Access answers to Maths RD Sharma Solutions For Class 12 Chapter 3 – Binary Operations

Exercise 3.1 Page No: 3.4

1. Determine whether the following operation define a binary operation on the given set or not:

(i) '*' on N defined by a * b = a^b for all a, b ∈ N.

(ii) 'O' on Z defined by a O b = a^b for all a, b ∈ Z.

(iii) '*' on N defined by a * b = a + b – 2 for all a, b ∈ N

(iv) '$\times_6$' on S = {1, 2, 3, 4, 5} defined by a $\times_6$ b = Remainder when a b is divided by 6.

(v) '$+_6$' on S = {0, 1, 2, 3, 4, 5} defined by a $+_6$ b

$$= \begin{cases} a+b, \ if \ a+b<6 \\ a+b-6, \ if \ a+b \geq 6 \end{cases}$$

(vi) '⊙' on N defined by a ⊙ b= $a^b + b^a$ for all a, b ∈ N

(vii) '*' on Q defined by a * b = (a – 1)/ (b + 1) for all a, b ∈ Q

Solution:

(i) Given '*' on N defined by a * b = a^b for all a, b ∈ N.

Let a, b ∈ N. Then,

a^b ∈ N [∵ $a^b \neq 0$ and a, b is positive integer]

⇒ a * b ∈ N

Therefore,

a * b ∈ N, ∀ a, b ∈ N

Thus, * is a binary operation on N.

(ii) Given 'O' on Z defined by a O b = a^b for all a, b ∈ Z.

Both a = 3 and b = -1 belong to Z.

⇒ a * b = 3^{-1}

= 1/3 ∉ Z

Thus, * is not a binary operation on Z.

(iii) Given '*' on N defined by a * b = a + b – 2 for all a, b ∈ N

If a = 1 and b = 1,

a * b = a + b – 2

= 1 + 1 – 2

= 0 ∉ N

Thus, there exist a = 1 and b = 1 such that a * b ∉ N

So, * is not a binary operation on N.

(iv) Given ‘$\times_6$‘ on S = {1, 2, 3, 4, 5} defined by a $\times_6$ b = Remainder when a b is divided by 6.

Consider the composition table,

X_6	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Here all the elements of the table are not in S.

⇒ For a = 2 and b = 3,

a $\times_6$ b = 2 $\times_6$ 3 = remainder when 6 divided by 6 = 0 ≠ S

Thus, $\times_6$ is not a binary operation on S.

(v) Given ‘$+_6$’ on S = {0, 1, 2, 3, 4, 5} defined by a $+_6$ b

$$= \begin{cases} a+b, \ if \ a+b<6 \\ a+b-6, \ if \ a+b \geq 6 \end{cases}$$

Consider the composition table,

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0

2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Here all the elements of the table are not in S.

⇒ For a = 2 and b = 3,

$a \times_6 b = 2 \times_6 3$ = remainder when 6 divided by 6 = 0 ≠
Thus, $\times_6$ is not a binary operation on S.

(vi) Given '⊙' on N defined by a ⊙ b= $a^b + b^a$ for all a, b ∈ N

Let a, b ∈ N. Then,

$a^b, b^a \in N$

⇒ $a^b + b^a \in N$ [∵Addition is binary operation on N]

⇒ a ⊙ b ∈ N

Thus, ⊙ is a binary operation on N.

(vii) Given '*' on Q defined by a * b = (a – 1)/ (b + 1) for all a, b ∈ Q

If a = 2 and b = -1 in Q,

a * b = (a – 1)/ (b + 1)

= (2 – 1)/ (- 1 + 1)

= 1/0 [which is not defined]

For a = 2 and b = -1

a * b does not belongs to Q

So, * is not a binary operation in Q.

2. Determine whether or not the definition of * given below gives a binary operation. In the event that * is not a binary operation give justification of this.

(i) On Z^+, defined * by a * b = a – b

(ii) On Z^+, define * by a*b = ab

(iii) On R, define * by a*b = ab^2

(iv) On Z^+ define * by a * b = |a − b|

(v) On Z^+ define * by a * b = a

(vi) On R, define * by a * b = a + $4b^2$

Here, Z^+ denotes the set of all non-negative integers.

