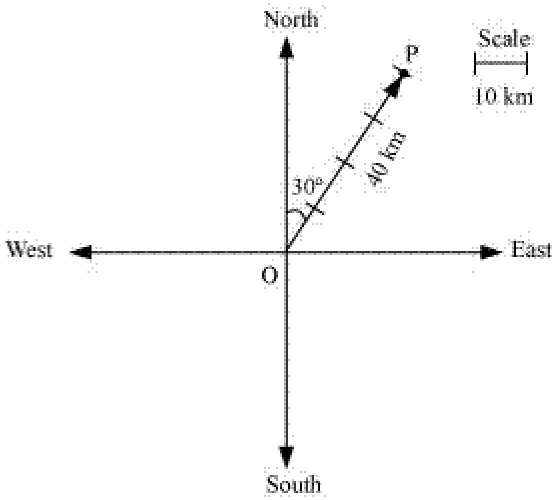
NCERT Solutions for Class 12-science Maths Chapter 10 - Vector Algebra

Chapter 10 - Vector Algebra Exercise Ex. 10.1

Solution 1



Here, vector $\overrightarrow{OP}$ represents the displacement of 40 km, 30° East of North.

Solution 2

(i) 10 kg is a scalar quantity because it involves only magnitude.

(ii) 2 meters north-west is a vector quantity as it involves both magnitude and direction.

(iii) 40° is a scalar quantity as it involves only magnitude.

(iv) 40 watts is a scalar quantity as it involves only magnitude.

(v) 10^{-19} coulomb is a scalar quantity as it involves only magnitude.

(vi) 20 m/s^2 is a vector quantity as it involves magnitude as well as direction.

Solution 3

(i) Time period is a scalar quantity as it involves only magnitude.

(ii) Distance is a scalar quantity as it involves only magnitude.

(iii) Force is a vector quantity as it involves both magnitude and direction.

(iv) Velocity is a vector quantity as it involves both magnitude as well as direction.

(v) Work done is a scalar quantity as it involves only magnitude.

Solution 4

(i) Vectors $\vec{a}$ and $\vec{d}$ are coinitial because they have the same initial point.

(ii) Vectors $\vec{b}$ and $\vec{d}$ are equal because they have the same magnitude and direction.

(iii) Vectors $\vec{a}$ and $\vec{c}$ are collinear but not equal. This is because although they are parallel, their directions are not the same.

Solution 5

(i) True.

Vectors $\vec{a}$ and $-\vec{a}$ are parallel to the same line.

(ii) False.

Collinear vectors are those vectors that are parallel to the same line.

(iii) False.

It is not necessary for two vectors having the same magnitude to be parallel to the same line.

(iv) False.

Two vectors are said to be equal if they have the same magnitude and direction, regardless of the positions of their initial points.

Chapter 10 - Vector Algebra Exercise Ex. 10.2

Solution 1

The given vectors are:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}; \quad \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \qquad \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

$$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$|\vec{b}| = \sqrt{(2)^2 + (-7)^2 + (-3)^2}$$

$$= \sqrt{4 + 49 + 9}$$

$$= \sqrt{62}$$

$$|\vec{c}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2}$$

$$= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$$

Solution 2

Consider $\vec{a} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right)$ and $\vec{b} = \left(2\hat{i} + \hat{j} - 3\hat{k}\right)$.

It can be observed that $\left|\vec{a}\right| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$ and

$\left|\vec{b}\right| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{4+1+9} = \sqrt{14}.$

Hence, $\vec{a}$ and $\vec{b}$ are two different vectors having the same magnitude. The vectors are different because they have different directions

Solution 3

Consider $\vec{p} = \left(\hat{i} + \hat{j} + \hat{k}\right)$ and $\vec{q} = \left(2\hat{i} + 2\hat{j} + 2\hat{k}\right)$.

The direction cosines of $\vec{p}$ are given by,

$$l = \frac{1}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}},\ m = \frac{1}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}},\ \text{and } n = \frac{1}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}.$$

The direction cosines of $\vec{q}$ are given by

$$l = \frac{2}{\sqrt{2^2+2^2+2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}},\ m = \frac{2}{\sqrt{2^2+2^2+2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}},$$

$$\text{and } n = \frac{2}{\sqrt{2^2+2^2+2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

The direction cosines of $\vec{p}$ and $\vec{q}$ are the same. Hence, the two vectors have the same direction.

Solution 4

The two vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ will be equal if their corresponding components are equal.

Hence, the required values of x and y are 2 and 3 respectively.

Solution 5

The vector with the initial point P (2, 1) and terminal point Q (–5, 7) can be given by,

$$\overrightarrow{PQ} = (-5-2)\hat{i} + (7-1)\hat{j}$$

$$\Rightarrow \overrightarrow{PQ} = -7\hat{i} + 6\hat{j}$$

Hence, the required scalar components are –7 and 6 while the vector components are $-7\hat{i}$ and $6\hat{j}$.

Solution 6

The given vectors are $\vec{a}=\hat{i}-2\hat{j}+\hat{k},\ \vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c}=\hat{i}-6\hat{j}-7\hat{k}$.

$$\begin{aligned}\therefore \vec{a}+\vec{b}+\vec{c} &= (1-2+1)\hat{i}+(-2+4-6)\hat{j}+(1+5-7)\hat{k}\\ &= 0\cdot\hat{i}-4\hat{j}-1\cdot\hat{k}\\ &= -4\hat{j}-\hat{k}\end{aligned}$$

Solution 7

The unit vector $\hat{a}$ in the direction of vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ is given by $\hat{a}=\frac{\vec{a}}{|a|}$.

$$|\vec{a}|=\sqrt{1^2+1^2+2^2}=\sqrt{1+1+4}=\sqrt{6}$$

$$\therefore \hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}=\frac{1}{\sqrt{6}}\hat{i}+\frac{1}{\sqrt{6}}\hat{j}+\frac{2}{\sqrt{6}}\hat{k}$$

Solution 8

The given points are P (1, 2, 3) and Q (4, 5, 6).

$$\therefore \overrightarrow{PQ}=(4-1)\hat{i}+(5-2)\hat{j}+(6-3)\hat{k}=3\hat{i}+3\hat{j}+3\hat{k}$$

$$|\overrightarrow{PQ}|=\sqrt{3^2+3^2+3^2}=\sqrt{9+9+9}=\sqrt{27}=3\sqrt{3}$$

Hence, the unit vector in the direction of $\overrightarrow{PQ}$ is

$$\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}=\frac{3\hat{i}+3\hat{j}+3\hat{k}}{3\sqrt{3}}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$$

Solution 9

The given vectors are $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}-\hat{k}$.

$$\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$$
$$\vec{b}=-\hat{i}+\hat{j}-\hat{k}$$
$$\therefore \vec{a}+\vec{b}=(2-1)\hat{i}+(-1+1)\hat{j}+(2-1)\hat{k}=1\hat{i}+0\hat{j}+1\hat{k}=\hat{i}+\hat{k}$$
$$|\vec{a}+\vec{b}|=\sqrt{1^2+1^2}=\sqrt{2}$$

Hence, the unit vector in the direction of $(\vec{a}+\vec{b})$ is

$$\frac{(\vec{a}+\vec{b})}{|\vec{a}+\vec{b}|}=\frac{\hat{i}+\hat{k}}{\sqrt{2}}=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}$$

Solution 10

Let $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$.

