NCERT Solutions for Class 12-science Maths Chapter 1 - Relations and Functions

Chapter 1 - Relations and Functions Exercise Ex. 1.1

Solution 1

(i) $A = \{1, 2, 3 \ldots 13, 14\}$

$R = \{(x, y): 3x - y = 0\}$

Therefore, $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$

R is not reflexive since $(1, 1), (2, 2) \ldots (14, 14) \notin R$.

Also, R is not symmetric as $(1, 3) \in R$, but $(3, 1) \notin R$. $[3(3) - 1 \neq 0]$

Also, R is not transitive as $(1, 3), (3, 9) \in R$, but $(1, 9) \notin R$.

$[3(1) - 9 \neq 0]$

Hence, R is neither reflexive, nor symmetric, nor transitive.

(ii) $R = \{(x, y): y = x + 5 \text{ and } x < 4\} = \{(1, 6), (2, 7), (3, 8)\}$

It is seen that $(1, 1) \notin R$.

Therefore, R is not reflexive.

$(1, 6) \in R$

But, $(6,1) \notin R$.

Therefore, R is not symmetric.

Now, since there is no pair in R such that (x, y) and $(y, z) \in R$, then (x, z) cannot belong to R.

Therefore, R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(iii) $A = \{1, 2, 3, 4, 5, 6\}$

R = $\{(x, y): y$ is divisible by $x\}$

We know that any number (x) is divisible by itself.

$\Rightarrow (x, x) \in R$

Therefore, R is reflexive.

Now, $(2, 4) \in R$ [as 4 is divisible by 2]

But, $(4, 2) \notin R$. [as 2 is not divisible by 4]

Therefore, R is not symmetric.

Let $(x, y), (y, z) \in R$. Then, y is divisible by x and z is divisible by y.

Therefore, z is divisible by x.

$\Rightarrow (x, z) \in R$

Therefore, R is transitive.

Hence, R is reflexive and transitive but not symmetric.

(iv) R = $\{(x, y): x - y$ is an integer$\}$

Now, for every $x \in \mathbf{Z}$, $(x, x) \in$R as $x - x = 0$ is an integer.

Therefore, R is reflexive.

Now, for every $x, y \in \mathbf{Z}$ if $(x, y) \in$ R, then $x - y$ is an integer.

$\Rightarrow -(x - y)$ is also an integer.

$\Rightarrow (y - x)$ is an integer.

Therefore, $(y, x) \in$ R

Therefore, R is symmetric.

Now,

Let (x, y) and $(y, z) \in$R, where $x, y, z \in \mathbf{Z}$.

$\Rightarrow (x - y)$ and $(y - z)$ are integers.

$\Rightarrow x - z = (x - y) + (y - z)$ is an integer.

Therefore, $(x, z) \in$R

Therefore, R is transitive.

Hence, R is reflexive, symmetric, and transitive.

(v) (a) R = $\{(x, y): x$ and y work at the same place$\}$

$\Rightarrow (x, x) \in$ R

Therefore, R is reflexive.

If $(x, y) \in$ R, then x and y work at the same place.

$\Rightarrow y$ and x work at the same place.

$\Rightarrow (y, x) \in$ R.

Therefore, R is symmetric.

Now, let $(x, y), (y, z) \in$ R

$\Rightarrow x$ and y work at the same place and y and z work at the same place.

$\Rightarrow x$ and z work at the same place.

$\Rightarrow (x, z) \in$R

Therefore, R is transitive.

Hence, R is reflexive, symmetric, and transitive.

(b) R = $\{(x, y): x$ and y live in the same locality$\}$

Clearly $(x, x) \in$ R as x and x is the same human being.

Therefore, R is reflexive.

If $(x, y) \in$R, then x and y live in the same locality.

$\Rightarrow y$ and x live in the same locality.

$\Rightarrow (y, x) \in$ R

Therefore, R is symmetric.

Now, let $(x, y) \in$ R and $(y, z) \in$ R.

$\Rightarrow x$ and y live in the same locality and y and z live in the same locality.

$\Rightarrow x$ and z live in the same locality.

$\Rightarrow (x, z) \in$ R

Therefore, R is transitive.

Hence, R is reflexive, symmetric, and transitive.

(c) R = $\{(x, y): x$ is exactly 7 cm taller than $y\}$

Now, $(x, x) \notin$ R

Since human being x cannot be taller than himself.

Therefore, R is not reflexive.

Now, let $(x, y) \in$R.

$\Rightarrow x$ is exactly 7 cm taller than y.

Then, y is not taller than x.

Therefore, $(y, x) \notin$R

Indeed if x is exactly 7 cm taller than y, then y is exactly 7 cm shorter than x.

Therefore, R is not symmetric.

Now,

Let $(x, y), (y, z) \in$ R.

$\Rightarrow x$ is exactly 7 cm taller than y and y is exactly 7 cm taller than z.

$\Rightarrow x$ is exactly 14 cm taller than z .

Therefore, $(x, z) \notin$R

Therefore, R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(d) R = $\{(x, y): x$ is the wife of $y\}$

Now, $(x, x) \notin$ R

Since x cannot be the wife of herself.

Therefore, R is not reflexive.

Now, let $(x, y) \in$ R

$\Rightarrow x$ is the wife of y.

Clearly y is not the wife of x.

Therefore, $(y, x) \notin$ R

Indeed if x is the wife of y, then y is the husband of x.

Therefore, R is not symmetric.

Let $(x, y), (y, z) \in$ R

$\Rightarrow x$ is the wife of y and y is the wife of z.

This case is not possible. Also, this does not imply that x is the wife of z.

Therefore, $(x, z) \notin$ R

Therefore, R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(e) R = $\{(x, y): x$ is the father of $y\}$

$(x, x) \notin$ R

As x cannot be the father of himself.

Therefore, R is not reflexive.

Now, let $(x, y) \in$R.

$\Rightarrow x$ is the father of y.

$\Rightarrow y$ cannot be the father of y.

Indeed, y is the son or the daughter of y.

Therefore, $(y, x) \notin$ R

Therefore, R is not symmetric.

Now, let $(x, y) \in$ R and $(y, z) \in$ R.

$\Rightarrow x$ is the father of y and y is the father of z.

$\Rightarrow x$ is not the father of z.

Indeed x is the grandfather of z.

Therefore, $(x, z) \notin$ R

Therefore, R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Solution 2

$R = \{(a, b): a \leq b^2\}$

Therefore, R is not reflexive.

Now, $(1, 4) \in R$ as $1 < 4^2$

But, 4 is not less than 1^2.

Therefore, $(4, 1) \notin R$

Therefore, R is not symmetric.

Now, $(3, 2), (2, 1.5) \in R$ (as $3 < 2^2 = 4$ and $2 < (1.5)^2 = 2.25$)

But, $3 > (1.5)^2 = 2.25$

Therefore, $(3, 1.5) \notin R$

Therefore, R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Concept Insights:

Here the result is disproved using some specific examples. In order to prove a result we must prove it in generality and in order to disprove a result we can just provide one example where the condition is false.It is important to pick up the examples suitably since there are certain ordered pairs like (1,1) for which the relation is reflexive.

Solution 3

Let $A = \{1, 2, 3, 4, 5, 6\}$.

A relation R is defined on set A as:

$R = \{(a, b): b = a + 1\}$

Therefore, $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

We find $(a, a) \notin R$, where $a \in A$.

For instance, $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) \notin R$

Therefore, R is not reflexive.

It can be observed that $(1, 2) \in R$, but $(2, 1) \notin R$.

Therefore, R is not symmetric.

Now, $(1, 2), (2, 3) \in \mathbf{R}$

But, $(1, 3) \notin R$

Therefore, R is not transitive

Hence, R is neither reflexive, nor symmetric, nor transitive.

Solution 4

R = {(a, b); $a \leq b$}

Clearly (a, a) $\in$ R as a = a.

Therefore, R is reflexive.

Now, (2, 4) $\in$ R (as 2 < 4)

But, (4, 2) $\notin$ R as 4 is greater than 2.

Therefore, R is not symmetric.

Now, let (a, b), (b, c) $\in$ R.

Then, $a \leq b$ and $b \leq c$

$\Rightarrow a \leq c$

$\Rightarrow$ (a, c) $\in$ R

Therefore, R is transitive.

Hence, R is reflexive and transitive but not symmetric.

Solution 5

R = {(a, b): $a \leq b^3$}

It is observed that $\left(\frac{1}{2}, \frac{1}{2}\right) \notin \text{R as } \frac{1}{2} > \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

Therefore, R is not reflexive.

Now, (1, 2) $\in$ R (as 1 < 2^3 = 8)

But, (2, 1) $\notin$ R (as 2^3 > 1)

Therefore, R is not symmetric.

We have

$$\left(3, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{6}{5}\right) \in \text{R as } 3 < \left(\frac{3}{2}\right)^3 \text{ and } \frac{3}{2} < \left(\frac{6}{5}\right)^3.$$

But $\left(3, \frac{6}{5}\right) \notin \text{R as } 3 > \left(\frac{6}{5}\right)^3$.

Therefore, R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Solution 6

Let A = {1, 2, 3}.

A relation R on A is defined as R = {(1, 2), (2, 1)}.

It is seen that (1, 1), (2, 2), (3, 3) $\notin$R.

Therefore, R is not reflexive.

Now, as (1, 2) $\in$ R and (2, 1) $\in$ R, then R is symmetric.

Now, (1, 2) and (2, 1) $\in$ R

However, (1, 1) $\notin$ R

Therefore, R is not transitive.

Hence, R is symmetric but neither reflexive nor transitive.

Concept Insights:

Solution 7

Set A is the set of all books in the library of a college.

R = {(x, y): x and y have the same number of pages}

Now, R is reflexive since (x, x) ∈ R as x and x has the same number of pages.

Let (x, y) ∈ R ⇒ x and y have the same number of pages.

⇒ y and x have the same number of pages.

⇒ (y, x) ∈ R

Therefore, R is symmetric.

Now, let (x, y) ∈R and (y, z) ∈ R.

⇒ x and y and have the same number of pages and y and z have the same number of pages.

⇒ x and z have the same number of pages.

⇒ (x, z) ∈ R

Therefore, R is transitive.

Hence, R is an equivalence relation.

Solution 8

A = {1, 2, 3, 4, 5}

R = {(a, b):|a - b| is even}

It is clear that for any element a ∈A, we have |a − a| = 0 (which is even).

Therefore, R is reflexive.

Let (a, b) ∈ R.

⇒ |a − b| is even,

⇒ |-(a − b)| = |b − a| is also even

⇒ (b, a) ∈ R

Therefore, R is symmetric.

Now, let (a, b) ∈ R and (b, c) ∈ R.

⇒ |a − b| is even and |b − c| is even

⇒ (a − b) is even and (b − c) is even

⇒ (a − c) = (a − b) + (b − c) is even [Sum of two even integers is even]

⇒ |a − c| is even

⇒ (a, c) ∈ R

Therefore, R is transitive.