Solution:

(i) Given On Z^+, defined * by a * b = a – b

If a = 1 and b = 2 in Z^+, then

a * b = a – b

= 1 – 2

= -1 $\notin Z^+$ [because Z^+ is the set of non-negative integers]

For a = 1 and b = 2,

a * b $\notin Z^+$

Thus, * is not a binary operation on Z^+.

(ii) Given Z^+, define * by a*b = a b

Let a, b $\in Z^+$

$\Rightarrow$ a, b $\in Z^+$

$\Rightarrow$ a * b $\in Z^+$

Thus, * is a binary operation on R.

(iii) Given on R, define by a*b = ab^2

Let a, b $\in$ R

$\Rightarrow$ a, $b^2 \in$ R

$\Rightarrow$ $ab^2 \in$ R

$\Rightarrow$ a * b $\in$ R

Thus, * is a binary operation on R.

(iv) Given on Z^+ define * by a * b = |a − b|

Let a, b $\in Z^+$

$\Rightarrow$ | a – b | $\in Z^+$

$\Rightarrow$ a * b $\in Z^+$

Therefore,

a * b $\in Z^+$, $\forall$ a, b $\in Z^+$

Thus, * is a binary operation on Z^+.

(v) Given on Z^+ define * by a * b = a

Let a, b $\in Z^+$

$\Rightarrow$ a $\in Z^+$

⇒ a * b ∈ Z^+

Therefore, a * b ∈ Z^+ ∀ a, b ∈ Z^+

Thus, * is a binary operation on Z^+.

(vi) Given On R, define * by a * b = a + $4b^2$

Let a, b ∈ R

⇒ a, $4b^2$ ∈ R

⇒ a + $4b^2$ ∈ R

⇒ a * b ∈ R

Therefore, a *b ∈ R, ∀ a, b ∈ R

Thus, * is a binary operation on R.

3. Let * be a binary operation on the set I of integers, defined by a * b = 2a + b − 3. Find the value of 3 * 4.

Solution:

Given *a* * *b* = 2*a* + *b* – 3

3 * 4 = 2 (3) + 4 – 3

= 6 + 4 – 3

= 7

4. Is * defined on the set {1, 2, 3, 4, 5} by a * b = LCM of a and b a binary operation? Justify your answer.

Solution:

LCM	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	5	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

In the given composition table, all the elements are not in the set {1, 2, 3, 4, 5}.

If we consider a = 2 and b = 3, a * b = LCM of a and b = 6 ∉ {1, 2, 3, 4, 5}.

Thus, * is not a binary operation on {1, 2, 3, 4, 5}.

5. Let S = {a, b, c}. Find the total number of binary operations on *S*.

Solution:

Number of binary operations on a set with n elements is nn2
Here, S = {a, b, c}

Number of elements in S = 3

Number of binary operations on a set with 3 elements is 332

Exercise 3.2 Page No: 3.12

1. Let '*' be a binary operation on N defined by a * b = l.c.m. (a, b) for all a, b ∈ N
(i) Find 2 * 4, 3 * 5, 1 * 6.

(ii) Check the commutativity and associativity of '*' on N.

Solution:

(i) Given a * b = 1.c.m. (a, b)

2 * 4 = l.c.m. (2, 4)

= 4

3 * 5 = l.c.m. (3, 5)

= 15

1 * 6 = l.c.m. (1, 6)

= 6

(ii) We have to prove commutativity of *

Let a, b ∈ N

a * b = l.c.m (a, b)

= l.c.m (b, a)

= b * a

Therefore

a * b = b * a ∀ a, b ∈ N

Thus * is commutative on N.

Now we have to prove associativity of *

Let a, b, c ∈ N

a * (b * c) = a * l.c.m. (b, c)

= l.c.m. (a, (b, c))

= l.c.m (a, b, c)

(a * b) * c = l.c.m. (a, b) * c

= l.c.m. ((a, b), c)

= l.c.m. (a, b, c)

Therefore

(a * (b * c) = (a * b) * c, ∀ a, b , c ∈ N

Thus, * is associative on N.

2. Determine which of the following binary operation is associative and which is commutative:

(i) * on N defined by a * b = 1 for all a, b ∈ N

(ii) * on Q defined by a * b = (a + b)/2 for all a, b ∈ Q

Solution:

(i) We have to prove commutativity of *

Let a, b ∈ N

a * b = 1

b * a = 1

Therefore,

a * b = b * a, for all a, b ∈ N

Thus * is commutative on N.

