RD SHARMA Solutions for Class 12-science Maths Chapter 21 - Areas of Bounded Regions

Chapter 21 - Areas of bounded regions Exercise Ex. 21.1

Question 1

Using integration, find the area of the region bounded between the line $x = 2$ and the parabola $y^2 = 8x$.

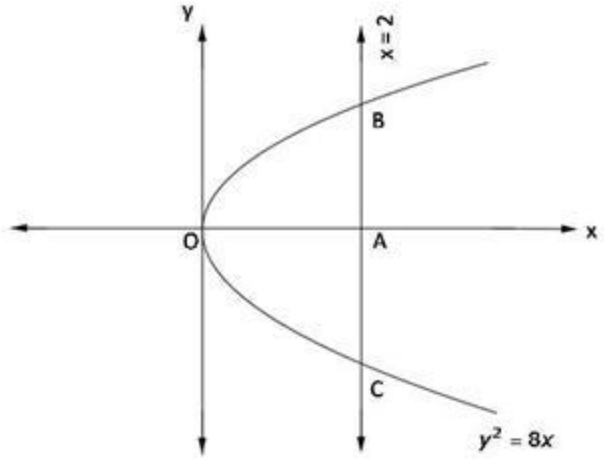
Solution 1

Given equations are

$x = 2$ ---(1)

and $y^2 = 8x$ ---(2)

Equation (1) represents a line parallel to y-axis and equation (2) represents a parabola with vertex at origin and x-axis as its axis, A rough sketch is given as below:-



We have to find the area of shaded region . We sliced it in vertical rectangle width of rectangle = Δx,

Length = $(y - 0) = y$

Area of rectangle = $y\,\Delta x$

This rectangle can move horizontal from $x = 0$ to $x = 2$

Required area = Shaded region $OCBO$

$$= 2(\text{Shaded region } OABO)$$

$$= 2\int_0^2 y\,dx$$

$$= 2\int_0^2 \sqrt{8x}\,dx$$

$$= 2.2\sqrt{2}\int_0^2 \sqrt{x}\,dx$$

$$= 4\sqrt{2}\left[\frac{2}{3}x\sqrt{x}\right]_0^2$$

$$= 4\sqrt{2}\left[\left(\frac{2}{3}.2\sqrt{2}\right) - \left(\frac{2}{3}.0.\sqrt{0}\right)\right]$$

$$= 4\sqrt{2}\left(\frac{4\sqrt{2}}{3}\right)$$

Required area = $\frac{32}{3}$ square units

Question 2

Using integration, find the area of the region bounded by the line $y - 1 = x$, the x-axis and the ordinates $x = -2$ and $x = 3$.

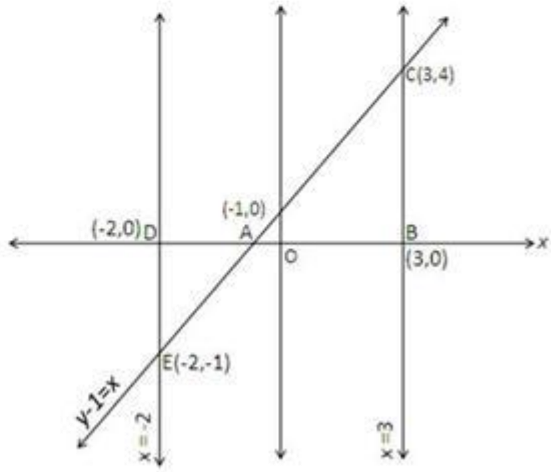
Solution 2

To find area of region bounded by x-axis the ordinates $x = -2$ and $x = 3$ and

$y - 1 = x$ $\quad ---(1)$

Equation (1) is a line that meets at axes at $(0,1)$ and $(-1,0)$.

A rough sketch of the curve is as under:-



Shaded region is required area.

Required area = Region $ABCA$ + Region $ADEA$

$$A = \int_{-1}^{3} y\,dx + \left|\int_{-2}^{-1} y\,dx\right|$$

$$= \int_{-1}^{3}(x+1)\,dx + \left|\int_{-2}^{-1}(x+1)\,dx\right|$$

$$= \left(\frac{x^2}{2}+x\right)_{-1}^{3} + \left|\left(\frac{x^2}{2}+x\right)_{-2}^{-1}\right|$$

$$= \left[\left(\frac{9}{2}+3\right)-\left(\frac{1}{2}-1\right)\right] + \left|\left(\frac{1}{2}-1\right)-(2-2)\right|$$

$$= \left[\frac{15}{2}+\frac{1}{2}\right] + \left|-\frac{1}{2}\right|$$

$$= 8 + \frac{1}{2}$$

$$A = \frac{17}{2} \text{ sq. units}$$

Question 3

Find the area of the region bounded by the parabola $y^2 = 4ax$ and the line $x = a$.

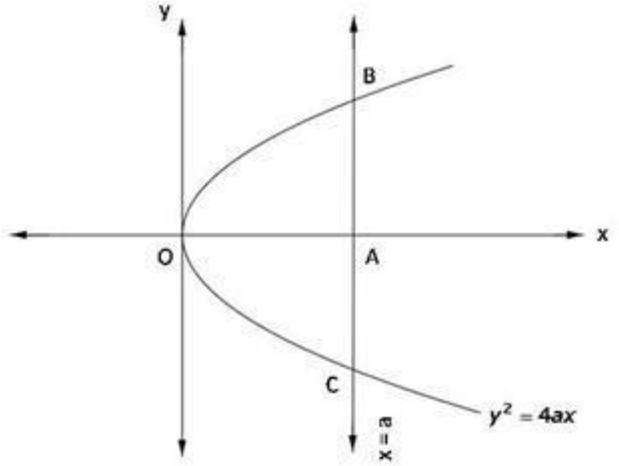
Solution 3

We have to find the area of the region bounded by

$x = a$ $\quad ---(1)$

and $y^2 = 4ax$ $\quad ---(2)$

Equation (1) represents a line parallel to y-axis and equation (2) represents a parabola with vertex at origin and axis as x-axis. A rough sketch of the two curves is as below:-



We have to find the area of the shaded region. Now, we slice it in rectangles.
Width $= \Delta x$, Length $= y - 0 = y$

Area rectangle $= y\Delta x$

This approximating rectangle can move from $x = 0$ to $x = a$.

$$\begin{aligned}
\text{Required area} &= \text{Region } OCBO \\
&= 2(\text{Region } OABO) \\
&= 2\int_0^a \sqrt{4ax}\, dx \\
&= 2.2\sqrt{a}\int_0^a \sqrt{x}\,dx \\
&= 4\sqrt{a}.\left(\frac{2}{3}x\sqrt{x}\right)_0^a \\
&= 4\sqrt{a}.\left(\frac{2}{3}a\sqrt{a}\right)
\end{aligned}$$

Required area $= \frac{8}{3}a^2$ square units

Question 4

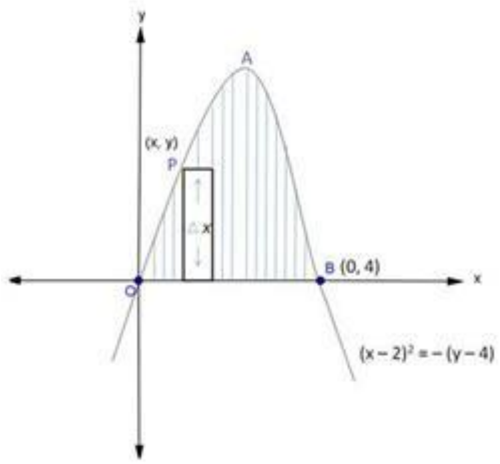
Find the area lying above the x-axis and under the parabola $y = 4x - x^2$.

Solution 4

We have to find area bounded by x-axis and parabola
$y = 4x - x^2$

$$\Rightarrow \quad x^2 - 4x + 4 = -y + 4$$
$$\Rightarrow \quad (x-2)^2 = -(y-4) \qquad ---(1)$$

Equation (1) represents a downward parabola with vertex $(2,4)$ and passing through $(0,0)$ and $(0,4)$. A rough sketch is as below:-



the shaded region represents the required area. We slice the region in approximation rectangles with width $= \Delta x$, length $= y - 0 = y$

Area of rectangle $= y\Delta x$.

This approximation rectangle slide from $x = 0$ to $x = a$, so

Required area = Region $OABO$

$$= \int_0^4 \left(4x - x^2\right) dx$$
$$= \left(4\frac{x^2}{2} - \frac{x^3}{3}\right)_0^4$$
$$= \left(\frac{4 \times 16}{2} - \frac{64}{3}\right) - (0 - 0)$$
$$= \frac{64}{6}$$

Required area $= \frac{32}{3}$ square units

Question 5

Draw a rough sketch to indicate the region bounded between the curve $y^2 = 4x$ and the line $x = 3$. Also, find the area of this region.

Solution 5

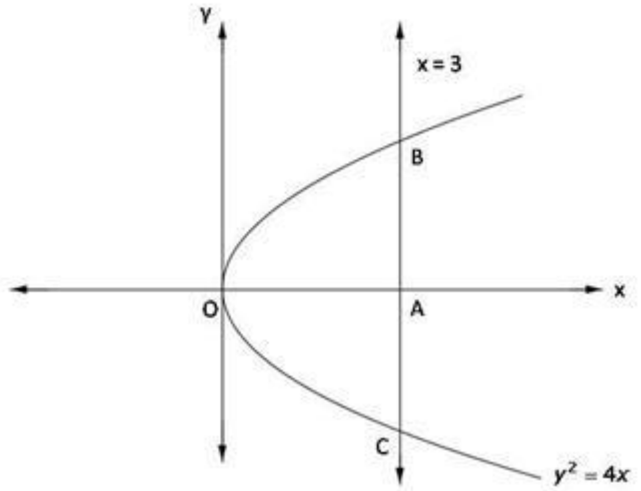
To find area bounded by

$y^2 = 4x$ --- (1)

and $x = 3$ --- (2)

Equation (1) represents a parabola with vertex at origin and axis as x-axis and equation (2) represents a line parallel to y-axis.

A rough sketch of the equations is as below:-



Shaded region represents the required area we slice this area with approximation rectangles with Width $= \Delta x$, length $= y - 0 = y$

Area of rectangle $= y\Delta x$.

This approximation rectangle can slide from $x = 0$ to $x = 3$, so

$$
\begin{aligned}
\text{Required area} &= \text{Region } OCBO \\
&= 2\left(\text{Region } OABO\right) \\
&= 2\int_0^3 y\,dx \\
&= 2\int_0^3 \sqrt{4x}\,dx \\
&= 4\int_0^3 \sqrt{x}\,dx \\
&= 4\left(\frac{2}{3}x\sqrt{x}\right)_0^3 \\
&= \frac{8}{3}.3\sqrt{3}
\end{aligned}
$$

Required area $= 8\sqrt{3}$ square units

Question 6

Make a rough sketch of the graph of the function $y = 4 - x^2$, $0 \le x \le 2$ and determine the area enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 2$.

Solution 6

We have to find the area enclosed by

$$y = 4 - x^2$$

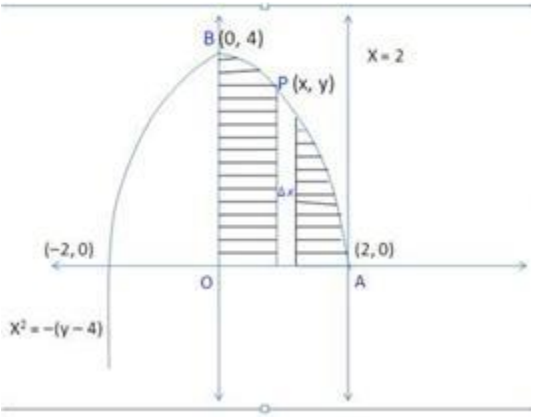
$$\Rightarrow \quad x^2 = -(y-4) \qquad ---(1)$$

$$x = 0 \qquad ---(2)$$

$$x = 2 \qquad ---(3)$$

Equation (1) represent a downward parabola with vertex at $(0,4)$ and passing through $(2,0), (-2,0)$. Equation (2) represents y-axis and equation (3) represents a line parallel to y-axis.

A rough sketch of curves is as below:-



Shaded region represents the required area. We slice this region into approximation rectangles with Width $= \Delta x$, length $= y - 0 = y$

Area of rectangle $= y\Delta x$.

This approximation rectangle move from $x = 0$ to $x = 2$, so

$$\begin{aligned}\text{Required area} &= (\text{Region } OABO) \\ &= \int_0^2 \left(4 - x^2\right) dx \\ &= \left(4x - \frac{x^3}{3}\right)_0^2 \\ &= \left[4(2) - \frac{(2)^3}{3}\right] - [0] \\ &= \left[\frac{24 - 8}{3}\right]\end{aligned}$$

Required area $= \dfrac{16}{3}$ square units

Question 7

Sketch the graph of $y = \sqrt{x+1}$ in $[0,4]$ and determine the area of the region enclosed by the curve, the x-axis and the lines $x = 0$, $x = 4$.

Solution 7

We have to find area enclosed by x-axis and

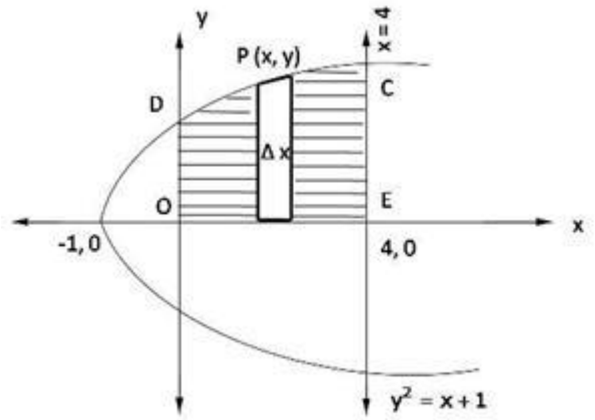
$$y = \sqrt{x+1}$$

$$\Rightarrow \quad y^2 = x + 1 \qquad ---(1)$$

$$\text{and } x = 0 \qquad ---(2)$$

$$x = 4 \qquad ---(3)$$

Equation (1) represent a parabola with vertex at $(-1,0)$ and passing through $(0,1)$ and $(0,-1)$. Equation (2) is y-axis and equation (3) is a line parallel to y-axis passing through $(4,0)$. So rough sketch of the curve is as below:-



We slice the required region in approximation rectangle with its Width $= \Delta x$, and length $= y - 0 = y$

Area of rectangle $= y\Delta x$.

Approximation rectangle moves from $x = 0$ to $x = 4$. So

Required area = Shaded region

$$= (\text{Region } OECDO)$$

$$= \int_0^4 y\,dx$$

$$= \int_0^4 \sqrt{x+1}\,dx$$

$$= \left(\frac{2}{3}(x+1)\sqrt{x+1}\right)_0^4$$

$$= \frac{2}{3}\left[\left((4+1)\sqrt{4+1}\right) - \left((0+1)\sqrt{0+1}\right)\right]$$

Required area $= \frac{2}{3}\left[5\sqrt{5} - 1\right]$ square units

Thus, Required area = $\frac{2}{3}\left(5^{\frac{3}{2}} - 1\right)$ square units

Question 8

Find the area under the curve $y = \sqrt{6x + 4}$ above x-axis from $x = 0$ to $x = 2$. Draw a sketch of the curve also.

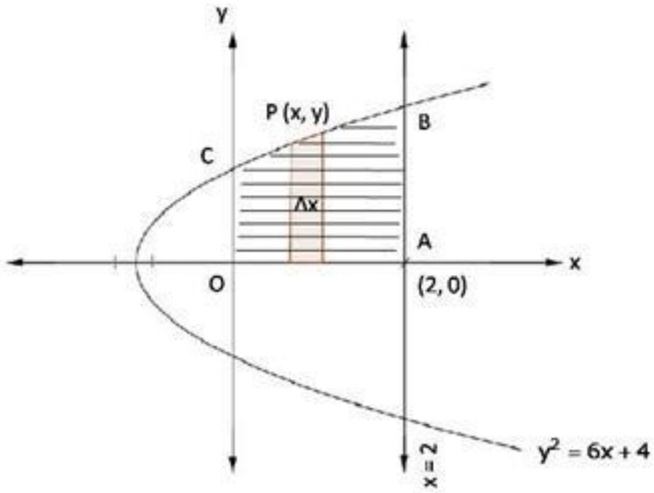
Solution 8

We have to find area enclosed by x-axis

$x = 0,\ x = 2$ --- (1)

and $y^2 = 6x + 4$ --- (2)

Equation (1) represents y-axis and a line parallel to y-axis passing through (2,0) respectively. Equation (2) represents a parabola with vertex at $\left(-\frac{2}{3}, 0\right)$ and passes through the points $(0,2), (0,-2)$, so rough sketch of the curves is as below:-



Shaded region represents the required area. It is sliced in approximation rectangle with its Width $= \Delta x$, and length $= (y - 0) = y$

Area of rectangle $= y\Delta x$.

This approximation rectangle slide from $x = 0$ to $x = 2$, so

Required area = Region $OABCO$

$$= \int_0^2 \sqrt{6x+4}\,dx$$

$$= \left\{\frac{2}{3}\frac{(6x+4)\sqrt{6x+4}}{6}\right\}_0^2$$

$$= \frac{1}{9}\left[\left((12+4)\sqrt{12+4}\right) - \left((0+4)\sqrt{0+4}\right)\right]$$

$$= \frac{1}{9}\left[16\sqrt{16} - 4\sqrt{4}\right]$$

$$= \frac{1}{9}(64 - 8)$$

Required area $= \frac{56}{9}$ square units

Question 9

Draw the rough sketch of $y^2+1=x$, $x \le 2$. find the areaenclosed by the curve and the line $x=2$.

Solution 9

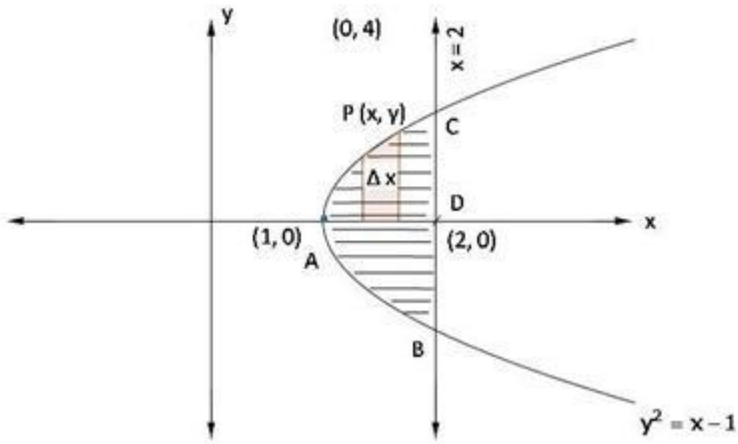
We have to find area enclosed by

$y^2 = x - 1$ ---(1)

and $x = 2$ ---(2)

Equation (1) is a parabola with vertex at (1,0) and axis as x-axis. Equation (2) represents a line parallel to y-axis passing through (2,0).

A rough sketch of curves is as below:-



Shaded region shows the required area. We slice it in approximation rectangle with its Width $=\Delta x$ and length $= y - 0 = y$

Area of the rectangle $= y\Delta x$.

This rectangle can slide from $x = 1$ to $x = 2$, so

Required area = Region $ABCA$

$= 2$ (Region $AOCA$)

$= 2\int_1^2 y\,dx$

$= 2\int_1^2 \sqrt{x-1}\,dx$

$= 2\left(\frac{2}{3}(x-1)\sqrt{x-1}\right)_1^2$

$= \frac{4}{3}\left[\left((2-1)\sqrt{2-1}\right) - \left((1-1)\sqrt{1-1}\right)\right]$

$= \frac{4}{3}(1-0)$

Required area $= \frac{4}{3}$ square units

Question 11

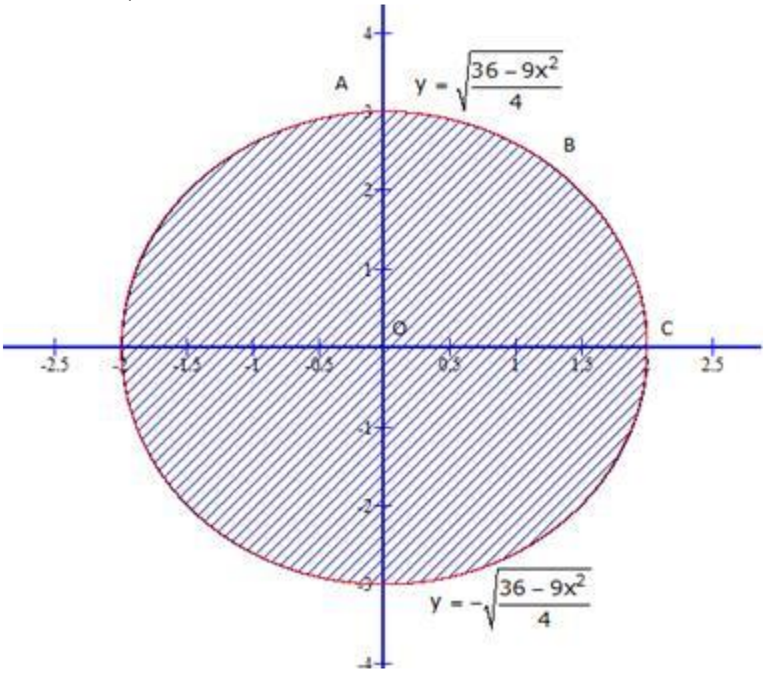
Sketch the region $\{(x, y): 9x^2 + 4y^2 = 36\}$ and find the area enclosed by it, using integration.

Solution 11

$9x^2 + 4y^2 = 36$

$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\Rightarrow y = \pm\sqrt{\frac{36 - 9x^2}{4}}$$



Area of Sector OABCO =

$$\int_0^2 \sqrt{\frac{36 - 9x^2}{4}}dx$$

$$= \frac{3}{2}\int_0^2 \sqrt{4 - x^2}dx$$

$$= \frac{3}{2}\left[\frac{x\sqrt{4 - x^2}}{2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \frac{3}{2}\left[\frac{2\sqrt{4 - 2^2}}{2} + \frac{2^2}{2}\sin^{-1}\left(\frac{2}{2}\right)\right] - \frac{3}{2}\left[\frac{0\sqrt{4 - 0^2}}{2} + \frac{2^2}{2}\sin^{-1}\left(\frac{0}{2}\right)\right]$$

$$= \frac{3}{2}.2.\frac{\pi}{2} - 0$$

$$= \frac{3\pi}{2}\text{ sq. units}$$

Area of the whole figure = 4 × Ar. D OABCO

$$= 4 \times \frac{3\pi}{2}$$

= 6p sq. units

Question 12

Draw a rough sketch of the graph of the function $y = 2\sqrt{1 - x^2}$, $x \in [0,1]$ and evaluate the area enclosed between the curve and the x-axis.

Solution 12

We have to find area enclosed between the curve and x-axis.

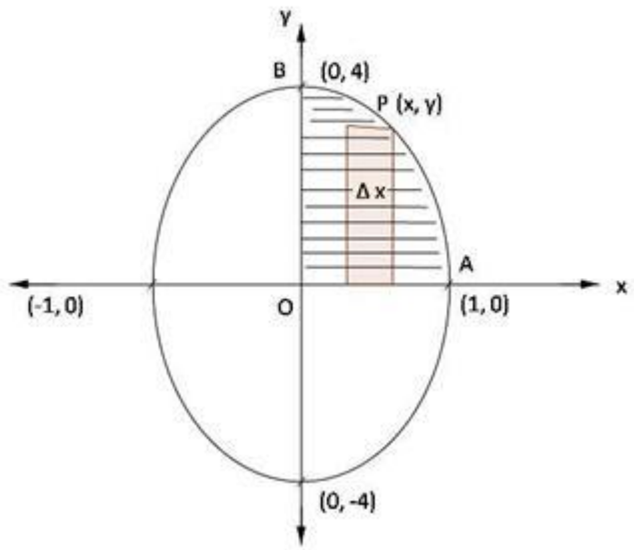
$y = 2\sqrt{1-x^2},\ x \in [0,1]$

$$\Rightarrow \qquad y^2 + 4x^2 = 4,\ x \in [0,1]$$

$$\Rightarrow \qquad \frac{x^2}{1} + \frac{y^2}{4} = 1,\ x \in [0,1] \qquad ---(1)$$

Equation (1) represents an ellipse with centre at origin and passes through $(\pm 1, 0)$ and $(0, \pm 2)$ and $x \in [0,1]$ as represented by region between y-axis and line $x = 1$.

A rough sketch of curves is as below:-



Shaded region represents the required. We slice it into approximation rectangles of Width $= \Delta x$ and length $= y$

Area of the rectangle $= y\Delta x$.

The approximation rectangle slides from $x = 0$ to $x = 1$, so

Required area = Region $OAPBO$

$$= \int_0^1 y\,dx$$

$$= \int_0^1 2\sqrt{1 - x^2}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_0^1$$

$$= 2\left[\left(\frac{1}{2}\sqrt{1-1} + \frac{1}{2}\sin^{-1}(1)\right) - (0+0)\right]$$

$$= 2\left[0 + \frac{1}{2}\cdot\frac{\pi}{2}\right]$$

Required area $= \frac{\pi}{2}$ square units

Question 13

Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included between the lines $x = 0$ and $x = a$.

Solution 13

To find area under the curves

$y = \sqrt{a^2 - x^2}$

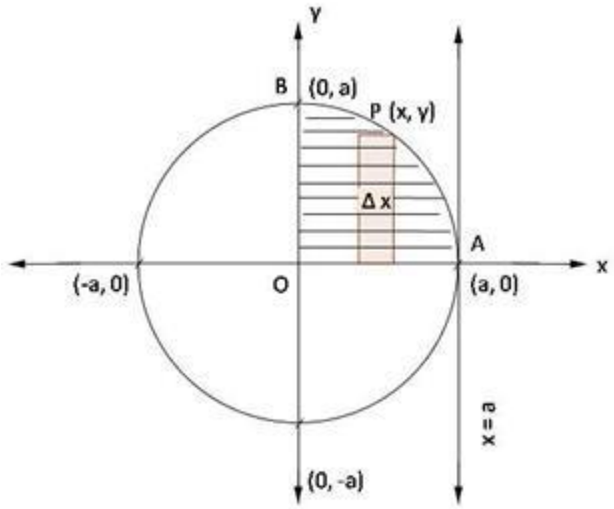
$\Rightarrow \quad x^2 + y^2 = a^2 \qquad ---(1)$

Between $x = 0 \qquad ---(2)$

$x = a \qquad ---(3)$

Equation (1) represents a circle with centre $(0,0)$ and passes axes at $(0, \pm a)$ $(\pm a, 0)$ equation (2) represents y-axis and equation $x = a$ represent a line parallel to y-axis passing through $(a,0)$.

A rough sketch of the curves is as below:-



Shaded region represents the required area. We slice it into approximation rectangles of Width $= \Delta x$ and length $= y - 0 = y$

Area of the rectangle $= y \Delta x$.

The approximation rectangle can slide from $x = 0$ to $x = a$, so

Required area = Region $OAPBO$

$$= \int_0^a y\,dx$$

$$= \int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \left[\left(\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}(1)\right) - (0)\right]$$

$$= \left[0 + \frac{a^2}{2}.\frac{\pi}{2}\right]$$

Required area $= \frac{\pi}{4}a^2$ square units

Question 14

Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$.

Solution 14

To find area bounded by x-axis and

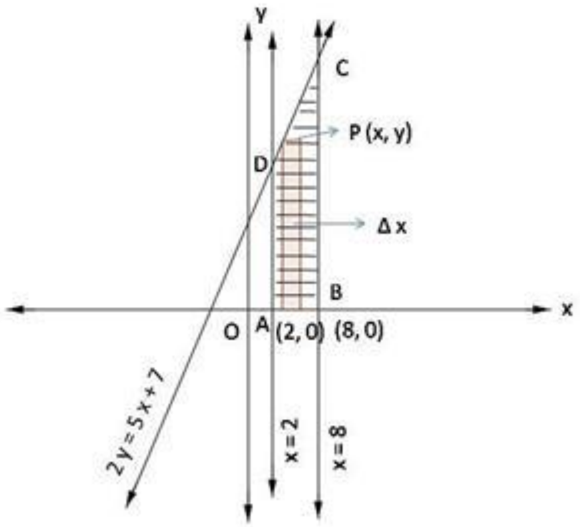
$$2y = 5x + 7 \qquad ---(1)$$

$$x = 2 \qquad ---(2)$$

$$x = 8 \qquad ---(3)$$

Equation (1) represents line passing through $\left(-\frac{7}{5}, 0\right)$ and $\left(0, \frac{7}{2}\right)$ equation $(2), (3)$ shows line parallel to y-axis passing through $(2,0), (8,0)$ respectively.

A rough sketch of curves is as below:-



Shaded region represents the required area. We slice the region into approximation rectangles of Width $= \Delta x$ and length $= y$

Area of the rectangle $= y\Delta x$.

This approximation rectangle slides from $x = 2$ to $x = 8$, so

Required area $=$ (Region $ABCDA$)

$$= \int_2^8 \left(\frac{5x+7}{2}\right)dx$$

$$= \frac{1}{2}\left(\frac{5x^2}{2}+7x\right)_2^8$$

$$= \frac{1}{2}\left[\left(\frac{5(8)^2}{2}+7(8)\right)-\left(\frac{5(2)^2}{2}+7(2)\right)\right]$$

$$= \frac{1}{2}\left[(160+56)-(10+14)\right]$$

$$= \frac{192}{2}$$

Required area $=$ 96 square units

Question 15

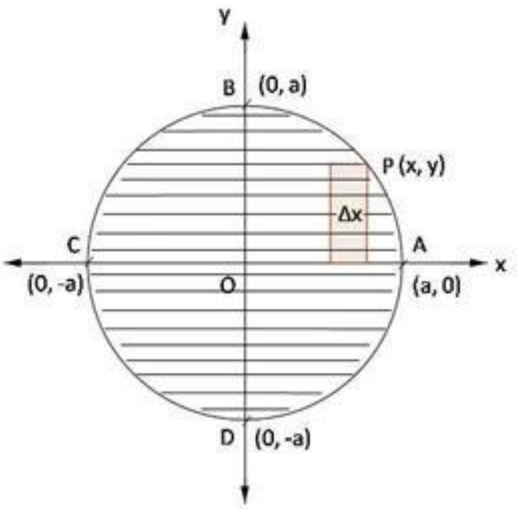
USing definite integrals, find the area of the circle $x^2 + y^2 = a^2$.

Solution 15

We have to find the area of circle

$x^2 + y^2 = a^2$ $\qquad\qquad\qquad ---(1)$

Equation (1) represents a circle with centre $(0,0)$ and radius a, so it meets the axes at $(\pm a, 0), (0, \pm a)$. A rough sketch of the curve is given below:-



Shaded region is the required area. We slice the region $AOBA$ in rectangles of width Δx and length $= y - 0 = y$

Area of rectangle $= y\Delta x$.

This approximation rectangle can slide from $x = 0$ to $x = a$, so

Required area = Region $ABCDA$

$= 4(\text{Region } ABOA)$

$= 4\left(\int_0^a y dx\right)$

$= 4\int_0^a \sqrt{a^2 - x^2} dx$

$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$= 4\left[\left(\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right) - (0 + 0)\right]$

$= 4\left[0 + \frac{a^2}{2}.\frac{\pi}{2}\right]$

$= 4\left(\frac{a^2\pi}{4}\right)$

Required area $= \pi a^2$ sq.units

Question 16

Using integration, find the area of the region bounded by the following curves, after making a rough sketch: $y = 1 + |x + 1|, x = -2, x = 3, y = 0$.

Solution 16

To find area enclosed by

$x = -2$, $x = 3$, $y = 0$ and $y = 1 + |x+1|$

$\Rightarrow \quad y = 1 + x + 1$, if $x + 1 \geq 0$

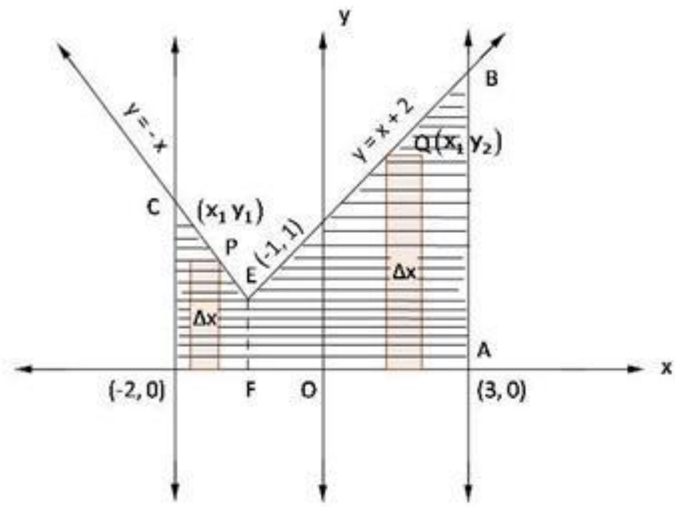
$\Rightarrow \quad y = 2 + x \qquad ---(1)$, if $x \geq -1$

And $\quad y = 1 - (x+1)$, if $x + 1 < 0$

$\Rightarrow \quad y = 1 - x - 1$, if $x < -1$

$\Rightarrow \quad y = -x \qquad ---(2)$, if $x < -1$

So, equation (1) is a straight line that passes through $(0,2)$ and $(-1,1)$. Equation (2) is a line passing through $(-1,1)$ and $(-2,2)$ and it is enclosed by line $x = 2$ and $x = 3$ which are lines parallel to y-axis and pass through $(2,0)$ and $(3,0)$ respectively $y = 0$ is x-axis. So, a rough sketch of the curves is given as:-



Shaded region represents the required area.

So, required area = Region $(ABECDFA)$

Required area = (region $ABEFA$ + region $ECDFE$) $\qquad ---(1)$

region $ECDFE$ is sliced into approximation rectangle with width Δx and length y_1.
Area of those approximation rectangle is $y_1 \Delta x$ and these slids from $x = -2$ to $x = -1$.

Region $ABEFA$ is sliced into approximation rectangle with width Δx and length y_2.
Area of those rectangle is $y_2 \Delta x$ which slides from $x = -1$ to $x = 3$. So, using equation (1),

$$\text{Required area} = \int_{-2}^{-1} y_1 dx + \int_{-1}^{3} y_2 dx$$
$$= \int_{-2}^{-1} (-x) dx + \int_{-1}^{3} (x+2) dx$$
$$= -\left[\frac{x^2}{2}\right]_{-2}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$$
$$= -\left[\frac{1}{2} - \frac{4}{2}\right] + \left[\left(\frac{9}{2} + 6\right) - \left(\frac{1}{2} - 2\right)\right]$$
$$= \frac{3}{2} + \left(\frac{21}{2} + \frac{3}{2}\right)$$
$$= \frac{27}{2}$$

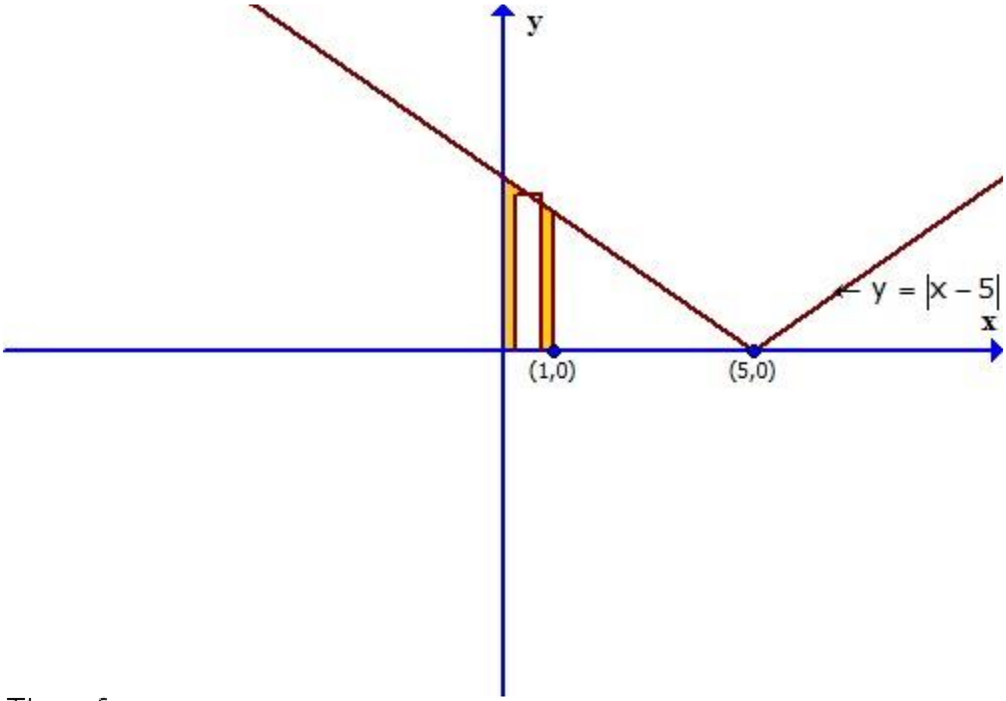
Required area $= \frac{27}{2}$ sq.units

Question 17

Sketch the graph $y = |x - 5|$. *Evaluate* $\int_0^1 |x - 5| dx$. *What does this value of the integral represent on the graph?*

Solution 17

Consider the sketch of the given graph: $y = |x - 5|$



Therefore,

$$\text{Required area} = \int_0^1 y\,dx$$

$$= \int_0^1 |x - 5|\,dx$$

$$= \int_0^1 -(x - 5)\,dx$$

$$= \left[\frac{-x^2}{2} + 5x\right]_0^1$$

$$= \left[-\frac{1}{2} + 5\right]$$

$$= \frac{9}{2}\,sq.\,units$$

Therefore, the given integral represents the area bounded by the curves, $x = 0, y = 0$, $x = 1$ *and* $y = -(x - 5)$.

