Access answers to RD Sharma Solutions for Class 11 Maths Chapter 3 – Functions

EXERCISE 3.1 PAGE NO: 3.7

1. Define a function as a set of ordered pairs.

Solution:

Let A and B be two non-empty sets. A relation from A to B, i.e., a subset of A×B, is called a function (or a mapping) from A to B, if

(i) for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$

(ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

2. Define a function as a correspondence between two sets.

Solution:

Let A and B be two non-empty sets. Then a function 'f' from set A to B is a rule or method or correspondence which associates elements of set A to elements of set B such that:

(i) all elements of set A are associated to elements in set B.

(ii) an element of set A is associated to a unique element in set B.

3. What is the fundamental difference between a relation and a function? Is every relation a function?

Solution:

Let 'f' be a function and R be a relation defined from set X to set Y.

The domain of the relation R might be a subset of the set X, but the domain of the function f must be equal to X. This is because each element of the domain of a function must have an element associated with it, whereas this is not necessary for a relation.

In relation, one element of X might be associated with one or more elements of Y, while it must be associated with only one element of Y in a function.

Thus, not every relation is a function. However, every function is necessarily a relation.

4. Let A = {–2, –1, 0, 1, 2} and f: A → Z be a function defined by $f(x) = x^2 – 2x – 3$. Find:
(i) range of f i.e. f (A)
(ii) pre-images of 6, –3 and 5

Solution:

Given:

A = {–2, –1, 0, 1, 2}

f : A → Z such that $f(x) = x^2 – 2x – 3$

(i) Range of f i.e. f (A)

A is the domain of the function f. Hence, range is the set of elements f(x) for all $x \in A$.

Substituting x = –2 in f(x), we get

$f(–2) = (–2)^2 – 2(–2) – 3$

= 4 + 4 – 3

= 5

Substituting x = –1 in f(x), we get

$f(–1) = (–1)^2 – 2(–1) – 3$

= 1 + 2 – 3

= 0

Substituting x = 0 in f(x), we get

$f(0) = (0)^2 – 2(0) – 3$

$= 0 – 0 – 3$

$= – 3$

Substituting $x = 1$ in $f(x)$, we get

$f(1) = 1^2 – 2(1) – 3$

$= 1 – 2 – 3$

$= – 4$

Substituting $x = 2$ in $f(x)$, we get

$f(2) = 2^2 – 2(2) – 3$

$= 4 – 4 – 3$

$= –3$

Thus, the range of f is $\{-4, -3, 0, 5\}$.

(ii) pre-images of 6, –3 and 5

Let x be the pre-image of 6 $\Rightarrow f(x) = 6$

$x^2 – 2x – 3 = 6$

$x^2 – 2x – 9 = 0$

$x = [-(-2) \pm \sqrt{((-2)^2 – 4(1)(-9))}] / 2(1)$

$= [2 \pm \sqrt{(4+36)}] / 2$

$= [2 \pm \sqrt{40}] / 2$

$= 1 \pm \sqrt{10}$

However, $1 \pm \sqrt{10} \notin A$

Thus, there exists no pre-image of 6.

Now, let x be the pre-image of –3 $\Rightarrow f(x) = –3$

$x^2 – 2x – 3 = –3$

$x^2 – 2x = 0$

$x(x – 2) = 0$

$x = 0$ or 2

Clearly, both 0 and 2 are elements of A.

Thus, 0 and 2 are the pre-images of –3.

Now, let x be the pre-image of 5 $\Rightarrow f(x) = 5$

$x^2 – 2x – 3 = 5$

$x^2 – 2x – 8= 0$

$x^2 – 4x + 2x – 8= 0$

$x(x – 4) + 2(x – 4) = 0$

$(x + 2)(x – 4) = 0$

$x = –2$ or 4

However, $4 \notin A$ but $–2 \in A$

Thus, –2 is the pre-images of 5.

$\therefore \varnothing, \{0, 2\}, -2$ are the pre-images of 6, -3, 5

5. If a function f: R → R be defined by

$$f(x) = \begin{cases} 3x - 2, x < 0 \\ 1, x = 0 \\ 4x + 1, x > 0 \end{cases}$$

Find: f (1), f (–1), f (0), f (2).

Solution:

Given:

Let us find f (1), f (–1), f (0) and f (2).

