0$, $u(4,t) = 0$, $u(x,0) = x(4-x)$, taking $\Delta x = 1$ and $\Delta t = 1$. Find the value of u upto $t = 5$ using Bender-Schmidt's explicit finite difference Scheme.

Given:

$$u_{xx} = 2u_t \quad \Rightarrow \quad a = 2$$
$$h = \Delta x = 1 \quad \text{and} \quad k = \Delta t = 1$$
$$\lambda = \frac{k}{ah^2} \quad \Rightarrow \quad \lambda = \frac{1}{2(1)^2} = \frac{1}{2}$$

Using the boundary conditions

$$u(0,t) = 0 \quad \Rightarrow \quad u_{0,j} = 0 \quad \text{First Column}$$
$$u(4,t) = 0 \quad \Rightarrow \quad u_{4,j} = 0 \quad \text{Last Column}$$

x varies from 0 to 4 with $h = 1$.

x :	0	1	2	3	4
i :	0	1	2	3	4

Find the value of u upto $t = 5$ with $k = 1$.

$t:$	0	1	2	3	4	5
$j:$	0	1	2	3	4	5

Initial condition is given as,

$$u(x,0) = x(4-x)$$

The value of first row is given by,

$$\begin{aligned} u(0,0) &= 0 \\ u(1,0) &= 1(4-1) = 3 \\ u(2,0) &= 2(4-2) = 4 \\ u(3,0) &= 3(4-3) = 3 \\ u(4,0) &= 4(4-4) = 0 \end{aligned}$$

General formula of Bender-Schmidt explicit method is

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

The values of $u_{i,j}$ are tabulated below:

		$x=0$	$x=1$	$x=2$	$x=3$	$x=4$
	j \ i	0	1	2	3	4
$t=0$	0	0	3	4	3	0
$t=1$	1	0				0
$t=2$	2	0				0
$t=3$	3	0				0
$t=4$	4	0				0
$t=5$	5	0				0

		$x=0$	$x=1$	$x=2$	$x=3$	$x=4$
	j \ i	0	1	2	3	4
$t=0$	0	0	3	4	3	0
$t=1$	1	0	2	3	2	0
$t=2$	2	0	1.5	2	1.5	0
$t=3$	3	0	1	1.5	1	0
$t=4$	4	0	0.75	1	0.75	0
$t=5$	5	0	0.5	0.75	0.50	0

Example 5.6.2

Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ given $u(0,t) = 0$, $u(5,t) = 0$, $u(x,0) = x^2(25 - x^2)$. Compute $u(x,t)$ upto $t = 5$ with $\Delta x = 1$ using Benders-Schmidt's method.

Ans: Since λ value is not given, let us take $\lambda = \frac{1}{2}$.

Given:

$$u_{xx} = u_t \quad \Rightarrow \quad a = 1$$

Since, $h = \Delta x = 1$

$$\frac{k}{ah^2} = \lambda \quad \Rightarrow \quad \frac{k}{1(1)^2} = \frac{1}{2} \quad \Rightarrow \quad k = 0.5$$

Using the boundary conditions

$$u(0,t) = 0 \quad \Rightarrow \quad u_{0,j} = 0 \quad \text{First Column}$$
$$u(5,t) = 0 \quad \Rightarrow \quad u_{5,j} = 0 \quad \text{Last Column}$$

x varies from 0 to 5 with $h = 1$.

$x:$	0	1	2	3	4	5
$i:$	0	1	2	3	4	5

Find the value of u upto $t = 5$ with $k = 0.5$.

$t:$	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$j:$	0	1	2	3	4	5	6	7	8	9	10

Next,

$$u(x,0) = x^2(25 - x^2)$$

The value of first row is given by,

$$u(0,0) = 0^2(25 - 0^2) = 0$$
$$u(1,0) = 1^2(25 - 1^2) = 24$$
$$u(2,0) = 2^2(25 - 2^2) = 84$$
$$u(3,0) = 3^2(25 - 3^2) = 144$$
$$u(4,0) = 4^2(25 - 4^2) = 144$$
$$u(5,0) = 5^2(25 - 5^2) = 0$$

General formula of Bender-Schmidt explicit method is

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

The values of $u_{i,j}$ are tabulated below:

		$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$
	j \ i	0	1	2	3	4	5
$t = 0$	0	0	24	84	144	144	0
$t = 0.5$	1						
$t = 1$	2						
$t = 1.5$	3						
$t = 2$	4						
$t = 2.5$	5						
$t = 3$	6						
$t = 3.5$	7						
$t = 4$	8						
$t = 4.5$	9						
$t = 5$	10						

Crank-Nicolson method is used to solve the heat equation and similar partial equations.

