

Access answers to Maths RD Sharma Solutions For Class 12 Chapter 15 – Mean Value Theorems

Exercise 15.1 Page No: 15.9

1. Discuss the applicability of Rolle's Theorem for the following functions on the indicated intervals:

(i) $f(x) = 3 + (x - 2)^{\frac{2}{3}}$ on $[1, 3]$

Solution:

Given function is

$$\Rightarrow f(x) = 3 + (x - 2)^{\frac{2}{3}} \text{ on } [1, 3]$$

Let us check the differentiability of the function $f(x)$.

Now we have to find the derivative of $f(x)$,

$$\Rightarrow f'(x) = \frac{d}{dx} \left(3 + (x - 2)^{\frac{2}{3}} \right)$$

$$\Rightarrow f'(x) = \frac{d(3)}{dx} + \frac{d\left((x-2)^{\frac{2}{3}}\right)}{dx}$$

$$\Rightarrow f'(x) = 0 + \frac{2}{3}(x - 2)^{\frac{2}{3}-1}$$

$$\Rightarrow f'(x) = \frac{2}{3}(x - 2)^{-\frac{1}{3}}$$

$$\Rightarrow f'(x) = \frac{2}{3(x-2)^{\frac{1}{3}}}$$

Now we have to check differentiability at the value of $x = 2$

$$\Rightarrow \lim_{x \rightarrow 2} f'(x) = \lim_{x \rightarrow 2} \frac{2}{3(x-2)^{\frac{1}{3}}}$$

$$\Rightarrow \lim_{x \rightarrow 2} f'(x) = \frac{2}{3(2-2)^{\frac{1}{3}}}$$

$$\Rightarrow \lim_{x \rightarrow 2} f'(x) = \frac{2}{3(0)}$$

$$\Rightarrow \lim_{x \rightarrow 2} f'(x) = \text{undefined}$$

$\therefore f$ is not differentiable at $x = 2$, so it is not differentiable in the closed interval $(1, 3)$.

So, Rolle's theorem is not applicable for the function f on the interval $[1, 3]$.

(ii) $f(x) = [x]$ for $-1 < x \leq 1$, where $[x]$ denotes the greatest integer not exceeding x

Solution:

Given function is $f(x) = [x]$, $-1 \leq x \leq 1$ where $[x]$ denotes the greatest integer not exceeding x .

Let us check the continuity of the function f .

Here in the interval $x \in [-1, 1]$, the function has to be Right continuous at $x = 1$ and left continuous at $x = 1$.

$$\Rightarrow \lim_{x \rightarrow 1+} f(x) = \lim_{x \rightarrow 1+} [x]$$

$$\Rightarrow \lim_{x \rightarrow 1+} f(x) = \lim_{x \rightarrow 1+h} [x] \text{ Where } h > 0.$$

$$\Rightarrow \lim_{x \rightarrow 1+} f(x) = \lim_{h \rightarrow 0} 1$$

$$\Rightarrow \lim_{x \rightarrow 1+} f(x) = 1 \dots\dots (1)$$

$$\Rightarrow \lim_{x \rightarrow 1-} f(x) = \lim_{x \rightarrow 1-} [x]$$

$$\Rightarrow \lim_{x \rightarrow 1-} f(x) = \lim_{x \rightarrow 1-h} [x], \text{ where } h > 0$$

$$\Rightarrow \lim_{x \rightarrow 1-} f(x) = \lim_{h \rightarrow 0} 0$$

$$\Rightarrow \lim_{x \rightarrow 1-} f(x) = 0 \dots\dots (2)$$

From (1) and (2), we can see that the limits are not the same so, the function is not continuous in the interval $[-1, 1]$.

\therefore Rolle's Theorem is not applicable for the function f in the interval $[-1, 1]$.

(iii) $f(x) = \sin 1/x$ for $-1 \leq x \leq 1$

Solution:

Given function is $f(x) = \sin\left(\frac{1}{x}\right)$ for $-1 \leq x \leq 1$

Let us check the continuity of the function 'f' at the value of $x = 0$. We cannot directly find the value of limit at $x = 0$, as the function is not valid at $x = 0$. So, we take the limit on either sides of $x = 0$, and we check whether they are equal or not.

So consider RHL:

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sin\left(\frac{1}{x}\right)$$

We assume that the limit $\lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right) = k$, $k \in [-1, 1]$.

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+ h} \sin\left(\frac{1}{x}\right), \text{ where } h > 0$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \sin\left(\frac{1}{h+0}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = k \quad \dots\dots (1)$$

Now consider LHL:

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \sin\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^- h} \sin\left(\frac{1}{x}\right), \text{ where } h > 0$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} \sin\left(\frac{1}{0-h}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} \sin\left(\frac{1}{-h}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} -\sin\left(\frac{1}{h}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = -\lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = -k \dots\dots (2)$$

From (1) and (2), we can see that the Right hand and left – hand limits are not equal, so the function 'f' is not continuous at $x = 0$.

\therefore Rolle's Theorem is not applicable to the function 'f' in the interval $[-1, 1]$.

(iv) $f(x) = 2x^2 - 5x + 3$ on $[1, 3]$

Solution:

Given function is $f(x) = 2x^2 - 5x + 3$ on $[1, 3]$

Since given function f is a polynomial. So, it is continuous and differentiable everywhere.

Now, we find the values of function at the extreme values.

$$\Rightarrow f(1) = 2(1)^2 - 5(1) + 3$$

$$\Rightarrow f(1) = 2 - 5 + 3$$

$$\Rightarrow f(1) = 0 \dots\dots (1)$$

$$\Rightarrow f(3) = 2(3)^2 - 5(3) + 3$$

$$\Rightarrow f(3) = 2(9) - 15 + 3$$

$$\Rightarrow f(3) = 18 - 12$$

$$\Rightarrow f(3) = 6 \dots\dots (2)$$

From (1) and (2), we can say that, $f(1) \neq f(3)$

\therefore Rolle's Theorem is not applicable for the function f in interval $[1, 3]$.

(v) $f(x) = x^{2/3}$ on $[-1, 1]$

Solution:

Given function is $f(x) = x^{\frac{2}{3}}$ on $[-1, 1]$

Now we have to find the derivative of the given function:

$$\Rightarrow f'(x) = \frac{d\left(x^{\frac{2}{3}}\right)}{dx}$$

$$\Rightarrow f'(x) = \frac{2}{3}x^{\frac{2}{3}-1}$$

$$\Rightarrow f'(x) = \frac{2}{3}x^{\frac{2}{3}-1}$$

$$\Rightarrow f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

$$\Rightarrow f'(x) = \frac{2}{3x^{\frac{1}{3}}}$$

Now we have to check the differentiability of the function at $x = 0$.

$$\Rightarrow \lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \frac{2}{3x^{\frac{1}{3}}}$$

$$\Rightarrow \lim_{x \rightarrow 0} f'(x) = \frac{2}{3(0)^{\frac{1}{3}}}$$

$$\Rightarrow \lim_{x \rightarrow 0} f'(x) = \text{undefined}$$

Since the limit for the derivative is undefined at $x = 0$, we can say that f is not differentiable at $x = 0$.

\therefore Rolle's Theorem is not applicable to the function ' f ' on $[-1, 1]$.

$$(vi) f(x) = \begin{cases} -4x + 5, & 0 \leq x \leq 1 \\ 2x - 3, & 1 < x \leq 2 \end{cases}$$

Solution:

Given function is $f(x) = \begin{cases} -4x + 5, & 0 \leq x \leq 1 \\ 2x - 3, & 1 < x \leq 2 \end{cases}$

Now we have to check the continuity at $x = 1$ as the equation of function changes.

Consider LHL:

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} -4x + 5$$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = -4(1) + 5$$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = 1 \quad \dots\dots (1)$$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = 1 \quad \dots\dots (1)$$

Now consider RHL:

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2x - 3$$

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = 2(1) - 3$$

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = -1 \quad \dots\dots (2)$$

From (1) and (2), we can see that the values of both side limits are not equal.
So, the function 'f' is not continuous at $x = 1$.

\therefore Rolle's Theorem is not applicable to the function 'f' in the interval $[0, 2]$.

2. Verify the Rolle's Theorem for each of the following functions on the indicated intervals:

(i) $f(x) = x^2 - 8x + 12$ on $[2, 6]$

Solution:

Given function is $f(x) = x^2 - 8x + 12$ on $[2, 6]$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on \mathbb{R} .

Let us find the values at extremes:

$$\Rightarrow f(2) = 2^2 - 8(2) + 12$$

$$\Rightarrow f(2) = 4 - 16 + 12$$

$$\Rightarrow f(2) = 0$$

$$\Rightarrow f(6) = 6^2 - 8(6) + 12$$

$$\Rightarrow f(6) = 36 - 48 + 12$$

$$\Rightarrow f(6) = 0$$

$\therefore f(2) = f(6)$, Rolle's theorem applicable for function f on $[2, 6]$.

Now we have to find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(x^2 - 8x + 12)}{dx}$$

$$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(8x)}{dx} + \frac{d(12)}{dx}$$

$$\Rightarrow f'(x) = 2x - 8 + 0$$

$$\Rightarrow f'(x) = 2x - 8 + 0$$

$$\Rightarrow f'(x) = 2x - 8$$

We have $f'(c) = 0 \in [2, 6]$, from the above definition

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 2c - 8 = 0$$

$$\Rightarrow 2c = 8$$

$$\Rightarrow c = \frac{8}{2}$$

$$\Rightarrow c = 4 \in [2, 6]$$

\therefore Rolle's Theorem is verified.

(ii) $f(x) = x^2 - 4x + 3$ on $[1, 3]$

Solution:

Given function is $f(x) = x^2 - 4x + 3$ on $[1, 3]$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on \mathbb{R} . Let us find the values at extremes:

$$\Rightarrow f(1) = 1^2 - 4(1) + 3$$

$$\Rightarrow f(1) = 1 - 4 + 3$$

$$\Rightarrow f(1) = 0$$

$$\Rightarrow f(3) = 3^2 - 4(3) + 3$$

$$\Rightarrow f(3) = 9 - 12 + 3$$

$$\Rightarrow f(3) = 0$$

$\therefore f(1) = f(3)$, Rolle's theorem applicable for function ' f ' on $[1, 3]$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(x^2 - 4x + 3)}{dx}$$

$$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(4x)}{dx} + \frac{d(3)}{dx}$$

$$\Rightarrow f'(x) = 2x - 4 + 0$$

$$\Rightarrow f'(x) = 2x - 4$$

We have $f'(c) = 0$, $c \in (1, 3)$, from the definition of Rolle's Theorem.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 2c - 4 = 0$$

$$\Rightarrow 2c = 4$$

$$\Rightarrow c = \frac{4}{2}$$

$$\Rightarrow c = 2 \in (1, 3)$$

\therefore Rolle's Theorem is verified.

$$\Rightarrow f'(x) = 2x - 4$$

We have $f'(c) = 0$, $c \in (1, 3)$, from the definition of Rolle's Theorem.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 2c - 4 = 0$$

$$\Rightarrow 2c = 4$$

$$\Rightarrow c = 4/2$$

$$\Rightarrow C = 2 \in (1, 3)$$

\therefore Rolle's Theorem is verified.

(iii) $f(x) = (x - 1)(x - 2)^2$ on $[1, 2]$

Solution:

Given function is $f(x) = (x - 1)(x - 2)^2$ on $[1, 2]$

Since, given function f is a polynomial it is continuous and differentiable everywhere that is on \mathbb{R} .