When x > 0, f (x) = 4x + 1

Substituting x = 1 in the above equation, we get

f (1) = 4(1) + 1

= 4 + 1

= 5

When x < 0, f(x) = 3x – 2

Substituting x = –1 in the above equation, we get

f (–1) = 3(–1) – 2

= –3 – 2

= –5

When x = 0, f(x) = 1

Substituting x = 0 in the above equation, we get

f (0) = 1

When x > 0, f(x) = 4x + 1

Substituting x = 2 in the above equation, we get

f (2) = 4(2) + 1

= 8 + 1

= 9

∴ f (1) = 5, f (–1) = –5, f (0) = 1 and f (2) = 9.

6. A function f: R → R is defined by f(x) = x^2. Determine
(i) range of f
(ii) {x: f(x) = 4}
(iii) {y: f(y) = –1}

Solution:

Given:

f : R → R and f(x) = x^2.

(i) range of f

Domain of f = R (set of real numbers)

We know that the square of a real number is always positive or equal to zero.

∴ range of f = $R^+ \cup \{0\}$

(ii) {x: f(x) = 4}

Given:

f(x) = 4

we know, $x^2 = 4$

$x^2 - 4 = 0$

$(x - 2)(x + 2) = 0$

∴ $x = \pm 2$

∴ $\{x: f(x) = 4\} = \{-2, 2\}$

(iii) $\{y: f(y) = -1\}$

Given:

$f(y) = -1$

$y^2 = -1$

However, the domain of f is R, and for every real number y, the value of y^2 is non-negative.

Hence, there exists no real y for which $y^2 = -1$.

∴$\{y: f(y) = -1\} = \emptyset$

7. Let f: $R^+ \rightarrow R$, where R^+ is the set of all positive real numbers, be such that $f(x) = \log_e x$. Determine
(i) the image set of the domain of f
(ii) $\{x: f(x) = -2\}$
(iii) whether $f(xy) = f(x) + f(y)$ holds.

Solution:

Given f: $R^+ \rightarrow R$ and $f(x) = \log_e x$.

(i) the image set of the domain of f

Domain of f = R^+ (set of positive real numbers)

We know the value of logarithm to the base e (natural logarithm) can take all possible real values.

∴ The image set of f = R

(ii) $\{x: f(x) = -2\}$

Given $f(x) = -2$

$\log_e x = -2$

∴ $x = e^{-2}$ [since, $\log_b a = c \Rightarrow a = b^c$]

∴ $\{x: f(x) = -2\} = \{e^{-2}\}$

(iii) Whether $f(xy) = f(x) + f(y)$ holds.

We have $f(x) = \log_e x \Rightarrow f(y) = \log_e y$

Now, let us consider $f(xy)$

$F(xy) = \log_e (xy)$

$f(xy) = \log_e (x \times y)$ [since, $\log_b (a \times c) = \log_b a + \log_b c$]

$f(xy) = \log_e x + \log_e y$

$f(xy) = f(x) + f(y)$

∴ the equation $f(xy) = f(x) + f(y)$ holds.

8. Write the following relations as sets of ordered pairs and find which of them are functions:
(i) $\{(x, y): y = 3x, x \in \{1, 2, 3\}, y \in \{3, 6, 9, 12\}\}$
(ii) $\{(x, y): y > x + 1, x = 1, 2$ and $y = 2, 4, 6\}$
(iii) $\{(x, y): x + y = 3, x, y \in \{0, 1, 2, 3\}\}$

Solution:

(i) $\{(x, y): y = 3x, x \in \{1, 2, 3\}, y \in \{3, 6, 9, 12\}\}$

When $x = 1$, $y = 3(1) = 3$

When $x = 2$, $y = 3(2) = 6$

When x = 3, y = 3(3) = 9

∴ R = {(1, 3), (2, 6), (3, 9)}

Hence, the given relation R is a function.

(ii) {(x, y): y > x + 1, x = 1, 2 and y = 2, 4, 6}

When x = 1, y > 1 + 1 or y > 2 ⇒ y = {4, 6}

When x = 2, y > 2 + 1 or y > 3 ⇒ y = {4, 6}

∴ R = {(1, 4), (1, 6), (2, 4), (2, 6)}

Hence, the given relation R is not a function.

(iii) {(x, y): x + y = 3, x, y ∈ {0, 1, 2, 3}}

When x = 0, 0 + y = 3 ⇒ y = 3

When x = 1, 1 + y = 3 ⇒ y = 2

When x = 2, 2 + y = 3 ⇒ y = 1

When x = 3, 3 + y = 3 ⇒ y = 0

∴ R = {(0, 3), (1, 2), (2, 1), (3, 0)}

Hence, the given relation R is a function.