Now we have to prove associativity of *

Let a, b, c ∈ N

Then a * (b * c) = a * (1)

= 1

(a * b) *c = (1) * c

= 1

Therefore a * (b * c) = (a * b) *c for all a, b, c ∈ N

Thus, * is associative on N.

(ii) First we have to prove commutativity of *

Let a, b ∈ N

a * b = (a + b)/2

= (b + a)/2

= b * a

Therefore, a * b = b * a, ∀ a, b ∈ N

Thus * is commutative on N.

Now we have to prove associativity of *

Let a, b, c ∈ N

a * (b * c) = a * (b + c)/2

= [a + (b + c)]/2

= (2a + b + c)/4

Now, (a * b) * c = (a + b)/2 * c

= [(a + b)/2 + c] /2

= (a + b + 2c)/4

Thus, a * (b * c) ≠ (a * b) * c

If a = 1, b= 2, c = 3

1 * (2 * 3) = 1 * (2 + 3)/2

= 1 * (5/2)

= [1 + (5/2)]/2

= 7/4

(1 * 2) * 3 = (1 + 2)/2 * 3

= 3/2 * 3

= [(3/2) + 3]/2

= 4/9

Therefore, there exist a = 1, b = 2, c = 3 ∈ N such that a * (b * c) ≠ (a * b) * c

Thus, * is not associative on N.

3. Let A be any set containing more than one element. Let '*' be a binary operation on A defined by a * b = b for all a, b ∈ A Is '*' commutative or associative on A?

Solution:

Let a, b ∈ A

Then, a * b = b

b * a = a

Therefore a * b ≠ b * a

Thus, * is not commutative on A

Now we have to check associativity:

Let a, b, c ∈ A

a * (b * c) = a * c

= c

Therefore

a * (b * c) = (a * b) * c, ∀ a, b, c ∈ A

Thus, * is associative on A

4. Check the commutativity and associativity of each of the following binary operations:

(i) '*' on Z defined by a * b = a + b + a b for all a, b ∈ Z

(ii) '*' on N defined by a * b = 2^{ab} for all a, b ∈ N

(iii) '*' on Q defined by a * b = a – b for all a, b ∈ Q

(iv) '⊙' on Q defined by a ⊙ b = $a^2 + b^2$ for all a, b ∈ Q

(v) 'o' on Q defined by a o b = (ab/2) for all a, b ∈ Q

(vi) '*' on Q defined by a * b = ab^2 for all a, b ∈ Q

(vii) '*' on Q defined by a * b = a + a b for all a, b ∈ Q

(viii) '*' on R defined by a * b = a + b -7 for all a, b ∈ R

(ix) '*' on Q defined by a * b = $(a – b)^2$ for all a, b ∈ Q

(x) '*' on Q defined by a * b = a b + 1 for all a, b ∈ Q

(xi) '*' on N defined by a * b = a^b for all a, b ∈ N

(xii) '*' on Z defined by a * b = a – b for all a, b ∈ Z

(xiii) '*' on Q defined by a * b = (ab/4) for all a, b ∈ Q

(xiv) '*' on Z defined by a * b = a + b – ab for all a, b ∈ Z

(xv) '*' on Q defined by a * b = gcd (a, b) for all a, b ∈ Q

Solution:

(i) First we have to check commutativity of *

Let a, b ∈ Z

Then a * b = a + b + ab

= b + a + ba

= b * a

Therefore,

a * b = b * a, ∀ a, b ∈ Z

Now we have to prove associativity of *

Let a, b, c ∈ Z, Then,

a * (b * c) = a * (b + c + b c)

= a + (b + c + b c) + a (b + c + b c)

= a + b + c + b c + a b + a c + a b c

(a * b) * c = (a + b + a b) * c

= a + b + a b + c + (a + b + a b) c

= a + b + a b + c + a c + b c + a b c

Therefore,

a * (b * c) = (a * b) * c, ∀ a, b, c ∈ Z

Thus, * is associative on Z.

