

Access answers to Maths RD Sharma Solutions For Class 12 Chapter 6 – Determinants

Exercise 6.1 Page No: 6.10

1. Write the minors and cofactors of each element of the first column of the following matrices and hence evaluate the determinant in each case:

$$(i) A = \begin{bmatrix} 5 & 20 \\ 0 & -1 \end{bmatrix}$$

$$(ii) A = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$(iii) A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}$$

$$(iv) A = \begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}$$

$$(v) A = \begin{bmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{bmatrix}$$

$$(vi) A = \begin{bmatrix} a & h & g \\ h & b & f \\ f & f & c \end{bmatrix}$$

$$(vii) A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 5 & 0 \end{bmatrix}$$

Solution:

(i) Let M_{ij} and C_{ij} represents the minor and co-factor of an element, where i and j represent the row and column. The minor of the matrix can be obtained for a particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$

Given,

$$A = \begin{bmatrix} 5 & 20 \\ 0 & -1 \end{bmatrix}$$

From the given matrix we have,

$$M_{11} = -1$$

$$M_{21} = 20$$

$$C_{11} = (-1)^{1+1} \times M_{11}$$

$$= 1 \times -1$$

$$= -1$$

$$C_{21} = (-1)^{2+1} \times M_{21}$$

$$= 20 \times -1$$

$$= -20$$

Now expanding along the first column we get

$$|A| = a_{11} \times C_{11} + a_{21} \times C_{21}$$

$$= 5 \times (-1) + 0 \times (-20)$$

$$= -5$$

(ii) Let M_{ij} and C_{ij} represents the minor and co-factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

$$\text{Also, } C_{ij} = (-1)^{i+j} \times M_{ij}$$

Given

$$A = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$$

From the above matrix we have

$$M_{11} = 3$$

$$M_{21} = 4$$

$$C_{11} = (-1)^{1+1} \times M_{11}$$

$$= 1 \times 3$$

$$= 3$$

$$C_{21} = (-1)^{2+1} \times 4$$

$$= -1 \times 4$$

$$= -4$$

Now expanding along the first column we get

$$|A| = a_{11} \times C_{11} + a_{21} \times C_{21}$$

$$= -1 \times 3 + 2 \times (-4)$$

$$= -11$$

(iii) Let M_{ij} and C_{ij} represents the minor and co-factor of an element, where i and j represent the row and column. The minor of the matrix can be obtained for a particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

$$\text{Also, } C_{ij} = (-1)^{i+j} \times M_{ij}$$

Given,

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}$$

From given matrix we have,

$$\Rightarrow M_{11} = \begin{bmatrix} -1 & 2 \\ 5 & 2 \end{bmatrix}$$

$$M_{11} = -1 \times 2 - 5 \times 2$$

$$M_{11} = -12$$

$$\Rightarrow M_{21} = \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}$$

$$M_{21} = -3 \times 2 - 5 \times 2$$

$$M_{21} = -16$$

$$\Rightarrow M_{31} = \begin{bmatrix} -3 & 2 \\ -1 & 2 \end{bmatrix}$$

$$M_{31} = -3 \times 2 - (-1) \times 2$$

$$M_{31} = -4$$

$$C_{11} = (-1)^{1+1} \times M_{11}$$

$$= 1 \times -12$$

$$= -12$$

$$C_{21} = (-1)^{2+1} \times M_{21}$$

$$= -1 \times -16$$

$$= 16$$

$$C_{31} = (-1)^{3+1} \times M_{31}$$

$$= 1 \times -4$$

$$= -4$$

Now expanding along the first column we get

$$|A| = a_{11} \times C_{11} + a_{21} \times C_{21} + a_{31} \times C_{31}$$

$$= 1 \times (-12) + 4 \times 16 + 3 \times (-4)$$

$$= -12 + 64 - 12$$

$$= 40$$

(iv) Let M_{ij} and C_{ij} represents the minor and co-factor of an element, where i and j represent the row and column. The minor of the matrix can be obtained for a particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

$$\text{Also, } C_{ij} = (-1)^{i+j} \times M_{ij}$$

Given,

$$A = \begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}$$

$$\Rightarrow M_{11} = \begin{bmatrix} b & ca \\ c & ab \end{bmatrix}$$

$$M_{11} = b \times ab - c \times ca$$

$$M_{11} = ab^2 - ac^2$$

$$\Rightarrow M_{21} = \begin{bmatrix} a & bc \\ c & ab \end{bmatrix}$$

$$M_{21} = a \times ab - c \times bc$$

$$M_{21} = a^2b - c^2b$$

$$\Rightarrow M_{31} = \begin{bmatrix} a & bc \\ b & ca \end{bmatrix}$$

$$M_{31} = a \times ca - b \times bc$$

$$M_{31} = a^2c - b^2c$$

$$C_{11} = (-1)^{1+1} \times M_{11}$$

$$= 1 \times (ab^2 - ac^2)$$

$$= ab^2 - ac^2$$

$$C_{21} = (-1)^{2+1} \times M_{21}$$

$$= -1 \times (a^2b - c^2b)$$

$$= c^2b - a^2b$$

$$C_{31} = (-1)^{3+1} \times M_{31}$$

$$= 1 \times (a^2c - b^2c)$$

$$= a^2c - b^2c$$

Now expanding along the first column we get

$$|A| = a_{11} \times C_{11} + a_{21} \times C_{21} + a_{31} \times C_{31}$$

$$= 1 \times (ab^2 - ac^2) + 1 \times (c^2b - a^2b) + 1 \times (a^2c - b^2c)$$

$$= ab^2 - ac^2 + c^2b - a^2b + a^2c - b^2c$$

(v) Let M_{ij} and C_{ij} represents the minor and co-factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

$$\text{Also, } C_{ij} = (-1)^{i+j} \times M_{ij}$$

Given,

$$A = \begin{bmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{bmatrix}$$

From the above matrix we have,

$$\Rightarrow M_{11} = \begin{bmatrix} 5 & 0 \\ 7 & 1 \end{bmatrix}$$

$$M_{11} = 5 \times 1 - 7 \times 0$$

$$M_{11} = 5$$

$$\Rightarrow M_{21} = \begin{bmatrix} 2 & 6 \\ 7 & 1 \end{bmatrix}$$

$$M_{21} = 2 \times 1 - 7 \times 6$$

$$M_{21} = -40$$

$$\Rightarrow M_{31} = \begin{bmatrix} 2 & 6 \\ 5 & 0 \end{bmatrix}$$

$$M_{31} = 2 \times 0 - 5 \times 6$$

$$M_{31} = -30$$

$$C_{11} = (-1)^{1+1} \times M_{11}$$

$$= 1 \times 5$$

$$= 5$$

$$C_{21} = (-1)^{2+1} \times M_{21}$$

$$= -1 \times -40$$

$$= 40$$

$$C_{31} = (-1)^{3+1} \times M_{31}$$

$$= 1 \times -30$$

$$= -30$$

Now expanding along the first column we get

$$|A| = a_{11} \times C_{11} + a_{21} \times C_{21} + a_{31} \times C_{31}$$

$$= 0 \times 5 + 1 \times 40 + 3 \times (-30)$$

$$= 0 + 40 - 90$$

$$= 50$$

(vi) Let M_{ij} and C_{ij} represents the minor and co-factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

$$\text{Also, } C_{ij} = (-1)^{i+j} \times M_{ij}$$

Given,

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

From the given matrices we have,

$$\Rightarrow M_{11} = \begin{bmatrix} b & f \\ f & c \end{bmatrix}$$

$$M_{11} = b \times c - f \times f$$

$$M_{11} = bc - f^2$$

$$\Rightarrow M_{21} = \begin{bmatrix} h & g \\ f & c \end{bmatrix}$$

$$M_{21} = h \times c - f \times g$$

$$M_{21} = hc - fg$$

$$\Rightarrow M_{31} = \begin{bmatrix} h & g \\ b & f \end{bmatrix}$$

$$M_{31} = h \times f - b \times g$$

$$M_{31} = hf - bg$$

$$C_{11} = (-1)^{1+1} \times M_{11}$$

$$= 1 \times (bc - f^2)$$

$$= bc - f^2$$

$$C_{21} = (-1)^{2+1} \times M_{21}$$

$$= -1 \times (hc - fg)$$

$$= fg - hc$$

$$C_{31} = (-1)^{3+1} \times M_{31}$$

$$= 1 \times (hf - bg)$$

$$= hf - bg$$

Now expanding along the first column we get

$$|A| = a_{11} \times C_{11} + a_{21} \times C_{21} + a_{31} \times C_{31}$$

$$= a \times (bc - f^2) + h \times (fg - hc) + g \times (hf - bg)$$

$$= abc - af^2 + hgf - h^2c + ghf - bg^2$$

(vii) Let M_{ij} and C_{ij} represents the minor and co-factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

$$\text{Also, } C_{ij} = (-1)^{i+j} \times M_{ij}$$

Given,

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 5 & 0 \end{bmatrix}$$

From the given matrix we have,

$$\Rightarrow M_{11} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ -1 & 5 & 0 \end{bmatrix}$$

$$M_{11} = 0(-1 \times 0 - 5 \times 1) - 1(1 \times 0 - (-1) \times 1) + (-2)(1 \times 5 - (-1) \times (-1))$$

$$M_{11} = -9$$

$$\Rightarrow M_{21} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & 5 & 0 \end{bmatrix}$$

$$M_{21} = -1(-1 \times 0 - 5 \times 1) - 0(1 \times 0 - (-1) \times 1) + 1(1 \times 5 - (-1) \times (-1))$$

$$M_{21} = 9$$

$$\Rightarrow M_{31} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -2 \\ -1 & 5 & 0 \end{bmatrix}$$

$$M_{31} = -1(1 \times 0 - 5 \times (-2)) - 0(0 \times 0 - (-1) \times (-2)) + 1(0 \times 5 - (-1) \times 1)$$

$$M_{31} = -9$$

$$\Rightarrow M_{41} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$M_{41} = -1(1 \times 1 - (-1) \times (-2)) - 0(0 \times 1 - 1 \times (-2)) + 1(0 \times (-1) - 1 \times 1)$$

$$M_{41} = 0$$

$$C_{11} = (-1)^{1+1} \times M_{11}$$

$$= 1 \times (-9)$$

$$= -9$$

$$C_{21} = (-1)^{2+1} \times M_{21}$$

$$= -1 \times 9$$

$$= -9$$

$$C_{31} = (-1)^{3+1} \times M_{31}$$

$$= 1 \times -9$$

$$= -9$$

$$C_{41} = (-1)^{4+1} \times M_{41}$$

$$= -1 \times 0$$

$$= 0$$

Now expanding along the first column we get

$$|A| = a_{11} \times C_{11} + a_{21} \times C_{21} + a_{31} \times C_{31} + a_{41} \times C_{41}$$

$$= 2 \times (-9) + (-3) \times -9 + 1 \times (-9) + 2 \times 0$$

$$= -18 + 27 - 9$$

$$= 0$$

2. Evaluate the following determinants:

$$(i) \begin{vmatrix} x & -7 \\ x & 5x + 1 \end{vmatrix}$$

$$(ii) \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix}$$

$$(iii) \begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$$

$$(iv) \begin{vmatrix} a + ib & c + id \\ -c + id & a - ib \end{vmatrix}$$

Solution:

(i) Given

$$\begin{vmatrix} x & -7 \\ x & 5x + 1 \end{vmatrix}$$

$$\Rightarrow |A| = x(5x + 1) - (-7)x$$

$$|A| = 5x^2 + 8x$$

(ii) Given

$$\begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix}$$

$$\Rightarrow |A| = \cos \theta \times \cos \theta - (-\sin \theta) \times \sin \theta$$

$$|A| = \cos^2 \theta + \sin^2 \theta$$

$$\text{We know that } \cos^2 \theta + \sin^2 \theta = 1$$

$$|A| = 1$$

(iii) Given

$$(iii) \begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$$

$$\Rightarrow |A| = \cos 15^\circ \times \cos 75^\circ + \sin 15^\circ \times \sin 75^\circ$$

$$\text{We know that } \cos (A - B) = \cos A \cos B + \sin A \sin B$$

$$\text{By substituting this we get, } |A| = \cos (75 - 15)^\circ$$

$$|A| = \cos 60^\circ$$

$$|A| = 0.5$$

(iv) Given

$$(iv) \begin{vmatrix} a + ib & c + id \\ -c + id & a - ib \end{vmatrix}$$

$$\Rightarrow |A| = (a + ib)(a - ib) - (c + id)(-c + id)$$

$$= (a + ib)(a - ib) + (c + id)(c - id)$$

$$= a^2 - i^2 b^2 + c^2 - i^2 d^2$$

$$\text{We know that } i^2 = -1$$

$$= a^2 - (-1) b^2 + c^2 - (-1) d^2$$

$$= a^2 + b^2 + c^2 + d^2$$

3. Evaluate:

$$\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}^2$$

Solution:

Since $|AB| = |A||B|$

$$|A| = \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

$$|A| = 2 \begin{vmatrix} 17 & 5 \\ 20 & 12 \end{vmatrix} - 3 \begin{vmatrix} 13 & 5 \\ 15 & 12 \end{vmatrix} + 7 \begin{vmatrix} 13 & 17 \\ 15 & 20 \end{vmatrix}$$

$$= 2(17 \times 12 - 5 \times 20) - 3(13 \times 12 - 5 \times 15) + 7(13 \times 20 - 15 \times 17)$$

$$= 2(204 - 100) - 3(156 - 75) + 7(260 - 255)$$

$$= 2 \times 104 - 3 \times 81 + 7 \times 5$$

$$= 208 - 243 + 35$$

$$= 0$$

Now $|A|^2 = |A| \times |A|$

$$|A|^2 = 0$$

4. Show that

$$\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix}$$

Solution:

Given

$$\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix}$$

Let the given determinant as A

Using $\sin(A+B) = \sin A \times \cos B + \cos A \times \sin B$

$$\Rightarrow |A| = \sin 10^\circ \times \cos 80^\circ + \cos 10^\circ \times \sin 80^\circ$$

$$|A| = \sin(10 + 80)^\circ$$

$$|A| = \sin 90^\circ$$

$$|A| = 1$$

Hence Proved

5. Evaluate $\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$ by two methods.

Solution:

Given,

$$|A| = \begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$

Expanding along the first row

$$|A| = 2 \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 3 \begin{vmatrix} 7 & -2 \\ -3 & 1 \end{vmatrix} - 5 \begin{vmatrix} 7 & 1 \\ -3 & 4 \end{vmatrix}$$

$$= 2(1 \times 1 - 4 \times (-2)) - 3(7 \times 1 - (-2) \times (-3)) - 5(7 \times 4 - 1 \times (-3))$$

$$= 2(1 + 8) - 3(7 - 6) - 5(28 + 3)$$

$$= 2 \times 9 - 3 \times 1 - 5 \times 31$$

$$= 18 - 3 - 155$$

$$= -140$$

Now by expanding along the second column

$$|A| = 2 \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 7 \begin{vmatrix} 3 & -5 \\ 4 & 1 \end{vmatrix} - 3 \begin{vmatrix} 3 & -5 \\ 1 & -2 \end{vmatrix}$$

$$= 2(1 \times 1 - 4 \times (-2)) - 7(3 \times 1 - 4 \times (-5)) - 3(3 \times (-2) - 1 \times (-5))$$

$$= 2(1 + 8) - 7(3 + 20) - 3(-6 + 5)$$

$$= 2 \times 9 - 7 \times 23 - 3 \times (-1)$$

$$= 18 - 161 + 3$$

$$= -140$$

$$6. \text{ Evaluate : } \Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$$

Solution:

Given

$$\Delta = \begin{vmatrix} 0 & \sin \alpha & -\cos \alpha \\ -\sin \alpha & 0 & \sin \beta \\ \cos \alpha & -\sin \beta & 0 \end{vmatrix}$$

Expanding along the first row

$$|A| = 0 \begin{vmatrix} 0 & \sin \beta \\ -\sin \beta & 0 \end{vmatrix} - \sin \alpha \begin{vmatrix} -\sin \alpha & \sin \beta \\ \cos \alpha & 0 \end{vmatrix} - \cos \alpha \begin{vmatrix} -\sin \alpha & 0 \\ \cos \alpha & -\sin \beta \end{vmatrix}$$

$$\Rightarrow |A| = 0 (0 - \sin \beta (-\sin \beta)) - \sin \alpha (-\sin \alpha \times 0 - \sin \beta \cos \alpha) - \cos \alpha ((-\sin \alpha) (-\sin \beta) - 0 \times \cos \alpha)$$

$$|A| = 0 + \sin \alpha \sin \beta \cos \alpha - \cos \alpha \sin \alpha \sin \beta$$

$$|A| = 0$$

Exercise 6.2 Page No: 6.57

1. Evaluate the following determinant:

$$(i) \begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix}$$

$$(ii) \begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$$

$$(iii) \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$(iv) \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$$

$$(v) \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$$

$$(vi) \begin{vmatrix} 6 & 3 & -2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

$$(vii) \begin{vmatrix} 1 & 3 & 9 & 27 \\ 3 & 9 & 27 & 1 \\ 9 & 27 & 1 & 3 \\ 27 & 1 & 3 & 9 \end{vmatrix}$$

$$(viii) \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

Solution:

(i) Given

$$(i) \begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 & 5 \\ 1 & 3 & 5 \\ 31 & 11 & 38 \end{vmatrix}$$

Now by applying, $R_2 \rightarrow R_2 - R_1$, we get,

$$\Rightarrow \Delta = 2 \begin{vmatrix} 1 & 3 & 5 \\ 0 & 0 & 0 \\ 31 & 11 & 38 \end{vmatrix} = 0$$

$$\text{So, } \Delta = 0$$

(ii) Given

$$\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$$

By applying column operation $C_1 \rightarrow C_1 - 4 C_3$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 4 & 19 & 21 \\ -3 & 13 & 14 \\ -3 & 24 & 26 \end{vmatrix}$$

Again by applying row operation, $R_1 \rightarrow R_1 + R_2$ and $R_3 \rightarrow R_3 - R_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 32 & 35 \\ -3 & 13 & 14 \\ 0 & 11 & 12 \end{vmatrix}$$

Now, applying $R_2 \rightarrow R_2 + 3 R_1$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 32 & 35 \\ 0 & 109 & 119 \\ 0 & 11 & 12 \end{vmatrix}$$

$$= 1[(109)(12) - (119)(11)]$$

$$= 1308 - 1309$$

$$= -1$$

$$\text{So, } \Delta = -1$$

(iii) Given,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= a(bc - f^2) - h(hc - fg) + g(hf - bg)$$

$$= abc - af^2 - ch^2 + fgh + fgh - bg^2$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2$$

$$\text{So, } \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

(iv) Given

$$= \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$$

By taking 2 as common we get,

$$\Rightarrow \Delta = 2 \begin{vmatrix} 1 & -3 & 1 \\ 4 & -1 & 1 \\ 3 & 5 & 1 \end{vmatrix}$$

Now by applying, row operation $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\Rightarrow \Delta = 2 \begin{vmatrix} 1 & -3 & 1 \\ 3 & 2 & 0 \\ 2 & 8 & 0 \end{vmatrix}$$

$$= 2[1(24 - 4)] = 40$$

$$\text{So, } \Delta = 40$$

(v) Given

$$\text{Let, } \Delta = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$$

By applying column operation $C_3 \rightarrow C_3 - C_2$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 4 & 5 \\ 4 & 9 & 7 \\ 9 & 16 & 9 \end{vmatrix}$$

Again by applying column operation $C_2 \rightarrow C_2 + C_1$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 5 & 5 \\ 4 & 13 & 7 \\ 9 & 25 & 9 \end{vmatrix}$$

Now by applying $C_2 \rightarrow C_2 - 5C_1$ and $C_3 \rightarrow C_3 - 5C_1$ we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 4 & -7 & -13 \\ 9 & -20 & -36 \end{vmatrix}$$

$$= 1[(-7)(-36) - (-20)(-13)]$$

$$= 252 - 260$$

$$= -8$$

$$\text{So, } \Delta = -8$$

(vi) Given,

$$\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

Applying row operations, $R_1 \rightarrow R_1 - 3R_2$ and $R_3 \rightarrow R_3 + 5R_2$ we get,

$$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & -4 \\ 2 & -1 & 2 \\ 0 & 0 & 12 \end{vmatrix} = 0$$

$$\text{So, } \Delta = 0$$

(vii) Given

$$\begin{vmatrix} 1 & 3 & 9 & 27 \\ 3 & 9 & 27 & 1 \\ 9 & 27 & 1 & 3 \\ 27 & 1 & 3 & 9 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1 & 3 & 9 & 27 \\ 3 & 9 & 27 & 1 \\ 9 & 27 & 1 & 3 \\ 27 & 1 & 3 & 9 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 3 & 3^2 & 3^3 & 1 \\ 3^2 & 3^3 & 1 & 3 \\ 3^3 & 1 & 3 & 3^2 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3 + C_4$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 + 3 + 3^2 + 3^3 & 3 & 3^2 & 3^3 \\ 1 + 3 + 3^2 + 3^3 & 3^2 & 3^3 & 1 \\ 1 + 3 + 3^2 + 3^3 & 3^3 & 1 & 3 \\ 1 + 3 + 3^2 + 3^3 & 1 & 3 & 3^2 \end{vmatrix}$$

$$\Rightarrow \Delta = (1 + 3 + 3^2 + 3^3) \begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 1 & 3^2 & 3^3 & 1 \\ 1 & 3^3 & 1 & 3 \\ 1 & 1 & 3 & 3^2 \end{vmatrix}$$

Now, applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$, $R_4 \rightarrow R_4 - R_1$, we get

$$\Rightarrow \Delta = (1 + 3 + 3^2 + 3^3) \begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 0 & 3^2 - 3 & 3^3 - 3^2 & 1 - 3^3 \\ 0 & 3^3 - 3 & 1 - 3^2 & 3 - 3^3 \\ 0 & 1 - 3 & 3 - 3^2 & 3^2 - 3^3 \end{vmatrix}$$

$$\Rightarrow \Delta = (1 + 3 + 3^2 + 3^3) \begin{vmatrix} 6 & 18 & -26 \\ 24 & -8 & -24 \\ -2 & -6 & -18 \end{vmatrix}$$

$$\Rightarrow \Delta = (1 + 3 + 3^2 + 3^3) 2^3 \begin{vmatrix} 3 & -9 & 13 \\ 12 & 4 & 12 \\ -1 & 3 & 9 \end{vmatrix}$$

Now, applying $R_1 \rightarrow R_1 + 3R_3$

$$\Rightarrow \Delta = (1 + 3 + 3^2 + 3^3) 2^3 \begin{vmatrix} 0 & 0 & 40 \\ 12 & 4 & 12 \\ -1 & 3 & 9 \end{vmatrix}$$

Now, applying $R_1 \rightarrow R_1 + 3R_3$

$$\Rightarrow \Delta = (1 + 3 + 3^2 + 3^3)2^3 \begin{vmatrix} 0 & 0 & 40 \\ 12 & 4 & 12 \\ -1 & 3 & 9 \end{vmatrix}$$

$$= (1 + 3 + 3^2 + 3^3)2^3 [40(36 - (-4))]$$

$$= (40)(8)(40)(40) = 512000$$

So, $\Delta = 512000$

(viii) Given,

$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$$\Rightarrow \Delta = 6 \begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

Applying $R_3 \rightarrow R_3 - R_1$, we get,

$$\Rightarrow \Delta = 6 \begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

So, $\Delta = 0$

2. Without expanding, show that the value of each of the following determinants is zero:

$$(i) \begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} 6 & 3 & -2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

$$(iii) \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

$$(iv) \begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ac \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$$

$$(v) \begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$$

$$(vi) \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ac \\ 1 & c & c^2 - ab \end{vmatrix}$$

$$(vii) \begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix}$$

$$(viii) \begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$$

$$(ix) \begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$$

$$(x) \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

$$(xi) \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

$$(xii) \begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$

$$(xiii) \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha + \delta) \\ \sin\beta & \cos\beta & \cos(\beta + \delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma + \delta) \end{vmatrix}$$

$$(xiv) \begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

$$(xv) \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & -\cos y \end{vmatrix}$$

$$(xvi) \begin{vmatrix} \sqrt{23} + \sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15} + \sqrt{46} & 5 & \sqrt{10} \\ 3 + \sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

$$(xvii) \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}, \text{ where } A, B, C \text{ are the angles of } \Delta ABC$$

Solution:

(i) Given,

$$\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$$

Now by applying row operation $R_3 \rightarrow R_3 - R_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 4 & 1 & -2 \end{vmatrix}$$

Again apply row operations $R_2 \rightarrow R_2 - R_1$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 8 & 2 & 7 \\ 4 & 1 & -2 \\ 4 & 1 & -2 \end{vmatrix}$$

As, $R_2 = R_3$, therefore the value of the determinant is zero.

(ii) Given,

$$\text{Let, } \Delta = \begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

Taking (-2) common from C_1 in above matrix we get,

$$\Rightarrow \Delta = \begin{vmatrix} -3 & -3 & 2 \\ -1 & -1 & 2 \\ 5 & 5 & 2 \end{vmatrix}$$

As, $C_1 = C_2$, hence the value of the determinant is zero.

(iii) Given,

$$\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

Now by applying column operation $C_3 \rightarrow C_3 - C_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 2 & 3 & 7 \end{vmatrix}$$

As, $R_1 = R_3$, so value so determinant is zero.

(iv) Given,

$$\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix}$$

Multiplying R_1 , R_2 and R_3 with a , b and c respectively we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix}$$

Now by taking, abc common from C_3 gives,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix}$$

As, $C_1 = C_3$ hence the value of determinant is zero.

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix}$$

Now by taking, abc common from C_3 gives,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix}$$

As, $C_1 = C_3$ hence the value of determinant is zero.