Hence, R is an equivalence relation.

Now, all elements of the set {1, 3, and 5} are related to each other as all the elements of this subset are odd. Thus, the modulus of the difference between any two elements will be even.

Similarly, all elements of the set {2, 4} are related to each other as all the elements of this subset are even.

Also, no element of the subset {1, 3, 5} can be related to any element of {2, 4} as all elements of {1, 3, 5} are odd and all elements of {2, 4} are even. Thus, the modulus of the difference between the two elements (from each of these two subsets) will not be even.

Solution 9

$A = \{x \in Z : 0 \le x \le 12\}$ $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

(i) $R = \{(a, b) : |a - b|$ is a multiple of $4\}$

For any element $a \in A$, we have $(a, a) \in R$ as $|a - a| = 0$ is a multiple of 4.

Therefore, R is reflexive.

Now, let $(a, b) \in R \Rightarrow |a - b|$ is a multiple of 4.

$|b - a| = |a - b|$ is a multiple of 4.

$\Rightarrow (b, a) \in R$

Therefore, R is symmetric.

Now, let $(a, b), (b, c) \in R$.

So $|a - b|$ and $|b - c|$ is a multiple of 4.

So $|a - c| = |(a - b) + (b - c)|$ is a multiple of 4.

$\Rightarrow (a, c) \in R$

Therefore, R is transitive.

Hence, R is an equivalence relation.

The set of elements related to 1 is $\{1, 5, 9\}$ since

$|1 - 1| = 0$ is a multiple of 4,

$|5 - 1| = 4$ is a multiple of 4, and

$|9 - 1| = 8$ is a multiple of 4.

(ii) $R = \{(a, b): a = b\}$

For any element $a \in A$, we have $(a, a) \in R$, since $a = a$.

Therefore, R is reflexive.

Now, let $(a, b) \in R$.

$\Rightarrow a = b$

$\Rightarrow b = a$

$\Rightarrow (b, a) \in R$

Therefore, R is symmetric.

Now, let $(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow a = b$ and $b = c$

$\Rightarrow a = c$

$\Rightarrow (a, c) \in R$

Therefore, R is transitive.

Hence, R is an equivalence relation.

The elements in R that are related to 1 will be those elements from set A which are equal to 1.

Hence, the set of elements related to 1 is $\{1\}$.

Solution 10

(i) Let $A = \{5, 6, 7\}$.

Define a relation R on A as R = {(5, 6), (6, 5)}.

Relation R is not reflexive as (5, 5), (6, 6), (7, 7) $\notin$ R.

Now, as (5, 6) $\in$ R and also (6, 5) $\in$ R, R is symmetric.

$\Rightarrow$ (5, 6), (6, 5) $\in$ R, but (5, 5) $\notin$ R

Therefore, R is not transitive.

Hence, relation R is symmetric but not reflexive or transitive.

(ii)Consider a relation R in **R** defined as:

R = $\{(a, b): a < b\}$

For any $a \in$ R, we have $(a, a) \notin$ R since a cannot be strictly less than a itself. In fact, $a = a$.

Therefore, R is not reflexive.

Now, (1, 2) $\in$ R (as $1 < 2$)

But, 2 is not less than 1.

Therefore, (2, 1) $\notin$ R

Therefore, R is not symmetric.

Now, let (a, b), $(b, c) \in$ R.

$\Rightarrow a < b$ and $b < c$

$\Rightarrow a < c$

$\Rightarrow (a, c) \in$ R

Therefore, R is transitive.

Hence, relation R is transitive but not reflexive and symmetric.

(iii)Let A = {4, 6, 8}.

Define a relation R on A as:

A = {(4, 4), (6, 6), (8, 8), (4, 6), (6, 4), (6, 8), (8, 6)}

Relation R is reflexive since for every $a \in A$, $(a, a) \in$R i.e., (4, 4), (6, 6), (8, 8)} $\in$ R.

Relation R is symmetric since $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in R$.

Relation R is not transitive since (4, 6), (6, 8) $\in$ R, but (4, 8) $\notin$ R.

Hence, relation R is reflexive and symmetric but not transitive.

(iv) Define a relation R in **R** as:

R = $\{a, b): a^3 \geq b^3\}$

Clearly $(a, a) \in R$ as $a^3 = a^3$.

Therefore, R is reflexive.

Now, (2, 1) $\in$ R (as $2^3 \geq 1^3$)

But, (1, 2) $\notin$ R (as $1^3 < 2^3$)

Therefore, R is not symmetric.

Now, Let $(a, b), (b, c) \in R$.

$\Rightarrow a^3 \geq b^3$ and $b^3 \geq c^3$

$\Rightarrow a^3 \geq c^3$

$\Rightarrow (a, c) \in R$

Therefore, R is transitive.

Hence, relation R is reflexive and transitive but not symmetric.

(v) Let A = {−5, −6}.

Define a relation R on A as:

R = {(−5, −6), (−6, −5), (−5, −5)}

Relation R is not reflexive as (−6, −6) $\notin$ R.

Relation R is symmetric as (−5, −6) $\in$ R and (−6, −5}$\in$R.

It is seen that (−5, −6), (−6, −5) $\in$ R. Also, (−5, −5) $\in$ R.

Therefore, the relation R is transitive.

Hence, relation R is symmetric and transitive but not reflexive.

Concept insight: For these examples always select a finite set of 2 or 3 elements, and then define the relation R with ordered pairs as required.

For example, A ={1,2,3}. like symmetric means include pairs like (1,1), for symmetric (1,2) and (2,1) , for transitive (1,2), (2,3), (1,3).

Solution 11

R = {(P, Q): distance of point P from the origin is the same as the distance of point Q from the origin}

Clearly, (P, P) ∈ R since the distance of point P from the origin is always the same as the distance of the same point P from the origin.

Therefore, R is reflexive.

Now, let (P, Q) ∈ R.

⇒ The distance of point P from the origin is the same as the distance of point Q from the origin.

⇒ The distance of point Q from the origin is the same as the distance of point P from the origin.

⇒ (Q, P) ∈ R

Therefore, R is symmetric.

Now, let (P, Q), (Q, S) ∈ R.

⇒ The distance of points P and Q from the origin is the same and also, the distance of points Q and S from the origin is the same.

⇒ The distance of points P and S from the origin is the same.

⇒ (P, S) ∈ R

Therefore, R is transitive.

Therefore, R is an equivalence relation.

The set of all points related to P ≠ (0, 0) will be those points whose distance from the origin is the same as the distance of point P from the origin.

In other words, if O (0, 0) is the origin and OP = k, then the set of all points related to P is at a distance of k from the origin.

Hence, this set of points forms a circle with the centre as the origin and this circle passes through point P.

Solution 12

R = {(T_1, T_2): T_1 is similar to T_2}

R is reflexive since every triangle is similar to itself.

Further, if (T_1, T_2) ∈ R, then T_1 is similar to T_2

⇒ T_2 is similar to T_1

⇒ (T_2, T_1) ∈R

Therefore, R is symmetric.

Now, let (T_1, T_2), (T_2, T_3) ∈ R.

⇒ T_1 is similar to T_2 and T_2 is similar to T_3

⇒ T_1 is similar to T_3

⇒ (T_1, T_3) ∈ R

Therefore, R is transitive.

Thus, R is an equivalence relation.

Now, we can observe that:

$$\frac{3}{6}=\frac{4}{8}=\frac{5}{10}\left(=\frac{1}{2}\right)$$

Therefore, the corresponding sides of triangles T_1 and T_3 are in the same ratio.

Then, triangle T_1 is similar to triangle T_3

Hence, T_1 is related to T_3.

Solution 13

R = {(P_1, P_2): P_1 and P_2 have the same number of sides}

R is reflexive since (P_1, P_1) ∈ R as the same polygon has the same number of sides with itself.

Let (P_1, P_2) ∈ R.

⇒ P_1 and P_2 have the same number of sides.

⇒ P_2 and P_1 have the same number of sides.

⇒ (P_2, P_1) ∈ R

Therefore, R is symmetric.

Now, let (P_1, P_2), (P_2, P_3) ∈ R.

⇒ P_1 and P_2 have the same number of sides. Also, P_2 and P_3 have the same number of sides.

⇒ P_1 and P_3 have the same number of sides.

⇒ (P_1, P_3) ∈ R

Therefore, R is transitive.

Hence, R is an equivalence relation.

The elements in *A* related to the right-angled triangle (*T*) with sides 3, 4, and 5 are those polygons which have 3 sides (since *T* is a polygon with 3 sides).

Hence, the set of all elements in *A* related to triangle *T* is the set of all triangles.

Solution 14

R = {(L_1, L_2): L_1 is parallel to L_2}

R is reflexive as any line L_1 is parallel to itself i.e., $(L_1, L_1) \in$ R.

Now, let $(L_1, L_2) \in$ R.

$\Rightarrow L_1$ is parallel to L_2

$\Rightarrow L_2$ is parallel to L_1

$\Rightarrow (L_2, L_1) \in$ R

Therefore, R is symmetric.

Now, let (L_1, L_2), $(L_2, L_3) \in$R.

$\Rightarrow L_1$ is parallel to L_2. Also, L_2 is parallel to L_3

$\Rightarrow L_1$ is parallel to L_3

Therefore, R is transitive.

Hence, R is an equivalence relation.

The set of all lines related to the line $y = 2x + 4$ is the set of all lines that are parallel to the line $y = 2x+ 4$.

Slope of line $y = 2x + 4$ is $m = 2$

It is known that parallel lines have the same slopes.

The line parallel to the given line is of the form $y = 2x + c$, where $c \in \mathbf{R}$.

Hence, the set of all lines related to the given line is given by $y = 2x + c$, where $c \in \mathbf{R}$.

Solution 15

R = {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}

It is seen that $(a, a) \in$ **R,** for every $a \in \{1, 2, 3, 4\}$.

Therefore, R is reflexive.

It is seen that (1, 2) $\in$ R, but (2, 1) $\notin$ R.

Therefore, R is not symmetric.

Also, it is observed that (a, b), $(b, c) \in$ R $\Rightarrow (a, c) \in$ R for all $a, b, c \in \{1, 2, 3, 4\}$.

Therefore, R is transitive.

Hence, R is reflexive and transitive but not symmetric.

The correct answer is B.

Solution 16

R = {(a, b): $a = b - 2$, $b > 6$}

Now, since $b > 6$, (2, 4) $\notin$ R

Also, as $3 \neq 8 - 2$, (3, 8) $\notin$ R

And, as $8 \neq 7 - 2$

Therefore, (8, 7) $\notin$ R

Now, consider (6, 8).

We have $8 > 6$ and also, $6 = 8 - 2$.

Therefore, (6, 8) $\in$ R

The correct answer is C.