(ii) First we have to check commutativity of *

Let a, b ∈ N

a * b = 2^{ab}

= 2^{ba}

= b * a

Therefore, a * b = b * a, ∀ a, b ∈ N

Thus, * is commutative on N

Now we have to check associativity of *

Let a, b, c ∈ N

Then, a * (b * c) = a * (2^{bc})

$=2a*2bc$

(a * b) * c = (2^{ab}) * c

$=2ab*2c$

Therefore, a * (b * c) ≠ (a * b) * c

Thus, * is not associative on N

(iii) First we have to check commutativity of *

Let a, b ∈ Q, then

$a * b = a - b$

$b * a = b - a$

Therefore, $a * b \neq b * a$

Thus, * is not commutative on Q

Now we have to check associativity of *

Let a, b, c ∈ Q, then

$a * (b * c) = a * (b - c)$

$= a - (b - c)$

$= a - b + c$

$(a * b) * c = (a - b) * c$

$= a - b - c$

Therefore,

$a * (b * c) \neq (a * b) * c$

Thus, * is not associative on Q

(iv) First we have to check commutativity of ⊙

Let a, b ∈ Q, then

$a \odot b = a^2 + b^2$

$= b^2 + a^2$

$= b \odot a$

Therefore, $a \odot b = b \odot a$, $\forall$ a, b ∈ Q

Thus, ⊙ on Q

Now we have to check associativity of ⊙

Let a, b, c ∈ Q, then

$a \odot (b \odot c) = a \odot (b^2 + c^2)$

$= a^2 + (b^2 + c^2)^2$

$= a^2 + b^4 + c^4 + 2b^2c^2$

$(a \odot b) \odot c = (a^2 + b^2) \odot c$

$= (a^2 + b^2)^2 + c^2$

$= a^4 + b^4 + 2a^2b^2 + c^2$

Therefore,

(a ⊙ b) ⊙ c ≠ a ⊙ (b ⊙ c)

Thus, ⊙ is not associative on Q.

(v) First we have to check commutativity of o

Let a, b ∈ Q, then

a o b = (ab/2)

= (b a/2)

= b o a

Therefore, a o b = b o a, ∀ a, b ∈ Q

Thus, o is commutative on Q

Now we have to check associativity of o

Let a, b, c ∈ Q, then

a o (b o c) = a o (b c/2)

= [a (b c/2)]/2

= [a (b c/2)]/2

= (a b c)/4

(a o b) o c = (ab/2) o c

= [(ab/2) c] /2

= (a b c)/4

Therefore a o (b o c) = (a o b) o c, ∀ a, b, c ∈ Q

Thus, o is associative on Q.

(vi) First we have to check commutativity of *

Let a, b ∈ Q, then

$a * b = ab^2$

$b * a = ba^2$

Therefore,

a * b ≠ b * a

Thus, * is not commutative on Q

Now we have to check associativity of *

Let a, b, c ∈ Q, then

$a * (b * c) = a * (bc^2)$

$= a (bc^2)^2$

$= ab^2 c^4$

$(a * b) * c = (ab^2) * c$

$= ab^2c^2$

Therefore $a * (b * c) \neq (a * b) * c$

Thus, * is not associative on Q.

(vii) First we have to check commutativity of *

Let a, b ∈ Q, then

$a * b = a + ab$

$b * a = b + ba$

$= b + ab$

Therefore, $a * b \neq b * a$

Thus, * is not commutative on Q.

Now we have to prove associativity on Q.

Let a, b, c ∈ Q, then

$a * (b * c) = a * (b + b c)$

$= a + a (b + b c)$

$= a + ab + a b c$

$(a * b) * c = (a + a b) * c$

$= (a + a b) + (a + a b) c$

$= a + a b + a c + a b c$

Therefore $a * (b * c) \neq (a * b) * c$

Thus, * is not associative on Q.

(viii) First we have to check commutativity of *

Let a, b ∈ R, then

$a * b = a + b – 7$

$= b + a – 7$

$= b * a$

Therefore,

$a * b = b * a$, for all a, b ∈ R

Thus, * is commutative on R

Now we have to prove associativity of * on R.

Let a, b, c ∈ R, then

$a * (b * c) = a * (b + c – 7)$

$= a + b + c -7 -7$

$= a + b + c – 14$

$(a * b) * c = (a + b – 7) * c$

$= a + b – 7 + c – 7$

$= a + b + c – 14$

Therefore,

$a * (b * c) = (a * b) * c$, for all a, b, c ∈ R

Thus, * is associative on R.