Let us find the values at extremes:

$$\Rightarrow f(1) = (1 - 1)(1 - 2)^2$$

$$\Rightarrow f(1) = 0(1)^2$$

$$\Rightarrow f(1) = 0$$

$$\Rightarrow f(2) = (2 - 1)(2 - 2)^2$$

$$\Rightarrow f(2) = 0^2$$

$$\Rightarrow f(2) = 0$$

$\therefore f(1) = f(2)$, Rolle's Theorem applicable for function ' f ' on $[1, 2]$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d((x-1)(x-2)^2)}{dx}$$

Differentiating by using product rule, we get

$$\Rightarrow f'(x) = (x-2)^2 \times \frac{d(x-1)}{dx} + (x-1) \times \frac{d((x-2)^2)}{dx}$$

$$\Rightarrow f'(x) = ((x-2)^2 \times 1) + ((x-1) \times 2 \times (x-2))$$

$$\Rightarrow f'(x) = x^2 - 4x + 4 + 2(x^2 - 3x + 2)$$

$$\Rightarrow f'(x) = 3x^2 - 10x + 8$$

We have $f'(c) = 0$ $c \in (1, 2)$, from the definition of Rolle's Theorem.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 3c^2 - 10c + 8 = 0$$

$$\Rightarrow c = \frac{10 \pm \sqrt{(-10)^2 - (4 \times 3 \times 8)}}{2 \times 3}$$

$$\Rightarrow c = \frac{10 \pm \sqrt{100 - 96}}{6}$$

$$\Rightarrow c = \frac{10 \pm 2}{6}$$

$$\Rightarrow 3c^2 - 10c + 8 = 0$$

$$\Rightarrow c = \frac{10 \pm \sqrt{(-10)^2 - (4 \times 3 \times 8)}}{2 \times 3}$$

$$\Rightarrow c = \frac{10 \pm \sqrt{100 - 96}}{6}$$

$$\Rightarrow c = \frac{10 \pm 2}{6}$$

$$\Rightarrow c = \frac{12}{6} \text{ or } c = \frac{8}{6}$$

$$\Rightarrow c = \frac{4}{3} \in (1, 2) \text{ (neglecting the value 2)}$$

\therefore Rolle's Theorem is verified.

(iv) $f(x) = x(x-1)^2$ on $[0, 1]$

Solution:

Given function is $f(x) = x(x - 1)^2$ on $[0, 1]$

Since, given function f is a polynomial it is continuous and differentiable everywhere that is, on \mathbb{R} .

Let us find the values at extremes

$$\Rightarrow f(0) = 0(0 - 1)^2$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(1) = 1(1 - 1)^2$$

$$\Rightarrow f(1) = 0^2$$

$$\Rightarrow f(1) = 0$$

$\therefore f(0) = f(1)$, Rolle's theorem applicable for function ' f ' on $[0, 1]$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(x(x-1)^2)}{dx}$$

Differentiating using product rule:

$$\Rightarrow f'(x) = (x - 1)^2 \times \frac{d(x)}{dx} + x \frac{d((x-1)^2)}{dx}$$

$$\Rightarrow f'(x) = ((x - 1)^2 \times 1) + (x \times 2 \times (x - 1))$$

$$\Rightarrow f'(x) = (x - 1)^2 + 2(x^2 - x)$$

$$\Rightarrow f'(x) = x^2 - 2x + 1 + 2x^2 - 2x$$

$$\Rightarrow f'(x) = 3x^2 - 4x + 1$$

We have $f'(c) = 0$ $c \in (0, 1)$, from the definition given above.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow f'(x) = (x-1)^2 + 2(x^2 - x)$$

$$\Rightarrow f'(x) = x^2 - 2x + 1 + 2x^2 - 2x$$

$$\Rightarrow f'(x) = 3x^2 - 4x + 1$$

We have $f'(c) = 0$ $c \in (0, 1)$, from the definition given above.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 3c^2 - 4c + 1 = 0$$

$$\Rightarrow c = \frac{4 \pm \sqrt{(-4)^2 - (4 \times 3 \times 1)}}{2 \times 3}$$

$$\Rightarrow c = \frac{4 \pm \sqrt{16 - 12}}{6}$$

$$\Rightarrow c = \frac{4 \pm \sqrt{4}}{6}$$

$$\Rightarrow c = \frac{6}{6} \text{ or } c = \frac{2}{6}$$

$$\Rightarrow c = \frac{1}{3} \in (0, 1)$$

\therefore Rolle's Theorem is verified.

(v) $f(x) = (x^2 - 1)(x - 2)$ on $[-1, 2]$

Solution:

Given function is $f(x) = (x^2 - 1)(x - 2)$ on $[-1, 2]$

Since, given function f is a polynomial it is continuous and differentiable everywhere that is on \mathbb{R} .

Let us find the values at extremes:

$$\Rightarrow f(-1) = ((-1)^2 - 1)(-1 - 2)$$

$$\Rightarrow f(-1) = (1 - 1)(-3)$$

$$\Rightarrow f(-1) = (0)(-3)$$

$$\Rightarrow f(-1) = 0$$

$$\Rightarrow f(2) = (2^2 - 1)(2 - 2)$$

$$\Rightarrow f(2) = (4 - 1)(0)$$

$$\Rightarrow f(2) = 0$$

$\therefore f(-1) = f(2)$, Rolle's theorem applicable for function f on $[-1, 2]$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d((x^2-1)(x-2))}{dx}$$

Differentiating using product rule,

$$\Rightarrow f'(x) = (x-2) \times \frac{d(x^2-1)}{dx} + (x^2-1) \frac{d(x-2)}{dx}$$

$$\Rightarrow f'(x) = ((x-2) \times 2x) + ((x^2-1) \times 1)$$

$$\Rightarrow f'(x) = 2x^2 - 4x + x^2 - 1$$

$$\Rightarrow f'(x) = 2x^2 - 4x - 1$$

We have $f'(c) = 0$ $c \in (-1, 2)$, from the definition of Rolle's Theorem.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 2c^2 - 4c - 1 = 0$$

$$\Rightarrow c = \frac{4 \pm \sqrt{(-4)^2 - (4 \times 2 \times -1)}}{2 \times 2}$$

$$\Rightarrow c = \frac{4 \pm \sqrt{16 + 8}}{4}$$

$$\Rightarrow c = \frac{4 \pm \sqrt{24}}{4}$$

$$\Rightarrow c = \frac{4 + 2\sqrt{6}}{4} \text{ or } c = \frac{4 - 2\sqrt{6}}{4}$$

$$\Rightarrow c = 1 + \frac{\sqrt{6}}{2} \text{ or } c = 1 - \frac{\sqrt{6}}{2}$$

$$\Rightarrow c = 1 - \frac{\sqrt{6}}{2} \in (-1, 2)$$

\therefore Rolle's Theorem is verified.

(vi) $f(x) = x(x-4)^2$ on $[0, 4]$

Solution:

Given function is $f(x) = x(x - 4)^2$ on $[0, 4]$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on \mathbb{R} .

Let us find the values at extremes:

$$\Rightarrow f(0) = 0(0 - 4)^2$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(4) = 4(4 - 4)^2$$

$$\Rightarrow f(4) = 4(0)^2$$

$$\Rightarrow f(4) = 0$$

$\therefore f(0) = f(4)$, Rolle's theorem applicable for function ' f ' on $[0, 4]$.

Let's find the derivative of $f(x)$:

$$\Rightarrow f'(x) = \frac{d(x(x-4)^2)}{dx}$$

Differentiating using product rule

$$\Rightarrow f'(x) = (x-4)^2 \times \frac{d(x)}{dx} + x \frac{d((x-4)^2)}{dx}$$

$$\Rightarrow f'(x) = ((x-4)^2 \times 1) + (x \times 2 \times (x-4))$$

$$\Rightarrow f'(x) = (x-4)^2 + 2(x^2 - 4x)$$

$$\Rightarrow f'(x) = x^2 - 8x + 16 + 2x^2 - 8x$$

$$\Rightarrow f'(x) = 3x^2 - 16x + 16$$

We have $f'(c) = 0$ $c \in (0, 4)$, from the definition of Rolle's Theorem.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 3c^2 - 16c + 16 = 0$$

$$\Rightarrow c = \frac{16 \pm \sqrt{(-16)^2 - (4 \times 3 \times 16)}}{2 \times 3}$$

$$\Rightarrow c = \frac{16 \pm \sqrt{256 - 192}}{6}$$

$$\Rightarrow c = \frac{16 \pm \sqrt{64}}{6}$$

$$\Rightarrow c = \frac{8}{6} \text{ or } c = \frac{24}{6}$$

$$\Rightarrow c = \frac{8}{6} \in (0, 4)$$

\therefore Rolle's Theorem is verified.

(vii) $f(x) = x(x-2)^2$ on $[0, 2]$

Solution:

Given function is $f(x) = x(x-2)^2$ on $[0, 2]$

Since, given function f is a polynomial it is continuous and differentiable everywhere that is on \mathbb{R} .

Let us find the values at extremes:

$$\Rightarrow f(0) = 0(0-2)^2$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(2) = 2(2 - 2)^2$$

$$\Rightarrow f(2) = 2(0)^2$$

$$\Rightarrow f(2) = 0$$

$f(0) = f(2)$, Rolle's theorem applicable for function f on $[0, 2]$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(x(x-2)^2)}{dx}$$

Differentiating using UV rule,

$$\Rightarrow f'(x) = (x-2)^2 \times \frac{d(x)}{dx} + x \frac{d((x-2)^2)}{dx}$$

$$\Rightarrow f'(x) = ((x-2)^2 \times 1) + (x \times 2 \times (x-2))$$

$$\Rightarrow f'(x) = (x-2)^2 + 2(x^2 - 2x)$$

$$\Rightarrow f'(x) = x^2 - 4x + 4 + 2x^2 - 4x$$

$$\Rightarrow f'(x) = 3x^2 - 8x + 4$$

We have $f'(c) = 0$ $c \in (0, 1)$, from the definition of Rolle's Theorem.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 3c^2 - 8c + 4 = 0$$

$$\Rightarrow c = \frac{8 \pm \sqrt{(-8)^2 - (4 \times 3 \times 4)}}{2 \times 3}$$

$$\Rightarrow 3c^2 - 8c + 4 = 0$$

$$\Rightarrow c = \frac{8 \pm \sqrt{(-8)^2 - (4 \times 3 \times 4)}}{2 \times 3}$$

$$\Rightarrow c = \frac{8 \pm \sqrt{64 - 48}}{6}$$

$$\Rightarrow c = \frac{8 \pm \sqrt{16}}{6}$$

$$\Rightarrow c = \frac{12}{6} \text{ or } c = \frac{6}{6}$$

$$\Rightarrow c = 1 \in (0, 2)$$

\therefore Rolle's Theorem is verified.

(viii) $f(x) = x^2 + 5x + 6$ on $[-3, -2]$

Solution:

Given function is $f(x) = x^2 + 5x + 6$ on $[-3, -2]$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on \mathbb{R} . Let us find the values at extremes:

$$\Rightarrow f(-3) = (-3)^2 + 5(-3) + 6$$

$$\Rightarrow f(-3) = 9 - 15 + 6$$

$$\Rightarrow f(-3) = 0$$

$$\Rightarrow f(-2) = (-2)^2 + 5(-2) + 6$$

$$\Rightarrow f(-2) = 4 - 10 + 6$$

$$\Rightarrow f(-2) = 0$$

$\therefore f(-3) = f(-2)$, Rolle's theorem applicable for function f on $[-3, -2]$.

Let's find the derivative of $f(x)$:

$$\Rightarrow f'(x) = \frac{d(x^2 + 5x + 6)}{dx}$$

$$\Rightarrow f'(x) = \frac{d(x^2)}{dx} + \frac{d(5x)}{dx} + \frac{d(6)}{dx}$$

$$\Rightarrow f'(x) = 2x + 5 + 0$$

$$\Rightarrow f'(x) = 2x + 5$$

We have $f'(c) = 0$ $c \in (-3, -2)$, from the definition of Rolle's Theorem

$$\Rightarrow f'(c) = 0$$

We have $f'(c) = 0$ $c \in (-3, -2)$, from the definition of Rolle's Theorem

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 2c + 5 = 0$$

$$\Rightarrow 2c = -5$$

$$\Rightarrow c = -\frac{5}{2}$$

$$\Rightarrow C = -2.5 \in (-3, -2)$$

\therefore Rolle's Theorem is verified.