Concept Insight:

To prove a relation is reflexive, check that (x, x) is in the relation R for all x in the domain.
To prove a relation is symmetric, check that (x, y) and (y, x) both belong to the relation R for all x in the domain.
To prove a relation is transitive, check that if (x, y) and (y, z) both belong to the relation R then, (x, z) is also in R, for all x in the domain.
On the other hand, if we have to prove that a relation is not reflexive, symmetric or transitive, it is sufficient to show this for any one element in the domain. Always choose simple elements like 1, 2, 3 etc to disprove.
Mathematically we say that one counter example is sufficient to disprove anything, but to prove we have to verify for all the elements in the domain.
A relation is an equivalence relation if it is reflexive, symmetric and transitive.

Chapter 1 - Relations and Functions Exercise Ex. 1.2

Solution 1

It is given that $f: \mathbf{R}_* \to \mathbf{R}_*$ is defined by $f(x)=\frac{1}{x}$.

One-one:

$f(x)=f(y)$

$\Rightarrow \frac{1}{x}=\frac{1}{y}$

$\Rightarrow x=y$

Therefore, f is one-one.

Onto:

It is clear that for $y \in \mathbf{R}_*$, there exists $x=\frac{1}{y} \in \mathrm{R}_*$ $(\text{Exists as } y \neq 0)$ such that

$f(x)=\frac{1}{\left(\frac{1}{y}\right)}=y.$

Therefore, f is onto.

Thus, the given function (f) is one-one and onto.

Now, consider function $g: \mathbf{N} \to \mathbf{R}_*$ defined by

$g(x)=\frac{1}{x}.$

We have,

$g(x_1)=g(x_2) \Rightarrow \frac{1}{x_1}=\frac{1}{x_2} \Rightarrow x_1=x_2$

Therefore, g is one-one.

Further, it is clear that g is not onto as for $1.2 \in \mathbf{R}_*$ there does not exist any x in $\mathbf{N}$ such that $g(x)=\frac{1}{1.2}$.

Hence, function g is one-one but not onto.

Solution 2

(i) f: **N** → **N** is given by, $f(x) = x^2$

It is seen that for $x, y \in$ **N**, $f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$.

Therefore, f is injective.

Now, 2 ∈ **N**. But, there does not exist any x in **N** such that $f(x) = x^2 = 2$.

Therefore, f is not surjective.

Hence, function f is injective but not surjective.

(ii) f: **Z** → **Z** is given by, $f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

Therefore, f is not injective.

Now, -2 ∈ **Z**. But, there does not exist any element $x \in$ **Z** such that $f(x) = x^2 = -2$.

Therefore, f is not surjective.

Hence, function f is neither injective nor surjective.

(iii) f: **R** → **R** is given by,

$f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

Therefore, f is not injective.

Now, -2 ∈ **R**. But, there does not exist any element $x \in$ **R** such that $f(x) = x^2 = -2$.

Therefore, f is not surjective.

Hence, function f is neither injective nor surjective.

(iv) f: **N** → **N** given by, $f(x) = x^3$

It is seen that for $x, y \in$ **N**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

Therefore, f is injective.

Now, 2 ∈ **N**. But, there does not exist any element x in domain **N** such that $f(x) = x^3 = 2$.

Therefore, f is not surjective

Hence, function f is injective but not surjective.

(v) f: **Z** → **Z** is given by, $f(x) = x^3$

It is seen that for $x, y \in$ **Z**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

Therefore, f is injective.

Now, 2 ∈ **Z**. But, there does not exist any element x in domain **Z** such that $f(x) = x^3 = 2$.

Therefore, f is not surjective.

Hence, function f is injective but not surjective.

Solution 3

f: **R** → **R** is given by, $f(x) = [x]$

It is seen that $f(1.2) = [1.2] = 1$, $f(1.9) = [1.9] = 1$.

Therefore, $f(1.2) = f(1.9)$, but $1.2 \neq 1.9$.

Therefore, f is not one-one.

Now, consider 0.7 ∈ **R**.

It is known that $f(x) = [x]$ is always an integer.

Thus, there does not exist any element $x \in$ **R** such that $f(x) = 0.7$.

Therefore, f is not onto.

Hence, the greatest integer function is neither one-one nor onto.

Solution 4

f: **R** → **R** is given by,

$$f(x)=|x|=\begin{cases}x, & \text{if } x\geq 0\\ -x, & \text{if } x<0\end{cases}$$

It is seen that f(-1) = |-1| = 1, f(1) = |1| = 1.

Therefore, $f(-1) = f(1)$, but $-1 \neq 1$.

Therefore, f is not one-one.

Now, consider $-1 \in$ **R**.

It is known that $f(x) = |x|$ is always non-negative. Thus, there does not exist any element x in domain **R** such that $f(x) = |x| = -1$.

Therefore, f is not onto.

Hence, the modulus function is neither one-one nor onto.

Solution 5

f: **R** → **R** is given by,

$$f(x)=\begin{cases}1, & \text{if } x>0\\ 0, & \text{if } x=0\\ -1, & \text{if } x<0\end{cases}$$

It is seen that $f(1) = f(2) = 1$, but $1 \neq 2$.

Therefore, f is not one-one.

Now, as $f(x)$ takes only 3 values (1, 0, or −1) for the element −2 in co-domain **R**, there does not exist any x in domain **R** such that $f(x) = -2$.

Therefore, f is not onto.

Hence, the signum function is neither one-one nor onto.

Solution 6

It is given that $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$.

$f: A \to B$ is defined as $f = \{(1, 4), (2, 5), (3, 6)\}$.

Therefore, $f(1) = 4$, $f(2) = 5$, $f(3) = 6$

It is seen that the images of distinct elements of A under f are distinct.

Hence, function f is one-one.

Solution 7

(i) f: **R** → **R** is defined as $f(x) = 3 - 4x$.

Let $x_1, x_2 \in$ R such that $f(x_1) = f(x_2)$.

$$\Rightarrow 3-4x_1 = 3-4x_2$$

$$\Rightarrow -4x_1 = -4x_2$$

$$\Rightarrow x_1 = x_2$$

Therefore, f is one-one.

For any real number (y) in **R**, there exists $\frac{3-y}{4}$ in **R** such that

$$f\left(\frac{3-y}{4}\right)=3-4\left(\frac{3-y}{4}\right)=y.$$

Therefore, f is onto.

Hence, f is bijective.

(ii) *f*: **R** → **R** is defined as

$f(x) = 1 + x^2$.

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$.

$\Rightarrow 1 + x_1^2 = 1 + x_2^2$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

For instance,

$f(1) = f(-1) = 2$

Therefore, $f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$

Therefore, *f* is not one-one.

Consider an element −2 in co-domain **R**.

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in$ **R**.

Thus, there does not exist any *x* in domain **R** such that $f(x) = -2$.

Therefore, *f* is not onto.

Hence, *f* is neither one-one nor onto.

Solution 8

$f: A \times B \to B \times A$ is defined as $f(a, b) = (b, a)$.

Let $(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$.

$\Rightarrow (b_1, a_1) = (b_2, a_2)$

$\Rightarrow b_1 = b_2$ and $a_1 = a_2$

$\Rightarrow (a_1, b_1) = (a_2, b_2)$

Therefore, *f* is one-one.

Now, let $(b, a) \in B \times A$ be any element.

Then, there exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$. [By definition of *f*]

Therefore, *f* is onto.

Hence, *f* is bijective.

Solution 9

f: **N** → **N** is defined as $f(n)=\begin{cases}\dfrac{n+1}{2}, & \text{if } n \text{ is odd}\\ \dfrac{n}{2}, & \text{if } n \text{ is even}\end{cases}$ for all $n \in \mathbf{N}$.

It can be observed that:

$f(1)=\dfrac{1+1}{2}=1$ and $f(2)=\dfrac{2}{2}=1$ [By definition of f]

$\therefore f(1)=f(2)$, where $1 \neq 2$.

Therefore, f is not one-one.

Consider a natural number (n) in co-domain **N**.

Case **I:** n is odd

Therefore, $n = 2r + 1$ for some $r \in$ **N.**

Then, there exists $4r + 1 \in$ **N** such that $f(4r+1)=\dfrac{4r+1+1}{2}=2r+1$.

Case **II:** n is even

Therefore, $n = 2r$ for some $r \in$ **N.**

Then, there exists $4r \in$ **N** such that $f(4r)=\dfrac{4r}{2}=2r$.

Therefore, f is onto.

f is not one-one but it is onto.

Hence, f is not a bijective function.

Solution 10

A = **R** − {3}, B = **R** − {1}

f: A → B is defined as $f(x)=\left(\frac{x-2}{x-3}\right)$.

Let $x,\ y \in \mathrm{A}$ such that $f(x)=f(y)$.

$$\Rightarrow \frac{x-2}{x-3}=\frac{y-2}{y-3}$$
$$\Rightarrow (x-2)(y-3)=(y-2)(x-3)$$
$$\Rightarrow xy-3x-2y+6=xy-3y-2x+6$$
$$\Rightarrow -3x-2y=-3y-2x$$
$$\Rightarrow 3x-2x=3y-2y$$
$$\Rightarrow x=y$$

Therefore, *f* is one-one.

Let $y \in$B = **R** − {1}.

Then, $y \neq 1$.

The function *f* is onto if there exists $x \in$A such that $f(x) = y$.

Now,

$$f(x)=y$$
$$\Rightarrow \frac{x-2}{x-3}=y$$
$$\Rightarrow x-2=xy-3y$$
$$\Rightarrow x(1-y)=-3y+2$$
$$\Rightarrow x=\frac{2-3y}{1-y}\in \mathrm{A} \qquad [y\neq 1]$$

Thus, for any $y \in$ B, there exists $\frac{2-3y}{1-y}\in \mathrm{A}$ such that

$$f\left(\frac{2-3y}{1-y}\right)=\frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3}=\frac{2-3y-2+2y}{2-3y-3+3y}=\frac{-y}{-1}=y.$$

$\therefore f$ is onto.

Hence, function *f* is one-one and onto.

Solution 11

f: **R** → **R** is defined as $f(x) = x^4$

Let x, $y \in$ **R** such that $f(x) = f(y)$.

$\Rightarrow x^4 = y^4$

$\Rightarrow x = \pm y$

Therefore, $f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$.

For instance, $f(1) = f(-1) = 1$

Therefore, f is not one-one.

Consider an element 2 in co-domain **R**. It is clear that there does not exist any x in domain **R** such that $f(x) = 2$.

Therefore, f is not onto.

Hence, function f is neither one-one nor onto.

The correct answer is D.

Solution 12

f: **R** → **R** is defined as $f(x) = 3x$.

Let x, $y \in$ **R** such that $f(x) = f(y)$.

$\Rightarrow 3x = 3y$

$\Rightarrow x = y$

Therefore, f is one-one.

Also, for any real number (y) in co-domain **R**, there exists $\frac{y}{3}$ in **R** such that

$f\left(\frac{y}{3}\right) = 3\left(\frac{y}{3}\right) = y$.

Therefore, f is onto.

Hence, function f is one-one and onto.

The correct answer is A.

Concept Insight:

1. A function f is one- one or injective, if every element x in the domain has only one image y in the co-domain. This can be proved very easily as follows:

If $f(x_1) = f(x_2) \rightarrow x_1 = x_2$, then f is one-one.