The general equation of heat equation is

$$\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0; \text{ where } \begin{cases} x & \rightarrow \text{ indicate distance} \\ y & \rightarrow \text{ indicate time} \end{cases}$$

Types:

- One step
- Two step

The Crank-Nicolson implicit formula is

$$u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{1-i,j+1} + u_{i+1,j} + u_{i-1,j}\right]$$

1. $\lambda = \frac{k}{ah^2}$ $[\because \lambda = 1]$, implies $\frac{k}{ah^2} = 1 \Rightarrow k = ah^2$.
2. $u_{xx} = au_t$
3. $h = \Delta x$
4. First row value can be found from $u(x, 0)$

Example 5.6.3

Use Crank-Nicholson scheme to solve $\frac{\partial^2 u}{\partial x^2} = 16\frac{\partial u}{\partial t}$, $0 < x < 1$ and $t > 0$ given $u(x,0) = 0$, $u(0,t) = 0$ and $u(1,t) = 100t$. Compute $u(x,t)$ for one time step, taking $\Delta x = \frac{1}{4}$.

Given, $u_{xx} = 16u_t$, implies $a = 16$.
Since $0 < x < 1$, x varies from 0 to 1 with $\Delta x = h = \frac{1}{4} = 0.25$. So,

$$\begin{array}{cccccccc} & x: & 0 & 0.25 & 0.50 & 0.75 & 1 \\ \Rightarrow & i: & 0 & 1 & 2 & 3 & 4 \end{array}$$

To find k,

$$\lambda = \frac{k}{ah^2} \qquad \text{[By default } \lambda = 1]$$

$$\frac{k}{ah^2} = 1 \quad \Rightarrow \quad k = ah^2$$

$$k = 16\left(\frac{1}{4}\right)^2 \quad \Rightarrow \quad k = 16\left(\frac{1}{16}\right) = 1$$

$$k = 1$$

To find the value of $u(x,t)$ upto one step time with $k = 1$.

$$t = 0, 1$$
$$\Rightarrow \; j = 0, 1$$

Using the boundary conditions (i) $u(0,t) = 0$ and (ii) $u(1,t) = 100t$.
From $u(0,t) = 0$, the values of first column is given by,

when	$t = 0,$	$u(0,0) = 0$
when	$t = 1,$	$u(0,1) = 0$

From $u(1,t) = 100t$, the values of last column is given by,

when	$t = 0,$	$u(1,0) = 100(0) = 0$
when	$t = 1,$	$u(1,1) = 100(1) = 100$

Using initial condition $u(x,0) = 0$, we can find the first row values.

$$u(0,0) = 0 \qquad u(0.75,0) = 0$$
$$u(0.25,0) = 0 \qquad u(1,0) = 0$$
$$u(0.50,0) = 0$$

The Crank-Nicolson implicit formula is

$$u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{1-i,j+1} + u_{i+1,j} + u_{i-1,j}\right] \tag{92}$$

		$x = 0$	$x = 0.25$	0.5	0.75	$x = 1$
	$j \backslash i$	0	1	2	3	4
$t = 0$	0	0	0	0	0	0
$t = 1$	1	0	u_1	u_2	u_3	100

From (92),

$$u_1 = \frac{1}{4}\left[0 + 0 + 0 + u_2\right] = \frac{1}{4}u_2 \tag{93}$$

$$u_2 = \frac{1}{4}\left[0 + 0 + u_1 + u_3\right] = \frac{1}{4}[u_1 + u_3] \tag{94}$$

$$u_3 = \frac{1}{4}\left[0 + 0 + u_2 + 100\right] = \frac{1}{4}[u_2 + 100] \tag{95}$$

Substitute equation (93) and (95) in (94)

$$\begin{aligned} u_2 &= \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}[u_2 + 100]\right] \\ &= \frac{1}{16}\left[u_2 + u_2 + 100\right] = \frac{2u_2 + 100}{16} \\ 16u_2 &= 2u_2 + 100 \\ u_2 &= \frac{100}{14} = 7.1429 \end{aligned} \tag{96}$$

From equation (93)

$$u_1 = \frac{1}{4}u_2 = \frac{1}{4}(7.1428) = 1.7857$$

From equation (95)

$$u_3 = \frac{1}{4}[u_2 + 100] = \frac{1}{4}(u_2 + 100) = \frac{7.1429 + 100}{4} = 26.7857$$

The values are

$$u_1 = 1.7857, \qquad u_2 = 7.1429, \qquad u_3 = 26.7857.$$

Example 5.6.4

Solve by Crank-Nicholson method the equation $u_{xx} = u_t$ subject to $u(0,t) = 0$, $u(1,t) = t$ and $u(x,0) = 0$ for 2 time steps with $h = \frac{1}{4}$, $k = \frac{1}{16}$.