(v) Given,

$$\begin{vmatrix} a + b & 2a + b & 3a + b \\ 2a + b & 3a + b & 4a + b \\ 4a + b & 5a + b & 6a + b \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a + b & 2a + b & 3a + b \\ 2a + b & 3a + b & 4a + b \\ 4a + b & 5a + b & 6a + b \end{vmatrix}$$

Now by applying column operation $C_3 \rightarrow C_3 - C_2$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} a + b & 2a + b & a \\ 2a + b & 3a + b & a \\ 4a + b & 5a + b & a \end{vmatrix}$$

Again applying column operation $C_2 \rightarrow C_2 - C_1$ gives,

$$\Rightarrow \Delta = \begin{vmatrix} a + b & a & a \\ 2a + b & a & a \\ 4a + b & a & a \end{vmatrix}$$

As, $C_2 = C_3$, so the value of the determinant is zero.

(vi) Given,

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ac \\ 1 & c & c^2 - ab \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ac \\ 1 & c & c^2 - ab \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 0 & b-a & (a-b)c \\ 0 & c-a & (a-c)b \end{vmatrix}$$

Taking $(b-a)$ and $(c-a)$ common from R_2 and R_3 respectively,

$$\Rightarrow \Delta = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} - (b-a)(c-a) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix}$$

$$= [(b-a)(c-a)] [(c+a) - (b+a) - (-b+c)]$$

$$= [(b-a)(c-a)] [c+a+b-a-b-c]$$

$$= [(b-a)(c-a)] [0] = 0$$

(vii) Given,

$$\begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix}$$

Now by applying column operation, $C_1 \rightarrow C_1 - 8C_3$ we get

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 2 & 2 & 3 \end{vmatrix}$$

As, $C_1 = C_2$ hence, the determinant is zero.

(viii) Given,

$$\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$$

Multiplying C_1 , C_2 and C_3 with z , y and x respectively we get,

$$\Rightarrow \Delta = \left(\frac{1}{xyz}\right) \begin{vmatrix} 0 & xy & yx \\ -xz & 0 & zx \\ -yz & -zy & 0 \end{vmatrix}$$

Now, taking y , x and z common from R_1 , R_2 and R_3 gives,

$$\Rightarrow \Delta = \left(\frac{1}{xyz}\right) \begin{vmatrix} 0 & x & x \\ -z & 0 & z \\ -y & -y & 0 \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_3$ gives,

$$\Rightarrow \Delta = \left(\frac{1}{xyz}\right) \begin{vmatrix} 0 & x & x \\ -z & -z & z \\ -y & -y & 0 \end{vmatrix}$$

As, $C_1 = C_2$, therefore determinant is zero.

(ix) Given,

$$\begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - 7C_3$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix}$$

As, $C_1 = C_2$, hence determinant is zero.

As, $C_1 = C_2$, hence determinant is zero

(x) Given,

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

Now we have to apply the column operation $C_3 \rightarrow C_3 - C_2$, and $C_4 \rightarrow C_4 - C_1$, then we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1^2 & 2^2 & 3^2 - 2^2 & 4^2 - 1^2 \\ 2^2 & 3^2 & 4^2 - 3^2 & 5^2 - 2^2 \\ 3^2 & 4^2 & 5^2 - 4^2 & 6^2 - 3^2 \\ 4^2 & 5^2 & 6^2 - 5^2 & 7^2 - 4^2 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1^2 & 2^2 & 5 & 15 \\ 2^2 & 3^2 & 7 & 21 \\ 3^2 & 4^2 & 9 & 27 \\ 4^2 & 5^2 & 11 & 33 \end{vmatrix}$$

Taking 3 common from C_4 we get,

$$\Rightarrow \Delta = 3 \begin{vmatrix} 1^2 & 2^2 & 5 & 5 \\ 2^2 & 3^2 & 7 & 7 \\ 3^2 & 4^2 & 9 & 9 \\ 4^2 & 5^2 & 11 & 11 \end{vmatrix}$$

As, $C_3 = C_4$ so, the determinant is zero.

(xi) Given,

$$\text{Let, } \Delta = \begin{vmatrix} a & b & c \\ a + 2x & b + 2y & c + 2z \\ x & y & z \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a & b & c \\ a + 2x & b + 2y & c + 2z \\ x & y & z \end{vmatrix}$$

Now by applying, $C_2 \rightarrow C_2 + C_1$ and $C_3 \rightarrow C_3 + C_1$, we get

$$\Rightarrow \Delta = \begin{vmatrix} a & b & c \\ 2a + 2x & 2b + 2y & 2c + 2z \\ a + x & b + y & c + z \end{vmatrix}$$

Taking 2 common from R_2 we get,

$$\Rightarrow \Delta = 2 \begin{vmatrix} a & b & c \\ a + x & b + y & c + z \\ a + x & b + y & c + z \end{vmatrix}$$

As, $R_2 = R_3$, hence value of determinant is zero.

(xii) Given,

$$\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 2^{2x} + 2^{-2x} + 2 & 2^{2x} + 2^{-2x} - 2 & 1 \\ 3^{2x} + 3^{-2x} + 2 & 3^{2x} + 3^{-2x} - 2 & 1 \\ 4^{2x} + 4^{-2x} + 2 & 4^{2x} + 4^{-2x} - 2 & 1 \end{vmatrix}$$

By applying, column operation $C_1 \rightarrow C_1 - C_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 4 & 2^{2x} + 2^{-2x} - 2 & 1 \\ 4 & 3^{2x} + 3^{-2x} - 2 & 1 \\ 4 & 4^{2x} + 4^{-2x} - 2 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = 4 \begin{vmatrix} 1 & 2^{2x} + 2^{-2x} - 2 & 1 \\ 1 & 3^{2x} + 3^{-2x} - 2 & 1 \\ 1 & 4^{2x} + 4^{-2x} - 2 & 1 \end{vmatrix}$$

As $C_1 = C_3$ hence determinant is zero.

(xiii) Given,

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix}$$

Multiplying C_1 with $\sin \delta$, C_2 with $\cos \delta$, we get

$$\Rightarrow \Delta = \frac{1}{\sin \delta \cos \delta} \begin{vmatrix} \sin \alpha \sin \delta & \cos \alpha \cos \delta & \cos(\alpha + \delta) \\ \sin \beta \sin \delta & \cos \beta \cos \delta & \cos(\beta + \delta) \\ \sin \gamma \sin \delta & \cos \gamma \cos \delta & \cos(\gamma + \delta) \end{vmatrix}$$

Now, by applying column operation, $C_2 \rightarrow C_2 - C_1$, we get,

$$\Rightarrow \Delta = \frac{1}{\sin \delta \cos \delta} \begin{vmatrix} \sin \alpha \sin \delta & \cos \alpha \cos \delta - \sin \alpha \sin \delta & \cos(\alpha + \delta) \\ \sin \beta \sin \delta & \cos \beta \cos \delta - \sin \beta \sin \delta & \cos(\beta + \delta) \\ \sin \gamma \sin \delta & \cos \gamma \cos \delta - \sin \gamma \sin \delta & \cos(\gamma + \delta) \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{1}{\sin \delta \cos \delta} \begin{vmatrix} \sin \alpha \sin \delta & \cos(\alpha + \delta) & \cos(\alpha + \delta) \\ \sin \beta \sin \delta & \cos(\beta + \delta) & \cos(\beta + \delta) \\ \sin \gamma \sin \delta & \cos(\gamma + \delta) & \cos(\gamma + \delta) \end{vmatrix}$$

As $C_2 = C_3$ hence determinant is zero.

(xiv) Given,

$$\begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} \sin^2 23^\circ + \sin^2 67^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ - \sin^2 23^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ + \sin^2 23^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

Using, $\sin(90 - A) = \cos A$, $\sin^2 A + \cos^2 A = 1$, and $\cos 180^\circ = -1$,

$$\Rightarrow \Delta = \begin{vmatrix} \sin^2 23^\circ + \cos^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -(\sin^2 67^\circ + \cos^2 67^\circ) & -\sin^2 23^\circ & \cos^2 180^\circ \\ -(1 - \sin^2 23^\circ) & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & \sin^2 67^\circ & -1 \\ -1 & -\sin^2 23^\circ & 1 \\ -\cos^2 23^\circ & \sin^2 23^\circ & \cos^2 23^\circ \end{vmatrix}$$

Taking, (-1) common from C_1 , we get

$$\Rightarrow \Delta = - \begin{vmatrix} -1 & \sin^2 67^\circ & -1 \\ 1 & -\sin^2 23^\circ & 1 \\ \cos^2 23^\circ & \sin^2 23^\circ & \cos^2 23^\circ \end{vmatrix}$$

Therefore, as $C_1 = C_3$ determinant is zero.

(xv) Given,

$$\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & -\cos y \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & -\cos y \end{vmatrix}$$

Multiplying R_2 with $\sin y$ and R_3 with $\cos y$ we get,

$$\Rightarrow \Delta = \frac{1}{\sin y \cos y} \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x \sin y & \cos x \sin y & \sin^2 y \\ -\cos x \cos y & \sin x \cos y & -\cos^2 y \end{vmatrix}$$

Now, by applying row operation $R_2 \rightarrow R_2 + R_3$, we get,

$$= \frac{1}{\sin y \cos y} \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x \sin y - \cos x \cos y & \cos x \sin y + \sin x \cos y & \sin^2 y - \cos^2 y \\ -\cos x \cos y & \sin x \cos y & -\cos^2 y \end{vmatrix}$$

Taking (-1) common from R_2 , we get

Taking (-1) common from R_2 , we get

$$= \frac{-1}{\sin y \cos y} \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ -\sin x \sin y + \cos x \cos y & -(\cos x \sin y + \sin x \cos y) & -\sin^2 y + \cos^2 y \\ -\cos x \cos y & \sin x \cos y & -\cos^2 y \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{-1}{\sin y \cos y} \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \cos(x+y) & -\sin(x+y) & \cos 2y \\ -\cos x \cos y & \sin x \cos y & -\cos^2 y \end{vmatrix}$$

As $R_1 = R_2$ hence determinant is zero.

(xvi) Given,

$$\begin{vmatrix} \sqrt{23} + \sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15} + \sqrt{46} & 5 & \sqrt{10} \\ 3 + \sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \sqrt{23} + \sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15} + \sqrt{46} & 5 & \sqrt{10} \\ 3 + \sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

Multiplying C_2 with $\sqrt{3}$ and C_3 with $\sqrt{23}$ we get,

$$\Rightarrow \Delta = \begin{vmatrix} \sqrt{23} + \sqrt{3} & \sqrt{15} & \sqrt{115} \\ \sqrt{15} + \sqrt{46} & 5\sqrt{3} & \sqrt{230} \\ 3 + \sqrt{115} & \sqrt{45} & 5\sqrt{23} \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} \sqrt{23} + \sqrt{3} & \sqrt{5}(\sqrt{3}) & \sqrt{5}(\sqrt{23}) \\ \sqrt{15} + \sqrt{46} & \sqrt{5}(\sqrt{15}) & \sqrt{5}(\sqrt{46}) \\ 3 + \sqrt{115} & \sqrt{5}(3) & \sqrt{5}(\sqrt{115}) \end{vmatrix}$$

Now taking $\sqrt{5}$ common from C_2 and C_3 we get,

$$\Rightarrow \Delta = \sqrt{5}\sqrt{5} \begin{vmatrix} \sqrt{23} + \sqrt{3} & (\sqrt{3}) & (\sqrt{23}) \\ \sqrt{15} + \sqrt{46} & (\sqrt{15}) & (\sqrt{46}) \\ 3 + \sqrt{115} & (3) & (\sqrt{115}) \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 + C_3$

Applying $C_2 \rightarrow C_2 + C_3$

$$\Rightarrow \Delta = 5 \begin{vmatrix} \sqrt{23} + \sqrt{3} & \sqrt{23} + \sqrt{3} & (\sqrt{23}) \\ \sqrt{15} + \sqrt{46} & \sqrt{15} + \sqrt{46} & (\sqrt{46}) \\ 3 + \sqrt{115} & 3 + \sqrt{115} & (\sqrt{115}) \end{vmatrix}$$

As $C_1 = C_2$ hence determinant is zero.

(xvii) Given,

$$\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

Now,

$$\Delta = \sin^2 A (\cot B - \cot C) - \cot A (\sin^2 B - \sin^2 C) + 1 (\sin^2 B \cot C - \cot B \sin^2 C)$$

As A, B and C are angles of a triangle,

$$A + B + C = 180^\circ$$

$$\Delta = \sin^2 A \cot B - \sin^2 A \cot C - \cot A \sin^2 B + \cot A \sin^2 C + \sin^2 B \cot C - \cot B \sin^2 C$$

By using formulae, we get

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Delta = 0$$

Hence proved.