A function is many-one, if there exists at least 2 elements in the domain which have the same image in the co-domain.

Hence to prove that a function is not injective, i.e many-one, show that two elements in the domain have same image. $f(x_1) = f(x_2)$ for $x_1 \neq x_2$.

For example, $f(x) = x^2$

Then $f(-2) = f(2) = 4$.

2. A function is onto or surjective if every element of the co-domain has a pre-image in the domain.

This means for any y ∈ Co-domain there exists x such that f(x) = y.

The simple way to do this is to take y = f(x) and then solve for x.

For example, f(x) = 3-5x. Therefore y = 3-5x. Hence x = (3-y)/5.

Now for y in the co-domain, x = (3-y)/5 such that y=f(x). Hence it is onto function.

3. If a function is both injective and surjective, it is said to be bijective.

Chapter 1 - Relations and Functions Exercise Ex. 1.3

Solution 1

The functions f: {1, 3, 4} → {1, 2, 5} and g: {1, 2, 5} → {1, 3} are defined as f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}.

$$gof(1) = g(f(1)) = g(2) = 3 \qquad [f(1) = 2 \text{ and } g(2) = 3]$$

$$gof(3) = g(f(3)) = g(5) = 1 \qquad [f(3) = 5 \text{ and } g(5) = 1]$$

$$gof(4) = g(f(4)) = g(1) = 3 \qquad [f(4) = 1 \text{ and } g(1) = 3]$$

$$\therefore gof = \{(1,3),(3,1),(4,3)\}$$

Solution 2

To prove: (f + g)oh = foh + goh

Consider:

$$((f+g)oh)(x)$$

$$=(f+g)(h(x))$$

$$=f(h(x))+g(h(x))$$

$$=(foh)(x)+(goh)(x)$$

$$=\{(foh)+(goh)\}(x)$$

$$\therefore ((f+g)oh)(x)=\{(foh)+(goh)\}(x) \qquad \forall x \in \mathbf{R}$$

Hence, $(f+g)oh = foh + goh$.

To prove:

$$(f \cdot g)oh = (foh)\cdot(goh)$$

Consider:

$$((f \cdot g)oh)(x)$$

$$=(f \cdot g)(h(x))$$

$$=f(h(x)).g(h(x))$$

$$=(foh)(x).(goh)(x)$$

$$=\{(foh).(goh)\}(x)$$

$$\therefore ((f \cdot g)oh)(x)=\{(foh).(goh)\}(x) \ \forall x \in \mathrm{R}$$

Hence, $(f \cdot g)oh = (foh)\cdot(goh)$.

Solution 3

(i) f(x) = |x| and g(x) = |5x − 2|

$$\therefore (gof)(x) = g(f(x)) = g(|x|) = |5|x| - 2|$$

$$(fog)(x) = f(g(x)) = f(|5x-2|) = ||5x-2|| = |5x-2|$$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

$$\therefore (gof)(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = 2x$$

$$(fog)(x) = f(g(x)) = f\left(x^{\frac{1}{3}}\right) = 8\left(x^{\frac{1}{3}}\right)^3 = 8x$$

Solution 4

It is given that $f(x) = \frac{(4x+3)}{(6x-4)},\ x \neq \frac{2}{3}$.

$$(fof)(x) = f(f(x)) = f\left(\frac{4x+3}{6x-4}\right)$$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} = \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34} = x$$

Therefore, $fof(x) = x$, for all $x \neq \frac{2}{3}$.

$\Rightarrow fof = \mathrm{I}$

Hence, the given function *f* is invertible and the inverse of *f* is *f* itself.

Solution 5

(i) *f*: {1, 2, 3, 4} → {10} defined as:
f = {(1, 10), (2, 10), (3, 10), (4, 10)}
From the given definition of *f*, we can see that *f* is a many one function as: *f*(1) = *f*(2) = *f*(3) = *f*(4) = 10
Therefore, *f* is not one-one.
Hence, function *f* does not have an inverse.

(ii) *g*: {5, 6, 7, 8} → {1, 2, 3, 4} defined as:
g = {(5, 4), (6, 3), (7, 4), (8, 2)}
From the given definition of *g*, it is seen that *g* is a many one function as: *g*(5) = *g*(7) = 4.
Therefore, *g* is not one-one,
Hence, function g does not have an inverse.
(iii) *h*: {2, 3, 4, 5} → {7, 9, 11, 13} defined as:
h = {(2, 7), (3, 9), (4, 11), (5, 13)}
It is seen that all distinct elements of the set {2, 3, 4, 5} have distinct images under *h*.
Therefore, Function *h* is one-one.
Also, *h* is onto since for every element *y* of the set {7, 9, 11, 13}, there exists an element *x* in the set {2, 3, 4, 5} such that *h*(*x*) = *y*.
Thus, *h* is a one-one and onto function. Hence, *h* has an inverse.

Solution 6

f: [−1, 1] → R is given as $f\left(x\right)=\dfrac{x}{\left(x+2\right)}$.

Let $f(x) = f(y)$.

$$\Rightarrow \frac{x}{x+2}=\frac{y}{y+2}$$

$$\Rightarrow xy+2x=xy+2y$$

$$\Rightarrow 2x=2y$$

$$\Rightarrow x=y$$

Therefore, f is a one-one function.
It is clear that f: [−1, 1] → Range f is onto.

Let y = $\dfrac{x}{x+2}$, xy + 2y = x so $x = \dfrac{2y}{1-y}$

So for every y in the range there exists x in the domain such that f(x) = y
Hence function f is onto.
Therefore, f: [−1, 1] → Range f is one-one and onto and therefore,
the inverse of the function: f: [−1, 1] → Range f exists.
Let g: Range f → [−1, 1] be the inverse of f.
Let y be an arbitrary element of range f.
Since f: [−1, 1] → Range f is onto, we have:

$$y=f\left(x\right)\text{ for same }x\in\left[-1,\ 1\right]$$

$$\Rightarrow y=\frac{x}{x+2}$$

$$\Rightarrow xy+2y=x$$

$$\Rightarrow x\left(1-y\right)=2y$$

$$\Rightarrow x=\frac{2y}{1-y},\ y\neq 1$$

Now, let us define g: Range f → [−1, 1] as

$$g\left(y\right)=\frac{2y}{1-y},\ y\neq 1.$$

$$\text{Now, }\left(gof\right)\left(x\right)=g\left(f\left(x\right)\right)=g\left(\frac{x}{x+2}\right)=\frac{2\left(\frac{x}{x+2}\right)}{1-\frac{x}{x+2}}=\frac{2x}{x+2-x}=\frac{2x}{2}=x$$

$$\left(fog\right)\left(y\right)=f\left(g\left(y\right)\right)=f\left(\frac{2y}{1-y}\right)=\frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2}=\frac{2y}{2y+2-2y}=\frac{2y}{2}=y$$

Therefore, $gof = I_{[-1,\ 1]}$ and $fog = I_{\text{Range}\,f}$
Therefore, $f^{-1} = g$

Therefore, $f^{-1}\left(y\right)=\dfrac{2y}{1-y}, y\neq 1$

Solution 7

f: $\mathbf{R} \to \mathbf{R}$ is given by, $f(x) = 4x + 3$
One-one:
Let $f(x) = f(y)$.

$\Rightarrow 4x + 3 = 4y + 3$

$\Rightarrow 4x = 4y$

$\Rightarrow x = y$

Therefore f is a one-one function.
Onto:
For $y \in \mathbf{R}$, let $y = 4x + 3$.

$\Rightarrow x = \dfrac{y-3}{4} \in \mathbf{R}$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \dfrac{y-3}{4} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y.$$

Therefore, f is onto.
Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define g: $\mathbf{R} \to \mathbf{R}$ by $g(x) = \dfrac{x-3}{4}$

$$\text{Now}, (gof)(x) = g(f(x)) = g(4x+3) = \frac{(4x+3)-3}{4} = x$$

$$(fo\,g)(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y - 3 + 3 = y$$

Therefore, gof = fog = I_R
Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{y-3}{4}.$$

Solution 8

f: $\mathbf{R}_+ \to [4, \infty)$ is given as $f(x) = x^2 + 4$.
One-one:
Let $f(x) = f(y)$.

$$\Rightarrow x^2 + 4 = y^2 + 4$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x = y \qquad [\text{as } x = y \in \mathbf{R}_+]$$

Therefore, f is a one-one function.
Onto:
For $y \in [4, \infty)$, let $y = x^2 + 4$.

$$\Rightarrow x^2 = y - 4 \geq 0 \qquad [\text{as } y \geq 4]$$

$$\Rightarrow x = \sqrt{y-4} \geq 0$$

Therefore, for any $y \in [4, \infty)$, there exists $x = \sqrt{y-4} \in \mathbf{R}_+$ such that

$$f(x) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = y - 4 + 4 = y.$$

Therefore, f is onto.
Thus, f is one-one and onto and therefore, f^{-1} exists.
Let us define g: $[4, \infty) \to \mathbf{R}_+$ by,

$$g(y) = \sqrt{y-4}$$

$$\text{Now, } gof(x) = g(f(x)) = g\left(x^2+4\right) = \sqrt{\left(x^2+4\right)-4} = \sqrt{x^2} = x$$

$$\text{And, } fog(y) = f(g(y)) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = (y-4) + 4 = y$$

Therefore, $gof = fog = I_R$
Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \sqrt{y-4}.$$

Solution 9

f: $\mathbf{R}_+ \to [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.
Let y be an arbitrary element of $[-5, \infty)$.
Let $y = 9x^2 + 6x - 5$.

$$\Rightarrow y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$$

$$\Rightarrow (3x+1)^2 = y+6$$

$$\Rightarrow 3x+1 = \sqrt{y+6} \qquad [\text{as } y \geq -5 \Rightarrow y+6 > 0]$$

$$\Rightarrow x = \frac{\sqrt{y+6}-1}{3}$$

Therefore, f is onto, thereby range $f = [-5, \infty)$.

Let us define g: $[-5, \infty) \to \mathbf{R}_+$ as $g(y) = \frac{\sqrt{y+6}-1}{3}$.

We now have:

$$(gof)(x) = g(f(x)) = g(9x^2+6x-5)$$

$$= g\left((3x+1)^2-6\right)$$

$$= \frac{\sqrt{(3x+1)^2-6+6}-1}{3}$$

$$= \frac{3x+1-1}{3} = x$$

$$\text{And,} (fog)(y) = f(g(y)) = f\left(\frac{\sqrt{y+6}-1}{3}\right)$$

$$= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right)+1\right]^2 - 6$$

$$= \left(\sqrt{y+6}\right)^2 - 6 = y+6-6 = y$$

Therefore, $gof = I_R$ and $fog = I_{[-5,\infty)}$
Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}.$$

Solution 10

Let f: $X \to Y$ be an invertible function.
Also, suppose f has two inverses (say g_1 and g_2).
Then, for all $y \in Y$, we have:

$$fog_1(y) = I_Y(y) = fog_2(y)$$

$$\Rightarrow f(g_1(y)) = f(g_2(y))$$

$$\Rightarrow g_1(y) = g_2(y) \qquad [f \text{ is invertible} \Rightarrow f \text{ is one-one}]$$

$$\Rightarrow g_1 = g_2 \qquad [g \text{ is one-one}]$$

Hence, f has a unique inverse.

