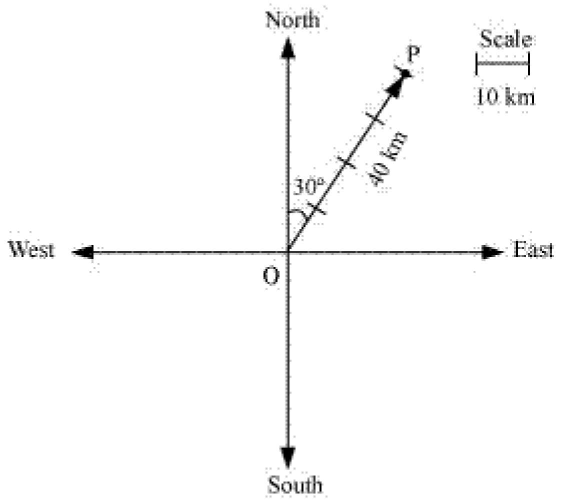
RD SHARMA Solutions for Class 12-science Maths Chapter 23 - Algebra of Vectors

Chapter 23 - Algebra of vectors Exercise Ex. 23.1

Question 1(i)

Represent graphically a displacement of 40 km, 30° east of north.

Solution 1(i)

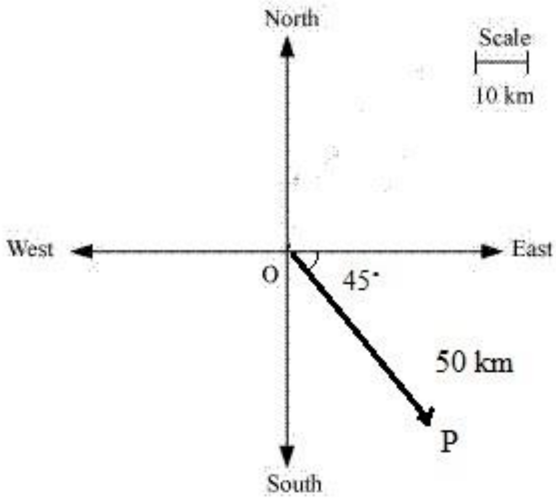


Here, vector $\overrightarrow{OP}$ represents the displacement of 40 km, 30° East of North.

Question 1(ii)

Represent graphically a displacement of 50 km, south-east

Solution 1(ii)

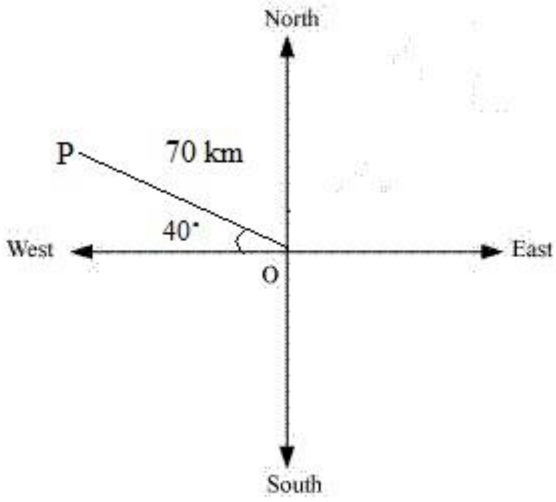


Here, vector $\overrightarrow{OP}$ represents the displacement of 50 km, south-east

Question 1(iii)

Represent graphically a displacement of 70 km, 40° north of west.

Solution 1(iii)



Here, vector $\overrightarrow{OP}$ represents the displacement of 7.0 km, 40° north of west

Question 2

Classify the following measures as scalars and vectors.

(i) 15 kg (ii) 20 kg weight (iii) 45°

(iv) 10 meters south-east (v) 50 m/s^2

Solution 2

(i) 15 kg is a scalar quantity because it involves only mass

(ii) 20 kg weight is a vector quantity as it involves both magnitude and direction.

(iii) 45° is a scalar quantity as it involves only magnitude.

(iv) 10 meters south-east is a vector quantity as it involve direction.

(v) 50 m/s^2 is a scalar quantity as it involves magnitude of acceleration.

Question 3

Classify the following as scalar and vector quantities.

(i) time period (ii) distance (iii) displacement (iv) force (v) work

(vi) velocity (vii) Acceleration

Solution 3

(i) Time period is a scalar quantity as it involves only magnitude.

(ii) Distance is a scalar quantity as it involves only magnitude.

(iii) Displacement is vector quantity as it involves both magnitude and direction.

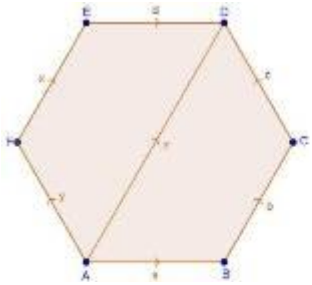
(iv) Force is a vector quantity as it involves both magnitude and direction.

(v) Work done is a scalar quantity as it involves only magnitude.

(vi) Velocity is a vector quantity as it involves both magnitude as well as direction.

(vii) Acceleration is a vector quantity because it involves both magnitude as well as direction.

Question 4



Which vectors are:

(i) Collinear (ii) Equal (iii) Coinitial

(iv) Collinear but not equal.

Solution 4

(i)

Collinear vectors are

$\vec{x}, \vec{z}$ and $\vec{b}$

$\vec{y}, \vec{c}$

$\vec{a}, \vec{d}$

(ii)

Equal vectors are

$\vec{y}$ and $\vec{c}$

$\vec{x}$ and $\vec{b}$

$\vec{a}$ and $\vec{d}$

(iii)

Coinitial vector are

$\vec{a}$, $\vec{y}$ and $\vec{z}$

(iv)

Collinear but not equal

$\vec{b}$ and $\vec{z}$

$\vec{x}$ and $\vec{z}$

Question 5

Answer the following as true or false:

(i) a and b are collinear .

(ii) Two collinear vectors are always equal in magnitude.

(iii) Zero vector is unique.

(iv) Two vectors having same magnitude are collinear.

(v) Two collinear vectors having the same magnitude are equal.

Solution 5

(i) a and b are collinear , it is true.

(ii) Two collinear vectors are may not be equal in magnitude, so it is false.

(iii) Zero vector may not be unique, so it is false.

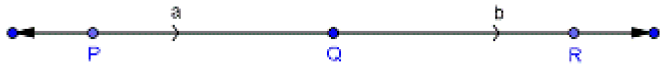
(iv) Two vectors having same magnitude are may not be collinear so it is false.

(v) Two collinear vectors having the same magnitude are may not be equal, so it is false.

Chapter 23 - Algebra of Vectors Exercise Ex. 23.2

Question 1

If P, Q and R are three collinear points such that $\overrightarrow{PQ} = \vec{a}$ and $\overrightarrow{QR} = \vec{b}$. Find the vector $\overrightarrow{PR}$.



Solution 1

Given that, P, Q, R are collinear.

It also given that, $\overrightarrow{PQ} = \vec{a}$ and $\overrightarrow{QR} = \vec{b}$

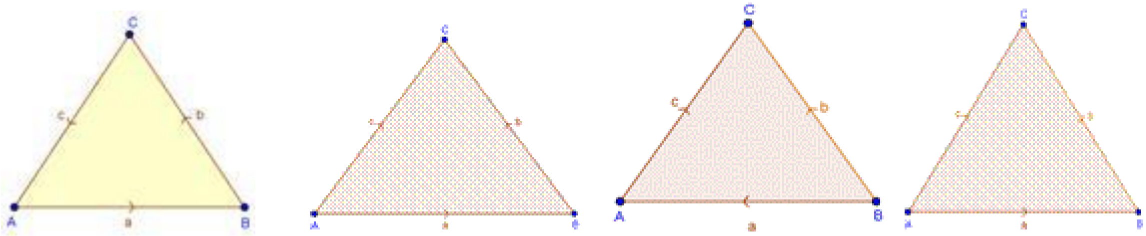
$$\begin{aligned}\overrightarrow{PR} &= \overrightarrow{PQ} + \overrightarrow{QR} \\ &= \vec{a} + \vec{b}\end{aligned}$$

$$\overrightarrow{PR} = \vec{a} + \vec{b}$$

Question 2

Give condition that three vectors $\vec{a}, \vec{b}$, and $\vec{c}$ form the three sides of a triangle. What are the other possibilities?

Solution 2



Given that, $\vec{a}, \vec{b}$, and $\vec{c}$ are three sides of a triangle.

$\vec{a} + \vec{b} + \vec{c}$

$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$

$= \overrightarrow{AC} + \overrightarrow{CA}$ $\left[\text{Since } \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}\right]$

$= \overrightarrow{AC} - \overrightarrow{AC}$ $\left[\text{Since } \overrightarrow{CA} = \overrightarrow{AC}\right]$

$= \vec{0}$

So, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

Triangle law says that, if vectors are represented in magnitude and direction by the two sides of triangle taken is same order, then their sum is represented by the third side taken in reverse order.

Thus,

$\vec{a} + \vec{b} = \vec{c}$

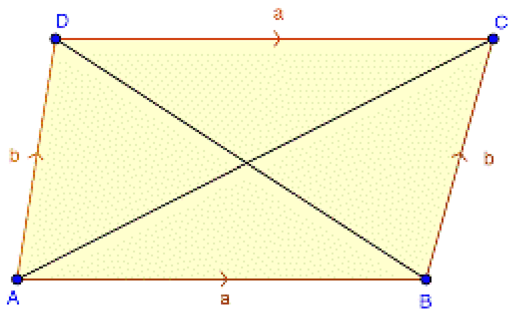
or

$\vec{a} + \vec{c} = \vec{b}$

$\vec{b} + \vec{c} = \vec{a}$

Question 3

If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors having the same initial point. What are the vectors represented by $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$?



Solution 3

Here, it is given that $\vec{a}$ and $\vec{b}$ are two non-collinear vectors having the same initial point.

Let $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AD}$, So we can draw a parallelogram ABCD as above.

By the properties of parallelogram
$\overrightarrow{BC} = \vec{b}$ and $\overrightarrow{DC} = \vec{a}$

In ΔABC,
Using triangle law,
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
$\vec{a} + \vec{b} = \overrightarrow{AC}$ $\quad - -$(i)

In ΔABD,
Using triangle law,
$\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$
$\vec{b} + \overrightarrow{DB} = \vec{a}$
$\overrightarrow{DB} = \vec{a} - \vec{b}$ $\quad - -$(ii)

From equation (i) and (ii), we get that

$\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are diagonals of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$

Question 4

If $\vec{a}$ is a vector and m is a scalar such that $m\vec{a} = \vec{0}$, then what are the alternatives for m and $\vec{a}$?

Solution 4

Given that m is a scalar and $\vec{a}$ is a vector such that
$m\vec{a} = \vec{0}$

$m\left(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}\right) = 0 \times \hat{i} + 0 \times \hat{j} + 0 \times \hat{k}$ $\left[\text{since let } \vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}\right]$

$ma_1\hat{i} + mb_1\hat{j} + mc_1\hat{k} = 0 \times \hat{i} + 0 \times \hat{j} + 0 \times \hat{k}$

Comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$ of LHS and RHS,

$ma_1 = 0 \Rightarrow m = 0$ or $a_1 = 0$ (i)

$mb_1 = 0 \Rightarrow m = 0$ or $b_1 = 0$ (ii)

$mc_1 = 0 \Rightarrow m = 0$ or $c_1 = 0$ (iii)

From (i), (ii) and (iii)

$m = 0$ or $a_1 = b_1 = c_1 = 0$

$\Rightarrow m = 0$ or $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{j} = 0$

$\Rightarrow m = 0$ or $\vec{a} = 0$

Question 5

If $\vec{a}$ $\vec{b}$ are two vectors, then write the truth value of the following statements:

(i) $\vec{a} = -\vec{b} \Rightarrow \left|\vec{a}\right| = \left|\vec{b}\right|$ (ii) $\left|\vec{a}\right| = \left|\vec{b}\right| \Rightarrow \vec{a} = \pm\vec{b}$

(iii) $\left|\vec{a}\right| = \left|\vec{b}\right| \Rightarrow \vec{a} = \vec{b}$

Solution 5

(i)

Let $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

Given that, $a = -b$

$a_1\hat{i} + b_1\hat{j} + c_1\hat{k} = -a_2\hat{i} - b_2\hat{j} - c_2\hat{k}$

Comparing the coefficients of i, j, k in LHS and RHS,

$a_1 = -a_2$ (1)

$b_1 = -b_2$ (2)

$c_1 = -c_2$ (3)

$|\vec{a}| = \sqrt{a_1^2 + b_1^2 + c_1^2}$

Using (1), (2)and (3),

$|\vec{a}| = \sqrt{(-a_2)^2 + (-b_2)^2 + (-c_2)^2}$

$|\vec{a}| = \sqrt{a_2^2 + b_2^2 + c_2^2}$

$\therefore |\vec{a}| = |\vec{b}|$

(ii)

Given a and b are two vectors such that $|\vec{a}| = |\vec{b}|$

It means magnitude of vector $\vec{a}$ is equal to the magnitude of vector $\vec{b}$, but we cannot conclude anything about the direction of the vector.

So,it is false that

$|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \pm\vec{b}$

(iii)

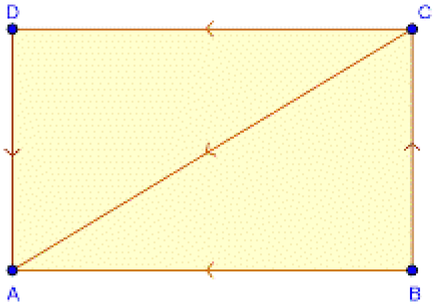
Given for any vector $\vec{a}$ and $\vec{b}$

$|\vec{a}| = |\vec{b}|$

It means magnitude of the vector $\vec{a}$ and $\vec{b}$ are equal but we cannot say any thing about the direction of the vector $\vec{a}$ and $\vec{b}$. And we know that $\vec{a} = \vec{b}$ means magnitude and same direction. So, it is false.

Question 6

$ABCD$ is a quadrilateral. Find the sum of the vectors $\overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{CD}$ and $\overrightarrow{DA}$.



Solution 6

Here it is given that $ABCD$ is a quadrilateral.

In ΔADC, using triangle law, we get

$\overrightarrow{CD} + \overrightarrow{DA} = \overrightarrow{CA}$ --(i)

In ΔABC, using triangle law, we get

$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$ --(ii)

Put value of $\overrightarrow{CA}$ in equation (ii),

$\overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \overrightarrow{BA}$

Adding $\overrightarrow{BA}$ on both the sides,

$\overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \overrightarrow{BA} + \overrightarrow{BA}$

$\therefore \overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = 2\overrightarrow{BA}$

Question 7

$ABCDE$ is a pentagon, prove that

(i) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} = 0$

(ii) $\overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC} = 3\overrightarrow{AC}$

Solution 7

(i)
Given that $ABCDE$ is a pentagon.

$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EA}$
$=\left(\overrightarrow{AB}+\overrightarrow{BC}\right)+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EA}$
$=\overrightarrow{AC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EA}$ $\quad$ $\left[\text{Using triangle law in } \Delta ABC,\ \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}\right]$
$=\left(\overrightarrow{AC}+\overrightarrow{CD}\right)+\overrightarrow{DE}+\overrightarrow{EA}$
$=\left(\overrightarrow{AD}\right)+\overrightarrow{DE}+\overrightarrow{EA}$ $\quad$ $\left[\text{Using triangle law in } \Delta ACD,\ \overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}\right]$
$=\overrightarrow{AD}+\overrightarrow{DA}$
$=\overrightarrow{AD}-(-\overrightarrow{AD})$
$=\vec{0}$

$\therefore\ \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EA}=\vec{0}$

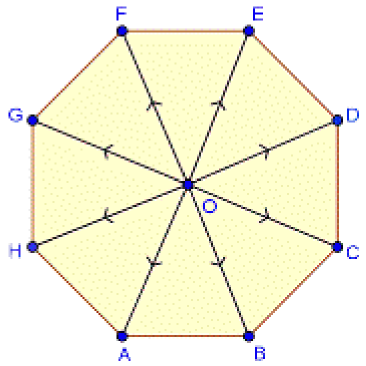
(ii)
It is given that ABCDE is a pentagon, So

$\overrightarrow{AB}+\overrightarrow{AE}+\overrightarrow{BC}+\overrightarrow{DC}+\overrightarrow{ED}+\overrightarrow{AC}$
$=\left(\overrightarrow{AB}+\overrightarrow{BC}\right)+\overrightarrow{AE}+\overrightarrow{DC}+\overrightarrow{ED}+\overrightarrow{AC}$
$=\overrightarrow{AC}+\overrightarrow{DC}+\left(\overrightarrow{AE}+\overrightarrow{ED}\right)+\overrightarrow{AC}$ $\quad$ $\left[\text{Using triangle law in } \Delta ABC,\ \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}\right]$
$=\overrightarrow{AC}+\overrightarrow{DC}+\left(\overrightarrow{AD}\right)+\overrightarrow{AC}$
$=\overrightarrow{AC}+\overrightarrow{DC}-\overrightarrow{DA}+\overrightarrow{AC}$
$=\overrightarrow{AC}+\overrightarrow{DC}+\overrightarrow{AD}+\overrightarrow{AC}$
$=\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{DC}+\overrightarrow{AC}$
$=\overrightarrow{AC}+\overrightarrow{AC}+\overrightarrow{AC}$
$=3\overrightarrow{AC}$

So,
$\overrightarrow{AB}+\overrightarrow{AE}+\overrightarrow{BC}+\overrightarrow{DC}+\overrightarrow{ED}+\overrightarrow{AC}=3\overrightarrow{AC}$

Question 8

Prove that the sum of all vectors drawn from the centre of a regular octagon to its vertices is the zero vector.



Solution 8

Let O be the centre of a regular octagon, we know that the centre of a regular octagon bisects all the diagonals passing through it.

Thus,

$\overrightarrow{OA} = -\overrightarrow{OE}$ (i)

$\overrightarrow{OB} = -\overrightarrow{OF}$ (ii)

$\overrightarrow{OC} = -\overrightarrow{OG}$ (iii)

$\overrightarrow{OD} = -\overrightarrow{OH}$ (iv)

Adding equation (i), (ii), and (iv),

$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = -\overrightarrow{OE} - \overrightarrow{OF} - \overrightarrow{OG} - \overrightarrow{OH}$

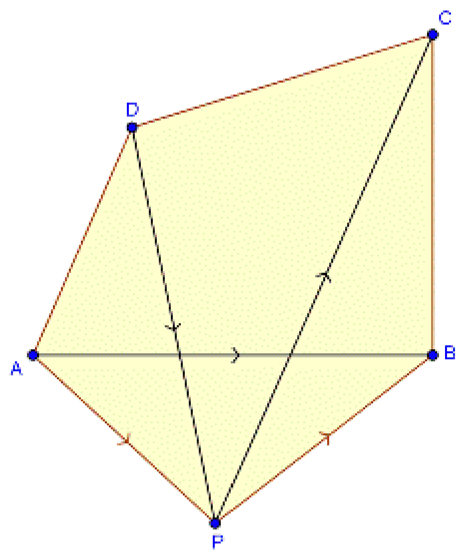
$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + = -\left(\overrightarrow{OE} + \overrightarrow{OF} + \overrightarrow{OG} + \overrightarrow{OH}\right)$

$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} + \overrightarrow{OG} + \overrightarrow{OH} = \vec{0}$

Question 9

If P is a point and $ABCD$ is a quadrilateral and $\overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PD} = \overrightarrow{PC}$, show that $ABCD$ is a parallelogram.

Solution 9



Given,

$$\overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PD} = \overrightarrow{PC}$$

$$\overrightarrow{AP} + \overrightarrow{PB} = \overrightarrow{PC} - \overrightarrow{PD}$$

$$\overrightarrow{AP} + \overrightarrow{PB} = \overrightarrow{PC} + \overrightarrow{DP} \quad \left[\text{Since } \overrightarrow{DP} = -\overrightarrow{PD}\right]$$

$$\overrightarrow{AP} + \overrightarrow{PB} = \overrightarrow{DP} + \overrightarrow{PC}$$

$$\overrightarrow{AB} = \overrightarrow{DC} \quad \left[\begin{array}{l}\text{Using triangle law in } \Delta APB,\ \overrightarrow{AP} + \overrightarrow{PB} = \overrightarrow{AB} \\ \text{Using triangle law in } \Delta DPC,\ \overrightarrow{DP} + \overrightarrow{PC} = \overrightarrow{DC}\end{array}\right]$$

Therefore, AB is parallel to DC and equal is magnitude.
Hence, $ABCD$ is a parallelogram.

Question 10

Five forces $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$ and $\overrightarrow{AF}$ act at the vertex of a regular hexagon $ABCDEF$. Prove that the resultant is 6 $\overrightarrow{AO}$, where O is the centre of hexagon.

Solution 10

We need to show that

$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 6\overrightarrow{AO}$$

We know that centre O of the hexagon bisects the diagonal $\overrightarrow{AD}$

$$\therefore \quad \overrightarrow{AO} = \frac{1}{2}\overrightarrow{AD};\ \overrightarrow{BO} = -\overrightarrow{EO};\ \overrightarrow{CO} = -\overrightarrow{FO}$$

Now

$$\overrightarrow{AB} + \overrightarrow{BO} = \overrightarrow{AO}$$

$$\overrightarrow{AC} + \overrightarrow{CO} = \overrightarrow{AO}$$

$$\overrightarrow{AD} + \overrightarrow{DO} = \overrightarrow{AO}$$

$$\overrightarrow{AE} + \overrightarrow{EO} = \overrightarrow{AO}$$

$$\overrightarrow{AF} + \overrightarrow{FO} = \overrightarrow{AO}$$

Adding these equations we get

$$\left(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}\right) + \left(\overrightarrow{BO} + \overrightarrow{CO} + \overrightarrow{DO} + \overrightarrow{EO} + \overrightarrow{FO}\right) = 5\ \overrightarrow{AO}$$

$$\Rightarrow \quad \left(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}\right) + \overrightarrow{DO} = 5\ \overrightarrow{AO}$$

But $\quad \overrightarrow{DO} = -\overrightarrow{AO}$

$$\therefore \quad \overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 6\ \overrightarrow{AO}.$$

Chapter 23 - Algebra of Vectors Exercise Ex. 23.3

Question 1

Find the position vector of a point R which divides the line joining the two points P and Q with position vectors

$\overrightarrow{OP} = 2\vec{a} + \vec{b}$ and $\overrightarrow{OQ} = \vec{a} - 2\vec{b}$, respectively in the ratio 1: 2 internally and externally.

Solution 1

Point R divides the line joining the two points P and Q in the ratio 1:2 internally.

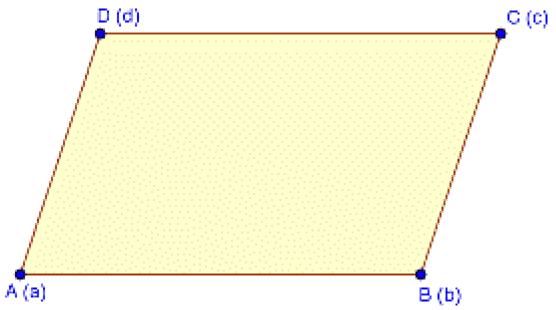
$$\text{Position vector of point R} = \frac{1(\vec{a} - 2\vec{b}) + 2(2\vec{a} + \vec{b})}{1+2} = \frac{5\vec{a}}{3}$$

Point R divides the line joining the two points P and Q in the ratio 1:2 externally.

$$\text{Position vector of point R} = \frac{1(\vec{a} - 2\vec{b}) - 2(2\vec{a} + \vec{b})}{1-2} = \frac{-3\vec{a} - 4\vec{b}}{-1} = 3\vec{a} + 4\vec{b}$$

Question 2

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of the four distinct points A, B, C, D. If $\vec{b} - \vec{a} = \vec{c} - \vec{d}$, then show that $ABCD$ is a parallelogram.



Solution 2

Here it is given that $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be the position vectors of the four distinct points A, B, C, D such that

$\vec{b} - \vec{a} = \vec{c} - \vec{d}$

Given that,

$\vec{b} - \vec{a} = \vec{c} - \vec{d}$

$\overrightarrow{AB} = \overrightarrow{DC}$

So, AB is parallel and equal to DC (in magnitude).

Hence,

$ABCD$ is a parallelogram.

Question 3

If a, b are the position vector of A, B respectively, find the position vector of a point C in AB produced such that $AC = 3AB$ and that a point D in BA produced such that $BD = 2BA$.

Solution 3

Here, it is given that $\vec{a}, \vec{b}$ are position vector of A and B.

Let C be a point in AB produced such that $AC = 3AB$.

It is clear that point C divides the line AB in ratio $3:2$ externally.
So position vector point C is given by

$$\vec{c} = \frac{m\vec{b} - n\vec{a}}{m - n}$$
$$= \frac{3\vec{b} - 2\vec{a}}{3 - 2}$$
$$\vec{c} = 3\vec{b} - 2\vec{a}$$

Again, let D be a point in BA produced such that $BD = 2BA$.

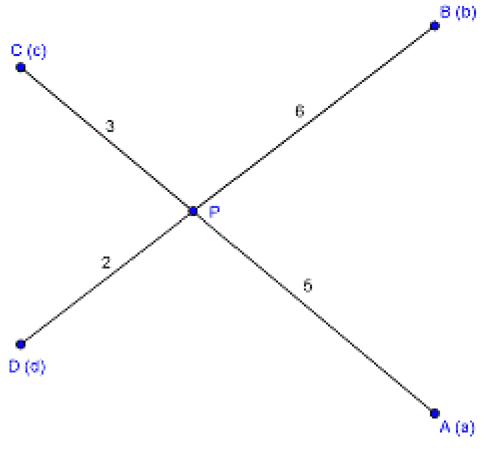
Let $\vec{d}$ be the position vector of D. It is clear that point D divides the line AB in $1:2$ externally. So position vector of D is given by

$$\vec{d} = \frac{m\vec{a} - n\vec{b}}{m - n}$$
$$= \frac{2\vec{a} - \vec{b}}{2 - 1}$$
$$\vec{d} = 2\vec{a} - \vec{b}$$

$$\vec{c} = 3\vec{b} - 2\vec{a}$$
$$\vec{d} = 2\vec{a} - \vec{b}$$

Question 4

Show that the four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively such that $3\vec{a} - 2\vec{b} + 5\vec{c} - 6\vec{d} = 0$, are coplanar. Also, find the position vector of the point of intersection of the lines AC and BD.



Solution 4

We have given that

$$3\vec{a} - 2\vec{b} + 5\vec{c} - 6\vec{d} = \vec{0}$$

$$3\vec{a} + 5\vec{c} = 2\vec{b} + 6\vec{d} \qquad \text{(i)}$$

Sum of the coefficients on both the sides of the equation (i) is 8, so divide equation (i) by 8 on both the sides,

$$\frac{3\vec{a} + 5\vec{c}}{8} = \frac{2\vec{b} + 6\vec{d}}{8}$$

$$\frac{3\vec{a} + 5\vec{c}}{3+5} = \frac{2\vec{b} + 6\vec{d}}{2+6}$$

It shows that position vector of a point p dividing AC in the ratio $3:5$, is same as that of a point dividing BD in the ratio of $2:6$.

Hence, point P is common to AC and BD. Therefore, P is the point of intersection of AC and BD.

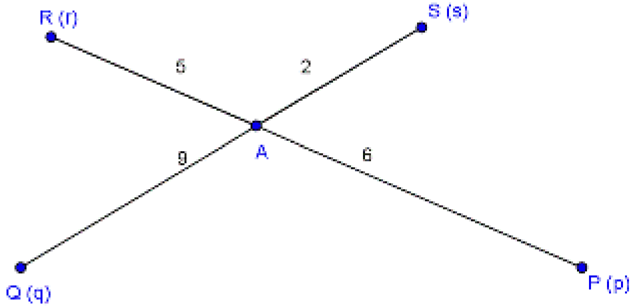
So, A, B, C and D are coplanar.

