For any given pair of values $(x_k, y_k),\ k = 0, 1, 2, 3, \cdots, n$ with equal-spaced abscissas of a function $y = f(x)$, we defined the *forward difference operator* Δ as follows: The first forward difference is usually expressed as

$$\Delta y_i = y_{i+1} - y_i,\ i = 0, 1, \cdots, (n-1) \tag{20}$$

To be explicit, we write

$$\begin{aligned} \Delta y_0 &= y_1 - y_0 \\ \Delta y_1 &= y_2 - y_1 \\ &\vdots \\ \Delta y_{n-1} &= y_n - y_{n-1} \end{aligned}$$

These differences are called *first differences of the function* y and are denoted by the symbol Δy_i. Here, Δ is called *forward difference operator*.

Similarly, the difference of the first differences are called *second differences,* defined by

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \quad \Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

Thus, in general

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

Here Δ^2 is called the *second difference operator*. Thus, continuing, we can define, r^{th} difference of y, as

$$\Delta^r y_t = \Delta^{r-1} y_{r+1} - \Delta^{r-1} y_i \tag{21}$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
$(= x_0 + h)$		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
$(= x_0 + 2h)$		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
$= (x_0 + 3h)$		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
$= (x_0 + 4h)$		Δy_4				
x_5	y_5					
$= (x_0 + 5h)$						

Figure 5: Forward difference table

It can be noted that the subscription remains constant along each diagonal of the table. The first term in the table, that is y_0 is called the *leading term*, while the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \cdots$ are called leading differences.

Example 2.2.1

Express $\Delta^2 y_0$ and $\Delta^3 y_0$ in terms of the values of the function y.

Noting that each higher order difference is defined in terms of the lower order difference, we have

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

and

$$\begin{aligned}
\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 = (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0) \\
&= (y_3 - y_2) - (y_2 - y_1) - (y_2 - y_1) + (y_1 - y_0) \\
&= y_3 - 3y_2 + 3y_1 - y_0. \\
\Delta^4 y_0 &= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.
\end{aligned}$$

Hence, we observed that the coefficients of the values of y, in the expansion of $\Delta^2 y_0$, $\Delta^3 y_0$ are binomial coefficients. Thus, in general, we arrive at the following result.

$$\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - {}^nC_3 y_{n-3} + \cdots + (-1)^n y_0$$

Example 2.2.2

Prove that

1. $\Delta(f(x)+g(x)) = \Delta f(x) + \Delta g(x)$.
2. $\Delta cy(n) = c\Delta y(n)$
3. $\Delta(cx_n + cy_n) = c\Delta x_n + c\Delta y_n$

$$\begin{aligned} \Delta(f(x)+g(x)) &= (f(x+1)+g(x+1)) - (f(x)+g(x)) \\ &= (f(x+1)-f(x)) + (g(x+1)-g(x)) \\ &= \Delta f(x) - \Delta g(x). \end{aligned} \tag{23}$$

$$\begin{aligned} \Delta cy(n) &= cy(n+1) - cy(n) \\ &= c(y(n+1)-y(n)) = c\Delta y(n) \end{aligned} \tag{24}$$

From Eq.23 and Eq.24, it is proved that $\Delta(cx_n + cy_n) = c\Delta x_n + c\Delta y_n$. That is Δ is a linear operator.

Example 2.3.1

Construct a forward difference table for the following values of x and y:

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
y	0.003	0.067	0.148	0.248	0.370	0.518	0.697

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0.1	0.003						
		0.064					
0.3	0.067		0.017				
		0.081		0.002			
0.5	0.148		0.019		0.001		
		0.1		0.003		0	
0.7	0.248		0.022		0.001		0
		0.122		0.004		0	
0.9	0.37		0.026		0.001		
		0.148		0.005			
1.1	0.518		0.031				
		0.179					
1.3	0.697						

Practice problem

1. Construct a forward difference table for the following set of values:

x_i	0	2	4	6	8	10	12	14
y_i	625	81	1	1	81	625	2401	65611

2. Construct a forward (diagonal) difference table for the following set of values:

x_i	1	2	3	4	5
y_i	4	13	34	73	136

Example 2.4.1

Using Newton's forward formula. find the value of $\sin 52^o$ from the following:

θ^o	45^o	50^o	55^o	60^o
$\sin\theta$	0.7071	0.7660	0.8192	0.8660

Solution:

θ^o	$\sin\theta$	Δ	Δ^2	Δ^3
45^o	0.7071			
		0.0589		
50^o	0.7660		-0.0057	
		0.0532		-0.0007
55^o	0.8192		-0.0064	
		0.0468		
60^o	0.8660			

Applying Newton's forward difference interpolation formula.