Solution 11

Function f: {1, 2, 3} → {a, b, c} is given by,
$f(1) = a$, $f(2) = b$, and $f(3) = c$
If we define g:
{a, b, c} → {1, 2, 3} as $g(a) = 1$, $g(b) = 2$, $g(c) = 3$, then we have:

$$(f o g)(a) = f(g(a)) = f(1) = a$$

$$(f o g)(b) = f(g(b)) = f(2) = b$$

$$(f o g)(c) = f(g(c)) = f(3) = c$$

And,

$$(g o f)(1) = g(f(1)) = g(a) = 1$$

$$(g o f)(2) = g(f(2)) = g(b) = 2$$

$$(g o f)(3) = g(f(3)) = g(c) = 3$$

Therefore, gof = I_X and fog = I_Y, where X = {1, 2, 3} and Y= {a, b, c}.
Thus, the inverse of f exists and $f^{-1} = g$.
Therefore, f^{-1}: {a, b, c} → {1, 2, 3} is given by,
$f^{-1}(a) = 1$, $f^{-1}(b) = 2$, $f^{-1}(c) = 3$
Let us now find the inverse of f^{-1} i.e., find the inverse of g.
If we define h: {1, 2, 3} → {a, b, c} as
$h(1) = a$, $h(2) = b$, $h(3) = c$, then we have:

$$(g o h)(1) = g(h(1)) = g(a) = 1$$

$$(g o h)(2) = g(h(2)) = g(b) = 2$$

$$(g o h)(3) = g(h(3)) = g(c) = 3$$

And,

$$(h o g)(a) = h(g(a)) = h(1) = a$$

$$(h o g)(b) = h(g(b)) = h(2) = b$$

$$(h o g)(c) = h(g(c)) = h(3) = c$$

Therefore, goh = I_X and hog = I_Y, where X = {1, 2, 3} and Y = {a, b, c}.
Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.
It can be noted that $h = f$.
Hence, $(f^{-1})^{-1} = f$.

Solution 12

Let f: $X \to Y$ be an invertible function.
Then, there exists a function g: $Y \to X$ such that $gof = I_X$ and $fog = I_Y$.
Here, $f^{-1} = g$.
Now, $gof = I_X$ and $fog = I_Y$
Therefore, $f^{-1}of = I_X$ and $fof^{-1} = I_Y$
Hence, f^{-1}: $Y \to X$ is invertible and f is the inverse of f^{-1}
i.e., $(f^{-1})^{-1} = f$.

Solution 13

f: **R** → **R** is given as $f(x)=(3-x^3)^{\frac{1}{3}}$.

$f(x)=(3-x^3)^{\frac{1}{3}}$

$$\therefore fof(x)=f(f(x))=f\left((3-x^3)^{\frac{1}{3}}\right)=\left[3-\left((3-x^3)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$=\left[3-(3-x^3)\right]^{\frac{1}{3}}=(x^3)^{\frac{1}{3}}=x$$

$\therefore fof(x)=x$

The correct answer is C.

Solution 14

It is given that $f:\mathbf{R}-\left\{-\frac{4}{3}\right\}\to\mathbf{R}$ is defined as $f(x)=\frac{4x}{3x+4}$.

Let y be an arbitrary element of Range f.

Then, there exists $x \in \mathbf{R}-\left\{-\frac{4}{3}\right\}$ such that y = f(x)

$$\Rightarrow y=\frac{4x}{3x+4}$$
$$\Rightarrow 3xy+4y=4x$$
$$\Rightarrow x(4-3y)=4y$$
$$\Rightarrow x=\frac{4y}{4-3y}$$

Let us define g: Range $f\to\mathbf{R}-\left\{-\frac{4}{3}\right\}$ as $g(y)=\frac{4y}{4-3y}$.

Now, $(g\circ f)(x)=g(f(x))=g\left(\frac{4x}{3x+4}\right)$

$$=\frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)}=\frac{16x}{12x+16-12x}=\frac{16x}{16}=x$$

And, $(f\circ g)(y)=f(g(y))=f\left(\frac{4y}{4-3y}\right)$

$$=\frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4}=\frac{16y}{12y+16-12y}=\frac{16y}{16}=y$$

Therefore, $gof=\mathrm{I}_{\mathbf{R}-\left\{-\frac{4}{3}\right\}}$ and $fog=\mathrm{I}_{\text{Range} f}$

Thus, g is the inverse of f i.e., $f^{-1}=g$.

Hence, the inverse of f is the map g: Range $f\to\mathbf{R}-\left\{-\frac{4}{3}\right\}$, which is given by

$$g(y)=\frac{4y}{4-3y}.$$

The correct answer is B.

Concept Insight:
1.Every function is not invertible. If f is a one- one and onto function, then it is invertible. So to check whether a function is invertible, determine whether the function is both one-one and onto.

2. Now let f(x) = y, and solve the equation for x . This will give us the formula for f^{-1}. For example, f(x) = (2x+3)/7. Let y = (2x+3)/7, hence x = (7y- 3)/2. So $f^{-1}(x) = (7x-3)/2$

3. verify that $fof^{-1} = f^{-1} o\ f = I$. i.e $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$.

Chapter 1 - Relations and Functions Exercise Ex. 1.4

Solution 1

(i) On $\mathbf{Z}^+$, * is defined by $a * b = a - b$.
It is not a binary operation as the image of (1, 2) under * is $1 * 2 = 1 - 2$
$= -1 \notin \mathbf{Z}^+$.

(ii) On $\mathbf{Z}^+$, * is defined by $a * b = ab$.
It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element ab in $\mathbf{Z}^+$.
This means that * carries each pair (a, b) to a unique element $a * b = ab$ in $\mathbf{Z}^+$.
Therefore, * is a binary operation.

(iii) On $\mathbf{R}$, * is defined by $a * b = ab^2$.
It is seen that for each $a, b \in \mathbf{R}$, there is a unique element ab^2 in $\mathbf{R}$.
This means that * carries each pair (a, b) to a unique element $a * b = ab^2$ in $\mathbf{R}$.
Therefore, * is a binary operation.

(iv) On $\mathbf{Z}^+$, * is defined by $a * b = |a - b|$.
It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element $|a - b|$ in $\mathbf{Z}^+$.
This means that * carries each pair (a, b) to a unique element $a * b =$
$|a - b|$ in $\mathbf{Z}^+$.
Therefore, * is a binary operation.

(v) On $\mathbf{Z}^+$, * is defined by $a * b = a$.
* carries each pair (a, b) to a unique element $a * b = a$ in $\mathbf{Z}^+$.
Therefore, * is a binary operation.

Solution 2

(i) On $\mathbf{Z}$, * is defined by $a * b = a - b$.
a-b ∈ Z, so the operation * is binary .
It can be observed that $1 * 2 = 1 - 2 = -1$ and $2 * 1 = 2 - 1 = 1$.
Therefore, $1 * 2 \neq 2 * 1$; where $1, 2 \in \mathbf{Z}$
Hence, the operation * is not commutative.
Also we have:
$(1 * 2) * 3 = (1 - 2) * 3 = -1 * 3 = -1 - 3 = -4$
$1 * (2 * 3) = 1 * (2 - 3) = 1 * -1 = 1 - (-1) = 2$
Therefore, $(1 * 2) * 3 \neq 1 * (2 * 3)$; where $1, 2, 3 \in \mathbf{Z}$
Hence, the operation * is not associative.

(ii) On $\mathbf{Q}$, * is defined by $a * b = ab + 1$.
ab+1 ∈ Q, so operation * is binary .
It is known that:
$ab = ba$ for $a, b \in \mathbf{Q}$
Therefore, $ab + 1 = ba + 1$ for $a, b \in \mathbf{Q}$
Therefore, $a * b = a * b$ for $a, b \in \mathbf{Q}$
Therefore, the operation * is commutative.
It can be observed that:
$(1 * 2) * 3 = (1 \times 2 + 1) * 3 = 3 * 3 = 3 \times 3 + 1 = 10$
$1 * (2 * 3) = 1 * (2 \times 3 + 1) = 1 * 7 = 1 \times 7 + 1 = 8$
Therefore, $(1 * 2) * 3 \neq 1 * (2 * 3)$; where $1, 2, 3 \in \mathbf{Q}$
Therefore, the operation * is not associative.

(iii) On **Q**, * is defined by $a * b = \frac{ab}{2}$.

ab/2 ∈ Q, so the operation * is binary.
It is known that:
ab = *ba* ,   *a*, *b* ∈ **Q**

Therefore, $\frac{ab}{2} = \frac{ba}{2}$ for *a*, *b* ∈ **Q**

Therefore, *a* * *b* = *b* * *a* for *a*, *b* ∈ **Q**
Therefore, the operation * is commutative.
For all *a*, *b*, *c* ∈ **Q**, we have:

$$(a*b)*c = \left(\frac{ab}{2}\right)*c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$$

$$a*(b*c) = a*\left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

Therefore,(a * b) * c = a * (b * c)
Therefore, the operation * is associative.

(iv) On $\mathbf{Z}^+$, * is defined by *a* * *b* = 2^{ab}.
2^{ab} ∈ Z^+ *so the operation * is binary.*
It is known that:
ab = *ba* for *a*, *b* ∈ $\mathbf{Z}^+$
Therefore, $2^{ab} = 2^{ba}$ for *a*, *b* ∈ $\mathbf{Z}^+$
Therefore, *a* * *b* = *b* * *a* for *a*, *b* ∈ $\mathbf{Z}^+$
Therefore, the operation * is commutative.
It can be observed that:

$$(1*2)*3 = 2^{(1\times 2)}*3 = 4*3 = 2^{4\times 3} = 2^{12}$$

$$1*(2*3) = 1*2^{2\times 3} = 1*2^{6} = 1*64 = 2^{64}$$

Therefore, (1 * 2) * 3 ≠ 1 * (2 * 3) ; where 1, 2, 3 ∈ $\mathbf{Z}^+$
Therefore, the operation * is not associative.

(v) On $\mathbf{Z}^+$, * is defined by $a * b = a^b$.
$a^b \in Z^+$, so the operation * is binary.
It can be observed that:
$1*2=1^2=1$ and $2*1=2^1=2$
Therefore, 1 * 2 ≠ 2 * 1 ; where 1, 2 ∈ $\mathbf{Z}^+$
Therefore, the operation * is not commutative.
It can also be observed that:

$$(2*3)*4=2^3*4=8*4=8^4=\left(2^3\right)^4=2^{12}$$

$$2*(3*4)=2*3^4=2*81=2^{81}$$

Therefore, (2 * 3) * 4 ≠ 2 * (3 * 4) ; where 2, 3, 4 ∈ $\mathbf{Z}^+$
Therefore, the operation * is not associative.