Position vector of point P is given by

$$\frac{3\vec{a} + 5\vec{c}}{8} \quad \text{or} \quad \frac{2\vec{b} + 6\vec{d}}{8}$$

Question 5

Show that the four points P, Q, R, S with position vectors $\vec{p}, \vec{q}, \vec{r}, \vec{s}$ respectively such that $5\vec{p} - 2\vec{q} + 6\vec{r} - 9\vec{s} = \vec{0}$, are coplanar. Also, find the position vector of the point of intersection of the lines PR and QS.



Solution 5

We have given that

$5\vec{p} - 2\vec{q} + 6\vec{r} - 9\vec{s} = \vec{0}$

Where $\vec{p}, \vec{q}, \vec{r}$ and $\vec{s}$ are the position vectors of point P, Q, R and S.

$5\vec{p} + 6\vec{r} = 2\vec{q} + 9\vec{s}$ (i)

Sum of the coefficients on both the sides of the equation (i) is 11. So divide equation (i) by 11 on both the sides.

$$\frac{5\vec{p} + 6\vec{r}}{11} = \frac{2\vec{q} + 9\vec{s}}{11}$$

$$\frac{5\vec{p} + 6\vec{r}}{5+6} = \frac{2\vec{q} + 9\vec{s}}{2+9}$$

It shows that position vector of a point A dividing PR in the ratio of 6 : 5 and QS in the ratio of 9 : 2. Thus, A is the common point to PR and QS and it is also point of intersection of PQ and QS.

So,

P, Q, R and S are coplanar

Position vector of point A is given by

$$\frac{5p + 6q}{11} \quad \text{or} \quad \frac{2\vec{q} + 9\vec{s}}{11}$$

Question 6

The vertices A, B, C of triangle ABC have respectively position vectors $\vec{a}, \vec{b}, \vec{c}$ with respect to a given origin O. Show that the point D where the bisector of $\angle A$ meets BC has position vector

$$\vec{d} = \frac{\beta\vec{b} + \gamma\vec{c}}{\beta + \gamma}, \text{ where } \beta = |\vec{c} - \vec{a}| = \gamma = |\vec{a} - \vec{b}|$$

Hence deduce that the incentre I has position vector $\dfrac{\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}}{\alpha + \beta + \gamma}$, where $\alpha = |\vec{b} - \vec{c}|$

Solution 6

Let ABC be a triangle.
Let the position vectors of A, B and C with respect to some origin, O be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.
Let D be the point on BC where the bisector of $\angle A$ meets.
Let $\vec{d}$ position vector of D which divides BC internally in the ratio β and γ,
where $\beta = |\overrightarrow{AC}|$ and $\gamma = |\overrightarrow{AB}|$
Thus, $\beta = |\vec{c} - \vec{a}|$ and $\gamma = |\vec{b} - \vec{a}|$
Thus, by section formula, the position vector of D is given by

$$\overrightarrow{OD} = \frac{\beta\vec{b} + \gamma\vec{c}}{\beta + \gamma}$$

Let $\alpha = |\vec{b} - \vec{c}|$
Incentre is the concurrent point of angle bisectors.
Thus, Incentre divides the line AD in the ratio $\alpha : \beta + \gamma$
Thus, the position vector of incentre is

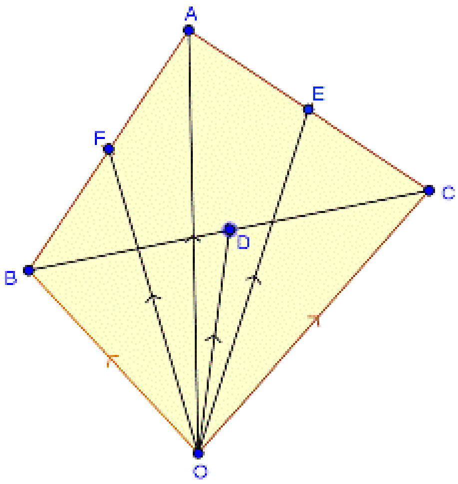
$$\text{equal to } \frac{\alpha\vec{a} + \frac{\beta\vec{b} + \gamma\vec{c}}{(\beta + \gamma)} \times (\beta + \gamma)}{\alpha + \beta + \gamma} = \frac{\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}}{\alpha + \beta + \gamma}$$

Chapter 23 - Algebra of Vectors Exercise Ex. 23.4

Question 1

If O is a point in space, ABC is a triangle and D, E, F are the mid-points of the sides BC, CA and AB respectively of the triangle, prove that
$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF}$

Solution 1



Here, in ΔABC, D, E, F are the mid points of the sides of BC, CA and AB respectively. And O is any point in space.

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}, \vec{f}$ be the position vector of point A, B, C, D, E, F with respect to O.

So, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$

$\overrightarrow{OD} = \vec{d}$, $\overrightarrow{OE} = \vec{e}$, $\overrightarrow{OF} = \vec{f}$

$$\vec{d} = \frac{\vec{b} + \vec{c}}{2}$$

$$\vec{e} = \frac{\vec{a} + \vec{c}}{2} \qquad \text{[Using mid point formula]}$$

$$\vec{f} = \frac{\vec{a} + \vec{b}}{2}$$

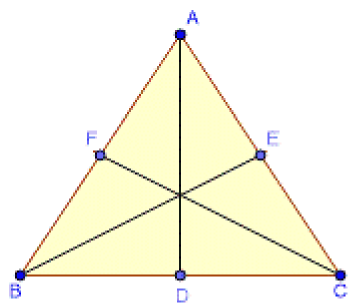
$$\begin{aligned}
\overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} &= \vec{d} + \vec{e} + \vec{f} \\
&= \frac{\vec{b} + \vec{c}}{2} + \frac{\vec{a} + \vec{c}}{2} + \frac{\vec{a} + \vec{b}}{2} \\
&= \frac{\vec{b} + \vec{c} + \vec{a} + \vec{c} + \vec{a} + \vec{b}}{2} \\
&= \frac{2\left(\vec{a} + \vec{b} + \vec{c}\right)}{2} \\
&= \vec{a} + \vec{b} + \vec{c} \\
&= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}
\end{aligned}$$

So,

$$\overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

Question 2

Show that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.



Solution 2

Here, we have to show that the sum of the three vectors ditermined by medians of a triangle directed from the vertices is zero.

Let ABC is triangle such that position vector of A, B and C are $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.

As AD, BE, CF are medians, D, E and F are mid points.

Position vector of $D = \dfrac{\vec{b}+\vec{c}}{2}$ [Using mid point formula]

Position vector of $E = \dfrac{\vec{c}+\vec{a}}{2}$ [Using mid point formula]

Position vector of $F = \dfrac{\vec{a}+\vec{b}}{2}$ [Using mid point formula]

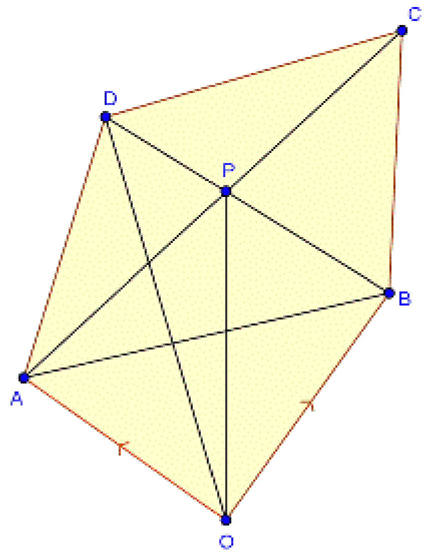
Now,

$$\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}$$
$$= \left(\frac{\vec{b}+\vec{c}}{2}-\vec{a}\right)+\left(\frac{\vec{c}+\vec{a}}{2}-\vec{b}\right)+\left(\frac{\vec{a}+\vec{b}}{2}-\vec{c}\right)$$
$$= \frac{\vec{b}+\vec{c}-2\vec{a}}{2}+\frac{\vec{c}+\vec{a}-2\vec{b}}{2}+\frac{\vec{a}+\vec{b}-2\vec{c}}{2}$$
$$= \frac{\vec{b}+\vec{c}-2\vec{a}+\vec{c}+\vec{a}-2\vec{b}+\vec{a}+\vec{b}-2\vec{c}}{2}$$
$$= \frac{2\vec{b}+2\vec{c}+2\vec{a}-2\vec{b}-2\vec{a}-2\vec{c}}{2}$$
$$= \frac{\vec{0}}{2}$$
$$= \vec{0}$$

$$\therefore\ \overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\vec{0}$$

Question 3

$ABCD$ is a parallelogram and P is the point of intersection of its diagonals.
If O is the origin of reference, show that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{PD}=4\overrightarrow{OP}$



Solution 3

Here, it is given that $ABCD$ is a parallelogram, P is the point of intersection of diagonals and O be the point of reference.

Using triangle law in ΔAOP,

$$\overrightarrow{OP} + \overrightarrow{PA} = \overrightarrow{OA} \qquad \text{(i)}$$

Using triangle law in ΔOBP,

$$\overrightarrow{OP} + \overrightarrow{PB} = \overrightarrow{OB} \qquad \text{(ii)}$$

Using triangle law in ΔOPC,

$$\overrightarrow{OP} + \overrightarrow{PC} = \overrightarrow{OC} \qquad \text{(iii)}$$

Using triangle law in ΔOPD,

$$\overrightarrow{OP} + \overrightarrow{PD} = \overrightarrow{OD} \qquad \text{(iv)}$$

Adding equation (i), (ii), (iii), and (iv),

$$\overrightarrow{OP} + \overrightarrow{PA} + \overrightarrow{OP} + \overrightarrow{PB} + \overrightarrow{OP} + \overrightarrow{PC} + \overrightarrow{OP} + \overrightarrow{PD} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$$

$$4\overrightarrow{OP} + \overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$$

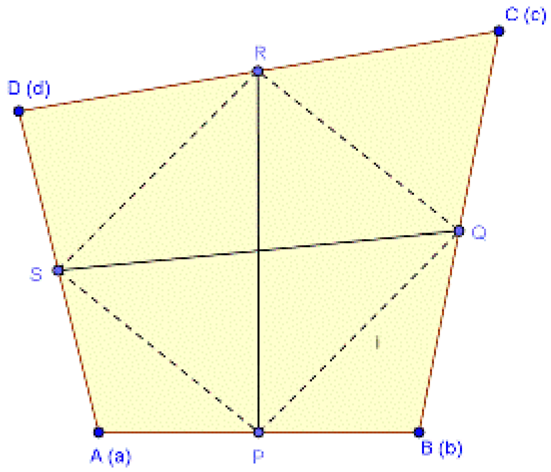
$$4\overrightarrow{OP} + \overrightarrow{PA} + \overrightarrow{PB} - \overrightarrow{PA} - \overrightarrow{PB} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} \qquad \left[\begin{array}{l}\text{Since } \overrightarrow{PC} = -\overrightarrow{PA} \text{ and } \overrightarrow{PD} = -\overrightarrow{PB} \\ \text{as } P \text{ is mid point of } AC, BD\end{array}\right]$$

$$4\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$$

Question 4

Show that the line segments joining the mid-points of opposite sides of a quadrilateral bisects each other.

Solution 4


C (c)
R
D (d)
Q
S
A (a)
P
B (b)

Let $ABCD$ be a quadrilateral and P, Q, R, S be the mid points of sides AB, BC, CD and DA respectively.

Let position vector of A, B, C and D be $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$.

So position vector of P, Q, R and S are $\left(\frac{\vec{a}+\vec{b}}{2}\right)$, $\left(\frac{\vec{b}+\vec{c}}{2}\right)$, $\left(\frac{\vec{c}+\vec{d}}{2}\right)$ and $\left(\frac{\vec{d}+\vec{a}}{2}\right)$ respectively.

Position vector of $\overrightarrow{PQ}$
= Position vector of Q − Position vector of P
$= \left(\frac{\vec{b}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{b}}{2}\right)$
$= \frac{\vec{b}+\vec{c}-\vec{a}-\vec{b}}{2}$
$= \frac{\vec{c}-\vec{a}}{2}$ (i)

Position vector of $\overrightarrow{SR}$
= Position vector of R − Position vector of S
$= \left(\frac{\vec{c}+\vec{d}}{2}\right) - \left(\frac{\vec{a}+\vec{d}}{2}\right)$
$= \frac{\vec{c}+\vec{d}-\vec{a}-\vec{d}}{2}$
$= \frac{\vec{c}-\vec{a}}{2}$ (ii)

Using (i) and (ii) ,
$\overrightarrow{PQ} = \overrightarrow{SR}$

So, $PQRS$ is a parallelogram.
Therefore, PR bisects QS [as diagonals of parallelogram]

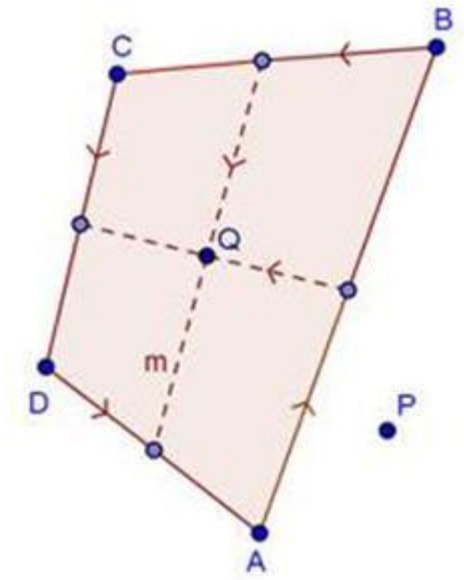
Line segment joining the mid point of opposite sides of a quadrilateral bisects each other.

Question 5

$ABCD$ are four points in a plane and Q is the point of intersection of the lines joining the mid-points of AB and CD; BC and AD. Show that

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 4\overrightarrow{PQ},$$ where P is any point.

Solution 5



Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of the points $A, B, C,$ and D respectively.
Then, position vector of

$$\text{mid point of } AB = \frac{\vec{a}+\vec{b}}{2}$$

$$\text{mid point of } BC = \frac{\vec{b}+\vec{c}}{2}$$

$$\text{mid point of } CD = \frac{\vec{c}+\vec{d}}{2}$$

$$\text{mid point of } DA = \frac{\vec{a}+\vec{d}}{2}$$

Q is the mid point of the line joining the mid points of AB and CD

$$\therefore \quad p.r. \text{ or } Q = \frac{\frac{\vec{a}+\vec{a}}{2}+\frac{\vec{c}+\vec{d}}{2}}{2}$$

$$= \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$$

Let $\vec{p}$ be the position vector of P.
Then,

$$\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}+\overrightarrow{PD}$$

$$= \vec{a}-\vec{p}+\vec{b}-\vec{p}+\vec{c}-\vec{p}+\vec{d}-\vec{p}$$

$$= \left(\vec{a}+\vec{b}+\vec{c}+\vec{d}\right)-4\vec{p}$$

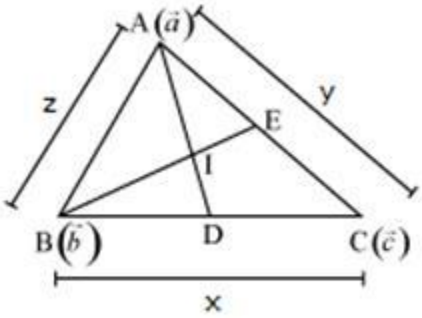
$$= 4\left(\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}-\vec{p}\right)$$

$$= 4\overrightarrow{PQ}$$

Question 6

Prove by vector method that the internal bisectors of the angles of a triangle are concurrent.

Solution 6



Let $A(\vec{a}), B(\vec{b})$ and $C(\vec{c})$ be the position vectors of the vertices of the ΔABC and the lengths of sides BC, CA and AB be x, y and z respectively.

The internal bisector of a triangle divides the opposite side in the ratio of the sides containing the angles.
Since AD is the internal bisector of $\angle ABC$,

$$\therefore \frac{BD}{DC} = \frac{AB}{AC} = \frac{z}{y} \ldots\ldots\ldots(1)$$

Position vector of D $= \dfrac{z\vec{c}+y\vec{b}}{z+y}$

Let the internal bisectors intersect at I.

$$\therefore \frac{ID}{AI} = \frac{BD}{AB} \ldots\ldots\ldots(2)$$

$$\frac{BD}{DC} = \frac{z}{y} \ldots\ldots [\text{from } (1)]$$

$$\therefore \frac{CD}{BD} = \frac{y}{z}$$

$$\Rightarrow \frac{BC}{BD} = \frac{y+z}{z}$$

$$\Rightarrow BD = \frac{xz}{y+z} \ldots\ldots(3)$$

From (2) and (3) we get

$$\frac{ID}{AI} = \frac{xz}{(y+z)z} = \frac{x}{y+z}$$

$$\therefore \text{Position vector of I } = \frac{\left(\frac{z\vec{c}+y\vec{b}}{z+y}\right)(y+z)x\vec{a}}{x+y+z} = \frac{x\vec{a}+y\vec{b}+z\vec{c}}{x+y+z}$$

Simillarly we can prove that I lie on internal bisector of angles B and C.
Hence three bisectros are concurrent.

Chapter 23 - Algebra of Vectors Exercise Ex. 23.5

Question 1

If the position vector of a point $(-4, -3)$ be $\vec{a}$, find $|\vec{a}|$.

Solution 1

Here $\vec{a} = -4\hat{i} - 3\hat{j}$

$$|\vec{a}| = \sqrt{(-4)^2 + (-3)^2}$$
$$= \sqrt{16 + 9}$$
$$= \sqrt{25}$$
$$= 5$$

$\therefore |\vec{a}| = 5$

Question 2

If the position vector $\vec{a}$ of a point $(12, n)$ is such that $|\vec{a}| = 13$, find the value(s) of n.

Solution 2

Here $\vec{a} = 12\hat{i} + n\hat{j}$

$$|\vec{a}| = \sqrt{(12)^2 + (n)^2}$$
$$13 = \sqrt{144 + n^2} \qquad \left[\text{Since } |\vec{a}| = 13\right]$$

Squaring both sides,

$$(13)^2 = \left(\sqrt{144 + n^2}\right)^2$$
$$169 = 144 + n^2$$
$$n^2 = 169 - 144$$
$$n^2 = 25$$
$$n = \pm\sqrt{25}$$
$$n = \pm 5$$

Question 3

Find a vector of magnitude 4 units which is parallel to the vector $\sqrt{3}\hat{i} + \hat{j}$.

Solution 3

Here, $\vec{a} = \sqrt{3}\hat{i} + \hat{j}$

Let $\vec{b}$ is any vector parallel to $\vec{a}$

So, $\vec{b} = \lambda\vec{a}$ (where λ is any scalar)

$$= \lambda\left(\sqrt{3}\hat{i} + \hat{j}\right)$$

$$\vec{b} = \lambda\sqrt{3}\hat{i} + \lambda\hat{j}$$

$$\left|\vec{b}\right| = \sqrt{\left(\lambda\sqrt{3}\right)^2 + \left(\lambda\right)^2}$$

$$= \sqrt{3\lambda^2 + \lambda^2}$$

$$= \sqrt{4\lambda^2}$$

$$\left|\vec{b}\right| = 2\lambda$$

$$4 = 2\lambda$$

$$\lambda = \frac{4}{2}$$

$$\lambda = 2$$

$$\therefore \vec{b} = \lambda\sqrt{3}\hat{i} + \lambda\hat{j}$$

$$\vec{b} = 2\sqrt{3}\hat{i} + 2\hat{j}$$

Question 4

Express $\overrightarrow{AB}$ in terms of unit vectors $\hat{i}$ and $\hat{j}$, when the points are:

(i) $A\left(4,-1\right)B\left(1,3\right)$ (ii) $A\left(-6,3\right)B\left(-2,-5\right)$

Solution 4

(i) Here, $A = (4, -1)$
$B = (1, 3)$

Position vector of $A = 4\hat{i} - \hat{j}$
Position vector of $B = \hat{i} + 3\hat{j}$

$$\begin{aligned}
\overrightarrow{AB} &= \text{Position vector of } B - \text{Position vector of } A \\
&= \left(\hat{i} + 3\hat{j}\right) - \left(4\hat{i} - \hat{j}\right) \\
&= \hat{i} + 3\hat{j} - 4\hat{i} + \hat{j} \\
\overrightarrow{AB} &= -3\hat{i} + 4\hat{j}
\end{aligned}$$

$$\begin{aligned}
\left|\overrightarrow{AB}\right| &= \sqrt{(-3)^2 + (4)^2} \\
&= \sqrt{9 + 16} \\
&= \sqrt{25}
\end{aligned}$$

$$\left|\overrightarrow{AB}\right| = 5$$
$$\overrightarrow{AB} = -3\hat{i} + 4\hat{j}$$

(ii) Here, $A = (-6, 3)$
$B = (-2, -5)$

Position vector of $A = -6\hat{i} + 3\hat{j}$
Position vector of $B = -2\hat{i} - 5\hat{j}$

$$\begin{aligned}
\overrightarrow{AB} &= \text{Position vector of } B - \text{Position vector of } A \\
&= \left(-2\hat{i} - 5\hat{j}\right) - \left(-6\hat{i} + 3\hat{j}\right) \\
&= -2\hat{i} - 5\hat{j} + 6\hat{i} - 3\hat{j} \\
\overrightarrow{AB} &= 4\hat{i} - 8\hat{j}
\end{aligned}$$

$$\begin{aligned}
\left|\overrightarrow{AB}\right| &= \sqrt{(4)^2 + (-8)^2} \\
&= \sqrt{16 + 64} \\
&= \sqrt{80} \\
&= \sqrt{16 \times 5} \\
&= 4\sqrt{5} \\
\left|\overrightarrow{AB}\right| &= 4\sqrt{5} \\
\overrightarrow{AB} &= 4\hat{i} - 8\hat{j}
\end{aligned}$$

Question 5

Find the coordinates of the tip of the position vector which is equivalent to $\overrightarrow{AB}$, where the coordinates of A and B are $(-1,3)$ and $(-2,1)$ respectively.

Solution 5

Here, $A = (-1,3)$
$B = (-2,1)$

Position vector of $A = -\hat{i} + 3\hat{j}$
Position vector of $B = -2\hat{i} + 1\hat{j}$

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$$= \left(-2\hat{i} + \hat{j}\right) - \left(-\hat{i} + 3\hat{j}\right)$$
$$= -2\hat{i} + \hat{j} + \hat{i} - 3\hat{j}$$
$$= -\hat{i} - 2\hat{j}$$

So,
Coordinate of the position vector equivalent to $\overrightarrow{AB} = (-1,-2)$

Question 6

$ABCD$ is a parallelogram. If the coordinates of A, B, C are $(-2,-1)(3,0)$ and $(1,-2)$ respectively find the coordinates of D.

Solution 6

Here, $A = (-2, -1)$

$B = (3, 0)$

$C = (1, -2)$

Let $D = (x, y)$

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= \left(3\hat{i} - 0 \times \hat{j}\right) - \left(-2\hat{i} - \hat{j}\right)$

$= 3\hat{i} - 0 \times \hat{j} + 2\hat{i} + \hat{j}$

$\overrightarrow{AB} = 5\hat{i} + \hat{j}$

$\overrightarrow{DC}$ = Position vector of C – Position vector of D

$= \left(\hat{i} - 2\hat{j}\right) - \left(x\hat{i} + y\hat{j}\right)$

$= \hat{i} - 2\hat{j} - x\hat{i} - y\hat{j}$

$\overrightarrow{DC} = (1 - x)\hat{i} + (-2 - y)\hat{j}$

Since $ABCD$ is a parallelogram, which have equal and parallel opposite sides.

So, $\overrightarrow{AB} = \overrightarrow{DC}$

$5\hat{i} + \hat{j} = (1 - x)\hat{i} + (-2 - y)\hat{j}$

Comparing components of LHS and RHS

$5 = 1 - x$

$x = 1 - 5$

$x = -4$

$1 = -2 - y$

$y = -2 - 1$

$y = -3$

So, coordinate of D is $(-4, -3)$

Question 7

If the position vectors of the points $A(3, 4)$, $B(5, -6)$ and $C(4, -1)$ are $\vec{a}, \vec{b}, \vec{c}$ respectively, compute $\vec{a} + 2\vec{b} - 3\vec{c}$.

Solution 7

Here, $A(3,4), B(5,-6), C(4,-1)$

$\vec{a} = 3\hat{i} + 4\hat{j}$
$\vec{b} = 5\hat{i} - 6\hat{j}$
$\vec{c} = 4\hat{i} - \hat{j}$

Now,

$$\begin{aligned}\vec{a} + 2\vec{b} - 3\vec{c} &= \left(3\hat{i} + 4\hat{j}\right) + 2\left(5\hat{i} - 6\hat{j}\right) - 3\left(4\hat{i} - \hat{j}\right) \\ &= 3\hat{i} + 4\hat{j} + 10\hat{i} - 12\hat{j} - 12\hat{i} + 3\hat{j} \\ &= \hat{i} - 5\hat{j}\end{aligned}$$

$\therefore\ \vec{a} + 2\vec{b} - 3\vec{c} = \hat{i} - 5\hat{j}$

Question 8

If $\vec{a}$ be the position vector whose tip is $(5,-3)$, find the coordinates of a point B such that $\overrightarrow{AB} = \vec{a}$, the coordinates of A being $(4,-1)$.

Solution 8

Here, we have

Coordinate of $A = (4,-1)$

Position vector of $A = 4\hat{i} - \hat{j}$

Position vector of $\vec{a} = 5\hat{i} - 3\hat{j}$

Let coordinate of point $B = (x, y)$

Position vector of $B = x\hat{i} + y\hat{j}$

Given that, $\overrightarrow{AB} = \vec{a}$

Position vector of B – Position vector of $A = \vec{a}$

$$\left(x\hat{i} + y\hat{j}\right) - \left(4\hat{i} - \hat{j}\right) = 5\hat{i} - 3\hat{j}$$

$$(x - 4)\hat{i} + (y + 1)\hat{j} = 5\hat{i} - 3\hat{j}$$

Comparing the coefficients of LHS and RHS

$x - y = 5$

$x = 5 + 4$

$x = 9$

$y + 1 = 3$

$y = -3 - 1$

$y = -4$

So, coordinate of $B = (9, -4)$

Question 9

Show that the points $2\hat{i}, -\hat{i} - 4\hat{j}$ and $-\hat{i} + 4j$ form an isosceles triangle.

Solution 9

$$\left|\overrightarrow{AB}\right| = 5 \text{ units}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{(8)^2}$$

$$\left|\overrightarrow{BC}\right| = 8 \text{ units}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{(-3)^2 + (8)^2}$$

$$= \sqrt{9 + 16}$$

$$= \sqrt{25}$$

$$\left|\overrightarrow{AC}\right| = 5 \text{ units}$$

Here, $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AC}\right|$

$5 = 5$

So, there are two sides AB, and BC of ΔABC have same length.

ΔABC is an isosceles triangle.

Question 10

Find a unit vector parallel to the vector $\vec{i} + \sqrt{3}\,\vec{j}$.