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

Here $y_n(x) = \sin 52^o$

$$y_0 = 0.7071, \Delta y_0 = 0.0589, \Delta^2 y_0 = -0.0057, \Delta^3 y_0 = -0.0007$$

$$p = \frac{x - x_0}{h} = \frac{52 - 45}{5} = \frac{7}{5} = 1.4$$

$$\sin 52^o = 0.7071 + (1.4)(0.0589) + \frac{1.4(0.4)}{2}(-0.0057)$$

$$+ \frac{1.4(0.4)(-0.6)}{6}(-0.0007)$$

$$= 0.7071 + 0.08246 - 0.0016 + 0.00004$$

$$\sin 52^o = 0.7880$$

Example 2.4.2

Using Newton's forward formula, find the value of $f(218)$ if,

X	100	150	200	250	300	350	400
100	10.63	13.03	15.04	16.81	18.42	19.90	21.27

X	$f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
100	10.63						
		2.4					
150	13.03		-0.39				
		2.01		0.15			
200	15.04		-0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.02			
350	19.90		-0.11				
		1.37					
400	21.27						

$$
\begin{aligned}
y_n(x) = y_0 \\
&+ p\Delta y_0 \\
&+ \frac{p(p-1)}{2!}\Delta^2 y_0 \\
&+ \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \\
&+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 \\
&+ \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0 \\
&+ \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{6!}\Delta^6 y_0
\end{aligned}
$$

Here

$$h = x_1 - x_0 = 150 - 100 = 50$$

$$y_n(x) = f(218), p = \frac{x - x_0}{h} = \frac{218 - 100}{50} = 2.36$$

$$
\begin{aligned}
f(218) = 10.63 &+ (2.36)(2.4) \\
&+ \frac{(2.36)(1.36)}{2}(-0.39) \\
&+ \frac{(2.36)(1.36)(0.36)}{6}(-0.39) \\
&+ \frac{(2.36)(1.36)(0.36)(-0.64)}{24}(0.15) \\
&+ \frac{(2.36)(1.36)(0.36)(-0.64)(-1.64)}{120}(-0.07) \\
&+ \frac{(2.36)(1.36)(0.36)(-0.64)(-1.64)(-2.64)}{720}(0.02) \\
&= 10.63 + 5.664 - 0.6259 + 0.0289 + 0.0022 + 0 + 0
\end{aligned}
$$

$$f(218) = 15.6993$$

Example 2.4.3

The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

$x = height$	100	150	200	250	300	350	400
$y = distance$	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when $x = 160ft.$.

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
100	10.63						
		2.4					
150	13.03		-0.39				
		2.01		0.15			
200	15.04		-0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.02			
350	19.90		-0.11				
		1.37					
400	21.27						

The value of x at $f(x) : x = 160$

$$h = x_1 - x_0 = 150 - 100 = 50$$

$$p = \frac{x - x_0}{h} = \frac{160 - 100}{50} = 1.2$$

$$y(160) = 10.63 + 2.88 - 0.0468 - 0.0048 - 0.001 + 0 + 0$$

$$y(160) = 13.4573$$

Example 2.4.4

Find the polynomial which satisfies the following table of values.

x	1	2	3	4	5	6	7	8
y	2	9	22	41	66	97	134	177

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7
1	2							
		7						
2	9		6					
		13		0				
3	22		6		0			
		19		0		0		
4	41		6		0		0	
		25		0		0		0
5	66		6		0		0	
		31		0		0		
6	97		6		0			
		37		0				
7	134		6					
		43						
8	177							

x	$f(x) = ax^2 + bx + c$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7
1	a+b+c							
		3a+b						
2	4a+2b+c		2a					
		5a+b		0				
3	9a+3b+c		2a		0			
		7a+b		0		0		
4	16a+4b+c		2a		0		0	
		9a+b		0		0		0
5	25a+5b+c		2a		0		0	
		11a+b		0		0		
6	36a+6b+c		2a		0			
		13a+b		0				
7	49a+7b+c		2a					
		15a+b						
8	64a+8b+c							

