Definition 1.5.1 (Diagonally Dominant Matrix)

A square matrix is called diagonally dominant if

$$|A_{i,i}| \geq \sum_{j=1,\ j\neq i}^{n} |A_{ij}|$$

A is called strictly diagonally dominant if

$$|A_{i,i}| > \sum_{j=1,\ j\neq i}^{n} |A_{ij}|$$

for all i.

Example 1.5.2

Find the solution to the following system of equations using the Gauss-Seidel method.

$$5x_1 - x_2 + 2x_3 = 12 \tag{11}$$

$$3x_1 + 8x_2 - 2x_3 = -25 \tag{12}$$

$$x_1 + x_2 + 4x_3 = 6 \tag{13}$$

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The coefficient matrix

$$[A] = \begin{bmatrix} 5 & -1 & 2 \\ 3 & 8 & -2 \\ 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is diagonally dominant as

$$\begin{aligned} |a_{11}| &= 5 & 12 &\geq |a_{12}| + |a_{13}| = |-1| + |2| = 3 \\ |a_{22}| &= 8 & 8 &\geq |a_{21}| + |a_{23}| = |3| + |-2| = 5 \\ |a_{33}| &= 4 & 4 &\geq |a_{31}| + |a_{32}| = |1| + |1| = 2 \end{aligned}$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidel method.

Since, no initial values are given, let us assume $x_1 = 0$, $x_2 = 0$ and $x = 0$ arbitrarily. Find the x_1, x_2 and x_3 values form Eqs (11), (12) and (13) correspondingly.

$$x_1 = \frac{12 + x_2 - 2x_3}{5} \tag{14}$$

$$x_2 = \frac{-25 - 3x_1 + 2x_3}{8} \tag{15}$$

$$x_3 = \frac{6 - x_1 - x_2}{4} \tag{16}$$

I^{st} Iteration:

Let us assume $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ in Eq (57), (15) and (16), Then form Eq.(57) we have

$$x_1 = \frac{12 + x_2 - 2x_3}{5} = \frac{12 + (0) - 2(0)}{5} = \frac{12}{5} = 2.4$$

Now the recent $x_1 = 2.4$. So, substitute $x_1 = 2.4$, $x_2 = 0$ and $x_3 = 0$ in Eq.(15), then we have,

$$x_2 = \frac{-25 - 3x_1 + 2x_3}{8} = \frac{-25 - 3(2.4) + 2(0)}{8} = -4.025$$

Now substitute the recent value of $x_1 = 2.4$, $x_2 = -4.025$ and $x_3 = 0$ in Eq.(16) we have

$$x_3 = \frac{6 - x_1 - x_2}{4} = \frac{6 - (2.4) - (-4.025)}{4} = 1.90625$$

2^{nd} Iteration:

Now the recent value of $x_1 = 2.4$, $x_2 = -4.025$ and $x_3 = 1.90625$. Substitute these values in Eq.(57), we get

$$x_1 = \frac{12 + x_2 - 2x_3}{5} = \frac{12 + (-4.025) - 2(1.90625)}{5} = 0.8325.$$

Substitute the recent value of $x_1 = 0.8325$, $x_2 = 4.025$ and $x_3 = 1.90625$ in Eq.(15), we get

$$x_2 = \frac{-25 - 3x_1 + 2x_3}{8} = \frac{-25 - 3(0.8325) + 2(1.90625)}{8} = -2.960625.$$

Substitute the recent values of $x_1 = 0.8325$, $x_2 = -2.960625$ and $x_3 = 1.90625$ in Eq.(16), we get

$$x_3 = \frac{6 - x_1 - x_2}{4} = \frac{6 - (0.835) - (-2.960625)}{4} = 2.03203125$$

3^{nd} Iteration:

Now the recent value of $x_1 = 0.8325$, $x_2 = -2.960625$ and $x_3 = 2.03203125$. Substitute these values in Eq.(57), we get

$$x_1 = \frac{12 + x_2 - 2x_3}{5} = \frac{12 + (-2.960625) - 2(2.03203125)}{5} = 0.9950625.$$