Solution 10

Let $\vec{a} = \hat{i} + \sqrt{3}\hat{j}$

Suppose $\vec{b}$ is any vector parallel to $\vec{a}$

$\vec{b} = \lambda\vec{a}$ where λ is a scalar

$= \lambda\left(\hat{i} + \sqrt{3}\hat{j}\right)$

$\vec{b} = \lambda\hat{i} + \sqrt{3}\lambda\hat{j}$

$$\left|\vec{b}\right| = \sqrt{(\lambda)^2 + \left(\sqrt{3}\lambda\right)^2}$$
$$= \sqrt{\lambda^2 + 3\lambda^2}$$
$$= \sqrt{4\lambda^2}$$
$$= 2\lambda$$

Unit vector of $\vec{b} = \dfrac{\vec{b}}{\left|\vec{b}\right|}$

$$\hat{b} = \frac{\lambda\hat{i} + \sqrt{3}\lambda\hat{j}}{2\lambda}$$

$$\hat{b} = \frac{\left(\hat{i} + \sqrt{3}\ \hat{j}\right)}{2}$$

$$\hat{b} = \frac{1}{2}\left(\hat{i} + \sqrt{3}\ \hat{j}\right)$$

$$\Rightarrow \hat{b} = \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$$

Question 11

Find the components along the coordinate axes of the position vector of each of the following points:

(i) $P\left(3,2\right)$ (ii) $Q\left(-5,1\right)$ (iii) $R\left(-11,-9\right)$ (iv) $S\left(4,-3\right)$

Solution 11

(i) Here, $P = (3, 2)$

Position vector of $P = 3\hat{i} + 2\hat{j}$

Component of P along x-axis $= 3\hat{i}$

Component of P along y-axis $= 2\hat{j}$

(ii) Here, $Q = (-5, 1)$

Position vector of $Q = -5\hat{i} + \hat{j}$

Component of Q along x-axis $= -5\hat{i}$

Component of Q along y-axis $= \hat{j}$

(ii) Here, $R = (-11, -9)$

Position vector of $R = -11\hat{i} - 9\hat{j}$

Component of R along x-axis $= -11\hat{i}$

Component of R along y-axis $= -9\hat{j}$

(iv) Here, $S = (4, -3)$

Position vector of $S = 4\hat{i} - 3\hat{j}$

Component of S along x-axis $= 4\hat{i}$

Component of S along y-axis $= -3\hat{j}$

Chapter 23 - Algebra of Vectors Exercise Ex. 23.6

Question 1

Find the magnitude of the vector $\vec{a} = 2\hat{i} + 3\hat{j} - 6\hat{k}$.

Solution 1

Magnitude of a vector $x\hat{i} + y\hat{j} + z\hat{k}$ is given by $\sqrt{(x)^2 + y^2 + z^2}$.

So,

$$\begin{aligned} |\vec{a}| &= \sqrt{(2)^2 + (3)^2 + (-6)^2} \\ &= \sqrt{4 + 9 + 36} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

$\therefore |\vec{a}| = 7$

Question 2

Find the unit vector in the direction of $3\hat{i} + 4\hat{j} - 12\hat{k}$.

Solution 2

Unit vector of $\vec{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$$\text{Unit vector of } 3\hat{i} + 4\hat{j} - 12\hat{k} = \frac{3\hat{i} + 4\hat{j} - 12\hat{k}}{\sqrt{(3)^2 + (4)^2 + (-12)^2}}$$

$$= \frac{3\hat{i} + 4\hat{j} - 12\hat{k}}{\sqrt{9 + 16 + 144}}$$

$$= \frac{3\hat{i} + 4\hat{j} - 12\hat{k}}{\sqrt{169}}$$

$$\text{Unit vector of } \left(3\hat{i} + 4\hat{j} - 12\hat{k}\right) = \frac{1}{13}\left(3\hat{i} + 4\hat{j} - 12\hat{k}\right)$$

$$\Rightarrow \text{Unit Vector of } (3\hat{i} + 4\hat{j} - 12\hat{k}) = \frac{3}{13}\hat{i} + \frac{4}{13}\hat{j} - \frac{12}{13}\hat{k}$$

Question 3

Find a unit vector in the direction of the resultant of the vectors

$\hat{i} - \hat{j} + 3\hat{k}$, $2\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} + 2\hat{j} - 2\hat{k}$

Solution 3

Let $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$

$\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

Let d be the resultant of $\vec{a}, \vec{b},$ and $\vec{c}$,

$$\vec{d} = \vec{a} + \vec{b} + \vec{c}$$

$$= \left(\hat{i} - \hat{j} + 3\hat{k}\right) + \left(2\hat{i} + \hat{j} - 2\hat{k}\right) + \left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$$

$$\vec{d} = 4\hat{i} + 2\hat{j} - \hat{k}$$

$$\text{Unit vector } \vec{d} = \frac{\vec{d}}{|\vec{d}|}$$

$$= \frac{4\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{(4)^2 + (2)^2 + (-1)^2}}$$

$$\vec{d} = \frac{4\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{16 + 4 + 1}}$$

$$\Rightarrow \vec{d} = \frac{1}{\sqrt{21}}(4\hat{i} + 2\hat{j} - \hat{k})$$

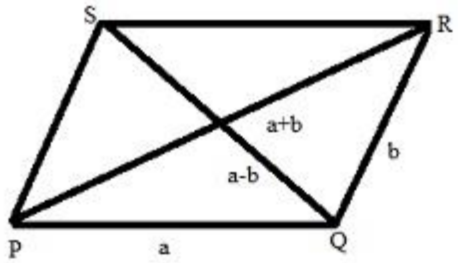
Question 4

The adjacent sides of a parallelogram are represented by the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} + 2\hat{k}$. Find unit vectors parallel to t5he diagonals of the parallelogram.

Solution 4

Let $PQRS$ be a parallelogram such that $PQ = \vec{a} = \hat{i} + \hat{j} - \hat{k}$ and

$QR = \vec{b} = -2\hat{i} + \hat{j} + 2\hat{k}$



In $\triangle PQR$

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

$$\overrightarrow{PR} = \vec{a} + \vec{b} = \hat{i} + \hat{j} - \hat{k} + \left(-2\hat{i} + \hat{j} + 2\hat{k}\right)$$

$$\overrightarrow{PR} = -\hat{i} + 2\hat{j} + \hat{k}$$

In $\triangle PSQ$

$$\overrightarrow{PS} + \overrightarrow{SQ} = \overrightarrow{PQ}$$

$$\overrightarrow{SQ} = \vec{a} - \vec{b} = \hat{i} + \hat{j} - \hat{k} - \left(-2\hat{i} + \hat{j} + 2\hat{k}\right)$$

$$\overrightarrow{SQ} = 3\hat{i} + 0\hat{j} - 3\hat{k}$$

The unit vector along $\overrightarrow{PR} = \dfrac{\overrightarrow{PR}}{\left|\overrightarrow{PR}\right|} = \dfrac{-\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{1+4+1}} = \dfrac{1}{\sqrt{6}}\left(-\hat{i} + 2\hat{j} + \hat{k}\right)$

The unit vector along $\overrightarrow{SQ} = \dfrac{\overrightarrow{SQ}}{\left|\overrightarrow{SQ}\right|} = \dfrac{3\hat{i} + 0\hat{j} - 3\hat{k}}{\sqrt{9+0+9}} = \dfrac{1}{\sqrt{2}}\left(\hat{i} - \hat{k}\right)$

Question 5

If $\vec{a} = 3\hat{i} - \hat{j} - 4\hat{k}, \vec{b} = -2\hat{i} + 4\hat{j} - 3\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$, find $\left|3\vec{a} - 2\vec{b} + 4\vec{c}\right|$.

Solution 5

$$3\vec{a} - 2\vec{b} + 4\vec{c} = 3\left(3\hat{i} - \hat{j} - 4\hat{k}\right) - 2\left(-2\hat{i} + 4\hat{j} - 3\hat{k}\right) + 4\left(\hat{i} + 2\hat{j} - \hat{k}\right)$$
$$= 9\hat{i} - 3\hat{j} - 12\hat{k} + 4\hat{i} - 8\hat{j} + 6\hat{k} + 4\hat{i} + 8\hat{j} - 4\hat{k}$$
$$= 17\hat{i} - 3\hat{j} - 10\hat{k}$$

$$\left|3\vec{a} - 2\vec{b} + 4\vec{c}\right| = \sqrt{(17)^2 + (-3)^2 + (10)^2}$$
$$= \sqrt{289 + 9 + 100}$$
$$= \sqrt{398}$$

$$\left|3\vec{a} - 2\vec{b} + 4\vec{c}\right| = \sqrt{398}$$

Question 6

If $\overrightarrow{PQ} = 3\hat{i} + 2\hat{j} - \hat{k}$ and the coordinates of P are $(1, -1, 2)$. Find the coordinates of Q.

Solution 6

Here, $\overrightarrow{PQ} = 3\hat{i} + 2\hat{j} - \hat{k}$

Position vector of $P = \hat{i} - \hat{j} + 2\hat{k}$

$\overrightarrow{PQ}$ = Position vector of Q – Position vector of P

$3\hat{i} + 2\hat{j} - \hat{k}$ = Position vector of $Q - \left(\hat{i} - \hat{j} + 2\hat{k}\right)$

Position vector of $Q = \left(3\hat{i} + 2\hat{j} - \hat{k}\right) + \left(\hat{i} - \hat{j} + 2\hat{k}\right)$

$$= 4\hat{i} + \hat{j} + \hat{k}$$

Coordinates of $Q = (4, 1, 1)$

Question 7

Prove that the points $\hat{i} - \hat{j}$, $4\hat{i} + 3\hat{j} + \hat{k}$ and $2\hat{i} - 4\hat{j} + 5\hat{k}$ are not the vertices of a right angled triangle.

Solution 7

Let $\vec{A} = \hat{i} - \hat{j}$
$\vec{B} = 4\hat{i} + 3\hat{j} + \hat{k}$
$\vec{C} = 2\hat{i} - 4\hat{j} + 5\hat{k}$

$$\begin{aligned}\overrightarrow{AB} &= \vec{B} - \vec{A} \\ &= \left(4\hat{i} + 3\hat{j} + \hat{k}\right) - \left(\hat{i} - \hat{j}\right) \\ &= 4\hat{i} + 3\hat{j} + \hat{k} - \hat{i} + \hat{j} \\ &= 3\hat{i} + 4\hat{j} + \hat{k}\end{aligned}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(3)^2 + (4)^2 + (1)^2} = \sqrt{9 + 16 + 1} = \sqrt{26}$$

$$\begin{aligned}\overrightarrow{BC} &= \vec{C} - \vec{B} \\ &= \left(2\hat{i} - 4\hat{j} + 5\hat{k}\right) - \left(4\hat{i} + 3\hat{j} + \hat{k}\right) \\ &= 2\hat{i} - 4\hat{j} + 5\hat{k} - 4\hat{i} - 3\hat{j} - \hat{k} \\ &= -2\hat{i} - 7\hat{j} + 4\hat{k}\end{aligned}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{(2)^2 + (-7)^2 + (4)^2} = \sqrt{4 + 49 + 16} = \sqrt{69}$$

$$\begin{aligned}\overrightarrow{CA} &= \vec{A} - \vec{C} \\ &= \hat{i} - \hat{j} - \left(2\hat{i} - 4\hat{j} + 5\hat{k}\right) \\ &= \hat{i} - \hat{j} - 2\hat{i} + 4\hat{j} - 5\hat{k} \\ &= -\hat{i} + 3\hat{j} - 5\hat{k}\end{aligned}$$

$$\left|\overrightarrow{CA}\right| = \sqrt{(-1)^2 + (3)^2 + (-5)^2} = \sqrt{1 + 9 + 25} = \sqrt{35}$$

Here, $\left|AB\right|^2 + \left|\overrightarrow{CA}\right|^2 = \left|\overrightarrow{BC}\right|^2$

$$\begin{aligned}26 + 35 &= 69 \\ 61 &\neq 69 \\ \text{LHS} &\neq \text{RHS}\end{aligned}$$

Since sum of square of two sides is not equal to the square of third sides. So, ΔABC is not a right triangle

Question 8

If the vertices of a triangle are the points with position vectors $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, what are the vectors determined by its sides? Find the length of these vectors.

Solution 8

Here,

Let vertex $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

vertex $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

vertex $\vec{C} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\text{Side } \overrightarrow{AB} = \vec{B} - \vec{A}$$
$$= \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) - \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$$
$$\overrightarrow{AB} = (b_1 - a_1)\,\hat{i} + (b_2 - a_2)\,\hat{j} + (b_3 - a_3)\,\hat{k}$$

$$\overrightarrow{BC} = \vec{C} - \vec{B}$$
$$= \left(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\right) - \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$
$$= c_1\hat{i} + c_2\hat{j} + c_3\hat{k} - b_1\hat{i} - b_2\hat{j} - b_3\hat{k}$$
$$\overrightarrow{BC} = (c_1 - b_1)\,\hat{i} + (c_2 - b_2)\,\hat{j} + (c_3 - b_3)\,\hat{k}$$

$$\overrightarrow{AC} = \left(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\right) - \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$$
$$= c_1\hat{i} + c_2\hat{j} + c_3\hat{k} - a_1\hat{i} - a_2\hat{j} - a_3\hat{k}$$
$$\overrightarrow{AC} = (c_1 - a_1)\,\hat{i} + (c_2 - a_2)\,\hat{j} + (c_3 - a_3)\,\hat{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{(c_1 - b_1)^2 + (c_2 - b_2)^2 + (c_3 - b_3)^2}$$
$$\left|\overrightarrow{AC}\right| = \sqrt{(c_1 - a_1)^2 + (c_2 - a_2)^2 + (c_3 - a_3)^2}$$

Question 9

Find the vector from the origin O to the centroid of the triangle whose vertices are $(1,-1,2)$, $(2,1,3)$ and $(-1,2,-1)$.

Solution 9

Here, given vertex $A = (1, -1, 2)$

$$\vec{A} = \hat{i} - \hat{j} + 2\hat{k}$$

vertex $B = (2, 1, 3)$

$$\vec{B} = 2\hat{i} + \hat{j} + 3\hat{k}$$

vertex $C = (-1, 2, -1)$

$$\vec{C} = -\hat{i} + 2\hat{j} - \hat{k}$$

$$\text{Centroid } \vec{O} = \frac{\vec{A} + \vec{B} + \vec{C}}{3}$$

$$= \frac{\left(\hat{i} - \hat{j} + 2\hat{k}\right) + \left(2\hat{i} + \hat{j} + 3\hat{k}\right) + \left(-\hat{i} + 2\hat{j} - \hat{k}\right)}{3}$$

$$= \frac{2\hat{i} + 2\hat{j} + 4\hat{k}}{3}$$

$$\text{Centroid } \vec{O} = \frac{2\hat{i}}{3} + \frac{2\hat{j}}{3} + \frac{4\hat{k}}{3}$$

Question 10

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ration 2: 1

(i) internally

(ii) externally

Solution 10

The position vector of point R dividing the line segment joining two points

P and Q in the ratio *m: n* is given by:

i. Internally:

$$\frac{m\vec{b}+n\vec{a}}{m+n}$$

ii. Externally:

$$\frac{m\vec{b}-n\vec{a}}{m-n}$$

Position vectors of P and Q are given as:

$$\overrightarrow{OP}=\hat{i}+2\hat{j}-\hat{k} \text{ and } \overrightarrow{OQ}=-\hat{i}+\hat{j}+\hat{k}$$

(i) The position vector of point R which divides the line joining two points P and Q internally in the ratio 2: 1 is given by,

$$\overrightarrow{OR}=\frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)+1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2+1}=\frac{\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)+\left(\hat{i}+2\hat{j}-\hat{k}\right)}{3}$$

$$=\frac{-\hat{i}+4\hat{j}+\hat{k}}{3}=-\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}$$

(ii) The position vector of point R which divides the line joining two points P and Q externally in the ratio 2: 1 is given by,

$$\overrightarrow{OR}=\frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)-1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2-1}=\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)-\left(\hat{i}+2\hat{j}-\hat{k}\right)$$

$$=-3\hat{i}+3\hat{k}$$

Question 11

Find the position vector of the mid-point of the vector joining the points $P\left(2\hat{i}-3\hat{j}+4\hat{k}\right)$, and $Q\left(4\hat{i}+\hat{j}-2\hat{k}\right)$.

Solution 11

Here, $P\left(2\hat{i}-3\hat{j}+4\hat{k}\right)$ and

$Q\left(4\hat{i}+\hat{j}-2\hat{k}\right)$

We know that,

If A and B are two points with position vector $\vec{a}$ and $\vec{b}$ then the position vector of mid point C is given by

$$\frac{\vec{a}+\vec{b}}{2}$$

Let R is the mid point of PQ.

Position vector of R $=\dfrac{\vec{P}+\vec{Q}}{2}$

$$\vec{R}=\frac{2\hat{i}-3\hat{j}+4\hat{k}+4\hat{i}+\hat{j}-2\hat{k}}{2}$$

$$=\frac{6\hat{i}-2\hat{j}+2\hat{k}}{2}$$

$$=\frac{2\left(3\hat{i}-\hat{j}+\hat{k}\right)}{2}$$

Position vector of mid point $=3\hat{i}-\hat{j}+\hat{k}$

Question 12

Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points $(1,2,3)$ and $(4,5,6)$.

Solution 12

Here, point $P = (1,2,3)$

$$\vec{P} = \hat{i} + 2\hat{j} + 3\hat{k}$$

Point $Q = (4,5,6)$

$$\vec{Q} = 4\hat{i} + 5\hat{j} + 6\hat{k}$$

$$\begin{aligned}\overrightarrow{PQ} &= \text{Position vector of } Q - \text{Position vector of } P\\ &= \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)\\ &= 4\hat{i} + 5\hat{j} + 6\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k}\\ &= 3\hat{i} + 3\hat{j} + 3\hat{k}\\ &= 3\left(i + \hat{j} + k\right)\end{aligned}$$

$$\begin{aligned}\left|\overrightarrow{PQ}\right| &= 3\sqrt{(1)^2 + (1)^2 + (1)^2}\\ &= 3\sqrt{1+1+1}\\ \left|\overrightarrow{PQ}\right| &= 3\sqrt{3}\end{aligned}$$

$$\begin{aligned}\text{Unit vector in the direction of } \overrightarrow{PQ} &= \frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|}\\ &= \frac{3\left(\hat{i} + \hat{j} + \hat{k}\right)}{3\sqrt{3}}\end{aligned}$$

$$\text{Unit vector is the direction of } \overrightarrow{PQ} = \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

Question 13

Show that the points $A\left(2\hat{i} - \hat{j} + \hat{k}\right), B\left(\hat{i} - 3\hat{j} - 5\hat{k}\right), C\left(3\hat{i} - 4\hat{j} - 4\hat{k}\right)$ are the vertices of a right angle triangle.

Solution 13

The position vectors of A, B and C are $2\hat{i}-\hat{j}+\hat{k}, \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively.
Therefore,
$\overrightarrow{AB}=-\hat{i}-2\hat{j}-6\hat{k}, \overrightarrow{BC}=2\hat{i}-\hat{j}+\hat{k}$ and $\overrightarrow{CA}=-\hat{i}+3\hat{j}+5\hat{k}$
Clearly, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$.
So, $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ form a triangle.
Now

$$\left|\overrightarrow{AB}\right|=\sqrt{1+4+36}=\sqrt{41}$$

$$\left|\overrightarrow{BC}\right|=\sqrt{4+1+1}=\sqrt{6}$$

$$\left|\overrightarrow{CA}\right|=\sqrt{1+9+25}=\sqrt{35}$$

Clearly,

$$\left|\overrightarrow{AB}\right|^2=\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{CA}\right|^2$$

$$AB^2=BC^2+CA^2$$

Hence ΔABC is a right triangle right angle at C.

Question 14

Find the position vector of the mid point of the vector joining the points P (2, 3, 4) and Q (4, 1, − 2).

Solution 14

Find the position vector of the mid point of the vector joining the points P (2, 3, 4) and Q (4, 1, − 2).

Solution 16:

The position vector of mid-point R of the vector joining points P (2, 3, 4) and Q (4, 1, − 2) is given by,

$$\overrightarrow{\text{OR}}=\frac{\left(2\hat{i}+3\hat{j}+4\hat{k}\right)+\left(4\hat{i}+\hat{j}-2\hat{k}\right)}{2}=\frac{(2+4)\hat{i}+(3+1)\hat{j}+(4-2)\hat{k}}{2}$$

$$=\frac{6\hat{i}+4\hat{j}+2\hat{k}}{2}=3\hat{i}+2\hat{j}+\hat{k}$$

Question 15

Find the value of x for which $x\left(\hat{i}+\hat{j}+\hat{k}\right)$ is a unit vectors.

Solution 15

$x\left(\hat{i}+\hat{j}+\hat{k}\right)$ is a unit vector if $\left|x\left(\hat{i}+\hat{j}+\hat{k}\right)\right|=1$.

Now,

$$\left|x\left(\hat{i}+\hat{j}+\hat{k}\right)\right|=1$$
$$\Rightarrow \sqrt{x^2+x^2+x^2}=1$$
$$\Rightarrow \sqrt{3x^2}=1$$
$$\Rightarrow \sqrt{3}\,x=1$$
$$\Rightarrow x=\pm\frac{1}{\sqrt{3}}$$

Hence, the required value of x is $\pm\frac{1}{\sqrt{3}}$.

Question 16

If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$, find a unit vector parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

Solution 16

We have,

$$\vec{a}=\hat{i}+\hat{j}+\hat{k},\ \vec{b}=2\hat{i}-\hat{j}+3\hat{k} \text{ and } \vec{c}=\hat{i}-2\hat{j}+\hat{k}$$

$$\begin{aligned}2\vec{a}-\vec{b}+3\vec{c}&=2\left(\hat{i}+\hat{j}+\hat{k}\right)-\left(2\hat{i}-\hat{j}+3\hat{k}\right)+3\left(\hat{i}-2\hat{j}+\hat{k}\right)\\&=2\hat{i}+2\hat{j}+2\hat{k}-2\hat{i}+\hat{j}-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k}\\&=3\hat{i}-3\hat{j}+2\hat{k}\end{aligned}$$

$$\left|2\vec{a}-\vec{b}+3\vec{c}\right|=\sqrt{3^2+\left(-3\right)^2+2^2}=\sqrt{9+9+4}=\sqrt{22}$$

Hence, the unit vector along $2\vec{a}-\vec{b}+3\vec{c}$ is

$$\frac{2\vec{a}-\vec{b}+3\vec{c}}{\left|2\vec{a}-\vec{b}+3\vec{c}\right|}=\frac{3\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{22}}=\frac{3}{\sqrt{22}}\hat{i}-\frac{3}{\sqrt{22}}\hat{j}+\frac{2}{\sqrt{22}}\hat{k}.$$

Question 17

If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=4\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$, find a vector of magnitude 6 units which is parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

Solution 17

Here, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$

$2\vec{a} - \vec{b} + 3\vec{c}$

$$= 2\left(\hat{i} + \hat{j} + \hat{k}\right) - \left(4\hat{i} - 2\hat{j} + 3\hat{k}\right) + 3\left(\hat{i} - 2\hat{j} + \hat{k}\right)$$

$$= 2\hat{i} + 2\hat{j} + 2\hat{k} - 4\hat{i} + 2\hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k}$$

$$= \hat{i} - 2\hat{j} + 2\hat{k}$$

Let $\vec{d}$ is a vector parallel to $2\vec{a} - \vec{b} + 3\vec{c}$

So, $\vec{d} = \lambda\left(2\vec{a} - \vec{b} + 3\vec{c}\right)$

Where λ is any scalar

$$= \lambda\left(\hat{i} - 2\hat{j} + 2\hat{k}\right)$$

$$\vec{d} = \lambda\hat{i} - \lambda 2\hat{j} + \lambda 2\hat{k} \qquad (i)$$

Given that $\left|\vec{d}\right| = 6$

$$\sqrt{(\lambda)^2 + (-2\lambda)^2 + (2\lambda)^2} = 6$$

$$\sqrt{\lambda^2 + 4\lambda^2 + 4\lambda^2} = 6$$

$$\sqrt{9\lambda^2} = 6$$

$$3\lambda = 6$$

$$\lambda = \frac{6}{3}$$

$$\lambda = 2$$

Put the value of λ in equation (i)

$$\vec{d} = 2\hat{i} - 2(2)\hat{j} + 2(2)\hat{k}$$

$$= 2\hat{i} - 4\hat{j} + 4\hat{k}$$

A vector of magnitude 6 which is parallel to $2\vec{a} - \vec{b} + 3\vec{c}$ is given by

$2\hat{i} - 4\hat{j} + 4\hat{k}$

Question 18

Find a vector of magnitude of 5 units parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

Solution 18

Given that

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$

and

$\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

Thus, Find a vector of magnitude of 5 *units parallel to the resultant of the vectors*

$\vec{a} + \vec{b} = 2\hat{i} + 3\hat{j} - \hat{k} + \hat{i} - 2\hat{j} + \hat{k}$

$\Rightarrow \vec{a} + \vec{b} = 3\hat{i} + \hat{j}$

$\Rightarrow \left|\vec{a} + \vec{b}\right| = \sqrt{9+1} = \sqrt{10}$

Thus, the unit vector along the resultant vector $\vec{a} + \vec{b}$ *is*

$$\frac{3\hat{i} + \hat{j}}{\sqrt{10}}$$

The vector of magnitude of 5 *units parallel to the resultant*

$$vector = \frac{3\hat{i} + \hat{j}}{\sqrt{10}} \times 5 = \sqrt{\frac{5}{2}}\left(3\hat{i} + \hat{j}\right)$$

Question 19

The two vectors $\hat{j} + \hat{i}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represent the sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively of triangle ABC. Find the lenght of the median through A.

Solution 19

Let D be the point at which median drawn from A touches side BC.

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of the vertices A, B and C.

Position vector of D $= \dfrac{\vec{b} + \vec{c}}{2}$..........[Since D is midpoint of B and C]

$$\overrightarrow{AD} = \frac{\vec{b} + \vec{c}}{2} - \vec{a} = \frac{\vec{b} + \vec{c} - 2\vec{a}}{2} = \frac{\vec{b} - \vec{a} + \vec{c} - \vec{a}}{2} = \frac{\overrightarrow{AB} + \overrightarrow{AC}}{2} = \frac{\hat{j} + \hat{i} + 3\hat{i} - \hat{j} + 4\hat{k}}{2}$$

$\overrightarrow{AD} = 2\,\hat{i} + 2\hat{k}$

$\left|\overrightarrow{AD}\right| = \sqrt{4+4} = 4\sqrt{2}$ units

Note : Answer given in the book is incorrect.

Chapter 23 - Algebra of Vectors Exercise Ex. 23.7

Question 1

Show that the points A, B, C with position vectors $\vec{a} - 2\vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$ and $-7\vec{b} + 10\vec{c}$ are collinear.

Solution 1

Here, position vector of A = Position vector of $A = \vec{a} - 2\vec{b} + 3\vec{c}$

position vector of B = Position vector of $B = 2\vec{a} + 3\vec{b} - 4\vec{c}$

position vector of C = Position vector of $C = -7\vec{b} + 10\vec{c}$

$\overrightarrow{AB}$ = position vector of B – position vector of A

$$= \left(2\vec{a} + 3\vec{b} - 4\vec{c}\right) - \left(\vec{a} - 2\vec{b} + 3\vec{c}\right)$$

$$= 2\vec{a} + 3\vec{b} - 4\vec{c} - \vec{a} + 2\vec{b} - 3\vec{c}$$

$$\overrightarrow{AB} = \vec{a} + 5\vec{b} - 7\vec{c}$$

$\overrightarrow{BC}$ = position vector of C – position vector of B

$$= \left(-7\vec{b} + 10\vec{c}\right) - \left(2\vec{a} + 3\vec{b} - 4\vec{c}\right)$$

$$= -7\vec{b} + 10\vec{c} - 2\vec{a} - 3\vec{b} + 4\vec{c}$$

$$\overrightarrow{BC} = -2\vec{a} - 10\vec{b} + 14\vec{c}$$

From $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we get

$$\overrightarrow{BC} = -2\left(\overrightarrow{AB}\right)$$

So, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel but $\vec{B}$ is a common vector. Hence, A, B, C are collinear.

Question 2 (i)

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, prove that the points having the position vectors $\vec{a}$, $\vec{b}$, $3\vec{a} - 2\vec{b}$ are collinear.