$\therefore |\vec{a}| = \sqrt{5^2 + (-1)^2 + 2^2} = \sqrt{25+1+4} = \sqrt{30}$

$\therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}$

Hence, the vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units is given by,

$$8\hat{a} = 8\left(\frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}\right) = \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}$$

Solution 11

Let $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$.

It is observed that $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k} = -2\left(2\hat{i} - 3\hat{j} + 4\hat{k}\right) = -2\vec{a}$

$\therefore \vec{b} = \lambda\vec{a}$

where,

$\lambda = -2$

Hence, the given vectors are collinear

Solution 12

Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$.

$\therefore |\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$

Hence, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$.

Solution 13

The given points are A (1, 2, −3) and B (−1, −2, 1).

$\therefore \overrightarrow{AB} = (-1-1)\hat{i} + (-2-2)\hat{j} + \{1-(-3)\}\hat{k}$

$\Rightarrow \overrightarrow{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$

$\therefore |\overrightarrow{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4+16+16} = \sqrt{36} = 6$

Hence, the direction cosines of $\overrightarrow{AB}$ are $\left(-\frac{2}{6}, -\frac{4}{6}, \frac{4}{6}\right) = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$.

Solution 14

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

Then,

$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Therefore, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Now, let α, β, and γ be the angles formed by $\vec{a}$ with the positive directions of x, y, and z axes.

Then, we have $\cos\alpha = \frac{1}{\sqrt{3}}, \cos\beta = \frac{1}{\sqrt{3}}, \cos\gamma = \frac{1}{\sqrt{3}}$.

Hence, the given vector is equally inclined to axes OX, OY, and OZ.

Solution 15

The position vector of point R dividing the line segment joining two points

P and Q in the ratio *m: n* is given by:

i. Internally:

$$\frac{m\vec{b} + n\vec{a}}{m+n}$$

ii. Externally:

$$\frac{m\vec{b} - n\vec{a}}{m-n}$$

Position vectors of P and Q are given as:

$\overrightarrow{OP} = \hat{i} + 2\hat{j} - \hat{k}$ and $\overrightarrow{OQ} = -\hat{i} + \hat{j} + \hat{k}$

(i) The position vector of point R which divides the line joining two points P and Q internally in the ratio 2:1 is given by,

$$\overrightarrow{OR} = \frac{2\left(-\hat{i} + \hat{j} + \hat{k}\right) + 1\left(\hat{i} + 2\hat{j} - \hat{k}\right)}{2+1} = \frac{\left(-2\hat{i} + 2\hat{j} + 2\hat{k}\right) + \left(\hat{i} + 2\hat{j} - \hat{k}\right)}{3}$$

$$= \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} = -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$$

(ii) The position vector of point R which divides the line joining two points P and Q externally in the ratio 2: 1 is given by,

$$\overrightarrow{OR} = \frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)-1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2-1} = \left(-2\hat{i}+2\hat{j}+2\hat{k}\right)-\left(\hat{i}+2\hat{j}-\hat{k}\right)$$
$$= -3\hat{i}+3\hat{k}$$

Solution 16

Find the position vector of the mid point of the vector joining the points P (2, 3, 4) and Q (4, 1, − 2).

Solution 16:

The position vector of mid-point R of the vector joining points P (2, 3, 4) and Q (4, 1, − 2) is given by,

$$\overrightarrow{OR} = \frac{\left(2\hat{i}+3\hat{j}+4\hat{k}\right)+\left(4\hat{i}+\hat{j}-2\hat{k}\right)}{2} = \frac{(2+4)\hat{i}+(3+1)\hat{j}+(4-2)\hat{k}}{2}$$
$$= \frac{6\hat{i}+4\hat{j}+2\hat{k}}{2} = 3\hat{i}+2\hat{j}+\hat{k}$$

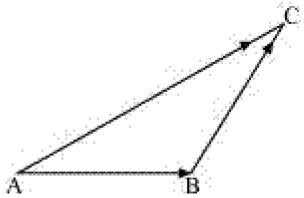
Solution 17

Position vectors of points A, B, and C are respectively given as:
$\vec{a} = 3\hat{i}-4\hat{j}-4\hat{k},\ \vec{b} = 2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}-3\hat{j}-5\hat{k}$

$$\vec{a} = 3\hat{i}-4\hat{j}-4\hat{k},\ \vec{b} = 2\hat{i}-\hat{j}+\hat{k} \text{ and } \vec{c} = \hat{i}-3\hat{j}-5\hat{k}$$
$$\therefore \overrightarrow{AB} = \vec{b}-\vec{a} = (2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k} = -\hat{i}+3\hat{j}+5\hat{k}$$
$$\overrightarrow{BC} = \vec{c}-\vec{b} = (1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k} = -\hat{i}-2\hat{j}-6\hat{k}$$
$$\overrightarrow{CA} = \vec{a}-\vec{c} = (3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k} = 2\hat{i}-\hat{j}+\hat{k}$$
$$\therefore \left|\overrightarrow{AB}\right|^2 = (-1)^2+3^2+5^2 = 1+9+25 = 35$$
$$\left|\overrightarrow{BC}\right|^2 = (-1)^2+(-2)^2+(-6)^2 = 1+4+36 = 41$$
$$\left|\overrightarrow{CA}\right|^2 = 2^2+(-1)^2+1^2 = 4+1+1 = 6$$
$$\therefore \left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{CA}\right|^2 = 35+6 = 41 = \left|\overrightarrow{BC}\right|^2$$

Hence, ABC is a right-angled triangle.

Solution 18



On applying the triangle law of addition in the given triangle, we have:

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ ...(1)

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=-\overrightarrow{CA}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$ ...(2)

$\therefore$ The equation given in alternative A is true.

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

$\therefore$ The equation given in alternative B is true.

From equation (2), we have:

$\overrightarrow{AB}-\overrightarrow{CB}+\overrightarrow{CA}=\vec{0}$

$\therefore$ The equation given in alternative D is true.

Now, consider the equation given in alternative C:

$\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{CA}=\vec{0}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{CA}$ ...(3)

From equations (1) and (3), we have:

$\overrightarrow{AC}=\overrightarrow{CA}$

$\Rightarrow \overrightarrow{AC}=-\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AC}+\overrightarrow{AC}=\vec{0}$

$\Rightarrow 2\overrightarrow{AC}=\vec{0}$

$\Rightarrow \overrightarrow{AC}=\vec{0}$, which is not true.

Hence, the equation given in alternative C is **incorrect**.

The correct answer is **C.**

Solution 19

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then they are parallel.