Substitute the recent value of $x_1 = 0.9950625$, $x_2 = -2.960625$ and $x_3 = 2.03203125$ in Eq.(15), we get

$$x_2 = \frac{-25 - 3x_1 + 2x_3}{8} = \frac{-25 - 3(0.9950625) + 2(2.03203125)}{8} = -2.99014$$

Substitute the recent values of $x_1 = 0.9950625$, we get $x_2 = -2.990140625$ and $x_3 = 2.03203125$ in Eq.(16)

$$x_3 = \frac{6 - x_1 - x_2}{4} = \frac{6 - (0.9950625) - (-2.990140625)}{4} = 1.99876953125$$

Three iteration values are tabulated below.

	x_1	x_2	x_3
I^{st} iteration	2.4	-4.025	1.90625
II^{nd} iteration	0.8325	-2.960625	2.03203125
III^{rd} iteration	0.9950625	-2.9901406525	1.9987953125

From the table one can conclude the value of $x_1 = 1$, $x_2 = -3$ and $x_3 = 2$.

Example 1.5.3

Find the solution to the following system of equations using the Gauss-Seidel method.

$$\begin{aligned} 12x_1 + 3x_2 - 5x_3 &= 1 \\ x_1 + 5x_2 + 3x_3 &= 28 \\ 3x_1 + 7x_2 + 13x_3 &= 76. \end{aligned}$$

Use $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as the initial guess and conduct three iterations.

The coefficient matrix

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is diagonally dominant as

$$\begin{aligned} |a_{11}| &= 12 & \qquad 12 &\geq |a_{12}| + |a_{13}| = |3| + |-5| = 8 \\ |a_{22}| &= 5 & \qquad 5 &\geq |a_{21}| + |a_{23}| = |1| + |3| = 4 \\ |a_{33}| &= 13 & \qquad 13 &\geq |a_{31}| + |a_{32}| = |3| + |7| = 10 \end{aligned}$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidel method.

Rewriting the equations, we get

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$
$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$
$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}.$$

Assuming an initial guess of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$

Iteration I

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12} = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5} = \frac{28 - (0.50000) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13} = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

Iteration II

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12} = \frac{1 - 3(4.9000) + 5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5} = \frac{28 - (0.14679) - 3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13} = \frac{76 - 3(0.14679) - 7(3.7153)}{13} = 3.8118$$

Iteration III

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12} = \frac{1 - 3(3.7153) + 5(3.8118)}{12} = 0.742758$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5} = \frac{28 - (0.742758) - 3(3.8118)}{5} = 3.164368$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13} = \frac{76 - 3(0.742758) - 7(3.164368)}{13} = 3.9708$$

Iteration	x_1	x_2	x_3
1	0.50000	4.9000	3.0923
2	0.14679	3.7153	3.8118
3	0.74275	3.1644	3.9708
4	0.94675	3.0281	3.9971

From the table we can approximate the solution $x_1 = 1$, $x_2 = 3$ and $x_3 = 4$.

Example 1.5.4

Given the system of equations

$$\begin{aligned} 3x_1 + 7x_2 + 13x_3 &= 76 \\ x_1 + 5x_2 + 3x_3 &= 28 \\ 12x_1 + 3x_2 - 5x_3 &= 1 \end{aligned}$$

find the solution using the Gauss-Seidel method. Use $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as the initial guess.

Rewriting the equations, we get

$$\begin{aligned} x_1 &= \frac{76 - 7x_2 - 13x_3}{3} \\ x_2 &= \frac{28 - x_1 - 3x_3}{5} \\ x_3 &= \frac{1 - 12x_1 - 3x_2}{-5} \end{aligned}$$

Assuming an initial guess of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ the next six iterative values are given in the table below.