Solution 2 (i)

Let the points be A, B, C

Position vector of $A = \vec{a}$
Position vector of $B = \vec{b}$
Position vector of $C = 3\vec{a} - 2\vec{b}$

$\overrightarrow{AB}$ = Position vector of B – Position vector of A
$= \vec{b} - \vec{a}$

$\overrightarrow{BC}$ = Position vector of C – Position vector of B
$= 3\vec{a} - 2\vec{b} - \vec{b}$
$= 3\vec{a} - 3\vec{b}$

Using $\overrightarrow{AB}$ and $\overrightarrow{BC}$

Let $\overrightarrow{BC} = \lambda\left(\overrightarrow{AB}\right)$ [where λ is and scalar]

$3\vec{a} - 3\vec{b} = \lambda\left(\vec{b} - \vec{a}\right)$
$3\vec{a} - 3\vec{b} = \lambda\vec{b} - \lambda\vec{a}$
$3\vec{a} - 3\vec{b} = \lambda\vec{a} + \lambda\vec{b}$

Comparing the coefficients of LHS and RHS, we get
$-\lambda = 3$
$\lambda = 3$
$\lambda = -3$

Since the value of λ are different.
Therefore,
A, B, C are not collinear.

Question 2 (ii)

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, prove that the points having the position vectors $\vec{a} + \vec{b} + \vec{c}$, $4\vec{a} + 3\vec{b}$, $10\vec{a} + 7\vec{b} - 2\vec{c}$ are collinear.

Solution 2 (ii)

Let the points be A, B, C

Position vector of $A = \vec{a} + \vec{b} + \vec{c}$
Position vector of $B = 4\vec{a} + 3\vec{b}$
Position vector of $C = 10\vec{a} + 7\vec{b} - 2\vec{c}$

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= \left(4\vec{a} + 3\vec{b}\right) - \left(\vec{a} + \vec{b} + \vec{c}\right)$

$= 4\vec{a} + 3\vec{b} - \vec{a} - \vec{b} - \vec{c}$

$\overrightarrow{AB} = 3\vec{a} + 2\vec{b} - \vec{c}$

$\overrightarrow{BC}$ = Position vector of C – Position vector of B

$= \left(10\vec{a} + 7\vec{b} - 2\vec{c}\right) - \left(4\vec{a} + 3\vec{b}\right)$

$= 10\vec{a} + 7\vec{b} - 2\vec{c} - 4\vec{a} - 3\vec{b}$

$\overrightarrow{BC} = 6\vec{a} + 4\vec{b} - 2\vec{c}$

Using $\overrightarrow{AB}$ and $\overrightarrow{BC}$

$\overrightarrow{BC} = 2\left(\overrightarrow{AB}\right)$

So, $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ but $\vec{B}$ is a common vector. Hence,
A,B,C are collinear.

Question 3

Prove that the points having position vectors $\hat{i} + 2\hat{j} + 3\hat{k}$, $3\hat{i} + 4\hat{j} + 7\hat{k}$, $-3\hat{i} - 2\hat{j} - 5\hat{k}$ are collinear.

Solution 3

Let the points be A, B, C

Position vector of $A = \hat{i} + 2\hat{j} + 3\hat{k}$

Position vector of $B = 3\hat{i} + 4\hat{j} + 7\hat{k}$

Position vector of $C = -3\hat{i} - 2\hat{j} - 5\hat{k}$

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= \left(3\hat{i} + 4\hat{j} + 7\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$

$= 3\hat{i} + 4\hat{j} + 7\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k}$

$\overrightarrow{AB} = 2\hat{i} + 2\hat{j} + 4\hat{k}$

$\overrightarrow{BC}$ = Position vector of C – Position vector of B

$= \left(-3\hat{i} - 2\hat{j} - 5\hat{k}\right) - \left(3\hat{i} + 4\hat{j} + 7\hat{k}\right)$

$= -3\hat{i} - 2\hat{j} - 5\hat{k} - 3\hat{i} - 4\hat{j} - 7\hat{k}$

$\overrightarrow{BC} = -6\hat{i} - 6\hat{j} - 12\hat{k}$

Using $\overrightarrow{AB}$ and $\overrightarrow{BC}$ we get

$\overrightarrow{BC} = -3\left(\overrightarrow{AB}\right)$

So, $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ but $\vec{B}$ is a common vector. Hence,
A, B, C are collinear.

Question 4

If the points with position vectors $10\hat{i} + 3\hat{j}$, $12\hat{i} - 5\hat{j}$ and $a\hat{i} + 11\hat{j}$ are collinear, find the value of a.

Solution 4

Let the points be A, B, C

Position vector of $A = 10\hat{i} + 3\hat{j}$

Position vector of $B = 12\hat{i} - 5\hat{j}$

Position vector of $C = a\hat{i} + 11\hat{j}$

Given that, A,B,C are collinear

$\Rightarrow \overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear

$\Rightarrow \overrightarrow{AB} = \lambda\left(\overrightarrow{BC}\right)$ (Where λ is same scalar)

$\Rightarrow$ Position vector of B – Position vector of A = λ – (Position vector of C – Position vector of B)

$$\Rightarrow \quad \left(12\hat{i} - 5\hat{j}\right) - \left(10\hat{i} + 3\hat{j}\right) = \lambda\left[\left(a\hat{i} + 11\hat{j}\right) - \left(12\hat{i} - 5\hat{j}\right)\right]$$

$$\Rightarrow \quad 12\hat{i} - 5\hat{j} - 10\hat{i} - 3\hat{j} = \lambda\left(a\hat{i} + 11\hat{j} - 12\hat{i} + 5\hat{j}\right)$$

$$\Rightarrow \quad 2\hat{i} - 8\hat{j} = \left(\lambda a - 12\lambda\right)\hat{i} = \left(11\lambda + 5\lambda\right)\hat{j}$$

Comparing the coefficients of LHS and RHS, we get

$\lambda a - 12\lambda = 2$ (i)

$-8 = 11\lambda + 5\lambda$ (ii)

$-8 = 16\lambda$

$\lambda = \frac{-8}{16}$

$\lambda = -\frac{1}{2}$

Put the value of λ in equation (i),

$$\lambda a - 12\lambda = 2$$

$$\left(-\frac{1}{2}\right)a - 12\left(-\frac{1}{2}\right) = 2$$

$$-\frac{1}{2}a + \frac{12}{2} = 2$$

$$-\frac{1}{2}a + 6 = 2$$

$$-\frac{1}{2}a = 2 - 6$$

$$-\frac{1}{2}a = -4$$

$$a = (-4) \times (-2)$$

$$a = 8$$

Question 5

If $\vec{a}, \vec{b}$ are two non-collinear vectors, prove that the points with position vectors $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$ and $\vec{a} + \lambda\vec{b}$ are collinear for all real values of λ.

Solution 5

Let A, B, C be the points then

Position vector of $A = \vec{a} + \vec{b}$

Position vector of $B = \vec{a} - \vec{b}$

Position vector of $C = \vec{a} + \lambda\vec{b}$

$$\overrightarrow{AB} = \text{Position vector of } B - \text{Position vector of } A$$
$$= \left(\vec{a} - \vec{b}\right) - \left(\vec{a} + \vec{b}\right)$$
$$= \vec{a} - \vec{b} - \vec{a} - \vec{b}$$
$$\overrightarrow{AB} = -2\vec{b}$$

$$\overrightarrow{BC} = \text{Position vector of } C - \text{Position vector of } B$$
$$= \left(\vec{a} + \lambda\vec{b}\right) - \left(\vec{a} - \vec{b}\right)$$
$$= \vec{a} + \lambda\vec{b} - \vec{a} + \vec{b}$$
$$= \lambda\vec{b} + \vec{b}$$
$$\overrightarrow{BC} = (\lambda + 1)\vec{b}$$

Using $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we get

$$\overrightarrow{AB} = \left[\frac{(\lambda+1)}{-2}\right]\left(\overrightarrow{BC}\right)$$

Let $\left(\frac{\lambda+1}{-2}\right) = \mu$

Since λ is a real number. So,
μ is also a real no.

So, $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, but $\vec{B}$ is a common vector. Hence,
A, B, C are collinear.

Question 6

If $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} + \overrightarrow{OC}$, prove that A, B, C are collinear points.

Solution 6

Here, $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} + \overrightarrow{OC}$
$$\overrightarrow{OA} - \overrightarrow{BO} = \overrightarrow{BO} - \overrightarrow{CO}$$
$$\overrightarrow{AB} = \overrightarrow{BC}$$

So, $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ but $\vec{B}$ is a common vector. Hence,
A, B, C are collinear.

Question 7

Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.

Solution 7

Let the given points be A and B

Position vector of $A = 2\hat{i} - 3\hat{j} + 4\hat{k}$

Position vector of $B = -4\hat{i} + 6\hat{j} - 8\hat{k}$

Let O be the initial point having postion vector

$$0 \times \hat{i} + 0 \times \hat{j} + 0 \times \hat{k}$$

$\overrightarrow{OA}$ = Position vector of A – Position vector of O

$$= \left(2\hat{i} - 3\hat{j} + 4\hat{k}\right) - \left(0 \times \hat{i} + 0 \times \hat{j} + 0 \times \hat{k}\right)$$

$$= 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$\overrightarrow{OB}$ = Position vector of B – Position vector of O

$$= \left(-4\hat{i} + 6\hat{j} - 8\hat{k}\right) - \left(0 \times \hat{i} + 0 \times \hat{j} + 0 \times \hat{k}\right)$$

$$\overrightarrow{OB} = -4\hat{i} + 6\hat{j} - 8\hat{k}$$

Using OA and OB, we get

$$\overrightarrow{OB} = -2\left(\overrightarrow{OA}\right)$$

Therefore, $\overrightarrow{OA}$ is parallel to $\overrightarrow{OB}$ but O is the common point to them. Hence, A and B are collinear.

Question 8

If the points $A(m, -1)$, $B(2, 1)$ $C(4, 5)$ are collinear, find the value of m.

Solution 8

Here, $A = (m, -1)$

$B = (2, 1)$

$C = (4, 5)$

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= \left(2\hat{i} + \hat{j}\right) - \left(m\hat{i} - \hat{j}\right)$

$= 2\hat{i} + \hat{j} - m\hat{i} + \hat{j}$

$= (2 - m)\hat{i} + 2\hat{j}$

$\overrightarrow{BC}$ = Position vector of C – Position vector of B

$= \left(4\hat{i} + 5\hat{j}\right) - \left(2\hat{i} + \hat{j}\right)$

$= 4\hat{i} + 5\hat{j} - 2\hat{i} - \hat{j}$

$\overrightarrow{BC} = 2\hat{i} + 4\hat{j}$

A, B, C are collinear. So, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear.

So, $\overrightarrow{AB} = \lambda\left(\overrightarrow{BC}\right)$

$(2 - m)\hat{i} + 2\hat{j} = \lambda\left(2\hat{i} + 4\hat{j}\right)$, for λ scalar

$(2 - m)\hat{i} + 2\hat{j} = 2\lambda i + 4\lambda\hat{j}$

Comparing the coefficient of LHS and RHS.

$2 - m = 2\lambda$

$\frac{2 - m}{2} = \lambda \qquad \text{(i)}$

$2 = 4\lambda$

$\frac{2}{4} = \lambda$

$\frac{1}{2} = \lambda \qquad \text{(ii)}$

Using (i) and (ii)

$\frac{2 - m}{2} = \frac{1}{2}$

$4 - 2m = 2$

$-2m = 2$

$-2m = 2 - 4$

$-2m = -2$

$m = \frac{-2}{-2}$

$m = 1$

$\therefore m = 1$

Question 9

Show that the points $(3,4)$, $(-5,16)$, $(5,1)$ are collinear.

Solution 9

Here, let $A = (3,4)$

$B = (-5,16)$

$C = (5,1)$

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= \left(-5\hat{i} + 16\hat{j}\right) - \left(3\hat{i} + 4\hat{j}\right)$

$= -5\hat{i} + 16\hat{j} - 3\hat{i} - 4\hat{j}$

$\overrightarrow{AB} = -8\hat{i} + 12\hat{j}$

$\overrightarrow{BC}$ = Position vector of C – Position vector of B

$= \left(5\hat{i} + \hat{j}\right) - \left(-5\hat{i} + 16\hat{j}\right)$

$= 5\hat{i} + \hat{j} + 5\hat{i} - 16\hat{j}$

$\overrightarrow{BC} = 10\hat{i} - 15\hat{j}$

So, $4\left(\overrightarrow{AB}\right) = -5\left(\overrightarrow{BC}\right)$

$\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ but B is a common point.

Hence, A, B, C are collinear.

Question 10

If the vectors $\vec{a} = 2\hat{i} - 3\hat{j}$ and $\vec{b} = -6\hat{i} + m\hat{j}$ are collinear, find the value of m.

Solution 10

Here, it is given that vectors
$a = 2\hat{i} - 3\hat{j}$ and $b = -6\hat{i} + m\hat{j}$ are collinear.

So, $a = \lambda b$, for a scalar λ

$2\hat{i} - 3\hat{j} = \lambda\left(-6\hat{i} + m\hat{j}\right)$

$2\hat{i} - 3\hat{j} = -6\lambda\hat{i} + \lambda m\hat{j}$

Comparing the coefficients of LHS and RHS,

$2 = -6\lambda$

$\lambda = \frac{2}{-6}$

$\lambda = \frac{-1}{3}$ (i)

$-3 = \lambda m$

$\lambda = \frac{-3}{m}$ (ii)

From (i) and (ii),

$\frac{-1}{3} = \frac{-3}{m}$

$m = 3 \times 3$

$= 9$

$\therefore m = 9$

Question 11

Show that the points A (1, −2, −8), B (5, 0, −2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.

Solution 11

The given points are A (1, −2, −8), B (5, 0, −2), and C (11, 3, 7).

$$\therefore \overrightarrow{AB} = (5-1)\hat{i} + (0+2)\hat{j} + (-2+8)\hat{k} = 4\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\overrightarrow{BC} = (11-5)\hat{i} + (3-0)\hat{j} + (7+2)\hat{k} = 6\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\overrightarrow{AC} = (11-1)\hat{i} + (3+2)\hat{j} + (7+8)\hat{k} = 10\hat{i} + 5\hat{j} + 15\hat{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{16+4+36} = \sqrt{56} = 2\sqrt{14}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{6^2 + 3^2 + 9^2} = \sqrt{36+9+81} = \sqrt{126} = 3\sqrt{14}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{10^2 + 5^2 + 15^2} = \sqrt{100+25+225} = \sqrt{350} = 5\sqrt{14}$$

$$\therefore \left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$$

Thus, the given points A, B, and C are collinear.

Now, let point B divide AC in the ratio $\lambda:1$. Then, we have:

$$\overrightarrow{OB} = \frac{\lambda\overrightarrow{OC} + \overrightarrow{OA}}{(\lambda+1)}$$

$$\Rightarrow 5\hat{i} - 2\hat{k} = \frac{\lambda\left(11\hat{i} + 3\hat{j} + 7\hat{k}\right) + \left(\hat{i} - 2\hat{j} - 8\hat{k}\right)}{\lambda+1}$$

$$\Rightarrow (\lambda+1)\left(5\hat{i} - 2\hat{k}\right) = 11\lambda\hat{i} + 3\lambda\hat{j} + 7\lambda\hat{k} + \hat{i} - 2\hat{j} - 8\hat{k}$$

$$\Rightarrow 5(\lambda+1)\hat{i} - 2(\lambda+1)\hat{k} = (11\lambda+1)\hat{i} + (3\lambda-2)\hat{j} + (7\lambda-8)\hat{k}$$

On equating the corresponding components, we get:

$$5(\lambda+1) = 11\lambda+1$$

$$\Rightarrow 5\lambda + 5 = 11\lambda + 1$$

$$\Rightarrow 6\lambda = 4$$

$$\Rightarrow \lambda = \frac{4}{6} = \frac{2}{3}$$

Hence, point B divides AC in the ratio $2:3$.

Question 12

Using vector show that the points $A(-2,3,5)$, $B(7,0,-1)$ *and* $C(-3,-2,-5)$ *and* $D(3,4,7)$ *are such that* AB *and* CD *intersect at the point* $P(1,2,3)$

Solution 12

We have

$\overrightarrow{AP} = \text{Position vector of } P - \text{Position vector of } A$

$\Rightarrow \overrightarrow{AP} = \hat{i} + 2\hat{j} + 3\hat{k} - \left(-2\hat{i} + 3\hat{j} + 5\hat{k}\right) = 3\hat{i} - \hat{j} - 2\hat{k}$

$\overrightarrow{PB} = \text{Position vector of } B - \text{Position vector of } P$

$\Rightarrow \overrightarrow{PB} = 7\hat{i} - \hat{k} - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 6\hat{i} - 2\hat{j} - 4\hat{k}$

Clearly, $\overrightarrow{PB} = 2\overrightarrow{AP}$

so vectors $\overrightarrow{AP}$ and $\overrightarrow{PB}$ are collinear.

But P is a point common to $\overrightarrow{AP}$ and $\overrightarrow{PB}$.

Hence P, A, B are collinear points.

Similarly, $\overrightarrow{CP} = \hat{i} + 2\hat{j} + 3\hat{k} - \left(-3\hat{i} - 2\hat{j} - 5\hat{k}\right) = 4\hat{i} + 4\hat{j} + 8\hat{k}$

and $\overrightarrow{PD} = 3\hat{i} + 4\hat{j} + 7\hat{k} - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 2\hat{i} + 2\hat{j} + 4\hat{k}$

So vectors $\overrightarrow{CP}$ and $\overrightarrow{PD}$ are collinear.

But P is a common point to $\overrightarrow{CP}$ and $\overrightarrow{CD}$.

Hence, C, P, D are collinear points.

Thus, A, B, C, D and P are points such that A, P, B and C, P, D are two sets of collinear points. Hence AB and CD intersect at the point P

Question 13

Using vectors, find the value of l such that the points
(l, - 10, 3), (1 -1, 3) and (3, 5, 3) are collinear.

Solution 13

Points (λ, - 10, 3), (1 -1, 3) and (3, 5, 3) are collinear.

$\therefore$ (λ, - 10, 3) = x(1 -1, 3) + y(3, 5, 3) for some scalars x and y.

$\Rightarrow \lambda = x + 3y,\ \ -10 = -x + 5y$ and $3 = 3x + 3y$

Solving $-10 = -x + 5y$ and $3 = 3x + 3y$ for x and y we get,

$x = \dfrac{5}{2}$ and $y = -\dfrac{3}{2}$

Now,

$\lambda = x + 3y$

$\Rightarrow \lambda = \dfrac{5}{2} + 3\left(-\dfrac{3}{2}\right) = -2$

Chapter 23 - Algebra of Vectors Exercise Ex. 23.8

Question 1

Show that the points whose position vectors are given, are collinear:

(i) $2\hat{i}+\hat{j}-\hat{k}$, $3\hat{i}-2\hat{j}+\hat{k}$ and $\hat{i}+4\hat{j}-3\hat{k}$

(ii) $3\hat{i}-2\hat{j}+4\hat{k}$, $\hat{i}+\hat{j}+\hat{k}$ and $-\hat{i}+4\hat{j}-2\hat{k}$

Solution 1

(i) Let P, Q, R be the points whose position vectors are $2\hat{i}+\hat{j}-\hat{k}$, $3\hat{i}-2\hat{j}+\hat{k}$ and $\hat{i}+4\hat{j}-3\hat{k}$ respectively.

$\overrightarrow{PQ}$ = Position vector of Q – Position vector of P

$= \left(3\hat{i}-2\hat{j}+\hat{k}\right)-\left(2\hat{i}+\hat{j}-\hat{k}\right)$

$= 3\hat{i}-2\hat{j}+\hat{k}-2\hat{i}-\hat{j}+\hat{k}$

$\overrightarrow{PQ} = \hat{i}-3\hat{j}+2\hat{k}$

$\overrightarrow{QR}$ = Position vector of R – Position vector of Q

$= \left(\hat{i}+4\hat{j}-3\hat{k}\right)-\left(3\hat{i}-2\hat{j}+\hat{k}\right)$

$= \hat{i}+4\hat{j}-3\hat{k}-3\hat{i}+2\hat{j}-\hat{k}$

$= -2\hat{i}+6\hat{j}-4\hat{k}$

$\overrightarrow{QR} = -2\overrightarrow{PQ}$

Therefore, $\overrightarrow{QR}$ is parallel to $\overrightarrow{PQ}$ but there is a common point Q. So, P, Q, R are collinear.

(ii) Let P, Q, R be the points represented be the vectors are $3\hat{i}-2\hat{j}+4\hat{k}$, $\hat{i}+\hat{j}+\hat{k}$ and $-\hat{i}+4\hat{j}-2\hat{k}$ respectively.

$\overrightarrow{PQ}$ = Position vector of Q – Position vector of P

$= \left(-\hat{i}+4\hat{j}-2\hat{k}\right)-\left(3\hat{i}-2\hat{j}+4\hat{k}\right)$

$= \hat{i}+\hat{j}+\hat{k}-3\hat{i}+2\hat{j}-4\hat{k}$

$= -2\hat{i}-3\hat{j}-3\hat{k}$

$\overrightarrow{QR}$ = Position vector of R – Position vector of Q

$= \left(-\hat{i}+4\hat{j}-2\hat{k}\right)-\left(\hat{i}+\hat{j}+\hat{k}\right)$

$= -\hat{i}+4\hat{j}-2\hat{k}-\hat{i}-\hat{j}-\hat{k}$

$= -2\hat{i}+3\hat{j}-3\hat{k}$

$\overrightarrow{PQ} = \overrightarrow{QR}$

So, $\overrightarrow{PQ}$ is parallel to $\overrightarrow{QR}$ but Q is the common point Q. So, P, Q, R are collinear.

Question 2 (i)

Using vector method, prove that $A(6,-7,-1)$ $B(2,-3,1)$ and $C(4,-5,0)$ are collinear.

Solution 2 (i)

Here, $\vec{A} = 6\hat{i} - 7\hat{j} - \hat{k}$

$\vec{B} = 2\hat{i} - 3\hat{j} + \hat{k}$

$\vec{C} = 4\hat{i} - 5\hat{j} - 0 \times \hat{k}$

$$\overrightarrow{AB} = \vec{B} - \vec{A}$$
$$= \left(2\hat{i} - 3\hat{j} + \hat{k}\right) - \left(6\hat{i} - 7\hat{j} - \hat{k}\right)$$
$$= 2\hat{i} - 3\hat{j} + \hat{k} - 6\hat{i} + 7\hat{j} + \hat{k}$$
$$\overrightarrow{AB} = -4\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\overrightarrow{BC} = \vec{C} - \vec{B}$$
$$= \left(4\hat{i} - 5\hat{j} - 0 \times \hat{k}\right) - \left(2\hat{i} - 3\hat{j} + \hat{k}\right)$$
$$= 4\hat{i} - 5\hat{j} - 0 \times \hat{k} - 2\hat{i} + 3\hat{j} - \hat{k}$$
$$\overrightarrow{BC} = 2\hat{i} - 2\hat{j} - \hat{k}$$

$$\overrightarrow{AB} = -2\left(\overrightarrow{BC}\right)$$

So, $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ but B is the common point. So, A, B, C are collinear.

Question 2 (ii)

Using vector method, prove that $A(2,-1,3)$ $B(4,3,1)$ and $C(3,1,2)$ are collinear.

Solution 2 (ii)

Here, $\vec{A} = 2\hat{i} - \hat{j} + 3\hat{k}$
$\vec{B} = 4\hat{i} + 3\hat{j} + \hat{k}$
$\vec{C} = 3\hat{i} + \hat{j} + 2\hat{k}$

$$\begin{aligned}
\overrightarrow{AB} &= \vec{B} - \vec{A} \\
&= \left(4\hat{i} + 3\hat{j} + \hat{k}\right) - \left(2\hat{i} - \hat{j} + 3\hat{k}\right) \\
&= 4\hat{i} + 3\hat{j} + \hat{k} - 2\hat{i} + \hat{j} - 3\hat{k} \\
\overrightarrow{AB} &= 2\hat{i} + 4\hat{j} - 2\hat{k}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{BC} &= \vec{C} - \vec{B} \\
&= \left(3\hat{i} + \hat{j} + 2\hat{k}\right) - \left(4\hat{i} + 3\hat{j} + \hat{k}\right) \\
&= 3\hat{i} + \hat{j} + 2\hat{k} - 4\hat{i} - 3\hat{j} - \hat{k} \\
\overrightarrow{BC} &= -\hat{i} - 2\hat{j} + \hat{k}
\end{aligned}$$

So, $\overrightarrow{AB} = -2\left(\overrightarrow{BC}\right)$

$\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ but $\vec{B}$ is a common vector.

Therefore, A, B, C are collinear.

Question 2 (iii)

Using vector method, prove that $A\left(1,2,7\right)$ $B\left(2,6,3\right)$ and $C\left(3,10,-1\right)$ are collinear.

Solution 2 (iii)

Here, $\vec{A} = \hat{i} + 2\hat{j} + 7\hat{k}$
$\vec{B} = 2\hat{i} + 6\hat{j} + 3\hat{k}$
$\vec{C} = 3\hat{i} + 10\hat{j} - \hat{k}$

$$\begin{aligned}
\overrightarrow{AB} &= \vec{B} - \vec{A} \\
&= \left(2\hat{i} + 6\hat{j} + 3\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 7\hat{k}\right) \\
&= 2\hat{i} + 6\hat{j} + 3\hat{k} - \hat{i} - 2\hat{j} - 7\hat{k} \\
\overrightarrow{AB} &= \hat{i} + 4\hat{j} - 4\hat{k}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{BC} &= \vec{C} - \vec{B} \\
&= \left(3\hat{i} + 10\hat{j} - \hat{k}\right) - \left(2\hat{i} + 6\hat{j} + 3\hat{k}\right) \\
&= 3\hat{i} + \hat{j} + 2\hat{k} - 2\hat{i} - 6\hat{j} - 3\hat{k} \\
\overrightarrow{BC} &= \hat{i} + 4\hat{j} - 4\hat{k}
\end{aligned}$$

$\overrightarrow{AB} = \overrightarrow{BC}$

So, $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ but $\vec{B}$ is a common vector.

So, A, B, C are collinear.

Question 2 (iv)

Using vector method, prove that $A(-3,-2,-5)$ $B(1,2,3)$ and $C(3,4,7)$ are collinear.

Solution 2 (iv)

Here, $\vec{A} = -3\hat{i} - 2\hat{j} - 5\hat{k}$

$\vec{B} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{C} = 3\hat{i} + 4\hat{j} + 7\hat{k}$

$$\overrightarrow{AB} = \vec{B} - \vec{A}$$
$$= \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) - \left(-3\hat{i} - 2\hat{j} - 5\hat{k}\right)$$
$$= \hat{i} + 2\hat{j} + 3\hat{k} + 3\hat{i} + 2\hat{j} + 5\hat{k}$$
$$\overrightarrow{AB} = 4\hat{i} + 4\hat{j} + 8\hat{k}$$

$$\overrightarrow{BC} = \vec{C} - \vec{B}$$
$$= \left(3\hat{i} + 4\hat{j} + 7\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$
$$= 3\hat{i} + 4\hat{j} + 7\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k}$$
$$\overrightarrow{BC} = 2\hat{i} + 2\hat{j} + 4\hat{k}$$

So, $\overrightarrow{AB} = 2\overrightarrow{BC}$

Hence, $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ but $\vec{B}$ is a common vector.

Therefore, A, B, C are collinear.

Question 3 (i)

If $\vec{a}, \vec{b}, \vec{c}$ are non-zero, non-coplanar vectors, prove that the vectors $5\vec{a} + 6\vec{b} + 7\vec{c}$, $7\vec{a} - 8\vec{b} + 9\vec{c}$, and $3\vec{a} + 20\vec{b} + 5\vec{c}$ are coplanar.

Solution 3 (i)

We know that,
Three vectors are coplanar if one of the vector can be expressed as the linear combination of other two.