9. Let f: R → R and g: C → C be two functions defined as f(x) = x^2 and g(x) = x^2. Are they equal functions?

Solution:

Given:

f: R → R ∈ f(x) = x^2 and g : R → R ∈ g(x) = x^2

f is defined from R to R, the domain of f = R.

g is defined from C to C, the domain of g = C.

Two functions are equal only when the domain and codomain of both the functions are equal.

In this case, the domain of f ≠ domain of g.

∴ f and g are not equal functions.

EXERCISE 3.2 PAGE NO: 3.11

1. If f (x) = x^2 – 3x + 4, then find the values of x satisfying the equation f (x) = f (2x + 1).

Solution:

Given:

f(x) = x^2 – 3x + 4.

Let us find x satisfying f (x) = f (2x + 1).

We have,

f (2x + 1) = $(2x + 1)^2$ – 3(2x + 1) + 4

= $(2x)^2$ + 2(2x) (1) + 1^2 – 6x – 3 + 4

= $4x^2$ + 4x + 1 – 6x + 1

= $4x^2$ – 2x + 2

Now, f (x) = f (2x + 1)

x^2 – 3x + 4 = $4x^2$ – 2x + 2

$4x^2 – 2x + 2 – x^2 + 3x – 4 = 0$

$3x^2 + x – 2 = 0$

$3x^2 + 3x – 2x – 2 = 0$

$3x(x + 1) – 2(x + 1) = 0$

$(x + 1)(3x – 2) = 0$

$x + 1 = 0$ or $3x – 2 = 0$

$x = –1$ or $3x = 2$

$x = –1$ or $2/3$

∴ The values of x are –1 and 2/3.

2. If $f (x) = (x – a)^2(x – b)^2$, find $f (a + b)$.

Solution:

Given:

$F (x) = (x – a)^2(x – b)^2$

Let us find f (a + b).

We have,

$f (a + b) = (a + b – a)^2(a + b – b)^2$

$f (a + b) = (b)^2 (a)^2$

∴ $f (a + b) = a^2b^2$

3. If y = f (x) = (ax – b) / (bx – a), show that x = f (y).

Solution:

Given:

$y = f (x) = (ax – b) / (bx – a) \Rightarrow f (y) = (ay – b) / (by – a)$

Let us prove that x = f (y).

We have,

$y = (ax – b) / (bx – a)$

By cross-multiplying,

$y(bx – a) = ax – b$

$bxy – ay = ax – b$

$bxy – ax = ay – b$

$x(by – a) = ay – b$

$x = (ay – b) / (by – a) = f (y)$

∴ x = f (y)

Hence proved.

4. If f (x) = 1 / (1 – x), show that f [f {f (x)}] = x.

Solution:

Given:

$f (x) = 1 / (1 – x)$

Let us prove that f [f {f (x)}] = x.

Firstly, let us solve for f {f (x)}.

$f \{f (x)\} = f \{1/(1 – x)\}$

= 1 / 1 – (1/(1 – x))

= 1 / [(1 – x – 1)/(1 – x)]

= 1 / (-x/(1 – x))

= (1 – x) / -x

= (x – 1) / x

∴ f {f (x)} = (x – 1) / x

Now, we shall solve for f [f {f (x)}]

f [f {f (x)}] = f [(x-1)/x]
= 1 / [1 – (x-1)/x]

= 1 / [(x – (x-1))/x]

= 1 / [(x – x + 1)/x]

= 1 / (1/x)

∴ f [f {f (x)}] = x

Hence proved.

5. If f (x) = (x + 1) / (x – 1), show that f [f (x)] = x.

Solution:

Given:

f (x) = (x + 1) / (x – 1)

Let us prove that f [f (x)] = x.

f [f (x)] = f [(x+1)/(x-1)]

= [(x+1)/(x-1) + 1] / [(x+1)/(x-1) – 1]

= [[(x+1) + (x-1)]/(x-1)] / [[(x+1) – (x-1)]/(x-1)]

= [(x+1) + (x-1)] / [(x+1) – (x-1)]

= (x+1+x-1)/(x+1-x+1)

= 2x/2

= x

∴ f [f (x)] = x

Hence proved.