Iteration	x_1	x_2	x_3
1	21.000	0.80000	50.680
2	–196.15	14.421	–462.30
3	1995.0	–116.02	4718.1
4	–20149	1204.6	–47636

You can see that this solution is not converging and the coefficient matrix is not diagonally dominant. The coefficient matrix

$$[A] = \begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

is not diagonally dominant as

$$|a_{11}| = |3| = 3 \ngeq |a_{12}| + |a_{13}| = |7| + |13| = 20$$

Hence, the Gauss-Seidel method may or may not converge. However, it is the same set of equations as the previous example and that converged. The only difference is that we exchanged first and the third equation with each other and that made the coefficient matrix not diagonally dominant.
Therefore, it is possible that a system of equations can be made diagonally dominant if one exchanges the equations with each other. However, it is not possible for all cases. For example, the following set of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 2x_2 + 3x_2 + 4x_3 &= 9 \\ x_1 + 7x_2 + x_3 &= 9 \end{aligned}$$

cannot be rewritten to make the coefficient matrix diagonally dominant.

Comparison of solving equation with algebraic method and matrix method

Solving by equation		Solving by matrix				
$Eq(1)$	$x_1 + x_2 - 2x_3 = 1$	$R(1)$	1	1	-2	1
$Eq(2)$	$2x_1 - 3x_2 + x_3 = -8$	$R(2)$	2	-3	1	-8
$Eq(3)$	$3x_1 + x_2 + 4x_3 = 7$	$R(3)$	3	1	4	7
	$x_1 + x_2 - 2x_3 = 1$		1	1	-2	1
$Eq(2) - 2Eq(1)$	$-5x_2 + 5x_3 = -10$	$R(2) - 2R(1)$	0	-5	5	-10
$Eq(3) - 3Eq(1)$	$-2x_2 + 10x_3 = 4$	$R(3) - 3R(1)$	0	-2	10	4
	$x_1 + x_2 - 2x_3 = 1$		1	1	-2	1
$(-1/5)Eq(2)$	$x_2 - x_3 = 2$	$(1/5)R(2)$	0	1	-1	2
$(-1/2)Eq(3)$	$x_2 - 5x_3 = -2$	$(1/2)R(3)$	0	1	-5	-2
	$x_1 + x_2 - 2x_3 = 1$		1	1	-2	1
	$x_2 - x_3 = 2$		0	1	-1	2
$Eq(3) - Eq(2)$	$-4x_3 = -4$	$R(3) - R(2)$	0	0	-4	-4
	$x_1 + x_2 - 2x_3 = 1$		1	1	-2	1
	$x_2 - x_3 = 2$		0	1	-1	2
$(-1/4)Eq(3)$	$x_3 = 1$	$(-1/4)R(3)$	0	0	[illegible]	1

Solving by equation		Solving by matrix	
$Eq(1)$	$x_1 + x_2 - 2x_3 = 1$	$R(1)$	$1 \quad 1 \quad -2 \quad 1$
$Eq(2)$	$2x_1 - 3x_2 + x_3 = -8$	$R(2)$	$2 \quad -3 \quad 1 \quad -8$
$Eq(3)$	$3x_1 + x_2 + 4x_3 = 7$	$R(3)$	$3 \quad 1 \quad 4 \quad 7$
	$x_1 + x_2 = 3$		$1 \quad 1 \quad 0 \quad 3$
$Eq(1) + 2Eq(3)$	$x_2 = 3$	$R(1) + 2R(3)$	$0 \quad 1 \quad 0 \quad 3$
$Eq(2) + Eq(3)$	$x_3 = 1$	$R(2) + R(3)$	$0 \quad 0 \quad 1 \quad 1$
	$x_1 = 0$		$1 \quad 0 \quad 0 \quad 0$
	$x_2 = 3$		$0 \quad 1 \quad 0 \quad 3$
$Eq(1) - Eq(2)$	$x_3 = 1$	$R(1) - R(2)$	$0 \quad 0 \quad 1 \quad 1$

Example 1.6.1

Solve the system of equations

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ -6x_1 + x_2 - x_3 &= 3 \\ 3x_1 + x_2 + x_3 &= 4 \end{aligned}$$

using Gaussian elimination method (GEM) with partial pivoting and backward substitution.