Question 18

Sketch the graph of $y = |x + 3|$ and evaluate $\int_{-6}^{0} |x + 3|\,dx$

What dose this integral represent on the graph?.

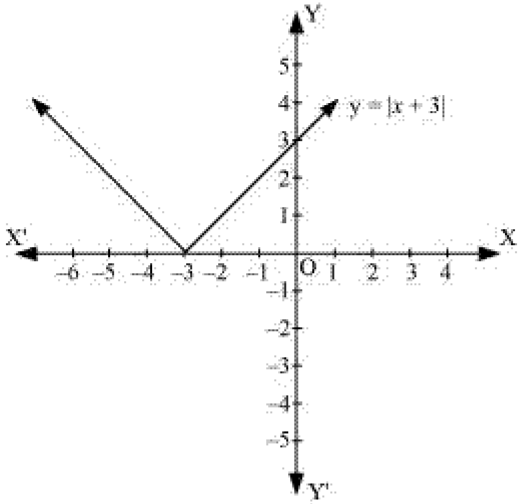
Solution 18

The given equation is $y = |x+3|$

The corresponding values of x and y are given in the following table.

x	-6	-5	-4	-3	-2	-1	0
y	3	2	1	0	1	2	3

On plotting these points, we obtain the graph of $y = |x+3|$ as follows.



It is known that, $(x+3) \le 0$ for $-6 \le x \le -3$ and $(x+3) \ge 0$ for $-3 \le x \le 0$

$$\therefore \int_{-6}^{0} |(x+3)| dx = -\int_{-6}^{-3}(x+3)dx + \int_{-3}^{0}(x+3)dx$$

$$= -\left[\frac{x^2}{2} + 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$$

$$= -\left[\left(\frac{(-3)^2}{2} + 3(-3)\right) - \left(\frac{(-6)^2}{2} + 3(-6)\right)\right] + \left[0 - \left(\frac{(-3)^2}{2} + 3(-3)\right)\right]$$

$$= -\left[-\frac{9}{2}\right] - \left[-\frac{9}{2}\right]$$

$$= 9$$

Question 19

Sketch the graph $y = |x+1|$. Evaluate $\int_{-4}^{2} |x+1| dx$. What does the value of this integral represent on this graph?

Solution 19

We have,

$$y = |x+1| = \begin{cases} x+1, & \text{if } x+1 \geq 0 \\ -(x+1), & \text{if } x+1 < 0 \end{cases}$$

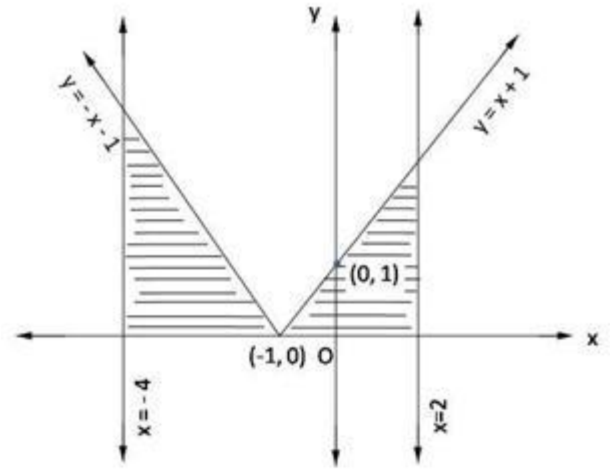
$$y = \begin{cases} (x+1), & \text{if } x \geq -1 \\ -x-1, & \text{if } x < -1 \end{cases}$$

$$\Rightarrow \quad y = x+1 \qquad (1)$$

$$\text{and } y = -x-1 \qquad (2)$$

Equation (1) represents a line which meets axes at $(0,1)$ and $(-1,0)$. Equation (2) represents a line passing through $(0,-1)$ and $(-1,0)$

A rough sketch is given below:-



$$\int_{-4}^{2} |x+1| dx = \int_{-4}^{-1} -(x+1) dx + \int_{-1}^{2} (x+1) dx$$

$$= -\left[\frac{x^2}{2}+x\right]_{-4}^{-1} + \left[\frac{x^2}{2}+x\right]_{-1}^{2}$$

$$= -\left[\left(\frac{1}{2}-1\right)-\left(\frac{16}{2}-4\right)\right]+\left[\left(\frac{4}{2}+2\right)-\left(\frac{1}{2}-1\right)\right]$$

$$= -\left[\left(-\frac{1}{2}-4\right)\right]+\left[4+\frac{1}{2}\right]$$

$$= \frac{9}{2}+\frac{9}{2}$$

$$= \frac{18}{2}$$

Required area = 9 sq. unit

Question 20

Find the area of the region bounded by the curve $xy - 3x - 2y - 10 = 0$, x-axis and the lines $x = 3$, $x = 4$.

Solution 20

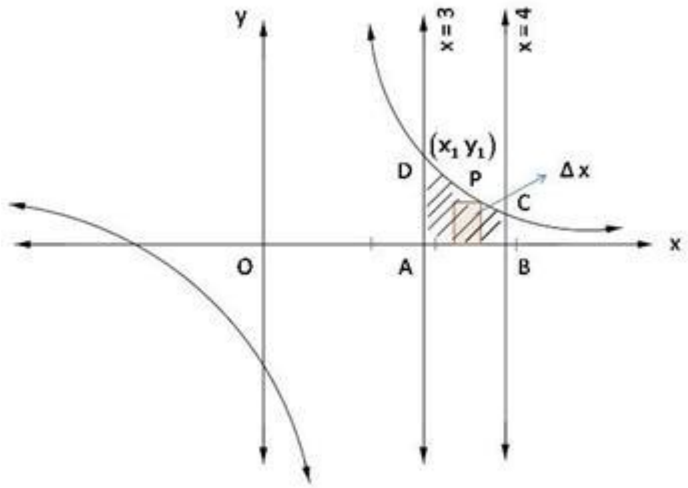
To find the area bounded by

x axis, $x = 3$, $x = 4$ and $xy - 3x - 2y - 10 = 0$

$\Rightarrow \quad y(x-2) = 3x + 10$

$\Rightarrow \quad y = \dfrac{3x+10}{x-2}$

A rough sketch of the curves is given below:-



Shaded region is required region.

It is sliced in rectangle with width $=\Delta x$ and length $= y$

Area of rectangle $= y\Delta x$

This approximation rectangle slide from $x = 3$ to $x = 4$. So,

Required area = Region $ABCDA$

$$= \int_3^4 y\,dx$$

$$= \int_3^4 \left(\frac{3x+10}{x-2}\right) dx$$

$$= \int_3^4 \left(3 + \frac{16}{x-2}\right) dx$$

$$= \left(3x\right)_3^4 + 16\left\{\log|x-2|\right\}_3^4$$

$$= (12 - 9) + 16\left(\log 2 - \log 1\right)$$

Required area $= (3 + 16\log 2)$ sq. units

Question 21

Draw a rough sketch of the curve $y = \frac{\pi}{2} + 2\sin^2 x$ and find the area between x-axis, the curve and the ordinates $x = 0, x = \pi$.

Solution 21

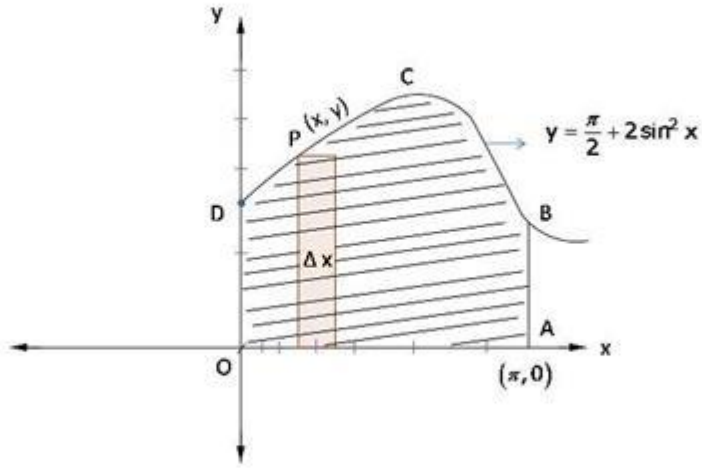
To find area bounded by $y = \frac{\pi}{2} + 2\sin^2 x$,

x-axis, $x = 0$ and $x = \pi$

A table for values of $y = \frac{\pi}{2} + 2\sin^2 x$ is:-

X	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\frac{\pi}{2} + 2\sin^2 x$	1.57	2.07	2.57	3.07	3.57	3.07	2.57	2.07	1.57

A rough sketch of the curves is given below:-



Shaded region represents required area. We slice it into rectangles of width $= \Delta x$ and length $= y$

Area of rectangle $= y\Delta x$

The approximation rectangle slides from $x = 0$ to $x = \pi$. So,

Required area $= (\text{Region } ABCDO)$

$$= \int_0^{\pi} y\,dx$$

$$= \int_0^{\pi} \left(\frac{\pi}{2} + 2\sin^2 x\right) dx$$

$$= \int_0^{\pi} \left(\frac{\pi}{2} + 1 - \cos 2x\right) dx$$

$$= \left[\frac{\pi}{2}x + x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \left\{\left(\frac{\pi^2}{2} + \pi - \frac{\sin 2x}{2}\right) - (0)\right\}$$

$$= \frac{\pi^2}{2} + \pi$$

Required area $= \frac{\pi}{2}(\pi + 2)$ sq. units

Question 22

Draw a rough sketch of the curve $y = \frac{x}{\pi} + 2\sin^2 x$ and find the area between the x-axis, the curve and the ordinates $x = 0$, $x = \pi$.

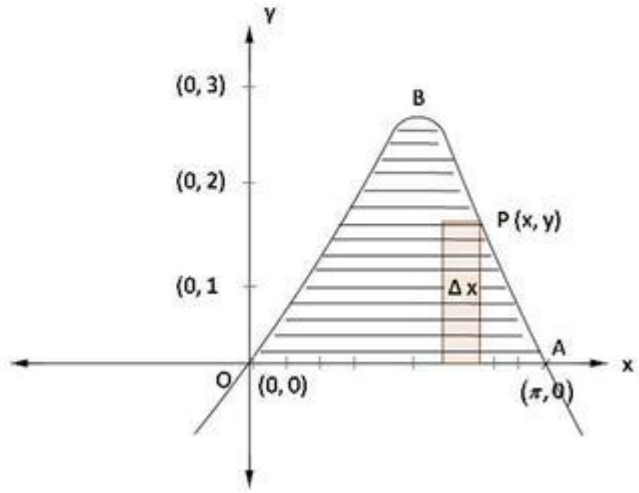
Solution 22

To find area between by x-axis, $x = 0$, $x = \pi$ and

$$y = \frac{x}{\pi} + 2\sin^2 x \qquad ---(1)$$

The table for equation (1) is:-

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
y	0	0.66	1.25	1.88	2.5	1.88	1.25	0.66	0



Shaded region is the required area. We slice the area into rectangles with width $= \Delta x$, length $= y$

Area of rectangle $= y\Delta x$

The approximation rectangle slides from $x = 0$ to $x = \pi$. So,

Required area $= (\text{Region } ABOA)$

$$= \int_0^\pi y\,dx$$

$$= \int_0^\pi \left(\frac{x}{\pi} + 2\sin^2 x\right)dx$$

$$= \int_0^\pi \left(\frac{x}{\pi} + 1 - \cos 2x\right)dx$$

$$= \left[\frac{x^2}{2\pi} + x - \frac{\sin 2x}{2}\right]_0^\pi$$

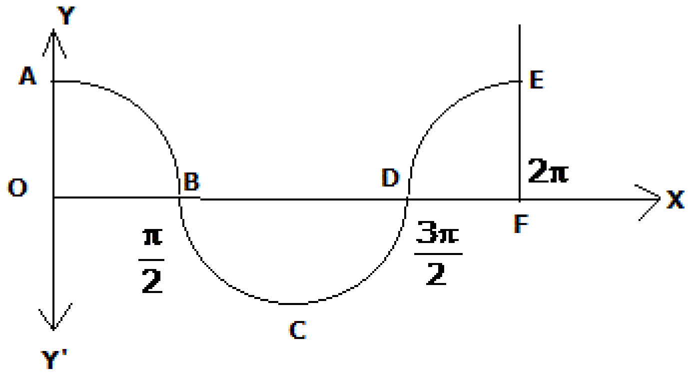
$$= \left(\frac{\pi^2}{2\pi} + \pi - 0\right) - (0)$$

Required area $= \dfrac{3\pi}{2}$ sq. units

Question 23

Find the area bounded by the curve $y=\cos x$ between $x=0$ and $x=2\pi$

Solution 23



From the figure, we notice that

The required area= area of the region OABO + area of the region BCDB + area of the region DEFD

Thus, the reqd. area = $\int_0^{\pi/2} \cos x \, dx + \left|\int_{\pi/2}^{3\pi/2} \cos x \, dx\right| + \int_{3\pi/2}^{2\pi} \cos x \, dx$

$= [\sin x]_0^{\pi/2} + \left|[\sin x]_{\pi/2}^{3\pi/2}\right| + [\sin x]_{3\pi/2}^{2\pi}$

$= \left[\sin\frac{\pi}{2} - \sin 0\right] + \left|\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right| + \left[\sin 2\pi - \sin\frac{3\pi}{2}\right]$

$= 1 + 2 + 1 = 4$ square units

Question 24

Show that the areas under the curves $y = \sin x$ and $y = \sin 2x$ between $x = 0$ and $x = \frac{\pi}{3}$ are in the ratio 2:3.

Solution 24

To find area under the curve

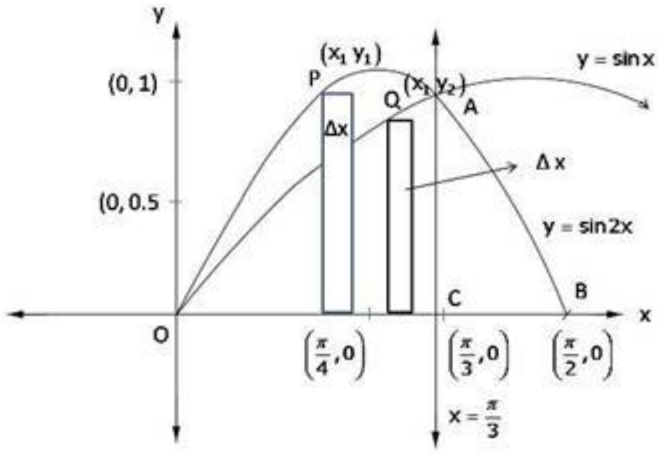
$y = \sin x$ $\quad ---(1)$

and $y = \sin 2x$ $\quad ---(2)$

between $x = 0$ and $x = \frac{\pi}{3}$.

X	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$Y=\sin x$	0	0.5	0.7	0.8	1
$Y=\sin 2x$	0	0.8	1	0.8	0

A rough sketch of the curve is given below:-



Area under curve $y = \sin 2x$

It is sliced in rectangles with width $= \Delta x$ and length $= y_1$

Area of rectangle $= y_1 \Delta x$

This approximation rectangle slides from $x = 0$ to $x = \frac{\pi}{3}$. So,

Required area = Region $OPACO$

$$A_1 = \int_0^{\frac{\pi}{3}} y_1 dx$$

$$= \int_0^{\frac{\pi}{3}} \sin 2x \, dx$$

$$= \left[\frac{-\cos 2x}{2}\right]_0^{\frac{\pi}{3}}$$

$$= -\left[-\frac{1}{4} - \frac{1}{2}\right]$$

$$A_1 = \frac{3}{4} \text{ sq.units}$$

Area under curve $y = \sin x$:

It is sliced in rectangles with width Δx and langth y_2

Area of rectangle = $y_2 \Delta x$

This approximation rectangle slides from $x = 0$ to $x = \frac{\pi}{3}$. So,

Required area = Region $OQACO$

$$= \int_0^{\frac{\pi}{3}} y_2 dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \, dx$$

$$= \left[-\cos x\right]_0^{\frac{\pi}{3}}$$

$$= -\left[\cos\frac{\pi}{3} - \cos 0\right]$$

$$= -\left(\frac{1}{2} - 1\right)$$

$A_2 = \frac{1}{2}$ sq.units

So,

$A_2 : A_1 = \frac{1}{2} : \frac{3}{4}$

$A_2 : A_1 = 2 : 3$

Question 25

Compare the areas under the curves $y = \cos^2 x$ and $y = \sin^2 x$ between $x = 0$ and $x = \pi$.

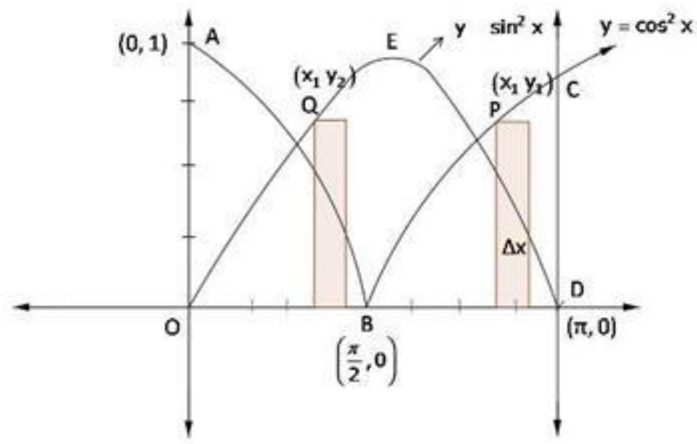
Solution 25

To compare area under curves

$y = \cos^2 x$ and $y = \sin^2 x$ between $x = 0$ and $x = \pi$.

Table for $y = \cos^2 x$ and $y = \sin^2 x$ is

X	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$Y=\cos^2 x$	1	0.75	0.5	0.25	0	0.25	0.5	0.75	1
$Y=\sin^2 x$	0	0.25	0.5	0.75	1	0.75	0.5	0.25	0



Area of region enclosed by $y = \cos^2 x$ and axis

A_1 = Region $OABO$ + Region $BCDB$

$$= 2\left(\text{Region } BCDB\right)$$
$$= 2\int_{\frac{\pi}{2}}^{\pi} \cos^2 x \, dx$$
$$= 2\int_{\frac{\pi}{2}}^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx$$
$$= \left[x - \frac{\sin 2x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$
$$= \left[\left(\pi - 0\right) - \left(\frac{\pi}{2} - 0\right)\right]$$

$$= \pi - \frac{\pi}{2}$$

$A_1 = \frac{\pi}{2}$ sq.units $\qquad ---(1)$

Area of region enclosed by $y = \sin^2 x$ and axis

A_2 = Region $OEDO$

$= \int_0^{\pi} \sin^2 x\, dx$

$= \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$

$= \frac{1}{2}\left[(\pi - 0) - (0)\right]$

$A_2 = \frac{\pi}{2}$ sq. units ---(2)

From equation (1) and (2),

$A_1 = A_2$

So,

Area enclosed by $y = \cos^2 x$ = Area enclosed by $y = \sin^2 x$

Question 26

Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates x = 0 and x = ae, where, b^2 =

Solution 26

The required area fig., of the region BOB'RFSB is enclosed by the ellipse and the lines x = 0 and x = ae.

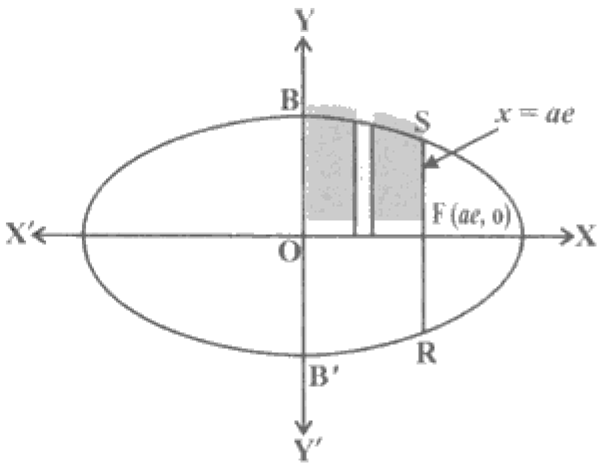
Note that the area of the region BOB'RFSB

$= 2\int_0^{ae} y\,dx = 2\frac{b}{a}\int_0^{ae}\sqrt{a^2 - x^2}\,dx$

$= \frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{ae}$

$= \frac{2b}{2a}\left[ae\sqrt{a^2 - a^2e^2} + a^2\sin^{-1} e\right]$

$= ab\left[e\sqrt{1 - e^2} + \sin^{-1} e\right]$

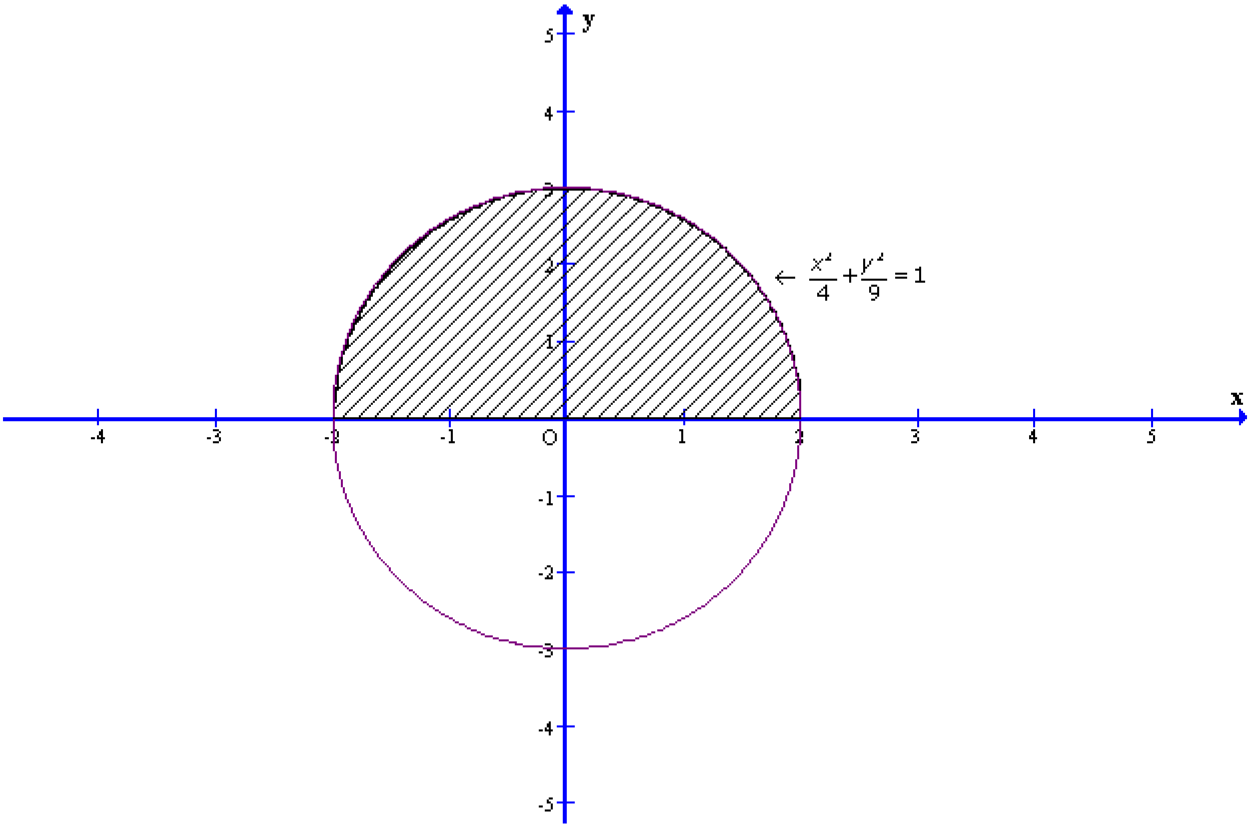


Question 10

Draw a rough sketch of the graph of the curve $\frac{x^2}{4}+\frac{y^2}{9}=1$

and evaluate the area of the region under the curve and above the x-axis.

Solution 10



It can be observed that ellipse is symmetrical about x-axis.

Area bounded by ellipse $= 2\int_0^2 y\,dx$

$= 2\int_0^2 3\sqrt{1-\frac{x^2}{4}}\,dx$

$= 3\int_0^2 \sqrt{4-x^2}\,dx$

$= 3\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$

$= 3\left[1(0)+2\sin^{-1}(1)-0-2\sin^{-1}(0)\right]$

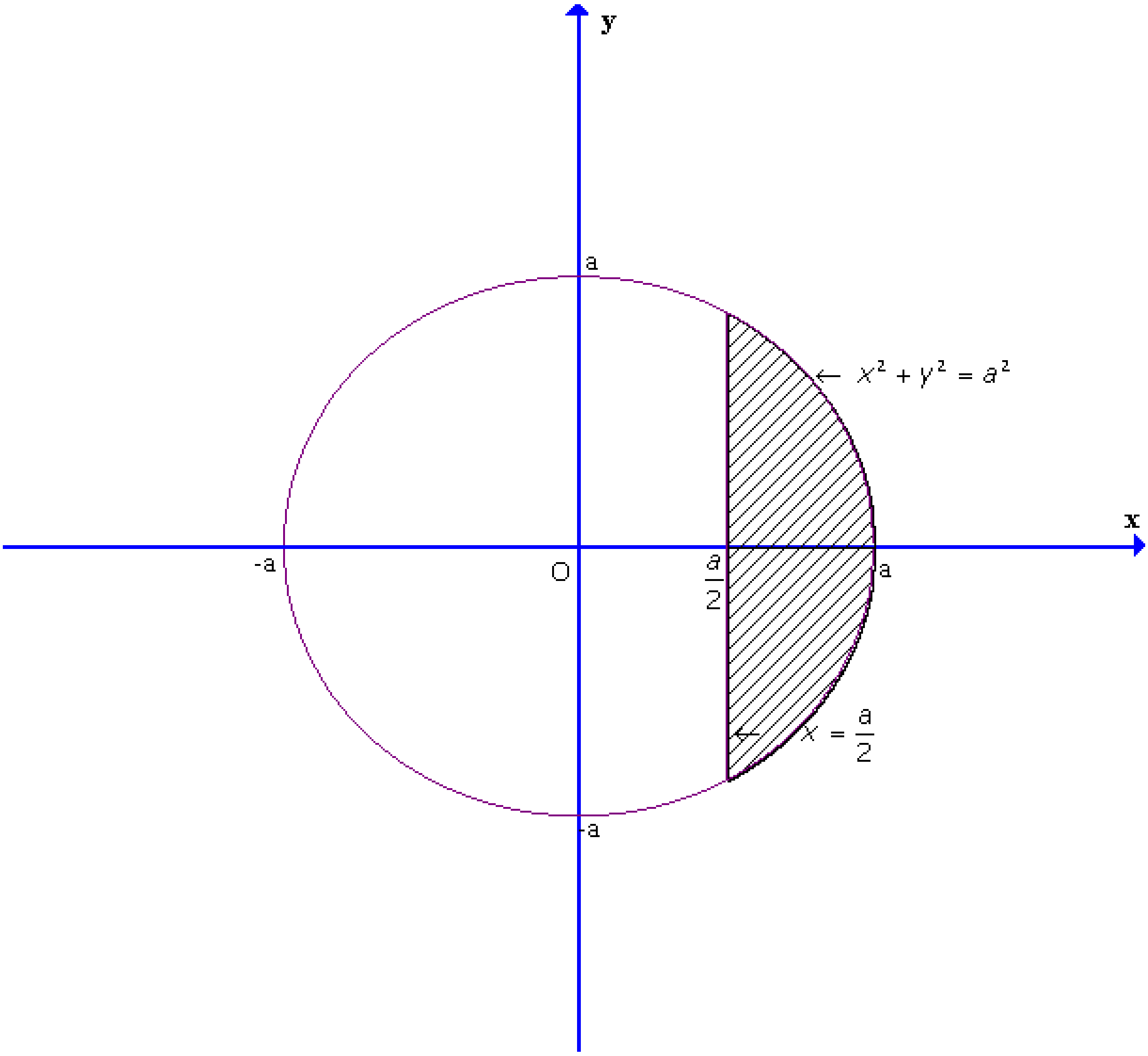
$= 3[\pi]$

$= 3\pi$ sq. units

Question 27

Find the area of the minor segment of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{2}$.

Solution 27



Area of the minor segment of the circle

$$=2\int_{\frac{a}{2}}^{a}\sqrt{a^2-x^2}\,dx$$

$$=2\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{2}\right]_{\frac{a}{2}}^{a}$$

$$=2\left[\frac{a}{2}(0)+\frac{a^2}{2}\sin^{-1}\left(\frac{a}{2}\right)-\frac{a}{4}\sqrt{a^2-\frac{a^2}{4}}-\frac{a^2}{2}\sin^{-1}\frac{a}{4}\right]$$

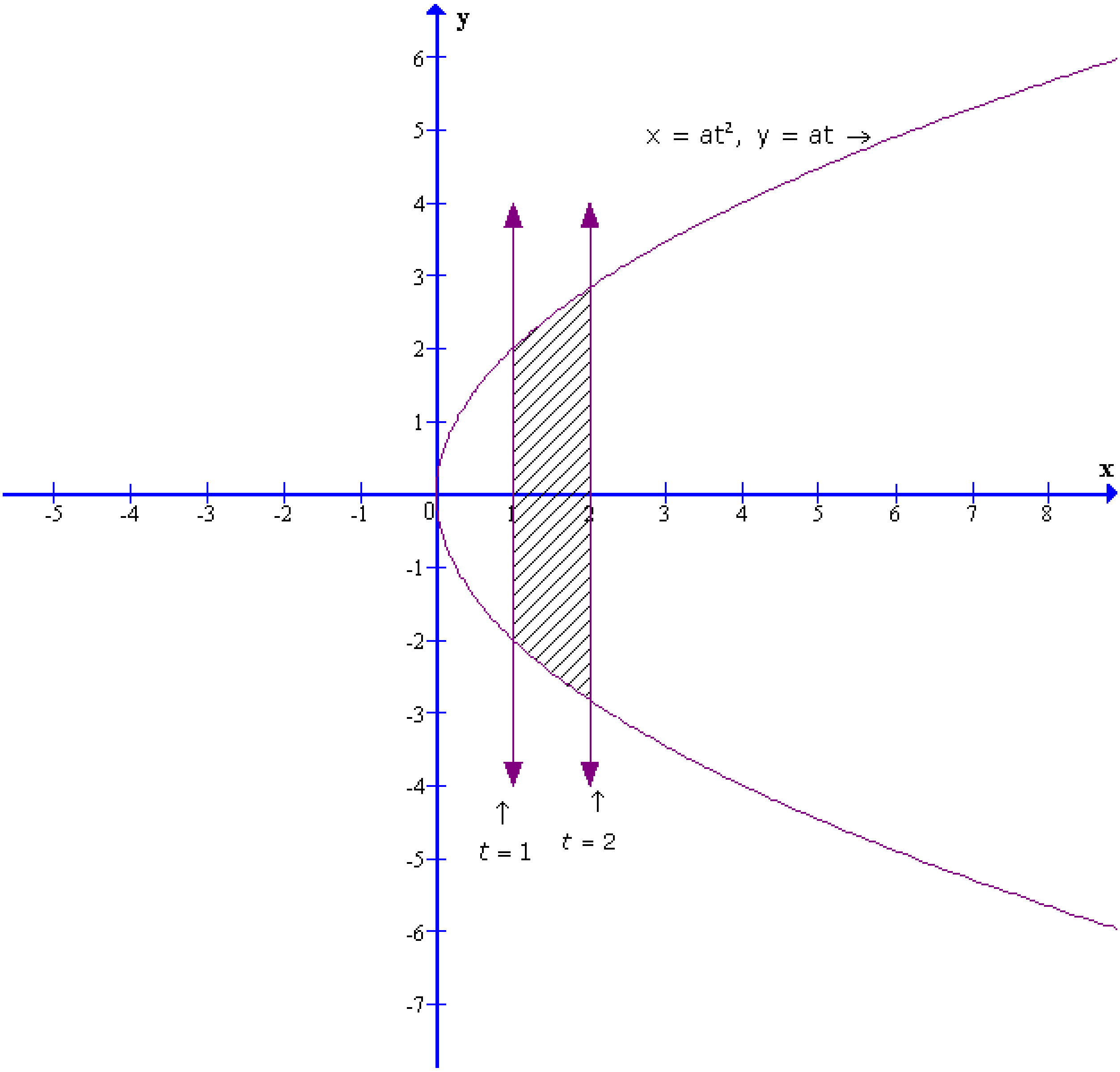
$$=2\left[\frac{a^2}{2}\sin^{-1}\left(\frac{a}{2}\right)-\frac{a}{4}\sqrt{a^2-\frac{a^2}{4}}-\frac{a^2}{2}\sin^{-1}\frac{a}{4}\right]$$

$$=\frac{a^2}{12}\left(4\pi-3\sqrt{3}\right)\text{sq. units}$$

Question 28

Find the area of the region bounded by the curve $x = at^2$, $y = 2at$ between the ordinates corresponding $t = 1$ and $t = 2$.

Solution 28



Area of the bounded region

$$= 2\int_1^2 y\,\frac{dx}{dt}dt$$

$$= 2\int_1^2 (2at)(2at)dt$$

$$= 8a^2\int_1^2 t^2 dt$$

$$= 8a^2\left[\frac{t^3}{3}\right]_1^2$$

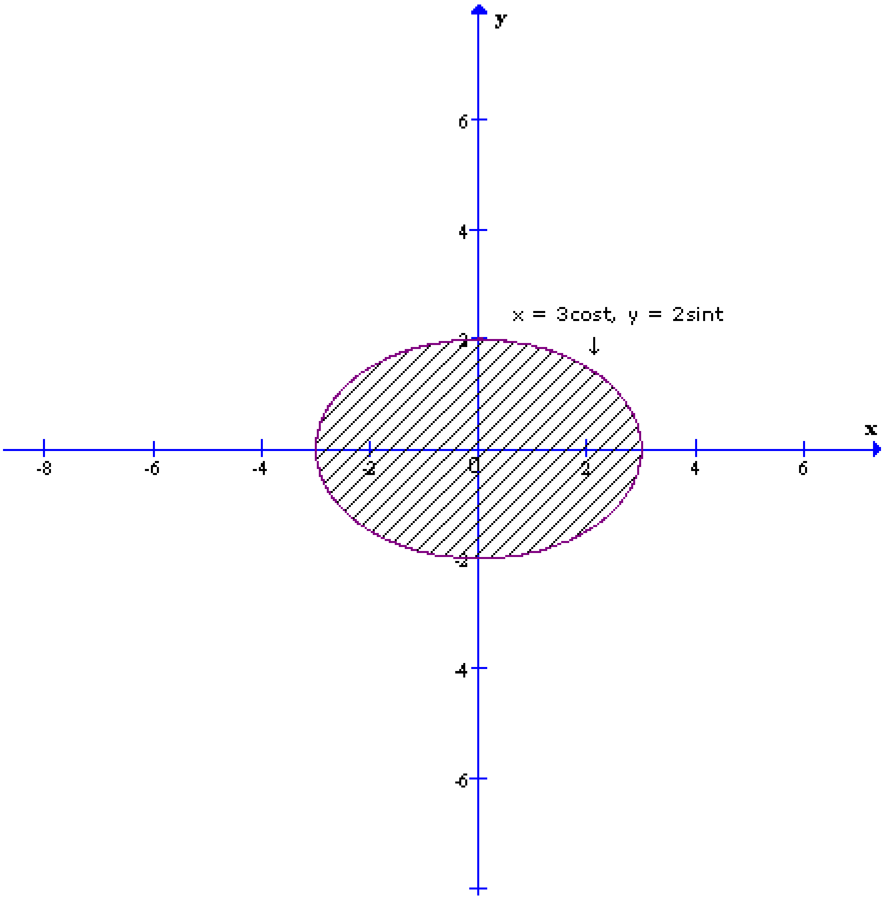
$$= 8a^2\left[\frac{8}{3}-\frac{1}{3}\right]$$

$$= \frac{56a^2}{3} \text{ sq. units}$$

Question 29

Find the area enclosed by the curve x = 3 cost,
y = 2 sin t.

Solution 29



Area of the bounded region

$$= 4\int_0^{\frac{\pi}{2}} 2\sin t\, dt$$

$$= -8[\cos t]_0^{\frac{\pi}{2}}$$

$$= -8[0-1]$$

$$= 8\text{sq units}$$

Note: Answer given in the book is incorrect.