Let,

$$5\vec{a}+6\vec{b}+7\vec{c}=x\left(7\vec{a}-8\vec{b}+9\vec{c}\right)+y\left(3\vec{a}+20\vec{b}+5\vec{c}\right)$$

$$5\vec{a}+6\vec{b}+7\vec{c}=7\vec{a}x-8\vec{b}x+9\vec{c}x+3\vec{a}y+20\vec{b}y+5\vec{c}y$$

$$5\vec{a}+6\vec{b}+7\vec{c}=\left(7x+3y\right)\vec{a}+\left(-8x+20y\right)\vec{b}+\left(9x+5y\right)\vec{c}$$

Comparing the LHS and RHS,

$$7x+3y=5 \qquad (i)$$

$$-8x+20y=6 \qquad (ii)$$

$$9x+5y=7 \qquad (iii)$$

For solving (i) and (ii),
Subtract $-8\times(i)$ from $7\times(ii)$,

$$-56x+140y=42$$

$$\underset{(+)}{-}56x\ \underset{(+)}{-}24y=\underset{(+)}{-}40$$

$$164y=82$$

$$y=\frac{82}{164}$$

$$y=\frac{1}{2}$$

Put $y = \frac{1}{2}$ in equation (i),

$$7x + 3y = 5$$
$$7x + 3\left(\frac{1}{2}\right) = 5$$
$$7x + \frac{3}{2} = 5$$
$$7x = \frac{5}{1} - \frac{3}{2}$$
$$7x = \frac{10 - 3}{2}$$
$$7x = \frac{7}{2}$$
$$x = \frac{7}{14}$$
$$x = \frac{1}{2}$$

Now, put $x = \frac{1}{2}$ and $y = \frac{1}{2}$ in equation (iii),

$$9x + 5y = 7$$
$$9\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right) = 7$$
$$\frac{9}{2} + \frac{5}{2} = 7$$
$$\frac{14}{2} = 7$$
$$7 = 7$$

LHS = RHS

$\therefore$ The value of x,y satisfy equation (iii).

So,

$5\vec{a} + 6\vec{b} + 7\vec{c}$, $7\vec{a} - 8\vec{b} + 9\vec{c}$, $3\vec{a} + 20\vec{b} + 5\vec{c}$ are coplanar.

Question 3 (ii)

If $\vec{a}, \vec{b}, \vec{c}$ are non-zero, non-coplanar vectors, prove that the vectors $\vec{a} - 2\vec{b} + 3\vec{c}$, $-3\vec{b} + 5\vec{c}$, and $-2\vec{a} + 3\vec{b} - 4\vec{c}$ are coplanar.

Solution 3 (ii)

We know that,
Three vectors are coplanar if one of them can be expressed as the linear combination of other two.

Let

$$\vec{a}-2\vec{b}+3\vec{c}=x\left(-3\vec{b}+5\vec{c}\right)+y\left(-2\vec{a}+3\vec{b}-4\vec{c}\right)$$
$$\vec{a}-2\vec{b}+3\vec{c}=-3\vec{b}x+5\vec{c}x+2\vec{a}y+3\vec{b}y-4\vec{c}y$$
$$\vec{a}-2\vec{b}+3\vec{c}=\left(-2y\right)\vec{a}+\left(-3x+3y\right)\vec{b}+\left(5x-4y\right)\vec{c}$$

Comparing the LHS and RHS,

$$-2y=1 \qquad (i)$$
$$-3x+3y=-2 \qquad (ii)$$
$$5x-4y=3 \qquad (iii)$$

From solving (i) and $y=-\frac{1}{2}$

Put value of y in equation (ii),

$$-3x+3y=-2$$
$$-3x+3\left(-\frac{1}{2}\right)=-2$$
$$-3x-\frac{3}{2}=-2$$
$$-3x=\frac{-2}{1}+\frac{3}{2}$$
$$-3x=\frac{-4+3}{2}$$
$$-3x=\frac{-1}{2}$$
$$x=\frac{-1}{-6}$$
$$x=\frac{1}{6}$$

Put value of x and y in equation (iii)

$$5x-4y=3$$
$$5\left(\frac{1}{6}\right)-4\left(-\frac{1}{2}\right)=3$$
$$\frac{5}{6}+\frac{4}{2}=3$$
$$\frac{5+12}{6}=3$$
$$\frac{17}{6}=3$$
$$\text{LHS}\neq\text{RHS}$$

So, value of x and y do not satisfy the equation (iii).

So,
vectors $\vec{a}-2\vec{b}+3\vec{c}$, $-3\vec{b}+5\vec{c}$, and $-2\vec{a}+3\vec{b}-4\vec{c}$ are not coplanar.

Question 4

Show that the four points having position vectors $6\hat{i}-7\hat{j}$, $16\hat{i}-19\hat{j}-4\hat{k}$, $3\hat{j}-6\hat{k}$, $2\hat{i}-5\hat{j}+10\hat{k}$ are coplanar.

Solution 4

Here,

Position vector of $P = 6\hat{i}-7\hat{j}$

Position vector of $Q = 16\hat{i}-19\hat{j}-4\hat{k}$

Position vector of $R = 3\hat{j}-6\hat{k}$

Position vector of $S = 2\hat{i}-5\hat{j}+10\hat{k}$

$\overrightarrow{PQ}$ = Position vector of Q – Position vector of P

$$= \left(16\hat{i}-19\hat{j}-4\hat{k}\right)-\left(6\hat{i}-7\hat{j}\right)$$
$$= 16\hat{i}-19\hat{j}-4\hat{k}-6\hat{i}+7\hat{j}$$
$$= 10\hat{i}-12\hat{j}-4\hat{k}$$

$\overrightarrow{PR}$ = Position vector of R – Position vector of P

$$= \left(3\hat{j}-6\hat{k}\right)-\left(6\hat{i}-7\hat{j}\right)$$
$$= 3\hat{j}-6\hat{k}-6\hat{i}+7\hat{j}$$
$$= -6\hat{i}+10\hat{j}-6\hat{k}$$

$\overrightarrow{PS}$ = Position vector of S – Position vector of P

$$= \left(2\hat{i}-5\hat{j}+10\hat{k}\right)-\left(6\hat{i}-7\hat{j}\right)$$
$$= 2\hat{i}-5\hat{j}+10\hat{k}-6\hat{i}+7\hat{j}$$
$$= -4\hat{i}+2\hat{j}+10\hat{k}$$

Let $\overrightarrow{PQ} = x\overrightarrow{PR}+y\overrightarrow{PS}$

$$10\hat{i}-12\hat{j}-4\hat{k} = x\left(-6\hat{i}+10\hat{j}-6\hat{k}\right)+\left(-4\hat{i}+2\hat{j}+10\hat{k}\right)$$
$$= -6x\hat{i}+x10\hat{j}-6x\hat{k}-4y\hat{i}+2y\hat{j}+10y\hat{k}$$
$$10\hat{i}-12\hat{j}-4\hat{k} = \left(-6x-4y\right)\hat{i}+\left(10x+2y\right)\hat{j}+\left(-6x+10y\right)\hat{k}$$

Comparing the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$ of LHS and RHS,

$-6x-4y = 10$

$3x+2y = -5 \qquad \text{(i)}$

$10x+2y = -12 \qquad \text{(ii)}$

$-6x+10y = -4 \qquad \text{(iii)}$

Substracting (i) from (ii),

$$\begin{array}{r} 10x + 2y = -12 \\ 3x + 2y = -5 \\ (-)\quad (-) \qquad (+) \\ \hline 7x \qquad = -7 \end{array}$$

$$x = \frac{-7}{7}$$
$$x = -1$$

Put $x = -1$ in equation (i)

$$\begin{aligned} 3x + 2y &= -5 \\ 3(-1) + 2y &= -5 \\ -3 + 2y &= -5 \\ 2y &= -5 + 3 \\ 2y &= -2 \\ y &= \frac{-2}{2} \\ y &= -1 \end{aligned}$$

Put $x = -1$ and $y = -1$ in equation (iii),

$$\begin{aligned} -6x + 10y &= -4 \\ -6(-1) + 10(-1) &= -4 \\ 6 - 10 &= -4 \\ -4 &= -4 \\ \text{LHS} &= \text{RHS} \end{aligned}$$

Therefore,
P, Q, R, S are coplanar.

Question 5 (i)

Prove that the following vectors are coplanar

$2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$

Solution 5 (i)

We know that, three vectors are coplanar if one of the vector can be expressed as linear combination of other two.

Let,

$$2\hat{i} - \hat{j} + \hat{k} = x\left(\hat{i} - 3\hat{j} - 5\hat{k}\right) + y\left(3\hat{i} - 4\hat{j} - 4\hat{k}\right)$$

$$2\hat{i} - \hat{j} + \hat{k} = x\hat{i} - 3x\hat{j} - 5x\hat{k} + 3y\hat{i} - 4y\hat{j} - 4y\hat{k}$$

$$2\hat{i} - \hat{j} + \hat{k} = \left(x + 3y\right)\hat{i} + \left(-3x - 4y\right)\hat{j} + \left(-5x - 4y\right)\hat{k}$$

Comparing the coefficients of LHS and RHS,

$$x + 3y = 2 \qquad (i)$$

$$-3x - 4y = -1 \qquad (ii)$$

$$-5x - 4y = 1 \qquad (iii)$$

For solving equation (i) and (ii),

Add $3 \times (i)$ with equation (ii),

$$\begin{array}{r} 3x + 9y = 6 \\ \underline{-3x - 4y = -1} \\ 5y = 5 \end{array}$$

$$y = \frac{5}{5}$$

$$y = 1$$

Put y in equation (i),

$$x + 3y = 2$$

$$x + 3(1) = 2$$

$$x + 3 = 2$$

$$x = 2 - 3$$

$$x = -1$$

Put the value of x and y in equation (iii),

$$-5x - 4y = 1$$

$$-5(-1) - 4(1) = 1$$

$$5 - 4 = 1$$

$$1 = 1$$

$$\text{LHS} = \text{RHS}$$

So, the value of x and y satisfy equation (iii).

Hence, vectors are coplanar.

Question 5 (ii)

Prove that the following vectors are coplanar

$\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 3\hat{j} - \hat{k}$ and $-\hat{i} - 2\hat{j} + 2\hat{k}$

Solution 5 (ii)

We know that,

Three vectors are coplanar if one of the vector can be expressed as the linear combination of other two vectors.

Let,

$$\hat{i}+\hat{j}+\hat{k} = x\left(2\hat{i}+3\hat{j}-\hat{k}\right)+y\left(-\hat{i}-2\hat{j}+2\hat{k}\right)$$

$$\hat{i}+\hat{j}+\hat{k} = 2x\hat{i}+3x\hat{j}-x\hat{k}+-y\hat{i}-2y\hat{j}+2y\hat{k}$$

$$\hat{i}+\hat{j}+\hat{k} = \left(2x-y\right)\hat{i}+\left(3x-2y\right)\hat{j}+\left(-x+2y\right)\hat{k}$$

Comparing the coefficients of LHS and RHS,

$2x-y=1$ (i)

$3x-2y=1$ (ii)

$-x+2y=1$ (iii)

For solving (i) and (ii),

Subtracting $2\times$(i) from (ii),

$$\begin{array}{rcl} 3x-2y &=& 1 \\ 4x-2y &=& 2 \\ {\scriptstyle(-)}\quad{\scriptstyle(+)} & & {\scriptstyle(-)} \\ \hline -x &=& -1 \end{array}$$

$x=1$

Put the value of x in equation (i),

$2x-y=1$

$2(1)-y=1$

$2-y=1$

$-y=1-2$

$-y=-1$

$y=1$

Put the value of x and y in equation (iii),

$-x+2y=1$

$-(1)+2(1)=1$

$-1+2=1$

$1=1$

LHS = RHS

The value of x and y satisfy equation (iii).

Hence, vectors are coplanar.

Question 6 (i)

Prove that the vector $3\hat{i} + \hat{j} - \hat{k}$, $2\hat{i} - \hat{j} + 7\hat{k}$ and $7\hat{i} - \hat{j} + 23\hat{k}$ are non-coplanar.

Solution 6 (i)

We know that,
Three vectors are coplanar if one of them vector can be expressed as the linear combination of the other two.

Let,

$$\left(3\hat{i}+\hat{j}-\hat{k}\right) = x\left(2\hat{i}-\hat{j}+7\hat{k}\right)+y\left(7\hat{i}-\hat{j}+23\hat{k}\right)$$
$$= 2x\hat{i}-x\hat{j}+7x\hat{k}+7y\hat{i}-y\hat{j}+23y\hat{k}$$
$$\left(3\hat{i}+\hat{j}-\hat{k}\right) = \left(2x+7y\right)\hat{i}+\left(-x-y\right)\hat{j}+\left(7x+23y\right)\hat{k}$$

Equating the coefficients of LHS and RHS,

$2x+7y = 3 \qquad \text{(i)}$

$-x-y = 1 \qquad \text{(ii)}$

$7x+23y = -1 \qquad \text{(iii)}$

For solving (i) and (ii),
Add (i) and $2\times$(ii),

$$\begin{array}{r} 2x+7y = 3 \\ \underline{-2x-2y = 2} \\ 5y = 5 \end{array}$$

$$y = \frac{5}{5}$$
$$y = 1$$

Put the value of y in equation (i),

$$2x+7y = 3$$
$$2x+7(1) = 3$$
$$2x+7 = 3$$
$$2x = 3-7$$
$$2x = -4$$
$$x = \frac{-4}{2}$$
$$x = -2$$

Put the value of x and y in equation (iii),

$$7x+23y = -1$$
$$7(2)+23(1) = -1$$
$$14+23 = -1$$
$$37 = -1$$
$$\text{LHS} \neq \text{RHS}$$

The value of x and y do not satisfy the equation (iii).
Hence, vectors are non-coplanar.

Question 6 (ii)

Prove that the following vector are non-coplanar $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}+\hat{j}+3\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$ are non-coplanar.

Solution 6 (ii)

We know that,
Three vectors are coplanar if any one of the vector can be expressed as the linear combination of other two vectors.

Let,

$$\hat{i}+2\hat{j}+3\hat{k}=x\left(2\hat{i}+\hat{j}+3\hat{k}\right)+y\left(\hat{i}+\hat{j}+\hat{k}\right)=2x\hat{i}+x\hat{j}+3x\hat{k}+y\hat{i}+y\hat{j}+y\hat{k}$$

$$\therefore\ \hat{i}+2\hat{j}+3\hat{k}=(2x+y)\hat{i}+(x+2y)\hat{j}+(3x+y)\hat{k}$$

Comparing the coefficients of LHS and RHS,

$2x+y=1$ (i)

$x+2y=2$ (ii)

$3x+y=3$ (iii)

Subtracting $2\times$(ii) from equation (i),

$$\begin{array}{r} 2x+4y=4 \\ 2x+y=1 \\ \underline{(-)\quad(-)\quad(-)} \\ 3y=3 \end{array}$$

$y=\frac{3}{3}$

$y=1$

Put the value of y in equation (i),

$2x+y=1$

$2x+1=1$

$2x=1-1$

$2x=0$

$x=\frac{0}{2}$

$x=0$

Put the value of x and y in equation (iii),

$3x+y=3$

$3(0)+1=3$

$0+1=3$

$1=3$

LHS $\neq$ RHS

The value of x and y do not satisfy the equation (iii).
Hence, vectors are non-coplanar.

Question 7 (i)

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, prove that the given vectors are non-coplanar $2\vec{a} - \vec{b} + 3\vec{c}$, $\vec{a} + \vec{b} - 2\vec{c}$ and $\vec{a} + \vec{b} - 3\vec{c}$

Solution 7 (i)

We know that,

Three vectors are coplanar if any one of them can be expressed as the linear combination of other two vectors.

Let,

$$\left(2\vec{a} - \vec{b} + 3\vec{c}\right) = x\left(\vec{a} + \vec{b} - 2\vec{c}\right) + y\left(\vec{a} + \vec{b} - 3\vec{c}\right)$$

$$= \vec{a}x + \vec{b}x - 2\vec{c}x + \vec{a}y + \vec{b}y - 3\vec{c}y$$

$$\left(2\vec{a} - \vec{b} + 3\vec{c}\right) = \left(x + y\right)\vec{a} + \left(x + y\right)\vec{b} + \left(-2x - 3y\right)\vec{c}$$

Comparing the coefficients of LHS and RHS,

$x + y = 2 \qquad (i)$

$x + y = -1 \qquad (ii)$

$-2x - 3y = 3 \qquad (iii)$

For solving the equation (i) and (ii),

Subtracting (ii) from (i),

$$\begin{array}{l} x + y = 2 \\ \underset{(-)}{x} + \underset{(-)}{y} = \underset{(+)}{-1} \\ \hline \quad 0 = 3 \end{array}$$

There is no value of x and y that can satisfy the equation (iii).

Hence, vectors are non-coplanar.

Question 7 (ii)

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, prove that the given vectors are non-coplanar $\vec{a} + 2\vec{b} + 3\vec{c}$, $2\vec{a} + \vec{b} + 3\vec{c}$ and $\vec{a} + \vec{b} + \vec{c}$

Solution 7 (ii)

We know that,
Three vectors are coplanar if any one of them can be expressed as the linear combination of other two vectors.

Let

$$\vec{a}+2\vec{b}+3\vec{c}=x\left(2\vec{a}+\vec{b}+3\vec{c}\right)+y\left(\vec{a}+\vec{b}+\vec{c}\right)$$
$$=2\vec{a}x+\vec{b}x+3\vec{c}x+\vec{a}y+\vec{b}y+\vec{c}y$$
$$\vec{a}+2\vec{b}+3\vec{c}=\left(2x+y\right)\vec{a}+\left(x+y\right)\vec{b}+\left(3x+y\right)\vec{c}$$

Comparing the coefficients of LHS and RHS,

$2x+y=1 \qquad (i)$

$x+y=2 \qquad (ii)$

$3x+y=3 \qquad (iii)$

For solving the equation (i) and (ii),
Subtracting equation (i) from equation (ii),

$$\begin{array}{r} x+y=2 \\ 2x+y=1 \\ (-)\ \ (-)\ \ (-) \\ \hline -x\quad =1 \end{array}$$

$x=-1$

Put the value of x in equation (i)

$x+y=2$

$-1+y=2$

$y=2+1$

$y=3$

Put the x and y in equation (iii),

$3x+y=3$

$3\left(-1\right)+3=3$

$-3+3=3$

$0=3$

$\text{LHS}\neq\text{RHS}$

The value of x and y do not satisfy the equation (iii).
Hence, vectors are non-coplanar.

Question 8

Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ given by $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} + \hat{j} + \hat{k}$ are non-coplanar. Express vector $\vec{d} = 2\hat{i} - \hat{j} - 3\hat{k}$ as a linear combination of the vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

Solution 8

We know that,
Three vectors are coplanar if any one of them can be expressed as the linear combination of other two vectors.

Let

$$\vec{a} = x\vec{b} + y\vec{c}$$
$$= x\left(2\hat{i} + \hat{j} + 3\hat{k}\right) + y\left(\hat{i} + \hat{j} + \hat{k}\right)$$
$$= 2\hat{i}x + \hat{j}x + 3\hat{k}x + \hat{i}y + \hat{j}y + \hat{k}y$$

$$\hat{i} + 2\hat{j} + 3\hat{k} = (2x + y)\hat{i} + (x + y)\hat{j} + (3x + y)\hat{k}$$

Comparing the coefficient of LHS and RHS,

$2x + y = 1$ (i)

$x + y = 2$ (ii)

$3x + y = 3$ (iii)

For solving (i) and (ii),

Subtracting (i) from (ii),

$$\begin{array}{rcl} x + y &=& 2 \\ 2x + y &=& 1 \\ (-)\quad (-) && (-) \\ \hline -x &=& 1 \\ x &=& -1 \end{array}$$

Put the value of x in equation (i),

$x + y = 2$

$-1 + y = 2$

$y = 2 + 1$

$y = 3$

Put the values of x and y in equation (iii)

$3x + y = 3$

$3(-1) + 3 = 3$

$-3 + 3 = 3$

$0 = 3$

$\text{LHS} \neq \text{RHS}$

The values of x and y do not satisfy equation (iii).

Hence,

$\vec{a}, \vec{b}, \vec{c}$ are non coplanar.

Let,

$$\vec{d} = x\vec{b} + y\hat{j} + z\hat{k}$$

$$= x\left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + y\left(2\hat{i} + \hat{j} + 3\hat{k}\right) + z\left(\hat{i} + \hat{j} + \hat{k}\right)$$

$$= x\hat{i} + 2x\hat{j} + 3x\hat{k} + 2y\hat{i} + \hat{j}y + 3y\hat{k} + z\hat{i} + z\hat{j} + z\hat{k}$$

$$2\hat{i} - \hat{j} - 3\hat{k} = \left(x + 2y + z\right)\hat{i} + \left(2x + y + z\right)\hat{j} + \left(3x + 3y + z\right)\hat{k}$$

Comparing the coefficient of LHS and RHS,

$$x + 2y + z = 2 \qquad (i)$$

$$2x + y + z = -1 \qquad (ii)$$

$$3x + 3y + z = -3 \qquad (iii)$$

Subtracting equation (i) from (ii),

$$\begin{array}{l} 2x + y + z = -1 \\ \underline{\underset{(-)}{x} \underset{(-)}{+} 2y \underset{(-)}{+} z = \underset{(-)}{2}} \\ \quad x - y \quad = -3 \qquad (iv) \end{array}$$

Subtracting equation (ii) from (iii),

$$\begin{array}{l} 3x + 3y + z = -3 \\ \underline{\underset{(-)}{2x} \underset{(-)}{+} y \underset{(-)}{+} z = \underset{(+)}{-1}} \\ \quad x + 2y \quad = -2 \qquad (v) \end{array}$$

Subtracting (iv) from (v),

$$\begin{array}{l} x + 2y = -2 \\ \underline{\underset{(-)}{x} \underset{(+)}{-} y = \underset{(+)}{-3}} \\ \quad 3y = 1 \\ \qquad y = \frac{1}{3} \end{array}$$

Put y in equation (v),

$$x + 2y = -2$$

$$x + 2\left(\frac{1}{3}\right) = -2$$

$$2 + \frac{2}{3} = -2$$

$$x = \frac{-2}{1} - \frac{2}{3}$$

$$= \frac{-6 - 2}{3}$$

$x = \frac{-8}{3}$

Put value of x and y in equation (i),

$x + 2y + z = 2$

$\frac{-8}{3} + 2\left(\frac{1}{3}\right) + z = 2$

$\frac{-8}{3} + \frac{2}{3} + z = 2$

$z = \frac{2}{1} + \frac{8}{3} - \frac{2}{3}$

$z = \frac{6 + 8 - 2}{3}$

$z = \frac{14 - 2}{3}$

$z = \frac{12}{3}$

$z = 4$

So,

$\vec{d} = x\vec{a} + y\vec{b} + z\vec{c}$

$\vec{d} = \left(\frac{-8}{3}\right)\vec{a} + \left(\frac{1}{3}\right)\vec{b} + (4)\vec{c}$

Question 9

Prove that a necessary and sufficient condition for three vectors $\vec{a}, \vec{b},$ and $\vec{c}$ to be coplanar is that there exist scalars $l, m, n,$ not all zero simultaneously such that $l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$.

Solution 9

Necessary Condition: Let $\vec{a}, \vec{b}, \vec{c}$ are three coplanar vectors. Then one of them can be expressed as the linear combination of other two vectors.

Let, $\vec{c} = x\vec{a} + y\vec{b}$

$x\vec{a} + y\vec{b} - \vec{c} = 0$

Put $x = l$, $y = m$, $(-1) = n$

$l\vec{a} + m\vec{b} + n\vec{c} = 0$

Thus, if $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors,
then there exist scalars l, m, n
$l\vec{a} + m\vec{b} + n\vec{c} = 0$
Such that l,m,n are not all zero simultaneously.

Sufficient Condition: Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that there exist scalars l, m, n not all zero simultaneously satisfying $l\vec{a} + m\vec{b} + n\vec{c} = 0$

$l\vec{a} + m\vec{b} + n\vec{c} = 0$

$n\vec{c} = -l\vec{a} - m\vec{b}$

Dividing by n, both the sides

$$\frac{n\vec{c}}{n} = \frac{-l\vec{a}}{n} - \frac{m\vec{b}}{n}$$

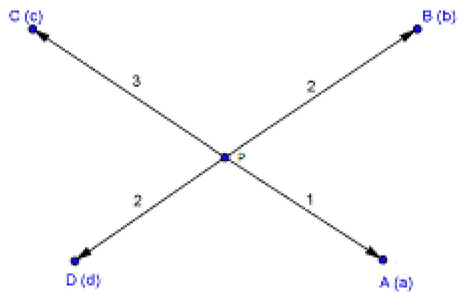
$$\vec{c} = \left(-\frac{l}{n}\right)\vec{a} + \left(-\frac{m}{n}\right)\vec{b}$$

$\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$

Hence, $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors.

Question 10

Show that the four points A, B, C and D with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively are coplanar if and only if $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$



Solution 10

Given that, A, B, C and D are four points with position vector $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

Let A, B, C, D are coplanar.

If so, there exists x, y, z, u not all zero such that
$x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = 0$
$x + y + z + u = 0$

Let, $x = 3$, $y = -2$, $z = 1$, $u = -2$
$3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = 0$

and, $x + y + z + u = 3 + (-2) + 1 + (-2)$
$= 4 - 4$
$= 0$

Thus, A, B, C, D are coplanar.
if $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = 0$

Let $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = 0$
$3\vec{a} + \vec{c} = 2\vec{b} + 2\vec{d}$

Divide by sum of the coefficients that is by 4 on both sides,

$$\frac{3\vec{a} + \vec{c}}{4} = \frac{2\vec{b} + 2\vec{d}}{4}$$

$$\frac{3\vec{a} + \vec{c}}{3 + 1} = \frac{2\vec{b} + 2\vec{d}}{2 + 2}$$

It shows that P is the point which divides AC in ratio $1:3$ internally as well as BD in ratio $2:2$ internally.

Thus, P is the point of intersection of AC and BD.

Hence,
A, B, C, D are coplanar.

We can say that,
A, B, C, D are coplanar if and only if Let $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$

Chapter 23 - Algebra of Vectors Exercise Ex. 23.9

Question 1

Can a vector have direction angles $45°, 60°, 120°$?

Solution 1

We know that, If l, m, n are the direction cosine of a vector and α, β, γ can the direction angle, then

$$l = \cos\alpha,\ m = \cos\beta\ n = \cos\gamma$$

and, $\quad l^2 + m^2 + n^2 = 1 \qquad (i)$

$\therefore \quad l = \cos 45°,\ m = \cos 60°,\ n = \cos 120°$

$$l = \frac{1}{\sqrt{2}},\ m = \frac{1}{2}, n = -\frac{1}{2}$$

Put l, m, n in equation (i)

$$l^2 + m^2 + n^2 = 1$$

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 = 1$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$$

$$\frac{2+1+1}{4} = 1$$

$$\frac{4}{4} = 1$$

$$1 = 1$$

$$\text{LHS} = \text{RHS}$$

Therefore, a vector can have direction angle 45°, 60°, 120°.

Question 2

Prove that 1,1,1 cannot be direction cosines of a straight line.

Solution 2

Here, $l = 1,\ m = 1,\ n = 1$

Put it in

$$l^2 + m^2 + n^2 = 1$$

$$(1)^2 + (1)^2 + (1)^2 = 1$$

$$1 + 1 + 1 = 1$$

$$3 = 1$$

$$\text{LHS} \neq \text{RHS}$$

Therefore,
1,1,1 can not be direction cosines of a straight line.

Question 3

A vector makes an angle of $\frac{\pi}{4}$ with each of x-axis and y-axis. Find the angle made by it with the z-axis.