Now, comparing Δ^2 column of both tables, then we

$$2a = 6$$
$$a = 3$$

Comparing Δ column of both tables, then we

$$3a + b = 7$$
$$9 + b = 7$$
$$b = -2$$

Comparing $f(x)$ column of both tables, then we

$$a + b + c = 2$$
$$3 - 2 + c = 2$$
$$c = 1$$

So, the desired polynomial is

$$f(x) = 3x^2 - 2x + 1$$

Example 2.4.5

Find the polynomial that satisfies the following table of values

x	-3	-2	-1	0	1	2	3
y	0	-4	0	6	8	0	-24

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
-3	0						
		-4					
-2	-4		8				
		4		-6			
-1	0		2		0		
		6		-6		0	
0	6		-4		0		0
		2		-6		0	
1	8		-10		0		
		-8		-6			
2	0		-24				
		-24					
3	-24						

$$a = -1;\ b = -2;\ c = 5;\ d = 6$$

$$f(x) = -x^3 - 2x^2 = 5x + 6$$

Example 2.4.6

Determine the value of constant finite differences

1. $f(x) = 3x^2 - x + 2$
2. $f(x) = -2x^3 + 3x^2 + x - 1$

Formula to find constant finite differences

Constant finite differences=(Degree of the polynomial)! $\times$ leading coefficient

1. Here,
Leading coefficient $= 3$ and Degree of the polynomial=2
Constant finite differences=$2! \times 3$

2. Here,
Leading coefficient $= -2$ and Degree of the polynomial=3
Constant finite differences=$3! \times -2 = -12$

Definition 2.5.1 (Backward Differences)

The differences $y_1 - y_0, y_2 - y_1, \cdots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \cdots, \nabla y_n$, respectively, are called first backward difference. Thus, the first backward differences are $\nabla y_r = y_r - y_{r-1}$. This formula is useful when the value of $f(x)$ is required near the end of the table. h is called the interval of difference and $u = \frac{x - x_n}{h}$, Here an is last term.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1 $(= x_0 + h)$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2 $(= x_0 + 2h)$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3 $(= x_0 + 3h)$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4 $(= x_0 + 4h)$	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5 $(= x_0 + 5h)$	y_5					

Figure 6: Backward difference table

Example 2.5.2

Given $\sin 45^o = 0.7071$, $\sin 50^o = 0.7660$, $\sin 55^o = 0.8192$, $\sin 60^o = 0.8660$. Find $\sin 57^o$ using on appropriate interpolation formula.

We can form a table of values as follows

x	45^o	50^o	55^o	60^o
y	0.7071	0.7660	0.8192	0.8660

Since $\sin 57^o$ is closer x_n value, we can choose Newton's backward interpolation formula.

x	y	∇	∇^2	∇^3
45^o	0.7071			
		0.0589		
50^o	0.7660		-0.0057	
		0.0532		-0.0007
55^o	0.8192		-0.0064	
		0.0468		
60^o	0.8660			

Here,

$$r = \frac{x - x_n}{h} = \frac{57 - 60}{5} = -0.6$$

We have newton's backward interpolation formula

$$y_r = y_n + r\nabla y_n + \frac{(r)(r+1)}{2!}\nabla^2 y_n + \frac{(r)(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

$$y_{57} = 0.8660 + (-0.6)(0.0468) + \frac{(-0.6)(-0.6+1)}{2!}(-0.0064)$$

$$+ \frac{(-0.6)(0.6+1)(0.6+2)}{3!}(-0.0007)$$

$$\sin(57) = 0.8387$$

Example 2.5.3

Find the value of $f(17)$ form following table of values

x	0	5	10	15	20
$f(x)$	1.0	1.6	3.8	8.2	15.4

x	$f(x)$	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$
0	1.0				
		6			
5	1.6		1.6		
		2.2		0.6	
10	3.8		2.2		0
		4.4		0.6	
15	8.2		4.4		
		7.2			
20	15.4				