Given, $u_{xx} = u_t$, implies $\alpha = 1$.
From $u(0,t) = 0$ and $u(1,t) = t$, x varies from 0 to 1 with $\Delta x = h = \frac{1}{4} = 0.25$.
So,

$$\begin{array}{rrccccc} & x: & 0 & 0.25 & 0.50 & 0.75 & 1 \\ \Rightarrow & i: & 0 & 1 & 2 & 3 & 4 \end{array}$$

To find k,

$$\lambda = \frac{k}{\alpha h^2} \qquad \text{[By default } \lambda = 1\text{]}$$

$$\frac{k}{\alpha h^2} = 1 \quad \Rightarrow \quad k = \alpha h^2$$

$$k = 1\left(\frac{1}{4}\right)^2 \quad \Rightarrow \quad k = 1\left(\frac{1}{16}\right) = \frac{1}{16}$$

Since, two time steps with $k = \frac{1}{16}$, we have $t = 0, 1\left(\frac{1}{16}\right), 2\left(\frac{1}{16}\right)$. i.e., $0, \frac{1}{16}, \frac{2}{16}$.

To find the value of $u(x,t)$ upto two step time with $k = \frac{1}{16}$.

$$t = 0, \frac{1}{16}, \frac{2}{16}$$
$$\Rightarrow j = 0, 1, 2$$

Using the boundary conditions (i) $u(0,t) = 0$ and (ii) $u(1,t) = t$.
From $u(0,t) = 0$, the values of first column is given by,

$$\text{when} \quad t = 0, \quad u(0,0) = 0$$
$$\text{when} \quad t = \frac{1}{16}, \quad u\left(0, \frac{1}{16}\right) = 0$$
$$\text{when} \quad t = \frac{2}{16}, \quad u\left(0, \frac{2}{16}\right) = 0$$

From $u(1,t) = t$, the values of last column is given by,

$$\text{when} \quad t = 0, \quad u(1,0) = (0) = 0$$

$$\text{when} \quad t = \frac{1}{16}, \quad u\left(1, \frac{1}{16}\right) = \left(\frac{1}{16}\right) = \frac{1}{16}$$

$$\text{when} \quad t = \frac{2}{16}, \quad u\left(1, \frac{2}{16}\right) = \left(\frac{2}{16}\right) = \frac{2}{16}$$

Using initial condition $u(x,0) = 0$, we can find the first row values.

$$u(0,0) = 0 \qquad u(0.75,0) = 0$$

$$u(0.25,0) = 0 \qquad u(1,0) = 0$$

$$u(0.50,0) = 0$$

The Crank-Nicolson implicit formula is

$$u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{1-i,j+1} + u_{i+1,j} + u_{i-1,j}\right] \tag{97}$$

		$x = 0$	$x = 0.25$	0.5	0.75	$x = 1$
	j \ i	0	1	2	3	4
$t = 0$	0	0	0	0	0	0
$t = \frac{1}{16}$	1	0	u_1	u_2	u_3	$\frac{1}{16}$
$t = \frac{2}{16}$	2	0	u_4	u_5	u_6	$\frac{2}{16}$

From (97),

$$u_1 = \frac{1}{4}\left[0 + 0 + 0 + u_2\right] = \frac{1}{4}u_2 \tag{98}$$

$$u_2 = \frac{1}{4}\left[0 + 0 + u_1 + u_3\right] = \frac{1}{4}[u_1 + u_3] \tag{99}$$

$$u_3 = \frac{1}{4}\left[0 + 0 + u_2 + \frac{1}{16}\right] = \frac{1}{4}\left[u_2 + \frac{1}{16}\right] \tag{100}$$

Substitute equation (98) and (100) in (99)

$$\begin{aligned} u_2 &= \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}\left[u_2 + \frac{1}{16}\right]\right] \\ &= \frac{1}{16}\left[u_2 + u_2 + \frac{1}{16}\right] = \frac{2u_2 + \frac{1}{16}}{16} \\ 16u_2 &= 2u_2 + \frac{1}{16} \\ u_2 &= \frac{1}{16} \times \frac{1}{14} = 0.0045 \end{aligned} \tag{101}$$

From equation (98)

$$u_1 = \frac{1}{4}u_2 = \frac{1}{4}(0.0045) = 0.0011$$

From equation (100)

$$u_3 = \frac{1}{4}\left[u_2 + \frac{1}{16}\right] = \frac{1}{4}\left(u_2 + \frac{1}{16}\right) = \frac{0.0045 + \frac{1}{16}}{4} = 0.0167$$

The values are $u_1 = 0.0011,\ u_2 = 0.0045,\ u_3 = 0.0167.$
Similarly $u_4 = 0.0059,\ u_5 = 0.0191,\ u_6 = 0.0528.$