Chapter 21 - Areas of Bounded Regions Exercise Ex. 21.2

Question 1

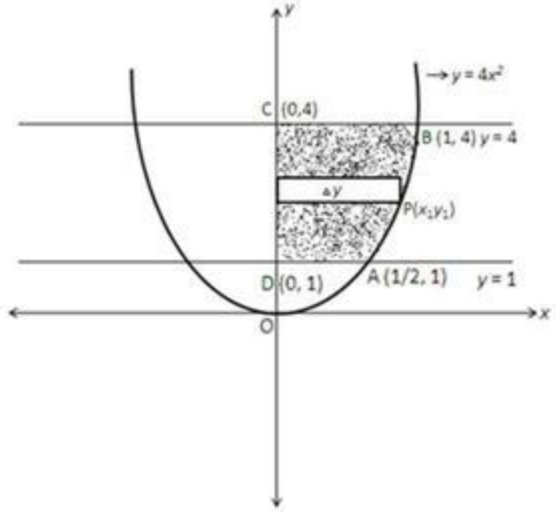
Find the area in the first quadrant bounded by the parabola $y = 4x^2$ and the lines $x = 0$, $y = 1$ and $y = 4$.

Solution 1

To find the area enclosed in first quadrant by

$x = 0$, $y = 1$, $y = 4$ and
$y = 4x^2$ $\qquad\qquad$ ---(1)

Equation (1) represents a parabola with vertex (0,0) and axis as y-axis. $x = 0$ is y-axis and $y = 1$, $y = 4$ are lines parallel to x-axis passing through (0,1) and (0, 4) respectively. A rough sketch of the curves is given as:-



Shaded region is required area and it is sliced into rectangles with area $x\Delta y$ it slides from $y = 1$ to $y = 4$, so

Required area = Region $ABCDA$

$$= \int_1^4 x\,dy$$

$$= \int_1^4 \sqrt{\frac{y}{4}}\,dy$$

$$= \frac{1}{2}\int_1^4 \sqrt{y}\,dy$$

$$= \frac{1}{2}\left[\frac{2}{3}y\sqrt{y}\right]_1^4$$

$$= \frac{1}{2}\left[\left(\frac{2}{3}.4.\sqrt{4}\right) - \left(\frac{2}{3}.1.\sqrt{1}\right)\right]$$

$$= \frac{1}{2}\left[\frac{16}{3} - \frac{2}{3}\right]$$

Required area $= \frac{7}{3}$ sq. units

Question 2

Find the area of the region bounded by $x^2 = 16y$, $y = 1$, $y = 4$ and the y-axis in the first quadrant.

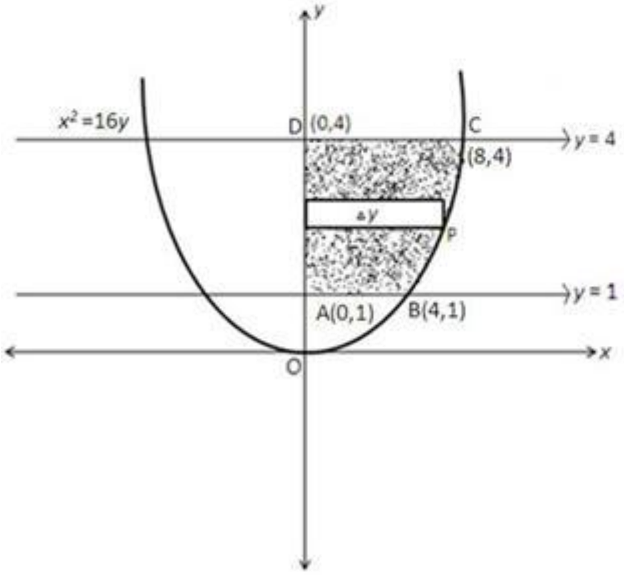
Solution 2

To find region in first quadrant bounded by $y = 1$, $y = 4$ and y-axis and

$x^2 = 16y$ ---(1)

Equation (1) represents a parabola with vertex (0,0) and axes as y-axis.

A rough sketch of the curves is as under:-



Shaded region is required area it is sliced in rectangles of area $x\Delta y$ which slides from $y = 1$ to $y = 4$, so

Required area = Region $ABCDA$

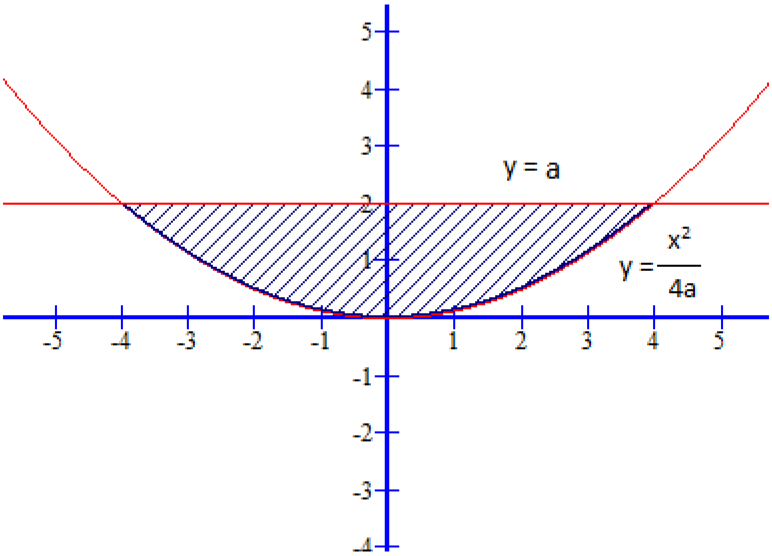
$$A = \int_1^4 x\,dy$$
$$= \int_1^4 4\sqrt{y}\,dy$$
$$= 4.\left[\frac{2}{3}y\sqrt{y}\right]_1^4$$
$$= 4.\left[\left(\frac{2}{3}.4\sqrt{4}\right) - \left(\frac{2}{3}.1.\sqrt{1}\right)\right]$$
$$= 4\left[\frac{16}{3} - \frac{2}{3}\right]$$

$$A = \frac{56}{3} \text{ sq. units}$$

Question 3

Find the area of the region bounded by $x^2 = 4ay$ and its latusrectum.

Solution 3



$$\text{Area of the region} = 2 \times \int_0^{2a} \left(a - \frac{x^2}{4a} \right) dx$$

$$= 2 \times \left[ax - \frac{x^3}{12a} \right]_0^{2a}$$

$$= 2\left[a(2a - 0) - \frac{(2a)^3 - 0^3}{12a} \right]$$

$$= 2\left[2a^2 - \frac{8a^3}{12a} \right]$$

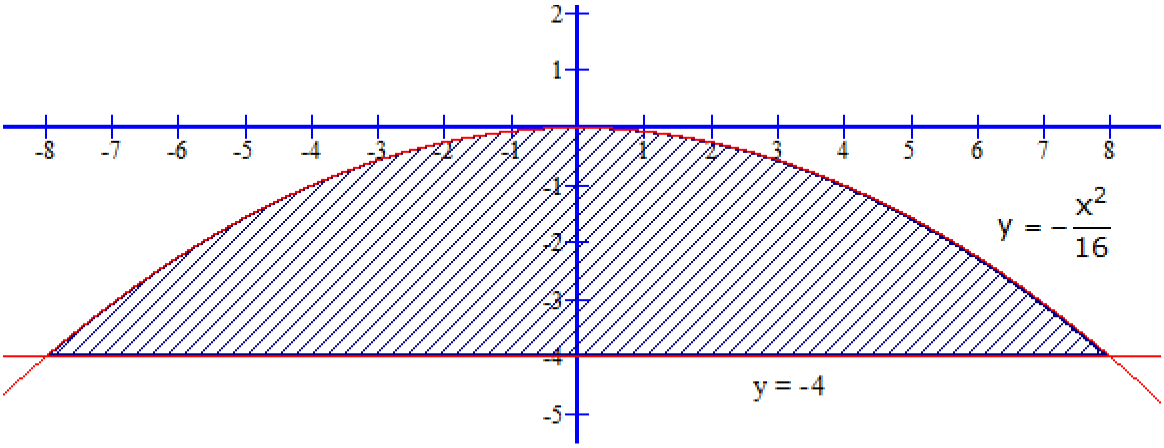
$$= 2\left[\frac{16a^3}{12a} \right]$$

$$= \frac{8}{3} a^2 \text{ sq. units}$$

Question 4

Find the area of the region bounded by $x^2 + 16y = 0$ and its latusrectum.

Solution 4



$$\text{Area of the region} = 2 \times \int_0^8 \left[-\frac{x^2}{16} - (-4)\right] dx$$

$$= 2 \times \left[-\frac{x^3}{48} + 4x\right]_0^8$$

$$= 2 \times \left[4x - \frac{x^3}{48}\right]_0^8$$

$$= 2 \times \left[4(8-0) - \frac{(8)^3 - 0^3}{48}\right]$$

$$= 2 \times \left[32 - \frac{512}{48}\right]$$

$$= 2 \times \left[32 - \frac{32}{3}\right]$$

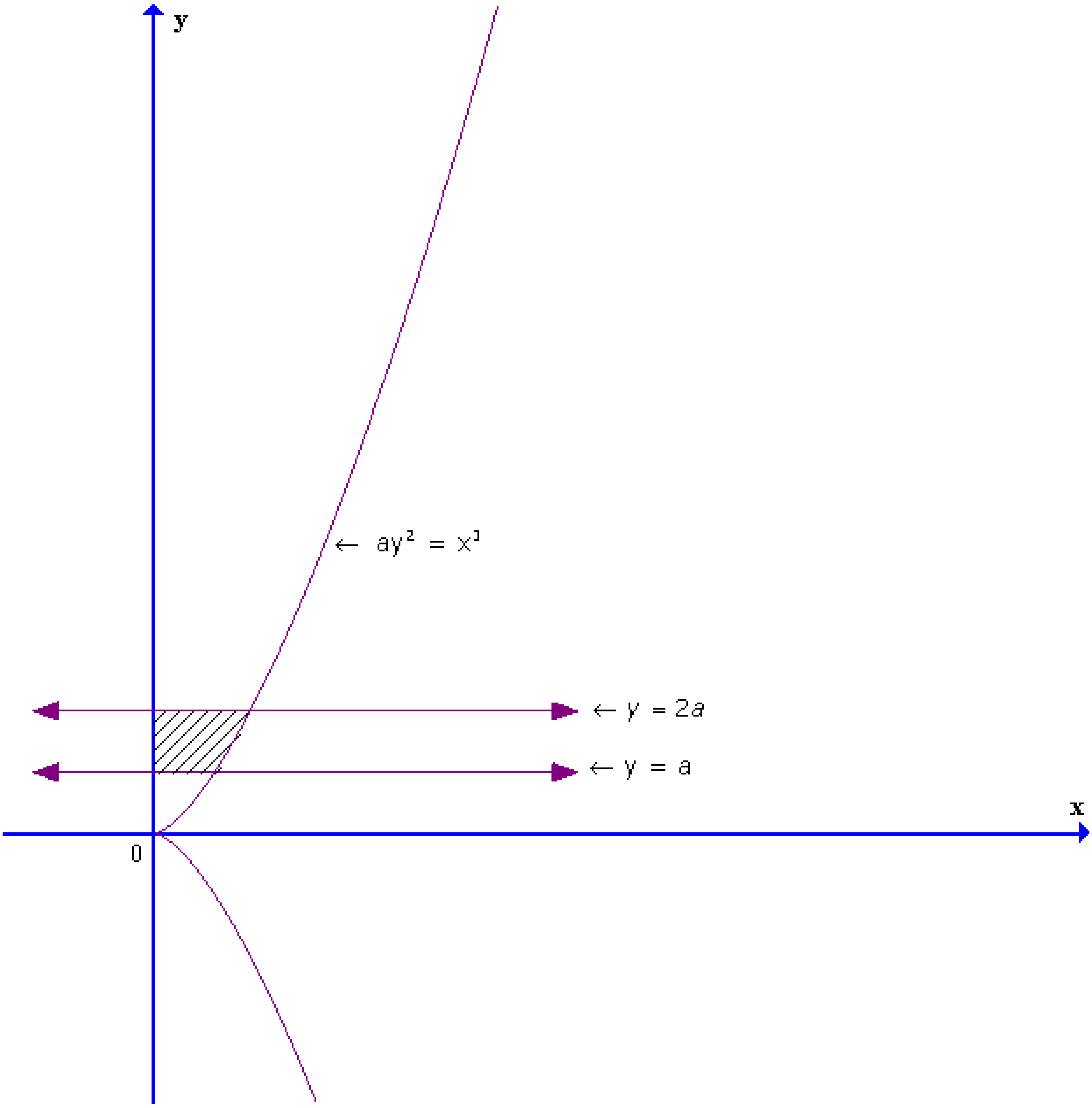
$$= 2 \times \left[\frac{96-32}{3}\right]$$

$$= 2 \times \frac{64}{3} = \frac{128}{3} \text{sq.units}$$

Question 5

Find the area of the region bounded by the curve $ay^2 = x^3$, the y-axis and the lines $y = a$ and $y = 2a$.

Solution 5



Area of the bounded region

$$= \int_a^{2a} \left(ay^2\right)^{\frac{1}{3}} dy$$

$$= a^{\frac{1}{3}} \int_a^{2a} y^{\frac{2}{3}} dy$$

$$= a^{\frac{1}{3}} \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^{2a}$$

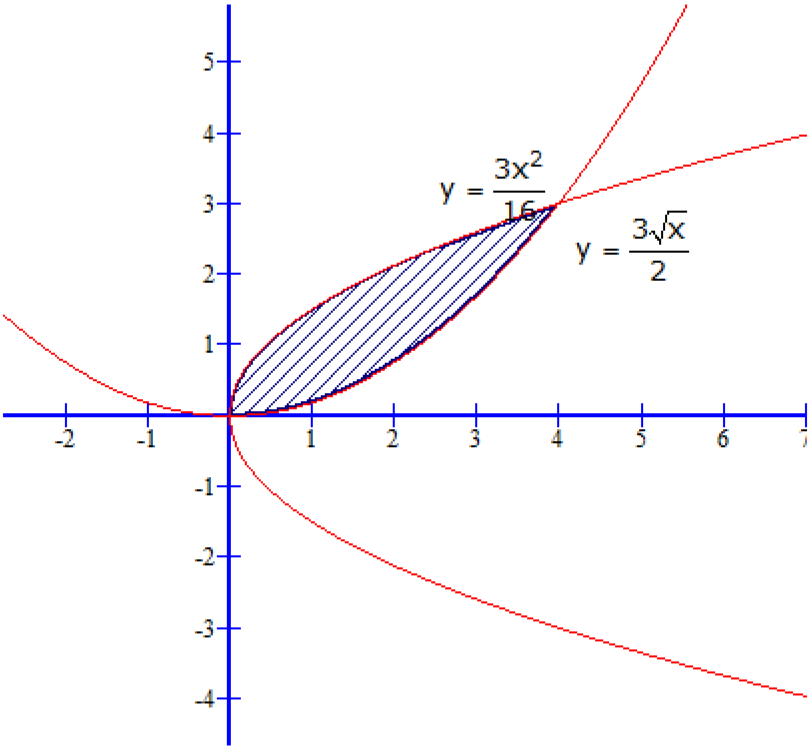
$$= \frac{3}{5}\left(2^{\frac{5}{3}} - 1\right) a^2 \text{ sq units}$$

Chapter 21 - Areas of Bounded Regions Exercise Ex. 21.3

Question 2

Find the area of the region common to the parabolas $4y^2 = 9x$ and $3x^2 = 16y$.

Solution 2



$$\text{Area of the region} = \int_0^4 \left[\frac{3\sqrt{x}}{2} - \frac{3x^2}{16}\right] dx$$

$$= \left[x^{3/2} - \frac{x^3}{16}\right]_0^4$$

$$= \left[(4)^{3/2} - \frac{(4)^3}{16}\right]$$

$$= \left[8 - \frac{64}{16}\right]$$

$$= [8 - 4] = 4 \text{ sq.units}$$

Question 3

Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.

Solution 3

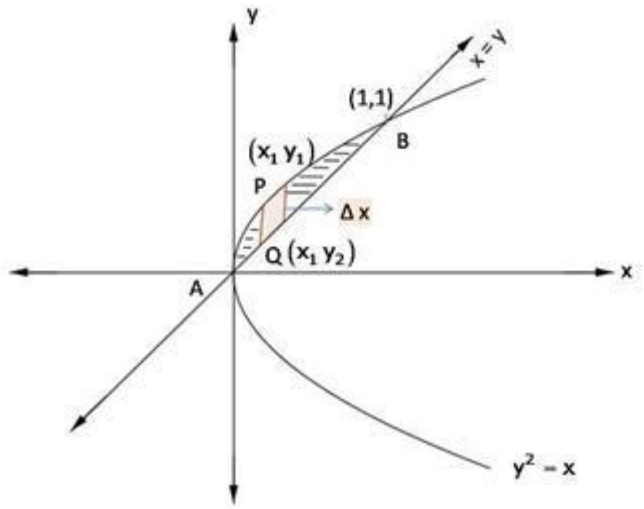
We have to find area of region bounded by

$y^2 = x \qquad\qquad ---(1)$

and $y = x \qquad\qquad ---(2)$

Equation (1) represents parabola with vertex $(0,0)$ and axis as x-axis and equation (2) represents a line passing through origin and intersecting parabola at $(0,0)$ and $(1,1)$.

A rough sketch of curves is as below:-



Shaded region represents the required area. We slice it in rectangle with Width $= \Delta x$, length $= y_1 - y_2$

Area of rectangle $= (y_1 - y_2)\Delta x$

The approximation triangle can slide from $x = 0$ to $x = 1$.

Required area = region $AOBPA$

$$= \int_0^1 (y_1 - y_2)\, dx$$
$$= \int_0^1 \left(\sqrt{x} - x\right) dx$$

$$= \left[\frac{2}{3}x\sqrt{x} - \frac{x^2}{2}\right]_0^1$$
$$= \left[\frac{2}{3}.1.\sqrt{1} - \frac{(1)^2}{2}\right] - [0]$$

$$= \left[\frac{2}{3} - \frac{1}{2}\right]$$

Required area $= \frac{1}{6}$ square units

Question 4

Find the area bounded by the curve $y = 4 - x^2$ and the lines $y = 0$, $y = 3$.

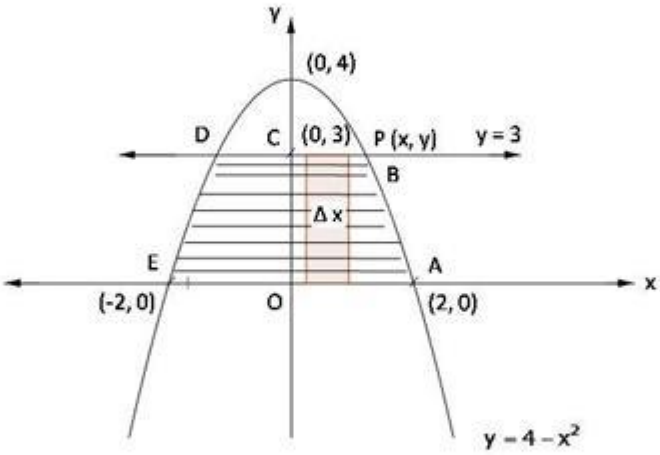
Solution 4

We have to find area bounded by the curves

$$y = 4 - x^2$$
$$\Rightarrow \quad x^2 = -(y - 4) \qquad ---(1)$$
$$\text{and } y = 0 \qquad ---(2)$$
$$y = 3 \qquad ---(3)$$

Equation (1) represents a parabola with vertex $(0,4)$ and passes through $(0,2)$, $(0,-2)$
Equation (1) is x-axis and equation (3) is a line parallel to x-axis passing through $(0,3)$.

A rough sketch of curves is below:-



Shaded region represents the required area. We slice it in approximation rectangle with its Width $=\Delta x$ and length $= y - 0 = y$

Area of the rectangle $= y\Delta x$.

This approximation rectangle can slide from $x = 0$ to $x = 2$ for region $OABCO$.

Required area = Region $ABDEA$

$$= 2\left(\text{Region } OABCO\right)$$

$$= 2\int_0^2 y\,dx$$

$$= 2\int_0^2 \left(4 - x^2\right)dx$$

$$= 2\left(4x - \frac{x^3}{3}\right)_0^2$$

$$= 2\left[\left(8 - \frac{8}{3}\right) - \left(0\right)\right]$$

Required area $= \frac{32}{3}$ square units

Question 5

Find the area of the region $\left\{\left(x, y\right) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \le \frac{x}{a} + \frac{y}{b}\right\}$.

Solution 5

Here to find area $\left\{(x,y): \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1 \le \frac{x}{a}+\frac{y}{b}\right\}$

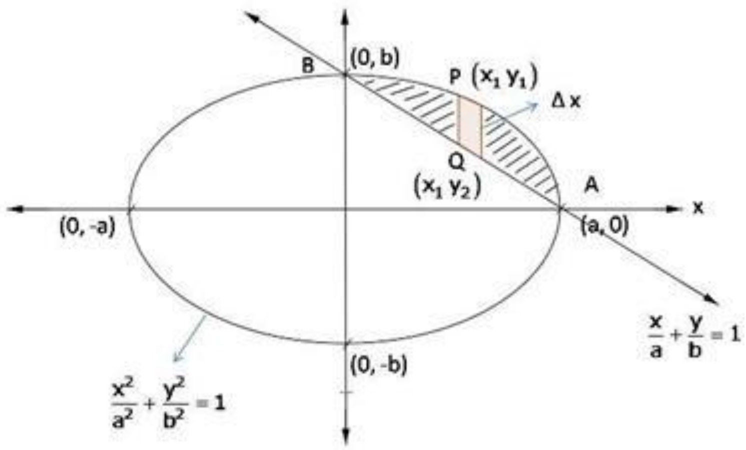
So,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ---(1)$$

$$\frac{x}{a}+\frac{y}{b}=1 \qquad ---(2)$$

Equation (1) represents ellipse with centre at origin and passing through $(\pm a,0)$, $(0,\pm b)$ equation (2) represents a line passing through $(a,0)$ and $(0,b)$.

A rough sketch of curves is below:- let $a > b$



Shaded region is the required region as by substituting $(0,0)$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$ gives a true statement and by substituting $(0,0)$ in $1\le \frac{x}{a}+\frac{y}{b}$ gives a false statement.

We slice the shaded region into approximation rectangles with Width $=\Delta x$,
length $=(y_1-y_2)$

Area of the rectangle $=(y_1-y_2)$

The approximation rectangle can slide from $x=0$ to $x=a$, so

$$\text{Required area} = \int_0^a\left[\frac{b}{a}\sqrt{a^2-x^2}-\frac{b}{a}(a-x)\right]dx$$
$$=\frac{b}{a}\int_0^a\left[\sqrt{a^2-x^2}-(a-x)\right]dx$$
$$=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)-ax+\frac{x^2}{2}\right]_0^a$$
$$=\frac{b}{a}\left[\left(\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}(1)-a^2+\frac{a^2}{2}\right)-(0+0+0+0)\right]$$
$$=\frac{b}{a}\left[\frac{a^2}{2}\cdot\frac{\pi}{2}-\frac{a^2}{2}\right]$$
$$=\frac{b}{a}\frac{a^2}{2}\left(\frac{\pi-2}{2}\right)$$

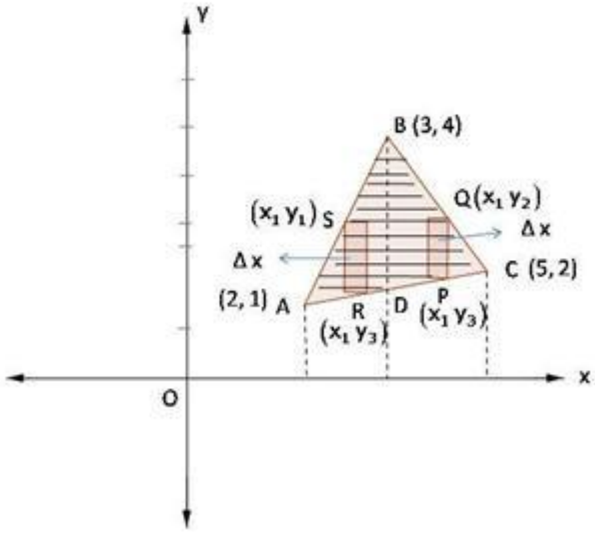
Required area $=\frac{ab}{4}(\pi-2)$ square units

Question 6

Using integration, find the area of the region bounded by the triangle whose vertices are $(2,1)$, $(3,4)$ and $(5,2)$.

Solution 6

Here we have find area of the triangle whose vertices are $A(2,1)$, $B(3,4)$ and $C(5,2)$



Equation of AB,

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - 1 = \left(\frac{4 - 1}{3 - 2}\right)(x - 2)$$

$$y - 1 = \frac{3}{1}(x - 2)$$

$$y = 3x - 6 + 1$$

$$y = 3x - 5 \qquad --- (1)$$

Equation of BC,

$$y - 4 = \left(\frac{2 - 4}{5 - 3}\right)(x - 3)$$

$$= \frac{-2}{2}(x - 3)$$

$$y - 4 = -x + 3$$

$$y = -x + 7 \qquad --- (2)$$

Equation of AC,

$$y - 1 = \left(\frac{2 - 1}{5 - 2}\right)(x - 2)$$

$$y - 1 = \frac{1}{3}(x - 2)$$

$$y = \frac{1}{3}x - \frac{2}{3} + 1$$

$$y = \frac{1}{3}x + \frac{1}{3} \qquad --- (3)$$

Shaded area $\triangle ABC$ is the required area.

$ar(\triangle \mathrm{ABC}) = ar(\triangle ABD) + ar(\triangle BDC)$

For $ar(\Delta ABD)$: we slice the region into approximation rectangle with width $= \Delta x$ and length $(y_1 - y_3)$ area of rectangle $= (y_1 - y_3)\Delta x$

This approximation rectangle slides from $x = 2$ to $x = 3$

$$ar(\Delta ABD) = \int_2^3 (y_1 - y_3)dx$$
$$= \int_2^3 \left[(3x - 5) - \left(\frac{1}{3}x + \frac{1}{3}\right)\right]dx$$
$$= \int_2^3 \left(3x - 5 - \frac{1}{3}x - \frac{1}{3}\right)dx$$
$$= \int_2^3 \left(\frac{8x}{3} - \frac{16}{3}\right)dx$$
$$= \frac{8}{3}\left(\frac{x^2}{2} - 12x\right)_2^3$$
$$= \frac{8}{3}\left[\left(\frac{9}{2} - 6\right) - (2 - 4)\right]$$
$$= \frac{8}{3}\left[-\frac{3}{2} + 2\right]$$
$$= \frac{8}{3} \times \frac{1}{2}$$
$$ar(\Delta ABD) = \frac{4}{3} \text{ sq. unit}$$

For $ar(\Delta BDC)$: we slice the region into rectangle with width $= \Delta x$ and length $(y_2 - y_3)$. Area of rectangle $= (y_2 - y_3)\Delta x$

The approximation rectangle slides from $x = 3$ to $x = 5$.

$$\text{Area}(\Delta BDC) = \int_3^5 (y_2 - y_3)dx$$
$$= \int_3^5 \left[(-x + 7) - \left(\frac{1}{3}x + \frac{1}{3}\right)\right]dx$$
$$= \int_3^5 \left(-x + 7 - \frac{1}{3}x - \frac{1}{3}\right)dx$$
$$= \int_3^5 \left(-\frac{4}{3}x + \frac{20}{3}\right)dx$$
$$= -\left(\frac{4x^2}{6} - \frac{20}{3}x\right)_3^5$$

$$= -\left[\left(\frac{4(5)^2}{6} + \frac{20(5)}{3}\right) - \left(\frac{4(3)^2}{6} - \frac{20}{3}(3)\right)\right]$$

$$= -\left[\left(\frac{50}{3} - \frac{100}{3}\right) - (6 - 20)\right]$$

$$= -\left[-\frac{50}{3} + 14\right]$$

$$= -\left[-\frac{8}{3}\right]$$

$ar(\triangle BDC) = \frac{8}{3}$ sq. units

So, $ar(\triangle ABC) = ar(\triangle ABD) + ar(\triangle BDC)$

$$= \frac{4}{3} + \frac{8}{3}$$

$$= \frac{12}{3}$$

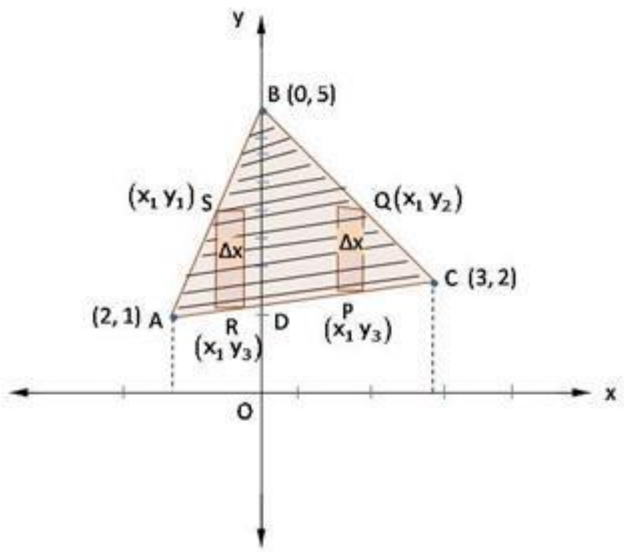
$ar(\triangle ABC) = 4$ sq. units

Question 7

Using integration, find the area of the region bounded by the triangle ABC whose vertices A, B, C are $(-1,1), (0,5)$ and $(3,2)$ respectively.

Solution 7

We have to find area of the triangle whose vertices are $A(-1,1)$, $B(0,5)$, $C(3,2)$



Equation of AB,

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - 1 = \left(\frac{5-1}{0+1}\right)(x+1)$$

$$y - 1 = \frac{4}{1}(x+1)$$

$$y = 4x + 4 + 1$$

$$y = 4x + 5 \qquad ---(1)$$

Equation of BC,

$$y - 5 = \left(\frac{2-5}{3-0}\right)(x - 0)$$

$$= \frac{-3}{3}(x - 0)$$

$$y - 5 = -x$$

$$y = 5 - x \qquad ---(2)$$

Equation of AC,

$$y - 1 = \left(\frac{2-1}{3+1}\right)(x+1)$$

$$y - 1 = \frac{1}{4}(x+1)$$

$$y = \frac{1}{4}x + \frac{1}{4} + 1$$

$$y = \frac{1}{4}(x+5) \qquad ---(3)$$

Shaded area $\triangle ABC$ is the required area.

$ar(\triangle \mathrm{ABC}) = ar(\triangle ABD) + ar(\triangle BDC)$

For $ar(\Delta ABD)$: we slice the region into approximation rectangle with width $= \Delta x$ and length $(y_1 - y_3)$ area of rectangle $= (y_1 - y_3)\Delta x$

This approximation rectangle slides from $x = -1$ to $x = 0$, so

$$
\begin{aligned}
ar(\Delta ABD) &= \int_{-1}^{0} (y_1 - y_3)\,dx \\
&= \int_{-1}^{0} \left[(4x+5) - \frac{1}{4}(x+5)\right]dx \\
&= \int_{-1}^{0} \left(4x + 5 - \frac{x}{4} - \frac{5}{4}\right)dx \\
&= \int_{-1}^{0} \left(\frac{15}{4}x + \frac{15}{4}\right)dx \\
&= \frac{15}{4}\left(\frac{x^2}{2} + x\right)_{-1}^{0} \\
&= \frac{15}{4}\left[(0) - \left(\frac{1}{2} - 1\right)\right] \\
&= \frac{15}{4} \times \frac{1}{2}
\end{aligned}
$$

$$ar(\Delta ABD) = \frac{15}{8} \text{ sq. units}$$

For $ar(\Delta BDC)$: we slice the region into rectangle with width $= \Delta x$ and length $(y_2 - y_3)$. Area of rectangle $= (y_2 - y_3)\Delta x$

The approximation rectangle slides from $x = 0$ to $x = 3$.

$$
\begin{aligned}
\text{Area}(\Delta BDC) &= \int_{0}^{3} (y_2 - y_3)\,dx \\
&= \int_{0}^{3} \left[(5 - x) - \left(\frac{1}{4}x + \frac{5}{4}\right)\right]dx \\
&= \int_{0}^{3} \left(5 - x - \frac{1}{4}x - \frac{5}{4}\right)dx \\
&= \int_{0}^{3} \left(-\frac{5}{4}x + \frac{15}{4}\right)dx \\
&= \frac{5}{4}\left(3x - \frac{x^2}{2}\right)_{0}^{3}
\end{aligned}
$$

$$= \frac{5}{4}\left[9 - \frac{9}{2}\right]$$

$$ar\left(\Delta BDC\right) = \frac{45}{8} \text{ sq. units}$$

So, $ar\left(\Delta ABC\right) = ar\left(\Delta ABD\right) + ar\left(\Delta BDC\right)$

$$= \frac{15}{8} + \frac{45}{8}$$
$$= \frac{60}{8}$$

$$ar\left(\Delta ABC\right) = \frac{15}{2} \text{ sq. units}$$

Question 8

Using integration, find the area of the triangular region, The equations of whose sides are $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Solution 8

To find area of triangular region bounded by

$y = 2x + 1$ (Say, line AB) ---(1)

$y = 3x + 1$ (Say, line BC) ---(2)

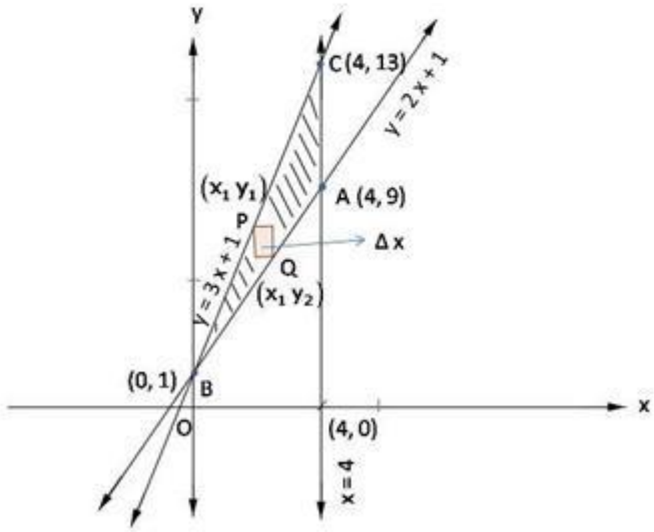
$y = 4$ (Say, line AC) ---(3)

equation (1) represents a line passing through points $(0,1)$ and $\left(-\frac{1}{2},0\right)$, equation (2) represents a line passing through points $(0,1)$ and $\left(-\frac{1}{3},0\right)$. Equation (3) represents a line parallel to y-axis passing through $(4,0)$.

Solving equation (1) and (2) gives point $B(0,1)$

Solving equation (2) and (3) gives point $C(4,13)$

Solving equation (1) and (3) gives point $A(4,9)$



Shaded region $ABCA$ gives required triangular region. We slice this region into approximation rectangle with width $=\Delta x$, length $= (y_1 - y_2)$.

Area of rectangle $= (y_1 - y_2)\Delta x$

This qpproximation rectangle slides from $x = 0$ to $x = 4$, so

Required area = (Region $ABCA$)

$= \int_0^4 (y_1 - y_2)dx$

$= \int_0^4 [(3x + 1) - (2x + 1)]dx$

$= \int_0^4 x dx$

$$= \left[\frac{x^2}{2}\right]_0^4$$

Required area = 8 sq. units

Question 9

Find the area of the region $\{(x, y) : y^2 \le 8x, x^2 + y^2 \le 9\}$.

Solution 9

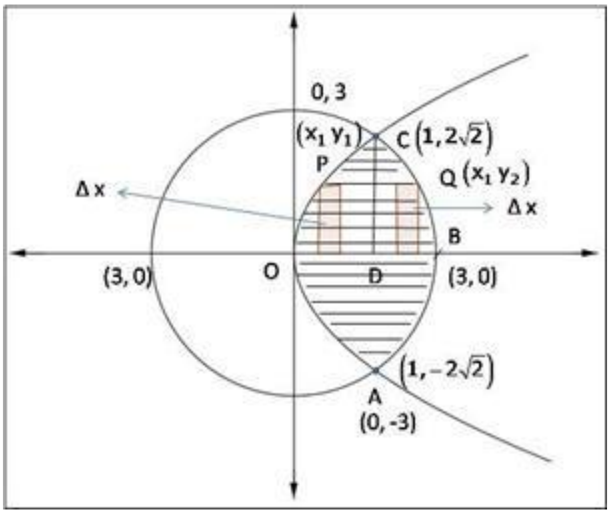
To find area $\left\{(x,y): y^2 \le 8x, x^2 + y^2 \le 9\right\}$ given equation is

$y^2 = 8x$ $\qquad\qquad$ ---(1)

$x^2 + y^2 = 9$ $\qquad\qquad$ ---(2)

Equation (1) represents a parabola with vertex (0,0) and axis as x-axis, equation (2) represents a circle with centre (0,0) and radius $\sqrt{9} = 3$, so it meets area at $(\pm 3, 0)$, $(0, \pm 3)$. point of intersection of parabola and circle is $\left(1, 2\sqrt{2}\right)$ and $\left(1, -2\sqrt{2}\right)$.

A rough sketch of the curves is as below:-



Shaded region is the required region.

