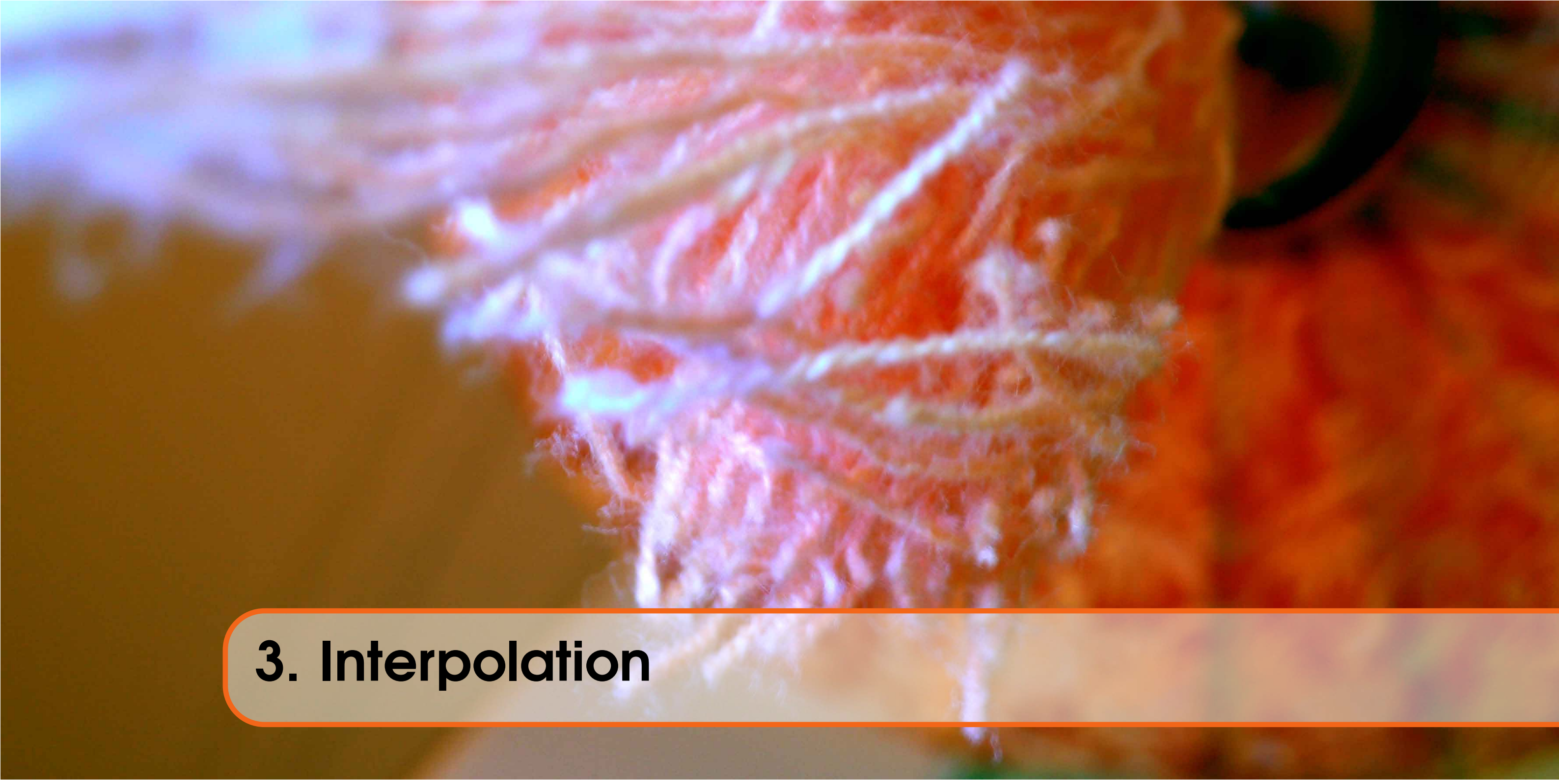

3. Interpolation

3.1 Cubic spline interpolation for a tabulated function with equally spaced data

In the interpolation methods so far explained, a single polynomial has been fitted to the tabulated points. If the given set of points belong to the polynomial, then this method works well, otherwise the results are rough approximations only. If we draw lines through every two closest points, the resulting graph will not be smooth. Similarly we may draw a quadratic curve through points A_i, A_{i+1} and another quadratic curve through A_{i+1}, A_{i+2}, such that the slopes of the two quadratic curves match at A,,, (Fig. 7.1). The resulting curve looks better but is not quite smooth. We can ensure this by drawing a cubic curve through $A_i, A+i+1$ and another cubic through A_{i+1}, A_{i+2}, such that the slopes and curvatures of the two curves match at A_{i+1}. Such a curve is called a cubic spline. We may use polynomials of higher order but the resulting graph is not better. As such, cubic splines are commonly used. This technique of 'spline-fitting' is of recent origin and has important applications.