(ix) First we have to check commutativity of *

Let a, b ∈ Q, then

$a * b = (a – b)^2$

$= (b – a)^2$

$= b * a$

Therefore,

$a * b = b * a$, for all a, b ∈ Q

Thus, * is commutative on Q

Now we have to prove associativity of * on Q

Let a, b, c ∈ Q, then

$a * (b * c) = a * (b – c)^2$

$= a * (b^2 + c^2 – 2 b c)$

$= (a – b^2 – c^2 + 2bc)^2$

$(a * b) * c = (a – b)^2 * c$

$= (a^2 + b^2 – 2ab) * c$

$= (a^2 + b^2 – 2ab – c)^2$

Therefore, $a * (b * c) \neq (a * b) * c$

Thus, * is not associative on Q.

(x) First we have to check commutativity of *

Let a, b ∈ Q, then

a * b = ab + 1

= ba + 1

= b * a

Therefore

a * b = b * a, for all a, b ∈ Q

Thus, * is commutative on Q

Now we have to prove associativity of * on Q

Let a, b, c ∈ Q, then

a * (b * c) = a * (bc + 1)

= a (b c + 1) + 1

= a b c + a + 1

(a * b) * c = (ab + 1) * c

= (ab + 1) c + 1

= a b c + c + 1

Therefore, a * (b * c) ≠ (a * b) * c

Thus, * is not associative on Q.

(xi) First we have to check commutativity of *

Let a, b ∈ N, then

a * b = a^b

b * a = b^a

Therefore, a * b ≠ b * a

Thus, * is not commutative on N.

Now we have to check associativity of *

a * (b * c) = a * (b^c)

=

a^{b^c}

(a * b) * c = (a^b) * c

= $(a^b)^c$

= a^{bc}

Therefore, a * (b * c) ≠ (a * b) * c

Thus, * is not associative on N

(xii) First we have to check commutativity of *

Let a, b ∈ Z, then

a * b = a – b

b * a = b – a

Therefore,

a * b ≠ b * a

Thus, * is not commutative on Z.

Now we have to check associativity of *

Let a, b, c ∈ Z, then

a * (b * c) = a * (b – c)

= a – (b – c)

= a – (b + c)

(a * b) * c = (a – b) – c

= a – b – c

Therefore, a * (b * c) ≠ (a * b) * c

Thus, * is not associative on Z

(xiii) First we have to check commutativity of *

Let a, b ∈ Q, then

a * b = (ab/4)

= (ba/4)

= b * a

Therefore, a * b = b * a, for all a, b ∈ Q

Thus, * is commutative on Q

Now we have to check associativity of *

Let a, b, c ∈ Q, then

a * (b * c) = a * (b c/4)

= [a (b c/4)]/4

= (a b c/16)

(a * b) * c = (ab/4) * c

= [(ab/4) c]/4

= a b c/16

Therefore,

a * (b * c) = (a * b) * c for all a, b, c ∈ Q

Thus, * is associative on Q.

(xiv) First we have to check commutativity of *

Let a, b ∈ Z, then

a * b = a + b – ab

= b + a – ba

= b * a

Therefore, a * b = b * a, for all a, b ∈ Z

Thus, * is commutative on Z.

Now we have to check associativity of *

Let a, b, c ∈ Z

a * (b * c) = a * (b + c – b c)

= a + b + c- b c – ab – ac + a b c

(a * b) * c = (a + b – a b) c

= a + b – ab + c – (a + b – ab) c

= a + b + c – ab – ac – bc + a b c

Therefore,

a * (b * c) = (a * b) * c, for all a, b, c ∈ Z

Thus, * is associative on Z.

(xv) First we have to check commutativity of *

Let a, b ∈ N, then

a * b = gcd (a, b)

= gcd (b, a)

= b * a

Therefore, a * b = b * a, for all a, b ∈ N

Thus, * is commutative on N.

Now we have to check associativity of *

Let a, b, c ∈ N

a * (b * c) = a * [gcd (a, b)]

= gcd (a, b, c)

(a * b) * c = [gcd (a, b)] * c

= gcd (a, b, c)

Therefore,

a * (b * c) = (a * b) * c, for all a, b, c ∈ N

Thus, * is associative on N.

5. If the binary operation o is defined by a0b = a + b – ab on the set Q – {-1} of all rational numbers other than 1, show that o is commutative on Q – [1].