Therefore, we have:

$\vec{b} = \lambda\vec{a}$ (For some scalar λ)

If λ = ±1, then $\vec{a} = \pm\vec{b}$.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{b} = \lambda\vec{a}.$$

$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$$

$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

$$\Rightarrow b_1 = \lambda a_1, b_2 = \lambda a_2, b_3 = \lambda a_3$$

$$\Rightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Thus, the respective components of $\vec{a}$ and $\vec{b}$ are proportional.

However, vectors $\vec{a}$ and $\vec{b}$ can have different directions.

Hence, the statement given in **D** is **incorrect**.

The correct answer is **D**.

Chapter 10 - Vector Algebra Exercise Ex. 10.3

Solution 1

It is given that,

$$|\vec{a}| = \sqrt{3},\ |\vec{b}| = 2 \text{ and, } \vec{a}\cdot\vec{b} = \sqrt{6}$$

Now, we know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \sqrt{6} = \sqrt{3}\times 2\times\cos\theta$$
$$\Rightarrow \cos\theta = \frac{\sqrt{6}}{\sqrt{3}\times 2}$$
$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$
$$\Rightarrow \theta = \frac{\pi}{4}$$

Hence, the angle between the given vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

Solution 2

The given vectors are $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$
$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$$
$$\begin{aligned}\text{Now, } \vec{a}\cdot\vec{b} &= (\hat{i} - 2\hat{j} + 3\hat{k})(3\hat{i} - 2\hat{j} + \hat{k}) \\ &= 1.3 + (-2)(-2) + 3.1 \\ &= 3+4+3 \\ &= 10\end{aligned}$$

Also, we know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore 10 = \sqrt{14}\sqrt{14}\cos\theta$$
$$\Rightarrow \cos\theta = \frac{10}{14}$$
$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Solution 3

Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{1+1}}\{1.1+(-1)(1)\} = \frac{1}{\sqrt{2}}(1-1) = 0$$

Hence, the projection of vector $\vec{a}$ on $\vec{b}$ is 0.

Solution 4

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\hat{b} = 7\hat{i} - \hat{j} + 8\hat{k}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b}) = \frac{1}{\sqrt{7^2+(-1)^2+8^2}}\{1(7)+3(-1)+7(8)\} = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$$

Solution 5

$$\text{Let } \vec{a} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right) = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k},$$

$$\vec{b} = \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k},$$

$$\vec{c} = \frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right) = \frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}.$$

$$|\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = 1$$

$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(-\frac{3}{7}\right)^2} = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = 1$$

Thus, each of the given three vectors is a unit vector.

$$\vec{a} \cdot \vec{b} = \frac{2}{7} \times \frac{3}{7} + \frac{3}{7} \times \left(\frac{-6}{7}\right) + \frac{6}{7} \times \frac{2}{7} = \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = 0$$

$$\vec{b} \cdot \vec{c} = \frac{3}{7} \times \frac{6}{7} + \left(\frac{-6}{7}\right) \times \frac{2}{7} + \frac{2}{7} \times \left(\frac{-3}{7}\right) = \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = 0$$

$$\vec{c} \cdot \vec{a} = \frac{6}{7} \times \frac{2}{7} + \frac{2}{7} \times \frac{3}{7} + \left(\frac{-3}{7}\right) \times \frac{6}{7} = \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = 0$$

Hence, the given three vectors are mutually perpendicular to each other.

Solution 6

$$\left(\vec{a}\cdot\vec{b}\right).\left(\vec{a}-\vec{b}\right)=8$$
$$\Rightarrow \vec{a}\cdot\vec{a}-\vec{a}.\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}=8$$
$$\Rightarrow |\vec{a}|^2-\left|\vec{b}\right|^2=8$$
$$\Rightarrow \left(8\left|\vec{b}\right|\right)^2-\left|\vec{b}\right|^2=8 \qquad \left[|\vec{a}|=8\left|\vec{b}\right|\right]$$
$$\Rightarrow 64\left|\vec{b}\right|^2-\left|\vec{b}\right|^2=8$$
$$\Rightarrow 63\left|\vec{b}\right|^2=8$$
$$\Rightarrow \left|\vec{b}\right|^2=\frac{8}{63}$$
$$\Rightarrow \left|\vec{b}\right|=\sqrt{\frac{8}{63}} \qquad [\text{Magnitude of a vector is non-negative}]$$
$$\Rightarrow \left|\vec{b}\right|=\frac{2\sqrt{2}}{3\sqrt{7}}$$

$$|\vec{a}|=8\left|\vec{b}\right|=\frac{8\times 2\sqrt{2}}{3\sqrt{7}}=\frac{16\sqrt{2}}{3\sqrt{7}}$$

Solution 7

$$\left(3\vec{a}-5\vec{b}\right)\cdot\left(2\vec{a}+7\vec{b}\right)$$
$$=3\vec{a}\cdot 2\vec{a}+3\vec{a}\cdot 7\vec{b}-5\vec{b}\cdot 2\vec{a}-5\vec{b}\cdot 7\vec{b}$$
$$=6\vec{a}\cdot\vec{a}+21\vec{a}\cdot\vec{b}-10\vec{a}\cdot\vec{b}-35\vec{b}\cdot\vec{b}$$
$$=6|\vec{a}|^2+11\vec{a}\cdot\vec{b}-35\left|\vec{b}\right|^2$$

Solution 8

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

It is given that $|\vec{a}|=\left|\vec{b}\right|$, $\vec{a}\cdot\vec{b}=\frac{1}{2}$, and $\theta=60°$. ...(1)

We know that $\vec{a}\cdot\vec{b}=|\vec{a}|\left|\vec{b}\right|\cos\theta$.

$$\therefore \frac{1}{2}=|\vec{a}||\vec{a}|\cos 60° \qquad [\text{Using (1)}]$$
$$\Rightarrow \frac{1}{2}=|\vec{a}|^2\times\frac{1}{2}$$
$$\Rightarrow |\vec{a}|^2=1$$
$$\Rightarrow |\vec{a}|=\left|\vec{b}\right|=1$$

Solution 9

$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$$
$$\Rightarrow \vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}-\vec{a}\cdot\vec{a}=12$$
$$\Rightarrow |\vec{x}|^2-|\vec{a}|^2=12$$
$$\Rightarrow |\vec{x}|^2-1=12 \qquad \left[|\vec{a}|=1 \text{ as } \vec{a} \text{ is a unit vector}\right]$$
$$\Rightarrow |\vec{x}|^2=13$$
$$\therefore |\vec{x}|=\sqrt{13}$$

Solution 10

The given vectors are $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k},\ \vec{b}=-\hat{i}+2\hat{j}+\hat{k},$ and $\vec{c}=3\hat{i}+\hat{j}.$

Now,

$$\vec{a}+\lambda\vec{b}=\left(2\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-\hat{i}+2\hat{j}+\hat{k}\right)=(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}$$

If $\left(\vec{a}+\lambda\vec{b}\right)$ is perpendicular to $\vec{c}$, then

$$\left(\vec{a}+\lambda\vec{b}\right)\cdot\vec{c}=0.$$
$$\Rightarrow \left[(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}\right]\cdot\left(3\hat{i}+\hat{j}\right)=0$$
$$\Rightarrow (2-\lambda)3+(2+2\lambda)1+(3+\lambda)0=0$$
$$\Rightarrow 6-3\lambda+2+2\lambda=0$$
$$\Rightarrow -\lambda+8=0$$
$$\Rightarrow \lambda=8$$

Hence, the required value of λ is 8.