Solution 3

Here, $\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{4}$, $\gamma = ?$

$l = \cos\alpha = \cos\frac{\pi}{4}$

$l = \frac{1}{\sqrt{2}}$

$m = \cos\beta = \cos\frac{\pi}{4}$

$m = \frac{1}{\sqrt{2}}$

$n = \cos\gamma$

Put value of $l, m,$ and n in

$$l^2 + m^2 + n^2 = 1$$
$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1$$
$$\frac{1}{2} + \frac{1}{2} + \cos^2\gamma = 1$$
$$1 + \cos^2\gamma = 1$$
$$\cos^2\gamma = 1 - 1$$
$$\cos^2\gamma = 0$$
$$\cos\gamma = 0$$
$$\gamma = \cos^{-1}(0)$$
$$\gamma = \frac{\pi}{2}$$

The angle made by the vector with the z-axis $= \frac{\pi}{2}$

Question 4

A vector $\vec{r}$ is inclined at equal acute angles to x-axis, y-axis and z-axis. If $\left|\vec{r}\right| = 6$ units, find $\vec{r}$.

Solution 4

Here, $\alpha = \beta = \gamma$

$\Rightarrow \quad \cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow \quad l = m = n = x \text{ (say)}$

We know that,

$$l^2 + m^2 + n^2 = 1$$
$$x^2 + x^2 + x^2 = 1$$
$$3x^2 = 1$$
$$x^2 = \frac{1}{3}$$
$$x = \pm\frac{1}{\sqrt{3}}$$

$$l = \pm\frac{1}{\sqrt{3}},\ m = \pm\frac{1}{\sqrt{3}},\ n = \pm\frac{1}{\sqrt{3}}$$

Hence, direction cosiner of $\vec{r}$ are,

$$\pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}}$$

Vector $\vec{r} = \left|\vec{r}\right|\left(l\hat{i} + m\hat{j} + n\hat{k}\right)$

$$= 6\left(\pm\frac{1}{\sqrt{3}}\hat{i} + \pm\frac{1}{\sqrt{3}}\hat{j} + \pm\frac{1}{\sqrt{3}}\hat{k}\right)$$

$$= \frac{\pm 6 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right) \qquad \left[\text{Rationalizing the denominator}\right]$$

$$= \frac{\pm 6\sqrt{3}}{3}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

$$\vec{r} = \pm 2\sqrt{3}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

Question 5

A vector $\vec{r}$ is inclined to x-axis at 45° and y-axis at 60°. If $\left|\vec{r}\right| = 8$ units, find $\vec{r}$.

Solution 5

Here, $\alpha = 45°,\ \beta = 60°,\ \gamma = \theta\text{ (say)}$

$$l = \cos\alpha$$
$$= \cos 45°$$
$$l = \frac{1}{\sqrt{2}}$$

$$m = \cos\beta$$
$$= \cos 60°$$
$$m = \frac{1}{2}$$

$$n = \cos\theta$$

Put l, m, and n in

$$l^2 + m^2 + n^2 = 1$$
$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\theta = 1$$
$$\frac{2+1}{4} + \cos^2\theta = 1$$
$$\frac{3}{4} + \cos^2\theta = 1$$
$$\cos^2\theta = \frac{1}{1} - \frac{3}{4}$$
$$= \frac{4-3}{4}$$
$$\cos^2\theta = \frac{1}{2}$$

So, $\quad l = \frac{1}{\sqrt{2}},\ m = \frac{1}{2},\ n = \pm\frac{1}{2}$

The required,

vector $\vec{r} = |\vec{r}|\left(l\hat{i} + m\hat{j} + n\hat{k}\right)$

$$= 8\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} \pm \frac{1}{2}\hat{k}\right)$$
$$= 8\,\frac{\sqrt{2}\hat{i} + \hat{j} \pm \hat{k}}{2}$$

$$\vec{r} = 4\left(\sqrt{2}\hat{i} + \hat{j} \pm \hat{k}\right)$$

Question 6

Find the direction cosines of the following vectors:

(i) $2\hat{i} + 2\hat{j} - \hat{k}$ (ii) $6\hat{i} - 2\hat{j} - 3\hat{k}$ (iii) $3\hat{i} - 4\hat{k}$

Solution 6

(i)

Here, the direction ratios of the vector

$2\hat{i} + 2\hat{j} - \hat{k} = 2, 2, -1$

The direction cosines of the vector

$$= \frac{2}{|\vec{r}|}, \frac{2}{|\vec{r}|}, \frac{-1}{|\vec{r}|}$$

$$= \frac{2}{\sqrt{(2)^2 + (2)^2 + (-1)^2}}, \frac{2}{\sqrt{(2)^2 + (2)^2 + (-1)^2}}, \frac{-1}{\sqrt{(2)^2 + (2)^2 + (-1)^2}}$$

$$= \frac{2}{\sqrt{4+4+1}}, \frac{2}{\sqrt{4+4+1}}, \frac{-1}{\sqrt{4+4+1}}$$

$$= \frac{2}{\sqrt{9}}, \frac{2}{\sqrt{9}}, \frac{-1}{\sqrt{9}}$$

$$= \frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$$

(ii)

Here, let $\vec{r} = 6\hat{i} - 2\hat{j} - 3\hat{k}$

and, $|\vec{r}| = \sqrt{(6)^2 + (-2)^2 + (-3)^2}$

$= \sqrt{36 + 4 + 9}$

$= \sqrt{49}$

$|\vec{r}| = 7$

The direction cosines of $\vec{r}$ are given by

$$= \frac{6}{|\vec{r}|}, \frac{-2}{|\vec{r}|}, \frac{-3}{|\vec{r}|}$$

$$= \frac{6}{7}, \frac{-2}{7}, \frac{-3}{7}$$

(iii)

Let, $\vec{r} = 3\hat{i} - 4\hat{k}$

$\vec{r} = 3\hat{i} + 0 \times \hat{k} - 4\hat{k}$

The direction ratios of vector $\vec{r} = 3, 0, -4$

And, $|\vec{r}| = \sqrt{(3)^2 + (0)^2 + (-4)^2}$

$= \sqrt{9 + 0 + 16}$

$= \sqrt{25}$

$|\vec{r}| = 5$

The direction cosines of the vector $\vec{r}$ are given by

$= \frac{3}{|\vec{r}|}, \frac{0}{|\vec{r}|}, \frac{-4}{|\vec{r}|}$

$= \frac{3}{5}, \frac{0}{5}, \frac{-4}{5}$

$= \frac{3}{5}, 0, \frac{-4}{5}$

Question 7 (i)

Find the angles at which the vector $\hat{i} - \hat{j} + \hat{k}$ is inclined to each of the coordinate axes.

Solution 7 (i)

Let, $\vec{r} = \hat{i} - \hat{j} + \hat{k}$

The direction ratios of the vector $\vec{r} = 1, -1, 1$

And, $|\vec{r}| = \sqrt{(1)^2 + (-1)^2 + (1)^2}$

$= \sqrt{1+1+1}$

$= \sqrt{3}$

The direction cosines of the vector $\vec{r}$

$= \frac{1}{|\vec{r}|}, \frac{-1}{|\vec{r}|}, \frac{1}{|\vec{r}|}$

$= \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

So, $l = \cos\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$m = \cos\beta = \frac{-1}{\sqrt{3}}$

$\beta = \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$n = \cos\gamma = \frac{1}{\sqrt{3}}$

$\gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Thus, angles made by $\vec{r}$ with the coordinate axes are given by

$\cos^{-1}\left(\frac{1}{\sqrt{3}}\right), \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right), \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Question 7 (ii)

Find the angles at which the vector $\hat{j} - \hat{k}$ is inclined to each of the coordinate axes.

Solution 7 (ii)

Let, $\vec{r} = \hat{j} - \hat{k}$

$\vec{r} = 0 \times \hat{i} + \hat{j} - \hat{k}$

The direction ratios of $\vec{r} = 0, 1, -1$

and, $\left|\vec{r}\right| = \sqrt{(0)^2 + (1)^2 + (-1)^2}$

$= \sqrt{0 + 1 + 1}$

$\left|\vec{r}\right| = \sqrt{2}$

The direction cosines of the $\vec{r}$ are given by

$$= \frac{0}{\left|\vec{r}\right|}, \quad \frac{1}{\left|\vec{r}\right|}, \quad \frac{-1}{\left|\vec{r}\right|}$$

$$= \frac{0}{\sqrt{2}}, \quad \frac{1}{\sqrt{2}}, \quad \frac{-1}{\sqrt{2}}$$

So, $l = \cos\alpha = 0$

$\alpha = \cos^{-1}(0)$

$\alpha = \frac{\pi}{2}$

$m = \cos\beta = \frac{1}{\sqrt{2}}$

$\beta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\beta = \frac{\pi}{4}$

$n = \cos\gamma = -\frac{1}{\sqrt{2}}$

$\gamma = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$\lambda = \pi - \frac{\pi}{7}$

$\gamma = \frac{3\pi}{4}$

So, angles made by the vector $\vec{r}$ with coordinate axes are given by

$\frac{\pi}{2}, \quad \frac{\pi}{4}, \quad \frac{3\pi}{4}$

Question 7 (iii)

Find the angles at which the vector $4\hat{i} + 8\hat{j} + \hat{k}$ is inclined to each of the coordinate axes.

Solution 7 (iii)

Let, $4\hat{i} + 8\hat{j} + \hat{k} = \vec{r}$

The direction ratios of $\vec{r}$ = 4, 8, 1

And, $|\vec{r}| = \sqrt{(4)^2 + (8)^2 + (1)^2}$

$= \sqrt{16 + 64 + 1}$

$= \sqrt{81}$

$|\vec{r}| = 9$

The direction cosines of the $\vec{r}$ are given by

$= \frac{4}{|\vec{r}|}, \quad \frac{8}{|\vec{r}|}, \quad \frac{1}{|\vec{r}|}$

$= \frac{4}{9}, \quad \frac{8}{9}, \quad \frac{1}{9}$

Now, $l = \cos\alpha = \frac{4}{9}$

$\alpha = \cos^{-1}\left(\frac{4}{9}\right)$

$m = \cos\beta = \frac{8}{9}$

$\beta = \cos^{-1}\left(\frac{8}{9}\right)$

$n = \cos\gamma = \frac{1}{9}$

$\gamma = \cos^{-1}\left(\frac{1}{9}\right)$

The angles made by the vector $\vec{r}$ with the coordinate axes are given by

$\cos^{-1}\left(\frac{4}{9}\right), \quad \cos^{-1}\left(\frac{8}{9}\right), \quad \cos^{-1}\left(\frac{1}{9}\right)$

Question 8

Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY, and OZ.

Solution 8

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

Then,

$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Therefore, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Now, let α, β, and γ be the angles formed by $\vec{a}$ with the positive directions of x, y, and z axes.

Then, we have $\cos\alpha = \frac{1}{\sqrt{3}}, \cos\beta = \frac{1}{\sqrt{3}}, \cos\gamma = \frac{1}{\sqrt{3}}$.

Hence, the given vector is equally inclined to axes OX, OY, and OZ.

Question 9

Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Solution 9

Let a vector be equally inclined to axes OX, OY, and OZ at angle α.

Then, the direction cosines of the vector are cos α, cos α, and cos α.

Now,

$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$\Rightarrow 3\cos^2\alpha = 1$

$\Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$

Hence, the direction cosines of the vector which are equally inclined to the axes are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Question 10

If a unit vector $\vec{a}$ makes an angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the compounds of $\vec{a}$.

Solution 10

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

$\Rightarrow \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$.

Also, it is given that $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$.

Then, we have:

$$\cos\frac{\pi}{3} = \frac{a_1}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{2} = a_1 \qquad \left[|\vec{a}| = 1\right]$$

$$\cos\frac{\pi}{4} = \frac{a_2}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = a_2 \qquad \left[|\vec{a}| = 1\right]$$

$$\text{Also, } \cos\theta = \frac{a_3}{|\vec{a}|}.$$

$$\Rightarrow a_3 = \cos\theta$$

Now,

$$|a| = 1$$

$$\Rightarrow \sqrt{a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2}} = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$$

$$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$$

$$\Rightarrow \frac{3}{4} + \cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\therefore a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$$

Hence, $\theta = \frac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

Question 11

Find a vector $\vec{r}$ of magnitude $3\sqrt{2}$ units which makes an angle of $\frac{\pi}{4}$ and $\frac{\pi}{2}$ with y and z-axes respctiovely.

Solution 11

Let l, m, n be the direction cosines of the vector $\vec{r}$.

$l = \cos\alpha$, $m = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $n = \cos\left(\frac{\pi}{2}\right) = 0$

$l^2 + m^2 + n^2 = 1$

$l^2 + \frac{1}{2} + 0 = 1$

$l = \pm\frac{1}{\sqrt{2}}$

$\vec{r} = |\vec{r}|\left(l\hat{i} + m\hat{j} + n\hat{k}\right)$

$\vec{r} = 3\sqrt{2}\left(\pm\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + 0\right)$

$\vec{r} = \pm 3\hat{i} + 3\hat{j}$

Question 12

A vector $\vec{r}$ is indined at equal angles to the three axes.
if the magnitude of $\vec{r}$ is $2\sqrt{3}$, find $\vec{r}$.

Solution 12

Let l, m, n be the direction cosines of the vector $\vec{r}$.

Vector $\vec{r}$ is indined at equal angles to the three axes.

$l = \cos\alpha$, $m = \cos\alpha$ and $n = \cos\alpha$

$\Rightarrow l = m = n$

$l^2 + m^2 + n^2 = 1$

$\Rightarrow l = m = n = \pm\dfrac{1}{\sqrt{3}}$

$\vec{r} = |\vec{r}|\left(l\hat{i} + m\hat{j} + n\hat{k}\right)$

$\vec{r} = 2\sqrt{3}\left(\pm\dfrac{1}{\sqrt{3}}\hat{i} \pm \dfrac{1}{\sqrt{3}}\hat{j} \pm \dfrac{1}{\sqrt{3}}\hat{k}\right)$

$\vec{r} = \pm 2\hat{i} \pm 2\hat{j} \pm 2\hat{k}$

Chapter 23 - Algebra of Vectors Exercise MCQ

Question 1

If in a ΔABC, A= (0,0), B = $(3,3\sqrt{3})$, = $(-3\sqrt{3},3)$, then the vector of magnitude $2\sqrt{2}$ units directed along AO, where O is the circumcentre of ΔABC is

(a) $(1-\sqrt{3})\hat{i} + (1+\sqrt{3})\hat{j}$

(b) $(1+\sqrt{3})\hat{i} + (1-\sqrt{3})\hat{j}$

(c) $(1+\sqrt{3})\hat{i} + (1-\sqrt{3})\hat{j}$

(d) none of these

Solution 1

Correct option:(a)

$\left|\overrightarrow{AO}\right| = 2\sqrt{2}$

O is the circumcentre.

$\Rightarrow \left|\overrightarrow{AO}\right| = \left|\overrightarrow{BO}\right| = \left|\overrightarrow{CO}\right| = 2\sqrt{2} = R$

Position vector of O be $x\hat{i} + y\hat{j}$.

$\Rightarrow \left|\overrightarrow{AO}\right| = \sqrt{x^2 + y^2}$

$\Rightarrow x^2 + y^2 = 8 \ldots\ldots (i)$

And

$\left|\overrightarrow{BO}\right| = \left|\overrightarrow{CO}\right|$

$$\sqrt{(x-3)^2 + \left(y - 3\sqrt{3}\right)^2} = \sqrt{\left(x + 3\sqrt{3}\right)^2 + (y-3)^2}$$

$$\Rightarrow y = \frac{x\left(1+\sqrt{3}\right)}{1-\sqrt{3}}$$

Put it in (i)

$x^2 + y^2 = 8$

$$x^2 + \left[\frac{x\left(1+\sqrt{3}\right)}{1-\sqrt{3}}\right]^2 = 8$$

$\Rightarrow x = 1 - \sqrt{3}$

$\Rightarrow y = 1 + \sqrt{3}$

$\overline{AO} = \left(1-\sqrt{3}\right)\hat{i} + \left(1+\sqrt{3}\right)\hat{j}$

Question 2

If $\vec{a}, \vec{b}$ are the vector forming consecutive side of a regular hexagon ABCDEF, then the vector representing side CD is

(a) $\vec{a} + \vec{b}$

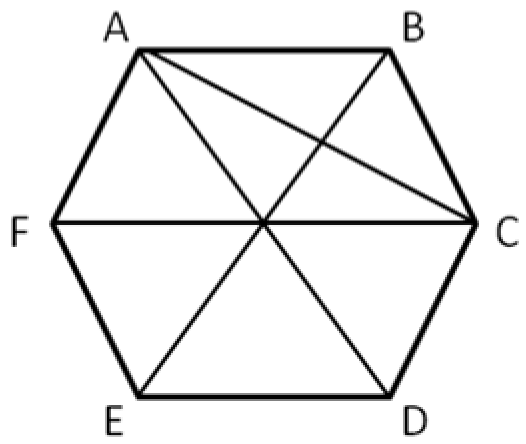
(b) $\vec{a} - \vec{b}$

(c) $\vec{b} - \vec{a}$

(d) $-(\vec{a} + \vec{b})$

Solution 2

Correct option:(c)



In regular hexagon $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$.
From the diagram we can say that AD = 2BC
as AD is parallel to BC.
In ΔABC,
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
In ΔACD
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
$\overrightarrow{AC} = \vec{a} + \vec{b}$
In ΔACD,
$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$
$\left(\vec{a} + \vec{b}\right) + \overrightarrow{CD} = 2\vec{b}$
$\Rightarrow \overrightarrow{CD} = 2\vec{b} - \vec{a} - \vec{b}$
$\Rightarrow \overrightarrow{CD} = \vec{b} - \vec{a}$

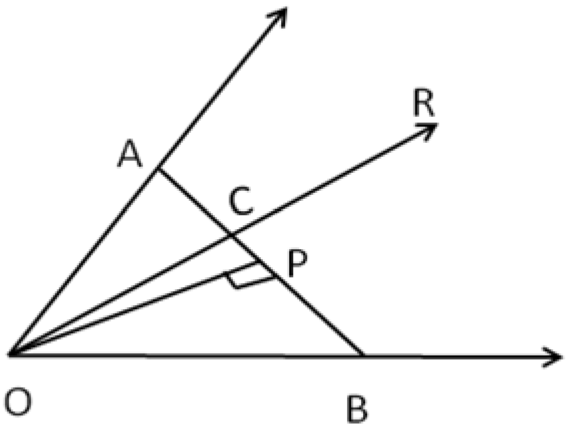
Question 3

Forces 3 $\overrightarrow{OA}$, 5 $\overrightarrow{OB}$ act along OA and OB. If their resultant passes through C on AB, then

a. C is a mid-point of AB
b. C divides AB in the ratio 2:1
c. 3 AC = 5CB
d. 2 AC = 3CB

Solution 3

Correct option: (c)



Contrsuction : OP perpendicular to AB.

Let, $\hat{i}$ be the unit vector along OP.

Resultant force $\vec{R} = 3\overrightarrow{OA} + 5\overrightarrow{OB}$

The angles between unit vector along x – axis and

$\vec{R}$, $3\overrightarrow{OA}$, $5\overrightarrow{OB}$ are $\angle COP$, $\angle AON$, $\angle BON$ respectively

$\Rightarrow R\cos\angle COP = 3OA \times \frac{OP}{OA} + 5OB \times \frac{OP}{OB}$

$\frac{R}{OD} = 3 + 5$

$R = 8\overrightarrow{OC}$

$\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{CA}$

$\Rightarrow 3\overrightarrow{OA} = 3\overrightarrow{OC} + 3\overrightarrow{CA}$.......(i)

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$\Rightarrow 5\overrightarrow{OB} = 5\overrightarrow{OC} + 5\overrightarrow{CB}$.........(ii)

$3\overrightarrow{OA} + 5\overrightarrow{OB} = 8\overrightarrow{OC} + 3\overrightarrow{CA} + 5\overrightarrow{CB}$

$\vec{R} = 8\overrightarrow{OC} + 3\overrightarrow{CA} + 5\overrightarrow{CB}$

$8\overrightarrow{OC} = 3\overrightarrow{CA} + 5\overrightarrow{CB}$

$\left|3\overrightarrow{CA}\right| = \left|5\overrightarrow{CB}\right|$

$\Rightarrow 3CA = 5CB$

Question 4

If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors, no two of which are collinear and the vector $\vec{a} + \vec{b}$ is collinear with $\vec{c}$, $\vec{b} + \vec{c}$ is collinear with $\vec{a}$, then $\vec{a} + \vec{b} + \vec{c} =$

(a) $\vec{a}$

(b) $\vec{b}$

(c) $\vec{c}$

(d) none of these

Solution 4

Correct option: (d)

$\vec{a}+\vec{b}$ is collinear with $\vec{c}$

$\Rightarrow \vec{a}+\vec{b}=u\vec{c}$..........(i)

where u is the scalar quantity and $u \neq 0$

$\vec{b}+\vec{c}$ is collinear with $\vec{a}$

$\Rightarrow \vec{b}+\vec{c}=v\vec{a}$..........(ii)

where v is the scalar quantity and $v \neq 0$

Subtracting (i) from (ii)

$\vec{b}+\vec{c}-\left(\vec{a}+\vec{b}\right)=v\vec{a}-u\vec{c}$

$\vec{c}-\vec{a}=v\vec{a}-u\vec{c}$

$\vec{c}+u\vec{c}=\vec{a}+v\vec{a}$

$(1+u)\vec{c}=\vec{a}(1+v)$

As $\vec{c}$, $\vec{a}$ are not collinear.

$\Rightarrow u+1=0 \Rightarrow u=-1$

and $1+v=0 \Rightarrow v=-1$

$\vec{a}+\vec{b}=u\vec{c} \Rightarrow \vec{a}+\vec{b}=-\vec{c}$

$\vec{b}+\vec{c}=-v\vec{a} \Rightarrow \vec{b}+\vec{c}=-\vec{a} \Rightarrow \vec{a}+\vec{b}+\vec{c}=0$

Question 5

If point A $(60\hat{i}+3\hat{j})$, B$(40\hat{i}-8\hat{j})$ and C $(a\hat{i}-52\hat{j})$ are collinear, then a is equal to

a. 40
b. -40
c. 20
d. -20

Solution 5

Correct option: (b)

Given that point A $(60\hat{i} + 3\hat{j})$, B$(40\hat{i} - 8\hat{j})$ and C $(a\hat{i} - 52\hat{j})$ are collinear

$\Rightarrow \overrightarrow{AB} = k\overrightarrow{BC}$

$\Rightarrow (40 - 60)\hat{i} + (-8 - 3)\hat{j} = k\left[(a - 40)\hat{i} + (-52 + 8)\hat{j}\right]$

$\Rightarrow -20\hat{i} - 11\hat{j} = k\left[(a - 40)\hat{i} - 44\hat{j}\right]$

$\Rightarrow -20\hat{i} - 11\hat{j} = k(a - 40)\hat{i} - 44k\hat{j}$

Comparing both sides

$-11 = -44k$

$\Rightarrow k = \frac{1}{4}$

and

$k(a - 40) = -20$

$\frac{1}{4}(a - 40) = -20$

$a - 40 = -80$

$a = -40$

Question 6

If G is the intersection of diagonals of a parallelogram ABCD and O is any point, then $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} =$

(a) $2\ \overrightarrow{OG}$

(b) $4\ \overrightarrow{OG}$

(c) $5\ \overrightarrow{OG}$

(d) $3\ \overrightarrow{OG}$

Solution 6

Correct option: (b)

Let O be the origin.

G is mid point of AC.

$\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OC}}{2}$

$2\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{OC}$

G is also mid point of BD

$\overrightarrow{OG} = \frac{\overrightarrow{OB} + \overrightarrow{OD}}{2}$

Adding both equations,

$4\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OB} + \overrightarrow{OD}$

Question 7

The vector $\cos\alpha \cos\beta\ \hat{i} + \cos\alpha \sin\beta\ \hat{j} + \sin\alpha \hat{k}$ is a

a. Null vector
b. Unit vector

c. Constant vector
d. None of these

Solution 7

Correct option: (b)

$\cos\alpha\cos\beta\,\hat{i} + \cos\alpha\sin\beta\,\hat{j} + \sin\alpha\hat{k}$

Magnitude of a vector

$= \sqrt{\cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha}$

$= \sqrt{\cos^2\alpha\left(\cos^2\beta + \sin^2\beta\right) + \sin^2\alpha}$

$= \sqrt{\cos^2\alpha + \sin^2\alpha}$

$= 1$

Hence, it is unit vector.

Question 8

In a regular hexagon ABCDEF , $\overrightarrow{AB} = a, \overrightarrow{BC} = \vec{b}$ and $\overrightarrow{CD} = \vec{c}$.

Then , $\overrightarrow{AE} =$

(a) $\vec{a} + \vec{b} + \vec{c}$

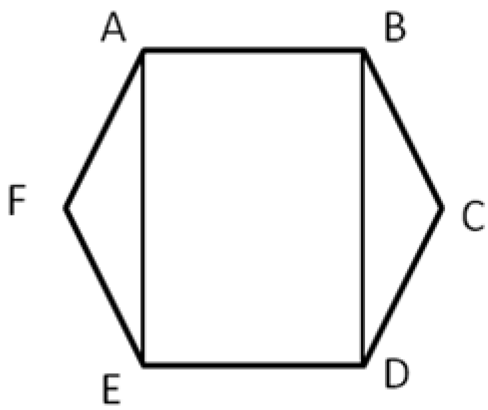
(b) $2\vec{a} + \vec{b} + \vec{c}$

(c) $\vec{b} + \vec{c}$

(d) $\vec{a} + 2\vec{b} + 2\vec{c}$

Solution 8

Correct option: (c)



In ΔBCD,

$\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$

Given that $\overrightarrow{BC} = \vec{b}$, $\overrightarrow{CD} = \bar{c}$

$\Rightarrow \overrightarrow{BD} = \vec{b} + \bar{c}$

And $\overrightarrow{BD}$ is parallel to $\overrightarrow{AE}$.

$\Rightarrow \overrightarrow{AE} = \vec{b} + \bar{c}$

Question 9

The vector equation of the plane passing through $\vec{a}, \vec{b}, \vec{c}$ is $\vec{r} = \alpha\ \vec{a} + \beta\ \vec{b} + \gamma\ \vec{c}$, provided that

(a) $\alpha + \beta + \gamma = 0$

(b) $\alpha + \beta + \gamma = 1$

(c) $\alpha + \beta = \gamma$

(d) $\alpha^2 + \beta^2 + \gamma^2 = 1$

Solution 9

Correct option: (b)

Given that a plane is passing through $\vec{a}, \vec{b}, \vec{c}$.

$\vec{a} - \vec{b}$ and $\vec{b} - \vec{c}$ lie on the same plane.

The parametric equation of the plane

$\vec{r} = \lambda_1\left(\vec{a} - \vec{b}\right) + \lambda_2\left(\vec{b} - \vec{c}\right) + \vec{c}$

$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \lambda_1\left(\vec{a} - \vec{b}\right) + \lambda_2\left(\vec{b} - \vec{c}\right) + \vec{c}$

$\Rightarrow \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \lambda_1\vec{a} - \lambda_1\vec{b} + \lambda_2\vec{b} - \lambda_2\vec{c} + \vec{c}$

$\Rightarrow \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \lambda_1\vec{a} + \vec{b}\left(-\lambda_1 + \lambda_2\right) + \vec{c}\left(-\lambda_2 + 1\right)$

$\Rightarrow \alpha = \lambda_1$

$\beta = -\lambda_1 + \lambda_2$

$\gamma = -\lambda_2 + 1$

$\Rightarrow \alpha + \beta + \gamma = 1$

Question 10

If O and O' are circumcentre and orthocentre of ΔABC, then $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$ equals

(a) $2\ \overrightarrow{OO}$

(b) $\overline{O\ O'}$

(c) $O'\ \vec{O}$

(d) $2\ O'\vec{O}$

Solution 10

Correct option: (b)

Given that O and O' are circumcentre and orthocentre of ΔABC.
G be the centroid of the triangle.
We know that $\overrightarrow{OO'} = 3\overrightarrow{OG}$.......(i)

$$\overrightarrow{OG} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

$$\frac{\overrightarrow{OO'}}{3} = \frac{\vec{a}+\vec{b}+\vec{c}}{3} \text{......from (i)}$$

$$\overrightarrow{OO'} = \vec{a}+\vec{b}+\vec{c}$$

$$\Rightarrow \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC} = \overrightarrow{OO'}$$

Question 11

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the position vectors of points A, B,C,D such that no three of them are collinear and $\vec{a}+\vec{c} = \vec{b}+\vec{d}$, then ABCD is a

a. Rhombus
b. Rectangle
c. Square
d. Parallelogram

Solution 11

Correct option: (d)

$$\vec{a}+\vec{c} = \vec{b}+\vec{d}$$

$$\Rightarrow \vec{a}-\vec{b} = \vec{d}-\vec{c}$$

$$\Rightarrow \overrightarrow{BA} = \overrightarrow{CD}$$

Also,

$$\vec{a}+\vec{c} = \vec{b}+\vec{d}$$

$$\frac{\vec{a}+\vec{c}}{2} = \frac{\vec{b}+\vec{d}}{2}$$

Position vector of mid point AC = Position vector of mid point BD
$\Rightarrow$ diagonals bisect each other.
ABCD is parallelogram.