Here,

$$r = \frac{17-20}{5} = -0.6$$

We have newton's backward interpolation formula

$$\begin{aligned} y_r &= y_n + r\nabla y_n + \frac{(r)(r+1)}{2!}\nabla^2 y_n \\ &\quad + \frac{(r)(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots \\ y_{57} &= 15.4 + (-0.6)(7.2) + \frac{(-0.6)(-0.6+1)}{2!}(4.4) \\ &\quad + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(0.6) \\ &\quad + \frac{(-0.6)(-0.6+1)(-0.6+2)(0.6+3)}{4!}(0) \\ f(17) &= 10.5184 \end{aligned}$$

Problem 2.5.4

Given the following data estimate $f(4.12)$ using Newton-Gregory backward difference interpolation polynomial:

x	0	1	2	3	4	5
$f(x)$	1	2	4	8	16	32

Suppose we have one point $(1, 3)$. How can we find a polynomial that could represent it?

$$P(x) = 3$$
$$P(x) = 3x$$
$$P(1) = 3$$

Suppose we have sequence of points: $(1, 3), (2, 4)$. How can we find a polynomial that could represent it?

$$P(x) = \frac{(x-2)}{(1-2)} \times 3 + \frac{(x-1)}{(2-1)} \times 4$$
$$P(x) = x + 2$$
$$P(1) = 3$$
$$P(2) = 4$$

Suppose we have sequence of points: $(1,3),(2,4),(7,11)$. How can we find a polynomial that could represent it?

$$P(x) = \frac{(x-2)(x-7)}{(1-2)(1-7)} \times 3 + \frac{(x-1)(x-7)}{(2-1)(2-7)} \times 4 + \frac{(x-1)(x-2)}{(7-1)(7-2)} \times 11$$
$$P(1) = 3$$
$$P(2) = 4$$
$$P(7) = 11$$

In a general form it looks like this:

$$P(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}y_3$$
$$P(x) = \sum_{1}^{3} P_i(x)y_i$$

Definition 2.6.1 (Formula to find the function value)

$$y = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1$$

$$+\cdots+\frac{(x-x_1)(x-x_1)\cdots(x-x_{n-1})(x_n-x_0)}{(x_0-x_1)\cdots(x_n-x_{n-1})}y_n$$

$$y = V_o(x_n)y_0 + V_1(x_n)y_1 + V_2(x_n)y_2 + V_3(x_n)y_3 + \cdots + V_n(x_n)y_n$$

Definition 2.6.2 (Find the Polynomial of the given data)

$$P(x) = \sum_{j=0}^{n} y_j \left(\prod_{i=0, i\neq j}^{n} \frac{x-x_i}{x_j-x_i} V_i(x) y_i \right)$$

Example 2.6.3

Using the Lagrange interpolation formula, find the value of y at $x = 0$ given some set of values $(-2, 5)$, $(1, 7)$, $(3, 11)$, $(7, 34)$?

Solution:

$$g(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_o-x_1)(x_o-x_2)(x_o-x_3)}f_o + \frac{(x-x_o)(x-x_2)(x-x_3)}{(x_1-x_o)(x_1-x_2)(x_1-x_3)}f_1$$
$$+ \frac{(x-x_o)(x-x_1)(x-x_3)}{(x_2-x_o)(x_2-x_1)(x_2-x_3)}f_2 + \frac{(x-x_o)(x-x_1)(x-x_2)}{(x_3-x_o)(x_3-x_1)(x_3-x_2)}f_3$$

Given the known values are, $x = 0; x_0 = -2; x_1 = 1; x_2 = 3; x_3 = 7; y_0 = 5;$ $y_1 = 7; y_2 = 11; y_3 = 34$.