Solution: Consider the augmented matrix of the given system of equations

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ -6 & 1 & -1 & 3 \\ 3 & 1 & 1 & 4 \end{array}\right]$$

Step: 1 Find the largest absolute value in the first column and shift the row into first row. Among $\{|1|, |-6|, |3|\}$, R_2 has the largest absolute value $|-6| = 6$. So, bring R_2 into first row.

$$R_2 \longleftrightarrow R_1 \Rightarrow \left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 1 & -1 & 1 & 2 \\ 3 & 1 & 1 & 4 \end{array}\right]$$

Step: 2 To obtain echelon form, let us make first element of R_2 into 0 using R_1.

$$R_2 + \left(\frac{1}{6}\right) R_1 \longrightarrow R_2 \Rightarrow \left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 0 & \frac{-5}{6} & \frac{5}{6} & \frac{5}{2} \\ 3 & 1 & 1 & 4 \end{array}\right]$$

Step: 3 To obtain echelon form, let us make first element of R_3 into 0 using R_1.

$$R_3 + \left(\frac{1}{2}\right) R_1 \longrightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 0 & \frac{-5}{6} & \frac{5}{6} & \frac{5}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} & \frac{11}{2} \end{array}\right]$$

Step: 4 Now, compare the absolute value of second element of R_2 and R_3. (i.e.) $\left|\frac{-5}{6}\right|, \left|\frac{3}{2}\right|$. Since absolute value of second element of 3^{rd} row is maximum. Swap R_3 and R_2. Then, we have,

$$R_3 + \left(\frac{1}{2}\right) R_1 \longrightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & \frac{11}{2} \\ 0 & \frac{-5}{6} & \frac{5}{6} & \frac{5}{2} \end{array}\right]$$

Step: 5 To obtain echelon form, let us make second element of R_3 into 0 using R_2 (Since first element of R_2 is zero).

$$R_3 + \left(\frac{5}{9}\right) R_2 \longrightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & \frac{11}{2} \\ 0 & 0 & \frac{10}{9} & \frac{50}{9} \end{array}\right]$$

Step: 6 Now, let us convert the augmented matrix into equations.

$$-6x_1 + 1x_2 - 1x_3 = 3 \tag{17}$$
$$\frac{3}{2}x_2 + \frac{1}{2}x_3 = \frac{11}{2} \tag{18}$$
$$\frac{10}{9}x_3 = \frac{50}{9} \tag{19}$$

Step: 7 Using backward substitution to solve the equations.
From Eq.(19), we have,

$$\frac{10}{9}x_3 = \frac{50}{9}$$
$$x_3 = \frac{50}{9} \times \frac{9}{10}$$
$$x_3 = 5$$

Substitute $x_3 = 5$ in Eq.(18), we have,

$$\begin{aligned} \frac{3}{2}x_2 + \frac{1}{2}x_3 &= \frac{11}{2} \\ \frac{3}{2}x_2 + \frac{1}{2}(5) &= \frac{11}{2} \\ \frac{3}{2}x_2 &= \frac{11}{2} - \frac{5}{2} = 3 \\ x_2 &= 3\left(\frac{2}{3}\right) \\ x_2 &= 2 \end{aligned}$$

Substitute $x_2 = 2$ and $x_3 = 5$ in Eq.(17), we have,

$$\begin{aligned} -6x_1 + x_2 - x_3 &= 3 \\ -6x_1 + (2) - (5) &= 3 \\ -6x_1 &= 3 + 3 = 6 \\ x_1 &= -1 \end{aligned}$$

So, the roots are $x_1 = -1$, $x_2 = 2$ and $x_3 = 5$.