Required area = Region $OABCO$

$= 2(\text{Region } OBCO)$

Required area $= 2(\text{region } ODCO + \text{region } DBCD)$

$$= 2\left[\int_0^1 \sqrt{8x}\,dx + \int_1^3 \sqrt{9-x^2}\,dx\right]$$

$$= 2\left[\left(2\sqrt{2}.\frac{2}{3}x\sqrt{x}\right)_0^1 + \left(\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right)_1^3\right]$$

$$= 2\left[\left(\frac{4\sqrt{2}}{3}.1.\sqrt{1}\right) + \left\{\left(\frac{3}{2}.\sqrt{9-9} + \frac{9}{2}\sin^{-1}(1)\right) - \left(\frac{1}{2}\sqrt{9-1} + \frac{9}{2}\sin^{-1}\frac{1}{3}\right)\right\}\right]$$

$$= 2\left[\frac{4\sqrt{2}}{3} + \left\{\left(\frac{9}{2}.\frac{\pi}{2}\right) - \left(\frac{2\sqrt{2}}{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)\right\}\right]$$

$$= 2\left[\frac{4\sqrt{2}}{3} + \frac{9\pi}{4} - \sqrt{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

Required area $= 2\left[\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$ square units

Question 10

Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

Solution 10

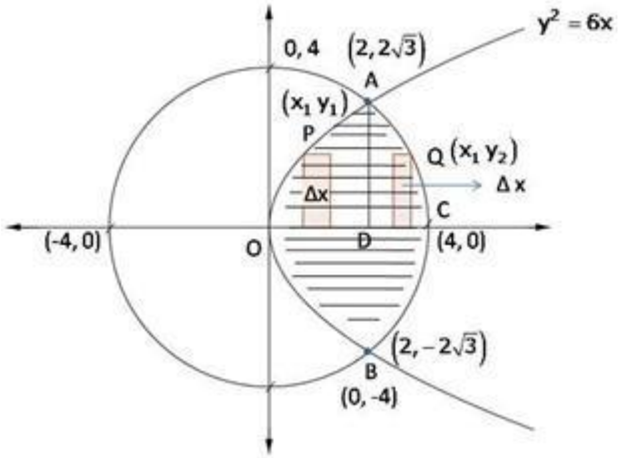
To find the area of common to

$x^2 + y^2 = 16$ ---(1)

$y^2 = 6x$ ---(2)

Equation (1) represents a parabola with vertex $(0,0)$ and axis as x-axis, equation (2) represents a circle with centre $(0,0)$ and radius $\sqrt{16} = 4$, so it meets areas at $(\pm 4, 0)$, $(0, \pm 4, 0)$. points of intersection of parabola and circle are $(2, 2\sqrt{3})$ and $(2, -2\sqrt{3})$.

A rough sketch of the curves is as below:-



Shaded region represents the required area.

Required area = Region $OBCAO$
Required area = 2 (region $ODAO$ + region $DCAD$) --- (1)

Region $ODAO$ is divided into approximation rectangle with area $y_1 \Delta x$ and slides from $x = 0$ to $x = 2$. And region $DCAD$ is divided into approximation rectangle with area $y_2 \Delta x$ and slides from $x = 2$ and $x = 4$. So using equation (1),

$$\text{Required area} = 2\left(\int_0^2 y_1 dx + \int_2^4 y_2 dx\right)$$
$$= 2\left[\int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16 - x^2}\,dx\right]$$
$$= 2\left[\left\{\sqrt{6}.\frac{2}{3}x\sqrt{x}\right\}_0^2 + \left\{\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right\}_2^4\right]$$
$$= 2\left[\left\{\sqrt{6}.\frac{2}{3}2.\sqrt{2}\right\} + \left\{\left(\frac{4}{2}\sqrt{16-16} + \frac{16}{2}\sin^{-1}\frac{4}{4}\right) - \left(\frac{2}{2}\sqrt{16-4} + \frac{16}{2}\sin^{-1}\frac{2}{4}\right)\right\}\right]$$
$$= 2\left[\frac{4}{3}\sqrt{12} + \left\{\left(0 + 8\sin^{-1}(1)\right) - \left(1.\sqrt{12} + 8\sin^{-1}\left(\frac{1}{2}\right)\right)\right\}\right]$$
$$= 2\left[\frac{8\sqrt{3}}{3} + \left\{\left(8.\frac{\pi}{2}\right) - \left(2\sqrt{3} + 8.\frac{\pi}{6}\right)\right\}\right]$$
$$= 2\left\{\frac{8\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right\}$$

$$= 2\left\{\frac{2\sqrt{3}}{3} + \frac{8\pi}{3}\right\}$$

$$\text{Required area} = \frac{4}{3}\left(4\pi + \sqrt{3}\right) \text{ sq.units}$$

Question 11
Find the area of the region between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

Solution 11

Equation of the given circles are

$$X^2 + y^2 = 4 \quad ...(1)$$

And $\quad (x - 2)^2 + y^2 = 4 \quad ...(2)$

Equation (1) is a circle with centre O at eh origin and radius 2. Equation (2) is a circle with centre C (2,0) and radius 2. Solving equations (1) and (2), we have

$$(x - 2)^2 + y^2 = x^2 + y^2$$

Or $\quad x^2 - 4x + 4 + y^2 = x^2 + y^2$

Or $\quad x = 1$ which gives $y \ \pm\sqrt{3}$

Thus, the points of intersection of the given circles are A $\left(1, \sqrt{3}\right)$ and A' $\left(1, -\sqrt{3}\right)$ as shown in the fig.,

Required area of the enclosed region OACA'O between circle

$= 2$ [area of the region ODCAO] (Why ?)

$= 2$ [area of the region ODAO + area of the region DCAD]

$$= 2\left[\int_0^1 y\,dx + \int_1^2 y\,dx\right]$$

$$= 2\left[\int_0^1 \sqrt{4 - (x - 2)^2}\,dx + \int_1^2 \sqrt{4 - x^2}\,dx\right] \quad \text{(Why ?)}$$

$$= 2\left[\frac{1}{2}(x - 2)\sqrt{4 - (x - 2)^2} + \frac{1}{2} \times 4\sin^{-1}\left(\frac{x - 2}{2}\right)\right]_0^1 + 2\left[\frac{1}{2} \times \sqrt{4 - x^2} + \frac{1}{2} \times 4\sin^{-1}\frac{x}{2}\right]_1^2$$

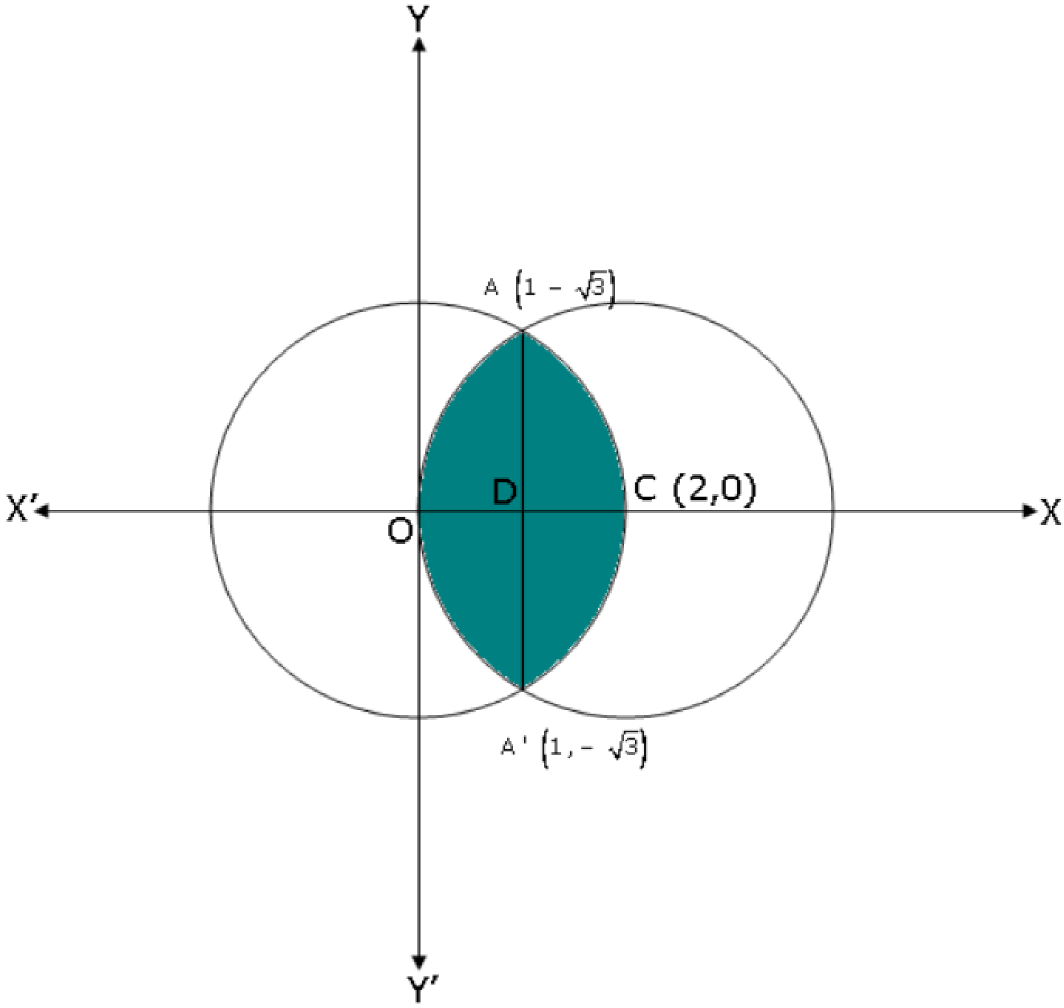
$$= \left[(x - 2)\sqrt{4 - (x - 2)^2} + 4\sin^{-1}\left(\frac{x - 2}{2}\right)\right]_0^1 + \left[x\sqrt{4 - x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[\left(-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right)\right) - 4\sin^{-1}(-1)\right] + \left[4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$$

$$= \left[\left(-\sqrt{3} - 4 \times \frac{\pi}{6}\right) + 4 \times \frac{\pi}{2}\right] + \left[4 \times \frac{\pi}{2} - \sqrt{3} - 4 \times \frac{\pi}{6}\right]$$

$$= \left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi\right) + \left(2\pi - \sqrt{3} - \frac{2\pi}{3}\right)$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \text{ square units}$$



Question 12

Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

Solution 12

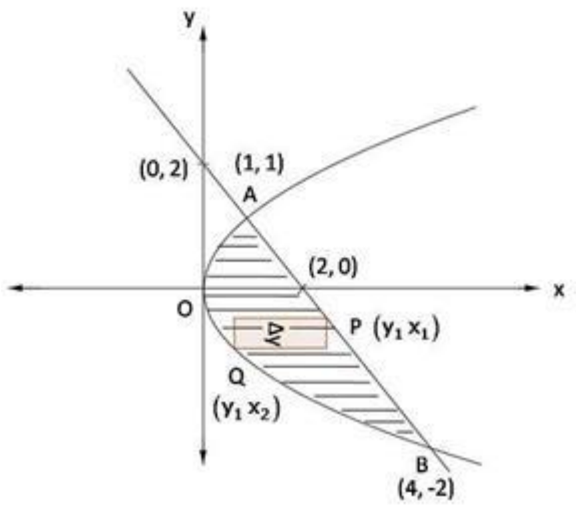
To find region enclosed by

$y^2 = x \quad\quad ---(1)$

$x + y = 2 \quad\quad ---(2)$

Equation (1) represents a parabola with vertex at origin and its axis as x-axis, equation (2) represents a line passing through $(2,0)$ and $(0,2)$. points of intersection of line and parabola are $(1,1)$ and $(4,-2)$.

A rough sketch of curves is as below:-



Shaded region represents the required area. We slice it in rectangles of width Δy and length $= (x_1 - x_2)$.

Area of rectangle $= (x_1 - x_2)\Delta y$.

This approximation rectangle slides from $y = -2$ to $y = 1$, so

Required area = Region $AOBA$

$$= \int_{-2}^{1}(x_1 - x_2)\,dy$$
$$= \int_{-2}^{1}(2 - y - y^2)\,dy$$
$$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$$
$$= \left[\left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)\right]$$
$$= \left[\left(\frac{12 - 3 - 2}{6}\right) - \left(\frac{-12 - 6 + 8}{3}\right)\right]$$
$$= \frac{7}{6} + \frac{10}{3}$$

Required area $= \frac{9}{2}$ sq.units

Question 13

Draw a rough sketch of the region $\{(x, y) : y^2 \le 3x, 3x^2 + 3y^2 \le 16\}$ and find the area enclosed by the region using method of integration.

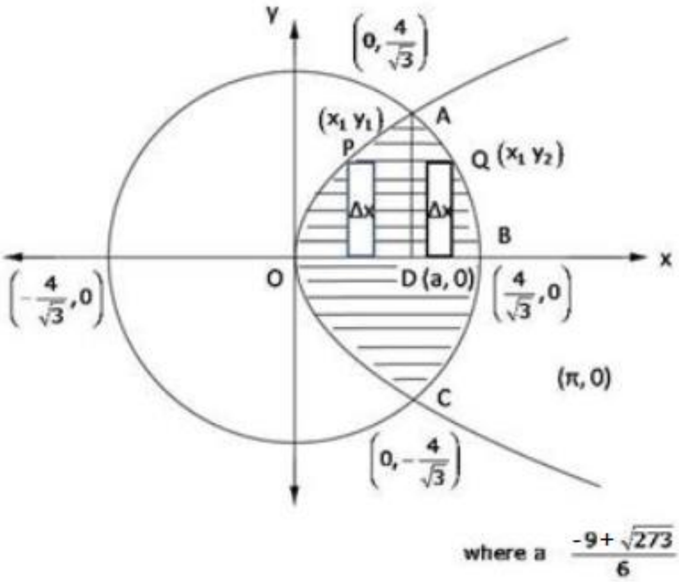
Solution 13

To find area $\{(x, y) : y^2 \le 3x, 3x^2 + 3y^2 \le 16\}$

$$\Rightarrow \quad y^2 = 3x \qquad ---(1)$$
$$3x^2 + 3y^2 = 16$$
$$x^2 + y^2 = \frac{16}{3} \qquad ---(2)$$

Equation (1) represents a parabola with vertex $(0,0)$ and axis as x-axis,

equation (2) represents a circle with centre $(0,0)$ and radius $\frac{4}{\sqrt{3}}$ and meets axes at $\left(\pm\frac{4}{\sqrt{3}}, 0\right)$ and $\left(0, \pm\frac{4}{\sqrt{3}}\right)$. A rough sketch of the curves is given below:-



$$\begin{aligned}
\text{Required area} &= \text{Region } OCBAO \\
&= 2\left(\text{Region } OBAO\right) \\
&= 2\left(\text{Region } ODAO + \text{Region } DBAD\right) \\
&= 2\left[\int_0^a \sqrt{3x}\,dx + \int_a^{\frac{4}{\sqrt{3}}} \sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - x^2}\,dx\right]
\end{aligned}$$

$$A = 2\left[\left(\sqrt{3}.\frac{2}{3}x\sqrt{x}\right)_0^a + \left(\frac{x}{2}\sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - x^2} + \frac{16}{6}\sin^{-1}\frac{x\sqrt{3}}{4}\right)_a^{\frac{4}{\sqrt{3}}}\right]$$

$$= 2\left[\left(\frac{2}{\sqrt{3}}a\sqrt{a}\right) + \left\{\left(0 + \frac{8}{3}\sin^{-1}(1)\right) - \left(\frac{a}{2}\sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - a^2} + \frac{8}{3}\sin^{-1}\frac{a\sqrt{3}}{4}\right)\right\}\right]$$

Thus, $A = \frac{4}{\sqrt{3}}a^{\frac{3}{2}} + \frac{8\pi}{3} - a\sqrt{\frac{16}{3} - a^2} - \frac{16}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)$

Where, $a = \frac{-9+\sqrt{273}}{6}$

Question 14

Draw a rough sketch of the region $\left\{(x, y) : y^2 \le 5x, 5x^2 + 5y^2 \le 36\right\}$ and find the area enclosed by the region using method of integration.

Solution 14

To find area $\left\{(x, y) : y^2 \le 5x, 5x^2 + 5y^2 \le 36\right\}$

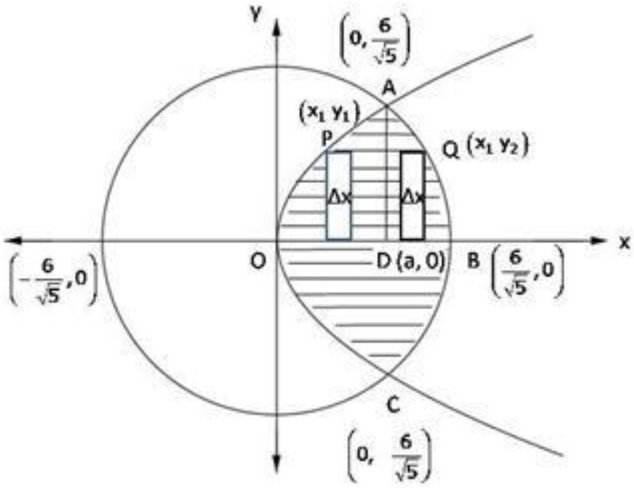
$$\Rightarrow \quad y^2 = 5x \qquad ---(1)$$

$$5x^2 + 5y^2 = 36$$

$$x^2 + y^2 = \frac{36}{5} \qquad ---(2)$$

Equation (1) represents a parabola with vertex $(0,0)$ and axis as x-axis.

Equation (2) represents a circle with centre $(0,0)$ and radius $\frac{6}{\sqrt{5}}$ and meets axes at $\left(\pm\frac{6}{\sqrt{5}}, 0\right)$ and $\left(0, \pm\frac{6}{\sqrt{5}}\right)$. x ordinate of point of intersection of circle and parabola is a where $a = \frac{-25+\sqrt{1345}}{10}$. A rough sketch of curves is:-



Required area = Region $OCBAO$

$$A = 2(\text{Region } OBAO)$$

$$= 2(\text{Region } ODAO + \text{Region } DBAD)$$

$$= 2\left[\int_0^a \sqrt{5x}\,dx + \int_a^{\frac{6}{\sqrt{5}}} \sqrt{\left(\frac{6}{\sqrt{5}}\right)^2 - x^2 dx}\right]$$

$$= 2\left[\left(\sqrt{5}.\frac{2}{3}x\sqrt{x}\right)_0^a + \left(\frac{x}{2}\sqrt{\left(\frac{6}{\sqrt{5}}\right)^2 - x^2} + \frac{36}{10}\sin^{-1}\left(\frac{x\sqrt{5}}{6}\right)\right)_a^{\frac{6}{\sqrt{5}}}\right]$$

$$= \frac{4\sqrt{5}}{3}a\sqrt{a} + 2\left\{\left(0 + \frac{18}{5}.\frac{\pi}{2}\right) - \left(\frac{a}{2}\sqrt{\left(\frac{6}{\sqrt{5}}\right)^2 - a^2} + \frac{18}{5}\sin^{-1}\left(\frac{a\sqrt{5}}{6}\right)\right)\right\}$$

Thus, $A = \frac{4\sqrt{5}}{a} a^{\frac{3}{2}} + \frac{18\pi}{5} - a\sqrt{\frac{36}{5} - a^2} - \frac{36}{5}\sin^{-1}\left(\frac{a\sqrt{5}}{6}\right)$

Where, $a = \frac{-25 + \sqrt{1345}}{10}$

Question 15

Draw a rough sketch and find the area of the region bounded by the two parabolas $y^2 = 4x$ and $x^2 = 4y$ by using methods of integration.

Solution 15

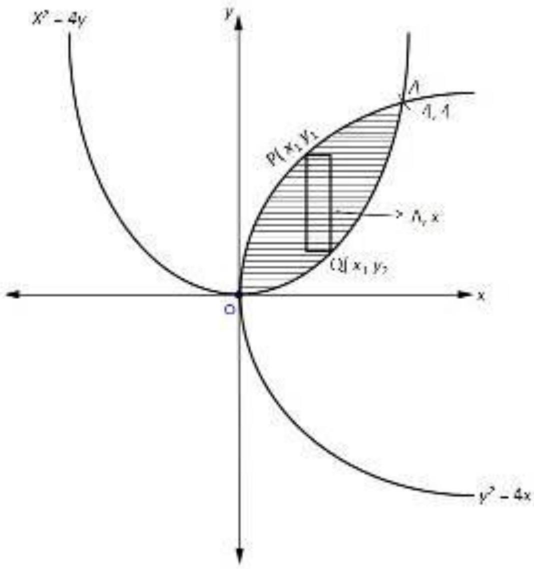
To find area bounded by

$$y^2 = 4x \qquad ---(1)$$

$$x^2 = 4y \qquad ---(2)$$

Equation (1) represents a parabola with vertex $(0,0)$ and axis as x-axis.
Equation (2) represents a parabola with vertex $(0,0)$ and axis as y-axis.
Points of intersection of parabolas are $(0,0)$ and $(4,4)$.

A rough sketch is given as:-



The shaded region is required area and it is sliced into rectangles with width Δx and length $(y_1 - y_2)$. Area of rectangle = $(y_1 - y_2)\Delta x$.

This approximation rectangle slide from $x = 0$ to $x = 4$, so

Required area = Region $OQAPO$

$$A = \int_0^4 (y_1 - y_2)dx$$
$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx$$
$$= \left[2.\frac{2}{3}x\sqrt{x} - \frac{x^3}{12}\right]_0^4$$
$$= \left[\left(\frac{4}{3}.4\sqrt{4} - \frac{64}{12}\right) - (0)\right]$$

$$A = \frac{32}{3} - \frac{16}{3}$$

$$A = \frac{16}{3} \text{ sq.units}$$

Question 16

Find the area included between the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

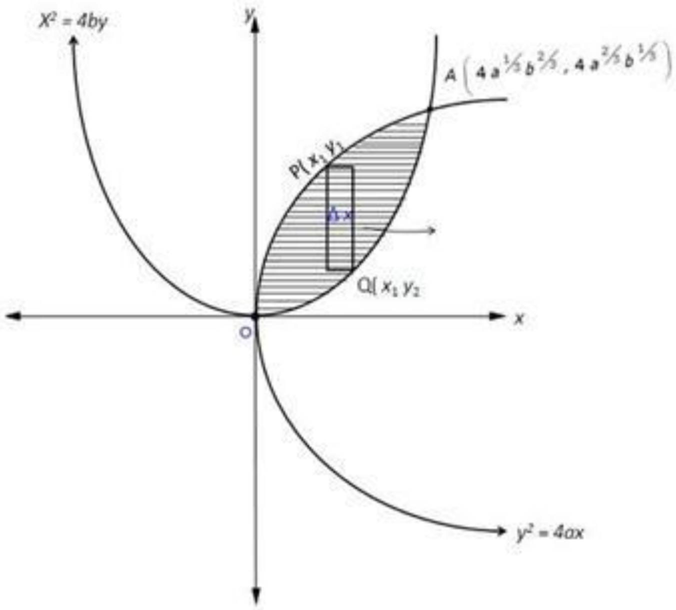
Solution 16

To find area enclosed by

$$y^2 = 4ax \qquad ---(1)$$
$$x^2 = 4by \qquad ---(2)$$

Equation (1) represents a parabola with vertex $(0,0)$ and axis as x-axis,
equation (2) represents a parabola with vertex $(0,0)$ and axis as y-axis,
points of intersection of parabolas are $(0,0)$ and $\left(4a\frac{1}{3}\ b\frac{2}{3},\ 4a\frac{2}{3}\ b\frac{1}{3}\right)$

A rough sketch is given as:-



The shaded region is required area and it is sliced into rectangles of width = Δx and length $(y_1 - y_2)$.

Area of rectangle = $(y_1 - y_2)\Delta x$.

This approximation rectangle slides from $x = 0$ to $x = 4a\frac{1}{3}\ b\frac{2}{3}$, so

Required area = Region $OQAPO$

$$= \int_0^{4a\frac{1}{3}\ b\frac{2}{3}} (y_1 - y_2)\, dx$$
$$= \int_0^{4a\frac{1}{3}\ b\frac{2}{3}} \left(2\sqrt{a}.\sqrt{x} - \frac{x^2}{4b}\right) dx$$
$$= \left[2\sqrt{a}.\frac{2}{3}x\sqrt{x} - \frac{x^3}{12b}\right]_0^{4a\frac{1}{3}\ b\frac{2}{3}}$$
$$= \frac{32\sqrt{a}}{3}.a\frac{1}{3}\ b\frac{2}{3}\ a\frac{1}{6}\ b\frac{1}{3} - \frac{64ab^2}{12b}$$
$$= \frac{32}{3}ab - \frac{16}{3}ab$$

$A = \frac{16}{3}ab$ sq.units

Question 17

Prove that the area in the first quadrant enclosed by the x-axis, the line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$ is $\frac{\pi}{3}$.

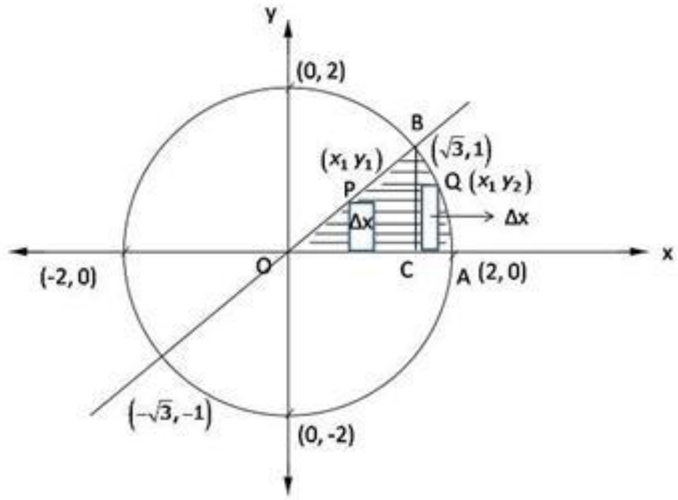
Solution 17

To find area in first quadrant enclosed by x-axis.

$$x = \sqrt{3}y \qquad ---(1)$$
$$x^2 + y^2 = 4 \qquad ---(2)$$

Equation (1) represents a line passing through $(0,0),\left(-\sqrt{3},-1\right),\left(\sqrt{3},1\right)$. Equation (2) represents a circle with centre $(0,0)$ and passing through $(\pm 2,0),(0,\pm 2)$. Points of intersection of line and circle are $\left(-\sqrt{3},-1\right)$ and $\left(\sqrt{3},1\right)$.

A rough sketch of curves is given below:-



Required area = Region $OABO$

$$A = \text{Region } OCBO + \text{Region } ABCA$$
$$= \int_0^{\sqrt{3}} y_1 dx + \int_{\sqrt{3}}^{2} y_2 dx$$
$$= \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}} dx + \int_{\sqrt{3}}^{2} \sqrt{4 - x^2} dx$$
$$= \left(\frac{x^2}{2\sqrt{3}}\right)_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$
$$= \left(\frac{3}{2\sqrt{3}} - 0\right) + \left[\left(0 + 2\sin^{-1}(1)\right) - \left(\frac{\sqrt{3}}{2}.1 + 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right]$$
$$= \frac{\sqrt{3}}{2} + 2.\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2.\frac{\pi}{3}$$
$$A = \frac{\pi}{3} \text{ sq.units}$$

Question 19

Find the area common to the circle $x^2 + y^2 = 16a^2$ and the parabola $y^2 = 6ax$.

Solution 19

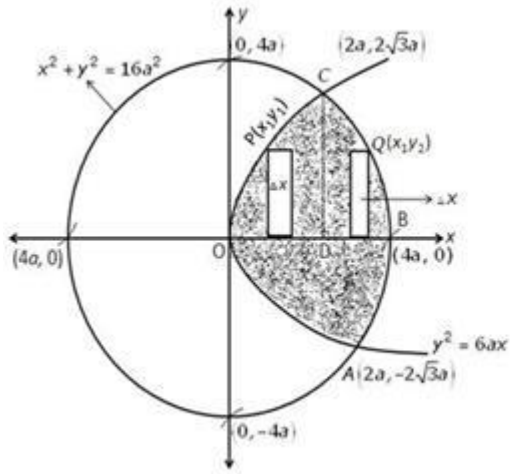
To find area in enclosed by

$x^2 + y^2 = 16a^2$ $\qquad\qquad$ ---(1)

and $y^2 = 6ax$ $\qquad\qquad$ ---(2)

Equation (1) represents a circle with centre $(0,0)$ and meets axes $(\pm 4a, 0), (0, \pm 4a)$. Equation(2) represents a parabola with vertex $(0,0)$ and axis as x-axis. Points of intersection of circle and parabola are $(2a, 2\sqrt{3}a), (2a, -2\sqrt{3}a)$.

A rough sketch of curves is given as:-



Region $ODCO$ is sliced into rectangles of area = $y_1 \Delta x$ and it slides from $x = 0$ to $x = 2a$.

Region $BCDB$ is sliced into rectangles of area = $y_2 \Delta x$ it slides from $x = 2a$ to $x = 4a$. So,

Required area = 2[Region $ODCO$ + Region $BCDB$]

$$= 2\left[\int_0^{2a} y_1 dx + \int_{2a}^{4a} y_2 dx\right]$$
$$= 2\left[\int_0^{2a} \sqrt{6ax}\,dx + \int_{2a}^{4a} \sqrt{16a^2 - x^2}\,dx\right]$$
$$= 2\left[\sqrt{6a}.\left(\frac{2}{3}x\sqrt{x}\right)_0^{2a} + \left[\frac{x}{2}\sqrt{16a^2 - x^2} + \frac{16a^2}{2}\sin^{-1}\left(\frac{x}{4a}\right)\right]_{2a}^{4a}\right]$$
$$= 2\left[\left(\sqrt{6a}.\frac{2}{3}2a\sqrt{2a}\right) + \left[\left(0 + 8a^2.\frac{\pi}{2}\right) - \left(a\sqrt{12a^2} + 8a^2.\frac{\pi}{6}\right)\right]\right]$$
$$= 2\left[\frac{8\sqrt{3}a^2}{3} + 4a^2\pi - 2\sqrt{3}a^2 - \frac{4}{3}a^2\pi\right]$$

$$= 2\left[\frac{2\sqrt{3}a^2}{3} + \frac{8a^2\pi}{3}\right]$$

$$A = \frac{4a^2}{3}\left(4\pi + \sqrt{3}\right) \text{ sq.units}$$

Question 20

Find the area, lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

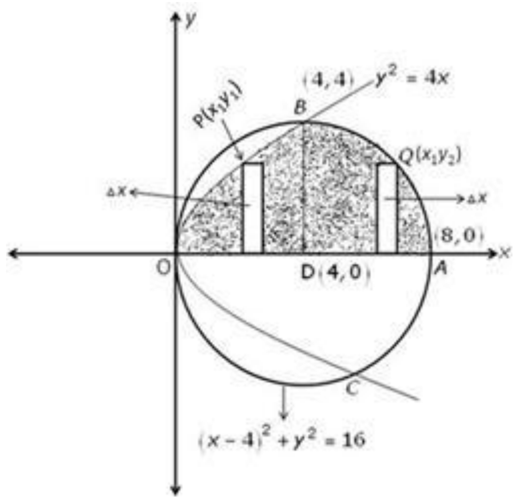
Solution 20

To find area lying above x-axis and included in the circle

$$x^2 + y^2 = 8x$$
$$(x-4)^2 + y^2 = 16 \qquad ---(1)$$
$$\text{and } y^2 = 4x \qquad ---(2)$$

Equation (1) represents a circle with centre (4,0) and meets axes at (0,0) and (8,0).
Equation(2) represent a parabola with vertex (0,0) and axis as x-axis. They intersect at (4,−4) and (4,4).

A rough sketch of the curves is as under:-



y
(4, 4)
$y^2 = 4x$
B
$P(x, y_1)$
$Q(x_1 y_2)$
Δx
Δx
(8, 0)
O
D(4, 0)
A
x
C
$(x-4)^2 + y^2 = 16$

Shaded region is the required region

Required area = Region $OABO$

Required area = Region $ODBO$ + Region $DABD$ $\quad --- (1)$

Region $ODBO$ is sliced into rectangles of area $y_1 \Delta x$. This approximation rectangle can slide from $x = 0$ to $x = 4$. So,

$$\text{Region } ODBO = \int_0^4 y_1 dx$$
$$= \int_0^4 2\sqrt{x}dx$$
$$= 2\left(\frac{2}{3}x\sqrt{x}\right)_0^4$$

$$\text{Region } ODBO = \frac{32}{3} \text{ sq. units} \quad --- (2)$$

Region $DABD$ is sliced into rectangles of area $y_2 \Delta x$. Which moves from $x = 4$ to $x = 8$. So,

$$\text{Region } DABD = \int_4^8 y_2 dx$$
$$= \int_4^8 \sqrt{16 - (x-4)^2}dx$$
$$= \left[\frac{(x-4)}{2}\sqrt{16-(x-4)^2} + \frac{16}{2}\sin^{-1}\left(\frac{x-4}{4}\right)\right]_4^8$$
$$= \left[\left(0 + 8.\frac{\pi}{2}\right) - (0+0)\right]$$

$$\text{Region } DABD = 4\pi \text{ sq. units} \quad --- (3)$$

Using (1), (2) and (3), we get

$$\text{Required area} = \left(\frac{32}{3} + 4\pi\right)$$
$$A = 4\left(\pi + \frac{8}{3}\right) \text{ sq.units}$$

Question 21

Find the area enclosed by the parabolas $y = 5x^2$ and $y = 2x^2 + 9$.

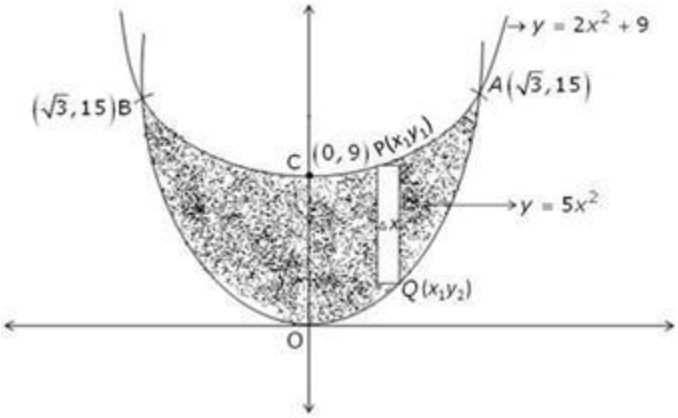
Solution 21

To find area enclosed by

$$y = 5x^2 \qquad ---(1)$$
$$y = 2x^2 + 9 \qquad ---(2)$$

Equation (1) represents a parabola with vertex $(0,0)$ and axis as y-axis. Equation (2) represents a parabola with vertex $(0,9)$ and axis as y-axis. Points of intersection of parabolas are $(\sqrt{3}, 15)$ and $(-\sqrt{3}, 15)$.

A rough sketch of curves is given as:-



Region $AOCA$ is sliced into rectangles with area $(y_1 - y_2)\Delta x$. It slides from $x = 0$ to $x = \sqrt{3}$, so

$$
\begin{aligned}
\text{Required area} &= \text{Region } AOBCA \\
&= 2(\text{Region } AOCA) \\
&= 2\int_0^{\sqrt{3}} (y_1 - y_2)\, dx \\
&= 2\int_0^{\sqrt{3}} (2x^2 + 9 - 5x^2)\, dx \\
&= 2\int_0^{\sqrt{3}} (9 - 3x^2)\, dx \\
&= 2\left[9x - x^3\right]_0^{\sqrt{3}} \\
&= 2\left[(9\sqrt{3} - 3\sqrt{3}) - (0)\right]
\end{aligned}
$$

Required area $= 12\sqrt{3}$ sq.units

Question 22

Prove that the area common to the two parabolas $y = 2x^2$ and $y = x^2 + 4$ is $\frac{32}{3}$ sq. units.

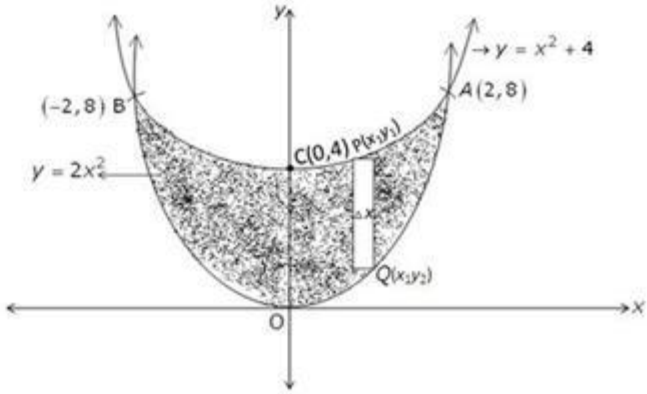
Solution 22

To find area enclosed by

$$y = 2x^2 \qquad ---(1)$$
$$y = x^2 + 4 \qquad ---(2)$$

Equation (1) represents a parabola with vertex (0,0) and axis as y-axis. Equation (2) represents a parabola with vertex (0,4) and axis as y-axis. Points of intersection of parabolas are (2,8) and (−2,8).