Evaluate the following (3 – 9):

$$3. \begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$$

Solution:

Given,

$$\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$$

Now by applying column operation $C_2 \rightarrow C_2 + C_1$

$$\Rightarrow \Delta = \begin{vmatrix} a & b+c+a & a^2 \\ b & c+a+b & b^2 \\ c & a+b+c & c^2 \end{vmatrix}$$

Taking, $(a+b+c)$ common,

$$\Rightarrow \Delta = (a+b+c) \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

Again by applying row operation $R_2 \rightarrow R_2 - R_1$, and $R_3 \rightarrow R_3 - R_1$

$$\Rightarrow \Delta = (a+b+c) \begin{vmatrix} a & 1 & a^2 \\ b-a & 0 & b^2-a^2 \\ c-a & 0 & c^2-a^2 \end{vmatrix}$$

Taking, $(b-a)$ and $(c-a)$ common,

$$\Rightarrow \Delta = (a+b+c)(b-a)(c-a) \begin{vmatrix} a & 1 & a^2 \\ 1 & 0 & b+a \\ 1 & 0 & c+a \end{vmatrix}$$

$$= (a+b+c)(b-a)(c-a)(b-c)$$

$$\text{So, } \Delta = (a+b+c)(b-a)(c-a)(b-c)$$

$$4. \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Solution:

Given,

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Now by applying row operation, $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-bc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & b(a-c) \end{vmatrix}$$

Taking $(a-b)$ and $(a-c)$ common we get,

$$\Rightarrow \Delta = (a-b)(a-c) \begin{vmatrix} 1 & a & bc \\ 0 & -1 & c \\ 0 & -1 & b \end{vmatrix}$$

$$= (a-b)(c-a)(b-c)$$

$$\text{So, } \Delta = (a-b)(b-c)(c-a)$$

$$5. \begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$$

Solution:

Given,

$$\begin{vmatrix} x + \lambda & x & x \\ x & x + \lambda & x \\ x & x & x + \lambda \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} x + \lambda & x & x \\ x & x + \lambda & x \\ x & x & x + \lambda \end{vmatrix}$$

Applying, $C_1 \rightarrow C_1 + C_2 + C_3$, we have,

$$\Rightarrow \Delta = \begin{vmatrix} 3x + \lambda & x & x \\ 3x + \lambda & x + \lambda & x \\ 3x + \lambda & x & x + \lambda \end{vmatrix}$$

Taking, $(3x + \lambda)$ common, we get

$$\Rightarrow \Delta = (3x + \lambda) \begin{vmatrix} 1 & x & x \\ 1 & x + \lambda & x \\ 1 & x & x + \lambda \end{vmatrix}$$

Applying, $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$, we get,

$$\Rightarrow \Delta = (3x + \lambda) \begin{vmatrix} 1 & x & x \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

$$= \lambda^2 (3x + \lambda)$$

$$\text{So, } \Delta = \lambda^2 (3x + \lambda)$$

$$6. \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Solution:

Given,

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Now we have to apply column operation, $C_1 \rightarrow C_1 + C_2 + C_3$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix}$$

Taking, $(a+b+c)$ we get,

$$\Rightarrow \Delta = (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix}$$

Now by applying row operation, $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$, we get,

$$\Rightarrow \Delta = (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & a-b & b-c \\ 0 & c-b & a-c \end{vmatrix}$$

$$= (a+b+c) [(a-b)(a-c) - (b-c)(c-b)]$$

$$= (a+b+c) [a^2 - ac - ab + bc + b^2 + c^2 - 2bc]$$

$$= (a+b+c) [a^2 + b^2 + c^2 - ac - ab - bc]$$

$$\text{So, } \Delta = (a+b+c) [a^2 + b^2 + c^2 - ac - ab - bc]$$

$$7. \begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

Solution:

Given,

$$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

Now by applying column operation, $C_1 \rightarrow C_1 + C_2 + C_3$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 2+x & 1 & 1 \\ 2+x & x & 1 \\ 2+x & 1 & x \end{vmatrix}$$

$$\Rightarrow \Delta = (2+x) \begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

Again by applying row operation, $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$, we get,

$$\Rightarrow \Delta = (2+x) \begin{vmatrix} 1 & 1 & 1 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix}$$

$$= (2+x)(x-1)^2$$

$$\text{So, } \Delta = (2+x)(x-1)^2$$

$$8. \begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ xz^2 & zy^2 & 0 \end{vmatrix}$$

Solution:

Given,

$$\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$$

On simplification we get,

$$= 0(0 - y^3z^3) - xy^2(0 - x^2yz^3) + xz^2(x^2y^3z - 0)$$

$$= 0 + x^3y^3z^3 + x^3y^3z^3$$

$$= 2x^3y^3z^3$$

$$\text{So, } \Delta = 2x^3y^3z^3$$

$$9. \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

Solution:

Given,

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

Now by applying row operation we get $R_1 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_3 - R_2$

$$\Rightarrow \Delta = \begin{vmatrix} a & -a & 0 \\ x & a+y & z \\ 0 & -a & a \end{vmatrix}$$

Again by applying column operation, $C_2 \rightarrow C_2 - C_1$

$$\Rightarrow \Delta = \begin{vmatrix} a & 0 & 0 \\ x & a+x+y & z \\ 0 & -a & a \end{vmatrix}$$

$$= a [a(a+x+y) + az] + 0 + 0$$

$$= a^2(a+x+y+z)$$

$$\text{So, } \Delta = a^2(a+x+y+z)$$

$$\Rightarrow \Delta = \begin{vmatrix} a & 0 & 0 \\ x & a+x+y & z \\ 0 & -a & a \end{vmatrix}$$

$$= a [a(a+x+y) + az] + 0 + 0$$

$$= a^2(a+x+y+z)$$

$$\text{So, } \Delta = a^2(a+x+y+z)$$

$$10. \text{ If } \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}, \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}, \text{ then prove that } \Delta + \Delta_1 = 0$$

Solution:

$$\text{Let, } \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$$

$$\text{As } |A| = |A|^T$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix}$$

If any two rows or columns of the determinant are interchanged, then determinant changes its sign

$$\Rightarrow \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} - \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & x^2 - yz \\ 0 & 0 & y^2 - zx \\ 0 & 0 & z^2 - xy \end{vmatrix} = 0$$

$$\text{So, } \Delta = 0$$

Hence the proof

Prove the following identities (11 – 45):

$$11. \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$$

Solution:

Given,

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

$$\text{L.H.S} = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

Apply $C_1 \rightarrow C_1 + C_2 + C_3$

$$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$$

Taking $(a+b+c)$ common from C_1 we get,

$$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

Applying, $R_3 \rightarrow R_3 - 2R_1$

$$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$$

$$= (a+b+c) [(b-c)(a+b-2c) - (c-a)(c+a-2b)]$$

$$= a^3 + b^3 + c^3 - 3abc$$

As, L.H.S = R.H.S

Hence, the proof.

$$12. \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$$

As $|A| = |A|^T$

$$\text{So, } \begin{vmatrix} b+c & c+a & a+b \\ a-b & b-c & c-a \\ a & b & c \end{vmatrix}$$

If any two rows or columns of the determinant are interchanged, then determinant changes its sign

$$- \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

Apply $C_1 \rightarrow C_1 + C_2 + C_3$

$$= - \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$$

Taking $(a+b+c)$ common from C_1 we get,

$$= -(a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

Applying, $R_3 \rightarrow R_3 - 2R_1$

$$= -(a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$$

$$= -(a+b+c) [(b-c)(a+b-2c) - (c-a)(c+a-2b)]$$

$$= 3abc - a^3 - b^3 - c^3$$

Therefore, L.H.S = R.H.S,

Hence the proof.

$$13. \begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Solution:

Given,

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$\text{L.H.S} = \begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$$

Now by applying, $C_1 \rightarrow C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \end{vmatrix}$$

$$= 2 \begin{vmatrix} (a+b+c) & b+c & c+a \\ (a+b+c) & c+a & a+b \\ (a+b+c) & a+b & b+c \end{vmatrix}$$

Again apply, $C_2 \rightarrow C_2 - C_1$, and $C_3 \rightarrow C_3 - C_1$, we have

$$= 2 \begin{vmatrix} (a+b+c) & -a & -b \\ (a+b+c) & -b & -c \\ (a+b+c) & -c & -a \end{vmatrix}$$

$$= 2 \begin{vmatrix} (a+b+c) & a & b \\ (a+b+c) & b & c \\ (a+b+c) & c & a \end{vmatrix}$$

By expanding, we get

$$= 2 \begin{vmatrix} (a+b+c) & -a & -b \\ (a+b+c) & -b & -c \\ (a+b+c) & -c & -a \end{vmatrix}$$

$$= 2 \begin{vmatrix} (a+b+c) & a & b \\ (a+b+c) & b & c \\ (a+b+c) & c & a \end{vmatrix}$$

By expanding, we get

$$= 2 \left(\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix} + \begin{vmatrix} a & a & b \\ b & b & c \\ c & c & a \end{vmatrix} + \begin{vmatrix} b & a & b \\ c & b & c \\ a & c & a \end{vmatrix} \right)$$

As in second and third determinant both have same column and its value is zero

Therefore,

$$= 2 \begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$$

$$= 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \text{R.H.S}$$

Hence, the proof.

$$14. \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix},$$

$$\text{R.H.S} = 2(a + b + c)^2$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we have

$$= \begin{vmatrix} 2(a + b + c) & a & b \\ 2(a + b + c) & b + c + 2a & b \\ 2(a + b + c) & a & c + a + 2b \end{vmatrix}$$

Taking, $2(a + b + c)$ common we get,

$$= 2(a + b + c) \begin{vmatrix} 1 & a & b \\ 1 & b + c + 2a & b \\ 1 & a & c + a + 2b \end{vmatrix}$$

Now, applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get,

$$= 2(a + b + c) \begin{vmatrix} 1 & a & b \\ 0 & b + c + a & 0 \\ 0 & 0 & c + a + b \end{vmatrix}$$

Thus, we have

$$\text{L.H.S} = 2(a + b + c) [1(a + b + c)^2]$$

$$= 2(a + b + c)^3 = \text{R.H.S}$$

$$15. \begin{vmatrix} a - b - c & 2a & 2a \\ 2b & b - c - a & 2b \\ 2c & 2c & c - a - b \end{vmatrix} = (a + b + c)^3$$

Solution:

Consider,

$$\text{L.H.S} =$$

$$\begin{vmatrix} a - b - c & 2a & 2a \\ 2b & b - c - a & 2b \\ 2c & 2c & c - a - b \end{vmatrix}$$

Now by applying, $R_1 \rightarrow R_1 + R_2 + R_3$, we get,

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a+b+c)$ common we get,

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get,

$$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & b+c+a & 0 \\ 2c & 0 & b+c+a \end{vmatrix}$$

$$= (a+b+c)^3 = \text{R.H.S}$$

Hence, proved.

$$16. \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

Now by applying, $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get,

$$= \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & a-b & a^2-b^2 \\ 0 & a-c & a^2-c^2 \end{vmatrix}$$

$$= (a-b)(a-c) \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{vmatrix}$$

Again by applying $R_3 \rightarrow R_3 - R_2$, we get,

$$= (a-b)(a-c) \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & 1 & a+b \\ 0 & 0 & c-a \end{vmatrix}$$

$$= (a-b)(a-c)(b-c) = \text{R.H.S}$$

Hence, the proof.

$$17. \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9(a+b)b^2$$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get,

$$= \begin{vmatrix} 3a+3b & 3a+3b & 3a+3b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Taking, $(3a+3b)$ common we get,

$$= (3a+3b) \begin{vmatrix} 1 & 1 & 1 \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Applying, $C_1 \rightarrow C_1 - C_2$ and $C_3 \rightarrow C_3 - C_2$, we get,

$$= (3a+3b) \begin{vmatrix} 0 & 1 & 0 \\ 2b & a & b \\ -b & a+2b & -2b \end{vmatrix}$$

$$= (3a+3b)b^2 \begin{vmatrix} 0 & 1 & 0 \\ 2 & a & 1 \\ -1 & a+2b & -2 \end{vmatrix}$$

$$= 3(a+b) b^2(3) = 9(a+b) b^2$$

= R.H.S

Hence, proved.

$$= (3a+3b) \begin{vmatrix} 0 & 1 & 0 \\ 2b & a & b \\ -b & a+2b & -2b \end{vmatrix}$$

$$= (3a+3b)b^2 \begin{vmatrix} 0 & 1 & 0 \\ 2 & a & 1 \\ -1 & a+2b & -2 \end{vmatrix}$$

$$= 3(a+b) b^2(3) = 9(a+b) b^2$$

= R.H.S

Hence, the proof.

$$18. \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Now by applying, $R_1 \rightarrow a R_1$, $R_2 \rightarrow b R_2$, $R_3 \rightarrow c R_3$

We get,

$$= \left(\frac{1}{abc}\right) \begin{vmatrix} a & a^2 & abc \\ b & b^2 & cab \\ c & c^2 & abc \end{vmatrix}$$

$$= \left(\frac{abc}{abc}\right) \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Hence, the proof.