$$y = \frac{(0-1)(0-3)(0-7)}{(-2-1)(-2-3)(-2-7)} \times 5 + \frac{(0-(-2))(0-3)(0-7)}{(1-(-2))(1-3)(1-7)} \times 7$$
$$+ \frac{(0-(-2))(0-1)(0-7)}{(3-(-2))(3-1)(3-7)} \times 11 + \frac{(0-(-2))(0-1)(0-3)}{(7-(-2))(7-1)(7-3)} \times 34$$
$$y = \frac{21}{27} + \frac{49}{6} + \frac{-77}{20} + \frac{51}{54}$$
$$y = \frac{1087}{180}$$

Example 2.6.4

Consider the following table of functional values, find $g(0.06)$ using Lagrange interpolation, where function generated with $f(x) = \log(x)$.

i	x_i	f_i
0	0.40	-0.916291
1	0.50	-0.693147
2	0.70	-0.356675
3	0.80	-0.223144

$$g(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_o-x_1)(x_o-x_2)(x_o-x_3)}f_o + \frac{(x-x_o)(x-x_2)(x-x_3)}{(x_1-x_o)(x_1-x_2)(x_1-x_3)}f_1$$
$$+ \frac{(x-x_o)(x-x_1)(x-x_3)}{(x_2-x_o)(x_2-x_1)(x_2-x_3)}f_2 + \frac{(x-x_o)(x-x_1)(x-x_2)}{(x_3-x_o)(x_3-x_1)(x_3-x_2)}f_3$$

$$y(0.06) = \frac{(0.06-0.50)(0.06-0.70)(0.06-0.80)}{(0.40-0.50)(0.40-0.70)(0.40-0.80)} \times -0.9163$$
$$+ \frac{(0.06-0.40)(0.06-0.70)(0.06-0.80)}{(0.50-0.40)(0.50-0.70)(0.50-0.80)} \times -0.6931$$
$$+ \frac{(0.06-0.40)(0.06-0.50)(0.06-0.80)}{(0.70-0.40)(0.70-0.50)(0.70-0.80)} \times -0.3567$$
$$+ \frac{(0.06-0.40)(0.06-0.50)(0.06-0.70)}{(0.80-0.40)(0.80-0.50)(0.80-0.70)} \times -0.2231$$

$$y(0.06)$$
$$= \frac{(-0.44)(-0.64)(-0.74)}{(-0.10)(-0.30)(-0.40)}(-0.9163) + \frac{(-0.34)(-0.64)(-0.74)}{(0.10)(-0.20)(-0.30)}(-0.6931)$$
$$+ \frac{(-0.34)(-0.44)(-0.74)}{(0.30)(0.20)(-0.10)}(-0.3567) + \frac{(-0.34)(-0.44)(-0.64)}{(0.40)(0.30)(0.10)}(-0.2231)$$

$$y(0.06) = \frac{-0.2084}{-0.012} \times -0.9163 + \frac{-0.161}{0.006} \times -0.6931$$
$$+ \frac{-0.1107}{-0.006} \times -0.3567 + \frac{-0.0957}{0.012} \times -0.2231$$
$$y(0.06) = -2.11$$

Solution of the polynomial at point 0.06 is $y(0.06) = -2.11$

Problem

Using Lagrange interpolation to find a polynomial P of degree < 4 satisfying

$$P_1(1) = 1, P_2(2) = 4, P_3(3) = 1, P_4(4) = 5,$$

what are the polynomials $P_1(x), P_2(x), P_3(x), P_4(x), P(x)$?

Solution:

Let $f(x) = (x-2)(x-3)(x-4)$. Then

$$f(1) = (-1)(-2)(-3) = -6, \text{ so } P_1(x) = \frac{-1}{6}(x-2)(x-3)(x-4).$$

Let $f(x) = (x-1)(x-3)(x-4)$. Then

$$f(2) = (1)(-1)(-2) = 2, \text{ so } P_2(x) = \frac{1}{2}(x-1)(x-3)(x-4).$$

Let $f(x) = (x-1)(x-2)(x-4)$. Then

$$f(3) = (2)(1)(-1) = -2, \text{ so } P_3(x) = \frac{-1}{2}(x-1)(x-2)(x-4).$$

Let $f(x) = (x-1)(x-2)(x-3)$. Then

$$f(4) = (3)(2)(1) = 6, \text{ so } P_4(x) = \frac{1}{6}(x-1)(x-2)(x-3).$$

Hence,

$$\begin{aligned} P(x) =& 1 \times \left(-\frac{1}{6}\right)(x-2)(x-3)(x-4) + 4 \times \frac{1}{2}(x-1)(x-3)(x-4) \\ &+ 1 \times \left(-\frac{1}{2}\right)(x-1)(x-2)(x-4) + 5 \times \frac{1}{6}(x-1)(x-2)(x-3). \end{aligned}$$

