

Algebra: JEE Maths

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1. For how many pairs of positive integers (x, y) is $x + 3y = 100$?

2. Let

$$S_n = n^2 + 20n + 12$$

n is a positive integer. What is the sum of all possible values of n for which S_n is a perfect square?

3. Suppose that

$$4^{X_1} = 5, 5^{X_2} = 6, 6^{X_3} = 7, \dots, 126^{X_{123}} = 127,$$

$$127^{X_{124}} = 128$$

What is the value of the product $X_1 X_2 \dots X_{124}$?

4. Let

$$P(n) = (n+1)(n+3)(n+5)(n+7)(n+9)$$

What is the largest integer that is a divisor of $P(n)$ for all positive even integers n?

5. If

$$a = b - c, b = c - d, c = d - a$$

and $abcd \neq 0$ then what is the value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$$

6. Let x_1, x_2, x_3 be the roots of the equation

$$x^3 + 3x + 5 = 0$$

What is the value of the expression

$$(x_1 + \frac{1}{x_1})(x_2 + \frac{1}{x_2})(x_3 + \frac{1}{x_3})$$

7. How many integer pairs (x, y) satisfy

$$x^2 + 4y^2 - 2xy - 2x - 4y - 8 = 0 \quad (7.1)$$

8. What is the sum of the squares of the roots of

the equation

$$x^2 - 7[x] + 5 = 0 \quad (8.1)$$

(Here $[x]$ denotes the greatest integer less than or equal to x. For example $[3.4] = 3$ and $[-2.3] = -3$.)

9. What is the smallest positive integer k such that

$$k(3^3 + 4^3 + 5^3) = a^n \quad (9.1)$$

for some positive integers a and n, with $n > 1$?

10. Let

$$S_n = \sum_{k=0}^n \frac{1}{\sqrt{k+1} + \sqrt{k}}$$

What is the value of

$$\sum_{n=1}^{90} \frac{1}{S_n + S_{n-1}}$$

11. It is given that the equation

$$x^2 + ax + 20 = 0 \quad (11.1)$$

has integer roots. What is the sum of all possible values of a?

12. Three real numbers x, y, z are such that

$$x^2 + 6y = -17 \quad (12.1)$$

$$y^2 + 4z = 1 \quad (12.2)$$

$$z^2 + 2x = 2 \quad (12.3)$$

What is the value of $x^2 + y^2 + z^2$?

13. Let

$$f(x) = x^3 - 3x + b$$

$$g(x) = x^2 + bx - 3$$

where b is a real number. What is the sum of the all possible values of b for which the equations $f(x) = 0$ and $g(x) = 0$ have a common root?

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14. A natural number k is such that

$$k^2 < 2014 < (k + 1)^2$$

What is the largest prime factor of k ?

15. If real numbers a, b, c, d, e satisfy

$$a + 1 = b + 2 = c + 3 = d + 4 \\ = e + 5 = a + b + c + d + e + 3,$$

What is the value of

$$a^2 + b^2 + c^2 + d^2 + e^2$$

16. What is the smallest possible natural number n for which the equation

$$x^2 - nx + 2014 = 0$$

has integer roots?

17. If $x^{x^4} = 4$, what is the value of $x^{x^2} + x^{x^8}$?

18. Natural numbers k, l, p and q are such that if a and b are roots of

$$x^2 - kx + 1 = 0$$

then $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of

$$x^2 - px + q = 0$$

What is the sum of all possible values of q ?

19. For natural numbers x and y , let (x, y) denote the greatest common divisor of x and y . How many pairs of natural numbers x and y with $x \leq y$ satisfy the equation $xy = x + y + (x, y)$?
20. For a natural number b , let $N(b)$ denote the number of natural numbers a for which the equation

$$x^2 + ax + b = 0$$

has integer roots. What is the smallest value of b for which $N(b) = 20$?

21. Positive integers a and b are such that $a + b = a/b + b/a$. What is the value of $a^2 + b^2$?
22. The equations

$$x^2 - 4x + k = 0$$

$$x^2 + kx - 4 = 0$$

where k is a real number, have exactly one common root. What is the value of k ?

23. Let $P(x)$ be a non-zero polynomial with integer coefficients. If $P(n)$ is divisible by n for each positive integer n , what is the value of $P(0)$?

24. Let a, b and c be real numbers such that

$$a - 7b + 8c = 4$$

$$8a + 4b - c = 7$$

What is the value of $a^2 - b^2 + c^2$?

25. If

$$3^x + 2y = 985$$

$$3^x - 2^y = 473$$

what is the value of xy ?

26. Let a, b and c be such that $a + b + c = 0$ and

$$P = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}$$

is defined. What is the value of P ?

27. Suppose a, b are positive real numbers such that

$$a\sqrt{a} + b\sqrt{b} = 183$$

$$a\sqrt{b} + b\sqrt{a} = 182$$

Find $\frac{9}{5}(a + b)$.