Definition 3.1.1 Consider the problem of interpolating between the data points $(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$ by means of spline fitting.

Then the cubic spline $f(x)$ is such that

1. $f(x)$ is a linear polynomial outside the interval (x_0, x_n),
2. $f(x)$ is a cubic polynomial in each of the subintervals,
3. $f'(x)$ and $f''(x)$ are continuous at each point.

Since $f(x)$ is cubic in each of the subintervals $f''(x)$ shall be linear.

$\therefore$ Taking equally-spaced values of x so that $x_{i+1} - x_i = h$, we can write

$$f''(x) = \frac{1}{h}\left[(x_{i+1} - x) f''(x_i) + (x - x_i) f''(x_{i+1})\right]$$

Integrating twice, we have

$$f(x) = \frac{1}{h}\left[\frac{(x_{i+1} - x)^3}{3!} f''(x_i) + \frac{(x - x_i)^3}{3!} f''(x_{i+1})\right] + a_i (x_{i+1} - x) + b_i (x - x_i). \tag{3.1}$$

The constants of integration a_i, b_i are determined by substituting the values of $y = f(x)$ at x_i and x_{i+1}. Thus

$$a_i = \frac{1}{h}\left[y_i - \frac{h^2}{3!}f''(x_i)\right] \text{ and } b_i = \frac{1}{h}\left[y_{i+1} - \frac{h^2}{3!}f''(x_{i+1})\right]$$

Substituting the values of a_i, b_i and writing $f''(x_i) = M_i$, (3.1) takes the form

$$f(x) = \frac{(x_{i+1}-x)^3}{6h}M_i + \frac{(x-x_i)}{6h}M_{i+1} + \frac{x_{i+1}-x}{h}\left(y_i - \frac{h^2}{6}M_i\right) + \frac{x-x_i}{h}\left(y_{i+1} - \frac{h^2}{6}M_{i+1}\right) \tag{3.2}$$

$$\therefore \quad f'(x) = -\frac{(x_{i+1}-x)^2}{2h}M_i + \frac{(x-x_i)^2}{h}M_{i+1} - \frac{h}{6}(M_{i+1}-M_i) + \frac{1}{h}(y_{i+1}-y_i)$$

To impose the condition of continuity of $f'(x)$, we get $f'(x-\varepsilon) = f'(x+\varepsilon)$ as $\varepsilon \to 0$, therefore

$$\frac{h}{6}(2M_i + M_{i-1}) + \frac{1}{h}(y_i - y_{i-1}) = -\frac{h}{6}(2M_i + M_{i+1}) + \frac{1}{h}(y_{i+1} - y_i)$$

or

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), i = 1 \text{ to } n-1 \tag{3.3}$$

Now since the graph is linear for $x < x_0$ and $x > x_n$, we have

$$M_0 = 0, M_n = 0 \tag{3.4}$$

Equations (3.3) and (3.4) give $(n+1)$ equations in $(n+1)$ unknowns $M_i (i = 0, 1, \ldots\ldots n)$ which can be solved. Substituting the value of M_i in (3.2) gives the concerned cubic spline.

■ **Example 3.1** Obtain the cubic spline for the following data: ■

$x:$	0	1	2	3
$y:$	2	-6	-8	2

Since the points are equispaced with $h = 1$ and $n = 3$, the cubic spline can be determined from $M_{i-1} + 4M_i + M_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1}), i = 1, 2$.

$$\begin{aligned}
\therefore \qquad & M_0 + 4M_1 + M_2 = 6(y_0 - 2y_1 + y_2) \\
& M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3) \\
\textit{i.e.,} \qquad & 4M_1 + M_2 = 36 \qquad [\because M_0 = 0, M_3 = 0] \\
& M_1 + 4M_2 = 72
\end{aligned}$$

Solving these, we get $M_1 = 4.8, M_2 = 16.8$.
Now the cubic spline in $(x_i \le x \le x_{i+1})$ is

$$f(x) = \frac{1}{6}(x_{i+1}-x)^3 M_i + \frac{1}{6}(x-x_i)^3 M_{i+1} + (x_{i+1}-x)\left(y_i - \frac{1}{6}M_i\right) + (x-x_i)\left(y_{i+1} - \frac{1}{6}M_{i+1}\right) \tag{3.5}$$