Solution 11

$$\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right)\cdot\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)$$
$$=|\vec{a}|^2\,\vec{b}\cdot\vec{b}-|\vec{a}||\vec{b}|\vec{b}\cdot\vec{a}+|\vec{b}||\vec{a}|\vec{a}\cdot\vec{b}-|\vec{b}|^2\,\vec{a}\cdot\vec{a}$$
$$=|\vec{a}|^2|\vec{b}|^2-|\vec{b}|^2|\vec{a}|^2$$
$$=0$$

Hence, $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$ are perpendicular to each other.

Solution 12

It is given that $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$.

Now,

$\vec{a}\cdot\vec{a}=0 \Rightarrow |\vec{a}|^2=0 \Rightarrow |\vec{a}|=0$

$\therefore \vec{a}$ is a zero vector.

Hence, vector $\vec{b}$ satisfying $\vec{a}\cdot\vec{b}=0$ can be any vector

Solution 13

Given: $\vec{a},\vec{b},\vec{c}$ are unit vectors

$\Rightarrow |\vec{a}|=|\vec{b}|=|\vec{c}|=1$

$\Rightarrow |\vec{a}|=|\vec{b}|=|\vec{c}|=1$

Also, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$$\Rightarrow \left(\vec{a}+\vec{b}+\vec{c}\right)^2=0$$

$$\Rightarrow \left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\left(\vec{a}+\vec{b}+\vec{c}\right)=0$$

$$\Rightarrow \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c}+2\vec{a}\cdot\vec{b}$$
$$+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}=0$$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+$$
$$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$$

$$\Rightarrow 1+1+1+2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)=0$$

$$\Rightarrow 3+2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)=0$$

$$\Rightarrow \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\frac{3}{2}$$

Solution 14

Consider $\vec{a} = 2\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 3\hat{j} - 6\hat{k}$.

Then,

$$\vec{a} \cdot \vec{b} = 2.3 + 4.3 + 3(-6) = 6 + 12 - 18 = 0$$

We now observe that:

$$|\vec{a}| = \sqrt{2^2 + 4^2 + 3^2} = \sqrt{29}$$
$$\therefore \vec{a} \neq \vec{0}$$
$$|\vec{b}| = \sqrt{3^2 + 3^2 + (-6)^2} = \sqrt{54}$$
$$\therefore \vec{b} \neq \vec{0}$$

Hence, the converse of the given statement need not be true.

Solution 15

The vertices of ΔABC are given as A (1, 2, 3), B (−1, 0, 0), and C (0, 1, 2).

Also, it is given that $\angle$ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \{1-(-1)\}\hat{i} + (2-0)\hat{j} + (3-0)\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$
$$\overrightarrow{BC} = \{0-(-1)\}\hat{i} + (1-0)\hat{j} + (2-0)\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$$
$$\therefore \overrightarrow{BA} \cdot \overrightarrow{BC} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) \cdot \left(\hat{i} + \hat{j} + 2\hat{k}\right) = 2\times1 + 2\times1 + 3\times2 = 2 + 2 + 6 = 10$$
$$\left|\overrightarrow{BA}\right| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4+4+9} = \sqrt{17}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{1+1+2^2} = \sqrt{6}$$

Now, it is known that:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\cos(\angle ABC)$$

$$\therefore 10 = \sqrt{17} \times \sqrt{6}\cos(\angle ABC)$$
$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17} \times \sqrt{6}}$$
$$\Rightarrow \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Solution 16

The given points are A (1, 2, 7), B (2, 6, 3), and C (3, 10, −1).

$$\therefore \overrightarrow{AB} = (2-1)\hat{i} + (6-2)\hat{j} + (3-7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$
$$\overrightarrow{BC} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$
$$\overrightarrow{AC} = (3-1)\hat{i} + (10-2)\hat{j} + (-1-7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k}$$
$$\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$
$$\left|\overrightarrow{AC}\right| = \sqrt{2^2 + 8^2 + 8^2} = \sqrt{4+64+64} = \sqrt{132} = 2\sqrt{33}$$
$$\therefore \left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$$

Hence, the given points A, B, and C are collinear.

Solution 17

Let vectors $2\hat{i} - \hat{j} + \hat{k},\ \hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ be position vectors of points A, B, and C respectively.

i.e., $\overrightarrow{OA} = 2\hat{i} - \hat{j} + \hat{k},\ \overrightarrow{OB} = \hat{i} - 3\hat{j} - 5\hat{k}$ and $\overrightarrow{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

Now, vectors $\overrightarrow{AB}, \overrightarrow{BC}$, and $\overrightarrow{AC}$ represent the sides of ΔABC.

i.e., $\overrightarrow{OA} = 2\hat{i} - \hat{j} + \hat{k},\ \overrightarrow{OB} = \hat{i} - 3\hat{j} - 5\hat{k}$, and $\overrightarrow{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

$$\therefore \overrightarrow{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$
$$\overrightarrow{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$
$$\overrightarrow{AC} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$
$$\left|\overrightarrow{AB}\right| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$
$$\left|\overrightarrow{AC}\right| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{1+9+25} = \sqrt{35}$$
$$\therefore \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AC}\right|^2 = 6 + 35 = 41 = \left|\overrightarrow{AB}\right|^2$$

Hence, ΔABC is a right-angled triangle.

Solution 18

Vector $\lambda\vec{a}$ is a unit vector if $\left|\lambda\vec{a}\right| = 1$.

Now,

$$\left|\lambda\vec{a}\right| = 1$$

$$\Rightarrow \left|\lambda\right|\left|\vec{a}\right| = 1$$

$$\Rightarrow \left|\vec{a}\right| = \frac{1}{\left|\lambda\right|} \qquad [\lambda \neq 0]$$

$$\Rightarrow a = \frac{1}{\left|\lambda\right|} \qquad \left[\left|\vec{a}\right| = a\right]$$

Hence, vector $\lambda\vec{a}$ is a unit vector if $a = \frac{1}{\left|\lambda\right|}$.

The correct answer is D.