A rough sketch of curves is given as:-



Region $AOCA$ is sliced into rectangles with area $(y_1 - y_2)\Delta x$. And it slides from $x = 0$ to $x = 2$

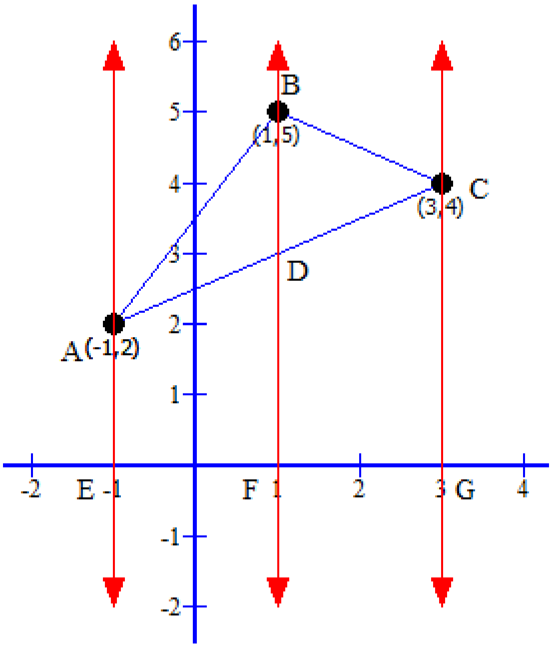
Required area = Region $AOBCA$

$$A = 2\left(\text{Region } AOCA\right)$$
$$= 2\int_0^2 (y_1 - y_2)\,dx$$
$$= 2\int_0^2 \left(x^2 + 4 - 2x^2\right)dx$$
$$= 2\int_0^2 \left(4 - x^2\right)dx$$
$$= 2\left[4x - \frac{x^3}{3}\right]_0^2$$
$$= 2\left[\left(8 - \frac{8}{3}\right) - (0)\right]$$
$$A = \frac{32}{3} \text{ sq.units}$$

Question 23

Using Integration, find the area of the region bounded by the triangle whose vertices are (- 1, 2), (1, 5) and (3, 4).

Solution 23



Equation of side AB,

$$\frac{x+1}{1+1}=\frac{y-2}{5-2}$$

$$\Rightarrow \frac{x+1}{2}=\frac{y-2}{3}$$

$$\Rightarrow 3x+3=2y-4$$

$$\Rightarrow 2y-3x=7$$

$$\therefore y=\frac{3x+7}{2}\text{.......(i)}$$

Equation of side BC,

$$\frac{x-1}{3-1}=\frac{y-5}{4-5}$$

$$\Rightarrow \frac{x-1}{2}=\frac{y-5}{-1}$$

$$\Rightarrow -x+1=2y-10$$

$$\Rightarrow 2y=11-x$$

$$\therefore y=\frac{11-x}{2}\text{......(ii)}$$

Equation of side AC,

$$\frac{x+1}{3+1}=\frac{y-2}{4-2}$$
$$\Rightarrow \frac{x+1}{4}=\frac{y-2}{2}$$
$$\Rightarrow \frac{x+1}{2}=\frac{y-2}{1}$$
$$\Rightarrow x+1=2y-4$$
$$\Rightarrow 2y=5+x$$
$$\therefore y=\frac{5+x}{2}$$

Area of required region
= Area of EABFE + Area of BFGCB - Area of AEGCA

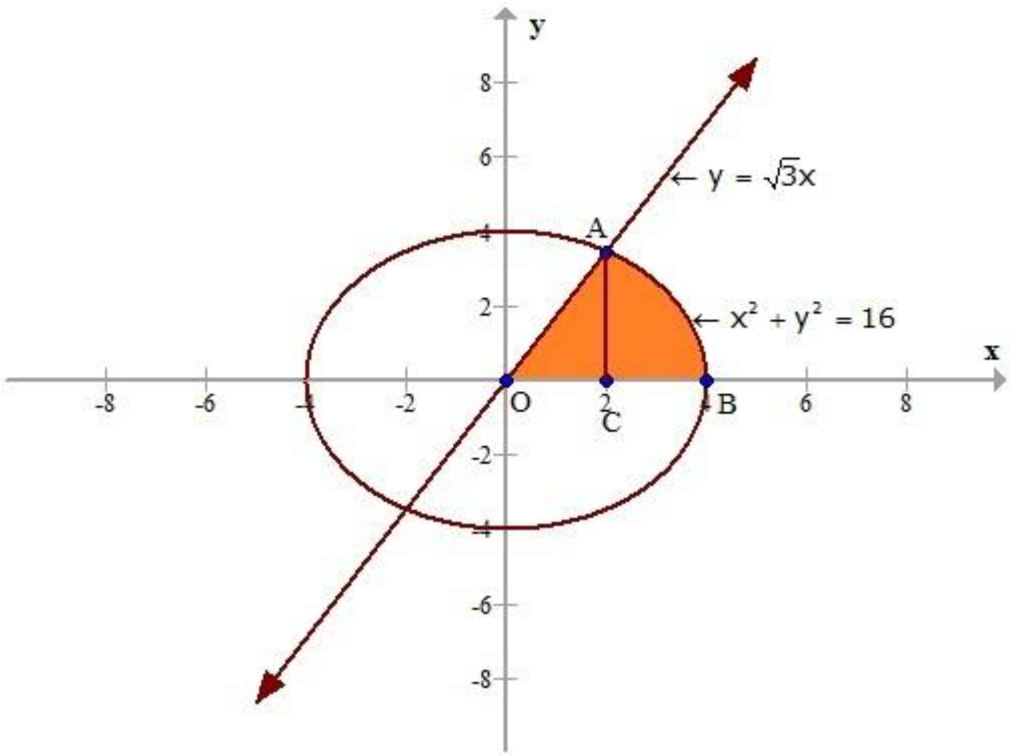
$$=\int_{-1}^{1} y_{AB}dx+\int_{1}^{3} y_{BC}dx-\int_{-1}^{3} y_{AC}dx$$
$$=\int_{-1}^{1}\frac{3x+7}{2}dx+\int_{1}^{3}\frac{11-x}{2}dx-\int_{-1}^{3}\frac{5+x}{2}dx$$
$$=\frac{1}{2}\left[\frac{3x^2}{2}+7x\right]_{-1}^{1}+\frac{1}{2}\left[11x-\frac{x^2}{2}\right]_{1}^{3}-\frac{1}{2}\left[5x+\frac{x^2}{2}\right]_{-1}^{3}$$
$$=\frac{1}{2}\left[\frac{3(1^2-1^2)}{2}+7(1-(-1))\right]+\frac{1}{2}\left[11(3-1)-\frac{(3)^2-1^2}{2}\right]$$
$$-\frac{1}{2}\left[5(3-(-1))+\frac{(3)^2-1^2}{2}\right]$$
$$=\frac{1}{2}[0+14]+\frac{1}{2}[22-4]-\frac{1}{2}[20+4]$$
$$=7+\frac{1}{2}\times 18-\frac{1}{2}\times 24$$
$$=7+9-12$$
$$=4 \text{ sq units}$$

Question 25

Find the area of the region in the first quadrant enclosed by x − axis, the line $y=\sqrt{3}x$ *and the circle* $x^2+y^2=16$

Solution 25

Consider the following graph.



y
x
$\leftarrow y = \sqrt{3}x$
$\leftarrow x^2 + y^2 = 16$
A
O
B
C
8
6
4
2
-2
-4
-6
-8

$We\ have,\ y=\sqrt{3}x$

$Substituting\ this\ value\ in\ x^2+y^2=16,$

$x^2+(\sqrt{3}x)^2=16$

$\Rightarrow x^2+3x^2=16$

$\Rightarrow 4x^2=16$

$\Rightarrow x^2=4$

$\Rightarrow x=\pm 2$

Since the shaded region is in the first quadrant, let us take the positive value of x.

Therefore, $x=2$ and $y=2\sqrt{3}$ are the coordinates of the intersection point A.

Thus, area of the shaded region $OAB = Area\ OAC + Area\ ACB$

$$\Rightarrow Area\ OAB=\int_0^2\sqrt{3}x\,dx+\int_2^4\sqrt{16-x^2}\,dx$$

$$\Rightarrow Area\ OAB=\left(\frac{\sqrt{3}x^2}{2}\right)_0^2+\frac{1}{2}\left[x\sqrt{16-x^2}+16\sin^{-1}\left(\frac{x}{4}\right)\right]_2^4$$

$$\Rightarrow Area\ OAB=\left(\frac{\sqrt{3}\times 4}{2}\right)+\frac{1}{2}\left[16\sin^{-1}\left(\frac{4}{4}\right)\right]-\frac{1}{2}\left[4\sqrt{16-12}+16\sin^{-1}\left(\frac{2}{4}\right)\right]$$

$$\Rightarrow Area\ OAB=2\sqrt{3}+\frac{1}{2}\left[16\times\frac{\pi}{2}\right]-\frac{1}{2}\left[4\sqrt{3}+16\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$\Rightarrow Area\ OAB=2\sqrt{3}+4\pi-2\sqrt{3}-\frac{4\pi}{3}$$

$$\Rightarrow Area\ OAB=4\pi-\frac{4\pi}{3}$$

$$\Rightarrow Area\ OAB=\frac{8\pi}{3}sq.\ units.$$

Question 26

Find the area of the region bounded by the parabola $y^2 = 2x + 1$ and the line $x - y - 1 = 0$.

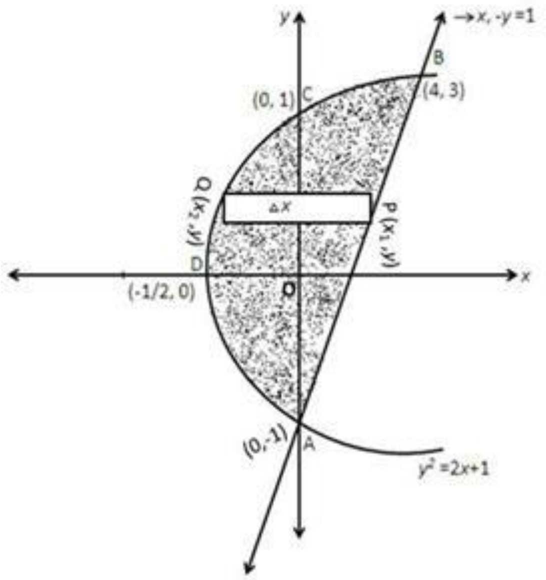
Solution 26

To find area bounded by

$$y^2 = 2x + 1 \qquad ---(1)$$
$$\text{and } x - y = 1 \qquad ---(2)$$

Equation (1) is a parabola with vertex $\left(-\frac{1}{2}, 0\right)$ and passes through $(0,1), (0,-1)$.
Equation (2) is a line passing through $(1,0)$ and $(0,-1)$. Points of intersection of parabola and line are $(3,2)$ and $(0,-1)$.

A rough sketch of the curves is given as:-



Shaded region represents the required area. It is sliced in rectangles of area $(x_1 - x_2)\Delta y$.
It slides from $y = -1$ to $y = 3$, so

Required area = Region $ABCDA$

$$= \int_{-1}^{3}(x_1 - x_2)dy$$

$$= \int_{-1}^{3}\left(1 + y - \frac{y^2 - 1}{2}\right)dy$$

$$= \frac{1}{2}\int_{-1}^{3}(2 + 2y - y^2 + 1)dy$$

$$= \frac{1}{2}\int_{-1}^{3}(3 + 2y - y^2)dy$$

$$= \frac{1}{2}\left[3y + y^2 - \frac{y^3}{3}\right]_{-1}^{3}$$

$$= \frac{1}{2}\left[(9 + 9 - 9) - \left(-3 + 1 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{2}\left[9 + \frac{5}{3}\right]$$

$$= \frac{32}{6}$$

Required area $= \frac{16}{3}$ sq. units

Question 27

Find the area of the region bounded by the curves $y = x - 1$ and $(y - 1)^2 = 4(x + 1)$

Solution 27

To find region bounded by curves

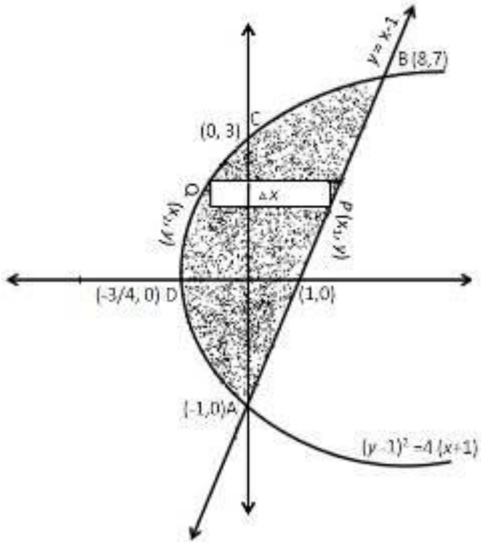
$$y = x - 1 \qquad ---(1)$$

$$\text{and } (y-1)^2 = 4(x+1) \qquad ---(2)$$

Equation (1) represents a line passing through $(1,0)$ and $(0,-1)$ equation (2) represents a parabola with vertex $(-1,1)$ passes through $(0,3)$, $(0,-1)$, $\left(-\frac{3}{4},0\right)$.

Their points of intersection $(0,-1)$ and $(8,7)$.

A rough sketch of curves is given as:-



Shaded region is required area. It is sliced in rectangles of area $(x_1 - x_2)\Delta y$.
It slides from $y = -1$ to $y = 7$, so

Required area = Region $ABCDA$

$$A = \int_{-1}^{7} (x_1 - x_2)\,dy$$

$$= \int_{-1}^{7} \left(y + 1 - \frac{(y-1)^2}{4} + 1 \right) dy$$

$$= \frac{1}{4}\int_{-1}^{7} \left(4y + 4 - y^2 - 1 + 2y + 4\right) dy$$

$$= \frac{1}{4}\int_{-1}^{7} \left(6y + 7 - y^2\right) dy$$

$$= \frac{1}{4}\left[3y^2 + 7y - \frac{y^3}{3}\right]_{-1}^{7}$$

$$= \frac{1}{4}\left[\left(147 + 49 - \frac{343}{3}\right) - \left(3 - 7 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[\frac{245}{3} + \frac{11}{3}\right]$$

$$A = \frac{64}{3} \text{ sq. units}$$

Question 29

Find the area bounded by the parabola $y = 2 - x^2$ and the straight line $y + x = 0$.

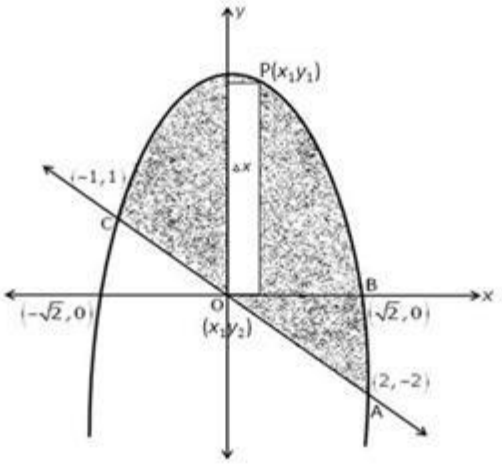
Solution 29

To find area bounded by

$$y = 2 - x^2 \qquad ---(1)$$

$$\text{and } y + x = 0 \qquad ---(2)$$

Equation (1) represents a parabola with vertex $(0,2)$ and downward, meets axes at $(\pm\sqrt{2}, 0)$.
Equation (2) represents a line passing through $(0,0)$ and $(2,-2)$. The points of intersection of line and parabola are $(2,-2)$ and $(-1,1)$.
A rough sketch of curves is as follows:-



Shaded region is sliced into rectangles with area = $(y_1 - y_2)\Delta x$. It slides from $x = -1$ to $x = 2$, so

Required area = Region $ABPCOA$

$$A = \int_{-1}^{2}(y_1 - y_2)dx$$

$$= \int_{-1}^{2}(2 - x^2 + x)dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$= \left[\left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)\right]$$

$$= \left[\frac{10}{3} + \frac{7}{6}\right]$$

$$= \frac{27}{6}$$

$A = \frac{9}{2}$ sq. units

Question 31

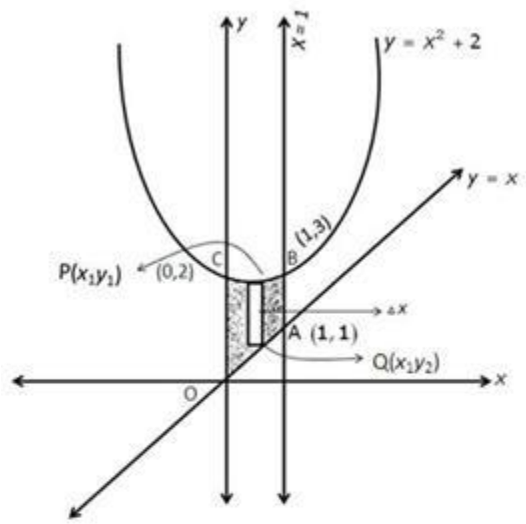
Sketch the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 1$. Also, find the area of this region.

Solution 31

To find area bounded by $x = 0$, $x = 1$
and

$$y = x \qquad ---(1)$$
$$y = x^2 + 2 \qquad ---(2)$$

Equation (1) is a line passing through $(2,2)$ and $(0,0)$. Equation (2) is a parabola upward with vertex at $(0,2)$. A rough sketch of curves is as under:-



Shaded region is sliced into rectangles of area $= (y_1 - y_2)\Delta x$. It slides from $x = 0$ to $x = 1$, so

Required area = Region $OABCO$

$$A = \int_0^1 (y_1 - y_2)dx$$
$$= \int_0^1 (x^2 + 2 - x)dx$$
$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^1$$
$$= \left[\left(\frac{1}{3} + 2 - \frac{1}{2}\right) - (0)\right]$$
$$= \left(\frac{2 + 12 - 3}{6}\right)$$
$$A = \frac{11}{6} \text{ sq. units}$$

Question 32

Find the area bounded by the curves $x = y^2$ and $x = 3 - 2y^2$.

Solution 32

To find area bounded by

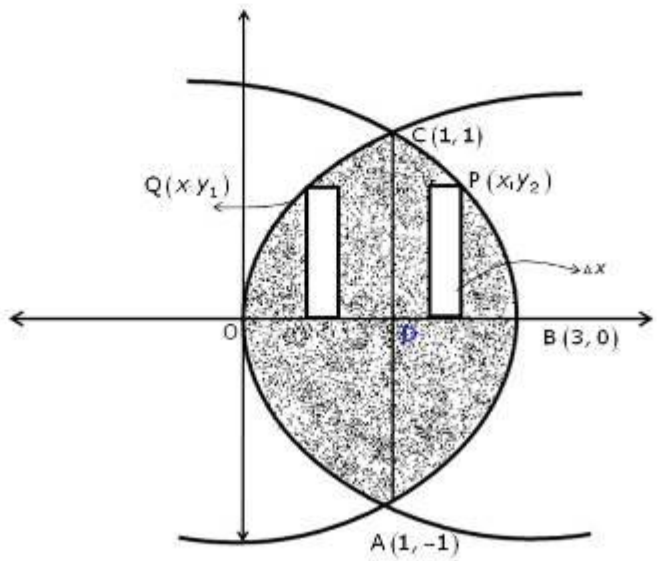
$x = y^2 \qquad ---(1)$

and

$x = 3 - 2y^2$

$2y^2 = -(x-3) \qquad ---(2)$

Equation (1) represents an upward parabola with vertex $(0,0)$ and axis -y. Equation (2) represents a parabola with vertex $(3,0)$ and axis as x-axis. They intersect at $(1,-1)$ and $(1,1)$. A rough sketch of the curves is as under:-



Required area = Region $OABCO$

$A = 2$ Region $OBCO$

$= 2[\text{Region } ODCO + \text{Region } BDCB]$

$= 2\left[\int_0^1 y_1 dx + \int_1^3 y_2 dx\right]$

$= 2\left[\int_0^1 \sqrt{x}dx + \int_1^3 \sqrt{\frac{3-x}{2}}dx\right]$

$= 2\left[\left(\frac{2}{3}x\sqrt{x}\right)_0^1 + \left(\frac{2}{3}.\left(\frac{3-x}{2}\right)\sqrt{\frac{3-x}{2}}.(-2)\right)_1^3\right]$

$= 2\left[\left(\frac{2}{3} - 0\right) + \left\{(0) - \left(\frac{2}{3}.1.1.(-2)\right)\right]\right]$

$= 2\left[\frac{2}{3} + \frac{4}{3}\right]$

$A = 4$ sq. units

Question 33

Using integration, find the area of the triangle ABC coordinates of whose vertices are

$A(4,1)$, $B(6,6)$ and $C(8,4)$.

Solution 33

To find area of $\triangle ABC$ with $A(4,1)$, $B(6,6)$ and $C(8,4)$.

Equation of AB,

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - 1 = \left(\frac{6-1}{6-4}\right)(x - 4)$$

$$y - 1 = \frac{5}{2}x - 10$$

$$y = \frac{5}{2}x - 9 \qquad ---(1)$$

Equation of BC,

$$y - 6 = \left(\frac{4-6}{8-6}\right)(x - 6)$$

$$= -1(x - 6)$$

$$y = -x + 12 \qquad ---(2)$$

Equation of AC,

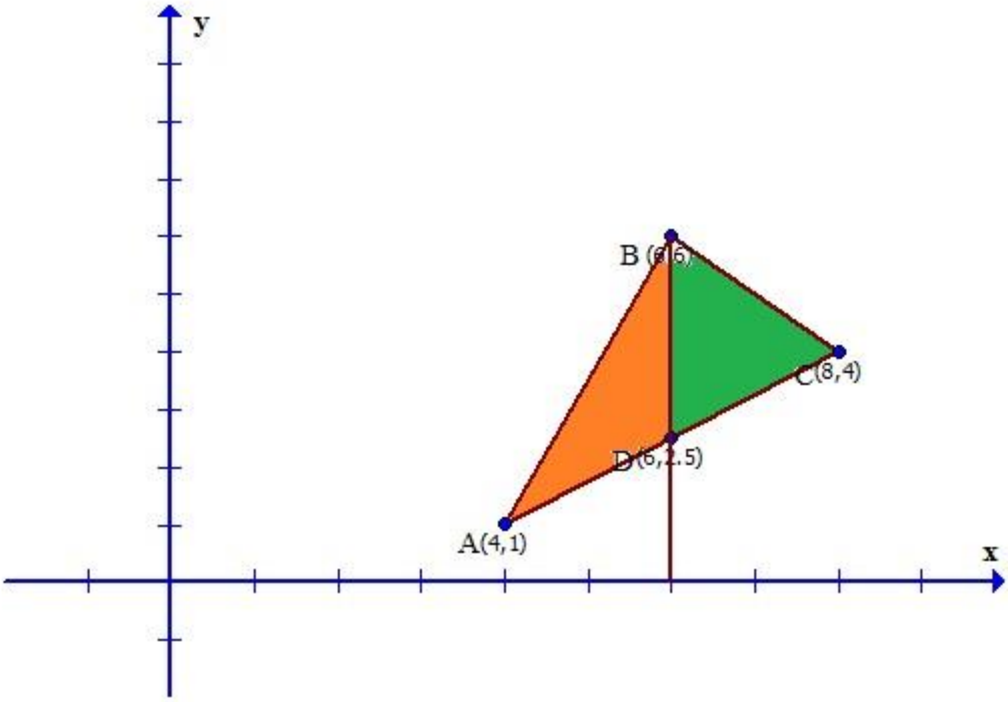
$$y - 1 = \left(\frac{4-1}{8-4}\right)(x - 4)$$

$$y - 1 = \frac{3}{4}(x - 4)$$

$$\Rightarrow \quad y = \frac{3}{4}x - 3 + 1$$

$$y = \frac{3}{4}x - 2 \qquad ---(3)$$

A rough sketch is as under:-



y
x
B (6,6)
C(8,4)
D(6,2.5)
A(4,1)

Clearly, Area of ΔABC = *Area ADB* + *Area BDC*

Area ADB: To find the area ADB, we slice it into vertical strips.

We observe that each vertical strip has its lower end on side AC and the upper end on AB. So the approximating rectangle has

Length $= y_2 - y_1$

Width $= \Delta x$

Area $= (y_2 - y_1)\Delta x$

Since the approximating rectangle can move from $x = 4$ *to* 6,

the area of the triangle ADB $= \int_4^6 (y_2 - y_1)dx$

$$\Rightarrow \text{ area of the triangle } ADB = \int_4^6 \left[\left(\frac{5x}{2} - 9\right) - \left(\frac{3}{4}x - 2\right)\right]dx$$

$$\Rightarrow \text{ area of the triangle } ADB = \int_4^6 \left(\frac{5x}{2} - 9 - \frac{3}{4}x + 2\right)dx$$

$$\Rightarrow \text{ area of the triangle } ADB = \int_4^6 \left(\frac{7x}{4} - 7\right)dx$$

$$\Rightarrow \text{ area of the triangle } ADB = \left(\frac{7x^2}{4 \times 2} - 7x\right)_4^6$$

$$\Rightarrow \text{ area of the triangle } ADB = \left(\frac{7 \times 36}{8} - 7 \times 6\right) - \left(\frac{7 \times 16}{8} - 7 \times 4\right)$$

$$\Rightarrow \text{ area of the triangle } ADB = \left(\frac{63}{2} - 42 - 14 + 28\right)$$

$$\Rightarrow \text{ area of the triangle } ADB = \left(\frac{63}{2} - 28\right)$$

Similarly, Area BDC $= \int_6^8 (y_4 - y_3)dx$

$$\Rightarrow \text{Area } BDC = \int_6^8 (y_4 - y_3)dx$$

$$\Rightarrow \text{Area } BDC = \int_6^8 \left[(-x + 12) - \left(\frac{3}{4}x - 2\right)\right]dx$$

$$\Rightarrow \text{Area } BDC = \int_6^8 \left[\frac{-7x}{4} + 14\right]dx$$

$$\Rightarrow \text{Area } BDC = \left[-\frac{7x^2}{8} + 14x\right]_6^8$$

$$\Rightarrow \text{Area } BDC = \left[-\frac{7 \times 64}{8} + 14 \times 8\right] - \left[-\frac{7 \times 36}{8} + 14 \times 6\right]$$

$$\Rightarrow \text{Area } BDC = \left[-56 + 112 + \frac{63}{2} - 84\right]$$

$$\Rightarrow \text{Area } BDC = \left(\frac{63}{2} - 28\right)$$

Thus, Area ABC = *Area ADB* + *Area BDC*

Question 34

Using integration, find the area of the region:

$$\left\{(x, y) : |x - 1| \le y \le \sqrt{5 - x^2}\right\}$$

Solution 34

To find area of region

$$\left\{(x,y): |x-1| \le y \le \sqrt{5-x^2}\right\}$$

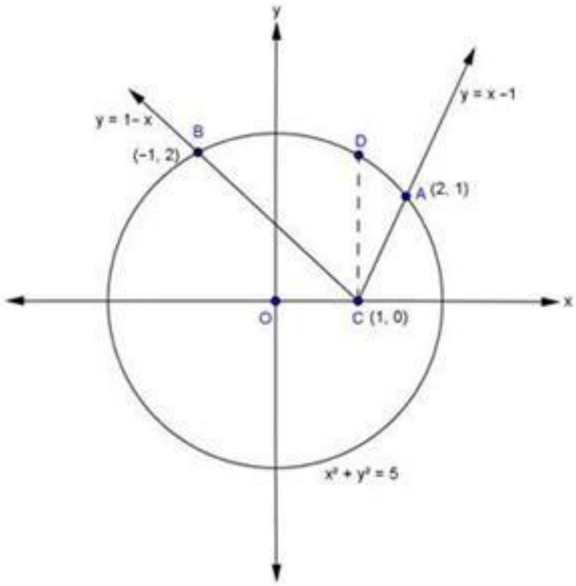
$\Rightarrow \quad |x-1| = y$

$\Rightarrow \quad y = \begin{cases} 1-x, \text{ if } x < 1 & ---(1) \\ x-1, \text{ if } x \ge 1 & ---(2) \end{cases}$

And $\quad x^2 + y^2 = 5 \qquad ---(3)$

Equation (1) and (2) represent straight lines and equation (3) is a circle with centre $(0,0)$, meets axes at $\left(\pm\sqrt{5},0\right)$ and $\left(0,\pm\sqrt{5}\right)$.

A rough sketch of the curves is as under:



Shaded region represents the required area.

Required area = Region $BCDB$ + Region $CADC$

$$A = \int_{-1}^{1}(y_1 - y_2)\,dx + \int_{1}^{2}(y_1 - y_2)\,dx$$
$$= \int_{-1}^{1}\left[\sqrt{5-x^2} - 1 + x\right]dx + \int_{1}^{2}\left(\sqrt{5-x^2} - x + 1\right)dx$$
$$= \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}} - x + \frac{x^2}{2}\right]_{-1}^{1} + \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}} - \frac{x^2}{2} + x\right]_{1}^{2}$$
$$= \left[\left(\frac{1}{2}.2 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - 1 + \frac{1}{2}\right) - \left(-\frac{1}{2}.2 - \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + 1 + \frac{1}{2}\right)\right]$$
$$+\left[\left(1.1. + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - 2 + 2\right) - \left(\frac{1}{2}.2 + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2} + 1\right)\right]$$

$$= \left[1 + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2} + 1 + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{3}{2}\right] + \left[1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - 1 - \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2}\right]$$

$$= 5\sin^{-1}\frac{1}{\sqrt{5}} + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2}$$

$$A = \left[\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right) - \frac{1}{2}\right] \text{ sq. units.}$$

Question 35

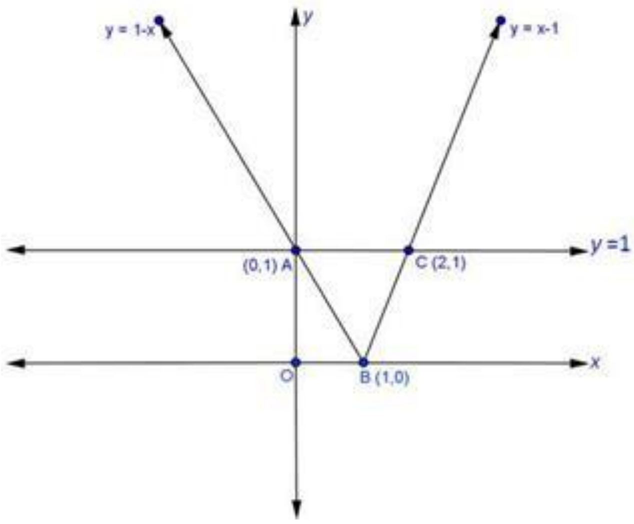
Find the area of the region bounded by $y = |x-1|$ and $y = 1$.

Solution 35

To find area bounded by $y = 1$ and

$$y = |x-1|$$
$$y = \begin{cases} x-1, \text{ if } x \ge 0 & ---(1) \\ 1-x, \text{ if } x < 0 & ---(2) \end{cases}$$

A rough sketch of the curve is as under:-



Shaded region is the required area. So

Required area = Region $ABCA$

$$
\begin{aligned}
A &= \text{Region } ABDA + \text{Region } BCDB \\
&= \int_0^1 (y_1 - y_2)\,dx + \int_1^2 (y_1 - y_3)\,dx \\
&= \int_0^1 (1 - 1 + x)\,dx + \int_1^2 (1 - x + 1)\,dx \\
&= \int_0^1 x\,dx + \int_1^2 (2 - x)\,dx \\
&= \left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2 \\
&= \left(\frac{1}{2} - 0\right) + \left[(4 - 2) - \left(2 - \frac{1}{2}\right)\right] \\
&= \frac{1}{2} + \left(2 - 2 + \frac{1}{2}\right)
\end{aligned}
$$

$A = 1$ sq. unit

Question 36

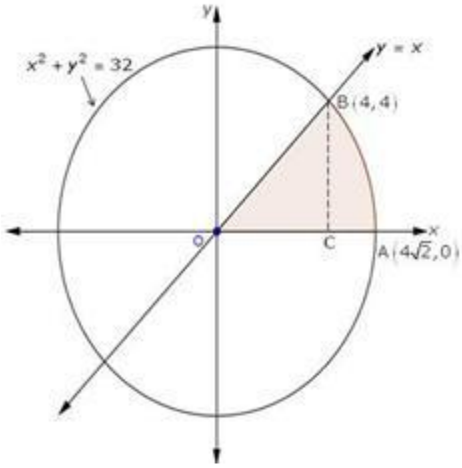
Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

Solution 36

To find area of in first quadrant enclosed by x-axis, the line $y = x$ and circle

$x^2 + y^2 = 32$ ---(1)

Equation (1) is a circle with centre (0,0) and meets axes at $(\pm 4\sqrt{2}, 0), (0, \pm 4\sqrt{2})$.
And $y = x$ is a line passes through (0,0) and intersect circle at (4,4).
A rough sketch of curve is as under:-



Required area is shaded region $OABO$

Region $OABO$ = Region $OCBO$ + Region $CABC$

$$= \int_0^4 y_1 dx + \int_4^{4\sqrt{2}} y_2 dx$$
$$= \int_0^4 x dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2} dx$$
$$= \left(\frac{x^2}{2}\right)_0^4 + \left[\frac{x}{2}\sqrt{32 - x^2} + \frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$
$$= (8 - 0) + \left[\left(0 + 16.\frac{\pi}{2}\right) - \left(8 + 16.\frac{\pi}{4}\right)\right]$$

$$= 8 + 8\pi - 8 - 4\pi$$

$A = 4\pi$ sq. units

Question 37

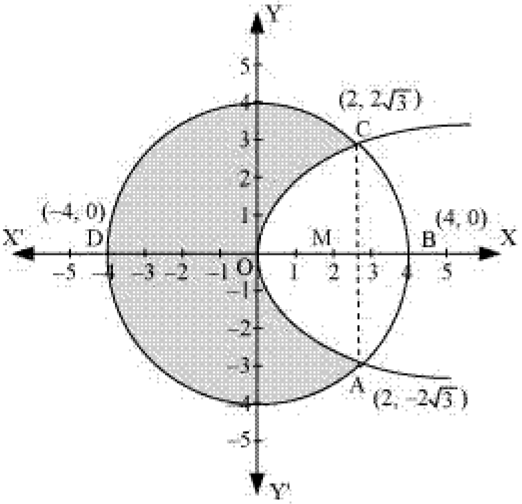
Find the area of the circle $x^2 + y^2 = 16$ which is exterior to the parabola $y^2 = 6x$.

Solution 37

The given equations are

$x^2 + y^2 = 16$ … (1)

$y^2 = 6x$ … (2)



Area bounded by the circle and parabola

$$= 2\left[\text{Area}(\text{OADO}) + \text{Area}(\text{ADBA})\right]$$

$$= 2\left[\int_0^2 \sqrt{16x}\,dx + \int_2^4 \sqrt{16 - x^2}\,dx\right]$$

$$= 2\left[\sqrt{6}\left\{\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right\}_0^2\right] + 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= 2\sqrt{6} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^2 + 2\left[8 \cdot \frac{\pi}{2} - \sqrt{16 - 4} - 8\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \frac{4\sqrt{6}}{3}\left(2\sqrt{2}\right) + 2\left[4\pi - \sqrt{12} - 8\frac{\pi}{6}\right]$$

$$= \frac{16\sqrt{3}}{3} + 8\pi - 4\sqrt{3} - \frac{8}{3}\pi$$

$$= \frac{4}{3}\left[4\sqrt{3} + 6\pi - 3\sqrt{3} - 2\pi\right]$$

$$= \frac{4}{3}\left[\sqrt{3} + 4\pi\right]$$

$$= \frac{4}{3}\left[4\pi + \sqrt{3}\right] \text{ square units}$$

$$\text{Area of circle} = \pi\,(r)^2$$

$$= \pi\,(4)^2 = 16\pi \text{ square units}$$

Thus, Required area $= 16\pi - \frac{4}{3}\left[4\pi + \sqrt{3}\right]$

$$= \frac{4}{3}\left[4 \times 3\pi - 4\pi - \sqrt{3}\right]$$

$$= \frac{4}{3}\left(8\pi - \sqrt{3}\right)$$

$$= \left(\frac{32}{3}\pi - \frac{4\sqrt{3}}{3}\right) \text{sq. units}$$

Question 39

Make a sketch of the region given below and find its area using integration.

$$\left\{(x,y) : 0 \le y \le x^2 + 3;\, 0 \le y \le 2x + 3;\, 0 \le x \le 3\right\}$$

Solution 39

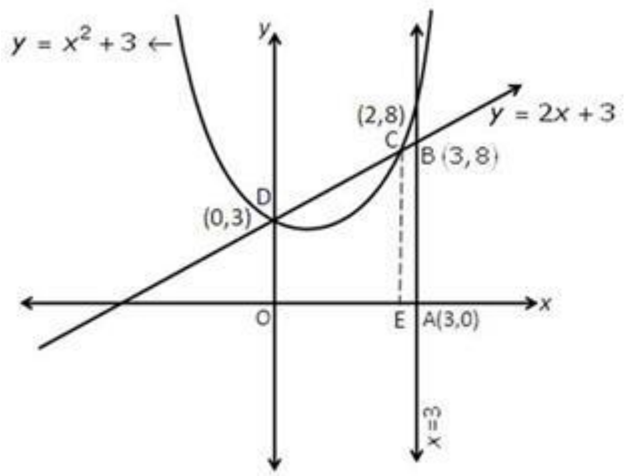
To find area of region

$$\left\{(x,y): 0 \le y \le x^2+3,\ 0 \le y \le 2x+3,\ 0 \le x \le 3\right\}$$

$$\Rightarrow \quad y = x^2 + 3 \qquad \text{---}(1)$$
$$y = 2x + 3 \qquad \text{---}(2)$$

and $\quad x = 0, x = 3$

Equation (1) represents a parabola with vertex $(3,0)$ and axis as y-axis. Equation (2) represents a line a passing through $(0,3)$ and $\left(-\frac{3}{2},0\right)$, a rough sketch of curve is as under:-



Required area = Region $ABCDOA$

A = Region $ABCEA$ + Region $ECDOE$

$$= \int_2^3 y_1 dx + \int_0^2 y_2 dx$$
$$= \int_2^3 (2x+3)\, dx + \int_0^2 \left(x^2+3\right) dx$$
$$= \left(x^2+3x\right)_2^3 + \left(\frac{x^3}{3}+x\right)_0^2$$
$$= \left[(9+9)-(4+6)\right] + \left[\left(\frac{8}{3}+2\right)-(0)\right]$$
$$= \left[18-10\right] + \left[\frac{14}{3}\right]$$

$$= 8 + \frac{14}{3}$$

$$A = \frac{38}{3} \text{ sq. units}$$

Question 40

Find the area of the region bounded by the curve $y = \sqrt{1 - x^2}$, line $y = x$ and the positive x-axis.