Solution:

Let a, b ∈ Q – {-1}.

Then aob = a + b – ab

= b+ a – ba

= boa

Therefore,

aob = boa for all a, b ∈ Q – {-1}

Thus, o is commutative on Q – {-1}

6. Show that the binary operation * on Z defined by a * b = 3a + 7b is not commutative?

Solution:

Let a, b ∈ Z

a * b = 3a + 7b

b * a = 3b + 7a

Thus, a * b ≠ b * a

Let a = 1 and b = 2

1 * 2 = 3 × 1 + 7 × 2

= 3 + 14

= 17

2 * 1 = 3 × 2 + 7 × 1

= 6 + 7

= 13

Therefore, there exist a = 1, b = 2 ∈ Z such that a * b ≠ b * a

Thus, * is not commutative on Z.

7. On the set Z of integers a binary operation * is defined by a 8 b = ab + 1 for all a, b ∈ Z. Prove that * is not associative on Z.

Solution:

Let a, b, c ∈ Z

a * (b * c) = a * (bc + 1)

= a (bc + 1) + 1

= a b c + a + 1

(a * b) * c = (ab+ 1) * c

= (ab + 1) c + 1

= a b c + c + 1

Thus, a * (b * c) ≠ (a * b) * c

Thus, * is not associative on Z.

Exercise 3.3 Page No: 3.15

1. Find the identity element in the set I^+ of all positive integers defined by a * b = a + b for all a, b ∈ I^+.

Solution:

Let e be the identity element in I^+ with respect to * such that

a * e = a = e * a, ∀ a ∈ I^+

a * e = a and e * a = a, ∀ a ∈ I^+

a + e = a and e + a = a, ∀ a ∈ I^+

e = 0, ∀ a ∈ I^+

Thus, 0 is the identity element in I^+ with respect to *.

2. Find the identity element in the set of all rational numbers except – 1 with respect to * defined by a * b = a + b + ab

Solution:

Let e be the identity element in I^+ with respect to * such that

a * e = a = e * a, ∀ a ∈ Q – {-1}

a * e = a and e * a = a, ∀ a ∈ Q – {-1}

a + e + ae = a and e + a + ea = a, ∀ a ∈ Q – {-1}

$e + ae = 0$ and $e + ea = 0$, $\forall a \in Q - \{-1\}$

$e (1 + a) = 0$ and $e (1 + a) = 0$, $\forall a \in Q - \{-1\}$

$e = 0$, $\forall a \in Q - \{-1\}$ [because a not equal to -1]

Thus, 0 is the identity element in $Q - \{-1\}$ with respect to *.

Exercise 3.4 Page No: 3.25

1. Let * be a binary operation on Z defined by a * b = a + b – 4 for all a, b ∈ Z.

(i) Show that * is both commutative and associative.

(ii) Find the identity element in Z

(iii) Find the invertible element in Z.

Solution:

(i) First we have to prove commutativity of *

Let $a, b \in Z$. then,

$a * b = a + b - 4$

$= b + a - 4$

$= b * a$

Therefore,

$a * b = b * a$, $\forall a, b \in Z$

Thus, * is commutative on Z.

Now we have to prove associativity of Z.

Let $a, b, c \in Z$. then,

$a * (b * c) = a * (b + c - 4)$

$= a + b + c -4 - 4$

$= a + b + c - 8$

$(a * b) * c = (a + b - 4) * c$

$= a + b - 4 + c - 4$

$= a + b + c - 8$

Therefore,

$a * (b * c) = (a * b) * c$, for all $a, b, c \in Z$

Thus, * is associative on Z.

(ii) Let e be the identity element in Z with respect to * such that

a * e = a = e * a ∀ a ∈ Z

a * e = a and e * a = a, ∀ a ∈ Z

a + e – 4 = a and e + a – 4 = a, ∀ a ∈ Z

e = 4, ∀ a ∈ Z

Thus, 4 is the identity element in Z with respect to *.

(iii) Let a ∈ Z and b ∈ Z be the inverse of a. Then,

a * b = e = b * a

a * b = e and b * a = e

a + b – 4 = 4 and b + a – 4 = 4

b = 8 – a ∈ Z

Thus, 8 – a is the inverse of a ∈ Z

2. Let * be a binary operation on Q_0 (set of non-zero rational numbers) defined by a * b = (3ab/5) for all a, b ∈ Q_0. Show that * is commutative as well as associative. Also, find its identity element, if it exists.