(vi) On $\mathbf{R} - \{-1\}$, * is defined by $a*b=\dfrac{a}{b+1}$.

$\dfrac{a}{b+1} \in R$ for $b \neq -1$, so the operation * is binary.

It can be observed that $1*2=\dfrac{1}{2+1}=\dfrac{1}{3}$ and $2*1=\dfrac{2}{1+1}=\dfrac{2}{2}=1$.

Therefore, 1 * 2 ≠ 2 * 1 ; where 1, 2 ∈ $\mathbf{R} - \{-1\}$
Therefore, the operation * is not commutative.
It can also be observed that:

$$(1*2)*3=\frac{1}{3}*3=\frac{\frac{1}{3}}{3+1}=\frac{1}{12}$$

$$1*(2*3)=1*\frac{2}{3+1}=1*\frac{2}{4}=1*\frac{1}{2}=\frac{1}{\frac{1}{2}+1}=\frac{1}{\frac{3}{2}}=\frac{2}{3}$$

Therefore, (1 * 2) * 3 ≠ 1 * (2 * 3) ; where 1, 2, 3 ∈ $\mathbf{R} - \{-1\}$
Therefore, the operation * is not associative.

Solution 3

The binary operation ^ on the set {1, 2, 3, 4, 5} is defined as a ^ b = min {a, b} for $a, b \in \{1, 2, 3, 4, 5\}$.
Thus, the operation table for the given operation ^ can be given as:

^	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Solution 4

(i) $(2 * 3) * 4 = 1 * 4 = 1$

$2 * (3 * 4) = 2 * 1 = 1$

(ii) For every $a, b \in \{1, 2, 3, 4, 5\}$, we have $a * b = b * a$. Therefore, the operation * is commutative.

(iii) $(2 * 3) = 1$ and $(4 * 5) = 1$

Therefore, $(2 * 3) * (4 * 5) = 1 * 1 = 1$

Solution 5

The binary operation *′ on the set {1, 2, 3 4, 5} is defined as $a *' b$ = H.C.F of a and b.

The operation table for the operation *′ can be given as:

*′	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

We observe that the operation tables for the operations * and *′ are the same.

Thus, the operation *′ is same as the operation*.

Solution 6

The binary operation * on **N** is defined as $a * b$ = L.C.M. of a and b.

(i) $5 * 7$ = L.C.M. of 5 and 7 = 35

$20 * 16$ = L.C.M of 20 and 16 = 80

(ii) It is known that:

L.C.M of a and b = L.C.M of b and a $a, b \in$ **N**.

Therefore, $a * b = b * a$

Thus, the operation * is commutative.

(iii) For $a, b, c \in$ **N**, we have:

$(a * b) * c$ = (L.C.M of a and b) $* c$ = LCM of a, b, and c

$a * (b * c) = a *$ (LCM of b and c) = L.C.M of a, b, and c

Therefore, $(a * b) * c = a * (b * c)$

Thus, the operation * is associative.

(iv) It is known that:

L.C.M. of a and 1 = a = L.C.M. 1 and a, $\forall$ $a \in$ **N**

$\Rightarrow a * 1 = a = 1 * a$, $\forall$ $a \in$ **N**

Thus, 1 is the identity of * in **N**.

(v) An element a in **N** is invertible with respect to the operation * if there exists an element b in **N**, such that $a * b = e = b * a$.

Here, $e = 1$

This means that:

L.C.M of a and b = 1 = L.C.M of b and a

This case is possible only when a and b are equal to 1.

Thus, 1 is the only invertible element of **N** with respect to the operation *.

Solution 7

The operation * on the set A = {1, 2, 3, 4, 5} is defined as

$a * b$ = L.C.M. of a and b.

2*3 = L.C.M of 2 and 3 = 6. But 6 does not belong to the given set.

Hence, the given operation * is not a binary operation.

Solution 8

The binary operation * on **N** is defined as:

$a * b$ = H.C.F. of a and b

It is known that:

H.C.F. of a and b = H.C.F. of b and a, $\forall$ $a, b \in$ **N**.

Therefore, $a * b = b * a$

Thus, the operation * is commutative.

For $a, b, c \in$ **N**, we have:

$(a * b) * c$ = (H.C.F. of a and b) * c = H.C.F. of a, b, and c

$a * (b * c) = a$ *(H.C.F. of b and c) = H.C.F. of a, b, and c

Therefore, $(a * b) * c = a * (b * c)$

Thus, the operation * is associative.

Now, an element $e \in$ **N** will be the identity for the operation * if $a * e = a = e * a$, $\forall$ $a \in$ **N**.

But this relation is not true for any $a \in$ **N**.

Thus, the operation * does not have any identity in **N**.

Solution 9

(i) On **Q**, the operation * is defined as $a * b = a - b$.

It can be observed that: for 2,3,4 $\in$ Q

2*3 =2-3 =-1 and 3*2=3-2=1.

2*3 ≠ 3*2

Thus, the operation * is not commutative.

It can also be observed that:

(2*3)*4 = (-1)*4 = -1-4=-5 and 2*(3*4) = 2*(-1) = 2-(-1) =3.

(2*3)*4 ≠ 2*(3*4)

Thus, the operation * is not associative.

(ii) On **Q**, the operation * is defined as $a * b = a^2 + b^2$.

For a, $b \in \mathbf{Q}$, we have:

$$a*b = a^2 + b^2 = b^2 + a^2 = b*a$$

Therefore, $a * b = b * a$

Thus, the operation * is commutative.

It can be observed that:

$$(1*2)*3 = (1^2+2^2)*3 = (1+4)*4 = 5*4 = 5^2+4^2 = 41$$

$$1*(2*3) = 1*(2^2+3^2) = 1*(4+9) = 1*13 = 1^2+13^2 = 170$$

$$\therefore (1*2)*3 \neq 1*(2*3) \text{ ; where } 1, 2, 3 \in \mathbf{Q}$$

Thus, the operation * is not associative.

(iii) On $\mathbf{Q}$, the operation * is defined as $a * b = a + ab$.

It can be observed that:

$$1*2=1+1\times 2=1+2=3$$

$$2*1=2+2\times 1=2+2=4$$

$$\therefore 1*2\neq 2*1 \text{ ; where } 1, 2 \in \mathbf{Q}$$

Thus, the operation * is not commutative.

It can also be observed that:

$$(1*2)*3=(1+1\times 2)*3=3*3=3+3\times 3=3+9=12$$

$$1*(2*3)=1*(2+2\times 3)=1*8=1+1\times 8=9$$

$$\therefore (1*2)*3\neq 1*(2*3) \text{ ; where } 1, 2, 3 \in \mathbf{Q}$$

Thus, the operation * is not associative.

(iv) On $\mathbf{Q}$, the operation * is defined by $a * b = (a - b)^2$.

For $a, b \in \mathbf{Q}$, we have:

$a * b = (a - b)^2$

$b * a = (b - a)^2 = [-(a - b)]^2 = (a - b)^2$

Therefore, $a * b = b * a$

Thus, the operation * is commutative.

It can be observed that:

$$(1*2)*3=(1-2)^2*3=(-1)^2*3=1*3=(1-3)^2=(-2)^2=4$$

$$1*(2*3)=1*(2-3)^2=1*(-1)^2=1*1=(1-1)^2=0$$

$$\therefore (1*2)*3\neq 1*(2*3) \text{ ; where } 1, 2, 3 \in \mathbf{Q}$$

Thus, the operation * is not associative.

(v) On $\mathbf{Q}$, the operation * is defined as $a * b = \frac{ab}{4}$.

For a, $b \in \mathbf{Q}$, we have:

$$a * b = \frac{ab}{4} = \frac{ba}{4} = b * a$$

Therefore, $a * b = b * a$

Thus, the operation * is commutative.

For a, b, $c \in \mathbf{Q}$, we have:

$$(a * b) * c = \frac{ab}{4} * c = \frac{\frac{ab}{4} \cdot c}{4} = \frac{abc}{16}$$

$$a * (b * c) = a * \frac{bc}{4} = \frac{a \cdot \frac{bc}{4}}{4} = \frac{abc}{16}$$

Therefore, $(a * b) * c = a * (b * c)$

Thus, the operation * is associative.

(vi) On $\mathbf{Q}$, the operation * is defined as $a * b = ab^2$

It can be observed that for 2,3 ∈ Q,:

$2*3 = 2.3^2 = 18$ and $3*2 = 3.2^2 = 12$

Hence $2*3 \neq 3*2$

$$\frac{1}{2} * \frac{1}{3} = \frac{1}{2} \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{2} \cdot \frac{1}{9} = \frac{1}{18}$$

$$\frac{1}{3} * \frac{1}{2} = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

$$\therefore \frac{1}{2} * \frac{1}{3} \neq \frac{1}{3} * \frac{1}{2} \text{ ; where } \frac{1}{2}, \frac{1}{3} \in \mathbf{Q}$$

Thus, the operation * is not commutative.

It can also be observed that for 1,2,3 ∈ Q:

$(1*2)*3 = (1.2^2)*3 = 4*3 = 4.3^2 = 36$

$1*(2*3) = 1*(2.3^2) = 1*18 = 1.18^2 = 324$

$(1*2)*3 \neq 1*(2*3)$

$$\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}=\left[\frac{1}{2}\cdot\left(\frac{1}{3}\right)^2\right]*\frac{1}{4}=\frac{1}{18}*\frac{1}{4}=\frac{1}{18}\cdot\left(\frac{1}{4}\right)^2=\frac{1}{18\times16}$$

$$\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)=\frac{1}{2}*\left[\frac{1}{3}\cdot\left(\frac{1}{4}\right)^2\right]=\frac{1}{2}*\frac{1}{48}=\frac{1}{2}\cdot\left(\frac{1}{48}\right)^2=\frac{1}{2\times48^2}$$

$$\therefore\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}\neq\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)\text{ ; where }\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}\in\mathbf{Q}$$

Thus, the operation * is not associative.

Hence, the operations defined in (ii), (iv), (v) are commutative and the operation defined in (v) is associative.

Solution 10

An element $e \in \mathbf{Q}$ will be the identity element for the operation * if $a * e = a = e * a,\ \forall a \in \mathbf{Q}$.

(i)a*b = a − b

This operation is not commutative, Hence it does not have identity element.

(ii)a*b = $a^2 + b^2$

If a*e = a, then $a^2+e^2=a$. For a =-2, $(-2)^4+ e^2=4 +e^2\neq -2$.

Hence there is no identity element.

(iii)a*b=a +ab

This operation is not commutative, Hence it does not have identity element.

(iv)a*b=$(a-b)^2$

If a*e =a, then $(a-e)^2 = a$. A square is always positive, so for a = -2, $(-2-e)^2 \neq -2$.

Hence there is no identity element.