Solution 40

To find area bounded by positive x-axis and curve

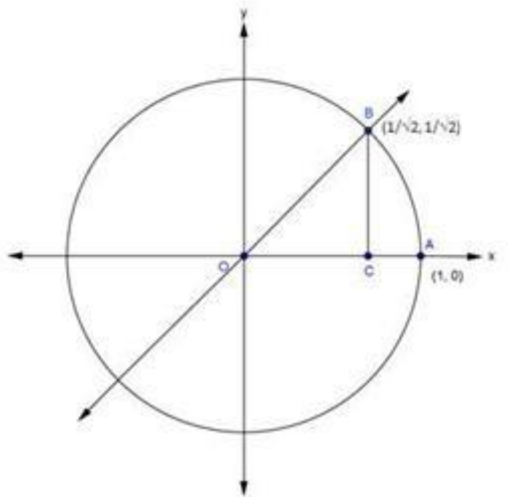
$y = \sqrt{1 - x^2}$

$x^2 + y^2 = 1 \qquad ---(1)$

$x = y \qquad ---(2)$

Equation (1) represents a circle with centre $(0,0)$ and meets axes at $(\pm1,0),(0,\pm1)$.

Equation (2) represents a line passing through $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ and they are also points of intersection. A rough sketch of the curve is as under:-



Required area = Region $OABO$

A = Region $OCBO$ + Region $CABC$

$$= \int_0^{\frac{1}{\sqrt{2}}} y_1 dx + \int_{\frac{1}{\sqrt{2}}}^1 y_2 dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x dx + \int_{\frac{1}{\sqrt{2}}}^1 \sqrt{1 - x^2} dx$$

$$= \left[\frac{x^2}{2}\right]_0^{\frac{1}{\sqrt{2}}} + \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_{\frac{1}{\sqrt{2}}}^1$$

$$= \left[\frac{1}{4} - 0\right] + \left[\left(0 + \frac{1}{2}.\frac{\pi}{2}\right) - \left(\frac{1}{2\sqrt{2}}.\frac{1}{\sqrt{2}} + \frac{1}{2}.\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{4} + \frac{\pi}{4} - \frac{1}{4} - \frac{\pi}{8}$$

$A = \frac{\pi}{8}$ sq. units

Question 41

Find the area bounded by the line $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$.

Solution 41

To find area bounded by lines

$y = 4x + 5$ (Say AB) ---(1)

$y = 5 - x$ (Say BC) ---(2)

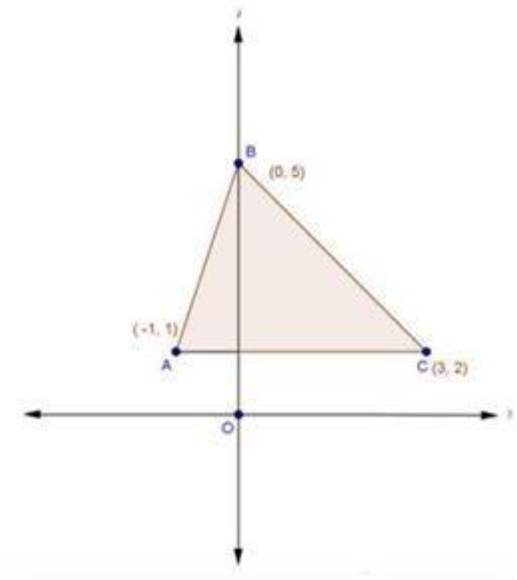
$4y = x + 5$ (Say AC) ---(3)

By solving equation (1) and (2), we get $B(0, 5)$

By solving equation (2) and (3), we get $C(3, 2)$

By solving equation (1) and (3), we get $A(-1, 1)$

A rough sketch of the curve is as under:-



Shaded area $\triangle ABC$ is the required area.

Required area $= ar\left(\triangle ABD\right) + ar\left(\triangle BDC\right) \qquad ---(1)$

$$ar\left(\triangle ABD\right) = \int_{-1}^{0}\left(y_1 - y_3\right)dx$$

$$= \int_{-1}^{0}\left(4x + 5 - \frac{x}{4} - \frac{5}{4}\right)dx$$

$$= \int_{-1}^{0}\left(\frac{15x}{4} + \frac{15}{4}\right)dx$$

$$= \frac{15}{4}\left(\frac{x^2}{2}+x\right)_{-1}^{0}$$

$$= \frac{15}{4}\left[\left(0\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{15}{4} \times \frac{1}{2}$$

$$ar\left(\triangle ABD\right) = \frac{15}{8} \text{ sq. units} \qquad ---(2)$$

$$ar\left(\triangle BDC\right) = \int_{0}^{3}\left(y_2 - y_3\right)dx$$

$$= \int_{0}^{3}\left[\left(5 - x\right) - \left(\frac{x}{4} + \frac{5}{4}\right)\right]dx$$

$$= \int_{0}^{3}\left[5 - x - \frac{x}{4} - \frac{5}{4}\right]dx$$

$$= \int_{0}^{3}\left(\frac{-5x}{4} + \frac{15}{4}\right)dx$$

$$= \frac{5}{4}\left(3x - \frac{x^2}{2}\right)$$

$$= \frac{5}{4}\left(9 - \frac{9}{2}\right)$$

$$ar\left(\triangle BDC\right) = \frac{45}{8} \text{ sq. units} \qquad ---(3)$$

Using equation $(1), (2)$ and (3),

$$ar\left(\triangle ABC\right) = \frac{15}{8} + \frac{45}{8}$$

$$= \frac{60}{8}$$

$$ar\left(\triangle ABC\right) = \frac{15}{2} \text{ sq. units}$$

Question 42

Find the area of the region enclosed between the two curves $x^2 + y^2 = 9$ and $(x - 3)^2 + y^2 = 9$.

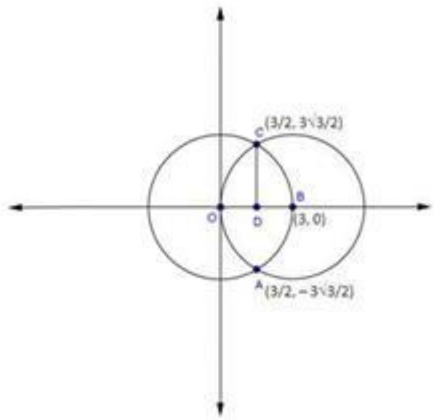
Solution 42

To find area enclosed by

$$x^2 + y^2 = 9 \qquad ---(1)$$

$$(x-3)^2 + y^2 = 9 \qquad ---(2)$$

Equation (1) represents a circle with centre $(0,0)$ and meets axes at $(\pm 3,0),(0,\pm 3)$.
Equation (2) is a circle with centre $(3,0)$ and meets axes at $(0,0),(6,0)$.
they intersect each other at $\left(\frac{3}{2},\frac{3\sqrt{3}}{2}\right)$ and $\left(\frac{3}{2},-\frac{3\sqrt{3}}{2}\right)$. A rough sketch of the curves is as under:



Shaded region is the required area.

Required area = Region $OABCO$

$$A = 2\left(\text{Region } OBCO\right)$$
$$= 2\left(\text{Region } ODCO + \text{Region } DBCD\right)$$
$$= 2\left[\int_0^{\frac{3}{2}}\sqrt{9-(x-3)^2}\,dx + \int_{\frac{3}{2}}^{3}\sqrt{9-x^2}\,dx\right]$$
$$= 2\left[\left\{\frac{(x-3)}{2}\sqrt{9-(x-3)^2}+\frac{9}{2}\sin^{-1}\frac{(x-3)}{3}\right\}_0^{\frac{3}{2}} + \left\{\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right\}_{\frac{3}{2}}^{3}\right]$$
$$= 2\left[\left\{\left(-\frac{3}{4}\sqrt{9-\frac{9}{4}}+\frac{9}{2}\sin^{-1}\left(-\frac{3}{6}\right)\right)-\left(0+\frac{9}{2}\sin^{-1}(-1)\right)\right\}+\left\{\left(0+\frac{9}{2}\sin^{-1}(1)\right)-\left(\frac{3}{4}\sqrt{9-\frac{9}{4}}+\frac{9}{2}\sin^{-1}\left(\frac{1}{2}\right)\right)\right\}\right]$$
$$= 2\left[\left\{-\frac{9\sqrt{3}}{8}-\frac{9}{2}.\frac{\pi}{6}+\frac{9}{2}.\frac{\pi}{2}\right\}+\left\{\frac{9}{2}.\frac{\pi}{2}-\frac{9\sqrt{3}}{8}-\frac{9}{2}.\frac{\pi}{6}\right\}\right]$$
$$= 2\left[-\frac{9\sqrt{3}}{8}-\frac{3\pi}{4}+\frac{9\pi}{4}+\frac{9\pi}{4}-\frac{9\sqrt{3}}{8}-\frac{3\pi}{4}\right]$$

$$= 2\left[\frac{12\pi}{4}-\frac{18\sqrt{3}}{8}\right]$$

$$A = \left(6\pi - \frac{9\sqrt{3}}{2}\right) \text{ sq. units}$$

Question 43

Find the area of the region, $\{(x,y): x^2+y^2 \le 4,\ x+y \ge 2\}$

Solution 43

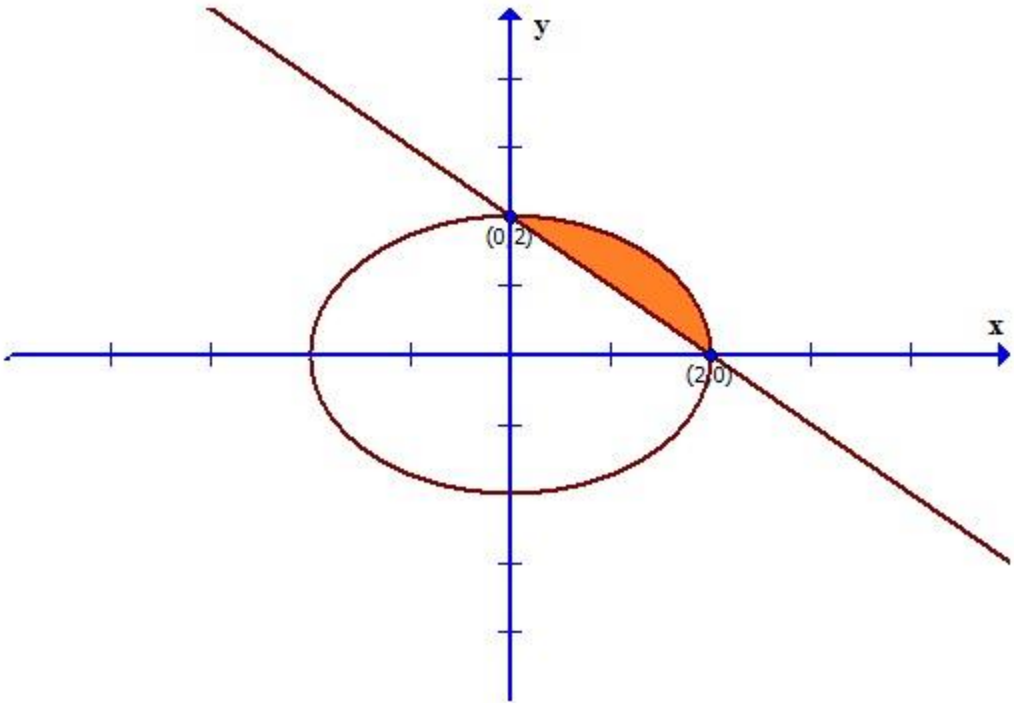
The equation of the given curves are

$x^2 + y^2 = 4 \ldots .(1)$

$x + y = 2 \ldots \ldots .(2)$

Clearly $x^2 + y^2 = 4$ *represents a circle and* $x + y = 2$ *is the equation of a straight line cutting x and y axes at* (0,2) *and* (2,0) *respectively.*

The smaller region bounded by these two curves is shaded in the following figure.



Length $= y_2 - y_1$

Width $= \Delta x$ *and*

Area $= (y_2 - y_1)\Delta x$

Since the approximating rectangle can move from $x = 0$ *to* $x = 2$, *the required area is given by*

$$A = \int_0^2 (y_2 - y_1)dx$$

We have $y_1 = 2 - x$ *and* $y_2 = \sqrt{4 - x^2}$

Thus,

$$A = \int_0^2 \left(\sqrt{4 - x^2} - 2 + x\right)dx$$

$$\Rightarrow A = \int_0^2 \left(\sqrt{4 - x^2}\right)dx - 2\int_0^2 dx + \int_0^2 xdx$$

$$\Rightarrow A = \left[\frac{x\sqrt{4 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 - 2(x)_0^2 + \left(\frac{x^2}{2}\right)_0^2$$

$$\Rightarrow A = \frac{4}{2}\sin^{-1}\left(\frac{2}{2}\right) - 4 + 2$$

$$\Rightarrow A = 2\sin^{-1}(1) - 2$$

$$\Rightarrow A = 2 \times \frac{\pi}{2} - 2$$

$$\Rightarrow A = \pi - 2 \text{ sq.units}$$

Question 44

Using integration, find the area of the following region:

$$\left\{(x, y) : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \leq \frac{x}{3} + \frac{y}{2}\right\}$$

Solution 44

To find area of region

$$\left\{(x, y) : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \leq \frac{x}{3} + \frac{y}{2}\right\}$$

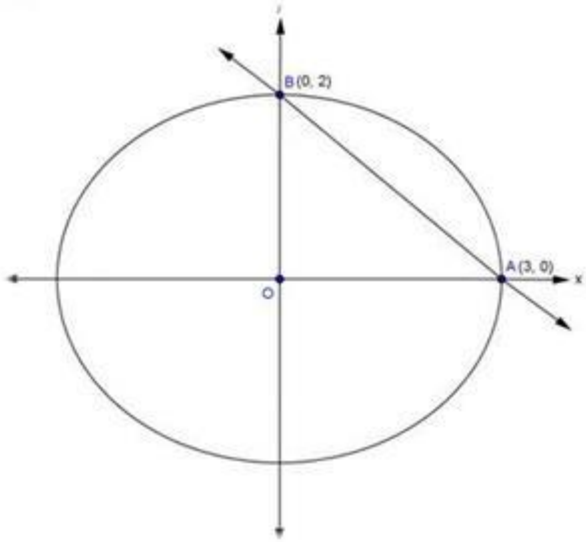
Here

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad ---(1)$$

$$\frac{x}{3} + \frac{y}{2} = 1 \qquad ---(2)$$

Equation (1) represents an ellipse with centre at origin and meets axes at $(\pm 3, 0)$, $(0, \pm 2)$. Equation (2) is a line that meets axes at $(3, 0), (0, 2)$.

A rough sketch is as under:



Shaded region represents required area. This is sliced into rectangles with area $\left(y_1 - y_2\right)\Delta x$ which slides from $x = 0$ to $x = 3$, so

Required area = Region $APBQA$

$$A = \int_0^3 \left(y_1 - y_2\right) dx$$

$$= \int_0^3 \left[\frac{2}{3}\sqrt{9 - x^2}dx - \frac{2}{3}\left(3 - x\right)dx\right]$$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - 3x + \frac{x^2}{2}\right]_0^3$$

$$= \frac{2}{3}\left[\left\{0 + \frac{9}{2}.\frac{\pi}{2} - 9 + \frac{9}{2}\right\} - \left\{0\right\}\right]$$

$$= \frac{2}{3}\left[\frac{9\pi}{4} - \frac{9}{2}\right]$$

$$A = \left(\frac{3\pi}{2} - 3\right) \text{ sq. units}$$

Question 45

Find the area enclosed by the curves $y = \left|x - 1\right|$ and $y = -\left|x - 1\right| + 1$.

Solution 45

To find area enclosed by

$$y = |x - 1|$$

$$\Rightarrow \quad y = \begin{cases} -(x-1), \text{ if } x - 1 < 0 \\ (x-1), \text{ if } x - 1 \geq 0 \end{cases}$$

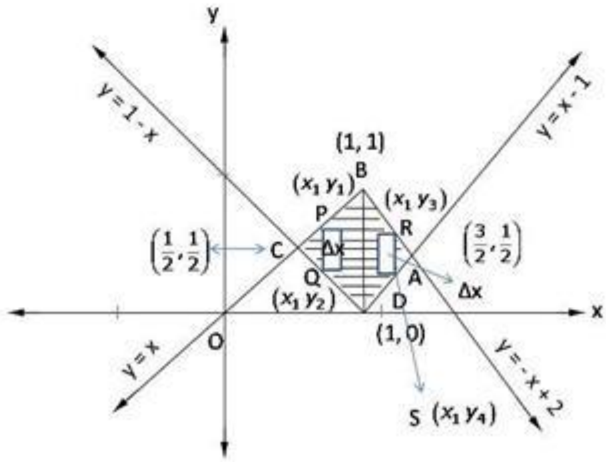
$$\Rightarrow \quad y = \begin{cases} 1 - x, \text{ if } x < 1 & ---(1) \\ x - 1, \text{ if } x \geq 1 & ---(2) \end{cases}$$

And $\quad y = -|x - 1| + 1$

$$\Rightarrow \quad y = \begin{cases} +(x-1)+1, \text{ if } x - 1 < 0 \\ -(x-1)+1, \text{ if } x - 1 \geq 0 \end{cases}$$

$$y = \begin{cases} x, \quad \text{ if } x < 1 & ---(3) \\ -x + 2, \text{ if } x \geq 1 & ---(4) \end{cases}$$

A rough sketch of equation of lines $(1),(2),(3),(4)$ is given as:



Shaded region is the required area.

Required area = Region $ABCDA$

Required area = Region $BDCB$ + Region $ABDA$ $\quad --- (1)$

Region $BDCB$ is sliced into rectangles of area = $(y_1 - y_2)\Delta x$ and it slides from $x = \frac{1}{2}$ to $x = 1$

Region $ABDA$ is sliced into rectangle of area = $(y_3 - y_4)\Delta x$ and it slides from $x = 1$ to $x = \frac{3}{2}$. So, using equation (1),

Required area = Region $BDCB$ + Region $ABDA$

$$= \int_{\frac{1}{2}}^{1}(y_1 - y_2)dx + \int_{1}^{\frac{3}{2}}(y_3 - y_4)dx$$

$$= \int_{\frac{1}{2}}^{1}(x - 1 + x)dx + \int_{1}^{\frac{3}{2}}(-x + 2 - x + 1)dx$$

$$= \int_{\frac{1}{2}}^{1}(2x - 1)dx + \int_{1}^{\frac{3}{2}}(3 - 2x)dx$$

$$= \left[x^2 - x\right]_{\frac{1}{2}}^{1} + \left[3x - x^2\right]_{1}^{\frac{3}{2}}$$

$$= \left[(1 - 1) - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[\left(\frac{9}{2} - \frac{9}{4}\right) - (3 - 1)\right]$$

$$= \frac{1}{4} + \frac{9}{4} - 2$$

$A = \frac{1}{2}$ sq.units

Question 46

Find the area enclosed by the curves $3x^2 + 5y = 32$ and $y = |x - 2|$.

Solution 46

To find area enclosed by

$$3x^2 + 5y = 32$$

$$3x^2 = -5\left(y - \frac{32}{5}\right) \qquad ---(1)$$

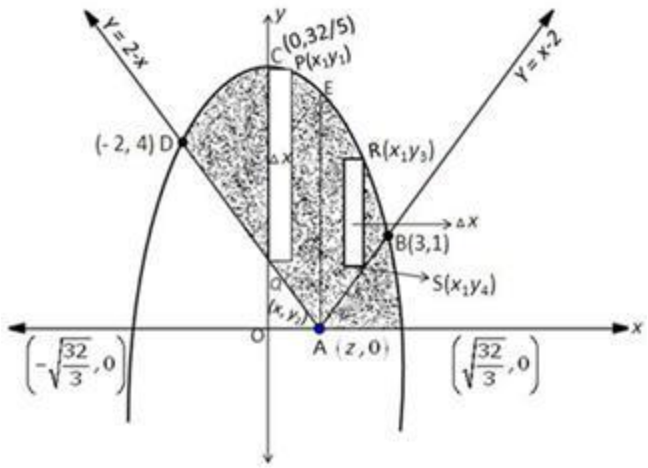
And

$$y = |x - 2|$$

$$\Rightarrow \qquad y = \begin{cases} -(x-2), \text{ if } x - 2 < 1 \\ (x-2), \text{ if } x - 2 \geq 1 \end{cases}$$

$$\Rightarrow \qquad y = \begin{cases} 2 - x, \text{ if } x < 2 \\ x - 2, \text{ if } x \geq 2 \end{cases} \qquad ---(2)$$

Equation (1) represents a downward parabola with vertex $\left(0, \frac{32}{5}\right)$ and equation (2) represents lines. A rough sketch of curves is given as:-



Required area = Region $ABECDA$

$$A = \text{Region } ABEA + \text{Region } AECDA$$
$$= \int_2^3 (y_3 - y_4)\,dx + \int_{-2}^2 (y_1 - y_2)\,dx$$
$$= \int_2^3 \left(\frac{32 - 3x^2}{5} - x + 2\right) dx + \int_{-2}^2 \left(\frac{32 - 3x^2}{5} - 2 + x\right) dx$$
$$= \int_2^3 \left(\frac{32 - 3x^2 - 5x + 10}{5}\right) dx + \int_{-2}^2 \left(\frac{32 - 3x^2 - 10 + 5x}{5}\right) dx$$
$$= \frac{1}{5}\left[\int_2^3 \left(42 - 3x^2 - 5x\right) dx + \int_{-2}^2 \left(22 - 3x^2 + 5x\right) dx\right]$$

$$A = \frac{1}{5}\left[\left(42x - x^3 - \frac{5x^2}{2}\right)_2^3 + \left(22x - x^3 + \frac{5x^2}{2}\right)_{-2}^2\right]$$
$$= \frac{1}{5}\left[\left\{\left(126 - 27 - \frac{45}{2}\right) - \left(84 - 8 - 10\right)\right\} + \left\{\left(44 - 8 + 10\right) - \left(-44 + 8 + 10\right)\right\}\right]$$
$$= \frac{1}{5}\left[\left\{\frac{153}{2} - 66\right\} + \left\{46 + 26\right\}\right]$$

$$= \frac{1}{5}\left[\frac{21}{2} + 72\right]$$

$$A = \frac{33}{2} \text{ sq. units}$$

Question 47

Find the area enclosed by the parabolas $y = 4x - x^2$ and $y = x^2 - x$.

Solution 47

To area enclosed by

$y = 4x - x^2$

$\Rightarrow \quad -y = x^2 - 4x + 4 - 4$

$\Rightarrow \quad -y + 4 = (x - 2)^2$

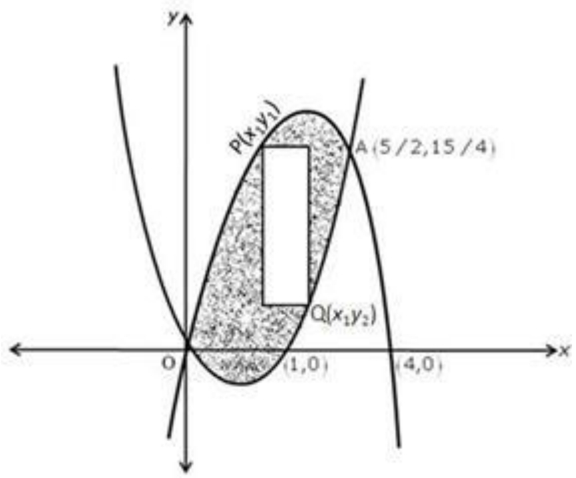
$\Rightarrow \quad -(y - 4) = (x - 2)^2 \qquad ---(1)$

and $y = x^2 - x$

$\left(y + \frac{1}{4}\right) = \left(x - \frac{1}{2}\right)^2 \qquad ---(2)$

Equation (1) represents a parabola downward with vertex at $(2,4)$ and meets axes at $(4,0),(0,0)$. Equation (2) represents a parabola upword whose vertex is $\left(\frac{1}{2}, -\frac{1}{4}\right)$ and meets axes at $(1,0),(0,0)$. Points of intersection of parabolas are $(0,0)$ and $\left(\frac{5}{2}, \frac{15}{4}\right)$.

A rough sketch of the curves is as under:-



Shaded region is required area it is sliced into rectangles with area = $(y_1 - y_2)\Delta x$. It slides from $x = 0$ to $x = \frac{5}{2}$, so

Required area = Region $OQAP$

$$A = \int_0^{\frac{5}{2}} (y_1 - y_2) dx$$

$$= \int_0^{\frac{5}{2}} \left[4x - x^2 - x^2 + x\right] dx$$

$$= \int_0^{\frac{5}{2}} \left[5x - 2x^2\right] dx$$

$$= \left[\frac{5x^2}{2} - \frac{2}{3}x^3\right]_0^{\frac{5}{2}}$$

$$= \left[\left(\frac{125}{8} - \frac{250}{24}\right) - (0)\right]$$

$$A = \frac{125}{24} \text{ sq. units}$$

Question 48

In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$?

Solution 48

Given curves are

$y = 4x - x^2$

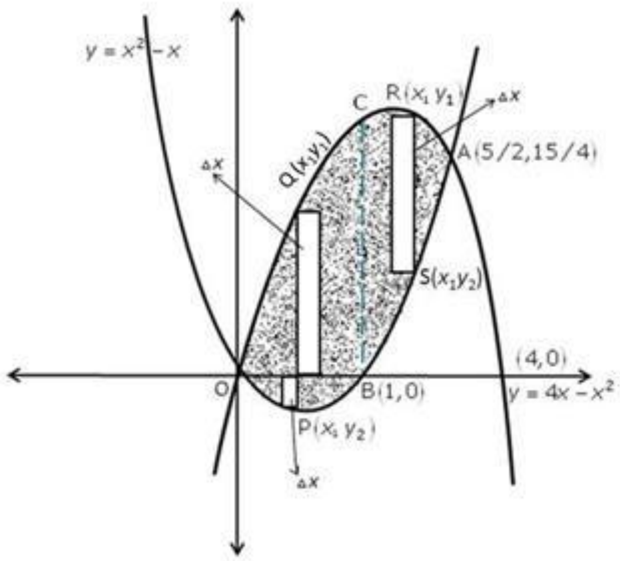
$\Rightarrow \quad -(y-4) = (x-2)^2 \qquad ---(1)$

and

$y = x^2 - x$

$\Rightarrow \quad \left(y+\frac{1}{4}\right)^2 = \left(x-\frac{1}{2}\right)^2 \qquad ---(2)$

Equation (1) represents a parabola downward with vertex at $(2,4)$ and meets axes at $(4,0),(0,0)$. Equation (2) represents a parabola upward whose vertex is $\left(\frac{1}{2},-\frac{1}{4}\right)$ and meets axes at $(1,0),(0,0)$ and $\left(\frac{5}{2},\frac{15}{4}\right)$. A rough sketch of the curves is as under:-



Area of the region above x-axis

A_1 = Area of region $OBACO$

$= \text{Region } OBCO + \text{Region } BACB$

$= \int_0^1 y_1 dx + \int_1^{\frac{5}{2}} \left(y_1 - y_2\right) dx$

$= \int_0^1 \left(4x - x^2\right) dx + \int_1^{\frac{5}{2}} \left(4x - x^2 - x^2 + x\right) dx$

$= \left(\frac{4x^2}{2} - \frac{x^3}{3}\right)_0^1 + \left[\frac{5x^2}{2} - \frac{2x^3}{3}\right]_1^{\frac{5}{2}}$

$= \left(2 - \frac{1}{3}\right) + \left[\left(\frac{125}{8} - \frac{250}{24}\right) - \left(\frac{5}{2} - \frac{2}{3}\right)\right]$

$= \frac{5}{3} + \frac{125}{24} - \frac{11}{6}$

$= \frac{121}{24}$ sq. units

Area of the region below x-axis

A_2 = Area of region $OPBO$

$= \text{Region } OBCO + \text{Region } BACB$

$= \left|\int_0^1 y_2 dx\right|$

$= \left|\int_0^1 \left(x^2 - x\right) dx\right|$

$= \left|\left(\frac{x^3}{3} - \frac{x^2}{2}\right)_0^1\right|$

$= \left|\left(\frac{1}{3} - \frac{1}{2}\right) - (0)\right|$

$= \left|-\frac{1}{6}\right|$

$A_2 = \frac{1}{6}$ sq. units

$A_1 : A_2 = \frac{121}{24} : \frac{1}{6}$

$\Rightarrow \quad A_1 : A_2 = \frac{121}{24} : \frac{4}{24}$

$\Rightarrow \quad A_1 : A_2 = 121 : 4$

Question 49

Find the area of the figure bounded by the curves $y = |x - 1|$ and $y = 3 - |x|$.

Solution 49

To find area bounded by the curve

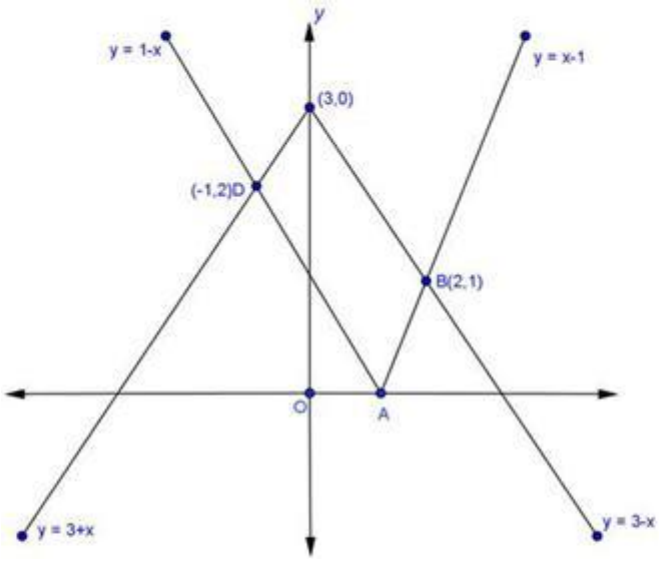
$$y = |x - 1|$$

$$\Rightarrow \quad y = \begin{cases} 1 - x, \text{ if } x < 1 & ---(1) \\ x - 1, \text{ if } x \geq 1 & ---(2) \end{cases}$$

and $\quad y = 3 - |x|$

$$\Rightarrow \quad y = \begin{cases} 3 + x, \text{ if } x < 0 & ---(3) \\ 3 - x, \text{ if } x \geq 0 & ---(4) \end{cases}$$

Drawing the rough sketch of lines $(1),(2),(3)$ and (4) as under:-



Shaded region is the required area

Required area = Region $ABCDA$

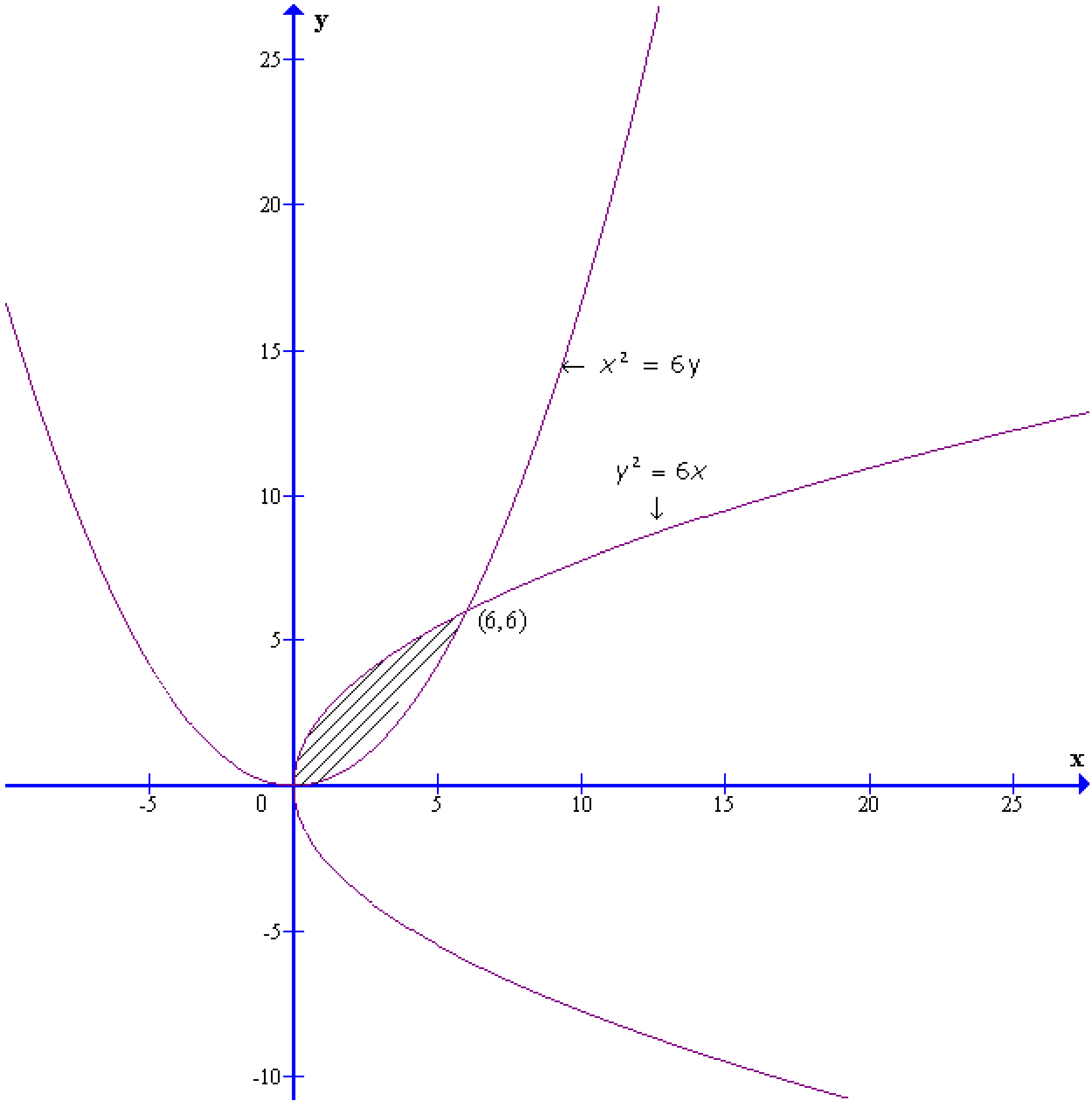
$$\begin{aligned} A &= \text{Region } ABFA + \text{Region } AFCEA + \text{Region } CDEC \\ &= \int_1^2 (y_1 - y_2)\,dx + \int_0^1 (y_1 - y_3)\,dx + \int_{-1}^0 (y_4 - y_3)\,dx \\ &= \int_1^2 (3 - x - x + 1)\,dx + \int_0^1 (3 - x - 1 + x)\,dx + \int_{-1}^0 (3 + x - 1 + x)\,dx \\ &= \int_1^2 (4 - 2x)\,dx + \int_0^1 2dx + \int_{-1}^0 (2 + 2x)\,dx \\ &= \left[4x - x^2\right]_1^2 + \left[2x\right]_0^1 + \left[2x + x^2\right]_{-1}^0 \\ &= \left[(8 - 4) - (4 - 1)\right] + \left[2 - 0\right] + \left[(0) - (-2 + 1)\right] \\ &= (4 - 3) + 2 + 1 \end{aligned}$$

$A = 4$ sq. unit

Question 1

Calculate the area of the region bounded by the parabolas $y^2 = 6x$ and $x^2 = 6y$.