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Hence, the proof.

$$19. \begin{vmatrix} z & x & y \\ z^2 & x^2 & y^2 \\ z^4 & x^4 & y^4 \end{vmatrix} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix} = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \\ x & y & z \end{vmatrix} = xyz(x-y)(y-z)(z-x)(x+y+z)$$

Solution:

Given,

$$\begin{vmatrix} z & x & y \\ z^2 & x^2 & y^2 \\ z^4 & x^4 & y^4 \end{vmatrix} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix} = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \\ x & y & z \end{vmatrix} \\ = xyz(x-y)(y-z)(z-x)(x+y+z)$$

Consider,

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix}$$

By taking xyz common

$$= xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= xyz \begin{vmatrix} 0 & 1 & 0 \\ x-y & y & z-y \\ x^3-y^3 & y^3 & z^3-y^3 \end{vmatrix}$$

$$= xyz(x-y)(z-y) \begin{vmatrix} 0 & 1 & 0 \\ 1 & y & 1 \\ x^2+y^2+xy & y^3 & z^2+y^2+zy \end{vmatrix}$$

$$= -xyz(x-y)(z-y)[z^2+y^2+zy-x^2-y^2-xy]$$

$$= -xyz(x-y)(z-y)[(z-x)(z+x)+y(z-x)]$$

$$= -xyz(x-y)(z-y)(z-x)(x+y+z)$$

$$= \text{R.H.S}$$

Hence, the proof.

$$20. \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

Applying, $C_1 \rightarrow C_1 + C_2 - 2C_3$

$$= \begin{vmatrix} (b+c)^2 - a^2 - 2bc & a^2 & bc \\ (c+a)^2 - b^2 - 2ca & b^2 & ca \\ (a+b)^2 - c^2 - 2ab & c^2 & ab \end{vmatrix}$$

$$= \begin{vmatrix} a^2 + b^2 + c^2 & a^2 & bc \\ a^2 + b^2 + c^2 & b^2 & ca \\ a^2 + b^2 + c^2 & c^2 & ab \end{vmatrix}$$

Taking $(a^2 + b^2 + c^2)$, common, we get,

$$= (a^2 + b^2 + c^2) \begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get,

$$= (a^2 + b^2 + c^2) \begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2 - a^2 & ca - bc \\ 0 & c^2 - a^2 & ab - bc \end{vmatrix}$$

$$= (a^2 + b^2 + c^2)(b-a)(c-a) \begin{vmatrix} 1 & a^2 & bc \\ 0 & b+a & -c \\ 0 & c+a & -b \end{vmatrix}$$

$$= (a^2 + b^2 + c^2) \begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2 - a^2 & ca - bc \\ 0 & c^2 - a^2 & ab - bc \end{vmatrix}$$

$$= (a^2 + b^2 + c^2)(b-a)(c-a) \begin{vmatrix} 1 & a^2 & bc \\ 0 & b+a & -c \\ 0 & c+a & -b \end{vmatrix}$$

$$= (a^2 + b^2 + c^2) (b-a) (c-a) [(b+a)(-b) - (-c)(c+a)]$$

$$= (a^2 + b^2 + c^2) (a-b) (c-a) (b-c) (a+b+c)$$

$$= R.H.S$$

Hence, the proof.

$$21. \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$$

Now by applying row operation, $R_3 \rightarrow R_3 - R_2$

$$= \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)2 & 1 & 0 \end{vmatrix}$$

Again by applying, $R_2 \rightarrow R_2 - R_1$

$$= \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)2 & 1 & 0 \\ (a+3)2 & 1 & 0 \end{vmatrix}$$

$$= [(2a+4)(1) - (1)(2a+6)]$$

$$= -2$$

$$= R.H.S$$

Hence, the proof.

$$22. \begin{vmatrix} a^2 & a^2 - (b-c)^2 & bc \\ b^2 & b^2 - (c-a)^2 & ca \\ c^2 & c^2 - (a-b)^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} a^2 & a^2 - (b-c)^2 & bc \\ b^2 & b^2 - (c-a)^2 & ca \\ c^2 & c - (a-b)^2 & ab \end{vmatrix}$$

Applying, $C_2 \rightarrow C_2 - 2C_1 - 2C_3$, we get,

$$= \begin{vmatrix} a^2 & a^2 - (b-c)^2 - 2a^2 - 2bc & bc \\ b^2 & b^2 - (c-a)^2 - 2b^2 - 2ca & ca \\ c^2 & c - (a-b)^2 - 2c^2 - 2ab & ab \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & -(a^2 + b^2 + c^2) & bc \\ b^2 & -(a^2 + b^2 + c^2) & ca \\ c^2 & -(a^2 + b^2 + c^2) & ab \end{vmatrix}$$

Taking, $-(a^2 + b^2 + c^2)$ common from C_2 we get,

$$= -(a^2 + b^2 + c^2) \begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$= -(a^2 + b^2 + c^2) \begin{vmatrix} a^2 & 1 & bc \\ b^2 - a^2 & 0 & ca - bc \\ c^2 - a^2 & 0 & ab - bc \end{vmatrix}$$

$$= -(a^2 + b^2 + c^2)(a-b)(c-a) \begin{vmatrix} a^2 & 1 & bc \\ -(b+a) & 0 & c \\ c+a & 0 & -b \end{vmatrix}$$

$$= -(a^2 + b^2 + c^2)(a-b)(c-a) \begin{vmatrix} a^2 & 1 & bc \\ -(b+a) & 0 & c \\ c+a & 0 & -b \end{vmatrix}$$

$$= -(a^2 + b^2 + c^2)(a-b)(c-a)[(-(b+a))(-b) - (c)(c+a)]$$

$$= (a-b)(b-c)(c-a)(a+b+c)(a^2 + b^2 + c^2)$$

$$= R.H.S$$

Hence, the proof.

$$23. \begin{vmatrix} 1 & a^2 + bc & a^3 \\ 1 & b^2 + ca & b^3 \\ 1 & c^2 + ab & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a^2 + b^2 + c^2)$$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} 1 & a^2 + bc & a^3 \\ 1 & b^2 + ca & b^3 \\ 1 & c^2 + ab & c^3 \end{vmatrix}$$

Applying, $R_2 \rightarrow R_2 - R_1$, and $R_3 \rightarrow R_3 - R_1$

$$= \begin{vmatrix} 1 & a^2 + bc & a^3 \\ 0 & b^2 + ca - a^2 - bc & b^3 - a^3 \\ 0 & c^2 + ab - a^2 - bc & c^3 - a^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 + bc & a^3 \\ 0 & b^2 - a^2 - c(b - a) & b^3 - a^3 \\ 0 & c^2 - a^2 + b(c - a) & c^3 - a^3 \end{vmatrix}$$

$$= (b - a)(c - a) \begin{vmatrix} 1 & a^2 + bc & a^3 \\ 0 & b + a - c & b^2 + a^2 + ab \\ 0 & c + a + b & c^2 + a^2 + ac \end{vmatrix}$$

$$= (b - a)(c - a) [((b + a - c))(c^2 + a^2 + ac) - (b^2 + a^2 + ab)(c^2 + a^2 + ac)]$$

$$= -(a - b)(c - a)(b - c)(a^2 + b^2 + c^2)$$

$$= R.H.S$$

Hence, proved.

$$= R.H.S$$

Hence, the proof.

$$24. \begin{vmatrix} a^2 & bc & ac + c^2 \\ a^2 + ab & b^2 & ac \\ ab & b^2 + bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} a^2 & bc & ac + c^2 \\ a^2 + ab & b^2 & ac \\ ab & b^2 + bc & c^2 \end{vmatrix}$$

Taking, a, b, c common from C₁, C₂, C₃ respectively we get,

$$= abc \begin{vmatrix} a & c & a + c \\ a + b & b & a \\ b & b + c & c \end{vmatrix}$$

Applying, C₁→C₁ + C₂ + C₃, we get,

$$= abc \begin{vmatrix} 2(a + c) & c & a + c \\ 2(a + b) & b & a \\ 2(b + c) & b + c & c \end{vmatrix}$$

$$= 2abc \begin{vmatrix} (a + c) & c & a + c \\ (a + b) & b & a \\ (b + c) & b + c & c \end{vmatrix}$$

Applying, C₂→C₂ - C₁ and C₃→C₃ - C₁, we get,

$$= 2abc \begin{vmatrix} (a + c) & -a & 0 \\ (a + b) & -a & -b \\ (b + c) & 0 & -b \end{vmatrix}$$

Applying, C₁→C₁ + C₂ + C₃, we get,

$$= 2abc \begin{vmatrix} c & -a & 0 \\ 0 & -a & -b \\ c & 0 & -b \end{vmatrix}$$

$$= 2abc \begin{vmatrix} c & -a & 0 \\ 0 & -a & -b \\ c & 0 & -b \end{vmatrix}$$

Taking c, a, b common from C₁, C₂, C₃ respectively, we get,

$$= 2a^2b^2c^2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{vmatrix}$$

Applying, R₃ → R₃ - R₁, we have

$$= 2a^2b^2c^2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= 2a^2b^2c^2 (2)$$

$$= 4a^2b^2c^2 = \text{R.H.S}$$

Hence, proved.

$$25. \begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = 16(3x+4)$$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

Applying, $C_1 \rightarrow C_1 + C_2 + C_3$, we get,

$$= \begin{vmatrix} 3x+4 & x & x \\ 3x+4 & x+4 & x \\ 3x+4 & x & x+4 \end{vmatrix}$$

Taking $(3x+4)$ common we get,

$$= (3x+4) \begin{vmatrix} 1 & x & x \\ 1 & x+4 & x \\ 1 & x & x+4 \end{vmatrix}$$

Now by applying, $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get,

$$= (3x+4) \begin{vmatrix} 1 & x & x \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

$$= 16(3x+4)$$

Hence the proof.

$$26. \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

Solution:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

Let

We know that the value of a determinant remains same if we apply the operation $R_i \rightarrow R_i + kR_j$ or $C_i \rightarrow C_i + kC_j$.

Applying $C_2 \rightarrow C_2 - pC_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1+p-p(1) & 1+p+q \\ 2 & 3+2p-p(2) & 4+3p+2q \\ 3 & 6+3p-p(3) & 10+6p+3q \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1+p+q \\ 2 & 3 & 4+3p+2q \\ 3 & 6 & 10+6p+3q \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 - qC_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p+q-q(1) \\ 2 & 3 & 4+3p+2q-q(2) \\ 3 & 6 & 10+6p+3q-q(3) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 4+3p \\ 3 & 6 & 10+6p \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 - pC_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p-p(1) \\ 2 & 3 & 4+3p-p(3) \\ 3 & 6 & 10+6p-p(6) \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p+q-q(1) \\ 2 & 3 & 4+3p+2q-q(2) \\ 3 & 6 & 10+6p+3q-q(3) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 4+3p \\ 3 & 6 & 10+6p \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 - pC_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p-p(1) \\ 2 & 3 & 4+3p-p(3) \\ 3 & 6 & 10+6p-p(6) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1-1 & 1 \\ 2 & 3-2 & 4 \\ 3 & 6-3 & 10 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 3 & 3 & 10 \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 1-1 \\ 2 & 1 & 4-2 \\ 3 & 3 & 10-3 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix}$$

Expanding the determinant along R_1 , we have

$$\Delta = 1[(1)(7) - (3)(2)] - 0 + 0$$

$$\therefore \Delta = 7 - 6 = 1$$

Thus,

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

Hence the proof.

Exercise 6.3 Page No: 6.71

1. Find the area of the triangle with vertices at the points:

- (i) (3, 8), (-4, 2) and (5, -1)**
- (ii) (2, 7), (1, 1) and (10, 8)**
- (iii) (-1, -8), (-2, -3) and (3, 2)**
- (iv) (0, 0), (6, 0) and (4, 3)**

Solution:

(i) Given (3, 8), (-4, 2) and (5, -1) are the vertices of the triangle.

We know that, if vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by:

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, substituting given value in above formula

$$\Delta = \frac{1}{2} \begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & -1 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2} \left[3 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - 8 \begin{vmatrix} -4 & 1 \\ 5 & 1 \end{vmatrix} + 1 \begin{vmatrix} -4 & 2 \\ 5 & -1 \end{vmatrix} \right]$$

$$= \frac{1}{2} [3(3) - 8(-9) + 1(-6)]$$

$$= \frac{1}{2} [9 + 72 - 6]$$

$$= \frac{75}{2} \text{ Square units}$$

Thus area of triangle is $\frac{75}{2}$ square units

(ii) Given (2, 7), (1, 1) and (10, 8) are the vertices of the triangle.

We know that if vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by:

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, substituting given value in above formula

$$\Delta = \frac{1}{2} \begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2} \left[2 \begin{vmatrix} 1 & 1 \\ 10 & 1 \end{vmatrix} - 7 \begin{vmatrix} 1 & 1 \\ 10 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 10 & 8 \end{vmatrix} \right]$$

$$= \frac{1}{2} [2(-7) - 7(-9) + 1(-2)]$$

$$= \frac{1}{2} [-14 + 63 - 2]$$

$$= \frac{47}{2} \text{ Square units}$$

Thus area of triangle is $\frac{47}{2}$ square units

(iii) Given $(-1, -8)$, $(-2, -3)$ and $(3, 2)$ are the vertices of the triangle.

We know that if vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by:

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, substituting given value in above formula

$$\Delta = \frac{1}{2} \begin{vmatrix} -1 & -8 & 1 \\ -2 & -3 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2} \left[-1 \begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix} - 8 \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} + 1 \begin{vmatrix} -2 & -3 \\ 3 & 2 \end{vmatrix} \right]$$

$$= \frac{1}{2} [-1(-5) - 8(-5) + 1(5)]$$

$$= \frac{1}{2} [5 - 40 + 5]$$

$$= \frac{-30}{2} \text{ Square units}$$

$$= \frac{1}{2} \left[-1 \begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix} - 8 \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} + 1 \begin{vmatrix} -2 & -3 \\ 3 & 2 \end{vmatrix} \right]$$

$$= \frac{1}{2} [-1(-5) - 8(-5) + 1(5)]$$

$$= \frac{1}{2} [5 - 40 + 5]$$

$$= \frac{-30}{2} \text{ Square units}$$

As we know area cannot be negative. Therefore, 15 square unit is the area

Thus area of triangle is 15 square units

(iv) Given $(-1, -8)$, $(-2, -3)$ and $(3, 2)$ are the vertices of the triangle.