3. Verify the Rolle's Theorem for each of the following functions on the indicated intervals:

(i) $f(x) = \cos 2(x - \pi/4)$ on $[0, \pi/2]$

Solution:

Given function is $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ on $\left[0, \frac{\pi}{2}\right]$

We know that cosine function is continuous and differentiable on \mathbb{R} .

Let's find the values of the function at an extreme,

$$\Rightarrow f(0) = \cos 2\left(0 - \frac{\pi}{4}\right)$$

$$\Rightarrow f(0) = \cos 2\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow f(0) = \cos\left(-\frac{\pi}{2}\right)$$

We know that $\cos(-x) = \cos x$

$$\Rightarrow \underline{f(0)} = 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{4}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$$

We get $f(0) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d\left(\cos 2\left(x - \frac{\pi}{4}\right)\right)}{dx}$$

$$\Rightarrow f'(x) = -\sin\left(2\left(x - \frac{\pi}{4}\right)\right) \frac{d\left(2\left(x - \frac{\pi}{4}\right)\right)}{dx}$$

$$\Rightarrow f'(x) = -2 \sin 2\left(x - \frac{\pi}{4}\right)$$

We have $f'(c) = 0$,

$$\Rightarrow -2 \sin 2\left(c - \frac{\pi}{4}\right) = 0$$

$$\Rightarrow c - \frac{\pi}{4} = 0$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

\therefore Rolle's Theorem is verified.

(ii) $f(x) = \sin 2x$ on $[0, \pi/2]$

Solution:

Given function is $f(x) = \sin 2x$ on $\left[0, \frac{\pi}{2}\right]$

We know that sine function is continuous and differentiable on \mathbb{R} . Let's find the values of function at extreme,

$$\Rightarrow f(0) = \sin 2(0)$$

$$\Rightarrow f(0) = \sin 0$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin 2\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f(0) = \sin 0$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin 2\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin(\pi)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$$

We have $f(0) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(\sin 2x)}{dx}$$

$$\Rightarrow f'(x) = \cos 2x \frac{d(2x)}{dx}$$

$$\Rightarrow f'(x) = 2\cos 2x$$

We have $f'(c) = 0$,

$$\Rightarrow 2\cos 2c = 0$$

$$\Rightarrow 2c = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

\therefore Rolle's Theorem is verified.

(iii) $f(x) = \cos 2x$ on $[-\pi/4, \pi/4]$

Solution:

Given function is $\cos 2x$ on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

We know that cosine function is continuous and differentiable on \mathbb{R} . Let's find the values of the function at an extreme,

$$\Rightarrow f\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow f(0) = \cos\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow f(0) = \cos\left(-\frac{\pi}{2}\right)$$

We know that $\cos(-x) = \cos x$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$$

We have $f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$, so there exist a $c \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(\cos 2x)}{dx}$$

$$\Rightarrow f'(x) = -\sin 2x \frac{d(2x)}{dx}$$

$$\Rightarrow f'(x) = -2\sin 2x$$

We have $f'(c) = 0$,

$$\Rightarrow -2\sin 2c = 0$$

$$\Rightarrow 2c = 0$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

\therefore Rolle's Theorem is verified.

(iv) $f(x) = e^x \sin x$ on $[0, \pi]$

Solution:

Given function is $f(x) = e^x \sin x$ on $[0, \pi]$

We know that exponential and sine functions are continuous and differentiable on \mathbb{R} . Let's find the values of the function at an extreme,

$$\Rightarrow f(0) = e^0 \sin(0)$$

$$\Rightarrow f(0) = 1 \times 0$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(\pi) = e^\pi \sin(\pi)$$

$$\Rightarrow f(\pi) = e^\pi \times 0$$

$$\Rightarrow f(\pi) = 0$$

We have $f(0) = f(\pi)$, so there exist a $c \in (0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(e^x \sin x)}{dx}$$

$$\Rightarrow f'(x) = \sin x \frac{d(e^x)}{dx} + e^x \frac{d(\sin x)}{dx}$$

$$\Rightarrow f'(x) = e^x (\sin x + \cos x)$$

We have $f'(c) = 0$,

$$\Rightarrow e^c (\sin c + \cos c) = 0$$

$$\Rightarrow \sin c + \cos c = 0$$

$$\Rightarrow \frac{1}{\sqrt{2}} \sin c + \frac{1}{\sqrt{2}} \cos c = 0$$

$$\Rightarrow \sin\left(\frac{\pi}{4}\right) \sin c + \cos\left(\frac{\pi}{4}\right) \cos c = 0$$

$$\Rightarrow \cos\left(c - \frac{\pi}{4}\right) = 0$$

$$\Rightarrow c - \frac{\pi}{4} = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{3\pi}{4} \in (0, \pi)$$

\therefore Rolle's Theorem is verified.

(v) $f(x) = e^x \cos x$ on $[-\pi/2, \pi/2]$

Solution:

Given function is $f(x) = e^x \cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

We know that exponential and cosine functions are continuous and differentiable on \mathbb{R} . Let's find the values of the function at an extreme,

$$\Rightarrow f\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}} \cos\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}} \times 0$$

$$\Rightarrow f\left(-\frac{\pi}{2}\right) = 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f(\pi) = e^{\frac{\pi}{2}} \times 0$$

$$\Rightarrow f(\pi) = 0$$

We have $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(e^x \cos x)}{dx}$$

$$\Rightarrow f'(x) = \cos x \frac{d(e^x)}{dx} + e^x \frac{d(\cos x)}{dx}$$

$$\Rightarrow f'(x) = e^x (-\sin x + \cos x)$$

We have $f'(c) = 0$,

$$\Rightarrow e^c (-\sin c + \cos c) = 0$$

$$\Rightarrow -\sin c + \cos c = 0$$

$$\Rightarrow \frac{-1}{\sqrt{2}} \sin c + \frac{1}{\sqrt{2}} \cos c = 0$$

$$\Rightarrow -\sin\left(\frac{\pi}{4}\right) \sin c + \cos\left(\frac{\pi}{4}\right) \cos c = 0$$

$$\Rightarrow \cos\left(c + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow -\sin\left(\frac{\pi}{4}\right) \sin c + \cos\left(\frac{\pi}{4}\right) \cos c = 0$$

$$\Rightarrow \cos\left(c + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow c + \frac{\pi}{4} = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

\therefore Rolle's Theorem is verified.

(vi) $f(x) = \cos 2x$ on $[0, \pi]$

Solution:

Given function is $f(x) = \cos 2x$ on $[0, \pi]$

We know that cosine function is continuous and differentiable on \mathbb{R} . Let's find the values of function at extreme,

$$\Rightarrow f(0) = \cos 2(0)$$

$$\Rightarrow f(0) = \cos(0)$$

$$\Rightarrow f(0) = 1$$

$$\Rightarrow f(\pi) = \cos 2(\pi)$$

$$\Rightarrow f(\pi) = \cos(2\pi)$$

$$\Rightarrow f(\pi) = 1$$

We have $f(0) = f(\pi)$, so there exist a c belongs to $(0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(\cos 2x)}{dx}$$

$$\Rightarrow f'(x) = -\sin 2x \frac{d(2x)}{dx}$$

$$\Rightarrow f'(x) = -2\sin 2x$$

We have $f'(c) = 0$,

$$\Rightarrow -2\sin 2c = 0$$

$$\Rightarrow 2c = 0$$

$$\Rightarrow c = \frac{\pi}{4} \in (0, \pi)$$

\therefore Rolle's Theorem is verified.

Hence Rolle's Theorem is verified.

$$(vii) f(x) = \frac{\sin x}{e^x} \text{ on } 0 \leq x \leq \pi$$

Solution:

Given function is $f(x) = \frac{\sin x}{e^x}$ on $[0, \pi]$

This can be written as

$$\Rightarrow f(x) = e^{-x} \sin x \text{ on } [0, \pi]$$

We know that exponential and sine functions are continuous and differentiable on \mathbb{R} . Let's find the values of the function at an extreme,

$$\Rightarrow f(0) = e^{-0} \sin(0)$$

$$\Rightarrow f(0) = 1 \times 0$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(\pi) = e^{-\pi} \sin(\pi)$$

$$\Rightarrow f(\pi) = e^{-\pi} \times 0$$

$$\Rightarrow f(\pi) = 0$$

We have $f(0) = f(\pi)$, so there exist a c belongs to $(0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(e^{-x} \sin x)}{dx}$$

$$\Rightarrow f'(x) = \sin x \frac{d(e^{-x})}{dx} + e^{-x} \frac{d(\sin x)}{dx}$$

$$\Rightarrow f'(x) = \sin x (-e^{-x}) + e^{-x} (\cos x)$$

$$\Rightarrow f'(x) = e^{-x} (-\sin x + \cos x)$$

We have $f'(c) = 0$,

$$\Rightarrow e^{-c} (-\sin c + \cos c) = 0$$

$$\Rightarrow -\sin c + \cos c = 0$$

$$\Rightarrow e^{-c}(-\sin c + \cos c) = 0$$

$$\Rightarrow -\sin c + \cos c = 0$$

$$\Rightarrow -\frac{1}{\sqrt{2}}\sin c + \frac{1}{\sqrt{2}}\cos c = 0$$

$$\Rightarrow -\sin\left(\frac{\pi}{4}\right)\sin c + \cos\left(\frac{\pi}{4}\right)\cos c = 0$$

$$\Rightarrow \cos\left(c + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow c + \frac{\pi}{4} = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{\pi}{4} \in (0, \pi)$$

\therefore Rolle's Theorem is verified.

(viii) $f(x) = \sin 3x$ on $[0, \pi]$

Solution:

Given function is $f(x) = \sin 3x$ on $[0, \pi]$

We know that sine function is continuous and differentiable on \mathbb{R} . Let's find the values of function at extreme,

$$\Rightarrow f(0) = \sin 3(0)$$

$$\Rightarrow f(0) = \sin 0$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(\pi) = \sin 3(\pi)$$

$$\Rightarrow f(\pi) = \sin(3\pi)$$

$$\Rightarrow f(\pi) = 0$$

We have $f(0) = f(\pi)$, so there exist a c belongs to $(0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(\sin 3x)}{dx}$$

$$\Rightarrow f'(x) = \cos 3x \frac{d(3x)}{dx}$$

$$\Rightarrow f'(x) = 3\cos 3x$$

We have $f'(c) = 0$,

$$\Rightarrow 3\cos 3c = 0$$

$$\Rightarrow 3c = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{\pi}{6} \in (0, \pi)$$

\therefore Rolle's Theorem is verified.

We have $f'(c) = 0$,

$$\Rightarrow 3\cos 3c = 0$$

$$\Rightarrow 3c = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{\pi}{6} \in (0, \pi)$$

\therefore Rolle's Theorem is verified.

(ix) $f(x) = e^{1-x^2}$ on $[-1, 1]$

Solution:

Given function is $f(x) = e^{1-x^2}$ on $[-1, 1]$

We know that exponential function is continuous and differentiable over \mathbb{R} .

Let's find the value of function f at extremes,

$$\Rightarrow f(-1) = e^{1-(-1)^2}$$

$$\Rightarrow f(-1) = e^{1-1}$$

$$\Rightarrow f(-1) = e^0$$

$$\Rightarrow f(-1) = 1$$

$$\Rightarrow f(1) = e^{1-1^2}$$

$$\Rightarrow f(1) = e^{1-1}$$

$$\Rightarrow f(1) = e^0$$

$$\Rightarrow f(1) = 1$$

We got $f(-1) = f(1)$ so, there exists a $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's find the derivative of the function f :

$$\Rightarrow f'(x) = \frac{d(e^{1-x^2})}{dx}$$

$$\Rightarrow f'(x) = e^{1-x^2} \frac{d(1-x^2)}{dx}$$

$$\Rightarrow f'(x) = e^{1-x^2} (-2x)$$

$$\Rightarrow f'(x) = e^{1-x^2} \frac{d(1-x^2)}{dx}$$

$$\Rightarrow f'(x) = e^{1-x^2} (-2x)$$

We have $f'(c) = 0$

$$\Rightarrow e^{1-c^2} (-2c) = 0$$

$$\Rightarrow 2c = 0$$

$$\Rightarrow c = 0 \in [-1, 1]$$

\therefore Rolle's Theorem is verified.