(v)a*b =ab/4

If a*e = a, then ae/4=a . hence e =4 is the identity element.

a*4 = 4*a = 4a/4=a.

(vi)a*b=ab^2

This operation is not commutative, Hence it does not have identity element.

Therefore only (v) has an identity element.

Solution 11

A = **N** × **N**

* is a binary operation on A and is defined by:

(*a*, *b*) * (*c*, *d*) = (*a* + *c*, *b* + *d*)

Let (*a*, *b*), (*c*, *d*) ∈ A

Then, *a*, *b*, *c*, *d* ∈ **N**

We have:

(*a*, *b*) * (*c*, *d*) = (*a* + *c*, *b* + *d*)

(*c*, *d*) * (*a*, *b*) = (*c* + *a*, *d* + *b*) = (*a* + *c*, *b* + *d*)

[Addition is commutative in the set of natural numbers]

Therefore, (*a*, *b*) * (*c*, *d*) = (*c*, *d*) * (*a*, *b*)

Therefore, the operation * is commutative.

Now, let (*a*, *b*), (*c*, *d*), (*e*, *f*) ∈A

Then, *a*, *b*, *c*, *d*, *e*, *f* ∈ **N**

We have:

$$\left((a,b)*(c,d)\right)*(e,f)=(a+c,\ b+d)*(e,f)=(a+c+e,\ b+d+f)$$

$$(a,b)*\left((c,d)*(e,f)\right)=(a,b)*(c+e,\ d+f)=(a+c+e,\ b+d+f)$$

$$\therefore \left((a,b)*(c,d)\right)*(e,f)=(a,b)*\left((c,d)*(e,f)\right)$$

Therefore, the operation * is associative.

An element $e=(e_1,e_2)\in \mathrm{A}$ will be an identity element for the operation * if $a*e=a=e*a\ \forall\ a=(a_1,a_2)\in \mathrm{A}$, i.e., $(a_1+e_1,a_2+e_2)=(a_1,a_2)=(e_1+a_1,e_2+a_2)$, which is not true for any element in A.

Note that a+e = a for e = 0, but 0 does not belong to N.

Therefore, the operation * does not have any identity element.

Solution 12

(i) Define an operation * on **N** as:

a * *b* = *a* + *b* ∀ *a*, *b* ∈ **N**

Then, in particular, for *b* = *a* = 3, we have:

3 * 3 = 3 + 3 = 6 ≠ 3

Therefore, statement (i) is false.

(ii) R.H.S. = (*c* * *b*) * *a*

= (*b* * *c*) * *a* [* is commutative]

= *a* * (*b* * *c*) [Again, as * is commutative]

= L.H.S.

Therefore, *a* * (*b* * *c*) = (*c* * *b*) * *a*

Therefore, statement (ii) is true.

Solution 13

On **N**, the operation * is defined as $a * b = a^3 + b^3$.

For, a, b, $\in$ **N**, we have:

$a * b = a^3 + b^3 = b^3 + a^3 = b * a$ [Addition is commutative in **N**]

Therefore, the operation * is commutative.

It can be observed that:

$$(1*2)*3=(1^3+2^3)*3=9*3=9^3+3^3=729+27=756$$

$$1*(2*3)=1*(2^3+3^3)=1*(8+27)=1\times 35=1^3+35^3=1+(35)^3=1+42875=42876$$

Therefore, $(1 * 2) * 3 \neq 1 * (2 * 3)$; where $1, 2, 3 \in$ **N**

Therefore, the operation * is not associative.

Hence, the operation * is commutative, but not associative.

Thus, the correct answer is B.

Concept Insight:

1.Binary operation is the operation of two elements of a set A such that the resulting element is also in the set A. For example, 'addition' of two Real numbers. The sum of two real numbers is again a real number. Thus we can define various operations on finite or infinite sets. They are binary if the resulting element also belongs to the same set. Note all operations are not binary.

2.The operations maybe commutative or associative. To prove commutative or associative, prove it for any element a,b,c in the set. But to disprove, show that it is not true for any two elements in the set. For example 'subtraction' in Real numbers, $2-3 \neq 3-2$ so 'subtraction' is not commutative.

3. In Real numbers we have, $a+0 = a = 0+a$ and $a.1= a =1.a$, so '0' is the additive identity and '1' is the multiplicative identity.

Thus for any operation *, if $a*e=a=e*a$, then 'e' is the identity element.

Chapter 1 - Relations and Functions Exercise Misc. Ex.

Solution 1

It is given that f: **R** → **R** is defined as $f(x) = 10x + 7$.

One-one:

Let $f(x) = f(y)$, where x, $y \in$ **R**.

$\Rightarrow 10x + 7 = 10y + 7$

$\Rightarrow x = y$

Therefore, f is a one-one function.

Onto:

For $y \in$ **R**, let $y = 10x + 7$.

$$\Rightarrow x = \frac{y-7}{10} \in \mathbf{R}$$

Therefore, for any $y \in$ **R**, there exists $x = \frac{y-7}{10} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y.$$

Therefore, f is onto.

Therefore, f is one-one and onto.

Thus, f is an invertible function.

Let us define g: **R** → **R** as $g(y) = \frac{y-7}{10}$.

Now, we have:

$$gof(x) = g(f(x)) = g(10x+7) = \frac{(10x+7)-7}{10} = \frac{10x}{10} = x$$

And,

$$fog(y) = f(g(y)) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$\therefore gof = I_{\mathbf{R}}$ and $fog = I_{\mathbf{R}}$

Hence, the required function g: **R** → **R** is defined as $g(y) = \frac{y-7}{10}$.

Solution 2

It is given that:

f: W → W is defined as $f(n)=\begin{cases} n-1, \text{ if } n \text{ is odd} \\ n+1, \text{ if } n \text{ is even} \end{cases}$

One-one:

Let $f(n) = f(m)$.

It can be observed that if n is odd and m is even, then we will have $n - 1 = m + 1$.

$\Rightarrow n - m = 2$

However, this is impossible.

Similarly, the possibility of n being even and m being odd can also be ignored under a similar argument.

Therefore, both n and m must be either odd or even.

Now, if both n and m are odd, then we have:

$f(n) = f(m) \Rightarrow n - 1 = m - 1 \Rightarrow n = m$

Again, if both n and m are even, then we have:

$f(n) = f(m) \Rightarrow n + 1 = m + 1 \Rightarrow n = m$

Therefore, f is one-one.

It is clear that any odd number $2r + 1$ in co-domain **N** is the image of $2r$ in domain **N** and any even number $2r$ in co-domain **N** is the image of $2r + 1$ in domain **N**.

Therefore, f is onto.

Hence, f is an invertible function.

Let us define g: W → W as:

$$g(m)=\begin{cases} m+1, \text{ if } m \text{ is even} \\ m-1, \text{ if } m \text{ is odd} \end{cases}$$

Now, when n is odd:

$$gof(n)=g(f(n))=g(n-1)=n-1+1=n$$

Note when n is odd, n-1 is even.

And, when n is even:

$$gof(n)=g(f(n))=g(n+1)=n+1-1=n$$

[when n is even, n+1 is odd.

Similarly, when m is odd:

$$fog(m)=f(g(m))=f(m-1)=m-1+1=m$$

When m is even:

$$fog(m)=f(g(m))=f(m+1)=m+1-1=m$$

$$\therefore gof=\mathrm{I_W} \text{ and } fog=\mathrm{I_W}$$

Thus, f is invertible and the inverse of f is given by $f^{-1} = g$, which is the same as f.

Hence, the inverse of f is f itself.

Solution 3

It is given that f: $\mathbf{R} \to \mathbf{R}$ is defined as $f(x) = x^2 - 3x + 2$.

$$f\left(f\left(x\right)\right) = f\left(x^2 - 3x + 2\right)$$
$$= \left(x^2 - 3x + 2\right)^2 - 3\left(x^2 - 3x + 2\right) + 2$$
$$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$$
$$= x^4 - 6x^3 + 10x^2 - 3x$$

Solution 4

It is given that f: $\mathbf{R} \to \{x \in \mathbf{R}: -1 < x < 1\}$ is defined as $f(x) = \frac{x}{1+|x|}$, $x \in\mathbf{R}$

Suppose $f(x) = f(y)$, where $x, y \in \mathbf{R}$.

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

It can be observed that if x is positive and y is negative, then we have:

$$\frac{x}{1+x} = \frac{y}{1-y} \Rightarrow 2xy = x - y$$

Since x is positive and y is negative:

$x > y \Rightarrow x - y > 0$

But, $2xy$ is negative.

Then, $2xy \neq x - y$.

Thus, the case of x being positive and y being negative can be ruled out.

Under a similar argument, x being negative and y being positive can also be ruled out

Therefore, x and y have to be either positive or negative.

When x and y are both positive, we have:

$$f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$$

When x and y are both negative, we have:

$$f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x - xy = y - yx \Rightarrow x = y$$

Therefore, f is one-one.

Now, let $y \in \mathbf{R}$ such that $-1 < y < 1$.

If y is negative, then there exists $x = \frac{y}{1+y} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y.$$

If y is positive, then there exists $x = \frac{y}{1-y} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{1+\left|\left(\frac{y}{1-y}\right)\right|} = \frac{\frac{y}{1-y}}{1+\frac{y}{1-y}} = \frac{y}{1-y+y} = y.$$

Therefore, f is onto.

Hence, f is one-one and onto.

Concept Insight:

1.Every function is not invertible. If f is a one- one and onto function, then it is invertible. So to check whether a function is invertible, determine whether the function is both one-one and onto.

2. Now let f(x) = y, and solve the equation for x . This will give us the formula for f^{-1}. For example, f(x) = (2x+3)/7. Let y = (2x+3)/7, hence x = (7y- 3)/2. So $f^{-1}(x) = (7x-3)/2$

3. verify that $fof^{-1} = f^{-1} o f = I$. i.e $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$.

Solution 5

f: **R** → **R** is given as $f(x) = x^3$.

Suppose $f(x) = f(y)$, where $x, y \in$ **R**.

$\Rightarrow x^3 = y^3$ ... (1)

Now, we need to show that $x = y$.

Suppose $x \neq y$, their cubes will also not be equal.

$x^3 \neq y^3$

However, this will be a contradiction to (1).

Therefore, $x = y$

Hence, f is injective.

Concept Insight:

One – one function can be proved in 2 ways.

(i) $f(x_1) = f(x_2) \rightarrow x_1 = x_2$ or (ii) $x_1 \neq x_2 \rightarrow f(x_2) \neq f(x_2)$.

Here the second method is useful i.e $x \neq y$, hence $x^3 \neq y^3$

Solution 6

Define f: **N** → **Z** as $f(x) = x$ and g: **Z** → **Z** as $g(x) = |x|$.

We first show that g is not injective.

It can be observed that:

$g(-1) = |-1| = 1$

$g(1) = |1| = 1$

Therefore, $g(-1) = g(1)$, but $-1 \neq 1$.

Therefore, g is not injective.

Now, gof: **N** → **Z** is defined as $gof(x) = g(f(x)) = g(x) = |x|$.