We know that if vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by:

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, substituting given value in above formula

$$\Delta = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2} \left[0 \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} - 0 \begin{vmatrix} 6 & 1 \\ 4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 6 & 0 \\ 4 & 3 \end{vmatrix} \right]$$

$$= \frac{1}{2} [0 - 0 + 1(18)]$$

$$= \frac{1}{2} [18]$$

= 9 square units

Thus area of triangle is 9 square units

2. Using the determinants show that the following points are collinear:

(i) (5, 5), (-5, 1) and (10, 7)

(ii) (1, -1), (2, 1) and (10, 8)

(iii) (3, -2), (8, 8) and (5, 2)

(iv) (2, 3), (-1, -2) and (5, 8)

Solution:

(i) Given (5, 5), (-5, 1) and (10, 7)

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Now, substituting given value in above formula

$$\Delta = \frac{1}{2} \begin{vmatrix} 5 & 5 & 1 \\ -5 & 1 & 1 \\ 10 & 7 & 1 \end{vmatrix} = 0$$

$$\frac{1}{2} \begin{vmatrix} 5 & 5 & 1 \\ -5 & 1 & 1 \\ 10 & 7 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2} \left[5 \begin{vmatrix} 1 & 1 \\ 7 & 1 \end{vmatrix} - 5 \begin{vmatrix} -5 & 1 \\ 10 & 1 \end{vmatrix} + 1 \begin{vmatrix} -5 & 1 \\ 10 & 7 \end{vmatrix} \right]$$

$$= \frac{1}{2} [5(-6) - 5(-15) + 1(-45)]$$

$$= \frac{1}{2} [-35 + 75 - 45]$$

$$= 0$$

Since, Area of triangle is zero

Hence, points are collinear

(ii) Given (1, -1), (2, 1) and (10, 8)

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by,

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Now, by substituting given value in above formula

$$\Delta = \frac{1}{2} \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 0$$

$$\frac{1}{2} \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2} \left[1 \begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} \right]$$

$$= \frac{1}{2} [1 - 5 + 2 - 4 + 10 - 4]$$

$$= \frac{1}{2} [0]$$

$$= 0$$

Since, Area of triangle is zero.

Hence, points are collinear.

(iii) Given (3, -2), (8, 8) and (5, 2)

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by,

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Now, by substituting given value in above formula

$$\Delta = \frac{1}{2} \begin{vmatrix} 3 & -2 & 1 \\ 8 & 8 & 1 \\ 5 & 2 & 1 \end{vmatrix} = 0$$

$$\frac{1}{2} \begin{vmatrix} 3 & -2 & 1 \\ 8 & 8 & 1 \\ 5 & 2 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2} \left[3 \begin{vmatrix} 8 & 1 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 8 & 1 \\ 5 & 1 \end{vmatrix} + 1 \begin{vmatrix} 8 & 8 \\ 5 & 2 \end{vmatrix} \right]$$

$$= \frac{1}{2} [3(6) - 2(3) + 1(-24)]$$

$$= \frac{1}{2} [0]$$

$$= 0$$

Since, Area of triangle is zero

Hence, points are collinear.

(iv) Given (2, 3), (-1, -2) and (5, 8)

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by,

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Now, by substituting given value in above formula

$$\Delta = \frac{1}{2} \begin{vmatrix} 2 & 3 & 1 \\ -1 & -2 & 1 \\ 5 & 8 & 1 \end{vmatrix} = 0$$

$$\frac{1}{2} \begin{vmatrix} 2 & 3 & 1 \\ -1 & -2 & 1 \\ 5 & 8 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2} \left[2 \begin{vmatrix} -2 & 1 \\ 8 & 1 \end{vmatrix} - 3 \begin{vmatrix} -1 & 1 \\ 5 & 1 \end{vmatrix} + 1 \begin{vmatrix} -1 & -2 \\ 5 & 8 \end{vmatrix} \right]$$

$$= \frac{1}{2} [2(-10) - 3(-1 - 5) + 1(-8 + 10)]$$

$$= \frac{1}{2} [-20 + 18 + 2]$$

$$= 0$$

$$= \frac{1}{2} \left[2 \begin{vmatrix} -2 & 1 \\ 8 & 1 \end{vmatrix} - 3 \begin{vmatrix} -1 & 1 \\ 5 & 1 \end{vmatrix} + 1 \begin{vmatrix} -1 & -2 \\ 5 & 8 \end{vmatrix} \right]$$

$$= \frac{1}{2} [2(-10) - 3(-1 - 5) + 1(-8 + 10)]$$

$$= \frac{1}{2} [-20 + 18 + 2]$$

$$= 0$$

Since, Area of triangle is zero

Hence, points are collinear.

3. If the points (a, 0), (0, b) and (1, 1) are collinear, prove that $a + b = ab$

Solution:

Given (a, 0), (0, b) and (1, 1) are collinear

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we

know that, vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by,

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Thus

$$\frac{1}{2} \begin{vmatrix} a & 0 & 1 \\ 0 & b & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow 0 = \frac{1}{2} \left[a \begin{vmatrix} b & 1 \\ 1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & b \\ 1 & 1 \end{vmatrix} \right]$$

$$\Rightarrow \frac{1}{2} [a(b - 1) - 0(-1) + 1(-b)] = 0$$

\Rightarrow

$$\frac{1}{2} [ab - a - b] = 0$$

$$\Rightarrow a + b = ab$$

Hence Proved

4. Using the determinants prove that the points (a, b) , (a', b') and $(a - a', b - b')$ are collinear if $a b' = a' b$.

Solution:

Given (a, b) , (a', b') and $(a - a', b - b')$ are collinear

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by,

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Thus

$$\frac{1}{2} \begin{vmatrix} a & b & 1 \\ a' & b' & 1 \\ a - a' & b - b' & 1 \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow 0 = \frac{1}{2} \left[a \begin{vmatrix} b' & 1 \\ b - b' & 1 \end{vmatrix} - b \begin{vmatrix} a' & 1 \\ a - a' & 1 \end{vmatrix} + 1 \begin{vmatrix} a' & b' \\ a - a' & b - b' \end{vmatrix} \right]$$

$$\Rightarrow \frac{1}{2} [a(b' - b + b') - b(a' - a + a') + 1(a'b - a'b' - ab' + a'b')] = 0$$

$$\Rightarrow \frac{1}{2} [a'b - ab + ab' - a'b + ab + a'b + a'b - a'b' - ab' + a'b'] = 0$$

$$\Rightarrow ab' - a'b = 0$$

$$\Rightarrow a b' = a' b$$

Hence, the proof.

5. Find the value of λ so that the points (1, -5), (-4, 5) and (λ , 7) are collinear.

Solution:

Given (1, -5), (-4, 5) and (λ , 7) are collinear

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by,

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Now, by substituting given value in above formula

$$\Delta = \frac{1}{2} \begin{vmatrix} 1 & -5 & 1 \\ -4 & 5 & 1 \\ \lambda & 7 & 1 \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow \frac{1}{2} \left[1 \begin{vmatrix} 5 & 1 \\ 7 & 1 \end{vmatrix} + 5 \begin{vmatrix} -4 & 1 \\ \lambda & 1 \end{vmatrix} + 1 \begin{vmatrix} -4 & 5 \\ \lambda & 7 \end{vmatrix} \right] = 0$$

$$\Rightarrow \frac{1}{2} [1(-2) + 5(-4 - \lambda) + 1(-28 - 5\lambda)] = 0$$

$$\Rightarrow \frac{1}{2} [-2 - 20 - 5\lambda - 28 - 5\lambda] = 0$$

$$\Rightarrow -50 - 10\lambda = 0$$

$$\Rightarrow \lambda = -5$$

6. Find the value of x if the area of Δ is 35 square cms with vertices (x, 4), (2, -6) and (5, 4).

Solution:

Given (x, 4), (2, -6) and (5, 4) are the vertices of a triangle.

We have the condition that three points to be collinear, the area of the triangle formed by these points will be zero. Now, we know that, vertices of a triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then the area of the triangle is given by,

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, by substituting given value in above formula

$$\Rightarrow 35 = \left| \frac{1}{2} \begin{vmatrix} x & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix} \right|$$

Removing modulus

$$\Rightarrow \pm 2 \times 35 = \begin{vmatrix} x & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix}$$

Expanding along R_1

$$\Rightarrow \left[x \begin{vmatrix} -6 & 1 \\ 4 & 1 \end{vmatrix} - 4 \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -6 \\ 5 & 4 \end{vmatrix} \right] = \pm 70$$

$$\Rightarrow \pm 2 \times 35 = \begin{vmatrix} x & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix}$$

Expanding along R_1

$$\Rightarrow \left[x \begin{vmatrix} -6 & 1 \\ 4 & 1 \end{vmatrix} - 4 \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -6 \\ 5 & 4 \end{vmatrix} \right] = \pm 70$$

$$\Rightarrow [x(-10) - 4(-3) + 1(8 - 30)] = \pm 70$$

$$\Rightarrow [-10x + 12 + 38] = \pm 70$$

$$\Rightarrow \pm 70 = -10x + 50$$

Taking positive sign, we get

$$\Rightarrow +70 = -10x + 50$$

$$\Rightarrow 10x = -20$$

$$\Rightarrow x = -2$$

Taking -negative sign, we get

$$\Rightarrow -70 = -10x + 50$$

$$\Rightarrow 10x = 120$$

$$\Rightarrow x = 12$$

Thus $x = -2, 12$

Exercise 6.4 Page No: 6.84

Solve the following system of linear equations by Cramer's rule:

1. $x - 2y = 4$

$-3x + 5y = -7$

Solution:

Given $x - 2y = 4$

$-3x + 5y = -7$

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

\vdots

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$x - 2y = 4$$

$$-3x + 5y = -7$$

So by comparing with the theorem, let's find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 1 & -2 \\ -3 & 5 \end{vmatrix}$$

$$\Rightarrow D = \begin{vmatrix} 1 & -2 \\ -3 & 5 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D = 5(1) - (-3)(-2)$$

$$\Rightarrow D = 5 - 6$$

$$\Rightarrow D = -1$$

Again,

$$\Rightarrow D_1 = \begin{vmatrix} 4 & -2 \\ -7 & 5 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = 5(4) - (-7)(-2)$$

$$\Rightarrow D_1 = 20 - 14$$

$$\Rightarrow D_1 = 6$$

And

$$\Rightarrow D_2 = \begin{vmatrix} 1 & 4 \\ -3 & -7 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = 1(-7) - (-3)(4)$$

$$\Rightarrow D_2 = -7 + 12$$

$$\Rightarrow D_2 = 5$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{6}{-1}$$

$$\Rightarrow x = -6$$

And

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{5}{-1}$$

$$\Rightarrow y = -5$$

$$\mathbf{2. \ 2x - y = 1}$$

$$\mathbf{7x - 2y = -7}$$

Solution:

Given $2x - y = 1$ and

$$7x - 2y = -7$$

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

\vdots

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$2x - y = 1$$

$$7x - 2y = -7$$

So by comparing with the theorem, let's find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 2 & -1 \\ 7 & -2 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = 1(-2) - (-7)(-1)$$

$$\Rightarrow D_1 = -2 - 7$$

$$\Rightarrow D_1 = -9$$

And

$$\Rightarrow D_2 = \begin{vmatrix} 2 & 1 \\ 7 & -7 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = 2(-7) - (7)(1)$$

$$\Rightarrow D_2 = -14 - 7$$

$$\Rightarrow D_2 = -21$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{-9}{3}$$

$$\Rightarrow x = -3$$

$$\text{And } \Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{-21}{3}$$

$$\Rightarrow y = -7$$

$$\mathbf{3. \ 2x - y = 17}$$

$$\mathbf{3x + 5y = 6}$$

Solution:

Given $2x - y = 17$ and

$$3x + 5y = 6$$

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

\vdots

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$2x - y = 17$$

$$3x + 5y = 6$$

So by comparing with the theorem, let's find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = 17(5) - (6)(-1)$$

$$\Rightarrow D_1 = 85 + 6$$

$$\Rightarrow D_1 = 91$$

$$\Rightarrow D_2 = \begin{vmatrix} 2 & 17 \\ 3 & 6 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = 2(6) - (17)(3)$$

$$\Rightarrow D_2 = 12 - 51$$

$$\Rightarrow D_2 = -39$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{91}{13}$$

$$\Rightarrow x = 7$$

$$\text{And } \Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{-39}{13}$$

$$\Rightarrow y = -3$$

$$4. \quad 3x + y = 19$$

$$3x - y = 23$$

Solution:

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

\vdots

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$3x + y = 19$$

$$3x - y = 23$$

So by comparing with the theorem, let's find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D = 3(-1) - (3)(1)$$

$$\Rightarrow D = -3 - 3$$

$$\Rightarrow D = -6$$

Again,

$$\Rightarrow D_1 = \begin{vmatrix} 19 & 1 \\ 23 & -1 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = 19(-1) - (23)(1)$$

$$\Rightarrow D_1 = -19 - 23$$

$$\Rightarrow D_1 = -42$$

$$\Rightarrow D_2 = \begin{vmatrix} 3 & 19 \\ 3 & 23 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = 3(23) - (19)(3)$$

$$\Rightarrow D_2 = 69 - 57$$

$$\Rightarrow D_2 = 12$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{-42}{-6}$$

$$\Rightarrow x = 7$$

$$\text{And } \Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{12}{-6}$$

$$\Rightarrow y = -2$$

$$\mathbf{5. \ 2x - y = -2}$$

$$\mathbf{3x + 4y = 3}$$

Solution:

Given $2x - y = -2$ and

$$3x + 4y = 3$$

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$2x - y = -2$$

$$3x + 4y = 3$$

So by comparing with the theorem, let's find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D = 2(4) - (3)(-1)$$

$$\Rightarrow D = 8 + 3$$

$$\Rightarrow D = 11$$

Again,

$$\Rightarrow D_1 = \begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = -2(4) - (3)(-1)$$

$$\Rightarrow D_1 = -8 + 3$$

$$\Rightarrow D_1 = -5$$

$$\Rightarrow D_2 = \begin{vmatrix} 2 & -2 \\ 3 & 3 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = 3(2) - (-2)(3)$$

$$\Rightarrow D_2 = 6 + 6$$

$$\Rightarrow D_2 = 12$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{-5}{11}$$

$$\text{And } \Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{12}{11}$$

$$\mathbf{6. \ 3x + ay = 4}$$

$$\mathbf{2x + ay = 2, \ a \neq 0}$$

Solution:

Given $3x + ay = 4$ and

$$2x + ay = 2, \ a \neq 0$$

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

$$3x + ay = 4$$

$$2x + ay = 2, a \neq 0$$

So by comparing with the theorem, let's find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 3 & a \\ 2 & a \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D = 3(a) - (2)(a)$$

$$\Rightarrow D = 3a - 2a$$

$$\Rightarrow D = a$$

Again,

$$\Rightarrow D_1 = \begin{vmatrix} 4 & a \\ 2 & a \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = 4(a) - (2)(a)$$

$$\Rightarrow D = 4a - 2a$$

$$\Rightarrow D = 2a$$

$$\Rightarrow D_2 = \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = 3(2) - (2)(4)$$

$$\Rightarrow D = 6 - 8$$

$$\Rightarrow D = -2$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{2a}{a}$$

$$\Rightarrow x = 2$$

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{-2}{a}$$

$$\mathbf{7. \ 2x + 3y = 10}$$

$$\mathbf{x + 6y = 4}$$

Solution:

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

\vdots

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$2x + 3y = 10$$

$$x + 6y = 4$$

So by comparing with the theorem, let's find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 2 & 3 \\ 1 & 6 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D = 2(6) - (3)(1)$$

$$\Rightarrow D = 12 - 3$$

$$\Rightarrow D = 9$$

Again,

$$\Rightarrow D_1 = \begin{vmatrix} 10 & 3 \\ 4 & 6 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = 10(6) - (3)(4)$$

$$\Rightarrow D = 60 - 12$$

$$\Rightarrow D = 48$$

$$\Rightarrow D_2 = \begin{vmatrix} 2 & 10 \\ 1 & 4 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = 2(4) - (10)(1)$$

$$\Rightarrow D_2 = 8 - 10$$

$$\Rightarrow D_2 = -2$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{48}{9}$$

$$\Rightarrow x = \frac{16}{3}$$

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{-2}{9}$$

$$\Rightarrow y = \frac{-2}{9}$$

$$\mathbf{8. \ 5x + 7y = -2}$$

$$\mathbf{4x + 6y = -3}$$

Solution:

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

\vdots

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$5x + 7y = -2$$

$$4x + 6y = -3$$

So by comparing with the theorem, let's find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 5 & 7 \\ 4 & 6 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D = 5(6) - (7)(4)$$

$$\Rightarrow D = 30 - 28$$

$$\Rightarrow D = 2$$

Again,

$$\Rightarrow D_1 = \begin{vmatrix} -2 & 7 \\ -3 & 6 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = -2(6) - (7)(-3)$$

$$\Rightarrow D_1 = -12 + 21$$

$$\Rightarrow D_1 = 9$$

$$\Rightarrow D_2 = \begin{vmatrix} 5 & -2 \\ 4 & -3 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = -3(5) - (-2)(4)$$

$$\Rightarrow D_2 = -15 + 8$$

$$\Rightarrow D_2 = -7$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{9}{2}$$

$$\Rightarrow x = \frac{9}{2}$$

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{-7}{2}$$

$$\Rightarrow y = \frac{-7}{2}$$

$$\mathbf{9. \ 9x + 5y = 10}$$

$$\mathbf{3y - 2x = 8}$$

Solution:

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$9x + 5y = 10$$

$$3y - 2x = 8$$

So by comparing with the theorem, let's find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 9 & 5 \\ -2 & 3 \end{vmatrix}$$

$$\Rightarrow D = \begin{vmatrix} 9 & 5 \\ -2 & 3 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D = 3(9) - (5)(-2)$$

$$\Rightarrow D = 27 + 10$$

$$\Rightarrow D = 37$$

Again,

$$\Rightarrow D_1 = \begin{vmatrix} 10 & 5 \\ 8 & 3 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = 10(3) - (8)(5)$$

$$\Rightarrow D_1 = 30 - 40$$

$$\Rightarrow D_1 = -10$$

$$\Rightarrow D_2 = \begin{vmatrix} 9 & 10 \\ -2 & 8 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = 9(8) - (10)(-2)$$

$$\Rightarrow D_2 = 72 + 20$$

$$\Rightarrow D_2 = 92$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{-10}{37}$$

$$\Rightarrow x = \frac{-10}{37}$$

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{92}{37}$$

$$\Rightarrow y = \frac{92}{37}$$

$$\mathbf{10. \ x + 2y = 1}$$

$$\mathbf{3x + y = 4}$$

Solution:

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$x + 2y = 1$$

$$3x + y = 4$$

So by comparing with theorem, now we have to find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D = 1(1) - (3)(2)$$

$$\Rightarrow D = 1 - 6$$

$$\Rightarrow D = -5$$

Again,

$$\Rightarrow D_1 = \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = 1(1) - (2)(4)$$

$$\Rightarrow D_1 = 1 - 8$$

$$\Rightarrow D_1 = -7$$

$$\Rightarrow D_2 = \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = 1(4) - (1)(3)$$

$$\Rightarrow D_2 = 4 - 3$$

$$\Rightarrow D_2 = 1$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{-7}{-5}$$

$$\Rightarrow x = \frac{7}{5}$$

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{1}{-5}$$

$$\Rightarrow y = -\frac{1}{5}$$

Solve the following system of linear equations by Cramer's rule:

$$11. \quad 3x + y + z = 2$$

$$2x - 4y + 3z = -1$$

$$4x + y - 3z = -11$$

Solution:

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$3x + y + z = 2$$

$$2x - 4y + 3z = -1$$

$$4x + y - 3z = -11$$

So by comparing with the theorem, let's find D , D_1 , D_2 and D_3

$$\Rightarrow D = \begin{vmatrix} 3 & 1 & 1 \\ 2 & -4 & 3 \\ 4 & 1 & -3 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D = 3[(-4)(-3) - (3)(1)] - 1[(2)(-3) - 12] + 1[2 - 4(-4)]$$

$$\Rightarrow D = 3[12 - 3] - [-6 - 12] + [2 + 16]$$

$$\Rightarrow D = 27 + 18 + 18$$

$$\Rightarrow D = 63$$

Again,

$$\Rightarrow D_1 = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -4 & 3 \\ -11 & 1 & -3 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = 2[(-4)(-3) - (3)(1)] - 1[(-1)(-3) - (-11)(3)] + 1[(-1) - (-4)(-11)]$$

$$\Rightarrow D_1 = 2[12 - 3] - 1[3 + 33] + 1[-1 - 44]$$

$$\Rightarrow D_1 = 2[9] - 36 - 45$$

$$\Rightarrow D_1 = 18 - 36 - 45$$

$$\Rightarrow D_1 = -63$$

Again

$$\Rightarrow D_2 = \begin{vmatrix} 3 & 2 & 1 \\ 2 & -1 & 3 \\ 4 & -11 & -3 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = 3[3 + 33] - 2[-6 - 12] + 1[-22 + 4]$$

$$\Rightarrow D_2 = 3[36] - 2(-18) - 18$$

$$\Rightarrow D_2 = 126$$

\Rightarrow

$$D_3 = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -4 & -1 \\ 4 & 1 & -11 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_3 = 3[44 + 1] - 1[-22 + 4] + 2[2 + 16]$$

$$\Rightarrow D_3 = 3[45] - 1(-18) + 2(18)$$

$$\Rightarrow D_3 = 135 + 18 + 36$$

$$\Rightarrow D_3 = 189$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{-63}{63}$$

$$\Rightarrow x = -1$$

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{126}{63}$$

$$\Rightarrow y = 2$$

$$\Rightarrow z = \frac{D_3}{D}$$

$$\Rightarrow z = \frac{189}{63}$$

$$\Rightarrow z = 3$$

$$12. \quad x - 4y - z = 11$$

$$2x - 5y + 2z = 39$$

$$-3x + 2y + z = 1$$

Solution:

Given,

$$x - 4y - z = 11$$

$$2x - 5y + 2z = 39$$

$$-3x + 2y + z = 1$$

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$x - 4y - z = 11$$

$$2x - 5y + 2z = 39$$

$$-3x + 2y + z = 1$$

So by comparing with theorem, now we have to find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 1 & -4 & -1 \\ 2 & -5 & 2 \\ -3 & 2 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D = 1[(-5)(1) - (2)(2)] + 4[(2)(1) + 6] - 1[4 + 5(-3)]$$

$$\Rightarrow D = 1[-5 - 4] + 4[8] - [-11]$$

$$\Rightarrow D = -9 + 32 + 11$$

$$\Rightarrow D = 34$$

Again,

$$\Rightarrow D_1 = \begin{vmatrix} 11 & -4 & -1 \\ 39 & -5 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_1 = 11[(-5)(1) - (2)(2)] + 4[(39)(1) - (2)(1)] - 1[2(39) - (-5)(1)]$$

$$\Rightarrow D_1 = 11[-5 - 4] + 4[39 - 2] - 1[78 + 5]$$

$$\Rightarrow D_1 = 11[-9] + 4(37) - 83$$

$$\Rightarrow D_1 = -99 - 148 - 45$$

$$\Rightarrow D_1 = -34$$

Again

$$\Rightarrow D_2 = \begin{vmatrix} 1 & 11 & -1 \\ 2 & 39 & 2 \\ -3 & 1 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_2 = 1[39 - 2] - 11[2 + 6] - 1[2 + 117]$$

$$\Rightarrow D_2 = 1[37] - 11(8) - 119$$

$$\Rightarrow D_2 = -170$$

And,

\Rightarrow

$$D_3 = \begin{vmatrix} 1 & -4 & 11 \\ 2 & -5 & 39 \\ -3 & 2 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1st row

$$\Rightarrow D_3 = 1[-5 - (39)(2)] - (-4)[2 - (39)(-3)] + 11[4 - (-5)(-3)]$$

$$\Rightarrow D_3 = 1[-5 - 78] + 4(2 + 117) + 11(4 - 15)$$

$$\Rightarrow D_3 = -83 + 4(119) + 11(-11)$$

$$\Rightarrow D_3 = 272$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{-34}{34}$$

$$\Rightarrow x = -1$$

Again,

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{-170}{34}$$

$$\Rightarrow y = -5$$

$$\Rightarrow z = \frac{D_3}{D}$$

$$\mathbf{13. \quad 6x + y - 3z = 5}$$

$$\mathbf{x + 3y - 2z = 5}$$

$$\mathbf{2x + y + 4z = 8}$$

Solution:

Given

$$6x + y - 3z = 5$$

$$x + 3y - 2z = 5$$

$$2x + y + 4z = 8$$

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$6x + y - 3z = 5$$

$$x + 3y - 2z = 5$$

$$2x + y + 4z = 8$$

So by comparing with theorem, now we have to find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 6 & 1 & -3 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D = 6[(4)(3) - (1)(-2)] - 1[(4)(1) + 4] - 3[1 - 3(2)]$$

$$\Rightarrow D = 6[12 + 2] - [8] - 3[-5]$$

$$\Rightarrow D = 84 - 8 + 15$$

$$\Rightarrow D = 91$$

Again, Solve D_1 formed by replacing 1st column by B matrices

Here

$$B = \begin{vmatrix} 5 \\ 5 \\ 8 \end{vmatrix}$$

$$\Rightarrow D_1 = \begin{vmatrix} 5 & 1 & -3 \\ 5 & 3 & -2 \\ 8 & 1 & 4 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D_1 = 5[(4)(3) - (-2)(1)] - 1[(5)(4) - (-2)(8)] - 3[(5) - (3)(8)]$$

$$\Rightarrow D_1 = 5[12 + 2] - 1[20 + 16] - 3[5 - 24]$$

$$\Rightarrow D_1 = 5[14] - 36 - 3(-19)$$

$$\Rightarrow D_1 = 70 - 36 + 57$$

$$\Rightarrow D_1 = 91$$

Again, Solve D_2 formed by replacing 1st column by B matrices

Here

$$B = \begin{vmatrix} 5 \\ 5 \\ 8 \end{vmatrix}$$

$$\Rightarrow D_2 = \begin{vmatrix} 6 & 5 & -3 \\ 1 & 5 & -2 \\ 2 & 8 & 4 \end{vmatrix}$$