(x) $f(x) = \log(x^2 + 2) - \log 3$ on $[-1, 1]$

Solution:

Given function is $f(x) = \log(x^2 + 2) - \log 3$ on $[-1, 1]$

We know that logarithmic function is continuous and differentiable in its own domain. We check the values of the function at the extreme,

$$\Rightarrow f(-1) = \log((-1)^2 + 2) - \log 3$$

$$\Rightarrow f(-1) = \log(1 + 2) - \log 3$$

$$\Rightarrow f(-1) = \log 3 - \log 3$$

$$\Rightarrow f(-1) = 0$$

$$\Rightarrow f(1) = \log(1^2 + 2) - \log 3$$

$$\Rightarrow f(1) = \log(1 + 2) - \log 3$$

$$\Rightarrow f(1) = \log 3 - \log 3$$

$$\Rightarrow f(1) = 0$$

We have got $f(-1) = f(1)$. So, there exists a c such that $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's find the derivative of the function f ,

$$\Rightarrow f'(x) = \frac{d(\log(x^2 + 2) - \log 3)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{x^2 + 2} \frac{d(x^2 + 2)}{dx} - 0$$

$$\Rightarrow f'(x) = \frac{2x}{x^2 + 2}$$

We have $f'(c) = 0$

$$\Rightarrow \frac{2c}{c^2 + 2} = 0$$

$$\Rightarrow 2c = 0$$

$$\Rightarrow c = 0 \in (-1, 1)$$

\therefore Rolle's Theorem is verified.

(xi) $f(x) = \sin x + \cos x$ on $[0, \pi/2]$

Solution:

Given function is $f(x) = \sin x + \cos x$ on $\left[0, \frac{\pi}{2}\right]$

We know that sine and cosine functions are continuous and differentiable on \mathbb{R} . Let's the value of function f at extremes:

$$\Rightarrow f(0) = \sin(0) + \cos(0)$$

$$\Rightarrow f(0) = 0 + 1$$

$$\Rightarrow f(0) = 1$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1 + 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1$$

We have $f(0) = f\left(\frac{\pi}{2}\right)$. So, there exists a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Let's find the derivative of the function f .

$$\Rightarrow f'(x) = \frac{d(\sin x + \cos x)}{dx}$$

$$\Rightarrow f'(x) = \cos x - \sin x$$

We have $f'(c) = 0$

$$\Rightarrow \cos c - \sin c = 0$$

We have $f'(c) = 0$

$$\Rightarrow \cos c - \sin c = 0$$

$$\Rightarrow \frac{1}{\sqrt{2}} \cos c - \frac{1}{\sqrt{2}} \sin c = 0$$

$$\Rightarrow \sin\left(\frac{\pi}{4}\right) \cos c - \cos\left(\frac{\pi}{4}\right) \sin c = 0$$

$$\Rightarrow \sin\left(\frac{\pi}{4} - c\right) = 0$$

$$\Rightarrow \frac{\pi}{4} - c = 0$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

\therefore Rolle's Theorem is verified.

(xii) $f(x) = 2 \sin x + \sin 2x$ on $[0, \pi]$

Solution:

Given function is $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$

We know that sine function continuous and differentiable over \mathbb{R} .

Let's check the values of function f at the extremes

$$\Rightarrow f(0) = 2\sin(0) + \sin 2(0)$$

$$\Rightarrow f(0) = 2(0) + 0$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(\pi) = 2\sin(\pi) + \sin 2(\pi)$$

$$\Rightarrow f(\pi) = 2(0) + 0$$

$$\Rightarrow f(\pi) = 0$$

We have $f(0) = f(\pi)$, so there exist a c belongs to $(0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of function f .

$$\Rightarrow f'(x) = \frac{d(2\sin x + \sin 2x)}{dx}$$

$$\Rightarrow f'(x) = 2\cos x + \cos 2x \frac{d(2x)}{dx}$$

$$\Rightarrow f'(x) = 2\cos x + 2\cos 2x$$

$$\Rightarrow f'(x) = 2\cos x + 2(2\cos^2 x - 1)$$

$$\Rightarrow f'(x) = 4\cos^2 x + 2\cos x - 2$$

We have $f'(c) = 0$,

$$\Rightarrow 4\cos^2 c + 2\cos c - 2 = 0$$

$$\Rightarrow 2\cos^2 c + \cos c - 1 = 0$$

$$\Rightarrow 2\cos^2 c + 2\cos c - \cos c - 1 = 0$$

$$\Rightarrow 2\cos c (\cos c + 1) - 1(\cos c + 1) = 0$$

$$\Rightarrow (2\cos c - 1)(\cos c + 1) = 0$$

$$\Rightarrow \cos c = \frac{1}{2} \text{ or } \cos c = -1$$

$$\Rightarrow c = \frac{\pi}{3} \in (0, \pi)$$

\therefore Rolle's Theorem is verified.

$$(xiii) f(x) = \frac{x}{2} - \sin \frac{\pi x}{6} \text{ on } [-1, 0]$$

Solution:

Given function is $f(x) = \frac{x}{2} - \sin\left(\frac{\pi x}{6}\right)$ on $[-1, 0]$

We know that sine function is continuous and differentiable over \mathbb{R} .

Now we have to check the values of 'f' at an extreme

$$\Rightarrow f(-1) = \frac{-1}{2} - \sin\left(\frac{\pi(-1)}{6}\right)$$

$$\Rightarrow f(-1) = -\frac{1}{2} - \sin\left(\frac{-\pi}{6}\right)$$

$$\Rightarrow f(-1) = -\frac{1}{2} - \left(-\frac{1}{2}\right)$$

$$\Rightarrow f(-1) = 0$$

$$\Rightarrow f(0) = \frac{0}{2} - \sin\left(\frac{\pi(0)}{6}\right)$$

$$\Rightarrow f(0) = 0 - \sin(0)$$

$$\Rightarrow f(0) = 0 - 0$$

$$\Rightarrow f(0) = 0$$

We have got $f(-1) = f(0)$. So, there exists a $c \in (-1, 0)$ such that $f'(c) = 0$.

Now we have to find the derivative of the function 'f'

$$\Rightarrow f'(x) = \frac{d\left(\frac{x}{2} - \sin\left(\frac{\pi x}{6}\right)\right)}{dx}$$

$$\Rightarrow f(0) = 0$$

We have got $f(-1) = f(0)$. So, there exists a $c \in (-1, 0)$ such that $f'(c) = 0$.

Now we have to find the derivative of the function 'f'

$$\Rightarrow f'(x) = \frac{d\left(\frac{x}{2} - \sin\left(\frac{\pi x}{6}\right)\right)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2} - \cos\left(\frac{\pi x}{6}\right) \frac{d\left(\frac{\pi x}{6}\right)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2} - \frac{\pi}{6} \cos\left(\frac{\pi x}{6}\right)$$

We have $f'(c) = 0$

$$\Rightarrow \frac{1}{2} - \frac{\pi}{6} \cos\left(\frac{\pi c}{6}\right) = 0$$

$$\Rightarrow \frac{\pi}{6} \cos\left(\frac{\pi c}{6}\right) = \frac{1}{2}$$

$$\Rightarrow \cos\left(\frac{\pi c}{6}\right) = \frac{1}{2} \times \frac{6}{\pi}$$

$$\Rightarrow \cos\left(\frac{\pi c}{6}\right) = \frac{3}{\pi}$$

$$\Rightarrow \frac{\pi c}{6} = \cos^{-1}\left(\frac{3}{\pi}\right)$$

$$\Rightarrow c = \frac{6}{\pi} \cos^{-1}\left(\frac{3}{\pi}\right)$$

Cosine is positive between $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, for our convenience we take the interval to be $-\frac{\pi}{2} \leq \theta \leq 0$, since the values of the cosine repeats.

We know that $\frac{3}{\pi}$ value is nearly equal to 1. So, the value of the c nearly equal to 0.

So, we can clearly say that $c \in (-1, 0)$.

\therefore Rolle's Theorem is verified.

$$(xiv). f(x) = \frac{6x}{\pi} - 4 \sin^2 x \text{ on } \left[0, \frac{\pi}{6}\right]$$

Solution:

Given function is $f(x) = \frac{6x}{\pi} - 4 \sin^2 x$ on $\left[0, \frac{\pi}{6}\right]$

We know that sine function is continuous and differentiable over \mathbb{R} .

Now we have to check the values of function 'f' at the extremes,

$$\Rightarrow f(0) = \frac{6(0)}{\pi} - 4 \sin^2(0)$$

$$\Rightarrow f(0) = 0 - 4(0)$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f\left(\frac{\pi}{6}\right) = \frac{6\left(\frac{\pi}{6}\right)}{\pi} - 4 \sin^2\left(\frac{\pi}{6}\right)$$

$$\Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{\pi} - 4\left(\frac{1}{2}\right)^2$$

$$\Rightarrow f\left(\frac{\pi}{6}\right) = 1 - 4\left(\frac{1}{4}\right)$$

$$\Rightarrow f\left(\frac{\pi}{6}\right) = 1 - 1$$

$$\Rightarrow f\left(\frac{\pi}{6}\right) = 0.$$

We have $f(0) = f\left(\frac{\pi}{6}\right)$. So, there exists a $c \in \left(0, \frac{\pi}{6}\right)$ such that $f'(c) = 0$.

We have to find the derivative of function 'f.'

$$\Rightarrow f'(x) = \frac{d\left(\frac{6x}{\pi} - 4 \sin^2 x\right)}{dx}$$

$$\Rightarrow f'(x) = \frac{6}{\pi} - 4 \times 2 \sin x \times \frac{d(\sin x)}{dx}$$

$$\Rightarrow f'(x) = \frac{6}{\pi} - 8 \sin x (\cos x)$$

$$\Rightarrow f'(x) = \frac{6}{\pi} - 4(2\sin x \cos x)$$

$$\Rightarrow f'(x) = \frac{6}{\pi} - 4\sin 2x$$

We have $f'(c) = 0$

$$\Rightarrow \frac{6}{\pi} - 4\sin 2c = 0$$

$$\Rightarrow 4\sin 2c = \frac{6}{\pi}$$

$$\Rightarrow \sin 2c = \frac{6}{4\pi}$$

We know $\frac{6}{4\pi} < \frac{1}{2}$

$$\Rightarrow \sin 2c < \frac{1}{2}$$

$$\Rightarrow 2c < \sin^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow 2c < \frac{\pi}{6}$$

$$\Rightarrow c < \frac{\pi}{12} \in \left(0, \frac{\pi}{6}\right)$$

\therefore Rolle's Theorem is verified.

(xv) $f(x) = 4^{\sin x}$ on $[0, \pi]$

Solution:

Given function is $f(x) = 4^{\sin x}$ on $[0, \pi]$

We that sine function is continuous and differentiable over \mathbb{R} .

Now we have to check the values of function 'f' at extremes

$$\Rightarrow f(0) = 4^{\sin(0)}$$

$$\Rightarrow f(0) = 4^0$$

$$\Rightarrow f(0) = 1$$

$$\Rightarrow f(\pi) = 4^{\sin \pi}$$

$$\Rightarrow f(\pi) = 4^0$$

$$\Rightarrow f(\pi) = 1$$

$$\Rightarrow f(\pi) = 4^{\sin \pi}$$

$$\Rightarrow f(\pi) = 4^0$$

$$\Rightarrow f(\pi) = 1$$

We have $f(0) = f(\pi)$. So, there exists a $c \in (0, \pi)$ such that $f'(c) = 0$.