Taking $i = 0$ in (3.5) the cubic spline in $(0 \le x \le 1)$ is

$$\begin{aligned}
f(x) &= \frac{1}{6}(1-x)^3(0) + \frac{1}{6}(x-0)^3(4.8) + (1-x)(x-0) + x\left[-6 - \frac{1}{6}(4.8)\right] \\
&= 0.8x^3 - 8.8x + 2 \quad (0 \le x \le 1)
\end{aligned}$$

Taking $i = 1$ in (3.5), the cubic spline in $(1 \leq x \leq 2)$ is

$$\begin{aligned} f(x) &= \frac{1}{6}(2-x)^3(4.8) + \frac{1}{6}(x-1)^3(16.8) + (2-x)\left[-6 - \frac{1}{6}(4.8)\right] \\ &= 2x^3 - 5.84x^2 - 1.68x + 0.8 \end{aligned}$$

Taking $i = 2$ in (3.5), the cubic spline in $(2 \leq x \leq 3)$ is

$$\begin{aligned} f(x) &= \frac{1}{6}(3-x)^3(4.8) + \frac{1}{6}(x-2)^3(0) + (3-x)\left[-8 - \frac{1}{6}(16.8)\right] + (x-2)\left[2 - \frac{1}{6}(2)\right] \\ &= -0.8x^3 + 2.64x^2 + 9.68x - 14.8 \end{aligned}$$

■ **Example 3.2** The following values of x and y are given : Find the cubic splines and evaluate

x:	1	2	3	4
y:	1	2	5	11

$y(1.5)$ and $y'(3)$. ■

Since the points are equispaced with $h = 1$ and $n = 3$, the cubic splines can be obtained from

$$\begin{aligned} &M_{i-1} + 4M_i + M_{i+1} = 6\left(y_{i-1} - 2y_i + y_{i+1}\right), i = 1, 2. \\ \therefore \quad &M_0 + 4M_1 + M_2 = 6\left(y_0 - 2y_1 + y_2\right) \\ &M_1 + 4M_2 + M_3 = 6\left(y_1 - 2y_2 + y_3\right) \\ &i.e., \quad 4M_1 + M_2 = 12, \\ &\qquad M_1 + 4M_2 = 18 \end{aligned}$$

which give

$$M_1 = 2, M_2 = 4.$$

Now the cubic spline in $(x_i \leq x \leq x_{i+1})$ is

$$f(x) = \frac{1}{6}\left[\left(x_{i+1} - x\right)^3 M_i + \left(x - x_i\right)^3 M_{i+1}\right] + \left(x_{i+1} - x\right)\left(y_i - \frac{1}{6}M_i\right) + \left(x - x_i\right)\left(y_{i+1} - \frac{1}{6}M_{i+1}\right) \tag{3.6}$$

Thus, taking $i = 0, i = 1, i = 2$ in (3.6), the cubic splines are

$$f(x) = \begin{cases} \frac{1}{3}\left(x^3 - 3x^2 + 5x\right), & 1 \leq x \leq 2 \\ \frac{1}{3}\left(x^3 - 3x^2 + 5x\right), & 2 \leq x \leq 3 \\ \frac{1}{3}\left(-2x^3 + 24x^2 - 76x + 81\right), & 3 \leq x \leq 4 \end{cases}$$

$$\therefore \qquad y(1.5) = f(1.5) = \frac{11}{8}.$$

Also $y'(3) = \frac{14}{3}$, from both the splines of intervals [2,3] and [3, 4] as they should be.

■ **Example 3.3** Find the cubic spline interpolation for the data: ■

$x:$	1	2	3	4	5
$f(x):$	1	0	1	0	1

Since the points are equispaced with $h=1, n=4$, the cubic spline can be found by means of

$$\begin{aligned}
& M_{i-1}+4M_i+M_{i+1}=6\left(y_{i-1}-2y_i+y_{i+1}\right), i=1,2,3 \\
\therefore\quad & M_0+4M_1+M_2=6\left(y_0-2y_1+y_2\right)=12 \\
& M_1+4M_2+M_3=6\left(y_1-2y_2+y_3\right)=-12 \\
& M_2+4M_3+M_4=6\left(y_2-2y_3+y_4\right)=12
\end{aligned}$$