Chapter 10 - Vector Algebra Exercise Ex. 10.4

Solution 1

We have,

$\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

$$= \hat{i}\left(-14 + 14\right) - \hat{j}\left(2 - 21\right) + \hat{k}\left(-2 + 21\right) = 19\hat{j} + 19\hat{k}$$

$$\therefore \left|\vec{a} \times \vec{b}\right| = \sqrt{\left(19\right)^2 + \left(19\right)^2} = \sqrt{2 \times \left(19\right)^2} = 19\sqrt{2}$$

Solution 2

We have,

$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

$$\therefore \vec{a} + \vec{b} = 4\hat{i} + 4\hat{j},\ \vec{a} - \vec{b} = 2\hat{i} + 4\hat{k}$$

$$\left(\vec{a} + \vec{b}\right) \times \left(\vec{a} - \vec{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = \hat{i}(16) - \hat{j}(16) + \hat{k}(-8) = 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$$\begin{aligned} \therefore \left|\left(\vec{a} + \vec{b}\right) \times \left(\vec{a} - \vec{b}\right)\right| &= \sqrt{16^2 + (-16)^2 + (-8)^2} \\ &= \sqrt{2^2 \times 8^2 + 2^2 \times 8^2 + 8^2} \\ &= 8\sqrt{2^2 + 2^2 + 1} = 8\sqrt{9} = 8 \times 3 = 24 \end{aligned}$$

Hence, the unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is given by the relation,

$$= \pm \frac{\left(\vec{a} + \vec{b}\right) \times \left(\vec{a} - \vec{b}\right)}{\left|\left(\vec{a} + \vec{b}\right) \times \left(\vec{a} - \vec{b}\right)\right|} = \pm \frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24}$$

$$= \pm \frac{2\hat{i} - 2\hat{j} - \hat{k}}{3} = \pm \frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{1}{3}\hat{k}$$

Solution 3

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

$\Rightarrow \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$.

Also, it is given that $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$.

Then, we have:

$$\cos\frac{\pi}{3} = \frac{a_1}{|\vec{a}|}$$
$$\Rightarrow \frac{1}{2} = a_1 \qquad \left[|\vec{a}| = 1\right]$$
$$\cos\frac{\pi}{4} = \frac{a_2}{|\vec{a}|}$$
$$\Rightarrow \frac{1}{\sqrt{2}} = a_2 \qquad \left[|\vec{a}| = 1\right]$$
$$\text{Also, } \cos\theta = \frac{a_3}{|\vec{a}|}.$$
$$\Rightarrow a_3 = \cos\theta$$

Now,

$$|a| = 1$$
$$\Rightarrow \sqrt{a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2}} = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$$

$$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$$

$$\Rightarrow \frac{3}{4} + \cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\therefore a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$$

Hence, $\theta = \frac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

Solution 4

$$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right)$$

$$= \left(\vec{a} - \vec{b}\right) \times \vec{a} + \left(\vec{a} - \vec{b}\right) \times \vec{b} \qquad \text{[By distributivity of vector product over addition]}$$

$$= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b} \qquad \text{[Again, by distributivity of vector product over addition]}$$

$$= \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{0}$$

$$= 2\vec{a} \times \vec{b}$$

Solution 5

$$\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$$

$$\Rightarrow\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 6 & 27\\ 1 & \lambda & \mu\end{vmatrix}=0\hat{i}+0\hat{j}+0\hat{k}$$

$$\Rightarrow\hat{i}\left(6\mu-27\lambda\right)-\hat{j}\left(2\mu-27\right)+\hat{k}\left(2\lambda-6\right)=0\hat{i}+0\hat{j}+0\hat{k}$$

On comparing the corresponding components, we have:

$$6\mu-27\lambda=0$$
$$2\mu-27=0$$
$$2\lambda-6=0$$

Now,

$$2\lambda-6=0\Rightarrow\lambda=3$$
$$2\mu-27=0\Rightarrow\mu=\frac{27}{2}$$

Hence, $\lambda=3$ and $\mu=\frac{27}{2}$.

Solution 6

$\vec{a}\cdot\vec{b}=0$

Then,

(i) Either $|\vec{a}|=0$ or $|\vec{b}|=0$, or $\vec{a}\perp\vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

$\vec{a}\times\vec{b}=0$

(ii) Either $|\vec{a}|=0$ or $|\vec{b}|=0$, or $\vec{a}\parallel\vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

But, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

Hence, $|\vec{a}|=0$ or $|\vec{b}|=0$.

Solution 7

We have,

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$$\left(\vec{b} + \vec{c}\right) = \left(b_1 + c_1\right)\hat{i} + \left(b_2 + c_2\right)\hat{j} + \left(b_3 + c_3\right)\hat{k}$$

$$\text{Now, } \vec{a} \times \left(\vec{b} + \vec{c}\right)\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$

$$= \hat{i}\left[a_2\left(b_3 + c_3\right) - a_3\left(b_2 + c_2\right)\right] - \hat{j}\left[a_1\left(b_3 + c_3\right) - a_3\left(b_1 + c_1\right)\right] + \hat{k}\left[a_1\left(b_2 + c_2\right) - a_2\left(b_1 + c_1\right)\right]$$
$$= \hat{i}\left[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2\right] + \hat{j}\left[-a_1b_3 - a_1c_3 + a_3b_1 + a_3c_1\right] + \hat{k}\left[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1\right] \quad \ldots(1)$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= \hat{i}\left[a_2b_3 - a_3b_2\right] + \hat{j}\left[b_1a_3 - a_1b_3\right] + \hat{k}\left[a_1b_2 - a_2b_1\right] \qquad (2)$$
$$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= \hat{i}\left[a_2c_3 - a_3c_2\right] + \hat{j}\left[a_3c_1 - a_1c_3\right] + \hat{k}\left[a_1c_2 - a_2c_1\right] \qquad (3)$$

On adding (2) and (3), we get:

$$\left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \times \vec{c}\right) = \hat{i}\left[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2\right] + \hat{j}\left[b_1a_3 + a_3c_1 - a_1b_3 - a_1c_3\right]$$
$$+ \hat{k}\left[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1\right] \qquad (4)$$

Now, from (1) and (4), we have:

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

Hence, the given result is proved.

Solution 8

Take any parallel non-zero vectors so that $\vec{a} \times \vec{b} = \vec{0}$.

Let $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{b} = 4\hat{i} + 6\hat{j} + 8\hat{k}$.

Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix} = \hat{i}(24-24) - \hat{j}(16-16) + \hat{k}(12-12) = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

It can now be observed that:

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$
$$\therefore \vec{a} \neq \vec{0}$$
$$|\vec{b}| = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116}$$
$$\therefore \vec{b} \neq \vec{0}$$

Hence, the converse of the given statement need not be true.

Solution 9

The vertices of triangle ABC are given as A (1, 1, 2), B (2, 3, 5), and C (1, 5, 5).

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of ΔABC are given as:

$$\overrightarrow{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{BC} = (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k} = -\hat{i} + 2\hat{j}$$

Area of ΔABC $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\therefore \left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{61}$$

Hence, the area of ΔABC is $\frac{\sqrt{61}}{2}$ square units.

Solution 10

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a}\times\vec{b}\right|$.