Now we have to find the derivative of 'f'

$$\Rightarrow f'(x) = \frac{d(4^{\sin x})}{dx}$$

$$\Rightarrow f'(x) = 4^{\sin x} \log 4 \frac{d(\sin x)}{dx}$$

$$\Rightarrow f'(x) = 4^{\sin x} \log 4 \cos x$$

We have $f'(c) = 0$

$$\Rightarrow 4^{\sin c} \log 4 \cos c = 0$$

$$\Rightarrow \cos c = 0$$

$$\Rightarrow c = \frac{\pi}{2} \in (0, \pi)$$

\therefore Rolle's Theorem is verified.

(xvi) $f(x) = x^2 - 5x + 4$ on $[0, \pi/6]$

Solution:

Given function is $f(x) = x^2 - 5x + 4$ on $[1, 4]$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on \mathbb{R} .

Let us find the values at extremes

$$\Rightarrow f(1) = 1^2 - 5(1) + 4$$

$$\Rightarrow f(1) = 1 - 5 + 4$$

$$\Rightarrow f(1) = 0$$

$$\Rightarrow f(4) = 4^2 - 5(4) + 4$$

$$\Rightarrow f(4) = 16 - 20 + 4$$

$$\Rightarrow f(4) = 0$$

We have $f(1) = f(4)$. So, there exists a $c \in (1, 4)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$:

$$\Rightarrow f'(x) = \frac{d(x^2 - 5x + 4)}{dx}$$

$$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(5x)}{dx} + \frac{d(4)}{dx}$$

$$\Rightarrow f'(x) = 2x - 5 + 0$$

$$\Rightarrow f'(x) = 2x - 5$$

We have $f'(c) = 0$

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 2c - 5 = 0$$

$$\Rightarrow 2c = 5$$

$$\Rightarrow c = \frac{5}{2}$$

$$\Rightarrow c = 2.5 \in (1, 4)$$

\therefore Rolle's Theorem is verified.

(xvii) $f(x) = \sin^4 x + \cos^4 x$ on $[0, \pi/2]$

Solution:

Given function is $f(x) = \sin^4 x + \cos^4 x$ on $\left[0, \frac{\pi}{2}\right]$

We know that sine and cosine functions are continuous and differentiable functions over \mathbb{R} .

Now we have to find the value of function 'f' at extremes

$$\Rightarrow f(0) = \sin^4(0) + \cos^4(0)$$

$$\Rightarrow f(0) = (0)^4 + (1)^4$$

$$\Rightarrow f(0) = 0 + 1$$

$$\Rightarrow f(0) = 1$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^4\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1^4 + 0^4$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^4\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1^4 + 0^4$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1 + 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1$$

We have $f(0) = f\left(\frac{\pi}{2}\right)$. So, there exists a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now we have to find the derivative of the function 'f'.

$$\Rightarrow f'(x) = \frac{d(\sin^4 x + \cos^4 x)}{dx}$$

$$\Rightarrow f'(x) = 4 \sin^3 x \frac{d(\sin x)}{dx} + 4 \cos^3 x \frac{d(\cos x)}{dx}$$

$$\Rightarrow f'(x) = 4 \sin^3 x \cos x - 4 \cos^3 x \sin x$$

$$\Rightarrow f'(x) = 4 \sin x \cos x (\sin^2 x - \cos^2 x)$$

$$\Rightarrow f'(x) = 2(2 \sin x \cos x) (-\cos 2x)$$

$$\Rightarrow f'(x) = -2(\sin 2x) (\cos 2x)$$

$$\Rightarrow f'(x) = -\sin 4x$$

We have $f'(c) = 0$

$$\Rightarrow -\sin 4c = 0$$

$$\Rightarrow \sin 4c = 0$$

$$\Rightarrow 4c = 0 \text{ or } \pi$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

\therefore Rolle's Theorem is verified.

(xviii) $f(x) = \sin x - \sin 2x$ on $[0, \pi]$

Solution:

Given function is $f(x) = \sin x - \sin 2x$ on $[0, \pi]$

We know that sine function is continuous and differentiable over \mathbb{R} .

Now we have to check the values of the function 'f' at the extremes.

$$\Rightarrow f(0) = \sin(0) - \sin^2(0)$$

$$\Rightarrow f(0) = 0 - \sin^2(0)$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(\pi) = \sin(\pi) - \sin^2(\pi)$$

$$\Rightarrow f(\pi) = 0 - \sin^2(\pi)$$

$$\Rightarrow f(\pi) = 0$$

We have $f(0) = f(\pi)$. So, there exists a $c \in (0, \pi)$ such that $f'(c) = 0$.

Now we have to find the derivative of the function 'f'

$$\Rightarrow f'(x) = \frac{d(\sin x - \sin 2x)}{dx}$$

$$\Rightarrow f'(x) = \cos x - \cos 2x \frac{d(2x)}{dx}$$

$$\Rightarrow f'(x) = \cos x - 2\cos 2x$$

$$\Rightarrow f'(x) = \cos x - 2(2\cos^2 x - 1)$$

$$\Rightarrow f'(x) = \cos x - 4\cos^2 x + 2$$

We have $f'(c) = 0$

$$\Rightarrow \cos c - 4\cos^2 c + 2 = 0$$

$$\Rightarrow \cos c = \frac{-1 \pm \sqrt{(1)^2 - (4 \times -4 \times 2)}}{2 \times -4}$$

$$\Rightarrow \cos c = \frac{-1 \pm \sqrt{1 + 33}}{-8}$$

$$\Rightarrow c = \cos^{-1}\left(\frac{-1 \pm \sqrt{33}}{-8}\right)$$

We can see that $c \in (0, \pi)$

\therefore Rolle's Theorem is verified.

4. Using Rolle's Theorem, find points on the curve $y = 16 - x^2$, $x \in [-1, 1]$, where tangent is parallel to x - axis.

Solution:

Given function is $y = 16 - x^2$, $x \in [-1, 1]$

We know that polynomial function is continuous and differentiable over \mathbb{R} .

Let us check the values of 'y' at extremes

$$\Rightarrow y(-1) = 16 - (-1)^2$$

$$\Rightarrow y(-1) = 16 - 1$$

$$\Rightarrow y(-1) = 15$$

$$\Rightarrow y(1) = 16 - (1)^2$$

$$\Rightarrow y(1) = 16 - 1$$

$$\Rightarrow y(1) = 15$$

We have $y(-1) = y(1)$. So, there exists a $c \in (-1, 1)$ such that $f'(c) = 0$.

We know that for a curve g , the value of the slope of the tangent at a point r is given by $g'(r)$.

Now we have to find the derivative of curve y

\Rightarrow

$$y' = \frac{d(16-x^2)}{dx}$$

$$\Rightarrow y' = -2x$$

We have $y'(c) = 0$

$$\Rightarrow -2c = 0$$

$$\Rightarrow c = 0 \in (-1, 1)$$

Value of y at $x = 1$ is

$$\Rightarrow y = 16 - 0^2$$

$$\Rightarrow y = 16$$

\therefore The point at which the curve y has a tangent parallel to x – axis (since the slope of x – axis is 0) is $(0, 16)$.

Exercise 15.2 Page No: 15.17

1. Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each case find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem:

(i) $f(x) = x^2 - 1$ on $[2, 3]$

Solution:

Given $f(x) = x^2 - 1$ on $[2, 3]$

We know that every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is a polynomial function. So it is continuous in $[2, 3]$ and differentiable in $(2, 3)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (2, 3)$ such that:

$$f'(c) = \frac{f(3) - f(2)}{3 - 2}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(2)}{1}$$

$$f(x) = x^2 - 1$$

Differentiating with respect to x

$$f'(x) = 2x$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2c$$

For $f(3)$, put the value of $x=3$ in $f(x)$:

$$f(3) = (3)^2 - 1$$

$$= 9 - 1$$

$$= 8$$

For $f(2)$, put the value of $x=2$ in $f(x)$:

$$f(2) = (2)^2 - 1$$

$$= 4 - 1$$

$$= 3$$

$$\therefore f'(c) = f(3) - f(2)$$

$$\Rightarrow 2c = 8 - 3$$

$$\Rightarrow 2c = 5$$

$$\Rightarrow c = \frac{5}{2} \in (2, 3)$$

Hence, Lagrange's mean value theorem is verified.

(ii) $f(x) = x^3 - 2x^2 - x + 3$ on $[0, 1]$

Solution:

Given $f(x) = x^3 - 2x^2 - x + 3$ on $[0, 1]$

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is a polynomial function. So it is continuous in $[0, 1]$ and differentiable in $(0, 1)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (0, 1)$ such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow f'(c) = \frac{f(1) - f(0)}{1}$$

$$f(x) = x^3 - 2x^2 - x + 3$$

Differentiating with respect to x

$$f'(x) = 3x^2 - 2(2x) - 1$$

$$= 3x^2 - 4x - 1$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$

$$f'(c) = 3c^2 - 4c - 1$$

For $f(1)$, put the value of $x = 1$ in $f(x)$

$$f(1) = (1)^3 - 2(1)^2 - (1) + 3$$

$$= 1 - 2 - 1 + 3$$

$$= 1$$

For $f(0)$, put the value of $x=0$ in $f(x)$

$$f(0) = (0)^3 - 2(0)^2 - (0) + 3$$

$$= 0 - 0 - 0 + 3$$

$$= 3$$

$$\therefore f'(c) = f(1) - f(0)$$

$$\Rightarrow 3c^2 - 4c - 1 = 1 - 3$$

$$\Rightarrow 3c^2 - 4c = 1 + 1 - 3$$

$$\Rightarrow 3c^2 - 4c = -1$$

$$\Rightarrow 3c^2 - 4c + 1 = 0$$

$$\Rightarrow 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow 3c(c - 1) - 1(c - 1) = 0$$

$$\Rightarrow (3c - 1)(c - 1) = 0$$

$$\Rightarrow c = \frac{1}{3}, 1$$

$$\Rightarrow c = \frac{1}{3} \in (0, 1)$$

Hence, Lagrange's mean value theorem is verified.

(iii) $f(x) = x(x - 1)$ on $[1, 2]$

Solution:

Given $f(x) = x(x - 1)$ on $[1, 2]$

$$= x^2 - x$$

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is

a polynomial function. So it is continuous in $[1, 2]$ and differentiable in $(1, 2)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (1, 2)$ such that:

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$
$$\Rightarrow f'(c) = \frac{f(2) - f(1)}{1}$$

$$f(x) = x^2 - x$$

Differentiating with respect to x

$$f'(x) = 2x - 1$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2c - 1$$

For $f(2)$, put the value of $x = 2$ in $f(x)$

$$f(2) = (2)^2 - 2$$

$$= 4 - 2$$

$$= 2$$

For $f(1)$, put the value of $x = 1$ in $f(x)$:

$$f(1) = (1)^2 - 1$$

$$= 1 - 1$$

$$= 0$$

$$\therefore f'(c) = f(2) - f(1)$$

$$\Rightarrow 2c - 1 = 2 - 0$$

$$\Rightarrow 2c = 2 + 1$$

$$\Rightarrow 2c = 3$$

$$\Rightarrow c = \frac{3}{2} \in (1, 2)$$

Hence, Lagrange's mean value theorem is verified.