Simplifying gives $P(x) = \frac{13}{6}x^3 - 16x^2 + \frac{215}{6}x - 21$.

Practice Problem

Given the following data:

$x_o = 3$	$f_o = 1$
$x_1 = 4$	$f_1 = 2$
$x_2 = 5$	$f_2 = 4$

Find the quadratic interpolating function $g(x)$.
Lagrange basis functions are

$$V_o(x) = \frac{(x-4)(x-5)}{(3-4)(3-5)}$$
$$V_1(x) = \frac{(x-3)(x-5)}{(4-3)(4-5)}$$
$$V_2(x) = \frac{(x-3)(x-4)}{(5-3)(5-4)}$$

Interpolating function $g(x)$ is: $g(x) = 1.0V_o(x) + 2.0V_1(x) + 4.0V_2(x)$.

x_i	f_i	$f[x_i, x_j]$	$f[x_i, x_j, x_k]$	$f[x_i, x_j, x_k, x_l]$
x_0	f_0			
		$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}$		
x_1	f_1		$f[x_0, x_1, x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0}$	
		$f[x_1, x_2] = \frac{f_2 - f_1}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \frac{f[x_1,x_2,x_3] - f[x_0,x_1,x_2]}{x_3 - x_0}$
x_2	f_2		$f[x_1, x_2, x_3] = \frac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1}$	
		$f[x_2, x_3] = \frac{f_3 - f_2}{x_3 - x_2}$		
x_3	f_3			

$$f(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$$

Example 2.7.1

Find solution of $f(x)$ when $x = 151.23855$ using Newton's divided difference interpolation formula for the following data.

x	300	304	305
f(x)	2.4771	2.4829	2.4843

x	y	1st order	2nd order
300	2.4771		
		$\frac{2.4829-2.4771}{304-300}$ =0.0014	
304	2.4829		$\frac{0.0014-0.0014}{305-300}$ =0
		$\frac{2.4843-2.4829}{305-304}$ =0.0014	
305	2.4843		

The value of x at you want to find the $f(x) : x = 151.23855.$
Newton's divided difference interpolation formula is

$$
\begin{aligned}
f(x) &= y_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\
y(151.23855) &= 2.4771 + (151.23855 - 300) \times 0.0014 \\
&\quad + (151.23855 - 300)(151.23855 - 304) \times 0 \\
y(151.23855) &= 2.4771 + (-148.7614) \times 0.0014 \\
&\quad + (-148.7614)(-152.7614) \times 0 \\
y(151.23855) &= 2.4771 - 0.2083 + 0 \\
y(151.23855) &= 2.2688
\end{aligned}
$$

Solution of divided difference interpolation method $y(151.23855) = 2.2688$

Problem 2.7.2

The upward velocity of a rocket is given as a function of time in the following Table

$t(s)$	0	10	15	20	22.5	30
$v(t)(m/s)$	0	227.04	362.78	517.35	602.97	901.67

(a) *Determine the value of the velocity at* $t = 16$ *seconds using Newton's divided difference polynomial method.*

(b) *Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from* $t = 11s$ *to* $t = 16s$.

(c) *Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at* $t = 16s$.