Example 1.6.2

Let us use the GEM with partial pivoting to solve the following system:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 5 \\ 4x_1 - 6x_2 &= -2 \\ -2x_1 + 7x_2 + 2x_3 &= 9. \end{aligned}$$

$$\left[\begin{array}{rrr|r} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

$$R_1 \longleftrightarrow R_2 \left[\begin{array}{rrr|r} 4 & -6 & 0 & -2 \\ 2 & 1 & 1 & 5 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

$$R_2 \to R_2 - (1/2)R_1, R_3 \to R_3 + (1/2)R_1 \left[\begin{array}{ccc|c} 4 & -6 & 0 & -2 \\ 0 & 4 & 1 & 6 \\ 0 & 4 & 2 & 8 \end{array}\right]$$

$$R_3 \Rightarrow R_3 - R_2 \left[\begin{array}{ccc|c} 4 & -6 & 0 & -2 \\ 0 & 4 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Thus $x_3 = 2$.
Then $4x_2 + x_3 = 6$ gives $x_2 = 1$, and $4x_1 - 6x_2 = -2$ gives $x_1 = 1$.
Hence $x_1 = 1, x_2 = 1, x_3 = 2$ is the (unique) solution of the given linear system.

Example 1.6.3

Solve the system equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 4x_1 + 3x_2 + 4x_3 &= 8 \\ 9x_1 + 3x_2 + 4x_3 &= 7 \end{aligned}$$

using Gauss elimination method with partial pivoting and backward substitution method.

Solution

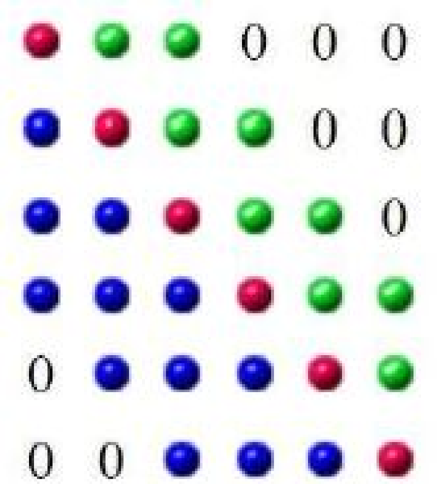
- $R_1 \longleftrightarrow R_1$
- $R_2 \to R_2 - 4R_3$, $R_3 \to R_1 - 9R_3$
- $R_3 \to -6R_2 + R_3$
- $x_1 = -\frac{1}{5}$, $x_2 = 4$ and $x_3 = -\frac{4}{5}$

Banded matrix

A band matrix is a sparse matrix whose non-zero entries are confined to a diagonal band, comprising the main diagonal and zero or more diagonals on either side.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 6 & 2 & 5 \\ 0 & 7 & 3 \end{bmatrix}$$

The matrix can be symmetric, having the same number of sub- and super- diagonals. If a matrix has only one sub- and one super-diagonal, we have a tridiagonal matrix etc. The number of super-diagonals is called the upper bandwidth (two in the example), and the number of sub-diagonals is the lower bandwidth (three in the example). The total number of diagonals, six in the example, is the bandwidth.



A tridiagonal matrix is a band matrix that has nonzero elements only on the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal.

Considering a 4 X 4 Matrix $\begin{bmatrix} a_1 & b_1 & 0 & 0 \\ c_2 & a_2 & b_2 & 0 \\ 0 & c_3 & a_3 & b_3 \\ 0 & 0 & c_4 & a_4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

- Thomas' algorithm, also called TriDiagonal Matrix Algorithm (TDMA) is essentially the result of applying Gaussian elimination to the tridiagonal system of equations.
- A system of simultaneous algebraic equations with nonzero coefficients only on the main diagonal, the lower diagonal, and the upper diagonal is called a tridiagonal system of equations.