(iv) $f(x) = x^2 - 3x + 2$ on $[-1, 2]$

Solution:

Given $f(x) = x^2 - 3x + 2$ on $[-1, 2]$

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is a polynomial function. So it is continuous in $[-1, 2]$ and differentiable in $(-1, 2)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (-1, 2)$ such that:

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(-1)}{2 + 1}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(-1)}{3}$$

$$f(x) = x^2 - 3x + 2$$

Differentiating with respect to x

$$f'(x) = 2x - 3$$

For $f'(c)$, put the value of $x = c$ in $f'(x)$:

$$f'(c) = 2c - 3$$

For $f(2)$, put the value of $x = 2$ in $f(x)$

$$f(2) = (2)^2 - 3(2) + 2$$

$$= 4 - 6 + 2$$

$$= 0$$

For $f(-1)$, put the value of $x = -1$ in $f(x)$:

$$f(-1) = (-1)^2 - 3(-1) + 2$$

$$= 1 + 3 + 2$$

$$= 6$$

$$f'(c) = \frac{f(2) - f(-1)}{3}$$

$$\Rightarrow 2c - 3 = \frac{0 - 6}{3}$$

$$\Rightarrow 2c = \frac{-6}{3} + 3$$

$$\Rightarrow 2c = -2 + 3$$

$$\Rightarrow 2c = -1$$

$$\Rightarrow c = \frac{-1}{2} \in (-1, 2)$$

Hence, Lagrange's mean value theorem is verified.

(v) $f(x) = 2x^2 - 3x + 1$ on $[1, 3]$

Solution:

Given $f(x) = 2x^2 - 3x + 1$ on $[1, 3]$

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is a polynomial function. So it is continuous in $[1, 3]$ and differentiable in $(1, 3)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (1, 3)$ such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

$$f(x) = 2x^2 - 3x + 1$$

Differentiating with respect to x

$$f'(x) = 2(2x) - 3$$

$$= 4x - 3$$

For $f'(c)$, put the value of $x = c$ in $f'(x)$:

$$f'(c) = 4c - 3$$

For $f(3)$, put the value of $x = 3$ in $f(x)$:

$$f(3) = 2(3)^2 - 3(3) + 1$$

$$= 2(9) - 9 + 1$$

$$= 18 - 9 = 9$$

For $f(1)$, put the value of $x = 1$ in $f(x)$:

$$f(1) = 2(1)^2 - 3(1) + 1$$

$$= 2(1) - 3 + 1$$

$$= 2 - 2 = 0$$

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow 4c - 3 = \frac{9 - 0}{2}$$

$$\Rightarrow 4c = \frac{9}{2} + 3$$

$$\Rightarrow 4c = 5 + 3$$

$$\Rightarrow 4c = 8$$

$$\Rightarrow c = \frac{8}{4} = 2 \in (1, 3)$$

Hence, Lagrange's mean value theorem is verified.

(vi) $f(x) = x^2 - 2x + 4$ on $[1, 5]$

Solution:

Given $f(x) = x^2 - 2x + 4$ on $[1, 5]$

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is a polynomial function. So it is continuous in $[1, 5]$ and differentiable in $(1, 5)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (1, 5)$ such that:

$$f'(c) = \frac{f(5) - f(1)}{5 - 1}$$
$$\Rightarrow f'(c) = \frac{f(5) - f(1)}{4}$$

$$f(x) = x^2 - 2x + 4$$

Differentiating with respect to x :

$$f'(x) = 2x - 2$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2c - 2$$

For $f(5)$, put the value of $x=5$ in $f(x)$:

$$f(5) = (5)^2 - 2(5) + 4$$
$$= 25 - 10 + 4$$
$$= 19$$

For $f(1)$, put the value of $x = 1$ in $f(x)$

$$f(1) = (1)^2 - 2(1) + 4$$
$$= 1 - 2 + 4$$
$$= 3$$

$$f'(c) = \frac{f(5) - f(1)}{4}$$

$$\Rightarrow 2c - 2 = \frac{19 - 3}{4}$$

$$\Rightarrow 2c = \frac{16}{4} + 2$$

$$\Rightarrow 2c = 4 + 2$$

$$\Rightarrow 2c = 6$$

$$\Rightarrow c = \frac{6}{2} = 3 \in (1, 5)$$

Hence, Lagrange's mean value theorem is verified.

(vii) $f(x) = 2x - x^2$ on $[0, 1]$

Solution:

Given $f(x) = 2x - x^2$ on $[0, 1]$

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is a polynomial function. So it is continuous in $[0, 1]$ and differentiable in $(0, 1)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (0, 1)$ such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow f'(c) = f(1) - f(0)$$

$$f(x) = 2x - x^2$$

Differentiating with respect to x :

$$f'(x) = 2 - 2x$$

For $f'(c)$, put the value of $x = c$ in $f'(x)$:

$$f'(c) = 2 - 2c$$

For $f(1)$, put the value of $x = 1$ in $f(x)$:

$$f(1) = 2(1) - (1)^2$$

$$= 2 - 1$$

$$= 1$$

For $f(0)$, put the value of $x = 0$ in $f(x)$:

$$f(0) = 2(0) - (0)^2$$

$$= 0 - 0$$

$$= 0$$

$$f'(c) = f(1) - f(0)$$

$$\Rightarrow 2 - 2c = 1 - 0$$

$$\Rightarrow -2c = 1 - 2$$

$$\Rightarrow -2c = -1$$

$$\Rightarrow c = \frac{-1}{-2} = \frac{1}{2} \in (0, 1)$$

Hence, Lagrange's mean value theorem is verified.

$$\text{(viii) } f(x) = (x - 1)(x - 2)(x - 3)$$

Solution:

Given $f(x) = (x - 1)(x - 2)(x - 3)$ on $[0, 4]$

$$= (x^2 - x - 2x + 3)(x - 3)$$

$$= (x^2 - 3x + 3)(x - 3)$$

$$= x^3 - 3x^2 + 3x - 3x^2 + 9x - 9$$

$$= x^3 - 6x^2 + 12x - 9 \text{ on } [0, 4]$$

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is a polynomial function. So it is continuous in $[0, 4]$ and differentiable in $(0, 4)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (0, 4)$ such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

$$f(x) = x^3 - 6x^2 + 12x - 9$$

Differentiating with respect to x:

$$f'(x) = 3x^2 - 6(2x) + 12$$

$$= 3x^2 - 12x + 12$$

For $f'(c)$, put the value of $x = c$ in $f'(x)$:

$$f'(c) = 3c^2 - 12c + 12$$

For $f(4)$, put the value of $x = 4$ in $f(x)$:

$$f(4) = (4)^3 - 6(4)^2 + 12(4) - 9$$

$$= 64 - 96 + 48 - 9$$

$$= 7$$

For $f(0)$, put the value of $x = 0$ in $f(x)$:

$$f(0) = (0)^3 - 6(0)^2 + 12(0) - 9$$

$$= 0 - 0 + 0 - 9$$

$$= -9$$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{7 - (-9)}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{7 + 9}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{16}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = 4$$

$$\Rightarrow 3c^2 - 12c + 8 = 0$$

We know that for quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 3 \times 8}}{2 \times 3}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{48}}{6}$$

$$\Rightarrow 3c^2 - 12c + 12 = 4$$

$$\Rightarrow 3c^2 - 12c + 8 = 0$$

We know that for quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 3 \times 8}}{2 \times 3}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{48}}{6}$$

$$\Rightarrow c = \frac{12 \pm 4\sqrt{3}}{6}$$

$$\Rightarrow c = \frac{12}{6} \pm \frac{4\sqrt{3}}{6}$$

$$\Rightarrow c = 2 \pm \frac{2\sqrt{3}}{3}$$

$$\Rightarrow c = 2 + \frac{2\sqrt{3}}{3}, 2 - \frac{2\sqrt{3}}{3} \in c$$

Hence, Lagrange's mean value theorem is verified.

(ix). $f(x) = \sqrt{25 - x^2}$ on $[-3, 4]$

Solution:

Given

$$f(x) = \sqrt{25 - x^2} \text{ on } [-3, 4]$$

$$\text{Here, } \sqrt{25 - x^2} > 0$$

$$\Rightarrow 25 - x^2 > 0$$

$$\Rightarrow x^2 < 25$$

$$\Rightarrow -5 < x < 5$$

$$\Rightarrow \sqrt{25 - x^2} \text{ has unique values for all } x \in (-5, 5)$$

$$\therefore f(x) \text{ is continuous in } [-3, 4]$$

$$f(x) = (25 - x^2)^{\frac{1}{2}}$$

Differentiating with respect to x:

$$f'(x) = \frac{1}{2} (25 - x^2)^{\left(\frac{1}{2} - 1\right)} \frac{d(25 - x^2)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2} (25 - x^2)^{-\frac{1}{2}} (-2x)$$

$$\Rightarrow f'(x) = \frac{-2x}{2 (25 - x^2)^{\frac{1}{2}}}$$

$$\Rightarrow 25 - x^2 > 0$$

$$\Rightarrow x^2 < 25$$

$$\Rightarrow -5 < x < 5$$

$$\Rightarrow \sqrt{25 - x^2} \text{ has unique values for all } x \in (-5, 5)$$

$\therefore f(x)$ is continuous in $[-3, 4]$

$$f(x) = (25 - x^2)^{\frac{1}{2}}$$

Differentiating with respect to x :

$$f'(x) = \frac{1}{2} (25 - x^2)^{\left(\frac{1}{2} - 1\right)} \frac{d(25 - x^2)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2} (25 - x^2)^{-\frac{1}{2}} (-2x)$$

$$\Rightarrow f'(x) = \frac{-2x}{2 (25 - x^2)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{-2x}{2 (25 - x^2)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

Here also,

$$\sqrt{25 - x^2} > 0$$

$$\Rightarrow -5 < x < 5$$

$\therefore f(x)$ is differentiable in $(-3, 4)$

So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point $c \in (-3, 4)$ such that:

$$f'(c) = \frac{f(4) - f(-3)}{4 - (-3)}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(-3)}{4 + 3}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(-3)}{7}$$

$$f(x) = (25 - x^2)^{\frac{1}{2}}$$

On differentiating with respect to x :

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

For $f'(c)$, put the value of $x = c$ in $f'(x)$:

$$f'(c) = \frac{-c}{\sqrt{25 - c^2}}$$

For $f(4)$, put the value of $x = 4$ in $f(x)$:

$$\Rightarrow f'(c) = \frac{f(4) - f(-3)}{4 + 3}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(-3)}{7}$$

$$f(x) = (25 - x^2)^{\frac{1}{2}}$$

On differentiating with respect to x:

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

For $f'(c)$, put the value of $x = c$ in $f'(x)$:

$$f'(c) = \frac{-c}{\sqrt{25 - c^2}}$$

For $f(4)$, put the value of $x = 4$ in $f(x)$:

$$f(4) = (25 - 4^2)^{\frac{1}{2}}$$

$$\Rightarrow f(4) = (25 - 16)^{\frac{1}{2}}$$

$$\Rightarrow f(4) = (9)^{\frac{1}{2}}$$

$$\Rightarrow f(4) = 3$$

For $f(-3)$, put the value of $x = -3$ in $f(x)$:

$$f(-3) = (25 - (-3)^2)^{\frac{1}{2}}$$

$$\Rightarrow f(-3) = (25 - 9)^{\frac{1}{2}}$$

$$\Rightarrow f(-3) = (16)^{\frac{1}{2}}$$

$$\Rightarrow f(-3) = 4$$

$$f'(c) = \frac{f(4) - f(-3)}{7}$$

$$\Rightarrow \frac{-c}{\sqrt{25-c^2}} = \frac{3-4}{7}$$

$$\Rightarrow \frac{-c}{\sqrt{25-c^2}} = \frac{-1}{7}$$

$$\Rightarrow -7c = -\sqrt{25-c^2}$$

Squaring on both sides:

$$\Rightarrow (-7c)^2 = (-\sqrt{25-c^2})^2$$

$$\Rightarrow 49c^2 = 25 - c^2$$

$$\Rightarrow 50c^2 = 25$$

$$\Rightarrow c^2 = \frac{25}{50}$$

$$\Rightarrow c^2 = \frac{1}{2}$$

$$\Rightarrow c = \pm \frac{1}{\sqrt{2}} \in (-3, 4)$$

Hence, Lagrange's mean value theorem is verified.