Question 12

Let G be the centroid of ΔABC. If $\overrightarrow{AB} = \vec{a}, \overrightarrow{AC} = \vec{b}$, then the bisector $\overrightarrow{AG}$,in terms of $\vec{a}$ and $\vec{b}$ is

(a) $\frac{2}{3}(\vec{a}+\vec{b})$

(b) $\frac{1}{6}(\vec{a}+\vec{b})$

(c) $\frac{1}{3}(\vec{a}+\vec{b})$

(d) $\frac{1}{2}(\vec{a}+\vec{b})$

Solution 12

Correct option: (a)

Consider A as origin.

Position vector of A, B, C are $0, \vec{a}, \vec{b}$ respectively.

position vector of centroid G is $\dfrac{\vec{a}+\vec{b}}{3}$.

$\Rightarrow AG = \dfrac{\vec{a}+\vec{b}}{3}$ as A is the origin.

NOTE: Answer not matching with back answer.

Question 13

If ABCDEF is a regular hexagon, then $\overrightarrow{AD}+\overrightarrow{EB}+\overrightarrow{FC}$ equals

(a) $2\overrightarrow{AB}$

(b) $\vec{0}$

(c) $3\overrightarrow{AB}$

(d) $4\overrightarrow{AB}$

Solution 13

Correct option: (d)

$AD = 2\overrightarrow{BC}$

$\overrightarrow{FC} = 2\overrightarrow{AB}$

$\overrightarrow{EB} = 2\overrightarrow{FA}$

$\Rightarrow$

$\overrightarrow{AD}+\overrightarrow{EB} = 2\overrightarrow{BC}+2\overrightarrow{FA}$

$\overrightarrow{AD}+\overrightarrow{EB} = 2\overrightarrow{AO}+2\overrightarrow{FA} \qquad \left(\because \overrightarrow{BC}=\overrightarrow{AO}\right)$

$\overrightarrow{AD}+\overrightarrow{EB} = 2\left(\overrightarrow{AO}+\overrightarrow{FA}\right)$

In ΔAOF,

$\overrightarrow{FA}+\overrightarrow{AO}+\overrightarrow{FO} = 0$

$\overrightarrow{FA}+\overrightarrow{AO} = -\overrightarrow{FO}$

$\overrightarrow{AD}+\overrightarrow{EB} = 2\overrightarrow{AB}$

$\overrightarrow{AD}+\overrightarrow{EB}+\overrightarrow{FC} = 4\overrightarrow{AB}$

Question 14

The position vectors of the points A,B,C are

$2\hat{i}+\hat{j}-\hat{k}, 3\hat{i}-2\hat{j}+\hat{k}$ and $\hat{i}+4\hat{j}-3\hat{k}$ respectively

these points

a. form an isosceles triangle
b. form a right triangle
c. are collinear

d. form a scalene triangle

Solution 14

Correct option: (a)

$A = 2\hat{i} + \hat{j} - \hat{k},$

$B = 3\hat{i} - 2\hat{j} + \hat{k}$ and

$C = \hat{i} + 4\hat{j} - 3\hat{k}$

$\overrightarrow{AB} = \hat{i} - 3\hat{j} + 2\hat{k}$

$\overrightarrow{BC} = -2\hat{i} + 6\hat{j} - 4\hat{k}$

$\overrightarrow{AC} = -\hat{i} + 3\hat{j} - 2\hat{k}$

$\Rightarrow \left|\overrightarrow{AB}\right| = \sqrt{1 + 9 + 4} = \sqrt{14}$

$\left|\overrightarrow{BC}\right| = \sqrt{4 + 36 + 16} = \sqrt{56}$

$\left|\overrightarrow{AC}\right| = \sqrt{1 + 9 + 4} = \sqrt{14}$

$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AC}\right|$

$\Rightarrow$ Isosceles triangle.

Question 15

If three point A, B and C have positive vectors

$\hat{i} + x\hat{j} + 3\hat{k}, 3\hat{i} + 4\hat{j} + 7\hat{k}$ and $y\hat{i} - 2\hat{j} - 5\hat{k}$

respectively are collinear, then $(x, y) =$

a. (2,-3)
b. (-2,3)
c. (-2,-3)
d. (2,3)

Solution 15

Correct option: (a)

$X = \hat{i} + x\hat{j} + 3\hat{k}$,

$Y = 3\hat{i} + 4\hat{j} + 7\hat{k}$ and

$Z = y\hat{i} - 2\hat{j} - 5\hat{k}$

Consider,

$XY = \lambda XZ$

$2\hat{i} + (4-x)\hat{j} + 4\hat{k} = \lambda\left((y-1)\hat{i} + (-2-x)\hat{j} - 8\hat{k}\right)$

$2\hat{i} + (4-x)\hat{j} + 4\hat{k} = \lambda(y-1)\hat{i} + \lambda(-2-x)\hat{j} - 8\lambda\hat{k}$

Comparing both sides,

$-8\lambda = 4$

$\Rightarrow \lambda = \frac{-1}{2}$

and $2 = \lambda(y-1)$

$2 = \frac{-1}{2}(y-1)$

$-4 = y - 1$

$y = -3$

And

$\lambda(-2-x) = 4 - x$

$\frac{-1}{2}(-2-x) = 4 - x$

$2 + x = 8 - 2x$

$3x = 6$

$x = 2$

Question 16

ABCD is a parallelogram with AC and BD as diagonals.

Than, $\overrightarrow{AC} - \overrightarrow{BD} =$

(a) $4\,\overrightarrow{AB}$

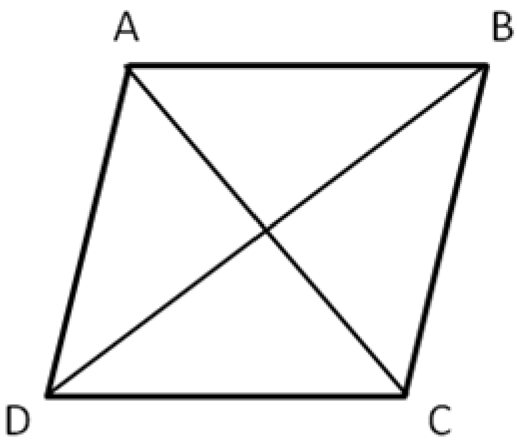
(b) $3\,\overrightarrow{AB}$

(c) $2\,\overrightarrow{AB}$

(d) $\overrightarrow{AB}$

Solution 16

Correct option: (c)



In the paralellogram ABCD diagonals are AC and BD.

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

also,

$\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}$

$\Rightarrow \overrightarrow{BD}=\overrightarrow{AD}-\overrightarrow{AB}$

$\overrightarrow{AC}-\overrightarrow{BD}=2\overrightarrow{AB}$

Question 17

If OACB is a parallelogram with $\overrightarrow{OC}=\vec{a}$ and $\overrightarrow{AB}=\vec{b}$, then $\overrightarrow{OA}=$

(a) $(\vec{a}+\vec{b})$

(b) $(\vec{a}-\vec{b})$

(c) $\frac{1}{2}(\overrightarrow{b-a})$

(d) $\frac{1}{2}(\vec{a}-\vec{b})$

Solution 17

Correct option: (d)

Given parallelogram OACB such that $\overrightarrow{OC}=\vec{a}$

$\overrightarrow{AB}=\vec{b}$

$\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}$

$\overrightarrow{OB}=\vec{a}-\overrightarrow{OA} \qquad (\because \overrightarrow{BC}=\overrightarrow{OA})$

$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}$

$\vec{a}-\overrightarrow{OA}=\overrightarrow{OA}+\vec{b}$

$\overrightarrow{OA}=\frac{\vec{a}-\vec{b}}{2}$

Question 18

If a and b are two collinear vectors, then which of the following are incorrect?

(a) $\vec{b} = \lambda \vec{a}$ for some scalar λ

(b) $\vec{a} = \pm \vec{b}$

(c) the respective components of $\vec{a}$ and $\vec{b}$ are proportional

(d) both the vectors $\vec{a}$ and $\vec{b}$ have the same direction but different magnitudes

Solution 18

Correct option: (d)

By definition of collinearity,

$\vec{b} = \lambda\vec{a}$ for some scalar λ.

$\vec{a} = \pm\vec{b}$

The respective components of $\vec{a}$ and $\vec{b}$ are proportional.

All are correct.

Option (d) is incorrect.

Question 19

In Figure 23.67, which of the following is not true?

(a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

(b) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

(c) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$

(d) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

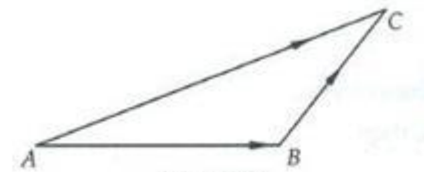


Fig. 23.68

Solution 19

Correct option: (c)

From the given diagram

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \overrightarrow{AC} - \overrightarrow{CA}$

$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = 2\overrightarrow{AC}$

Hence, $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = 0$ is not true.

Chapter 23 - Algebra of Vectors Exercise Ex. 23VSAQ

Question 1

Define "zero vector".

Solution 1

Zero vector is a vector whose initial point and terminal point are coincident.

It is represented by $\vec{0}$.

And, $\vec{0} = 0 \times \hat{i} + 0 \times \hat{j} + 0 \times \hat{k}$

$$\left|\vec{0}\right| = \sqrt{0^2 + 0^2 + 0^2}$$
$$= \sqrt{0}$$
$$\left|\vec{0}\right| = 0$$

Question 2

Define unit vector.

Solution 2

A vector whose modulus or magnitude or length is one unit is called unit vector.

The unit vector in the direction of a vector $\vec{a} = \dfrac{\vec{a}}{\left|\vec{a}\right|}$

It is represented by $\hat{a}$. "a cap".

$$\left|\hat{a}\right| = 1$$

Question 3

Define position vector of a point.

Solution 3

Position vector of a point P with respect to a point Q is represented as $\overrightarrow{PQ}$.

Here, P is called the initial point and Q is called the terminal point.

Question 4

Write $\overrightarrow{PQ} + \overrightarrow{RP} + \overrightarrow{QR}$ in the simplified form.

Solution 4

$$\overrightarrow{PQ} + \overrightarrow{RP} + \overrightarrow{QR}$$

= Position vector of $\vec{Q}$ – Position vector of P
+ Position vector of P – Position vector of R
+ Position vector of R – Position vector of Q
= $\vec{0}$

Here,

$$\overrightarrow{PQ} + \overrightarrow{RP} + \overrightarrow{QR} = \vec{0}$$

Question 5

Find $\vec{a}$ and $\vec{b}$ are two non-collinear vectors such that $x\vec{a} + y\vec{b} = \vec{0}$, then write the values of x and y.

Solution 5

Here,

$$x\vec{a} + y\vec{b} = \vec{0}$$
$$x\vec{a} + y\vec{b} = 0 \times \vec{a} + 0 \times \vec{b}$$

Comparing coefficients of $\vec{a}$ and $\vec{b}$ of LHS and RHS,

$$x = 0$$
$$y = 0$$

Question 6

Find $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a parallelogram, then write vectors representing its diagonals.

Solution 6

Given that $\vec{a}$, $\vec{b}$ represent the two adjasent sides of parallelogram
In ΔABC, using triangle law

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$
$$\vec{a} + \vec{b} = \overrightarrow{AC}$$

In ΔABD, using triangle law

$$\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$$
$$\vec{b} + \overrightarrow{DB} = \vec{a}$$
$$\vec{a} - \vec{b} = \overrightarrow{DB}$$

$\therefore$ Diagonals $\overrightarrow{AC} = \vec{a} + \vec{b}$

$\overrightarrow{DB} = \vec{a} - \vec{b}$

Question 7

If $\vec{a}, \vec{b}, \vec{c}$ represent the sides of a triangle taken in order, then write the value of $\vec{a} + \vec{b} + \vec{c}$.

Solution 7

Let, ABC be a triangle such that

$\overrightarrow{BC} = \vec{a}$

$\overrightarrow{CA} = \vec{b}$

$\overrightarrow{AB} = \vec{c}$

Then,

$\vec{a} + \vec{b} + \vec{c}$

$= \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}$

$= \overrightarrow{BA} + \overrightarrow{AB}$ $\quad \left[\text{Since } \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}\right]$

$= \overrightarrow{BA} - \overrightarrow{BA}$ $\quad \left[\text{Since } -\overrightarrow{BA} = -\vec{\ }\right]$

$= \vec{0}$

So, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

Question 8

If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of the vertices A, B and C respectively, of a triangle ABC, write the value of $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

Solution 8

Given $\vec{a}, \vec{b}$ and $\vec{c}$ are position vectors A, B, C respectively

$$\overrightarrow{AB} = \vec{b} - \vec{a} \qquad --(\text{i})$$
$$\overrightarrow{BC} = \vec{c} - \vec{b} \qquad --(\text{ii})$$
$$\overrightarrow{CA} = \vec{a} - \vec{c} \qquad --(\text{iii})$$

Adding (i), (ii), and (iii), we get

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{b} - \vec{a} + \vec{c} - \vec{b} + \vec{a} - \vec{c}$$
$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$$

Question 9

If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of the points A, B and C respectively, write the value of $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

Solution 9

If $\vec{a}, \vec{b}$ and $\vec{c}$ are position vectors of the point $A, B,$ and C respectively

$$\overrightarrow{AB} = \vec{b} - \vec{a}$$
$$\overrightarrow{BC} = \vec{c} - \vec{b}$$
$$\overrightarrow{AC} = \vec{a} - \vec{c}$$

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AC} = \vec{b} - \vec{a} + \vec{c} - \vec{b} + \vec{a} - \vec{c}$$
$$= 2\vec{c} - 2\vec{a}$$
$$= 2\left(\vec{c} - \vec{a}\right)$$
$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AC} = 2\left(\vec{c} - \vec{a}\right)$$

Question 10

If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of the vertices of a triangle, then write the position vector of its centroid.

Solution 10

Let $\vec{a}, \vec{b}, \vec{c}$ are position vectors of vertices A, B, C respectively of triangle.

Let D be the mid point of AB,

Position vector of $D = \dfrac{\vec{a} + \vec{b}}{2}$

Let O be the centroid of triangle.

We know that centroid is a point that divides AD in ratio 2:1.

Position vector of O

$$= \frac{2\left(\dfrac{\vec{a} + \vec{b}}{2}\right) + 1.\vec{c}}{2 + 1} \qquad \left[\text{Using intarnal section formula } \frac{mb + na}{m + n}\right]$$

$$O = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

Question 11

If G denotes the centroid of ΔABC, then write the value of $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC}$.

Solution 11

Here, G denotes the centroid of triangle ΔABC,

Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of points A, B and C respectively.

G is the centroid of the triangle ΔABC,

So position vector of $G\left(\vec{g}\right) = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

$$\overrightarrow{GA} = \vec{a} - \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)$$
$$= \frac{3\vec{a}-\vec{a}-\vec{b}-\vec{c}}{3}$$
$$\overrightarrow{GA} = \frac{2\vec{a}-\vec{b}-\vec{c}}{3} \qquad --(\text{i})$$

$$\overrightarrow{GB} = \vec{b} - \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)$$
$$\overrightarrow{GB} = \frac{3\vec{b}-\vec{a}-\vec{b}-\vec{c}}{3}$$
$$\overrightarrow{GB} = \frac{2\vec{b}-\vec{a}-\vec{c}}{3} \qquad --(\text{ii})$$

$$\overrightarrow{GC} = \vec{c} - \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)$$
$$= \frac{3\vec{c}-\vec{a}-\vec{b}-\vec{c}}{3}$$
$$\overrightarrow{GC} = \frac{2\vec{c}-\vec{a}-\vec{b}}{3} \qquad --(\text{iii})$$

Adding (i), (ii) and (iii),

$$\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC} = \frac{2\vec{a}-\vec{b}-\vec{c}}{3}+\frac{2\vec{b}-\vec{a}-\vec{c}}{3}+\frac{2\vec{c}-\vec{a}-\vec{b}}{3}$$
$$= \frac{2\vec{a}-\vec{b}-\vec{c}+2\vec{b}-\vec{a}-\vec{c}+2\vec{c}-\vec{a}-\vec{b}}{3}$$
$$= \frac{2\vec{a}-2\vec{a}+2\vec{b}-2\vec{b}+2\vec{c}-2\vec{c}}{3}$$
$$= \frac{\vec{0}}{3}$$
$$\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC} = \vec{0}$$

Question 12

If $\vec{a}$ and $\vec{b}$ denote the position vectors of points A and B respectively and C is a point on AB such that $3AC = 2AB$, then write the position vector of C.

Solution 12

Here, given that $\vec{a}$ and $\vec{b}$ denote the position vectors of points A and B respectively.
C is a point on AB such that

$$3AC = 2AB$$

$$\frac{AC}{AB} = \frac{2}{3}$$

Thus, C divides AB in 2:1

Therefore, position vector of C

$$\vec{c} = \frac{2\vec{b} + \vec{a}}{2+1} \qquad \left[\text{Using internal section formula } \frac{m\vec{b} + n\vec{a}}{(m+n)}\right]$$

$$\vec{c} = \frac{2\vec{b} + \vec{a}}{3}$$

Question 13

If D is the mid-point of side BC of a triangle ABC such that $AB + AC = \lambda AD$, write the value of λ.

Solution 13

Here, given that D is the mid-point of the side BC of a triangle ABC

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of A, B and C respectively.
D is mid-point of BC. Then

Position vector of $D\ (\vec{d}) = \dfrac{\vec{b}+\vec{c}}{2}$ $\left[\text{Using mid-point formula } \dfrac{\vec{a}+\vec{b}}{2}\right]$

$$\overrightarrow{AB} = \vec{b} - \vec{a} \qquad --(\text{i})$$

$$\overrightarrow{AC} = \vec{c} - \vec{a} \qquad --(\text{ii})$$

$$\overrightarrow{AD} = \vec{d} - \vec{a}$$
$$= \frac{\vec{b}+\vec{c}}{2} - \vec{a}$$
$$\overrightarrow{AD} = \frac{\vec{b}+\vec{c}-2\vec{a}}{2} \qquad --(\text{iii})$$

It is given that,
$$\overrightarrow{AB} + \overrightarrow{AC} = \lambda \overrightarrow{AD}$$
$$\left(\vec{b}-\vec{a}\right) + \left(\vec{c}-\vec{a}\right) = \lambda\, \frac{\vec{b}+\vec{c}-2\vec{a}}{2}$$
$$\frac{\vec{b}+\vec{c}-2\vec{a}}{1} = \lambda\left(\frac{\vec{b}+\vec{c}-2\vec{a}}{2}\right)$$

Cross multiplying, we get
$$2\left(\vec{b}+\vec{c}-2\vec{a}\right) = \lambda\left(\vec{b}+\vec{c}-2\vec{a}\right)$$
$$\frac{2\left(\vec{b}+\vec{c}-2\vec{a}\right)}{\left(\vec{b}+\vec{c}-2\vec{a}\right)} = \lambda$$
$$\lambda = 2$$

Question 14

If D, E, F are the mid-point of the sides BC, CA and AB respectively of a triangle ABC, write the value of $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}$.

Solution 14

Given that D, E, F are the mid-point of BC, CA and AB respectively of a triangle ABC.

Let $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of A, B, C respectively.

Position vector of $D = \frac{\vec{b}+\vec{c}}{2}$

Position vector of $E = \frac{\vec{a}+\vec{c}}{2}$

Position vector of $F = \frac{\vec{a}+\vec{b}}{2}$ $\qquad \left[\text{Using mid point formula, } \frac{\vec{a}+\vec{b}}{2}\right]$

$\overrightarrow{AD}$ = Position vector of D – Position vector of A

$$= \frac{\vec{b}+\vec{c}}{2} - \frac{\vec{a}}{1}$$

$$\overrightarrow{AD} = \frac{\vec{b}+\vec{c}-2\vec{a}}{2} \qquad ---(\text{i})$$

$\overrightarrow{BE}$ = Position vector of E – Position vector of B

$$= \frac{\vec{a}+\vec{c}}{2} - \frac{\vec{b}}{1}$$

$$\overrightarrow{BE} = \frac{\vec{b}+\vec{c}-2\vec{b}}{2} \qquad ---(\text{ii})$$

$\overrightarrow{CF}$ = Position vector of F – Position vector of C

$$= \frac{\vec{a}+\vec{b}}{2} - \vec{c}$$

$$\overrightarrow{CF} = \frac{\vec{a}+\vec{b}-2\vec{c}}{2} \qquad ---(\text{iii})$$

Adding (i), (ii) and (iii),

$$\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF} = \frac{\vec{b}+\vec{c}-2\vec{a}}{2} + \frac{\vec{b}+\vec{c}-2\vec{b}}{2} + \frac{\vec{a}+\vec{b}-2\vec{c}}{2}$$

$$= \frac{2\vec{b}-2\vec{b}+2\vec{a}-2\vec{a}+2\vec{c}-2\vec{c}}{2}$$

$$\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF} = \vec{0}$$

Question 15

If $\vec{a}$ is a non-zero vector of modulus a and m is a non-zero scalar such that $m\vec{a}$ is a unit vector, write the value of m.

Solution 15

Given that,

$$\left|\vec{a}\right| = a$$

and m is a non-zero scalar such that $m\vec{a}$ is a unit vector

So,

$$\left|m\vec{a}\right| = 1$$

$$\left|m\right|\left|\vec{a}\right| = 1$$

$$\left|m\right|a = 1$$

$$\left|m\right| = \frac{1}{a}$$

$$m = \pm\frac{1}{a}$$

Question 16

If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices of an equilateral triangle whose orthocentre is at the origin, then write the value of $\vec{a} + \vec{b} + \vec{c}$

Solution 16

In an equilateral triangle, the centroid, orthocentre, incentre and circumcentre coincide.

Since the orthocentre is at the origin, centroid is also at the origin.

Position vector of Centroid is $\dfrac{\vec{a} + \vec{b} + \vec{c}}{3} = \vec{0}$

Thus, we have, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

Question 17

Write a unit vector making equal acute angles with the coordinates axes.

Solution 17

Let α, β, γ be the angles that the vector makes with $x, y,$ and z axis respectively.

It is given that, $\alpha = \beta = \gamma$

$$\cos\alpha = \cos\beta = \cos\gamma$$

$$l = m = n = x \text{ (say)}$$

We know that,

$$l^2 + m^2 + n^2 = 1$$

$$x^2 + x^2 + x^2 = 1$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3}$$

$$x = \pm\frac{1}{\sqrt{3}}$$

$$\therefore \quad l = \pm\frac{1}{\sqrt{3}}$$

$$\cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Case I- $\quad \cos\alpha = \frac{1}{\sqrt{3}}$

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Case II- $\quad \cos\alpha = -\frac{1}{\sqrt{3}}$

$$\alpha = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

It is in second quadrant, therefore it is an obtuse angle but it is geiven that α, β, γ are acute angles, so,

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

$$l = m = n = \frac{1}{\sqrt{3}}$$

Let $\vec{r}$ be the required vector, then given that $|\vec{r}| = 1$ (as unit vector)

$$\vec{r} = |\vec{r}|\left(l\hat{i} + m\hat{j} + n\hat{k}\right)$$

$$= \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

The required vector $= \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$

Question 18

If a vector makes angles α, β, γ with OX, OY and OZ respectively, then write the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

Solution 18

Hence, it is given that α, β, γ are the angles that a vector makes with OX, OY and OZ respectively, so

$$l = \cos\alpha$$
$$m = \cos\beta$$
$$n = \cos\gamma$$

We know that,

$$l^2 + m^2 + n^2 = 1$$
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$
$$\left(1 - \sin^2\alpha\right) + \left(1 - \sin^2\beta\right) + \left(1 - \sin^2\gamma\right) = 1$$
$$3 - \left(\sin^2\alpha + \sin^2\beta + \sin^2\gamma\right) = 1$$
$$-\left(\sin^2\alpha + \sin^2\beta + \sin^2\gamma\right) = 1 - 3$$
$$-\left(\sin^2\alpha + \sin^2\beta + \sin^2\gamma\right) = -2$$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

Question 19

Write a vector of magnitude 12 units which makes 45° angle with X-axes, 60° angle with Y-axis and an obtuse angle with Z-axis.

Solution 19

Let the required vector $= \vec{r}$

If α, β, γ are angles that $\vec{r}$ makes with the coordinate axes x, y and z respectively then

$$l = \cos\alpha$$
$$= \cos 45°$$
$$l = \frac{1}{\sqrt{2}}$$

$$m = \cos\beta$$
$$= \cos 60°$$
$$m = \frac{1}{2}$$

$$n = \cos\gamma$$

We have,

$$l^2 + m^2 + n^2 = 1$$
$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\cos^2\gamma\right) = 1$$
$$\frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$$
$$\cos^2\gamma = 1 - \frac{1}{2} - \frac{1}{4}$$
$$= \frac{4 - 2 - 1}{4}$$
$$= \frac{1}{4}$$
$$\cos\gamma = \pm\frac{1}{2}$$
$$\cos\gamma = \frac{1}{2}$$
$$\gamma = \cos^{-1}\left(\frac{1}{2}\right)$$
$$\gamma = \frac{\pi}{3}$$

Neglecting it as γ is an obtuse angle.

$$\cos\gamma = -\frac{1}{2}$$

$$\gamma = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$= \pi - \frac{\pi}{3}$$

$$\gamma = \frac{2\pi}{3}$$

So, $n = \cos\gamma$

$$= \cos\left(\frac{2\pi}{3}\right)$$

$$= \cos\left(\pi - \frac{\pi}{3}\right)$$

$$= -\cos\frac{\pi}{3}$$

$$n = -\frac{1}{2}$$

Thus,

vector $\vec{r} = \left|\vec{r}\right|\left(l\hat{i} + m\hat{j} + n\hat{k}\right)$

$$= 12\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} - \frac{1}{2}n\hat{k}\right)$$

$$= 12\left(\frac{\sqrt{2}\hat{i} + \hat{j} - \hat{k}}{2}\right)$$

$$= 6\left(\sqrt{2}\hat{i} + \hat{j} - \hat{k}\right)$$

$$\vec{r} = 6\left(\sqrt{2}\hat{i} + \hat{j} - \hat{k}\right)$$

Question 20

Write the length (magnitude) of a vector whose projections on the coordinate axes are 12, 3 and 4 units.