Adjacent sides are given as:

$$\vec{a}=\hat{i}-\hat{j}+3\hat{k} \text{ and } \vec{b}=2\hat{i}-7\hat{j}+\hat{k}$$

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}=\hat{i}\left(-1+21\right)-\hat{j}\left(1-6\right)+\hat{k}\left(-7+2\right)=20\hat{i}+5\hat{j}-5\hat{k}$$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{20^2+5^2+5^2}=\sqrt{400+25+25}=15\sqrt{2}$$

Hence, the area of the given parallelogram is $15\sqrt{2}$ square units.

Solution 11

It is given that $\left|\vec{a}\right|=3$ and $\left|\vec{b}\right|=\frac{\sqrt{2}}{3}$.

We know that $\vec{a}\times\vec{b}=\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta\,\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Now, $\vec{a}\times\vec{b}$ is a unit vector if $\left|\vec{a}\times\vec{b}\right|=1$.

$$\left|\vec{a}\times\vec{b}\right|=1$$

$$\Rightarrow \left|\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta\,\hat{n}\right|=1$$

$$\Rightarrow \left|\vec{a}\right|\left|\vec{b}\right|\left|\sin\theta\right|=1$$

$$\Rightarrow 3\times\frac{\sqrt{2}}{3}\times\sin\theta=1$$

$$\Rightarrow \sin\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta=\frac{\pi}{4}$$

Hence, $\vec{a}\times\vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

The correct answer is B.

Solution 12

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$$\overline{OA} = -\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k},\ \overline{OB} = \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k},\ \overline{OC} = \hat{i} - \frac{1}{2}\hat{j} + 4\hat{k},\ \overline{OD} = -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$$

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of the given rectangle are given as:

$$\overrightarrow{AB} = (1+1)\hat{i} + \left(\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = 2\hat{i}$$

$$\overrightarrow{BC} = (1-1)\hat{i} + \left(-\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = -\hat{j}$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = \hat{k}(-2) = -2\hat{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-2)^2} = 2$$

Now, it is known that the area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a} \times \vec{b}\right|$.

Hence, the area of the given rectangle is $\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = 2$ square units.

The correct answer is C.

Chapter 10 - Vector Algebra Exercise Misc. Ex.

Solution 1

If $\vec{r}$ is a unit vector in the XY-plane, then $\vec{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$.

Here, θ is the angle made by the unit vector with the positive direction of the x-axis.

Therefore, for θ = 30°:

$$\vec{r} = \cos 30°\hat{i} + \sin 30°\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

Hence, the required unit vector is $\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$.

Solution 2

The vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ can be obtained by,

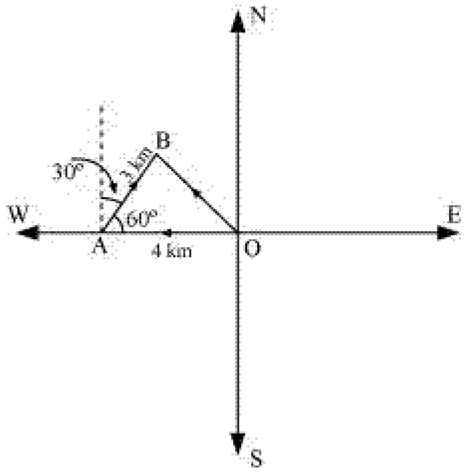
$$\overrightarrow{PQ} = \text{Position vector of Q} - \text{Position vector of P}$$
$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$
$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Hence, the scalar components and the magnitude of the vector joining the given points are respectively $\{(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)\}$ and $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Solution 3

Let O and B be the initial and final positions of the girl respectively.

Then, the girl's position can be shown as:



Now, we have:

$$\overrightarrow{OA} = -4\hat{i}$$
$$\overrightarrow{AB} = \hat{i}\left|\overrightarrow{AB}\right|\cos 60° + \hat{j}\left|\overrightarrow{AB}\right|\sin 60°$$
$$= \hat{i}3 \times \frac{1}{2} + \hat{j}3 \times \frac{\sqrt{3}}{2}$$
$$= \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

By the triangle law of vector addition, we have:

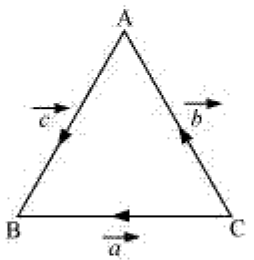
$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$
$$= \left(-4\hat{i}\right) + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}\right)$$
$$= \left(-4 + \frac{3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$
$$= \left(\frac{-8+3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$
$$= \frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

Hence, the girl's displacement from her initial point of departure is

$$\frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}.$$

Solution 4

In ΔABC, let $\overrightarrow{CB} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, and $\overrightarrow{AB} = \vec{c}$ (as shown in the following figure).



Now, by the triangle law of vector addition, we have $\vec{a} = \vec{b} + \vec{c}$.

It is clearly known that $|\vec{a}|$, $|\vec{b}|$, and $|\vec{c}|$ represent the sides of ΔABC.

Also, it is known that the sum of the lengths of any two sides of a triangle is greater than the third side.

$$\therefore |\vec{a}| < |\vec{b}| + |\vec{c}|$$

Hence, it is not true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$.

Solution 5

$x\left(\hat{i}+\hat{j}+\hat{k}\right)$ is a unit vector if $\left|x\left(\hat{i}+\hat{j}+\hat{k}\right)\right|=1$.

Now,

$$\left|x\left(\hat{i}+\hat{j}+\hat{k}\right)\right|=1$$
$$\Rightarrow \sqrt{x^2+x^2+x^2}=1$$
$$\Rightarrow \sqrt{3x^2}=1$$
$$\Rightarrow \sqrt{3}\,x=1$$
$$\Rightarrow x=\pm\frac{1}{\sqrt{3}}$$

Hence, the required value of x is $\pm\frac{1}{\sqrt{3}}$.

Solution 6

We have,

$\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$

Let $\vec{c}$ be the resultant of $\vec{a}$ and $\vec{b}$.

Then,

$$\vec{c}=\vec{a}+\vec{b}=(2+1)\hat{i}+(3-2)\hat{j}+(-1+1)\hat{k}=3\hat{i}+\hat{j}$$
$$\therefore |\vec{c}|=\sqrt{3^2+1^2}=\sqrt{9+1}=\sqrt{10}$$
$$\therefore \hat{c}=\frac{\vec{c}}{|\vec{c}|}=\frac{\left(3\hat{i}+\hat{j}\right)}{\sqrt{10}}$$

Hence, the vector of magnitude 5 units and parallel to the resultant of vectors $\vec{a}$ and $\vec{b}$ is

$$\pm 5\cdot\hat{c}=\pm 5\cdot\frac{1}{\sqrt{10}}\left(3\hat{i}+\hat{j}\right)=\pm\frac{3\sqrt{10}\hat{i}}{2}\pm\frac{\sqrt{10}}{2}\hat{j}.$$