(x) $f(x) = \tan^{-1}x$ on $[0, 1]$

Solution:

Given $f(x) = \tan^{-1}x$ on $[0, 1]$

$\tan^{-1}x$ has unique value for all x between 0 and 1.

$\therefore f(x)$ is continuous in $[0, 1]$

$f(x) = \tan^{-1}x$

Differentiating with respect to x :

$$f'(x) = \frac{1}{1+x^2}$$

x^2 always has value greater than 0.

$$\Rightarrow 1 + x^2 > 0$$

$\therefore f(x)$ is differentiable in $(0, 1)$

So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point $c \in (0, 1)$ such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow f'(c) = f(1) - f(0)$$

$$f(x) = \tan^{-1} x$$

Differentiating with respect to x:

$$f'(x) = \frac{1}{1+x^2}$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = \frac{1}{1+c^2}$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = \tan^{-1} 1$$

$$\Rightarrow f(1) = \frac{\pi}{4}$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = \tan^{-1} 0$$

$$\Rightarrow f(0) = 0$$

$$f'(c) = f(1) - f(0)$$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4} - 0$$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4}$$

$$\Rightarrow 4 = \pi(1+c^2)$$

$$\Rightarrow 4 = \pi + \pi c^2$$

$$\Rightarrow -\pi c^2 = \pi - 4$$

$$\Rightarrow c^2 = \frac{\pi - 4}{-\pi}$$

$$\Rightarrow c^2 = \frac{4 - \pi}{\pi}$$

$$\Rightarrow c^2 = \frac{\pi - 4}{-\pi}$$

$$\Rightarrow c^2 = \frac{4 - \pi}{\pi}$$

$$\Rightarrow c = \sqrt{\frac{4}{\pi} - 1} \approx 0.52 \in (0, 1)$$

Hence, Lagrange's mean value theorem is verified.

$$(xi) f(x) = x + \frac{1}{x} \text{ on } [1, 3]$$

Solution:

Given

$$f(x) = x + \frac{1}{x} \text{ on } [1, 3]$$

$f(x)$ has unique values for all $x \in (1, 3)$

$\therefore f(x)$ is continuous in $[1, 3]$

$$f(x) = x + \frac{1}{x} \text{ on } [1, 3]$$

Differentiating with respect to x

$$f'(x) = 1 + (-1)(x)^{-2}$$

$$\Rightarrow f'(x) = 1 - \frac{1}{x^2}$$

$$\Rightarrow f'(x) = \frac{x^2 - 1}{x^2}$$

Here, $x^2 \neq 0$

$\Rightarrow f'(x)$ exists for all values except 0

$\therefore f(x)$ is differentiable in $(1, 3)$

So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point $c \in (1, 3)$ such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point $c \in (1, 3)$ such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

$$f(x) = x + \frac{1}{x}$$

On differentiating with respect to x :

$$f'(x) = \frac{x^2 - 1}{x^2}$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = \frac{c^2 - 1}{c^2}$$

For $f(3)$, put the value of $x = 3$ in $f(x)$:

$$f(3) = 3 + \frac{1}{3}$$

$$\Rightarrow f(3) = \frac{9+1}{3}$$

$$\Rightarrow f(3) = \frac{10}{3}$$

For $f(1)$, put the value of $x = 1$ in $f(x)$:

$$f(1) = 1 + \frac{1}{1}$$

$$\Rightarrow f(1) = 2$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

$$\Rightarrow \frac{c^2 - 1}{c^2} = \frac{\frac{10}{3} - 2}{2}$$

$$\Rightarrow 2(c^2 - 1) = c^2 \left(\frac{10}{3} - 2 \right)$$

$$\Rightarrow 2(c^2 - 1) = c^2 \left(\frac{10}{3} - 2 \right)$$

$$\Rightarrow 2(c^2 - 1) = c^2 \left(\frac{10 - 6}{3} \right)$$

$$\Rightarrow 2(c^2 - 1) = c^2 \left(\frac{4}{3} \right)$$

$$\Rightarrow 6(c^2 - 1) = 4c^2$$

$$\Rightarrow 6c^2 - 6 = 4c^2$$

$$\Rightarrow 6c^2 - 4c^2 = 6$$

$$\Rightarrow 2c^2 = 6$$

$$\Rightarrow c^2 = \frac{6}{2}$$

$$\Rightarrow c^2 = 3$$

$$\Rightarrow c = \pm \sqrt{3} \in (-3, 4)$$

Hence, Lagrange's mean value theorem is verified.

(xii) $f(x) = x(x + 4)^2$ on $[0, 4]$

Solution:

Given $f(x) = x(x + 4)^2$ on $[0, 4]$

$$= x[(x)^2 + 2(4)(x) + (4)^2]$$

$$= x(x^2 + 8x + 16)$$

$$= x^3 + 8x^2 + 16x \text{ on } [0, 4]$$

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is a polynomial function. So it is continuous in $[0, 4]$ and differentiable in $(0, 4)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (0, 4)$ such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

$$f(x) = x^3 + 8x^2 + 16x$$

Differentiating with respect to x:

$$f'(x) = 3x^2 + 8(2x) + 16$$

$$= 3x^2 + 16x + 16$$

For $f'(c)$, put the value of $x = c$ in $f'(x)$:

$$f'(c) = 3c^2 + 16c + 16$$

For $f(4)$, put the value of $x = 4$ in $f(x)$:

$$f(4) = (4)^3 + 8(4)^2 + 16(4)$$

$$= 64 + 128 + 64$$

$$= 256$$

For $f(0)$, put the value of $x = 0$ in $f(x)$:

$$f(0) = (0)^3 + 8(0)^2 + 16(0)$$

$$= 0 + 0 + 0$$

$$= 0$$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = \frac{256 - 0}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = \frac{256}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = 64$$

$$\Rightarrow 3c^2 + 16c + 16 - 64 = 0$$

$$\Rightarrow 3c^2 + 16c - 48 = 0$$

For quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{-(16) \pm \sqrt{(16)^2 - 4 \times 3 \times (-48)}}{2 \times 3}$$

$$\Rightarrow c = \frac{-16 \pm \sqrt{256 + 576}}{6}$$

$$\Rightarrow c = \frac{-16 \pm \sqrt{832}}{6}$$

$$\Rightarrow c = \frac{-16 \pm 8\sqrt{13}}{6}$$

$$\Rightarrow c = \frac{-16 \pm \sqrt{832}}{6}$$

$$\Rightarrow c = \frac{-16 \pm 8\sqrt{13}}{6}$$

$$\Rightarrow c = \frac{-16}{6} \pm \frac{8\sqrt{13}}{6}$$

$$\Rightarrow c = \frac{-8}{3} \pm \frac{4\sqrt{13}}{3}$$

$$\Rightarrow c = \frac{-8}{3} + \frac{4\sqrt{13}}{3}, \frac{-8}{3} - \frac{4\sqrt{13}}{3} \in c$$

Hence, Lagrange's mean value theorem is verified.

(xiii) $f(x) = \sqrt{x^2 - 4}$ on $[2, 4]$

Solution:

Given

$$f(x) = \sqrt{x^2 - 4} \text{ on } [2, 4]$$

Here,

$$\sqrt{x^2 - 4} > 0$$

$$\Rightarrow x^2 - 4 > 0$$

$$\Rightarrow x^2 > 4$$

$\Rightarrow f(x)$ exists for all values except $(-2, 2)$

$\therefore f(x)$ is continuous in $[2, 4]$

$$f(x) = \sqrt{x^2 - 4}$$

Differentiating with respect to x :

$$f'(x) = \frac{1}{2} (x^2 - 4)^{\left(\frac{1}{2} - 1\right)} \frac{d(x^2 - 4)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2} (x^2 - 4)^{-\frac{1}{2}} (2x)$$

$$\Rightarrow f'(x) = \frac{2x}{2 (x^2 - 4)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{1}{2} (x^2 - 4)^{-\frac{1}{2}} (2x)$$

$$\Rightarrow f'(x) = \frac{2x}{2 (x^2 - 4)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{x}{\sqrt{x^2 - 4}}$$

Here also, $\sqrt{x^2 - 4} > 0$

$\Rightarrow f'(x)$ exists for all values of x except $(2, -2)$

$\therefore f(x)$ is differentiable in $(2, 4)$

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (2, 4)$ such that:

$$f'(c) = \frac{f(4) - f(2)}{4 - 2}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(2)}{2}$$

$$f(x) = \sqrt{x^2 - 4}$$

On differentiating with respect to x :

$$f'(x) = \frac{x}{\sqrt{x^2 - 4}}$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = \frac{c}{\sqrt{c^2 - 4}}$$

For $f(4)$, put the value of $x = 4$ in $f(x)$:

$$f(4) = \sqrt{4^2 - 4}$$

$$\Rightarrow f(4) = (16 - 4)^{\frac{1}{2}}$$

$$\Rightarrow f(4) = \sqrt{12}$$

$$\Rightarrow f(4) = (16 - 4)^{\frac{1}{2}}$$

$$\Rightarrow f(4) = \sqrt{12}$$

$$\Rightarrow f(4) = 2\sqrt{3}$$

For $f(2)$, put the value of $x = 2$ in $f(x)$:

$$f(2) = \sqrt{2^2 - 4}$$

$$\Rightarrow f(2) = (4 - 4)^{\frac{1}{2}}$$

$$\Rightarrow f(2) = 0$$

$$\Rightarrow f'(c) = \frac{f(4) - f(2)}{2}$$

$$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \frac{2\sqrt{3} - 0}{2}$$

$$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \sqrt{3}$$

$$\Rightarrow c = (\sqrt{3})\sqrt{c^2 - 4}$$

Squaring both sides:

$$\Rightarrow (c)^2 = ((\sqrt{3})\sqrt{c^2 - 4})^2$$

$$\Rightarrow c^2 = 3(c^2 - 4)$$

$$\Rightarrow c^2 = 3c^2 - 12$$

$$\Rightarrow -2c^2 = -12$$

$$\Rightarrow c^2 = \frac{-12}{-2}$$

$$\Rightarrow c^2 = 6$$

$$\Rightarrow c = \pm\sqrt{6}$$

$$\Rightarrow c = \sqrt{6} \in (2, 4)$$

Hence, Lagrange's mean value theorem is verified.

$$\Rightarrow c = \sqrt{6} \in (2, 4)$$

Hence, Lagrange's mean value theorem is verified.

(xiv) $f(x) = x^2 + x - 1$ on $[0, 4]$

Solution:

Given $f(x) = x^2 + x - 1$ on $[0, 4]$

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is a polynomial function. So it is continuous in $[0, 4]$ and differentiable in $(0, 4)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (0, 4)$ such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$
$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

$$f(x) = x^2 + x - 1$$

Differentiating with respect to x :

$$f'(x) = 2x + 1$$

For $f'(c)$, put the value of $x = c$ in $f'(x)$:

$$f'(c) = 2c + 1$$

For $f(4)$, put the value of $x = 4$ in $f(x)$:

$$f(4) = (4)^2 + 4 - 1$$

$$= 16 + 4 - 1$$

$$= 19$$

For $f(0)$, put the value of $x = 0$ in $f(x)$:

$$f(0) = (0)^2 + 0 - 1$$

$$= 0 + 0 - 1$$

$$= -1$$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 2c + 1 = \frac{19 - (-1)}{4}$$

$$\Rightarrow 2c + 1 = \frac{20}{4}$$

$$\Rightarrow 2c + 1 = 5$$

$$\Rightarrow 2c = 5 - 1$$

$$\Rightarrow 2c = 4$$

$$\Rightarrow c = \frac{4}{2} = 2 \in (0, 4)$$

Hence, Lagrange's mean value theorem is verified.

$$\Rightarrow 2c + 1 = 5$$

$$\Rightarrow 2c = 5 - 1$$

$$\Rightarrow 2c = 4$$

$$\Rightarrow c = \frac{4}{2} = 2 \in (0, 4)$$

Hence, Lagrange's mean value theorem is verified.