28. Let a, b be integers such that all the roots of the equation

$$(x^2 + ax + 20)(x^2 + 17x + b) = 0$$

are negative integers. What is the smallest possible value of $a + b$?

29. Let u, v, w be real numbers in geometric progression such that $u > v > w$. Suppose $u^4 + v^4 + w^4 = 0$. Find the value of n .

30. Let the sum

$$\sum_{n=1}^9 \frac{1}{n(n+1)(n+2)}$$

written in its lowest terms be $\frac{p}{q}$. Find the value of $q - p$.

31. Find the number of positive integers n , such that

$$\sqrt{n} + \sqrt{n+1} < 11$$

32. Suppose x is a positive real number such that $\{x\}$, $[x]$ and x are in a geometric progression. Find the least positive integer n such that $x^n > 100$. (Here $[x]$ denotes the integer part of x and $\{x\} = x - [x]$.)

33. Integers 1, 2, 3,.....n, where $n > 2$, are written on a board. Two numbers m, k such that $1 < m < n, 1 < k < n$ are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers?

34. If the real numbers x, y, z are such that

$$x^2 + 4y^2 + 16z^2 = 48 \quad (34.1)$$

$$xy + 4yz + 2zx = 24 \quad (34.2)$$

What is the value of $x^2 + y^2 + z^2$?

35. Suppose 1, 2, 3 are the roots of the equation

$$x^2 + ax^2 + bx = c \quad (35.1)$$

Find the value of c .

36. What is the number of triples (a, b, c) of positive integers such that

a) $a < b < c < 10$

b) $a, b, c, 10$ forms the sides of a quadrilateral?

37. Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$.

38. Suppose an integer x , a natural number n and a prime number p satisfy the equation

$$7x^2 - 44x + 12 = p^n \quad (38.1)$$

Find the largest value of p .

39. Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of $3pq$ for all positive integers n . Find the least possible value of $p + q$?

40. The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$ (that is 1, 6, 5, 8, 9, 0 are digits in base b in the above equation). Find the sum of all possible values of $b \geq 10$ satisfying the equation.

41. Integers a, b, c satisfy

$$a + b - c = 1 \quad (41.1)$$

$$a^2 + b^2 - c^2 = -1 \quad (41.2)$$

What is the sum of all possible values of $a^2 + b^2 + c^2$?

42. Suppose a, b are integers and $a + b$ is a root of

$$x^2 + ax + b = 0 \quad (42.1)$$

What is the maximum possible value of b^2 ?

43. If

$$x = \cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 89^\circ$$

$$y = \cos 2^\circ \cos 6^\circ \cos 10^\circ \dots \cos 86^\circ$$

then what is the integer nearest $\frac{2}{7} \log_2(y/x)$?

44. Let a and b be natural numbers such that $2z - b, a - 2b$ and $a + b$ are all distinct squares. What is the smallest possible value of b ?

45. What is the value of

$$\sum_{1 \leq i < j \leq 10(\text{odd})} (i + j) - \sum_{1 \leq i < j \leq 10(\text{even})} (i + j)$$

46. If $a, b, c \geq 4$ are integers, not all equal and

$$4abc = (a + 3)(b + 3)(c + 3)$$

then what is the value of $a + b + c$?

47. Determine the sum of all possible positive integers n , the product of whose digits equals $n^2 - 15n - 27$?

48. What is the largest positive integer n such that

$$\frac{a^2}{\frac{b}{29} + \frac{c}{31}} + \frac{b^2}{\frac{c}{29} + \frac{a}{31}} + \frac{c^2}{\frac{a}{29} + \frac{b}{31}} \geq n(a + b + c)$$

49. Let

$$P(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

be a polynomial in which a_i is a non-negative integer for each $i \in \{0, 1, 2, 3, \dots\}$. If $P(1) = 4$ and $P(5) = 136$, what is the value of $P(3)$?

50. Let

$$f(x) = x^2 + ax + b$$

If for all non-zero real x

$$f\left(x + \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

and the roots of $f(x) = 0$ are integers, What is the value of $a^2 + b^2$?

51. Let \overline{abc} be a three digit numbers with non-zero digits such that $a^2 + b^2 \equiv c^2$. What is the largest possible prime factor \overline{abc} ?

52. How many positive integers n are there such that $3 \leq n \leq 100$ and $x^{2^n} + x + 1$ is divisible by $x^2 + x + 1$?

53. A natural number $k > 1$ is called good if there exist natural numbers

$$a_1 < a_2 < \dots < a_k$$

such that

$$\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \dots + \frac{1}{\sqrt{a_k}} = 1$$

Let $f(n)$ be the sum of the first n good numbers, $n \geq 1$. Find the sum of all values of n for which $f(n+5)/f(n)$ is an integer.

54. Find the number of ordered triples (a, b, c) of positive integers such that

$$30a + 50b + 70c \leq 343$$

55. Positive integers x, y, z satisfy $xy + z = 160$. Compute the smallest possible value of $x + yz$?