Let $x, y \in$ **N** such that $gof(x) = gof(y)$.

$\Rightarrow |x| = |y|$

Since x and $y \in$ **N,** both are positive.

$\therefore |x| = |y| \Rightarrow x = y$

Hence, gof is injective

Solution 7

Define f: **N** → **N** by, $f(x) = x + 1$

And, g: **N** → **N** by,

$$g(x)=\begin{cases} x-1 \text{ if } x>1 \\ 1 \quad\ \ \text{ if } x=1 \end{cases}$$

We first show that f *is* not onto.

For this, consider element 1 in co-domain **N**. It is clear that this element is not an image of any of the elements in domain **N**.

Therefore, f is not onto.

Now, gof: **N** → **N** is defined by,

$$\begin{aligned} gof(x) = g(f(x)) = g(x+1) = (x+1)-1 \qquad & [x \in \mathbf{N} \Rightarrow (x+1) > 1] \\ = x \qquad \qquad \qquad & \end{aligned}$$

Then, it is clear that for $y \in$ **N**, there exists $x = y \in$ **N** such that $gof(x) = y$.

Hence, gof is onto.

Solution 8

Since every set is a subset of itself, ARA for all $A \in P(X)$.

Therefore, R is reflexive.

Let $ARB \Rightarrow A \subset B$.

This cannot be implied to $B \subset A$.

For instance, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it cannot be implied that B is related to A.

Therefore, R is not symmetric.

Further, if ARB and BRC, then $A \subset B$ and $B \subset C$.

$\Rightarrow A \subset C$

$\Rightarrow ARC$

Therefore, R is transitive.

Hence, R is not an equivalence relation since it is not symmetric.

Solution 9

It is given that

$* : P(X) \times P(X) \to P(X)$ is defined as $A * B = A \cap B \ \forall \ A, B \in P(X)$.

We know that .

$$\Rightarrow A * X = A = X * A \ \forall \ A \in P(X)$$

Thus, X is the identity element for the given binary operation $*$.

Now, an element $A \in P(X)$ is invertible if there exists $B \in P(X)$ such that

$A * B = X = B * A$. $\qquad$ (As X is the identity element)

i.e.,

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

Thus, X is the only invertible element in $P(X)$ with respect to the given operation$*$.

Hence, the given result is proved.

Solution 10

Onto functions from the set $\{1, 2, 3, \ldots, n\}$ to itself is simply a permutation on n symbols $1, 2, \ldots, n$.

Thus, the total number of onto maps from $\{1, 2, \ldots, n\}$ to itself is the same as the total number of permutations on n symbols $1, 2, \ldots, n$, which is $n!$.

Solution 11

$S = \{a, b, c\}$, $T = \{1, 2, 3\}$

(i) F: $S \to T$ is defined as:

$F = \{(a, 3), (b, 2), (c, 1)\}$

$\Rightarrow F(a) = 3$, $F(b) = 2$, $F(c) = 1$

Therefore, $F^{-1}: T \to S$ is given by

$F^{-1} = \{(3, a), (2, b), (1, c)\}$.

(ii) F: $S \to T$ is defined as:

$F = \{(a, 2), (b, 1), (c, 1)\}$

Since $F(b) = F(c) = 1$, F is not one-one.

Hence, F is not invertible i.e., F^{-1} does not exist.

<u>Concept insight:</u>

If $f: A \to B$ is a one –one and onto function, then $f^{-1}: B \to A$ is the inverse function. For finite sets A and B, if (a, b) is in f then (b, a) is in f^{-1}.

Solution 12

It is given that *: **R** × **R** → **R** and o: **R** × **R** → **R** is defined as $a*b=|a-b|$ and *a* o *b* = *a*, ∀ *a*, *b* ∈ **R**.

For *a*, *b* ∈ **R**, we have:

$a*b=|a-b|$

$b*a=|b-a|=|-(a-b)|=|a-b|$

Therefore, *a* * *b* = *b* * *a*

Therefore, the operation * is commutative.

It can be observed that,

$(1*2)*3=(|1-2|)*3=1*3=|1-3|=2$

$1*(2*3)=1*(|2-3|)=1*1=|1-1|=0$

$\therefore (1*2)*3\neq 1*(2*3)$ (where $1, 2, 3\in \mathbf{R}$)

Therefore, the operation * is not associative.

Now, consider the operation o:

It can be observed that 1 o 2 = 1 and 2 o 1 = 2.

Therefore, 1 o 2 ≠ 2 o 1 (where 1, 2 ∈ **R**)

Therefore, the operation o is not commutative.

Let *a*, *b*, *c* ∈ **R**. Then, we have:

(*a* o *b*) o *c* = *a* o *c* = *a*

a o (*b* o *c*) = *a* o *b* = *a*

⇒ (*a* o *b*) o c = *a* o (*b* o *c*)

Therefore, the operation o is associative.

Now, let *a*, *b*, *c* ∈ **R**, then we have:

a * (*b* o *c*) = a * *b* = |a - b|

(*a* * *b*) o (*a* * *c*) = $(|a-b|)\text{o}(|a-c|)=|a-b|$

Hence, *a* * (*b* o *c*) = (*a* * *b*) o (*a* * *c*).

Now,

1 o (2 * 3) = $1\text{ o}(|2-3|)=1\text{ o }1=1$

(1 o 2) * (1 o 3) = 1 * 1 = $|1-1|=0$

Therefore, 1 o (2 * 3) ≠ (1 o 2) * (1 o 3) (where 1, 2, 3 ∈ **R**)

Therefore, the operation o does not distribute over *.

Solution 13

It is given that $*: P(X) \times P(X) \to P(X)$ is defined as

$A * B = (A - B) \cup (B - A)$ $A, B \in P(X)$.

Let $A \in P(X)$. Then, we have:

$A * \Phi = (A - \Phi) \cup (\Phi - A) = A \cup \Phi = A$

$\Phi * A = (\Phi - A) \cup (A - \Phi) = \Phi \cup A = A$

Therefore, $A * \Phi = A = \Phi * A$, $A \in P(X)$

Thus, Φ is the identity element for the given operation*.

Now, an element $A \in P(X)$ will be invertible if there exists $B \in P(X)$ such that

$A * B = \Phi = B * A$. (As Φ is the identity element)

Now, we observed that $A * A = (A - A) \cup (A - A) = \phi \cup \phi = \phi \ \forall A \in P(X)$.

Hence, all the elements A of $P(X)$ are invertible with $A^{-1} = A$.

Solution 14

Let $X = \{0, 1, 2, 3, 4, 5\}$.

The operation * on X is defined as:

$$a * b = \begin{cases} a+b & \text{if } a+b<6 \\ a+b-6 & \text{if } a+b \geq 6 \end{cases}$$

An element $e \in X$ is the identity element for the operation *, if

$a * e = a = e * a \ \forall a \in X$.

For $a \in X$, we observed that:

$$a * 0 = a + 0 = a \qquad [a \in X \Rightarrow a + 0 < 6]$$

$$0 * a = 0 + a = a \qquad [a \in X \Rightarrow 0 + a < 6]$$

$$\therefore a * 0 = a = 0 * a \ \forall a \in X$$

Thus, 0 is the identity element for the given operation *.

An element $a \in X$ is invertible if there exists $b \in X$ such that $a * b = 0 = b * a$.

$$\text{i.e.,} \begin{cases} a+b=0=b+a, & \text{if } a+b<6 \\ a+b-6=0=b+a-6, & \text{if } a+b \geq 6 \end{cases}$$

i.e.,

$a = -b$ or $b = 6 - a$

But, $X = \{0, 1, 2, 3, 4, 5\}$ and $a, b \in X$. Then, $a \neq -b$.

Therefore, $b = 6 - a$ is the inverse of a $a \in X$.

Hence, the inverse of an element $a \in X$, $a \neq 0$ is $6 - a$ i.e., $a^{-1} = 6 - a$.

Solution 15

It is given that $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$.

Also, it is given that $f, g: A \rightarrow B$ are defined by $f(x) = x^2 - x$, $x \in A$ and

$$g(x) = 2\left|x - \frac{1}{2}\right| - 1,\ x \in A.$$

It is observed that:

$$f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$$

$$g(-1) = 2\left|(-1) - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(-1) = g(-1)$$

$$f(0) = (0)^2 - 0 = 0$$

$$g(0) = 2\left|0 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(0) = g(0)$$

$$f(1) = (1)^2 - 1 = 1 - 1 = 0$$

$$g(1) = 2\left|1 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(1) = g(1)$$

$$f(2) = (2)^2 - 2 = 4 - 2 = 2$$

$$g(2) = 2\left|2 - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(2) = g(2)$$

$$\therefore f(a) = g(a)\ \forall a \in A$$

Hence, the functions f and g are equal.

Solution 16

This is because relation R is reflexive as (1, 1), (2, 2), (3, 3) ∈ R.

Relation R is symmetric since (1, 2), (2, 1) ∈R and (1, 3), (3, 1) ∈R.

But relation R is not transitive as (3, 1), (1, 2) ∈ R, but (3, 2) ∉ R.

Now, if we add any one of the two pairs (3, 2) and (2, 3) (or both) to relation R, then relation R will become transitive.

Hence, the total number of desired relations is one.

The correct answer is A.

Solution 17

It is given that $A = \{1, 2, 3\}$.

An equivalence relation is reflexive, symmetric and transitive.

The smallest equivalence relation containing (1, 2) is given by,

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

Now, we are left with only four pairs i.e., (2, 3), (3, 2), (1, 3), and (3, 1).

If we add any one pair [say (2, 3)] to R_1, then for symmetry we must add (3, 2).

Also, for transitivity we are required to add (1, 3) and (3, 1).

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing (1, 2) is two.

The correct answer is B.

Solution 18

It is given that,

f: $\mathbf{R} \to \mathbf{R}$ is defined as $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

Also, g: $\mathbf{R} \to \mathbf{R}$ is defined as $g(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x.

Now, let $x \in (0, 1]$.

Then, we have:

$[x] = 1$ if $x = 1$ and $[x] = 0$ if $0 < x < 1$.

$$\therefore\ fog(x) = f(g(x)) = f([x]) = \begin{cases} f(1), & \text{if } x = 1 \\ f(0), & \text{if } x \in (0,1) \end{cases} = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in (0,1) \end{cases}$$

$$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(1) \qquad [x > 0] \\ &= [1] = 1 \end{aligned}$$

Thus, when $x \in (0, 1)$, we have $fog(x) = 0$and $gof(x) = 1$.

But fog(1) = gof(1)

Hence, fog and gof do not coincide in (0, 1].

Solution 19

A binary operation * on $\{a, b\}$ is a function from $\{a, b\} \times \{a, b\} \to \{a, b\}$

i.e., * is a function from $\{(a, a), (a, b), (b, a), (b, b)\} \to \{a, b\}$.

Hence, the total number of binary operations on the set $\{a, b\}$ is 2^4 i.e., 16.

The correct answer is B.