Solution 1



Area of the bounded region

$$= \int_0^6 \sqrt{6x} - \frac{x^2}{6} \, dx$$

$$= \left[\sqrt{6} \, \frac{x^{3/2}}{3/2} - \frac{x^3}{18} \right]_0^6$$

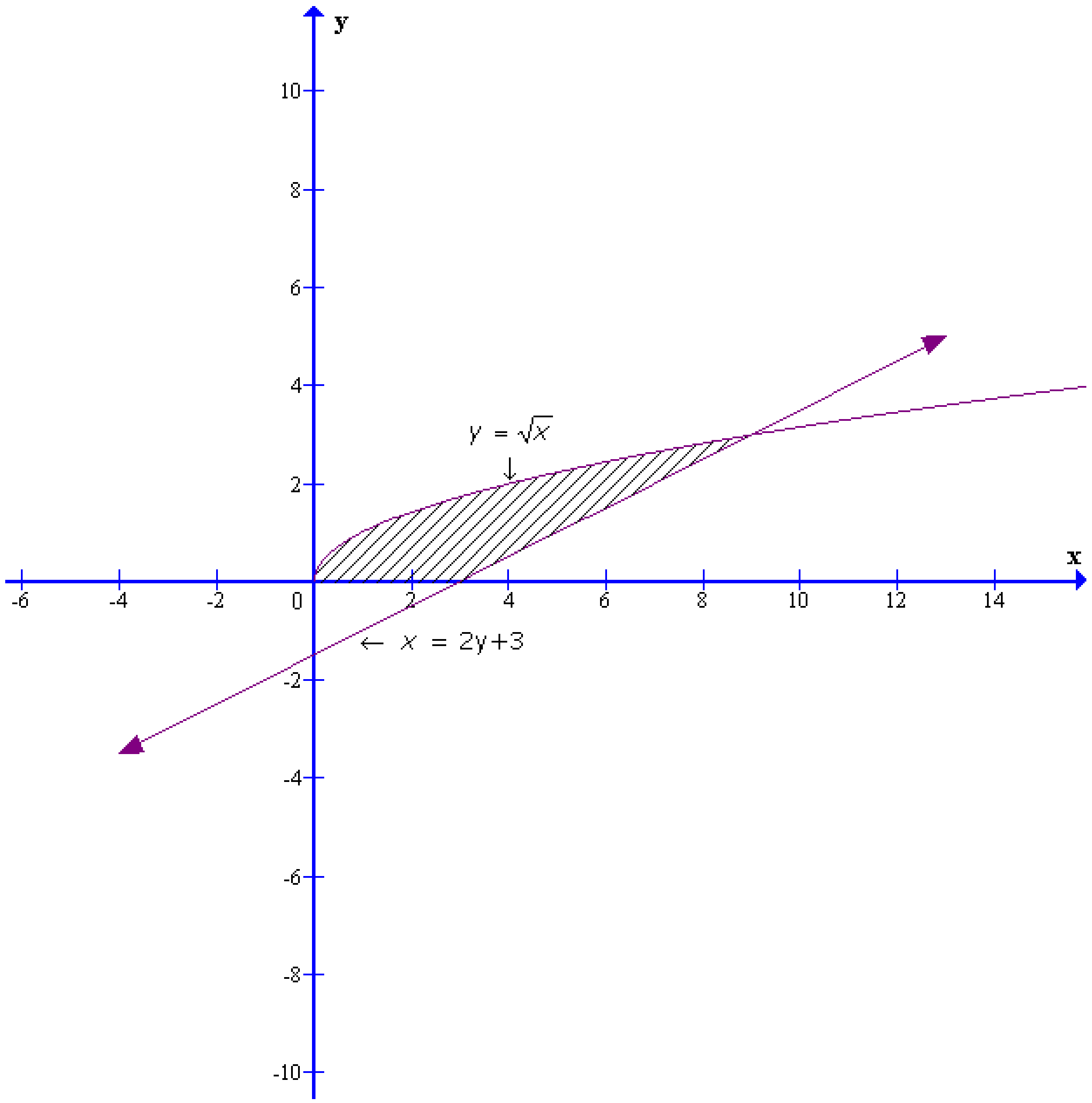
$$= \left[\sqrt{6} \, \frac{(6)^{3/2}}{3/2} - \frac{(6)^3}{18} - 0 \right]$$

= 12 sq. units

Question 18

Find the area of the region bounded by y = $\sqrt{x}$, x = 2y + 3 in the first quadrant and x-axis.

Solution 18



Area of the bounded region

$$= \int_0^3 \sqrt{x}\ dx + \int_3^9 \sqrt{x} - \left(\frac{x-3}{2}\right) dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_0^3 + \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{4} + \frac{3x}{2}\right]_3^9$$

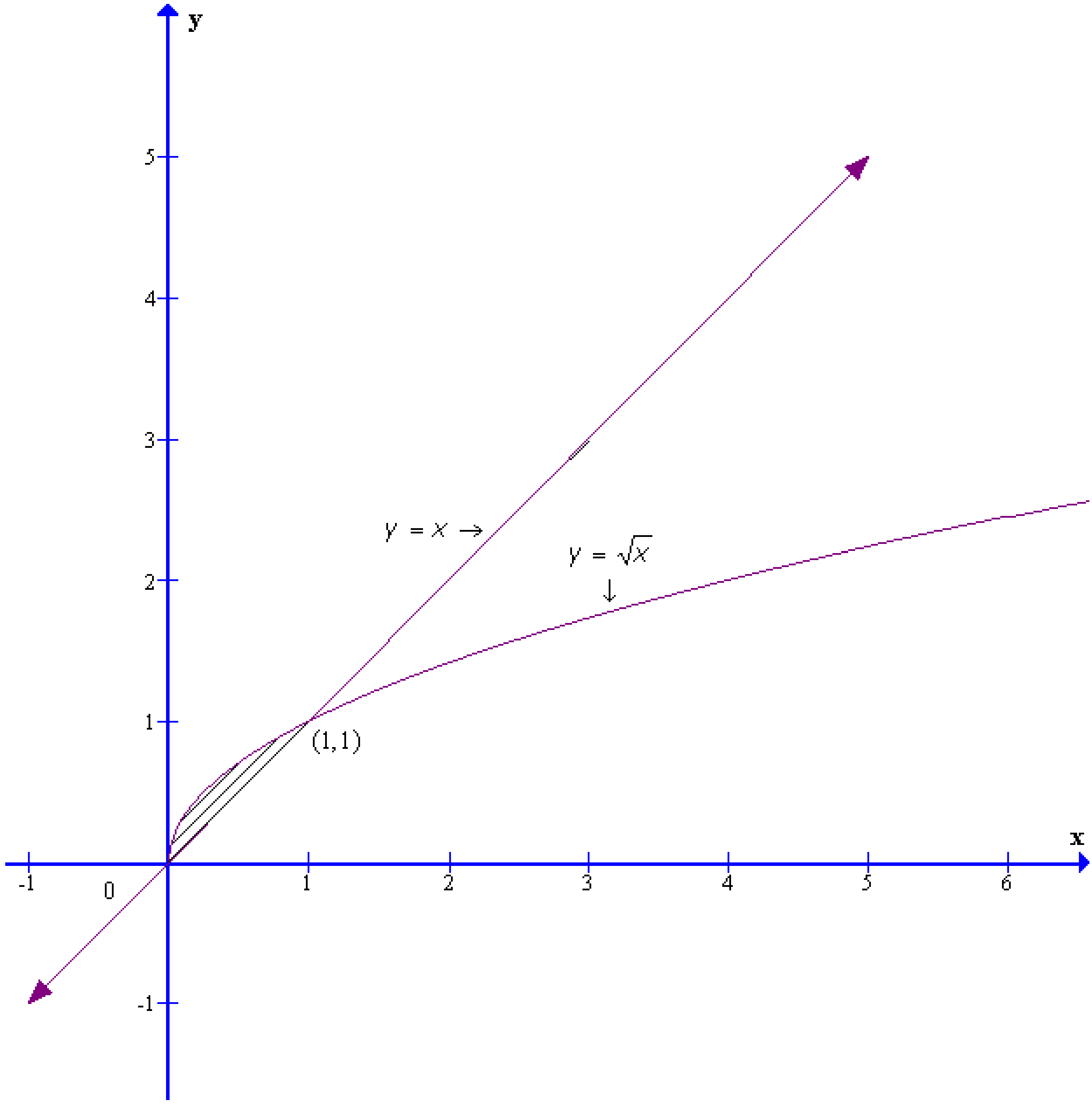
$$= \left[\frac{(3)^{3/2}}{3/2} - 0\right] + \left[\frac{(9)^{3/2}}{3/2} - \frac{(9)^2}{4} + \frac{3(9)}{2} - \frac{(3)^{3/2}}{3/2} + \frac{(3)^2}{4} - \frac{3(3)}{2}\right]$$

= 9 sq. units

Question 24

Find the area of the bounded by y = $\sqrt{x}$ and y = x.

Solution 24



Area of the bounded region

$$= \int_0^1 \sqrt{x} - x \, dx$$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} \right]_0^1$$

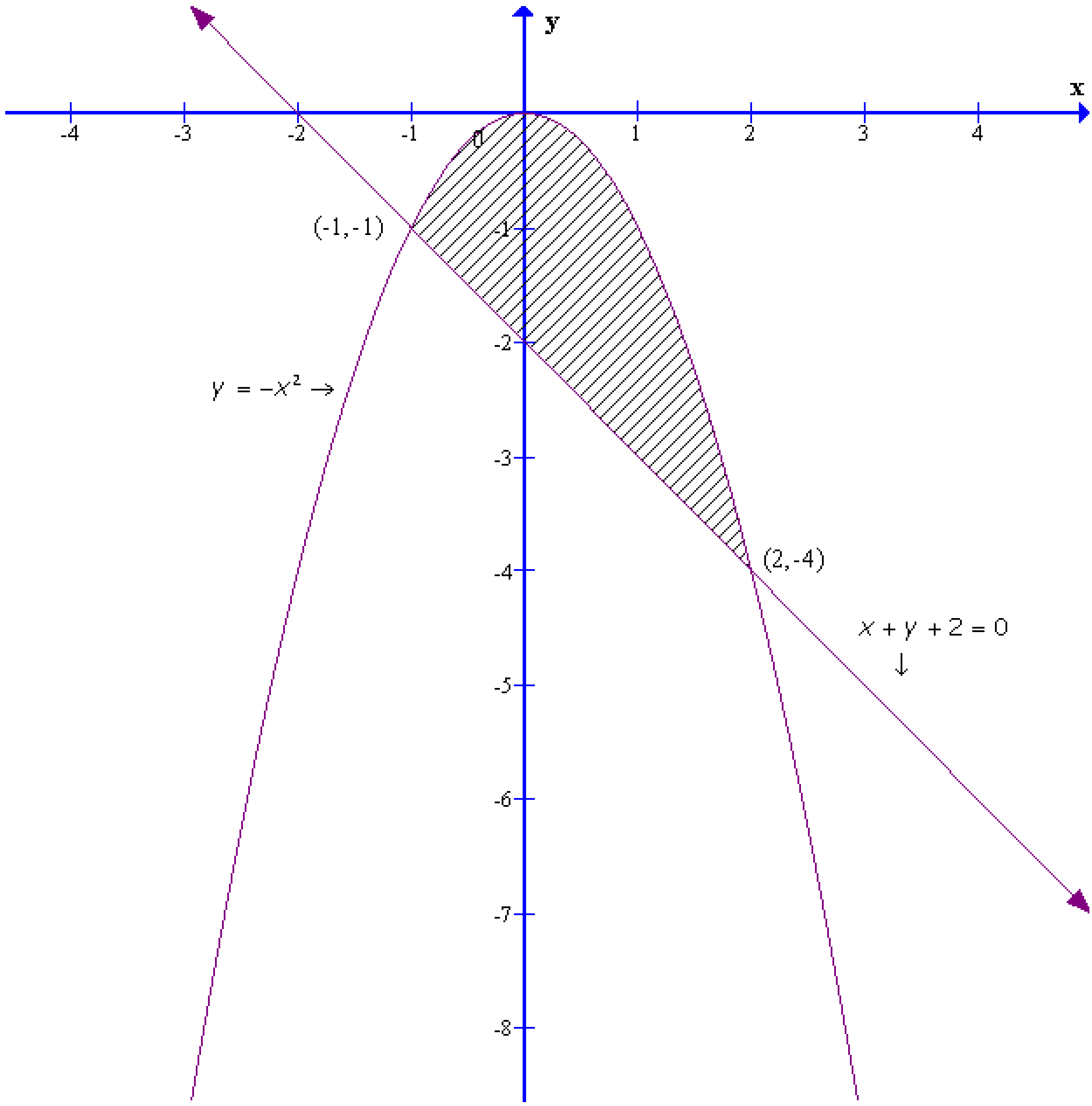
$$= \left[\frac{2}{3} - \frac{1}{2} \right]$$

$$= \frac{1}{6} \text{ sq. units}$$

Question 28

Find the area enclosed by the curve $y = -x^2$ and the straight line $x + y + 2 = 0$.

Solution 28



Area of the bounded region

$$= \int_{-1}^{2} -x^2 - (-2-x)\, dx$$

$$= \left[-\frac{x^3}{3} + 2x + \frac{x^2}{2} \right]_{-1}^{2}$$

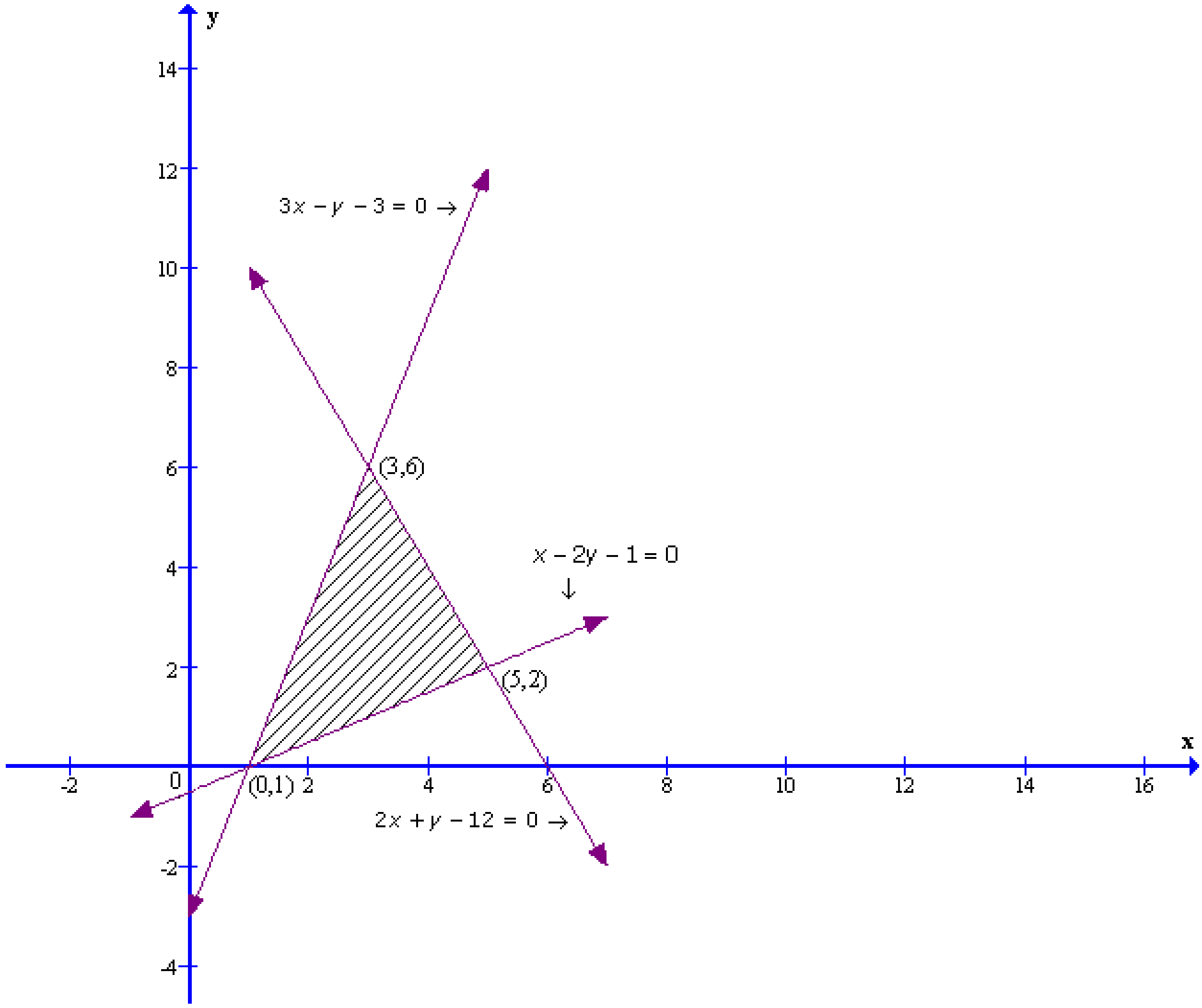
$$= \left[-\frac{8}{3} + 6 \right] - \left(\frac{1}{3} + \frac{1}{2} - 2 \right)$$

$$= \frac{9}{2} \text{ sq.units}$$

Question 30

Using the method of integration, find the area of the region bounded by the following lines: $3x - y - 3 = 0$, $2x + y - 12 = 0$, $x - 2y - 1 = 0$.

Solution 30



Area of the bounded region

$$= \int_0^3 3x - 3 - \left(\frac{x-1}{2}\right)dx + \int_3^5 12 - 2x - \left(\frac{x-1}{2}\right)dx$$

$$= \left[\frac{3x^2}{2} - 3x - \frac{x^2}{4} + \frac{1}{2}x\right]_0^3 + \left[12x - 2\frac{x^2}{2} - \frac{x^2}{4} + \frac{1}{2}x\right]_3^5$$

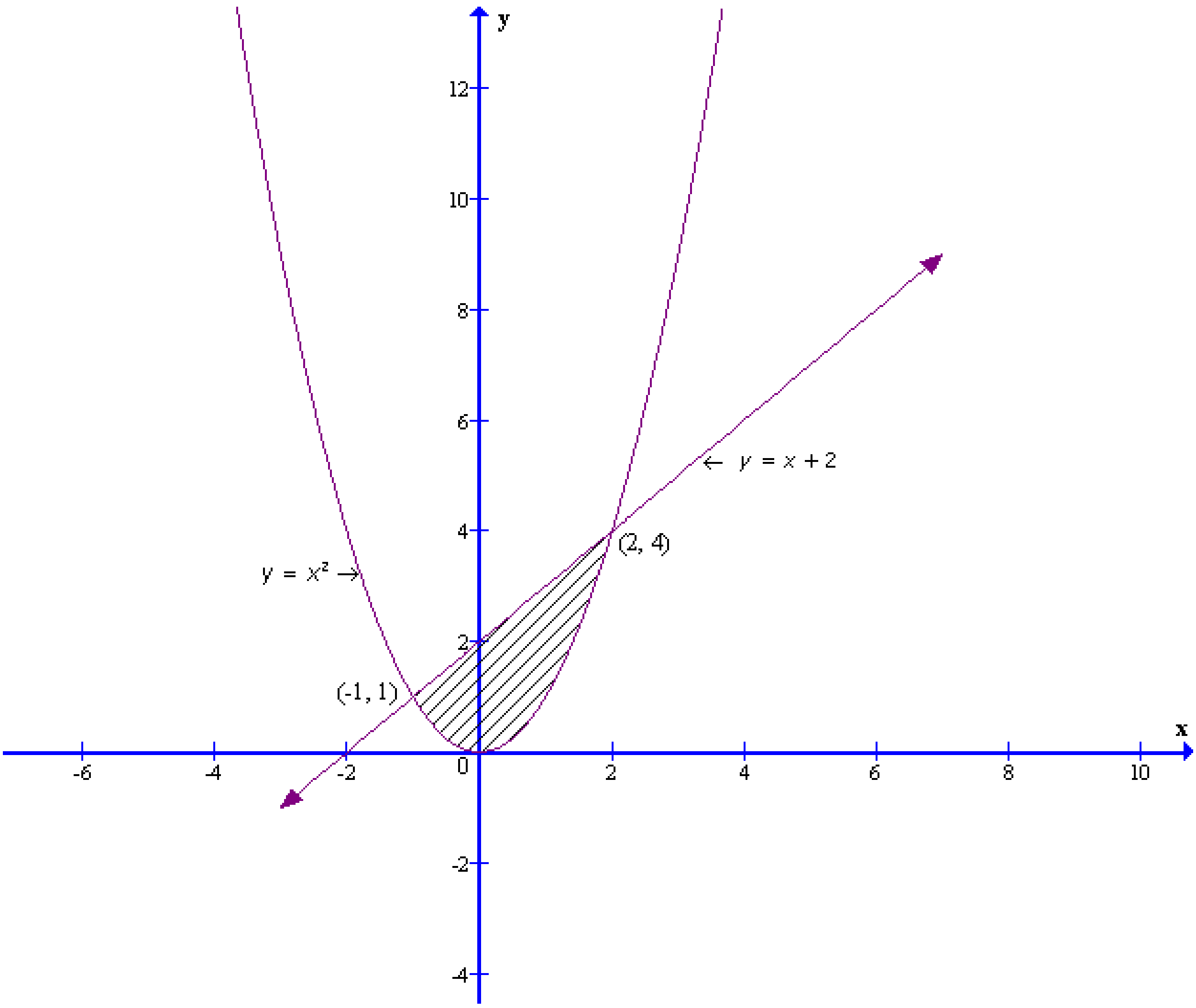
$$= \left[\frac{27}{2} - 9 - \frac{9}{4} + \frac{3}{2}\right] + \left[60 - 25 - \frac{25}{4} + \frac{5}{2} - 36 + 9 + \frac{9}{4} - \frac{3}{2}\right]$$

$= 11$ sq.units

Question 38

Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$.

Solution 38



Area of the bounded region

$$= \int_{-1}^{2} x+2-x^2 \, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

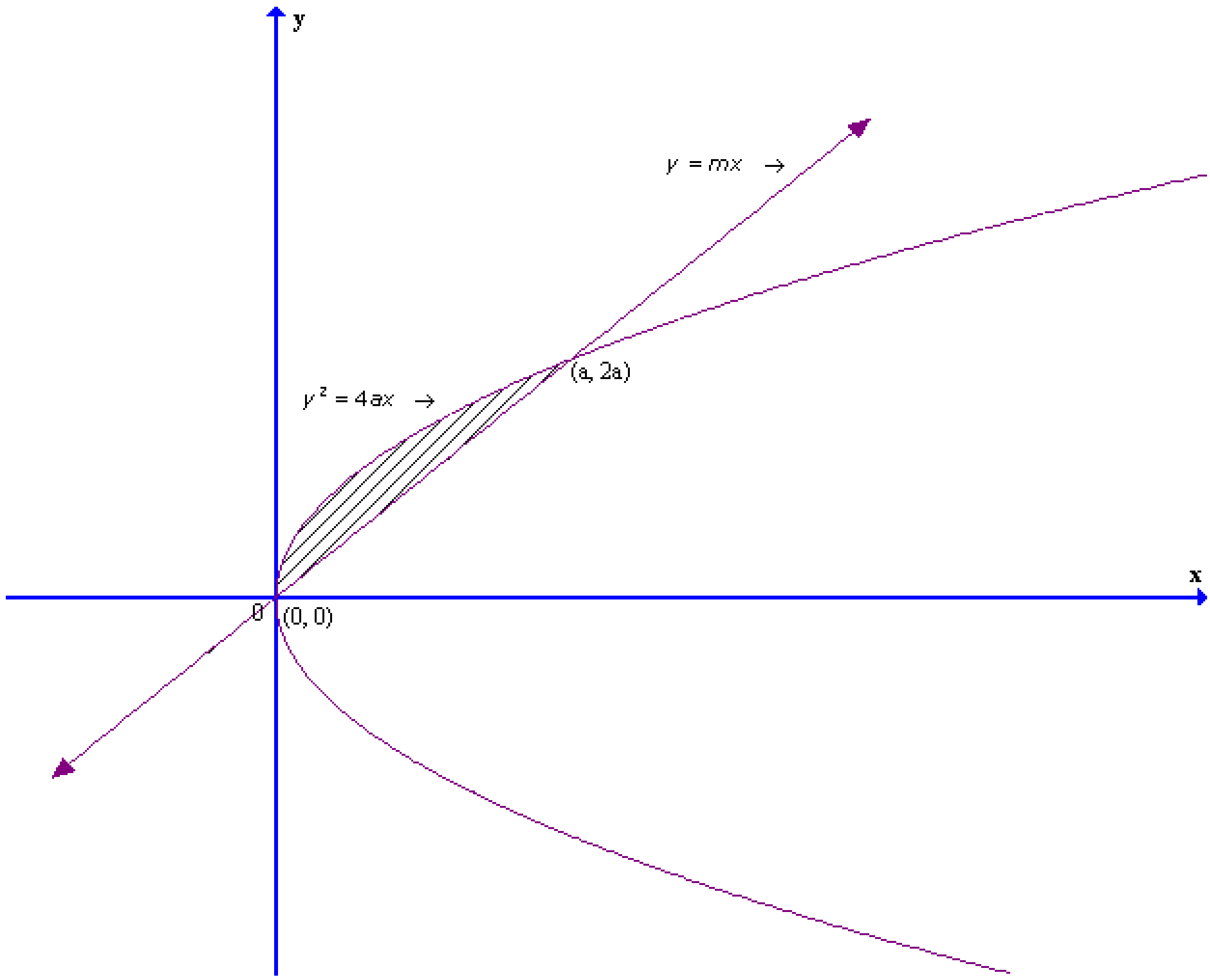
$$= \frac{4}{2} + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \frac{9}{2} \text{ sq.units}$$

Question 50

If the area bounded by the parabola $y^2 = 4ax$ and the line $y = mx$ is $\frac{a^2}{12}$ sq. units, then using integration, find the value of m.

Solution 50



Area of the bounded region $= \dfrac{a^2}{12}$

$$\frac{a^2}{12} = \int_0^a \sqrt{4ax} - mx\,dx$$

$$\frac{a^2}{12} = \left[2\sqrt{a}\frac{x^{3/2}}{3/2} - m\frac{x^2}{2}\right]_0^a$$

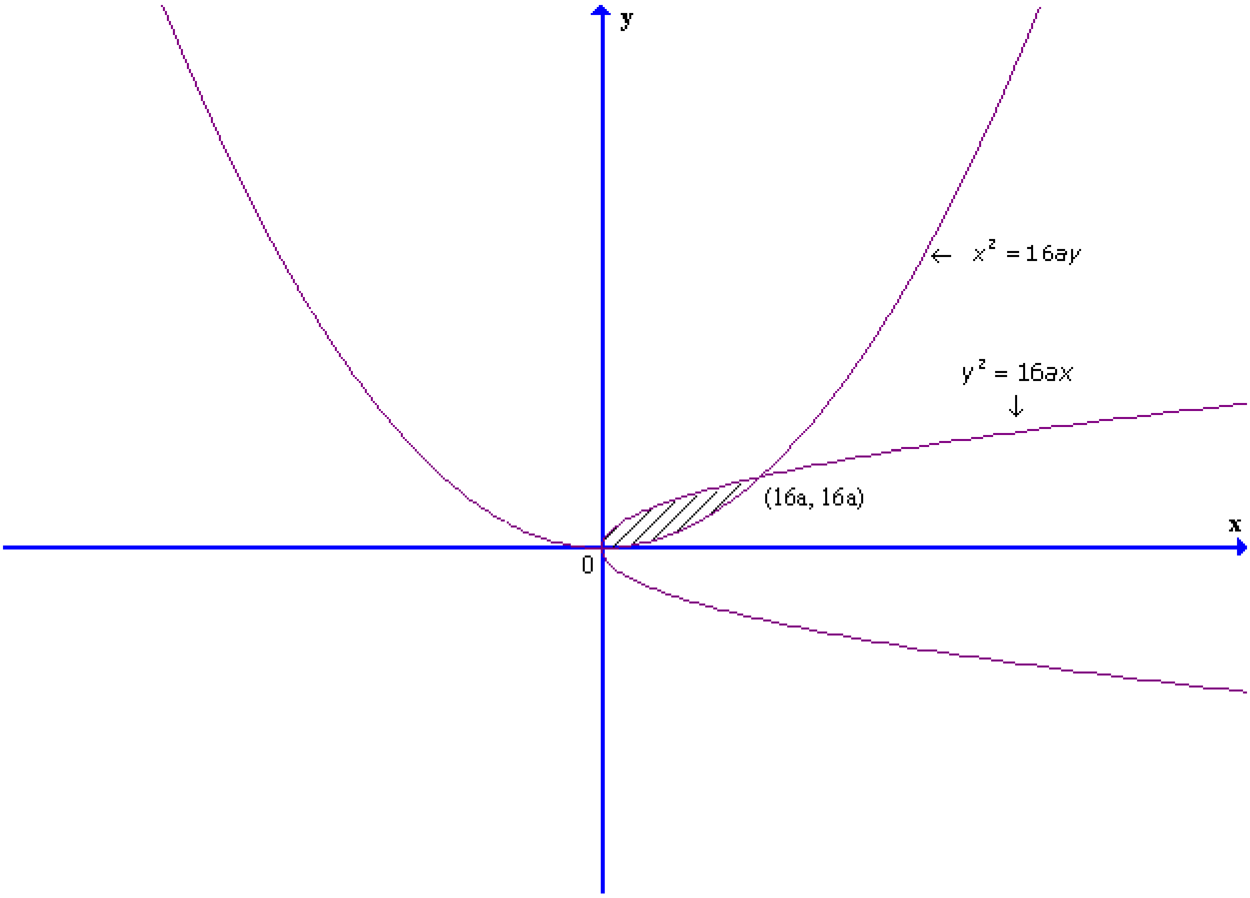
$$\frac{a^2}{12} = \frac{4a^2}{3} - m\frac{a^2}{2}$$

$m = 2$

Question 51

If the area bounded by the parabola $y^2 = 16ax$ and $x^2 = 16ay$, $a > 0$ is $\dfrac{1024}{3}$ squareunits, find the value of a.

Solution 51



$$\text{Area of the bounded region} = \frac{1024}{3}$$

$$\frac{1024}{3} = \int_0^{16a} \sqrt{16ax} - \frac{x^2}{16a}\, dx$$

$$\frac{1024}{3} = \left[4\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{48a}\right]_0^{16a}$$

$$\frac{1024}{3} = \frac{(16a)^2 \times 2}{3} - \frac{(16a)^3}{48a}$$

$$a = 2$$

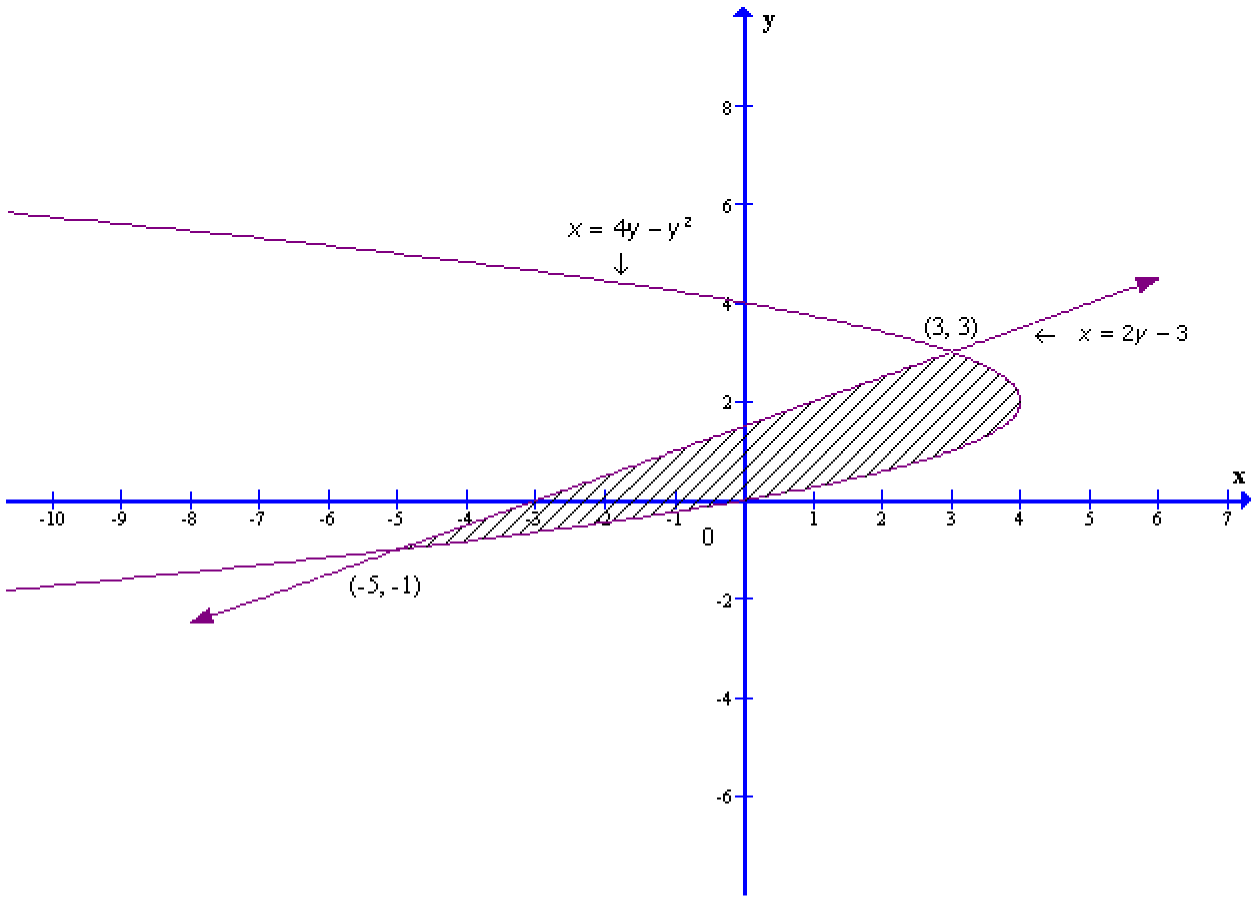
Note: Answer given in the book is incorrect.

Chapter 21 - Areas of Bounded Regions Exercise Ex. 21.4

Question 1

Find the area of the region between the parabola $x = 4y - y^2$ and the line $x = 2y - 3$.

Solution 1



Area of the bounded region

$$= \int_{-1}^{3} \left(4y - y^2 - 2y + 3\right)\, dy$$

$$= \left[2\frac{y^2}{2} - \frac{y^3}{3} + 3y\right]_{-1}^{3}$$

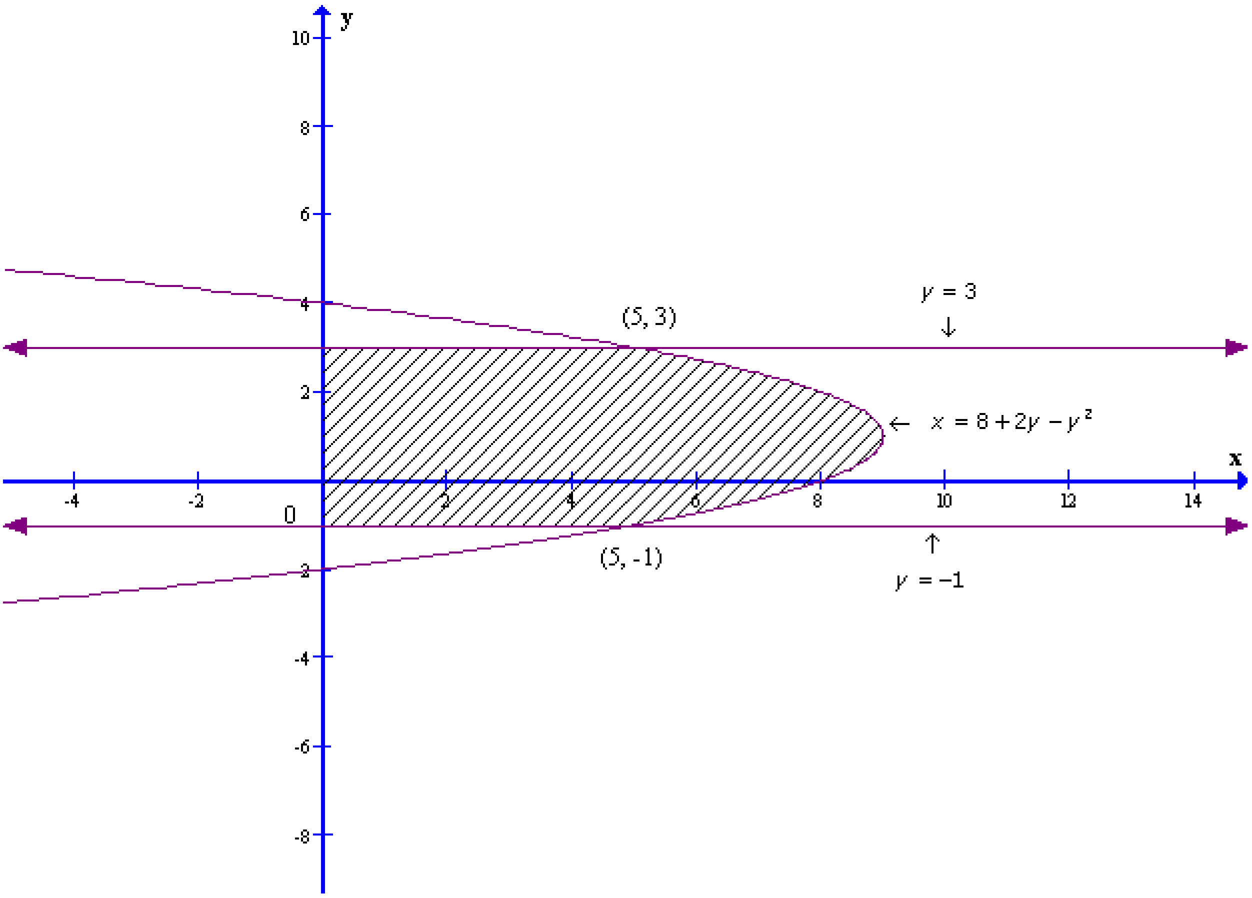
$$= 9 - 9 + 9 - 1 - \frac{1}{3} + 3 - \frac{(16a)^3}{48a}$$

$$= \frac{32}{3} sq.\ \text{units}$$

Question 2

Find the area bounded by the parabola $x = 8 + 2y - y^2$; the y-axis and the lines $y = -1$ and $y = 3$.