Solution 7

We have,

$\vec{a}=\hat{i}+\hat{j}+\hat{k},\ \vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$

$$\begin{aligned}2\vec{a}-\vec{b}+3\vec{c}&=2\left(\hat{i}+\hat{j}+\hat{k}\right)-\left(2\hat{i}-\hat{j}+3\hat{k}\right)+3\left(\hat{i}-2\hat{j}+\hat{k}\right)\\&=2\hat{i}+2\hat{j}+2\hat{k}-2\hat{i}+\hat{j}-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k}\\&=3\hat{i}-3\hat{j}+2\hat{k}\end{aligned}$$

$$\left|2\vec{a}-\vec{b}+3\vec{c}\right|=\sqrt{3^2+\left(-3\right)^2+2^2}=\sqrt{9+9+4}=\sqrt{22}$$

Hence, the unit vector along $2\vec{a}-\vec{b}+3\vec{c}$ is

$$\frac{2\vec{a}-\vec{b}+3\vec{c}}{\left|2\vec{a}-\vec{b}+3\vec{c}\right|}=\frac{3\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{22}}=\frac{3}{\sqrt{22}}\hat{i}-\frac{3}{\sqrt{22}}\hat{j}+\frac{2}{\sqrt{22}}\hat{k}.$$

Solution 8

The given points are A (1, -2, -8), B (5, 0, -2), and C (11, 3, 7).

$$\therefore\ \overrightarrow{AB}=\left(5-1\right)\hat{i}+\left(0+2\right)\hat{j}+\left(-2+8\right)\hat{k}=4\hat{i}+2\hat{j}+6\hat{k}$$

$$\overrightarrow{BC}=\left(11-5\right)\hat{i}+\left(3-0\right)\hat{j}+\left(7+2\right)\hat{k}=6\hat{i}+3\hat{j}+9\hat{k}$$

$$\overrightarrow{AC}=\left(11-1\right)\hat{i}+\left(3+2\right)\hat{j}+\left(7+8\right)\hat{k}=10\hat{i}+5\hat{j}+15\hat{k}$$

$$\left|\overrightarrow{AB}\right|=\sqrt{4^2+2^2+6^2}=\sqrt{16+4+36}=\sqrt{56}=2\sqrt{14}$$

$$\left|\overrightarrow{BC}\right|=\sqrt{6^2+3^2+9^2}=\sqrt{36+9+81}=\sqrt{126}=3\sqrt{14}$$

$$\left|\overrightarrow{AC}\right|=\sqrt{10^2+5^2+15^2}=\sqrt{100+25+225}=\sqrt{350}=5\sqrt{14}$$

$$\therefore\left|\overrightarrow{AC}\right|=\left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right|$$

Thus, the given points A, B, and C are collinear.

Now, let point B divide AC in the ratio $\lambda:1$. Then, we have:

$$\overrightarrow{OB}=\frac{\lambda\overrightarrow{OC}+\overrightarrow{OA}}{(\lambda+1)}$$

$$\Rightarrow 5\hat{i}-2\hat{k}=\frac{\lambda\left(11\hat{i}+3\hat{j}+7\hat{k}\right)+\left(\hat{i}-2\hat{j}-8\hat{k}\right)}{\lambda+1}$$

$$\Rightarrow(\lambda+1)\left(5\hat{i}-2\hat{k}\right)=11\lambda\hat{i}+3\lambda\hat{j}+7\lambda\hat{k}+\hat{i}-2\hat{j}-8\hat{k}$$

$$\Rightarrow 5(\lambda+1)\hat{i}-2(\lambda+1)\hat{k}=(11\lambda+1)\hat{i}+(3\lambda-2)\hat{j}+(7\lambda-8)\hat{k}$$

On equating the corresponding components, we get:

$$5(\lambda+1)=11\lambda+1$$

$$\Rightarrow 5\lambda+5=11\lambda+1$$

$$\Rightarrow 6\lambda=4$$

$$\Rightarrow \lambda=\frac{4}{6}=\frac{2}{3}$$

Hence, point B divides AC in the ratio $2:3$.

Solution 9

It is given that $\overrightarrow{OP}=2\vec{a}+\vec{b},\ \overrightarrow{OQ}=\vec{a}-3\vec{b}$.

It is given that point R divides a line segment joining two points P and Q externally in the ratio 1: 2. Then, on using the section formula, we get:

$$\overrightarrow{OR}=\frac{2\left(2\vec{a}+\vec{b}\right)-\left(\vec{a}-3\vec{b}\right)}{2-1}=\frac{4\vec{a}+2\vec{b}-\vec{a}+3\vec{b}}{1}=3\vec{a}+5\vec{b}$$

Therefore, the position vector of point R is $3\vec{a}+5\vec{b}$.

Position vector of the mid-point of RQ $=\dfrac{\overrightarrow{OQ}+\overrightarrow{OR}}{2}$

$$=\frac{\left(\vec{a}-3\vec{b}\right)+\left(3\vec{a}+5\vec{b}\right)}{2}$$

$$=2\vec{a}+\vec{b}$$

$$=\overrightarrow{OP}$$

Hence, P is the mid-point of the line segment RQ.

Solution 10

Adjacent sides of a parallelogram are given as: $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$

Then, the diagonal of a parallelogram is given by $\vec{a} + \vec{b}$.

$$\vec{a} + \vec{b} = (2+1)\hat{i} + (-4-2)\hat{j} + (5-3)\hat{k} = 3\hat{i} - 6\hat{j} + 2\hat{k}$$

Thus, the unit vector parallel to the diagonal is

$$\frac{\vec{a}+\vec{b}}{\left|\vec{a}+\vec{b}\right|} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{3^2 + (-6)^2 + 2^2}} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{9+36+4}} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{7} = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}.$$

$\therefore$ Area of parallelogram ABCD = $\left|\vec{a} \times \vec{b}\right|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= \hat{i}(12+10) - \hat{j}(-6-5) + \hat{k}(-4+4)$$

$$= 22\hat{i} + 11\hat{j}$$

$$= 11(2\hat{i} + \hat{j})$$

$$\therefore \left|\vec{a} \times \vec{b}\right| = 11\sqrt{2^2 + 1^2} = 11\sqrt{5}$$

Hence, the area of the parallelogram is $11\sqrt{5}$ square units.

Solution 11

Let a vector be equally inclined to axes OX, OY, and OZ at angle α.

Then, the direction cosines of the vector are cos α, cos α, and cos α.

Now,

$$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$

$$\Rightarrow 3\cos^2\alpha = 1$$

$$\Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$$

Hence, the direction cosines of the vector which are equally inclined to the axes are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Solution 12

Let $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$.

Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, we have:

$$\vec{d} \cdot \vec{a} = 0$$
$$\Rightarrow d_1 + 4d_2 + 2d_3 = 0 \qquad \text{...(i)}$$
And,
$$\vec{d} \cdot \vec{b} = 0$$
$$\Rightarrow 3d_1 - 2d_2 + 7d_3 = 0 \qquad \text{...(ii)}$$

Also, it is given that:

$$\vec{c} \cdot \vec{d} = 15$$
$$\Rightarrow 2d_1 - d_2 + 4d_3 = 15 \qquad \text{...(iii)}$$

On solving (i), (ii), and (iii), we get:

$$d_1 = \frac{160}{3}, d_2 = -\frac{5}{3} \text{ and } d_3 = -\frac{70}{3}$$
$$\therefore \vec{d} = \frac{160}{3}\hat{i} - \frac{5}{3}\hat{j} - \frac{70}{3}\hat{k} = \frac{1}{3}\left(160\hat{i} - 5\hat{j} - 70\hat{k}\right)$$

Hence, the required vector is $\frac{1}{3}\left(160\hat{i} - 5\hat{j} - 70\hat{k}\right)$.

Solution 13

$$\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$$
$$=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

Therefore, unit vector along $\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$ is given as:

$$\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{4+4\lambda+\lambda^2+36+4}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}$$

Scalar product of $\left(\hat{i}+\hat{j}+\hat{k}\right)$ with this unit vector is 1.

$$\Rightarrow \left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}=1$$
$$\Rightarrow \frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1$$
$$\Rightarrow \sqrt{\lambda^2+4\lambda+44}=\lambda+6$$
$$\Rightarrow \lambda^2+4\lambda+44=(\lambda+6)^2$$
$$\Rightarrow \lambda^2+4\lambda+44=\lambda^2+12\lambda+36$$
$$\Rightarrow 8\lambda=8$$
$$\Rightarrow \lambda=1$$

Hence, the value of λ is 1.

Solution 14

Since $\vec{a}, \vec{b},$ and $\vec{c}$ are mutually perpendicular vectors, we have

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0.$$

It is given that:

$$|\vec{a}| = |\vec{b}| = |\vec{c}|$$

Let vector $\vec{a} + \vec{b} + \vec{c}$ be inclined to $\vec{a}, \vec{b},$ and $\vec{c}$ at angles $\theta_1,\ \theta_2,$ and θ_3 respectively.

Then, we have:

$$\cos\theta_1 = \frac{\left(\vec{a} + \vec{b} + \vec{c}\right) \cdot \vec{a}}{\left|\vec{a} + \vec{b} + \vec{c}\right| |\vec{a}|} = \frac{\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}}{\left|\vec{a} + \vec{b} + \vec{c}\right| |\vec{a}|}$$

$$= \frac{|\vec{a}|^2}{\left|\vec{a} + \vec{b} + \vec{c}\right| |\vec{a}|} \qquad \left[\vec{b} \cdot \vec{a} = \vec{c} \cdot \vec{a} = 0\right]$$

$$= \frac{|\vec{a}|}{\left|\vec{a} + \vec{b} + \vec{c}\right|}$$

$$\cos\theta_2=\frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{b}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{b}\right|}=\frac{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\cdot\left|\vec{b}\right|}$$

$$=\frac{\left|\vec{b}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\cdot\left|\vec{b}\right|}\qquad\left[\vec{a}\cdot\vec{b}=\vec{c}\cdot\vec{b}=0\right]$$

$$=\frac{\left|\vec{b}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|}$$

$$\cos\theta_3=\frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{c}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|}=\frac{\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{c}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|}$$

$$=\frac{\left|\vec{c}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|}\qquad\left[\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}=0\right]$$

$$=\frac{\left|\vec{c}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|}$$

Now, as $\left|\vec{a}\right|=\left|\vec{b}\right|=\left|\vec{c}\right|$, $\cos\theta_1=\cos\theta_2=\cos\theta_3$.

$\therefore\theta_1=\theta_2=\theta_3$

Hence, the vector $\left(\vec{a}+\vec{b}+\vec{c}\right)$ is equally inclined to $\vec{a},\vec{b},$ and $\vec{c}$.

Solution 15

$\left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right)=\left|\vec{a}\right|^2+\left|\vec{b}\right|^2$

$\Leftrightarrow\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=\left|\vec{a}\right|^2+\left|\vec{b}\right|^2$ [Distributivity of scalar products over addition]

$\Leftrightarrow\left|\vec{a}\right|^2+2\vec{a}\cdot\vec{b}+\left|\vec{b}\right|^2=\left|\vec{a}\right|^2+\left|\vec{b}\right|^2$ $\left[\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}\ \text{(Scalar product is commutative)}\right]$

$\Leftrightarrow 2\vec{a}\cdot\vec{b}=0$

$\Leftrightarrow\vec{a}\cdot\vec{b}=0$

$\therefore\vec{a}$ and $\vec{b}$ are perpendicular. $\left[\vec{a}\neq\vec{0},\ \vec{b}\neq\vec{0}\ \text{(Given)}\right]$

Solution 16

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

It is known that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \vec{a}\cdot\vec{b}\geq 0$$
$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta \geq 0$$
$$\Rightarrow \cos\theta \geq 0 \qquad \left[|\vec{a}| \text{ and } |\vec{b}| \text{ are positive}\right]$$
$$\Rightarrow 0\leq\theta\leq\frac{\pi}{2}$$

Hence, $\vec{a}.\vec{b}\geq 0$ when $0\leq\theta\leq\frac{\pi}{2}$.

The correct answer is B.

Solution 17

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ be the angle between them.

Then, $|\vec{a}| = |\vec{b}| = 1$.

Now, $\vec{a}+\vec{b}$ is a unit vector if $|\vec{a}+\vec{b}| = 1$.

$$|\vec{a}+\vec{b}| = 1$$
$$\Rightarrow (\vec{a}+\vec{b})^2 = 1$$
$$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$$
$$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = 1$$
$$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 1$$
$$\Rightarrow 1^2 + 2|\vec{a}||\vec{b}|\cos\theta + 1^2 = 1$$
$$\Rightarrow 1 + 2.1.1\cos\theta + 1 = 1$$
$$\Rightarrow \cos\theta = -\frac{1}{2}$$
$$\Rightarrow \theta = \frac{2\pi}{3}$$

Hence, $\vec{a}+\vec{b}$ is a unit vector if $\theta = \frac{2\pi}{3}$.

Solution 18

$$\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}.(\hat{i}\times\hat{k}) + \hat{k}.(\hat{i}\times\hat{j})$$
$$= \hat{i}\cdot\hat{i} + \hat{j}\cdot(-\hat{j}) + \hat{k}\cdot\hat{k}$$
$$= 1 - \hat{j}\cdot\hat{j} + 1$$
$$= 1 - 1 + 1$$
$$= 1$$

The correct answer is C.

Solution 19

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors, so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

$$\left|\vec{a}\cdot\vec{b}\right| = \left|\vec{a}\times\vec{b}\right|$$

$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\sin\theta$$

$$\Rightarrow \cos\theta = \sin\theta \qquad \left[|\vec{a}| \text{ and } |\vec{b}| \text{ are positive}\right]$$

$$\Rightarrow \tan\theta = 1$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Hence, $\left|\vec{a}.\vec{b}\right| = \left|\vec{a}\times\vec{b}\right|$ when θ isequal to $\frac{\pi}{4}$.

The correct answer is B.