6. If

$$f(x) = \begin{cases} x^2, \text{when } x < 0 \\ x, \text{when } 0 \leq x < 1 \\ \frac{1}{x}, \text{when } x \geq 1 \end{cases}$$

Find:

(i) f (1/2)

(ii) f (-2)

(iii) f (1)

(iv) f (√3)

(v) f (√-3)

Solution:

(i) f (1/2)

When, $0 \le x \le 1$, f(x) = x

∴ f (1/2) = ½

(ii) f (-2)

When, x < 0, f(x) = x^2

f (–2) = $(–2)^2$

= 4

∴ f (–2) = 4

(iii) f (1)

When, $x \ge 1$, f (x) = 1/x

f (1) = 1/1

∴ f(1) = 1

(iv) f (√3)

We have √3 = 1.732 > 1

When, $x \ge 1$, f (x) = 1/x

∴ f (√3) = 1/√3

(v) f (√-3)

We know √-3 is not a real number and the function f(x) is defined only when $x \in R$.

∴ f (√-3) does not exist.

EXERCISE 3.3 PAGE NO: 3.18

1. Find the domain of each of the following real valued functions of real variable:

(i) f (x) = 1/x

(ii) f (x) = 1/(x-7)

(iii) f (x) = (3x-2)/(x+1)

(iv) f (x) = (2x+1)/(x^2-9)

(v) f (x) = (x^2+2x+1)/(x^2-8x+12)

Solution:

(i) f (x) = 1/x

We know, f (x) is defined for all real values of x, except for the case when x = 0.

∴ Domain of f = R – {0}

(ii) f (x) = 1/(x-7)

We know, f (x) is defined for all real values of x, except for the case when x – 7 = 0 or x = 7.

∴ Domain of f = R – {7}

(iii) f (x) = (3x-2)/(x+1)

We know, f(x) is defined for all real values of x, except for the case when x + 1 = 0 or x = –1.

∴ Domain of f = R – {–1}

(iv) f (x) = (2x+1)/(x^2-9)

We know, f (x) is defined for all real values of x, except for the case when $x^2 – 9 = 0$.

$x^2 – 9 = 0$

$x^2 – 3^2 = 0$

$(x + 3)(x – 3) = 0$

$x + 3 = 0$ or $x – 3 = 0$

$x = \pm 3$

∴ Domain of f = R – {–3, 3}

(v) $f(x) = (x^2+2x+1)/(x^2-8x+12)$

We know, f(x) is defined for all real values of x, except for the case when $x^2 – 8x + 12 = 0$.

$x^2 – 8x + 12 = 0$

$x^2 – 2x – 6x + 12 = 0$

$x(x – 2) – 6(x – 2) = 0$

$(x – 2)(x – 6) = 0$

$x – 2 = 0$ or $x – 6 = 0$

$x = 2$ or 6

∴ Domain of f = R – {2, 6}

2. Find the domain of each of the following real valued functions of real variable:

(i) $f(x) = \sqrt{(x-2)}$

(ii) $f(x) = 1/(\sqrt{(x^2-1)})$

(iii) $f(x) = \sqrt{(9-x^2)}$

(iv) $f(x) = \sqrt{(x-2)/(3-x)}$

Solution:

(i) $f(x) = \sqrt{(x-2)}$

We know the square of a real number is never negative.

f (x) takes real values only when $x – 2 \geq 0$

$x \geq 2$

$\therefore x \in [2, \infty)$

∴ Domain (f) = $[2, \infty)$

(ii) $f(x) = 1/(\sqrt{(x^2-1)})$

We know the square of a real number is never negative.

f (x) takes real values only when $x^2 – 1 \geq 0$

$x^2 – 1^2 \geq 0$

$(x + 1)(x – 1) \geq 0$

$x \leq –1$ or $x \geq 1$

$\therefore x \in (–\infty, –1] \cup [1, \infty)$

In addition, f (x) is also undefined when $x^2 – 1 = 0$ because denominator will be zero and the result will be indeterminate.