Solution 2



Area of the bounded region

$$= \int_{-1}^{3} (5-0)\ \ dy + \int_{-1}^{3} 8 + 2y - y^2 - 5\ \ dy$$

$$= \left[5y\right]_{-1}^{3} + \left[3y + y^2 - \frac{y^3}{3}\right]_{-1}^{3}$$

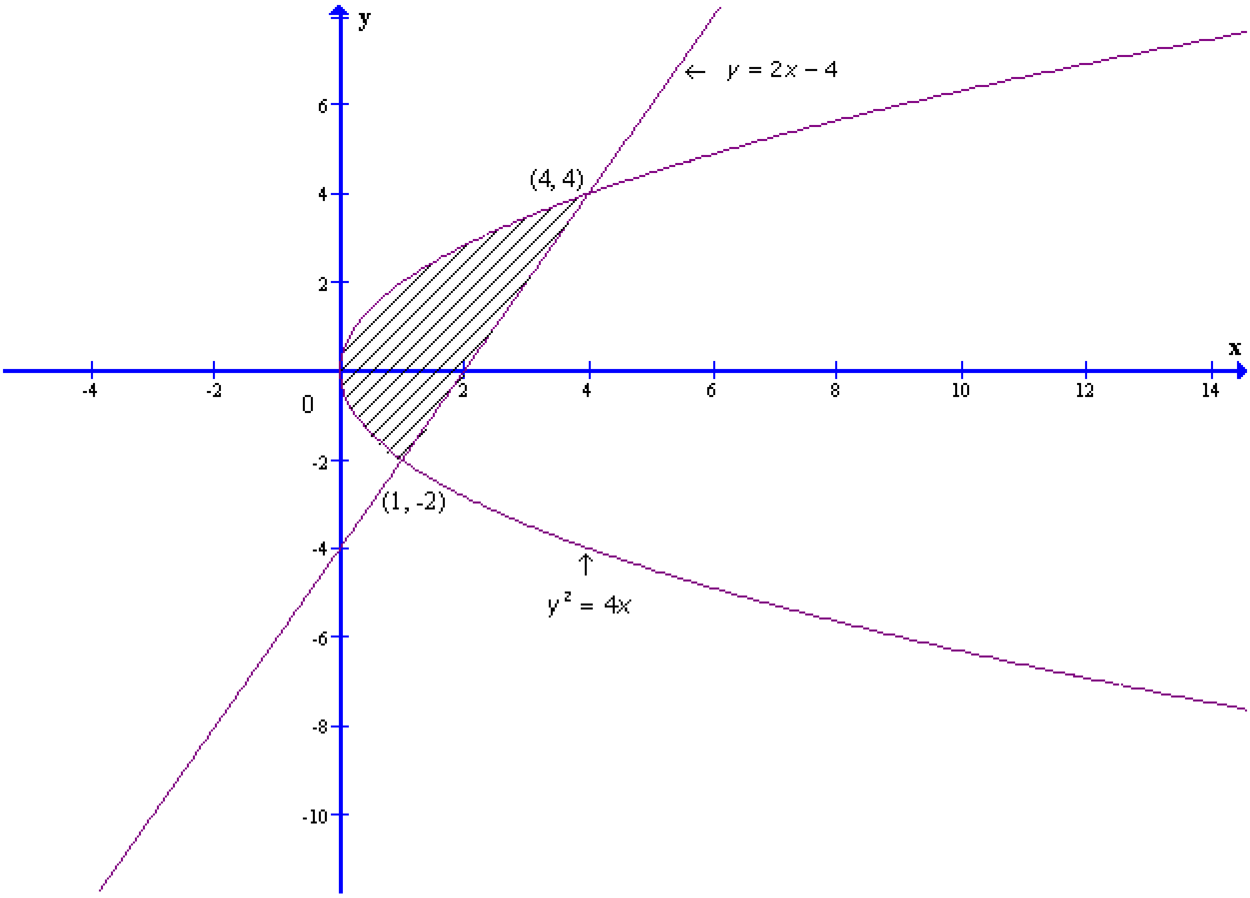
$$= 15 + 5 + 9 + 9 - \frac{27}{3} + 3 - 1 - \frac{1}{3}$$

$$= \frac{92}{3} sq.\ \text{units}$$

Question 3

Find the area bounded by the parabola $y^2 = 4x$ and the line
$y = 2x - 4$.
i. By using horizontal strips
ii. By using vertical strips

Solution 3



Area of the bounded region

$$= \int_{-2}^{4} \left(\frac{y+4}{2} - \frac{y^2}{4} \right) dy$$

$$= \left[\frac{y^2}{4} + 2y - \frac{y^3}{12} \right]_{-2}^{4}$$

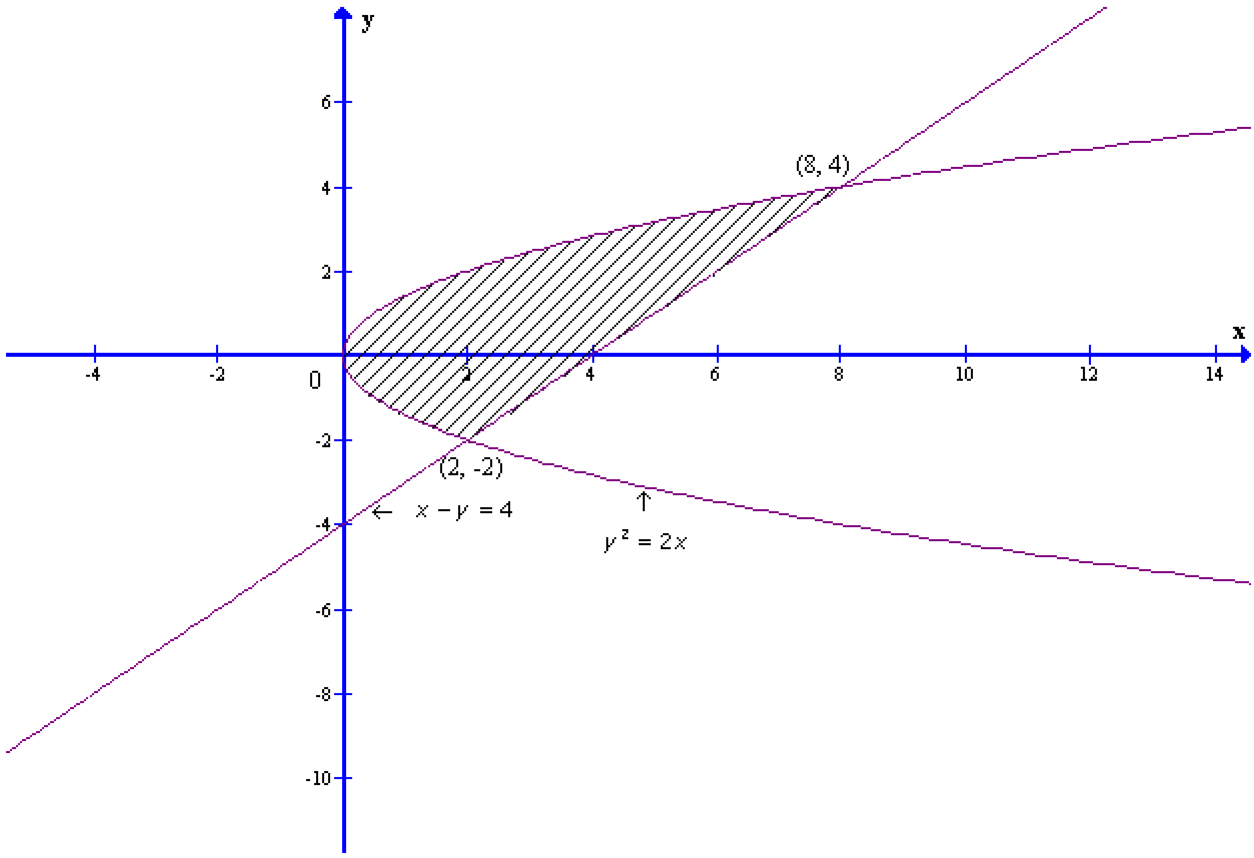
$$= 4 + 8 - \frac{16}{3} - 1 + 4 - \frac{2}{3}$$

$= 9$ sq. units

Question 4

Find the area of the region bounded the parabola $y^2 = 2x$ and straight line $x - y = 4$.

Solution 4



Area of the bounded region

$$= \int_{-2}^{4} \left(y + 4 - \frac{y^2}{2} \right) dy$$

$$= \left[\frac{y^2}{2} + 4y - \frac{y^3}{6} \right]_{-2}^{4}$$

$$= 8 + 16 - \frac{32}{3} - 2 + 8 - \frac{4}{3}$$

$= 18$ sq. units

Chapter 21 - Areas of Bounded Regions Exercise MCQ

Question 1

If the area above the x-axis, bounded by the curves $y = 2^{kx}$ and $x = 0$,

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $x = 2$ is $\frac{3}{\log_e 2}$, then the value of k is

a. 1/ 2
b. 1
c. -1
d. 2

Solution 1

Correct option: (b)

$$\int_0^2 y\,dx = \frac{3}{\log_e 2}$$

$$\int_0^2 2^{kx}\,dx = \frac{3}{\log_e 2}$$

$$\left[\frac{2^{kx}}{\log_e 2}\right]_0^2 = \frac{3}{\log_e 2}$$

$$2^{2k} - 1 = 3$$

$$2^{2k} = 4$$

$$2^{2k} = 2^2$$

$$\Rightarrow 2k = 2$$

$$\Rightarrow k = 1$$

Question 2

The area included between the parabolas $y^2=4x$ and $x^2 = 4y$ is (in square units)

a. 4/3
b. 1/3
c. 16/3
d. 8/3

Solution 2

Correct option: (c)

The area included in the parabolas $y^2 = 4x$ and $x^2 = 4y$.
To find intersection points,

$y^2 = 4x$ and $x^2 = 4y$

$\Rightarrow y^2 = 4x$ and $y^2 = \frac{x^4}{4}$

$\Rightarrow 4x = \frac{x^4}{4}$

$\Rightarrow 64x = x^4$

$\Rightarrow x^4 - 64x = 0$

$\Rightarrow x = 0$ or $x = 4$

$$\int_0^4 y\,dx = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx$$

$$\int_0^4 y\,dx = \left[\frac{2}{3} \times 4x^{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$$

$$\int_0^4 y\,dx = \frac{32}{3} - \frac{64}{12}$$

$$\int_0^4 y\,dx = \frac{16}{3} \text{ sq. units}$$

Question 3

The area bounded by the curve y= $\log_e x$ and x-axis and the straight line x =e is

a. e. sq. units
b. 1 sq. units
c. $1 - \frac{1}{e}$ sq.units
d. $1 + \frac{1}{e}$ sq.units

Solution 3

Correct option: (b)

$$A = \int_0^1 \left(e - e^y\right)dy$$

$$A = \left[ey - e^y\right]_0^1$$

$A = 1$ sq units

Question 4

The area bounded by $y=2-x^2$ and $x + y =0$ is

a. $\frac{7}{2}$ sq.units

b. $\frac{9}{2}$ sq.units

c. 9 sq.units

d. None of these

Solution 4

Correct option: (b)

The area bounded by $y = 2 - x^2$ and $x + y = 0 \Rightarrow y = -x$

$2 - x^2 = -x$

$x^2 - x - 2 = 0$

$\Rightarrow (x-2)(x+1) = 0$

$\Rightarrow x = 2$ or $x = -1$

$$\int_{-1}^{2}\left(2 - x^2 - x\right)dx$$

$$\left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$2(2+1) - \left(\frac{8}{3} + \frac{1}{3}\right) + \left(2 - \frac{1}{2}\right)$$

$$6 - 3 + \frac{3}{2}$$

$\frac{9}{2}$ sq. units

Question 5

The area bounded by the parabola $x = 4 - y^2$ and y-axis, in square units, is

a. $\frac{3}{32}$

b. $\frac{32}{3}$

c. $\frac{33}{2}$

d. $\frac{16}{3}$

Solution 5

Correct option: (b)

The area bounded by parabola $x = 4 - y^2$ and y – axis.
As parabola bounded by y – axis x = 0

$$\Rightarrow 4 - y^2 = 0$$

$$\Rightarrow y = \pm 2$$

$$\int_{-2}^{2}\left(4 - y^2\right)dy$$

$$\left[4y - \frac{y^3}{3}\right]_{-2}^{2}$$

$$16 - \left(\frac{8}{3} + \frac{8}{3}\right)$$

$$\frac{32}{3}$$

Question 6

If A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines x = 0, y =0 and x =π /4, then for x > 2

a. $A_n + A_{n-2} = \frac{1}{n-1}$

b. $A_n + A_{n-2} < \frac{1}{n-1}$

c. $A_n + A_{n-2} = \frac{1}{n-1}$

d. none of these

Solution 6

Correct option: (a)

Give $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$, $x = \frac{\pi}{4}$

$$A_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$$

$$A_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \left(\tan^2 x\right) dx$$

$$A_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \left(\sec^2 x - 1\right) dx$$

$$A_n = \int_0^{\frac{\pi}{4}} \left(\tan^{n-2} x \sec^2 x - \tan^{n-2} x\right) dx$$

$$A_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x dx$$

$$A_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x dx - A_{n-2}$$

$$A_n + A_{n-2} = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

x	0	$\frac{\pi}{4}$
t	0	1

$$A_n + A_{n-2} = \int_0^1 t^{n-2} dt$$

$$A_n + A_{n-2} = \left[\frac{t^{n-1}}{n-1}\right]_0^1$$

$$A_n + A_{n-2} = \frac{1}{n-1}$$

Question 7

The area of the region formed by $x^2+y^2-6x-4y+12 \leq x$ and $x \leq 5/2$ is

a. $\frac{\pi}{6} - \frac{\sqrt{3}+1}{8}$

b. $\frac{\pi}{6} + \frac{\sqrt{3}+1}{8}$

c. $\frac{\pi}{6} - \frac{\sqrt{3}-1}{8}$

d. none of these

Solution 7

Correct option: (c)

Consider given equations as

$$x^2 + y^2 - 6x - 4y + 12 = 0,\ y = x,\ x = \frac{5}{2}$$

$$\Rightarrow \text{intersection points } \left(\frac{5}{2}, \frac{5}{2}\right) \text{ and } (2,2)$$

$$x^2 + y^2 - 6x - 4y + 12 = 0$$

$$\Rightarrow (x-3)^2 + (y-2)^2 = 1$$

$$\Rightarrow y = \sqrt{1-(x-3)^2} + 2$$

$$A = \int_2^{\frac{5}{2}} \left(x - \sqrt{1-(x-3)^2} + 2 \right) dx$$

$$A = \frac{\pi}{6} - \frac{\sqrt{3}-1}{8}$$

Question 8

The area enclosed between the curves $y = \log_e(x+e)$ and $x = \log_e\left(\frac{1}{y}\right)$ and the x – axis is

a. 2
b. 1
c. 4
d. None of these

Solution 8

Correct option: (a)

Given two curves $y = \log_e(x+e)$ and $x = \log_e\left(\frac{1}{y}\right)$

As area enclosed on x – axis $x = 0$.

$\Rightarrow y = \log_e e = 1$

Intersection point is $(0, 1)$

Consider, $y = \log_e(x+e) \Rightarrow e^y - e = x$

Consider, $x = \log_e\left(\frac{1}{y}\right)$

Required area $= \int_0^1 \left(\log_e\left(\frac{1}{y}\right) - e^y - e\right) dy$

Required area $= \int_0^1 \log_e\left(\frac{1}{y}\right) dy - \left[e^y - ey\right]_0^1$

Required area $= I_1 - (e - 1 - e) = I_1 + 1$

Consider, $I_1 = \int_0^1 \log_e\left(\frac{1}{y}\right) dy$

Put, $\frac{1}{y} = t \Rightarrow \frac{1}{t} = y$

$\Rightarrow \frac{-1}{t^2} dt = dy$

$I_1 = \int -\frac{1}{t^2} \log_e t dt$

$I_1 = \left[\frac{1}{t}\log_e t + \frac{1}{t}\right]$

$I_1 = \left[y \log_e\left(\frac{1}{y}\right) + y\right]_0^1$

$I_1 = 1$

$\Rightarrow$ Requied area $= I_1 + 1 = 2$

Question 9

The area of the region bounded by the parabola $(y-2)^2 = x-1$, the tangent to it at the point with the ordinate 3 and the x-axis is

a. 3
b. 6
c. 7
d. None of these

Solution 9

Correct option: (d)

The area of the region bounded by the parabola $(y-2)^2 = x-1$ the tangent to it at the point with the ordinate 3 and y – axis

$y = 3 \Rightarrow x = 2$

$(y-2)^2 = x-1$

slope of tangent at $x = 2$

$$2(y-2)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2(y-2)}$$

$$\left|\frac{dy}{dx}\right|_{(2,3)} = \frac{1}{2}$$

Equation of tangent

$$y - 3 = \frac{1}{2}(x-2)$$

$$y = \frac{x}{2} + 2$$

$$A = \int_0^3 \left((y-2)^2 + 1 - 2(y-2)\right)dy$$

$$A = \int_0^3 (y-3)^2 dy$$

$$A = \left[\frac{(y-3)^3}{3}\right]_0^3$$

$A = 9$ sq units

NOTE: Answer not matching with back answer.

Question 10

The area bounded by the curves y = sin x between the ordinates x =0 , x =π and the x-axis is

a. 2 sq. units
b. 4 sq. units
c. 3 sq. units
d. 1 sq. units

Solution 10

Correct option: (a)

$$\int_0^{\pi} y\,dx = \int_0^{\pi} \sin x\,dx$$

$$\int_0^{\pi} y\,dx = -\left[\cos x\right]_0^{\pi}$$

$$\int_0^{\pi} y\,dx = -\left[-1-1\right]$$

$$\int_0^{\pi} y\,dx = 2$$

Question 11

The area bounded by the parabola $y^2 = 4ax$ and $x^2 = 4ay$ is

a. $\frac{8a^3}{3}$

b. $\frac{16a^2}{3}$

c. $\frac{32\alpha^2}{3}$

d. $\frac{64a^2}{3}$

Solution 11

Correct option: (b)

Given parabolas $y^2 = 4ax$ and $x^2 = 4ay$

$\Rightarrow y^2 = 4ax$ and $y^2 = \frac{x^4}{16a^2}$

$\Rightarrow 4ax = \frac{x^4}{16a^2}$

$\Rightarrow x = 0$ and $x = 4a$

$$A = \int_0^{4a} \sqrt{2ax}\,dx + \int_0^{4a} \frac{x^2}{4a}dx$$

$$A = \sqrt{2a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{4a} + \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$A = \frac{16a^2}{3}$$

Question 12

The area bounded by the curve $y=x^4-2x^3+x^2+3$ with x-axis and ordinates corresponding to the minima of y is

a. 1

b. $\frac{91}{30}$

c. $\frac{30}{9}$

d. 4

Solution 12

Correct option: (b)

The area bounded by the curve $y = x^4 - 2x^3 + x^2 + 3$ with x – axis and ordinates.

Minimum value of y when $x = 0$ is $y = 3$

Minimum value of y when $x = 1$ is $y = 3$

$$\Rightarrow \int_0^1 \left(x^4 - 2x^3 + x^2 + 3\right)dx$$

$$\Rightarrow \left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3} + 3x\right]_0^1$$

$$\Rightarrow \frac{1}{5} - \frac{2}{4} + \frac{1}{3} + 3$$

$$\Rightarrow \frac{91}{30}$$

Question 13

The area bounded by the parabola $y^2=4ax$, latus rectum and x-axis is

a. 0

b. $\frac{4}{3}a^2$

c. $\frac{2}{3}a^2$

d. $\frac{a^2}{3}$

Solution 13

Correct option: (b)

X – cordinate of latus rectum is a.

$$\Rightarrow \int_0^a ydx = \int_0^a 2\sqrt{a}\sqrt{x}dx$$

$$\Rightarrow \int_0^a ydx = 2\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$$

$$\Rightarrow \int_0^a ydx = \frac{4a^2}{3}$$

Question 14

The area of the region $\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$ is

a. $\frac{\pi}{5}$

b. $\frac{\pi}{4}$

c. $\frac{\pi}{2} - \frac{1}{2}$

d. $\frac{\pi^2}{2}$

Solution 14

Correct option: (c)

$$A = \int_0^1 \sqrt{1 - x^2}dx - \int_0^1 (1 - x)\,dx$$

$$A = \left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1} x - x + \frac{x^2}{2}\right]_0^1$$

$$A = \frac{\pi}{4} - \frac{1}{2}$$

NOTE: Answer not matching with back answer.

Question 15

The area common to the parabola $y = 2x^2$ and $y=x^2+4$ is

a. $\frac{2}{3}$ sq. units

b. $\frac{3}{2}$ sq. units

c. $\frac{32}{3}$ sq. units

d. $\frac{3}{32}$ sq. units

Solution 15

Correct option: (c)

Given two parabola $y = 2x^2$ and $y = x^2 + 4$

$\Rightarrow 2x^2 = x^2 + 4$

$\Rightarrow x = \pm 2$

$$A = \int_{-2}^{2} \left(x^2 - 4 - 2x^2\right) dx$$

$$A = \int_{-2}^{2} \left(4 - x^2\right) dx$$

$$A = 16 - \left(\frac{8}{3} + \frac{8}{3}\right)$$

$$A = \frac{32}{3} \text{ sq. units}$$

Question 16

The area of the region bounded by the parabola $y=x^2+1$ and the straight line x + y =3 is give by

a. $\frac{45}{7}$

b. $\frac{25}{4}$

c. $\frac{\pi}{18}$

d. $\frac{9}{2}$

Solution 16

Correct option: (d)

Given $y = x^2 + 1$ and $x + y = 3$

$\Rightarrow y = x^2 + 1$ and $y = 3 - x$

$\Rightarrow 3 - x = x^2 + 1$

$\Rightarrow x^2 + x - 2 = 0$

$\Rightarrow x = -2$ or $x = 1$

$\Rightarrow y = 5$ or $y = 2$

$$A = \int_{-2}^{1}\left(3 - x - \left[x^2 + 1\right]\right)dx$$

$$A = \int_{-2}^{1}\left(3 - x - x^2 - 1\right)dx$$

$$A = \int_{-2}^{1}\left(-x - x^2 + 2\right)dx$$

$$A = \left[-\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-2}^{1}$$

$$A = \frac{9}{2} \text{ sq. units}$$

Question 17

The ratio of the areas between the curves y= cos x and y = cos 2x and x-axis from x =0 to x = π/3 is

a. 1:2
b. 2:1
c. $\sqrt{3} : 1$
d. None of these

Solution 17

Correct options: (d)

Area of $y = \cos x$ is

$$\int_0^{\frac{\pi}{3}} \cos x dx = \frac{\sqrt{3}}{2}$$

Area between the curve $y = \cos 2x$ and x – axis from $x = 0$

and $x = \frac{\pi}{3}$

$$\int_0^{\frac{\pi}{4}} \cos 2x dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos 2x dx$$

$$\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} - \left[\frac{\sin 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$\frac{1}{2} - \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right)$$

$$\frac{4-\sqrt{3}}{4}$$

Ratio will be

$$\frac{\frac{\sqrt{3}}{2}}{\frac{4-\sqrt{3}}{4}} = \frac{2\sqrt{3}}{4-\sqrt{3}}$$

NOTE: Answer not matching with back answer.

Question 18

The area between x-axis and curve $y = \cos x$ when $0 \le x \le 2\pi$ is

a. 0
b. 2
c. 3
d. 4

Solution 18

Correct option: (d)

Given that $y = \cos x,\ 0 \le x \le 2\pi$

$$\Rightarrow \int_0^{2\pi} y\,dx = \int_0^{\frac{\pi}{2}} y\,dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} y\,dx + \int_{\frac{3\pi}{2}}^{2\pi} y\,dx$$

$$\Rightarrow \int_0^{2\pi} y\,dx = \int_0^{\frac{\pi}{2}} \cos x\,dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx + \int_{\frac{3\pi}{2}}^{2\pi} \cos x\,dx$$

$$\Rightarrow \int_0^{2\pi} y\,dx = \left[\sin x\right]_0^{\frac{\pi}{2}} - \left[\sin x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \left[\sin x\right]_{\frac{3\pi}{2}}^{2\pi}$$

$$\Rightarrow \int_0^{2\pi} y\,dx = 1 - 0 - (-1 - 1) + (0 + 1)$$

$$\Rightarrow \int_0^{2\pi} y\,dx = 4 \text{ sq units}$$

Question 19

Area bounded by parabola $y^2=x$ and staright line $2y = x$ is

a. 4/3
b. 1
c. 2/3
d. 1/3

Solution 19

Correct option: (a)

Given parabola and staright line

$y^2 = x$ and $2y = x$

$\Rightarrow y^2 = 2y$

$\Rightarrow y = 0$ or $y = 2$

$\Rightarrow x = 0$ or $x = 4$

$$A = \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)dx$$

$$A = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{4}\right]_0^4$$

$$A = \frac{2}{3} \times 8 - 4$$

$$A = \frac{4}{3} \text{sq units}$$

NOTE: Options are modified.

Question 20

The area bounded by the curve $y = 4x-x^2$ and x-axis is

a. $\frac{30}{7}$ sq. units

b. $\frac{31}{7}$ sq. units

c. $\frac{32}{3}$ sq. units

d. $\frac{34}{7}$ sq. units

Solution 20

Correct option: (c)

The area bounded by the curve $y = 4x - x^2$ and x - axis

$\Rightarrow y = 0$

$\Rightarrow 4x - x^2 = 0$

$x = 0$ or $x = 4$

$A = \int_0^4 \left(4x - x^2\right)dx$

$A = 32 - \frac{64}{3}$

$A = \frac{32}{3}$ sq units

Question 21

Area enclosed between the curve $y^2(2a-x)=x^3$ and the line $x = 2a$ above x-axis is

a. πa^2

b. $\frac{3}{2}\pi a^2$

c. $2\pi a^2$

d. $3\pi a^2$

Solution 21

Correct option: (b)

Area enclosed between the curves $y^2(2a - x) = x^3$
and the line $x = 2a$ above x – axis

$$\Rightarrow \int_0^{2a} \sqrt{\frac{x^3}{2a - x}}\,dx$$

Put $x = 2a\sin^2\theta \Rightarrow dx = 4a\sin\theta\cos\theta d\theta$

x	0	2a
θ	0	$\frac{\pi}{2}$

$$A = \int_0^{\frac{\pi}{2}} \sqrt{\frac{\left(2a\sin^2\theta\right)^3}{2a - 2a\sin^2\theta}}\,4a\sin\theta\cos\theta d\theta$$

$$A = \int_0^{\frac{\pi}{2}} \sqrt{\frac{8a^3\sin^6\theta}{2a\left(1 - \sin^2\theta\right)}}\,4a\sin\theta\cos\theta d\theta$$

$$A = \int_0^{\frac{\pi}{2}} 8a^2 \sqrt{\frac{\sin^6\theta}{\cos^2\theta}}\,4a\sin\theta\cos\theta d\theta$$

$$A = \int_0^{\frac{\pi}{2}} 8a^2 \sin^3\theta\sin\theta d\theta$$

$$A = 8a^2 \int_0^{\frac{\pi}{2}} \sin^4\theta d\theta$$

$$A = 8a^2\int_0^{\frac{\pi}{2}} \sin^4\theta d\theta$$

$$A = 8a^2\int_0^{\frac{\pi}{2}} \sin^2\theta\left(1-\cos^2\theta\right)d\theta$$

$$A = 8a^2\int_0^{\frac{\pi}{2}} \left(\sin^2\theta - \sin^2\theta\cos^2\theta\right)d\theta$$

$$A = 8a^2\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2\theta}{2} - \frac{1}{4}\sin^2 2\theta\right)d\theta$$

$$A = 8a^2\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2\theta}{2} - \frac{1}{4}\frac{\left(1-\cos 4\theta\right)}{2}\right)d\theta$$

$$A = 8a^2\int_0^{\frac{\pi}{2}} \left(\frac{3}{8} - \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{8}\right)d\theta$$

$$A = 8a^2\left[\frac{3}{8}x - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right]_0^{\frac{\pi}{2}}$$

$$A = 8a^2\left(\frac{3\pi}{16} - 0 + 0\right) = \frac{3\pi a^2}{2}$$

Question 22

The area of the region (in square units)bounded by the curve x^2=4y, line x =2 and x-axis is

a. 1
b. 2/3
c. 4/3
d. 8/3

Solution 22

Correct option: (b)

The area of the region bounded by

the curve $x^2 = 4y$ and line $x = 2$ and x − axis

$$\Rightarrow \int_0^2 y dx = \int_0^2 \frac{x^2}{4} dx$$

$$\Rightarrow \int_0^2 y dx = \left[\frac{x^3}{12}\right]_0^2$$

$$\Rightarrow \int_0^2 y dx = \frac{8}{12} = \frac{2}{3}$$

Question 23

The area bounded by the curve y=f (x), x-axis, and the ordinates x =1 and x=b is (b-1) sin (3b+4). Then, f (x) is

a. (x-1) cos (3x+4)
b. Sin (3x+4)

c. Sin (3x+4)+3(x-1)cos (3x+4)
d. None of these

Solution 23
Correct option: (c)
Given that area bounded by the curve x – axis,
x = 1 and x = b

$$\Rightarrow \int_1^b y\,dx = \int_1^b f(x)\,dx$$

$$\Rightarrow \int_1^b y\,dx = [A]_1^b$$

$$\Rightarrow \int_1^b f(x)\,dx = (b-1)\sin(3b+4)$$

$$\Rightarrow f(x) = \frac{d}{dx}\left[(x-1)\sin(3x+4)\right]$$

$$\Rightarrow 3(x-1)\cos(3x+4) + \sin(3x+4)$$

Question 24
The area bounded by the curve $y^2 = 8x$ and $x^2 = 8y$ is

a. $\frac{16}{3}$ sq. units
b. $\frac{3}{16}$ sq. units
c. $\frac{14}{3}$ sq. units
d. $\frac{3}{14}$ sq. units

Solution 24

The area bounded by the curve $y^2 = 8x$ and $x^2 = 8y$

$\Rightarrow y^2 = 8x$ and $x^4 = 64y^2$

$\Rightarrow 8x = \dfrac{x^4}{64}$

$\Rightarrow x = 0$ or $x = 8$

$$A = \int_0^8 \left(\sqrt{8x} - \frac{x^2}{8}\right) dx$$

$$A = 2\sqrt{2}\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^8 - \frac{1}{8}\left(\frac{x^3}{3}\right)_0^8$$

$$A = \frac{4\sqrt{2}}{3} \times 8\sqrt{8} - \frac{1}{24} \times 8^3$$

$$A = \frac{64}{3} \text{ sq units}$$

NOTE: Answer is not matching with back answer.

Question 25

The area bounded by the parabola $y^2=8x$, the x-axis, and the latus rectum is

a. $\dfrac{16}{3}$

b. $\dfrac{23}{3}$

c. $\dfrac{32}{3}$

d. $\dfrac{16\sqrt{2}}{3}$

Solution 25

Correct option: (a)

Given parabola $y^2 = 8x$

$\Rightarrow 4a = 8 \Rightarrow a = 2$

Parabola bounded by x – axis and latus rectum.

$$\Rightarrow \int_0^2 \sqrt{8x}dx = 2\sqrt{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 = \frac{16}{3}$$

NOTE: Answer is not matching with back answer.

Question 26

Area bounded by the curve $y=x^3$, the x-axis and the ordinates $x =-2$ and $x =1$ is

a. -9

b. $\frac{-15}{4}$

c. $\frac{15}{4}$

d. $\frac{17}{4}$

Solution 26

Correct option: (d)

Area bounded by the curve $y = x^3$ and x - axis and the ordinates $x = -2$ and $x = 1$

$$\Rightarrow A = \int_{-2}^{0} y dx + \int_{0}^{1} y dx$$

$$\Rightarrow A = \int_{-2}^{0} -x^3 dx + \int_{0}^{1} x^3 dx$$

$$\Rightarrow A = -\left(\frac{x^4}{4}\right)_{-2}^{0} + \left(\frac{x^4}{4}\right)_{0}^{1}$$

$$\Rightarrow A = 4 + \frac{1}{4}$$

$$\Rightarrow A = \frac{17}{4}$$

Question 27

The area bounded by the curve y = x |x| and the ordinates x =-1 and x = 1 is given by

a. 0

b. $\frac{1}{3}$

c. $\frac{2}{3}$

d. $\frac{4}{3}$

Solution 27

Correct option: (c)

The area bounded by the curve $y = x|x|$ and ordinates $x = -1$ and $x = 1$

$$A = \int_{-1}^{0} y dx + \int_{0}^{1} y dx$$

$$A = \int_{-1}^{0} -x^2 dx + \int_{0}^{1} x^2 dx$$

$$A = \left[\frac{-x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{1}$$

$$A = \left(0 + \frac{1}{3}\right) + \frac{1}{3} - 0$$

$$A = \frac{2}{3}$$

Question 28

The area bounded by the y – axis, $y = \cos x$ and $y = \sin x$ when $0 \le x \le \frac{\pi}{2}$ is

a. $2(\sqrt{2} - 1)$

b. $\sqrt{2} - 1$

c. $\sqrt{2} + 1$

d. $\sqrt{2}$

Solution 28

Correct option:(b)

Given that $y = \sin x$ and $y = \cos x$

$\Rightarrow \sin x = \cos x$

$\Rightarrow x = \frac{\pi}{4}$

$\Rightarrow y = \frac{1}{\sqrt{2}}$

$$A = \int_{0}^{\frac{1}{\sqrt{2}}} \sin^{-1} y dy + \int_{\frac{1}{\sqrt{2}}}^{1} \cos^{-1} y dy$$

Using partial integration

$A = \sqrt{2} - 1$ sq units

Question 29

The area of the circle $x^2 + y^2 = 16$ interior to the parabola $y^2 = 6x$ is

a. $\frac{4}{3}\left(4\pi-\sqrt{3}\right)$

b. $\frac{4}{3}\left(4\pi+\sqrt{3}\right)$

c. $\frac{4}{3}\left(8\pi-\sqrt{3}\right)$

d. $\frac{4}{3}\left(8\pi+\sqrt{3}\right)$

Solution 29

Correct option: (c)

$x^2 + y^2 = 16$ and $y^2 = 6x$

$\Rightarrow x^2 + 6x = 16$

$\Rightarrow x^2 + 6x - 16 = 0$

$\Rightarrow x = -8$ or $x = 2$

$x = -8$ will be outside the circle.

$\Rightarrow x = 2$

$\Rightarrow y = \pm 2\sqrt{3}$

circle of radius 4.

$$A = \int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16-x^2}\,dx$$

$$A = \frac{4}{3}\left(8\pi-\sqrt{3}\right) \text{ sq units}$$

Question 30

Smaller area enclosed by the circle $x^2+y^2=4$ and the line x + y =2 is

a. 2(π-2)
b. π-2
c. 2π-1
d. 2(π+2)

Solution 30

Correct option: (b)

Smaller area enclosed by the circle $x^2 + y^2 = 4$
and $x + y = 2$

$\Rightarrow (2 - x)^2 = 2 - x^2$

$\Rightarrow x = 0$ or $x = 2$

$$A = \int_0^2 y\,dx$$

$$A = \int_0^2 \sqrt{4 - x^2}\,dx + \int_0^2 (2 - x)\,dx$$

$$A = \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right) + 2x - \frac{x^2}{2}\right]_0^2$$

$A = \pi - 2$ sq units

Question 31

Area lying between the curves $y^2=4x$ and $y = 2x$ is

a. $\frac{2}{3}$

b. $\frac{1}{3}$

c. $\frac{1}{4}$

d. $\frac{3}{4}$

Solution 31

Correct option: (b)

Area lying between the curves $y^2 = 4x$ and $y = 2x$

$\Rightarrow y^2 = 4x$ and $y^2 = 4x^2$

$\Rightarrow 4x = 4x^2$

$\Rightarrow x = 0$ or $x = 1$

$$A = \int_0^1 2\left(\sqrt{x} - x\right)dx$$

$$A = 2\left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^2}{2}\right]_0^1$$

$$A = 2\left(\frac{1}{6}\right) = \frac{1}{3}$$

Question 32

Area lying in first quadrant and bounded by the circle $x^2+y^2=4$ and the lines $x=0$ and $x=2$, is

a. π

b. $\frac{\pi}{2}$

c. $\frac{\pi}{3}$

d. $\frac{\pi}{4}$

Solution 32

Correct option: (a)

Area lying in the first quadrant and bounded by the circle

$x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$

$$\Rightarrow A = \int_0^2 \sqrt{4 - x^2}\,dx$$

$$\Rightarrow A = \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{1}{2} x\ 4\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$\Rightarrow A = \pi$ sq units

Question 33

Area of the region bounded by the curve $y^2=4x$,y-axis and the line y =3, is

a. 2

b. $\frac{9}{4}$

c. $\frac{9}{3}$

d. $\frac{9}{2}$

Solution 33

Correct option: (b)

Area of the region bounded by the curve $y^2 = 4x$ and y axis,

and teh line $y = 3$

$$\Rightarrow x = \frac{y^2}{3}$$

$$\Rightarrow A = \int_0^3 \frac{y^2}{3}\,dy$$

$$\Rightarrow A = \left[\frac{y^3}{12}\right]_0^3$$

$$A = \frac{9}{4}$$