Since

$$M_0=y''=0 \text{ and } M_4=y''=0$$

Therefore

$$4M_1+M_2=12; M_1+4M_2+M_3=-12; M_1+4M_3=12$$

Solving these equations, we get

$$M_1=\frac{30}{7}, M_2=\frac{-36}{7}, M_3=\frac{30}{7}$$

Now the cubic spline in $(x_i \le x \le x_{i+1})$ is

$$f(x)=\frac{1}{6}\left(x_{i+1}-x\right)M_i+\left(x-x_i\right)M_{i+1}\left(x_{i+1}-x\right)\left(y_i-\frac{1}{6}M_i\right)+\left(x-x_i\right)\left(y_{i+1}-\frac{1}{6}M_{i+1}\right) \tag{3.7}$$

Taking $i=0$, in (3.7), the cubic spline in $(1 \le x \le 2)$ is

$$\begin{aligned}
y &= \frac{1}{6}\left[\left(x_1-x\right)^3 M_0+\left(x-x_0\right)^3 M_1\right]+\left(x_1-x\right)\left(y_0-\frac{1}{6}M_0\right)+\left(x-x_0\right)\left(y_1-\frac{1}{6}M_1\right) \\
&= \frac{1}{6}\left[(2-x)^3(0)+(x-1)^3(30/7)\right]+(2-x)\left[1-\frac{1}{6}(0)\right]+(x-1)\left[0-\frac{1}{6}\left(\frac{30}{7}\right)\right]
\end{aligned}$$

i.e.,

$$y=0.71x^3-2.14x^2+0.42x+2 \quad (1<x\le 2)$$

Taking $i=1$ in (3.7), the cubic spline in $(2 \le x \le 3)$ is

$$y=\frac{1}{6}\left[(3-x)^3\frac{30}{7}+(x-2)^3\left(-\frac{36}{7}\right)\right]+(3-x)\left[0-\frac{1}{6}\left(\frac{30}{7}\right)\right]+(x-2)\left[1-\frac{1}{6}\left(-\frac{36}{7}\right)\right]$$

i.e.,

$$y=-1.57x^3+11.57x^2-27x+20.28. \quad (2\le x\le 3)$$

Taking $i=2$ in (3.7), the cubic spline in $(3 \le x \le 4)$ is

$$y=\frac{1}{6}\left[(4-x)^3\left(-\frac{36}{7}\right)+(x-3)^3\left(\frac{30}{7}\right)\right]+(4-x)\left[1-\frac{1}{6}\left(-\frac{36}{7}\right)\right]+(x-3)\left(0-\frac{5}{7}\right)$$

i.e.,

$$y=1.57x^3-16.71x^2+57.86x-64.57 \quad (3\le x\le 4).$$

Taking $i=3$ in (3.7), the cubic spline in $(4 \le x \le 5)$ is

$$y=\frac{1}{6}\left[(1-x)^3\left(\frac{30}{7}\right)\right]+(5-x)\left(-\frac{5}{7}\right)+(x-4)(1)$$

i.e.,

$$y=-0.71x^3+2.14x^2-0.43x-6.86. \quad (4\le x\le 5)$$

3.2 Practice Problem

1. Find the cubic splines for the following table of values :

x:	1	2	3
y:	-6	-1	16

Hence evaluate $y(1.5)$ and $y'(2)$.

2. From the following table :

x:	1	2	3
y:	-8	-1	18

compute $y(1.5)$ and $y'(1)$ using cubic spline.

3. Obtain the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y_0'' = y_0''' = 0$:

x:	-1	0	1	2
y:	-1	1	3	35

4. The following values of x and y are given :

x :	1	2	3	4
y :	1	5	11	8

Using cubic splines, show that
   (a) $y(1.5) = 2.575$
   (b) $y'(3) = 2.067$

5. Fit a cubic spline curve for the points $(2,11), (3,49)$ and $(4,123)$. Hence find $y(2.5)$ and $y'(3.5)$. Assume that $y''(2) = 0$ and $y''(4) = 0$.
6. Find the cubic spline corresponding to the interval $[2,3]$ from the following table:

x :	1	2	3	4	5
y :	30	15	32	18	25

Hence compute
   (a) $y(2.5)$
   (b) $y'(3)$.

3.3 Interpolation

3.4 Finite Differences

3.4.1 Introduction

The calculus of finite differences deals with the changes that take place in the value of the function (dependent variable), due to finite changes in the independent variable. Through this, we also study the relations that exist between the values assumed by the function, whenever the independent variable changes by finite jumps whether equal or unequal. On the other hand, in infinitesimal calculus, we study those changes of the function which occur when the independent variable changes continuously in a given interval. In this chapter, we shall study the variations in the function when the independent variable changes by equal intervals.