Solution:

First we have to prove commutativity of *

Let a, b ∈ Q_0

a * b = (3ab/5)

= (3ba/5)

= b * a

Therefore, a * b = b * a, for all a, b ∈ Q_0

Now we have to prove associativity of *

Let a, b, c ∈ Q_0

a * (b * c) = a * (3bc/5)

= [a (3 bc/5)] /5

= 3 abc/25

(a * b) * c = (3 ab/5) * c

= [(3 ab/5) c]/ 5

= 3 abc /25

Therefore a * (b * c) = (a * b) * c, for all a, b, c ∈ Q_0

Thus * is associative on Q_0

Now we have to find the identity element

Let e be the identity element in Z with respect to * such that

a * e = a = e * a ∀ a ∈ Q_0

a * e = a and e * a = a, ∀ a ∈ Q_0

3ae/5 = a and 3ea/5 = a, ∀ a ∈ Q_0

e = 5/3 ∀ a ∈ Q_0 [because a is not equal to 0]

Thus, 5/3 is the identity element in Q_0 with respect to *.

3. Let * be a binary operation on Q – {-1} defined by a * b = a + b + ab for all a, b ∈ Q – {-1}. Then,

(i) Show that * is both commutative and associative on Q – {-1}

(ii) Find the identity element in Q – {-1}

(iii) Show that every element of Q – {-1} is invertible. Also, find inverse of an arbitrary element.

Solution:

(i) First we have to check commutativity of *

Let a, b ∈ Q – {-1}

Then a * b = a + b + ab

= b + a + ba

= b * a

Therefore,

a * b = b * a, ∀ a, b ∈ Q – {-1}

Now we have to prove associativity of *

Let a, b, c ∈ Q – {-1}, Then,

a * (b * c) = a * (b + c + b c)

= a + (b + c + b c) + a (b + c + b c)

= a + b + c + b c + a b + a c + a b c

(a * b) * c = (a + b + a b) * c

= a + b + a b + c + (a + b + a b) c

= a + b + a b + c + a c + b c + a b c

Therefore,

$a * (b * c) = (a * b) * c, \forall a, b, c \in Q – \{-1\}$

Thus, * is associative on Q – {-1}.

(ii) Let e be the identity element in I^+ with respect to * such that

$a * e = a = e * a, \forall a \in Q – \{-1\}$

$a * e = a$ and $e * a = a, \forall a \in Q – \{-1\}$

$a + e + ae = a$ and $e + a + ea = a, \forall a \in Q – \{-1\}$

$e + ae = 0$ and $e + ea = 0, \forall a \in Q – \{-1\}$

$e (1 + a) = 0$ and $e (1 + a) = 0, \forall a \in Q – \{-1\}$

$e = 0, \forall a \in Q – \{-1\}$ [because a not equal to -1]

Thus, 0 is the identity element in Q – {-1} with respect to *.

(iii) Let $a \in Q – \{-1\}$ and $b \in Q – \{-1\}$ be the inverse of a. Then,

$a * b = e = b * a$

$a * b = e$ and $b * a = e$

$a + b + ab = 0$ and $b + a + ba = 0$

$b (1 + a) = – a Q – \{-1\}$

$b = -a/1 + a Q – \{-1\}$ [because a not equal to -1]

Thus, $-a/1 + a$ is the inverse of $a \in Q – \{-1\}$

4. Let A = $R_0 \times R$, where R_0 denote the set of all non-zero real numbers. A binary operation 'O' is defined on A as follows: (a, b) O (c, d) = (ac, bc + d) for all (a, b), (c, d) $\in R_0 \times R$.

(i) Show that 'O' is commutative and associative on A

(ii) Find the identity element in A

(iii) Find the invertible element in A.

Solution:

(i) Let X = (a, b) and Y = (c, d) $\in$ A, $\forall a, c \in R_0$ and $b, d \in R$

Then, X O Y = (ac, bc + d)

And Y O X = (ca, da + b)

Therefore,

X O Y = Y O X, $\forall$ X, Y $\in$ A

Thus, O commutative on A.