(xv) $f(x) = \sin x - \sin 2x - x$ on $[0, \pi]$

Solution:

Given $f(x) = \sin x - \sin 2x - x$ on $[0, \pi]$

$\sin x$ and $\cos x$ functions are continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point $c \in (0, \pi)$ such that:

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

$$\Rightarrow f'(c) = \frac{f(\pi) - f(0)}{\pi}$$

$$f(x) = \sin x - \sin 2x - x$$

Differentiating with respect to x:

$$f(x) = \sin x - \sin 2x - x$$

$$\Rightarrow f'(x) = \cos x - \cos 2x \frac{d(2x)}{dx} - 1$$

$$\Rightarrow f'(x) = \cos x - 2\cos 2x - 1$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = \cos c - 2\cos 2c - 1$$

For $f(\pi)$, put the value of $x = \pi$ in $f(x)$:

$$f(\pi) = \sin \pi - \sin 2\pi - \pi$$

$$= 0 - 0 - \pi$$

$$= -\pi$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = \sin 0 - \sin 2(0) - 0$$

$$= -\pi$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = \sin 0 - \sin 2(0) - 0$$

$$= \sin 0 - \sin 0 - 0$$

$$= 0 - 0 - 0$$

$$= 0$$

$$f'(c) = \frac{f(\pi) - f(0)}{\pi}$$

$$\Rightarrow \cos c - 2\cos 2c - 1 = \frac{-\pi - 0}{\pi}$$

$$\Rightarrow \cos c - 2\cos 2c - 1 = -1$$

$$\Rightarrow \cos c - 2(2\cos^2 c - 1) = -1 + 1$$

$$\Rightarrow \cos c - 4\cos^2 c + 2 = 0$$

$$\Rightarrow 4\cos^2 c - \cos c - 2 = 0$$

For quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \cos c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 4 \times (-2)}}{2 \times 4}$$

$$\Rightarrow \cos c = \frac{1 \pm \sqrt{1 + 32}}{8}$$

$$\Rightarrow \cos c = \frac{1 \pm \sqrt{33}}{8}$$

$$\Rightarrow c = \cos^{-1} \left(\frac{1 \pm \sqrt{33}}{8} \right) \in (0, \pi)$$

Hence, Lagrange's mean value theorem is verified.

(xvi) $f(x) = x^3 - 5x^2 - 3x$ on $[1, 3]$

Solution:

Given $f(x) = x^3 - 5x^2 - 3x$ on $[1, 3]$

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is a polynomial function. So it is continuous in $[1, 3]$ and differentiable in $(1, 3)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (1, 3)$ such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

$$f(x) = x^3 - 5x^2 - 3x$$

Differentiating with respect to x :

$$f'(x) = 3x^2 - 5(2x) - 3$$

$$= 3x^2 - 10x - 3$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 3c^2 - 10c - 3$$

For $f(3)$, put the value of $x = 3$ in $f(x)$:

$$f(3) = (3)^3 - 5(3)^2 - 3(3)$$

$$= 27 - 45 - 9$$

$$= -27$$

For $f(1)$, put the value of $x = 1$ in $f(x)$:

$$f(1) = (1)^3 - 5(1)^2 - 3(1)$$

$$= 1 - 5 - 3$$

$$= -7$$

$$f'(c) = \frac{f(3) - f(1)}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{(-27) - (-7)}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{-27+7}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{-20}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = -10$$

$$\Rightarrow 3c^2 - 10c - 3 + 10 = 0$$

$$\Rightarrow 3c^2 - 10c + 7 = 0$$

$$\Rightarrow 3c^2 - 7c - 3c + 7 = 0$$

$$\Rightarrow c(3c - 7) - 1(3c - 7) = 0$$

$$\Rightarrow (3c - 7)(c - 1) = 0$$

$$\Rightarrow c = \frac{7}{3}, 1$$

$$\Rightarrow c = \frac{7}{3} \in (1, 3)$$

Hence, Lagrange's mean value theorem is verified.

$$\Rightarrow 3c^2 - 10c - 3 = -10$$

$$\Rightarrow 3c^2 - 10c - 3 + 10 = 0$$

$$\Rightarrow 3c^2 - 10c + 7 = 0$$

$$\Rightarrow 3c^2 - 7c - 3c + 7 = 0$$

$$\Rightarrow c(3c - 7) - 1(3c - 7) = 0$$

$$\Rightarrow (3c - 7)(c - 1) = 0$$

$$\Rightarrow c = \frac{7}{3}, 1$$

$$\Rightarrow c = \frac{7}{3} \in (1, 3)$$

Hence, Lagrange's mean value theorem is verified.

2. Discuss the applicability of Lagrange's mean value theorem for the function $f(x) = |x|$ on $[-1, 1]$.

Solution:

Given $f(x) = |x|$ on $[-1, 1]$

So $f(x)$ can be defined as
$$= \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

For differentiability at $x = 0$,

$$\text{LHD} = \lim_{x \rightarrow 0^-} \frac{f(0-h) - f(0)}{-h}$$

{Since $f(x) = -x, x < 0$ }

$$= \lim_{x \rightarrow 0^-} \frac{-(0-h) - 0}{-h}$$

$$= \lim_{x \rightarrow 0^-} \frac{h - 0}{-h}$$

$$= \lim_{x \rightarrow 0^-} \frac{h}{-h}$$

$$= -1$$

$$\text{RHD} = \lim_{x \rightarrow 0^+} \frac{f(0-h) - f(0)}{-h}$$

{Since $f(x) = x, x > 0$ }

{Since $f(x) = x, x > 0$ }

$$= \lim_{x \rightarrow 0^-} \frac{(0 - h) - 0}{-h}$$

$$= \lim_{x \rightarrow 0^-} \frac{-h - 0}{-h}$$

$$= \lim_{x \rightarrow 0^-} \frac{-h}{-h}$$

$$= 1$$

LHD \neq RHD

$\Rightarrow f(x)$ is not differential at $x=0$

\therefore Lagrange's mean value theorem is not applicable for the function $f(x) = |x|$ on $[-1, 1]$.

3. Show that the Lagrange's mean value theorem is not applicable to the function $f(x) = 1/x$ on $[-1, 1]$.

Solution:

Given

$$f(x) = \frac{1}{x} \text{ on } [-1, 1]$$

Here, $x \neq 0$

$\Rightarrow f(x)$ exists for all values of x except 0

$\Rightarrow f(x)$ is discontinuous at $x=0$

$\therefore f(x)$ is not continuous in $[-1, 1]$

Hence the Lagrange's mean value theorem is not applicable to the function $f(x) = 1/x$ on $[-1, 1]$

4. Verify the hypothesis and conclusion of Lagrange's mean value theorem for the function

$$f(x) = \frac{1}{4x - 1}, 1 \leq x \leq 4.$$

Solution:

Given

$$f(x) = \frac{1}{4x-1} \text{ on } [1, 4]$$

Where $4x - 1 > 0$

$f'(x)$ has unique values for all x except $\frac{1}{4}$

$\therefore f(x)$ is continuous in $[1, 4]$

$$f(x) = \frac{1}{4x-1}$$

Differentiating with respect to x :

$$f'(x) = (-1)(4x-1)^{-2}(4)$$

$$\Rightarrow f'(x) = -\frac{4}{(4x-1)^2}$$

Here, $4x - 1 > 0$

$f'(x)$ has unique values for all x except $\frac{1}{4}$

$\therefore f(x)$ is differentiable in $(1, 4)$

So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point $c \in (1, 4)$ such that:

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(1)}{3}$$

$$f(x) = \frac{1}{4x - 1}$$

On differentiating with respect to x:

$$f'(x) = -\frac{4}{(4x - 1)^2}$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = -\frac{4}{(4c - 1)^2}$$

For $f(4)$, put the value of $x = 4$ in $f(x)$:

$$f(c) = -\frac{4}{(4c - 1)^2}$$

For $f(4)$, put the value of $x = 4$ in $f(x)$:

$$f(4) = \frac{1}{4(4) - 1}$$

$$\Rightarrow f(4) = \frac{1}{16 - 1}$$

$$\Rightarrow f(4) = \frac{1}{15}$$

For $f(1)$, put the value of $x = 1$ in $f(x)$:

$$f(1) = \frac{1}{4(1) - 1}$$

$$\Rightarrow f(1) = \frac{1}{4 - 1}$$

$$\Rightarrow f(1) = \frac{1}{3}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(1)}{3}$$

$$\Rightarrow -\frac{4}{(4c - 1)^2} = \frac{\frac{1}{15} - \frac{1}{3}}{3}$$

$$\Rightarrow -3(4) = (4c - 1)^2 \left(\frac{1}{15} - \frac{1}{3} \right)$$

$$\Rightarrow -12 = (4c - 1)^2 \left(\frac{3 - 15}{45} \right)$$

$$\Rightarrow -12 = (4c - 1)^2 \left(\frac{-12}{45} \right)$$

$$\Rightarrow -12 \times \frac{45}{-12} = (4c - 1)^2$$

$$\Rightarrow -12 \times \frac{45}{-12} = (4c - 1)^2$$

$$\Rightarrow (4c - 1)^2 = 45$$

$$\Rightarrow (4c - 1) = \pm \sqrt{45}$$

$$\Rightarrow (4c - 1) = \pm 3\sqrt{5}$$

$$\Rightarrow c = \frac{\pm 3\sqrt{5} + 1}{4}$$

$$\Rightarrow c = \frac{3\sqrt{5} + 1}{4} \approx 1.92 \in (1, 4)$$

Hence, Lagrange's mean value theorem is verified.

5. Find a point on the parabola $y = (x - 4)^2$, where the tangent is parallel to the chord joining $(4, 0)$ and $(5, 1)$.

Solution:

Given $f(x) = (x - 4)^2$ on $[4, 5]$

This interval $[a, b]$ is obtained by x - coordinates of the points of the chord.

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, $f(x)$ is a polynomial function. So it is continuous in $[4, 5]$ and

differentiable in $(4, 5)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (4, 5)$ such that:

$$f'(c) = \frac{f(5) - f(4)}{5 - 4}$$

$$\Rightarrow f'(c) = \frac{f(5) - f(4)}{1}$$

$$f(x) = (x - 4)^2$$

Differentiating with respect to x :

$$f'(x) = 2(x - 4) \frac{d(x - 4)}{dx}$$

$$\Rightarrow f'(x) = 2(x - 4) \quad (1)$$

$$\Rightarrow f'(x) = 2(x - 4)$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2(c - 4)$$

For $f(5)$, put the value of $x=5$ in $f(x)$:

$$f(5) = (5 - 4)^2$$

$$= (1)^2$$

$$= 1$$

For $f(4)$, put the value of $x=4$ in $f(x)$:

$$f(4) = (4 - 4)^2$$

$$= (0)^2$$

$$= 0$$

$$f'(c) = f(5) - f(4)$$

$$\Rightarrow 2(c - 4) = 1 - 0$$

$$\Rightarrow 2c - 8 = 1$$

$$\Rightarrow 2c = 1 + 8$$

$$\Rightarrow c = \frac{9}{2} = 4.5 \in (4, 5)$$

We know that, the value of c obtained in Lagrange's Mean value Theorem is nothing but the value of x – coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points $(4, 0)$ and $(5, 1)$.

Now, put this value of x in $f(x)$ to obtain y :

$$y = (x - 4)^2$$

$$\Rightarrow y = \left(\frac{9}{2} - 4\right)^2$$

$$\Rightarrow y = \left(\frac{9-8}{2}\right)^2$$

$$\Rightarrow y = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow y = \frac{1}{4}$$

Hence, the required point is $\left(\frac{9}{2}, \frac{1}{4}\right)$