Discrete: Maths Olympiad

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Abstract—**In this letter, an algorithm for evaluating the exact analytical bit error rate (BER) for the piecewise linear (PL) combiner for multiple relays is presented. Previous results were available only for upto three relays. The algorithm is unique in the sense that the actual mathematical expressions, that are prohibitively large, need not be explicitly obtained. The diversity gain due to multiple relays is shown through plots of the analytical BER, well supported by simulations.**

1. Let a, b, c be the sides of a triangle, and T its area. Prove: $a^2 + b^2 + c^2 \geq 4\sqrt{3}T$.In what case does equality hold?
2. Let S = 1, 2, 3,...., 280. Find the smallest integer n such that each nelement subset of S contains five numbers which are pairwise relatively prime.

3. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges 1, 2, ... , k in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.
4. Let R denote the set of all real numbers. Find all functions f : $R \rightarrow R$ such that

   $f(x^2 + f(y)) = y + (f(x))^2$ for all x,y $\in$R.
5. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

   $$\left|S^2\right| \leq \left|S_x\right| . \left|S_y\right| . \left|S_z\right|$$

   where $\left|A\right|$ denotes the number of elements in the finite set $\left|A\right|$. (Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)
6. On an infinite chessboard, a game is played as follows. At the start, n^2 pieces are arranged on the chessboard in an n by n block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of n for which the game can end with only one piece remaining on the board.
7. Does there exist a function f : $N \rightarrow N$ such that f(1) = 2, f(f(n)) = f(n) + n for all $n \in N$, and $f(n) < f(n+1)$ for all $n \in N$?
8. For any positive integer k, let f(k) be the number of elements in the set k+1, k+2, ... , 2k whose base 2 representation has precisely three 1s.

   a) Prove that, for each positive integer m, there exists at least one positive integer k such that f(k) = m.
   b) Determine all positive integers m for which there exists exactly one k with f(k) = m.
9. Let S be the set of real numbers strictly greater than -1. Find all functions f : $S \rightarrow S$ satisfying the two conditions:

   a) f(x+f(y)+xf(y)) = y+f(x)+yf(x) for all x and y in S;
   b) $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$
10. Determine all integers $n > 3$ for which there exist n points $A_1, ..., A_n$ in the plane, no three collinear, and real numbers $r_1, ..., r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_iA_jA_k$ is $r_i + r_j + r_k$.

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11. Let p be an odd prime number. How many p-element subsets A of 1, 2, . . . 2p are there, the sum of whose elements is divisible by p?
12. Let S denote the set of non-negative integers. Find all functions f from S to itself such that

    f(m + f(n)) = f(f(m)) + f(n) $\forall m, n \in S$.

13. Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, ..., x_n)$ be an (n + 1) - tuple of integers satisfying the following conditions:

    a) $x_0 = x_n = 0$.
    b) For each i with $1 \leq i \leq n$, either $x_i - x_i - 1 = p$ or $x_i - x_{i-1} = -q$.

    Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.
14. An n x n matrix whose entries come from the set S = 1, 2, ... , 2n - 1 is called a silver matrix if, for each i = 1, 2, ... , n, the ith row and the ith column together contain all elements of S. Show that

    a) there is no silver matrix for n = 1997;
    b) silver matrices exist for infinitely many values of n.
15. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that k/a ≥ (b - 1)/(2b).
16. Consider all functions f from the set N of all positive integers into itself satisfying $f(t^2 f(s)) = s(f(t))^2$ for all s and t in N. Determine the least possible value of f(1998).
17. Determine all finite sets S of at least three points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

18. Let n be a fixed integer, with $n \geq 2$.

    a) Determine the least constant C such that the inequality

    $\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C (\sum_{1 \leq i \leq n} x_i)^4$

    holds for all real numbers $x_1, ..., x_n \geq 0$

    b) For this constant C, determine when equality holds.
19. Determine all functions f : $R \rightarrow R$ such that

    f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1 for all real numbers x, y.
20. 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

21. Can we find N divisible by just 2000 different primes, so that N divides $2^N + 1$? [N may be divisible by a prime power.]
22. Twenty-one girls and twenty-one boys took part in a mathematical contest.
    a) Each contestant solved at most six problems.
    b) For each girl and each boy, at least one problem was solved by both of them.

    Prove that there was a problem that was solved by at least three girls and at least three boys.

23. Let n be an odd integer greater than 1, and let $k_1, k_2, ..., k_n$ be given integers. For each of the n! permutations a = $(a_1, a_2, \ldots, a_n)$ of 1 , 2, . . . , n, let

    $S(a) = \sum_{i=1}^{n} k_i a_i$.

    Prove that there are two permutations b and c, $b \neq c$, such that n! is a divisor of S(b)-S(c).
24. Find all real-valued functions on the reals such that (f(x) + f(y))((f(u) + f(v)) = f(xu - yv) + f(xv + yu) for all x, y, u, v.
25. S is the set 1, 2, 3, . . . , 1000000. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets $a + x_i / a \in A$ are all pairwise disjoint.

26. Given $n > 2$ and reals $x_1 \leq x_2 \leq .... \leq x_n$, show

that $(\sum_{i,j} \left|x_i - x_j\right|)^2 \leq \frac{2}{3}(n^2 - 1) \sum_{i,j}(x_i - x_j)^2$. Show that we have equality iff the sequence is an arithmetic progression.

27. We call a positive integer alternating if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n such that n has a multiple which is alternating.