Now we have to check associativity of O

Let X = (a, b), Y = (c, d) and Z = (e, f), ∀ a, c, e ∈ R_0 and b, d, f ∈ R

X O (Y O Z) = (a, b) O (ce, de + f)

= (ace, bce + de + f)

(X O Y) O Z = (ac, bc + d) O (e, f)

= (ace, (bc + d) e + f)

= (ace, bce + de + f)

Therefore, X O (Y O Z) = (X O Y) O Z, ∀ X, Y, Z ∈ A

(ii) Let E = (x, y) be the identity element in A with respect to O, ∀ x ∈ R_0 and y ∈ R

Such that,

X O E = X = E O X, ∀ X ∈ A

X O E = X and EOX = X

(ax, bx +y) = (a, b) and (xa, ya + b) = (a, b)

Considering (ax, bx + y) = (a, b)

ax = a

x = 1

And bx + y = b

y = 0 [since x = 1]

Considering (xa, ya + b) = (a, b)

xa = a

x = 1

And ya + b = b

y = 0 [since x = 1]

Therefore (1, 0) is the identity element in A with respect to O.

(iii) Let F = (m, n) be the inverse in A ∀ m ∈ R_0 and n ∈ R

X O F = E and F O X = E

(am, bm + n) = (1, 0) and (ma, na + b) = (1, 0)

Considering (am, bm + n) = (1, 0)

am = 1

m = 1/a

And bm + n = 0

$n = -b/a$ [since $m = 1/a$]

Considering $(ma, na + b) = (1, 0)$

$ma = 1$

$m = 1/a$

And $na + b = 0$

$n = -b/a$

Therefore the inverse of $(a, b) \in A$ with respect to O is $(1/a, -1/a)$

Exercise 3.5 Page No: 3.33

1. Construct the composition table for $\times_4$ on set S = {0, 1, 2, 3}.

Solution:

Given that $\times_4$ on set $S = \{0, 1, 2, 3\}$

Here,

$1 \times_4 1$ = remainder obtained by dividing 1×1 by 4

= 1

$0 \times_4 1$ = remainder obtained by dividing 0×1 by 4

= 0

$2 \times_4 3$ = remainder obtained by dividing 2×3 by 4

= 2

$3 \times_4 3$ = remainder obtained by dividing 3×3 by 4

= 1

So, the composition table is as follows:

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	2
3	0	3	2	1

2. Construct the composition table for $+_5$ on set S = {0, 1, 2, 3, 4}

Solution:

$1 +_5 1$ = remainder obtained by dividing 1 + 1 by 5

= 2

$3 +_5 1$ = remainder obtained by dividing 3 + 1 by 5

= 2

$4 +_5 1$ = remainder obtained by dividing 4 + 1 by 5

= 3

So, the composition table is as follows:

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

3. Construct the composition table for $\times_6$ on set S = {0, 1, 2, 3, 4, 5}.

Solution:

Here,

$1 \times_6 1$ = remainder obtained by dividing 1 × 1 by 6

= 1

$3 \times_6 4$ = remainder obtained by dividing 3 × 4 by 6

= 0

$4 \times_6 5$ = remainder obtained by dividing 4 × 5 by 6

= 2

So, the composition table is as follows:

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4

3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

4. Construct the composition table for $\times_5$ on set Z_5 = {0, 1, 2, 3, 4}

Solution:

Here,

$1 \times_5 1$ = remainder obtained by dividing 1 × 1 by 5

= 1

$3 \times_5 4$ = remainder obtained by dividing 3 × 4 by 5

= 2

$4 \times_5 4$ = remainder obtained by dividing 4 × 4 by 5

= 1

So, the composition table is as follows:

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

5. For the binary operation $\times_{10}$ set S = {1, 3, 7, 9}, find the inverse of 3.

Solution:

Here,

$1 \times_{10} 1$ = remainder obtained by dividing 1 × 1 by 10

= 1

$3 \times_{10} 7$ = remainder obtained by dividing 3 × 7 by 10

= 1

$7 \times_{10} 9$ = remainder obtained by dividing 7 × 9 by 10

= 3

So, the composition table is as follows:

$\times_{10}$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

From the table we can observe that elements of first row as same as the top-most row.

So, $1 \in S$ is the identity element with respect to $\times_{10}$

Now we have to find inverse of 3

$3 \times_{10} 7 = 1$

So the inverse of 3 is 7.