Solution 20

We know that, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is a vector and l, m, n are direction cosines of the vector, then Projection of $\vec{r}$ as coordinate axes are given by

$l\left|\vec{r}\right|, m\left|\vec{r}\right|, n\left|\vec{r}\right|$

So, $\quad l\left|\vec{r}\right| = 12$

$$\frac{x}{\left|\vec{r}\right|}\left|\vec{r}\right| = 12$$

$$x = 12 \qquad \text{(i)}$$

$$m\left|\vec{r}\right| = 3$$

$$\frac{y}{\left|\vec{r}\right|}\left|\vec{r}\right| = 3$$

$$y = 3 \qquad \text{(ii)}$$

$$n\left|\vec{r}\right| = 4$$

$$\frac{z}{\left|\vec{r}\right|} \times \left|\vec{r}\right| = 4$$

$$z = 4 \qquad \text{(iii)}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Put x,y,z from (i),(ii) and (iii) respectively.

$$\vec{r} = 12\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\left|\vec{r}\right| = \sqrt{(12)^2 + (3)^2 + (4)^2}$$

$$= \sqrt{144 + 9 + 16}$$

$$= \sqrt{169}$$

$$= 13$$

$$\left|\vec{r}\right| = 13$$

Question 21

Write the position vector of a point dividing the line segment joining points A and B with position vectors $\vec{a}$ and $\vec{b}$ externally in the ratio 1 : 4, where $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$.

Solution 21

Here, $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{b} = -\hat{i} + \hat{j} + \hat{k}$

$m : n = 1 : 4$

We know that, if A and B are two points with postion vector $\vec{a}$ and $\vec{b}$ respectively, then the position vector of a point C dividing AB in $m : n$ externally is give by

$$= \frac{m\left(\vec{b}\right) - n\left(\vec{a}\right)}{m - n}$$

$$= \frac{1\left(-\hat{i} + \hat{j} + \hat{k}\right) - 4\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right)}{1 - 4}$$

$$= \frac{-\hat{i} + \hat{j} + \hat{k} - 8\hat{i} - 12\hat{j} - 16\hat{k}}{-3}$$

$$= \frac{-9\hat{i} - 11\hat{j} - 15\hat{k}}{-3}$$

$$= 3\hat{i} + \frac{11}{3}\hat{j} + 5\hat{k}$$

The required vector $= 3\hat{i} + \frac{11}{3}\hat{j} + 5\hat{k}$

Question 22

Write the direction cosines of the vector $\vec{r} = 6\hat{i} - 2\hat{j} + 3\hat{k}$.

Solution 22

Here, $\vec{r} = 6\hat{i} - 2\hat{j} + 3\hat{k}$

$$\left|\vec{r}\right| = \sqrt{(6)^2 + (-2)^2 + (3)^2}$$

$$= \sqrt{36 + 4 + 9}$$

$$= \sqrt{49}$$

$$\left|\vec{r}\right| = 7$$

Direction ratios of $\vec{r} = \frac{6}{\left|\vec{r}\right|}, \frac{-2}{\left|\vec{r}\right|}, \frac{3}{\left|\vec{r}\right|}$

Direction cosines of $\vec{r} = \frac{6}{7}, \frac{-2}{7}, \frac{3}{7}$

Question 23

If $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$ and $\vec{c} = \hat{k} + \hat{i}$, write unit vectors parallel to $\vec{a} + \vec{b} - 2\vec{c}$.

Solution 23

Here, $\vec{a} = \hat{i} + \hat{j}$

$\vec{b} = \hat{j} + \hat{k}$

$\vec{c} = \hat{k} + \hat{i}$

$$\vec{a} + \vec{b} - 2\vec{c} = \left(\hat{i} + \hat{j}\right) + \left(\hat{j} + \hat{k}\right) - 2\left(\hat{k} + \hat{i}\right)$$

$$= \hat{i} + \hat{j} + \hat{j} + \hat{k} - 2\hat{k} - 2\hat{i}$$

$$= -\hat{i} + 2\hat{j} - \hat{k}$$

Let $\vec{d}$ be a vector parallel to $\left(\vec{a} + \vec{b} - 2\vec{c}\right)$ and $\left|\vec{d}\right| = 1$, so,

$$\vec{d} = \lambda\left(\vec{a} + \vec{b} - 2\vec{c}\right)$$

$$= \lambda\left(-\hat{i} + 2\hat{j} - \hat{k}\right) \qquad ---(\text{i})$$

$$\vec{d} = -\lambda\hat{i} + 2\lambda\hat{j} - \lambda\hat{k}$$

$$\left|\vec{d}\right| = \sqrt{(-\lambda)^2 + (2\lambda)^2 + (-\lambda)^2}$$

$$1 = \sqrt{\lambda^2 + 4\lambda^2 + \lambda^2}$$

$$1 = \sqrt{6\lambda^2}$$

Squaring both the sides,

$$1 = 6\lambda^2$$

$$\lambda^2 = \frac{1}{6}$$

$$\lambda = \pm\frac{1}{\sqrt{6}}$$

Put the value of λ in equation (i),

$$\vec{d} = \pm\frac{1}{\sqrt{6}}\left(-\hat{i} + 2\hat{j} - \hat{k}\right)$$

Therefore,

Unit vector parallel to $\vec{a} + \vec{b} - 2\vec{c} = \pm\frac{1}{\sqrt{6}}\cdot\left(-\hat{i} + 2\hat{j} - \hat{k}\right)$

Question 24

If $\vec{a} = \hat{i} + 2\hat{j}$, $\vec{b} = \hat{j} + 2\hat{k}$ write a unit vectors along the vector $3\vec{a} - 2\vec{b}$.

Solution 24

Here, $\vec{a} = \hat{i} + 2\hat{j}$

$\vec{b} = \hat{j} + 2\hat{k}$

$$3\vec{a} - 2\vec{b} = 3\left(\hat{i} + 2\hat{j}\right) - 2\left(\hat{j} + 2\hat{k}\right)$$

$$= 3\hat{i} + 6\hat{j} - 2\hat{j} - 4\hat{k}$$

$$3\vec{a} - 2\vec{b} = 3\hat{i} + 4\hat{j} - 4\hat{k}$$

$$\left|3\vec{a} - 2\vec{b}\right| = \sqrt{(3)^2 + (4)^2 + (-4)^2}$$

$$= \sqrt{9 + 16 + 16}$$

$$= \sqrt{41}$$

Unit vector alogn $3\vec{a} - 2\vec{b} = \dfrac{3\vec{a} - 2\vec{b}}{\left|3\vec{a} - 2\vec{b}\right|}$

$$= \frac{1}{\sqrt{41}}\left(3\hat{i} + 4\hat{j} - 4\hat{k}\right)$$

Unit vector along $3\vec{a} - 2\vec{b} = \dfrac{1}{\sqrt{41}}\left(3\hat{i} + 4\hat{j} - 4\hat{k}\right)$

Question 25

Write the position vector of a point dividing the line segment joining points having position vectors $\hat{i} + \hat{j} - 2\hat{k}$ and $2\hat{i} - \hat{j} + 3\hat{k}$ externally in the ratio 2 : 3.

Solution 25

Let $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$

$\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$

$m : n = 2 : 3$

We know that, if A and B are two points with position vector $\vec{a}$ and $\vec{b}$ respectively, then the position vector of a point C dividing AB in $m : n$ externally is give by

$$= \frac{mb - na}{m - n}$$

$$= \frac{2\left(2\hat{i} - \hat{j} + 3\hat{k}\right) - \left(\hat{i} + \hat{j} - 2\hat{k}\right)}{2 - 3}$$

$$= \frac{4\hat{i} - 2\hat{j} + 6\hat{k} - 3\hat{i} - 3\hat{j} + 6\hat{k}}{-1}$$

$$= \frac{\hat{i} - 5\hat{j} + 12\hat{k}}{-1}$$

$$= -\hat{i} + 5\hat{j} - 12\hat{k}$$

The required vector $= -\hat{i} + 5\hat{j} - 12\hat{k}$

Question 26

If $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$, $\vec{c} = \hat{k} + \hat{i}$, find the unit vector in the direction of $\vec{a} + \vec{b} + \vec{c}$.

Solution 26

Here, $\vec{a} = \hat{i} + \hat{j}$

$\vec{b} = \hat{j} + \hat{k}$

$\vec{c} = \hat{k} + \hat{i}$

$$\vec{a} + \vec{b} + \vec{c} = \left(\hat{i} + \hat{j}\right) + \left(\hat{j} + \hat{k}\right) + \left(\hat{k} + \hat{i}\right)$$
$$= 2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\left|\vec{a} + \vec{b} + \vec{c}\right| = 2\sqrt{(1)^2 + (1)^2 + (1)^2}$$
$$= 2\sqrt{1+1+1}$$
$$= 2\sqrt{3}$$

Unit vector in the direction of $\vec{a} + \vec{b} + \vec{c}$

$$= \frac{\vec{a} + \vec{b} + \vec{c}}{\left|\vec{a} + \vec{b} + \vec{c}\right|}$$
$$= \frac{2\left(\hat{i} + \hat{j} + \hat{k}\right)}{2\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

So, unit vector in the direction of $\vec{a} + \vec{b} + \vec{c} = \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$

Question 27

If $\vec{a} = 3\hat{i} - \hat{j} - 4\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 3\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$, find $\left|3\vec{a} - 2\vec{b} + 4\vec{c}\right|$.

Solution 27

Here, $\vec{a} = 3\hat{i} - \hat{j} - 4\hat{k}$

$\vec{b} = -2\hat{i} + 4\hat{j} - 3\hat{k}$

$\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$

$$3\vec{a} - 2\vec{b} + 4\vec{c} = 3\left(3\hat{i} - \hat{j} - 4\hat{k}\right) - 2\left(-2\hat{i} + 4\hat{j} - 3\hat{k}\right) + 4\left(\hat{i} + 2\hat{j} - \hat{k}\right)$$
$$= 9\hat{i} - 3\hat{j} - 12\hat{k} + 4\hat{i} - 8\hat{j} + 6\hat{k} + 4\hat{i} + 8\hat{j} - 4\hat{k}$$
$$3\vec{a} - 2\vec{b} + 4\vec{c} = 17\hat{i} - 3\hat{j} - 10\hat{k}$$

$$\left|3\vec{a} - 2\vec{b} + 4\vec{c}\right| = \sqrt{(17)^2 + (-3)^2 + (-10)^2}$$
$$= \sqrt{289 + 9 + 100}$$
$$= \sqrt{398}$$

$$\left|3\vec{a} - 2\vec{b} + 4\vec{c}\right| = \sqrt{398}$$

Question 28

A unit vector $\vec{r}$ makes angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$ with $\hat{j}$ and $\hat{k}$ respectively and an acute angle θ with $\hat{i}$. Find θ.

Solution 28

Here, it is given that

Angle between $\vec{r}$ and $\hat{i} = \alpha = \theta$

Angle between $\vec{r}$ and $\hat{j} = \beta = \frac{\pi}{3}$

Angle between $\vec{r}$ and $\hat{k} = \gamma = \frac{\pi}{2}$

So, $l = \cos\alpha$

$l = \cos\theta$

And,

$m = \cos\beta$

$= \cos\frac{\pi}{3}$

$m = \frac{1}{2}$

And,

$n = \cos\gamma$

$= \cos\frac{\pi}{2}$

$n = 0$

We know that

$l^2 + m^2 + n^2 = 1$

$\cos^2\theta + \left(\frac{1}{2}\right)^2 + (0)^2 = 1$

$\cos^2\theta + \frac{1}{4} = 1$

$= 1 - \frac{1}{4}$

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$

So, $\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\theta = \frac{\pi}{6}$

$$\text{And,} \quad \theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$
$$= \pi - \frac{\pi}{6}$$
$$\theta = \frac{5\pi}{6}$$

Neglecting it at θ is an acute angle.

$$\text{So,} \quad \theta = \frac{\pi}{6}$$

Question 29

Write a unit vector in the direction of $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$.

Solution 29

Here, $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$$\left|\vec{a}\right| = \sqrt{(3)^2 + (-2)^2 + (6)^2}$$
$$= \sqrt{9 + 4 + 36}$$
$$= \sqrt{49}$$
$$\left|\vec{a}\right| = 7$$

Unit vector in the direction of $\vec{a} = \dfrac{\vec{a}}{\left|\vec{a}\right|}$

$$\hat{a} = \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{7}$$
$$\hat{a} = \frac{1}{7}\left(3\hat{i} - 2\hat{j} + 6\hat{k}\right)$$

Question 30

If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + 9\hat{k}$, find a unit vector parallel to $\vec{a} + \vec{b}$.

Solution 30

Here, $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$

$\vec{b} = 2\hat{i} + 4\hat{j} + 9\hat{k}$

$$\vec{a} + \vec{b} = \left(\hat{i} + 2\hat{j} - 3\hat{k}\right) + \left(2\hat{i} + 4\hat{j} + 9\hat{k}\right)$$
$$= 3\hat{i} + 6\hat{j} + 6\hat{k}$$

Let $\vec{r}$ be a unit vector parallel to $\left(\vec{a} + \vec{b}\right)$

Then, $\vec{r} = \lambda\left(\vec{a} + \vec{b}\right)$ [Wher λ is same scalar]

$$\lambda = \left(3\hat{i} + 6\hat{j} + 6\hat{k}\right)$$
$$\vec{r} = 3\lambda\hat{i} + 6\lambda\hat{j} + 6\lambda\hat{k} \quad ---(i)$$

And, $\left|\vec{r}\right| = 1$

$$\sqrt{(3\lambda)^2 + (6\lambda)^2 + (6\lambda)^2} = 1$$
$$\sqrt{9\lambda^2 + 36\lambda^2 + 36\lambda^2} = 1$$
$$\sqrt{81\lambda^2} = 1$$
$$9\lambda = 1$$
$$\lambda = \frac{1}{9}$$

Put the value of λ in equation (i)

$$\vec{r} = 3\lambda\hat{i} + 6\lambda\hat{j} + 6\lambda\hat{k}$$
$$= 3\left(\frac{1}{9}\right)\hat{i} + 6\left(\frac{1}{9}\right)\hat{j} + 6\left(\frac{1}{9}\right)\hat{k}$$
$$= \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$
$$\vec{r} = \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

Unit vector parallel to $\vec{a} + \vec{b} = \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

Question 31

Write a unit vector in the direction of $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$.

Solution 31

Here, $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$

$$\left|\vec{b}\right| = \sqrt{(2)^2 + (1)^2 + (2)^2}$$
$$= \sqrt{1 + 1 + 4}$$
$$= \sqrt{9}$$
$$\left|\vec{b}\right| = 3$$

Unit vector in the direction of $\vec{b} = \dfrac{\vec{b}}{\left|\vec{b}\right|}$

$$\hat{b} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

Unit vector in the direction of $\vec{b} = \dfrac{1}{3}\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$

Question 32

Find the position vector of the mid-point of the line segment AB, where A is the point $(3, 4, -2)$ and B is the point $(1, 2, 4)$.

Solution 32

Here, $A = (3, 4, -2)$
$B = (1, 2, 4)$

Position vector of $A = 3\hat{i} + 2\hat{j} - 2\hat{k}$

Position vector of $B = \hat{i} + 2\hat{j} + 4\hat{k}$

We know that, if $\vec{c}$ is the vector representing the mid point of vector $\vec{a}$ and $\vec{b}$, then,

$$\vec{c} = \frac{a + b}{2}$$
$$= \frac{\left(3\hat{i} + 2\hat{j} - 2\hat{k}\right) + \left(\hat{i} + 2\hat{j} + 4\hat{k}\right)}{2}$$
$$= \frac{4\hat{i} + 6\hat{j} + 2\hat{k}}{2}$$
$$= \frac{2\left(2\hat{i} + 3\hat{j} + \hat{k}\right)}{2}$$
$$= 2\hat{i} + 3\hat{j} + \hat{k}$$

Mid-point $C = (2, 3, 1)$

Question 33

Find a vector in the direction of $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$, which has magnitude of 6 units.

Solution 33

Here, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$

Let $\vec{r}$ be any vector in the direction of $\vec{a}$ and having magnitude 6 units, then

$\vec{r} = \lambda\vec{a}$ and

$|\vec{r}| = 6$

So, $\vec{r} = \lambda\left(2\hat{i} - \hat{j} + 2\hat{k}\right)$

$\vec{r} = 2\lambda\hat{i} - \lambda\hat{j} + 2\lambda\hat{k} \qquad ---(i)$

$$|\vec{r}| = \sqrt{(2\lambda)^2 + (-\lambda)^2 + (2\lambda)^2}$$
$$6 = \sqrt{4\lambda^2 + \lambda^2 + 4\lambda^2}$$
$$6 = \sqrt{9\lambda^2}$$
$$6 = 3\lambda$$
$$\lambda = \frac{6}{3}$$
$$\lambda = 2$$

Put the value of λ in equation (i),

$$\vec{r} = 2\lambda\hat{i} - \lambda\hat{j} + 2\lambda\hat{k}$$
$$= 2(2)\hat{i} - 2\hat{j} + 2(2)\hat{k}$$
$$= 4\hat{i} - 2\hat{j} + 4\hat{k}$$

The vector in the direction of $\vec{a}$ with magnitude 6 = $4\hat{i} - 2\hat{j} + 4\hat{k}$

Question 34

What is the cosine of the angle which the vector $\sqrt{2}\hat{i} + \hat{j} + \hat{k}$ makes with y-axis?

Solution 34

Let $\vec{r} = \sqrt{2}\hat{i} + \hat{j} + \hat{k}$

$$|\vec{r}| = \sqrt{(\sqrt{2})^2 + (1)^2 + (1)^2}$$
$$= \sqrt{2+1+1}$$
$$= \sqrt{4}$$
$$|\vec{r}| = 2$$

Direction ratios of $\vec{r} = \sqrt{2}, 1, 1$

Direction cosines of $\vec{r} = \frac{\sqrt{2}}{|\vec{r}|}, \frac{1}{|\vec{r}|}, \frac{1}{|\vec{r}|}$

$$= \frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2}$$

So, $l = \frac{\sqrt{2}}{2}$

$$m = \frac{1}{2}$$
$$n = \frac{1}{2}$$

If β be the angle that vector $\vec{r}$ makes with y-axis, then

$$m = \cos\beta = \frac{1}{2}$$
$$\cos\beta = \frac{1}{2}$$

Question 35

Write two different vectors having same magnitude.

Solution 35

Let us consider the two vectors $\vec{a}$ and $\vec{b}$ such that

$\vec{a} = 3\hat{i} - \hat{j} + 3\hat{k}$ *and*

$\vec{b} = \hat{i} + 3\hat{j} - 3\hat{k}$

The magnitude of a is $|\vec{a}| = \sqrt{3^2 + (-1)^2 + 3^2} = \sqrt{19}$

The magnitude of b is $|\vec{b}| = \sqrt{1^2 + 3^2 + (-3)^2} = \sqrt{19}$

It is clear that $|\vec{a}| = |\vec{b}|$

Question 36

Write two different vectors having same direction.

Solution 36

Let us consider the two vectors $\vec{a}$ *and* $\vec{b}$ *such that*

$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ *and*

$\vec{b} = 2\hat{i} - 2\hat{j} + 6\hat{k}$

It is clear that

$\vec{b} = 2\hat{i} - 2\hat{j} + 6\hat{k} = 2(\hat{i} - \hat{j} + 3\hat{k}) = k\vec{a}$

Thus, $\vec{a}$ *is parallel to* $\vec{b}$ *and hence in the same direction.*

Question 37

Write a vector in the directrion of vcetor $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude of 8 unit.

Solution 37

Let $\vec{r} = 5\hat{i} - \hat{j} + 2\hat{k}$

And, $\vec{p}$ be any vector in the direction of $\vec{r}$ having magnitude 6.

$$\vec{p} = \lambda\vec{r}$$
$$= \lambda\left(5\hat{i} - \hat{j} + 2\hat{k}\right)$$
$$\vec{p} = 5\lambda\hat{i} - \lambda\hat{j} + 2\lambda\hat{k} \qquad ---(i)$$

$$\left|\vec{p}\right| = \sqrt{(5\lambda)^2 + (-\lambda)^2 + (2\lambda)^2}$$
$$8 = \sqrt{25\lambda^2 + \lambda^2 + 4\lambda^2}$$
$$8 = \sqrt{30\lambda^2}$$

Substituting both the sides,

$$64 = 30\lambda^2$$
$$\lambda^2 = \frac{64}{30}$$
$$\lambda = \pm\frac{8}{\sqrt{30}}$$

Put λ in equation (i),

$$\vec{p} = \pm\frac{8}{\sqrt{30}}\left(5\hat{i} - \hat{j} + 2\hat{k}\right)$$

A vector in the direction of $\left(5\hat{i} - \hat{j} + 2\hat{k}\right)$ with

magnitude 8 $= \pm\frac{8}{\sqrt{30}}\left(5\hat{i} - \hat{j} + 2\hat{k}\right)$

Question 38

Write the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

Solution 38

Let $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k}$

Direction ratios of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$

$= 1, \quad 2, \quad 3$

Direction cosines of two vector

$$= \frac{1}{|\vec{r}|}, \quad \frac{2}{|\vec{r}|}, \quad \frac{3}{|\vec{r}|}$$

$$|\vec{r}| = \sqrt{(1)^2 + (2)^2 + (3)^2}$$
$$= \sqrt{1+4+9}$$
$$= \sqrt{14}$$

Direction cosines of the vector $= \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

Question 39

Find the unit vector in the direction of $\bar{a} = 2\bar{i} - 3\bar{j} + 6\bar{k}$

Solution 39

Here, $\bar{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$$|\bar{a}| = \sqrt{(3)^2 + (-2)^2 + (6)^2}$$
$$= \sqrt{4+9+36}$$
$$= \sqrt{49}$$
$$= 7$$

Unit vector in the direction of

$$\bar{a} = \frac{\bar{a}}{|\bar{a}|}$$
$$= \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{7}$$
$$= \frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$$

Question 40

For what value of a the vector $2\bar{i}-3\bar{j}+4\bar{k}$ and $a\bar{i}+6\bar{j}-8\bar{k}$ are collinear

Solution 40

Let the points be A,B respectively. Then
A,B are collinear.

$$A=\lambda B$$

$$2\hat{i}-3\hat{j}+4\hat{k}=\lambda\left(a\hat{i}+6\hat{j}-8\hat{k}\right)$$

$$(2-\lambda a)\hat{i}+(-3+\lambda 6)\hat{j}+(4-8\lambda)\hat{k}=0$$

Therefore,

$$\lambda=-\frac{1}{2}$$

And

$$(2-\lambda a)=0$$

$$2=\lambda a$$

$$2=\left(-\frac{1}{2}\right)a$$

$$a=-4$$

Question 41

Write the direction cosines of the vector $-2\bar{i}+\bar{j}-5\bar{k}$

Solution 41

Let $\vec{r} = -2\bar{i} + \bar{j} - 5\bar{k}$.

Direction ratios of the vector $-2\bar{i} + \bar{j} - 5\bar{k}$

$= -2, 1, -5$

Direction cosines of two vector

$= \frac{-2}{|\vec{r}|}, \frac{1}{|\vec{r}|}, \frac{-5}{|\vec{r}|}$

$|\vec{r}| = \sqrt{(-2)^2 + (1)^2 + (-5)^2}$

$= \sqrt{4 + 1 + 25}$

$= \sqrt{30}$

Direction cosines of the vector $= \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$

Question 42

Find the sum of the following vectors:

$\bar{a} = \hat{i} - 2\hat{j}, \bar{b} = 2\hat{i} - 3\hat{j}, \bar{c} = 2\hat{i} + 3\hat{k}$

Solution 42

We have,

$$\begin{aligned}
\bar{a} + \bar{b} + \bar{c} &= (\bar{a} + \bar{b}) + \bar{c} \\
&= \left\{(\hat{i} - 2\hat{j}) + (2\hat{i} - 3\hat{j})\right\} + (2\hat{i} + 3\hat{k}) \\
&= \left\{(1+2)\hat{i} + (-2-3)\hat{j}\right\} + (2\hat{i} + 3\hat{k}) \\
&= (3\hat{i} - 5\hat{j}) + (2\hat{i} + 3\hat{k}) \\
&= (3+2)\hat{i} - 5\hat{j} + 3\hat{k} \\
&= 5\hat{i} - 5\hat{j} + 3\hat{k}
\end{aligned}$$

Question 43

Find a unit vector in the direction of the vector

$\bar{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

Solution 43

Here, $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$$|\vec{a}| = \sqrt{(3)^2 + (-2)^2 + (6)^2}$$
$$= \sqrt{4+9+36}$$
$$= \sqrt{49}$$
$$= 7$$

Unit vector in the direction of

$$\bar{a} = \frac{\vec{a}}{|\vec{a}|}$$
$$= \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{7}$$
$$= \frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$$

Question 44

If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of $x + y + z$.

Solution 44

$\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors.

For equal vectors, the components are equal.

Hence, $x = 3$, $2 = -y$, $-z = 1$

$x + y + z = 3 - 2 - 1 = 0$

Question 45

Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$.

Solution 45

$\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$

$\vec{a} + \vec{b} = (2+2)\hat{i} + (2+1)\hat{j} - (5+7)\hat{k}$

$= 4\hat{i} + 3\hat{j} - 12\hat{k}$

Unit vector in the direction of $\vec{a} + \vec{b}$

$$= \frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{\sqrt{4^2 + 3^2 + 12^2}} = \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{\sqrt{169}} = \frac{4}{13}\hat{i} + \frac{3}{13}\hat{j} - \frac{12}{13}\hat{k}$$

Question 46

Find the value of 'p' for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel.

Solution 46

If two vectors $\vec{a}$ and $\vec{b}$ are parallel, then $\vec{a} = \lambda\vec{b}$.

Here λ is a constant.

$3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel.

Hence, $3\hat{i} + 2\hat{j} + 9\hat{k} = \lambda(\hat{i} - 2p\hat{j} + 3\hat{k})$

$\Rightarrow 3\hat{i} + 2\hat{j} + 9\hat{k} = \lambda\hat{i} - 2\lambda p\hat{j} + 3\lambda\hat{k}$

Equating the components we get

$\lambda = 3$ and $-2\lambda p = 2 \Rightarrow 6p = -2 \Rightarrow p = -\frac{1}{3}$

Question 47

Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an agle of $\frac{\pi}{4}$ with x−axis,

$\frac{\pi}{2}$ with y−axis and an acute angle θ with z−axis.

Solution 47

Vector $\vec{a}$ of magnitude $5\sqrt{2}$ makes an angle of $\frac{\pi}{4}$ with the x−axis.

Hence, the component along the x−axis is $5\sqrt{2}\cos\frac{\pi}{4} = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5$

Vector $\vec{a}$ of magnitude $5\sqrt{2}$ makes an angle of $\frac{\pi}{2}$ with the y−axis.

Hence, the component along the y−axis is $5\sqrt{2}\cos\frac{\pi}{2} = 5\sqrt{2} \times 0 = 0$

Vector $\vec{a}$ of magnitude $5\sqrt{2}$ makes an acute angle θ with the z−axis.

Hence, the component along the z−axis is $5\sqrt{2}\cos\theta$.

$(5\sqrt{2})^2 = 5^2 + 0^2 + (5\sqrt{2}\cos\theta)^2 = 50 = 25 + 50\cos^2\theta$

$\Rightarrow \cos^2\theta = \frac{25}{50} = \frac{1}{2}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$ ...($\because$ θ is acute)

Hence, $\vec{a} = 5\hat{i} + 5\hat{k}$

Question 48

Write a unit vector in the direction of $\overrightarrow{PQ}$, where P and Q are the points (1,3,0) and (4,5,6) respectively.

Solution 48

P and Q are the points with co−ordinates (1,3,0) and (4,5,6) respectively.

$\overrightarrow{PQ} = (4-1)\hat{i} + (5-3)\hat{j} + (6-0)\hat{k}$

$= 3\hat{i} + 2\hat{j} + 6\hat{k}$

Unit vector in the direction of $\overrightarrow{PQ} = \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{49}}$

$= \frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$

Question 49

Find a vector in the direction of vector $2\hat{i} - 3\hat{j} + 6\hat{k}$ which has magnitude 21 units.

Solution 49

Unit vector in the direction of vector $2\hat{i} - 3\hat{j} + 6\hat{k}$

$= \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$

Hence, a vector along the above vector

which has magnitude 21 units $= 21\left(\frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}\right)$

$= 6\hat{i} - 9\hat{j} + 18\hat{k}$