Consider a tridiagonal system of N equations with N unknowns, $u_1, u_2, u_3, \cdots, u_N$

as given below:
$$\begin{bmatrix} b_1 & c_1 & & & & & & \\ a_1 & b_2 & c_2 & & & & & \\ & a_3 & b_3 & c_3 & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & & & a_N & b_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{N-1} \\ d_N \end{bmatrix}$$

Example

Let us consider the system of equations

$$\begin{aligned} 3x_1 - x_2 + 0x_3 &= -1 \\ -x_1 + 3x_2 - x_3 &= 7 \\ 0x_1 - x_2 + 3x_3 &= 7 \end{aligned}$$

Matrix form is $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix}$

Stage I : (Converting $Mx = r$ into $U_x = \rho$)

Row 1 operation

$$3x_1 - x_2 = -1$$

Divide the Equation by a_1, in this case $a_1 = 3$

$$\Rightarrow x_1 - \frac{1}{3}x_2 = \frac{-1}{3}$$

Assuming the coefficient of x_2 as γ_1 and the remaining constants as ρ_1. Now the equations converts to,

$$\Rightarrow x_1 + \gamma_1 x_2 = \rho_1$$

$$\Rightarrow \gamma_1 = \frac{-1}{3}, \rho_1 = \frac{-1}{3}$$

After changing row 1

$$\begin{bmatrix} 1 & \gamma_1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ 7 \\ 7 \end{bmatrix}$$

Converting $Mx = r$ into $Ux = \rho \ldots$

Row 2 operation

Multiplying $a_2 = -1$ in Row 1 and eliminating x_1 Row 2

Row 2	$-x_1 + 3x_2 - x_3 = 7$
$a_2 \times$ Row 1	$-x_1 - \gamma_1 x_2 - 0x_3 = -\rho_1$
Subtracting	$x_2(3 + \gamma_1) - x_3 = 7 + \rho_1$

$$x_2(3 + \gamma_1) - x_3 = 7 + \rho_1$$

Divide by $(3 + \gamma_1)$, Equation becomes $x_2 + \gamma_2 x_3 = \rho_2$

$$\gamma_2 = \frac{-1}{3 + \gamma_1} = \frac{-1}{3 + \left(\frac{-1}{3}\right)} = 0.375,$$

$$\rho_2 = \frac{7 + \rho_1}{3 + \gamma_1} = \frac{7 + \left(\frac{-1}{3}\right)}{3 + \left(\frac{-1}{3}\right)} = 2.5$$

After changing row 2

$$\begin{bmatrix} 1 & \gamma_1 & 0 \\ 0 & 1 & \gamma_2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ 7 \end{bmatrix}$$

Converting $Mx = r$ into $Ux = \rho \ldots$

Row 3 operation

Multiplying $a_3 = -1$ in Row 2 and eliminating x_2 in Row 3

Row 3	$-x_2 + 3x_3 = 7$
$a_3 \times$ Row 2	$-x_2 - \gamma_2 x_3 = -\rho_2$
Subtracting	$(3 + \gamma_2)x_3 = 7 + \rho_2$

Divide by $3 + \gamma_2$, then the equation $x_3 = \frac{7+\rho_2}{3+\gamma_2} \Rightarrow \rho_3$ becomes,

$$x_3 = \rho_3$$

where,

$$\rho_3 = \frac{7 + \rho_2}{3 + \gamma_2} = \frac{7 + 2.5}{3 + 0.375} = 3.619$$
$$\Rightarrow x_3 = 3.619$$

After changing row 2 $\begin{bmatrix} 1 & \gamma_1 & 0 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}$

Stage II: (Backward Substitution)

Row 2:

$$x_2 + \gamma_2 x_3 = \rho_2$$
$$x_2 = \rho_2 - \gamma_2 x_3$$

Substituting

$$\rho_2 = 2.5$$
$$\gamma_2 = -0.375$$

$x_2 = 3.857$

$$\begin{bmatrix} 1 & \gamma_1 & 0 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}$$

Row 1:

$$x_1 + \gamma_1 x_2 = \rho_1$$
$$x_1 = \rho_1 - \gamma_1 x_2$$

Substituting

$$\rho_1 = -0.333$$
$$\gamma_1 = -0.333$$

$x_1 = 0.952$

$$x_1 = 0.952$$
$$x_2 = 3.857$$
$$x_3 = 3.619$$

- Algorithm is unstable when if the tridiagonal matrix is singular and in some cases non-singular also.
- condition for algorithm to be stable

$$|b_i| > |a_i| + |c_i| \, for \; every \; i.$$