(a)

x	y	1^{st} **order**	2^{nd} **order**	3^{rd} **order**	4^{th} **order**	5^{th} **ord**
$x_0 = 0$	$y_0 = 0$					
		22.704				
$x_1 = 10$	227.04		0.2963			
		27.148		0.004		
$x_2 = 15$	362.78		0.3766		0.0001	
		30.914		0.0054		0
$x_3 = 20$	517.35		0.4445		0.0001	
		34.248		0.0076		
$x_4 = 22.5$	602.97		0.5579			
		39.8267				
$x_5 = 30$	901.67					

The value of x at you want to find the $f(x)$: $x = 16$ Newton's divided difference interpolation formula is

$$
\begin{aligned}
f(x) = y_0 &+ (x - x_0)f[x_0, x_1] \\
&+ (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\
&+ (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \\
&+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x_0, x_1, x_2, x_3, x_4] \\
&+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)f[x_0, x_1, x_2, x_3, x_4, x_5]
\end{aligned}
$$

$$
\begin{aligned}
y(16) = 0 &+ (16 - 0) \times 22.704 + (16 - 0)(16 - 10) \times 0.2963 \\
&+ (16 - 0)(16 - 10)(16 - 15) \times 0.004 \\
&+ (16 - 0)(16 - 10)(16 - 15)(16 - 20) \times 0.0001 \\
&+ (16 - 0)(16 - 10)(16 - 15)(16 - 20)(16 - 22.5) \times 0
\end{aligned}
$$

$$
\begin{aligned}
y(16) = 0 &+ (16) \times 22.704 + (16)(6) \times 0.2963 + (16)(6)(1) \times 0.004 \\
&+ (16)(6)(1)(-4) \times 0.0001 + (16)(6)(1)(-4)(-6.5) \times 0
\end{aligned}
$$

$$y(16) = 0 + 363.264 + 28.4448 + 0.384 - 0.0384 + 0$$

$$y(16) = 392.0544$$

Solution of divided difference interpolation method $y(16) = 392.0544$

(b). The distance covered by the rocket between $t = 11s$ and $t = 16s$ can be calculated from the interpolating polynomial

$$\begin{aligned} v(t) &= y_1 + f[x_1, x_2](t - x_1) + f[x_1, x_2, x_3](t - x_1)(t - x_2) \\ &\quad + f[x_1, x_2, x_3, x_4](t - x_1)(t - x_2)(t - x_3) \\ v(t) &= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15) \\ &\quad + 0.005434 \times (t - 10)(t - 15)(t - 20) \\ &= -4.2541 + 21.265t + 0.13204t^2 + 0.005434t^3, \; 10 \le t \le 22.5 \end{aligned}$$

Note that the polynomial is valid between $t = 10$ and $t = 22.5$ and hence includes the limits of $t = 11$ and $t = 16$. So

$$
\begin{aligned}
& s(16) - s(11) \\
&= \int_{11}^{16} v(t)dt \\
&= \int_{11}^{16} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3)dt \\
&= \left[-4.2541t + 21.265\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4}\right]_{11}^{16} \\
&= \left[-4.2541(16) + 21.265\frac{(16)^2}{2} + 0.13204\frac{(16)^3}{3} + 0.0054347\frac{(16)^4}{4}\right] \\
&\quad - \left[-4.2541(11) + 21.265\frac{(11)^2}{2} + 0.13204\frac{(11)^3}{3} + 0.0054347\frac{(11)^4}{4}\right] \\
&= 1605m
\end{aligned}
$$

(c). The acceleration at $t = 16$ is given by

$$
\begin{aligned}
a(16) &= \frac{d}{dt} v(t)|_{t=16} \\
a(t) &= \frac{d}{dt} v(t) \\
&= \frac{d}{dt}(-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3) \\
&= (0) + 21.265(1) + 0.13204(2 \times t^{(2-1)}) + 0.0054347(3 \times t^{(3-1)}) \\
&= 21.265 + 0.26408t + 0.016304t^2 \\
a(16) &= 21.265 + 0.26408(16) + 0.016304(16)^2 \\
&= 29.664 m/s^2
\end{aligned}
$$

x	0	1	2	3	4
$y = f(x)$	2	3	6	11	18

Interpolation: Finding the value of $f(x)$ at some value of $x = 1.4$ in between two tabular values, e.g., between $f(x) = 3$ and $f(x) = 6$.

Inverse interpolation: If a value of $f(x)$ between $f(x) = 6$ and $f(x) = 11$ is known, inverse interpolation is to find the corresponding value of x.

Extrapolation: Determining the value of $f(x)$ at point $x = 5$ (outside the range of tabular values) .