Solving determinant

$$\Rightarrow D_2 = 6[20 + 16] - 5[4 - 2(-2)] + (-3)[8 - 10]$$

$$\Rightarrow D_2 = 6[36] - 5(8) + (-3)(-2)$$

$$\Rightarrow D_2 = 182$$

And, Solve D_3 formed by replacing 1st column by B matrices

Here

$$B = \begin{vmatrix} 5 \\ 5 \\ 8 \end{vmatrix}$$

$$\Rightarrow D_3 = \begin{vmatrix} 6 & 1 & 5 \\ 1 & 3 & 5 \\ 2 & 1 & 8 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D_3 = 6[24 - 5] - 1[8 - 10] + 5[1 - 6]$$

$$\Rightarrow D_3 = 6[19] - 1(-2) + 5(-5)$$

$$\Rightarrow D_3 = 114 + 2 - 25$$

$$\Rightarrow D_3 = 91$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{91}{91}$$

$$\Rightarrow x = 1$$

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{182}{91}$$

$$\Rightarrow y = 2$$

$$\Rightarrow z = \frac{D_3}{D}$$

$$\Rightarrow z = \frac{91}{91}$$

$$\Rightarrow z = 1$$

$$\mathbf{14. \ x + y = 5}$$

$$\mathbf{y + z = 3}$$

$$\mathbf{x + z = 4}$$

Solution:

$$\text{Given } x + y = 5$$

$$y + z = 3$$

$$x + z = 4$$

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$x + y = 5$$

$$y + z = 3$$

$$x + z = 4$$

So by comparing with theorem, now we have to find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D = 1[1] - 1[-1] + 0[-1]$$

$$\Rightarrow D = 1 + 1 + 0$$

$$\Rightarrow D = 2$$

Again, Solve D_1 formed by replacing 1st column by B matrices

Here

$$B = \begin{vmatrix} 5 \\ 3 \\ 4 \end{vmatrix}$$

$$\Rightarrow D_1 = \begin{vmatrix} 5 & 1 & 0 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D_1 = 5[1] - 1[(3)(1) - (4)(1)] + 0[0 - (4)(1)]$$

$$\Rightarrow D_1 = 5 - 1[3 - 4] + 0[-4]$$

$$\Rightarrow D_1 = 5 - 1[-1] + 0$$

$$\Rightarrow D_1 = 5 + 1 + 0$$

$$\Rightarrow D_1 = 6$$

Again, Solve D_2 formed by replacing 1st column by B matrices

Here

$$B = \begin{vmatrix} 5 \\ 3 \\ 4 \end{vmatrix}$$

$$\Rightarrow D_2 = \begin{vmatrix} 1 & 5 & 0 \\ 0 & 3 & 1 \\ 1 & 4 & 1 \end{vmatrix}$$

Solving determinant

$$\Rightarrow D_2 = 1[3 - 4] - 5[-1] + 0[0 - 3]$$

$$\Rightarrow D_2 = 1[-1] + 5 + 0$$

$$\Rightarrow D_2 = 4$$

And, Solve D_3 formed by replacing 1st column by B matrices

Here

$$B = \begin{vmatrix} 5 \\ 3 \\ 4 \end{vmatrix}$$

$$\Rightarrow D_3 = \begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D_3 = 1[4 - 0] - 1[0 - 3] + 5[0 - 1]$$

$$\Rightarrow D_3 = 1[4] - 1(-3) + 5(-1)$$

$$\Rightarrow D_3 = 4 + 3 - 5$$

$$\Rightarrow D_3 = 2$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{6}{2}$$

$$\Rightarrow x = 3$$

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{4}{2}$$

$$\Rightarrow y = 2$$

$$\Rightarrow z = \frac{D_3}{D}$$

$$\Rightarrow z = \frac{2}{2}$$

$$\Rightarrow z = 1$$

$$\mathbf{15. \ 2y - 3z = 0}$$

$$\mathbf{x + 3y = -4}$$

$$\mathbf{3x + 4y = 3}$$

Solution:

Given

$$2y - 3z = 0$$

$$x + 3y = -4$$

$$3x + 4y = 3$$

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$2y - 3z = 0$$

$$x + 3y = -4$$

$$3x + 4y = 3$$

So by comparing with theorem, now we have to find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 0 & 2 & -3 \\ 1 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D = 0[0] - 2[(0)(1) - 0] - 3[1(4) - 3(3)]$$

$$\Rightarrow D = 0 - 0 - 3[4 - 9]$$

$$\Rightarrow D = 0 - 0 + 15$$

$$\Rightarrow D = 15$$

Again, Solve D_1 formed by replacing 1st column by B matrices

Here

$$B = \begin{vmatrix} 0 \\ -4 \\ 3 \end{vmatrix}$$
$$\Rightarrow D_1 = \begin{vmatrix} 0 & 2 & -3 \\ -4 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D_1 = 0[0] - 2[(0)(-4) - 0] - 3[4(-4) - 3(3)]$$

$$\Rightarrow D_1 = 0 - 0 - 3[-16 - 9]$$

$$\Rightarrow D_1 = 0 - 0 - 3(-25)$$

$$\Rightarrow D_1 = 0 - 0 + 75$$

$$\Rightarrow D_1 = 75$$

Again, Solve D_2 formed by replacing 2nd column by B matrices

Here

$$B = \begin{vmatrix} 0 \\ -4 \\ 3 \end{vmatrix}$$
$$\Rightarrow D_2 = \begin{vmatrix} 0 & 0 & -3 \\ 1 & -4 & 0 \\ 3 & 3 & 0 \end{vmatrix}$$

Solving determinant

$$\Rightarrow D_2 = 0[0] - 0[(0)(1) - 0] - 3[1(3) - 3(-4)]$$

$$\Rightarrow D_2 = 0 - 0 + (-3)(3 + 12)$$

$$\Rightarrow D_2 = -45$$

And, Solve D_3 formed by replacing 3rd column by B matrices

Here

$$B = \begin{vmatrix} 0 \\ -4 \\ 3 \end{vmatrix}$$
$$\Rightarrow D_3 = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & -4 \\ 3 & 4 & 3 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D_3 = 0[9 - (-4) 4] - 2[(3) (1) - (-4) (3)] + 0[1 (4) - 3 (3)]$$

$$\Rightarrow D_3 = 0[25] - 2(3 + 12) + 0(4 - 9)$$

$$\Rightarrow D_3 = 0 - 30 + 0$$

$$\Rightarrow D_3 = -30$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{75}{15}$$

$$\Rightarrow x = 5$$

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{-45}{15}$$

$$\Rightarrow y = -3$$

$$\Rightarrow z = \frac{D_3}{D}$$

$$\Rightarrow z = \frac{-30}{15}$$

$$\Rightarrow z = -2$$

$$\mathbf{16. \ 5x - 7y + z = 11}$$

$$\mathbf{6x - 8y - z = 15}$$

$$\mathbf{3x + 2y - 6z = 7}$$

Solution:

Given

$$5x - 7y + z = 11$$

$$6x - 8y - z = 15$$

$$3x + 2y - 6z = 7$$

Let there be a system of n simultaneous linear equations and with n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$$

Then,

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D} \text{ Provided that } D \neq 0$$

Now, here we have

$$5x - 7y + z = 11$$

$$6x - 8y - z = 15$$

$$3x + 2y - 6z = 7$$

So by comparing with theorem, now we have to find D , D_1 and D_2

$$\Rightarrow D = \begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D = 5[(-8)(-6) - (-1)(2)] - 7[(-6)(6) - 3(-1)] + 1[2(6) - 3(-8)]$$

$$\Rightarrow D = 5[48 + 2] - 7[-36 + 3] + 1[12 + 24]$$

$$\Rightarrow D = 250 - 231 + 36$$

$$\Rightarrow D = 55$$

Again, Solve D_1 formed by replacing 1st column by B matrices

Here

$$B = \begin{vmatrix} 11 \\ 15 \\ 7 \end{vmatrix}$$
$$\Rightarrow D_1 = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D_1 = 11[(-8)(-6) - (2)(-1)] - (-7)[(15)(-6) - (-1)(7)] + 1[(15)(2) - (7)(-8)]$$

$$\Rightarrow D_1 = 11[48 + 2] + 7[-90 + 7] + 1[30 + 56]$$

$$\Rightarrow D_1 = 11[50] + 7[-83] + 86$$

$$\Rightarrow D_1 = 550 - 581 + 86$$

$$\Rightarrow D_1 = 55$$

Again, Solve D_2 formed by replacing 2nd column by B matrices

Here

$$B = \begin{vmatrix} 11 \\ 15 \\ 7 \end{vmatrix}$$
$$\Rightarrow D_2 = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D_2 = 5[(15)(-6) - (7)(-1)] - 11[(6)(-6) - (-1)(3)] + 1[(6)(7) - (15)(3)]$$

$$\Rightarrow D_2 = 5[-90 + 7] - 11[-36 + 3] + 1[42 - 45]$$

$$\Rightarrow D_2 = 5[-83] - 11(-33) - 3$$

$$\Rightarrow D_2 = -415 + 363 - 3$$

$$\Rightarrow D_2 = -55$$

And, Solve D_3 formed by replacing 3rd column by B matrices

Here

$$B = \begin{vmatrix} 11 \\ 15 \\ 7 \end{vmatrix}$$

$$\Rightarrow D_3 = \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix}$$

Solving determinant, expanding along 1st Row

$$\Rightarrow D_3 = 5[(-8)(7) - (15)(2)] - (-7)[(6)(7) - (15)(3)] + 11[(6)(2) - (-8)(3)]$$

$$\Rightarrow D_3 = 5[-56 - 30] - (-7)[42 - 45] + 11[12 + 24]$$

$$\Rightarrow D_3 = 5[-86] + 7[-3] + 11[36]$$

$$\Rightarrow D_3 = -430 - 21 + 396$$

$$\Rightarrow D_3 = -55$$

Thus by Cramer's Rule, we have

$$\Rightarrow x = \frac{D_1}{D}$$

$$\Rightarrow x = \frac{55}{55}$$

$$\Rightarrow x = 1$$

$$\Rightarrow y = \frac{D_2}{D}$$

$$\Rightarrow y = \frac{-55}{55}$$

$$\Rightarrow y = -1$$

$$\Rightarrow z = \frac{D_3}{D}$$

$$\Rightarrow z = \frac{-55}{55}$$

$$\Rightarrow z = -1$$

Exercise 6.5 Page No: 6.89

Solve each of the following system of homogeneous linear equations:

1. $x + y - 2z = 0$

$2x + y - 3z = 0$

$$5x + 4y - 9z = 0$$

Solution:

$$\text{Given } x + y - 2z = 0$$

$$2x + y - 3z = 0$$

$$5x + 4y - 9z = 0$$

Any system of equation can be written in matrix form as $AX = B$

Now finding the Determinant of these set of equations,

$$D = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$$

$$|A| = 1 \begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1 \begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$$

$$= 1(1 \times (-9) - 4 \times (-3)) - 1(2 \times (-9) - 5 \times (-3)) - 2(4 \times 2 - 5 \times 1)$$

$$= 1(-9 + 12) - 1(-18 + 15) - 2(8 - 5)$$

$$= 1 \times 3 - 1 \times (-3) - 2 \times 3$$

$$= 3 + 3 - 6$$

$$= 0$$

Since $D = 0$, so the system of equation has infinite solution.

Now let $z = k$

$$\Rightarrow x + y = 2k$$

$$\text{And } 2x + y = 3k$$

Now using the Cramer's rule

$$x = \frac{D_1}{D}$$

$$x = \frac{\begin{vmatrix} 2k & 1 \\ 3k & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}$$

$$x = \frac{-k}{-1}$$

$$x = k$$

Similarly,

$$y = \frac{D_2}{D}$$

$$y = \frac{\begin{vmatrix} 1 & 2k \\ 2 & 3k \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}$$

$$y = \frac{-k}{-1}$$

$$y = k$$

Hence, $x = y = z = k$.

$$\mathbf{2. \quad 2x + 3y + 4z = 0}$$

$$\mathbf{x + y + z = 0}$$

$$\mathbf{2x + 5y - 2z = 0}$$

Solution:

Given

$$2x + 3y + 4z = 0$$

$$x + y + z = 0$$

$$2x + 5y - 2z = 0$$

Any system of equation can be written in matrix form as $AX = B$

Now finding the Determinant of these set of equations,

$$D = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 5 & -2 \end{vmatrix}$$

$$|A| = 2 \begin{vmatrix} 1 & 1 \\ 5 & -2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} + 4 \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}$$

$$= 2(1 \times (-2) - 1 \times 5) - 3(1 \times (-2) - 2 \times 1) + 4(1 \times 5 - 2 \times 1)$$

$$= 2(-2 - 5) - 3(-2 - 2) + 4(5 - 2)$$

$$= 1 \times (-7) - 3 \times (-4) + 4 \times 3$$

$$= -7 + 12 + 12$$

$$= 17$$

Since $D \neq 0$, so the system of equation has infinite solution.

Therefore the system of equation has only solution as $x = y = z = 0$.