$x^2 – 1 = 0 \Rightarrow x = \pm 1$

So, $x \in (–\infty, –1] \cup [1, \infty) – \{–1, 1\}$

$x \in (–\infty, –1) \cup (1, \infty)$

∴ Domain (f) = $(–\infty, –1) \cup (1, \infty)$

(iii) $f(x) = \sqrt{(9-x^2)}$

We know the square of a real number is never negative.

f (x) takes real values only when $9 – x^2 \geq 0$

$9 \geq x^2$

$x^2 \leq 9$

$x^2 - 9 \leq 0$

$x^2 - 3^2 \leq 0$

$(x + 3)(x - 3) \leq 0$

$x \geq -3$ and $x \leq 3$

$x \in [-3, 3]$

∴ Domain (f) = $[-3, 3]$

(iv) f (x) = √(x-2)/(3-x)

We know the square root of a real number is never negative.

f (x) takes real values only when x – 2 and 3 – x are both positive and negative.

(a) Both x – 2 and 3 – x are positive

$x - 2 \geq 0$

$x \geq 2$

$3 - x \geq 0$

$x \leq 3$

Hence, $x \geq 2$ and $x \leq 3$

∴ $x \in [2, 3]$

(b) Both x – 2 and 3 – x are negative

$x - 2 \leq 0$

$x \leq 2$

$3 - x \leq 0$

$x \geq 3$

Hence, $x \leq 2$ and $x \geq 3$

However, the intersection of these sets is null set. Thus, this case is not possible.

Hence, $x \in [2, 3] - \{3\}$

$x \in [2, 3]$

∴ Domain (f) = $[2, 3]$

3. Find the domain and range of each of the following real valued functions:

(i) f (x) = (ax+b)/(bx-a)

(ii) f (x) = (ax-b)/(cx-d)

(iii) f (x) = √(x-1)

(iv) f (x) = √(x-3)

(v) f (x) = (x-2)/(2-x)

(vi) f (x) = |x-1|

(vii) f (x) = -|x|

(viii) f (x) = √(9-x²)

Solution:

(i) f (x) = (ax+b)/(bx-a)

f(x) is defined for all real values of x, except for the case when bx – a = 0 or x = a/b.

Domain (f) = R – (a/b)

Let f (x) = y

$(ax+b)/(bx-a) = y$

$ax + b = y(bx – a)$

$ax + b = bxy – ay$

$ax – bxy = –ay – b$

$x(a – by) = –(ay + b)$

∴ $x = – (ay+b)/(a-by)$

When $a – by = 0$ or $y = a/b$

Hence, f(x) cannot take the value a/b.

∴ Range (f) = R – (a/b)

(ii) $f (x) = (ax-b)/(cx-d)$

f(x) is defined for all real values of x, except for the case when $cx – d = 0$ or $x = d/c$. Domain (f) = R – (d/c)

Let f (x) = y

$(ax-b)/(cx-d) = y$

$ax – b = y(cx – d)$

$ax – b = cxy – dy$

$ax – cxy = b – dy$

$x(a – cy) = b – dy$

∴ $x = (b-dy)/(a-cy)$

When $a – cy = 0$ or $y = a/c$,

Hence, f(x) cannot take the value a/c.

∴ Range (f) = R – (a/c)

(iii) $f (x) = \sqrt{(x-1)}$

We know the square of a real number is never negative.

f(x) takes real values only when $x – 1 \geq 0$

$x \geq 1$

∴ $x \in [1, \infty)$

Thus, domain (f) = $[1, \infty)$

When $x \geq 1$, we have $x – 1 \geq 0$

Hence, $\sqrt{(x-1)} \geq 0 \Rightarrow f (x) \geq 0$

$f(x) \in [0, \infty)$

∴ Range (f) = $[0, \infty)$

(iv) $f (x) = \sqrt{(x-3)}$

We know the square of a real number is never negative.

f (x) takes real values only when $x – 3 \geq 0$

$x \geq 3$

∴ $x \in [3, \infty)$

Domain (f) = $[3, \infty)$

When $x \geq 3$, we have $x – 3 \geq 0$

Hence, $\sqrt{(x-3)} \geq 0 \Rightarrow f (x) \geq 0$

$f(x) \in [0, \infty)$

∴ Range (f) = $[0, \infty)$

(v) $f (x) = (x-2)/(2-x)$

f(x) is defined for all real values of x, except for the case when 2 – x = 0 or x = 2.

Domain (f) = R – {2}

We have, f (x) = (x-2)/(2-x)

f (x) = -(2-x)/(2-x)

= –1

When x ≠ 2, f(x) = –1

∴ Range (f) = {–1}

(vi) f (x) = |x-1|

$$we\ know\ |x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$$

Now we have,

$$|x-1| = \begin{cases} -(x-1), x-1 < 0 \\ x-1, x-1 \geq 0 \end{cases}$$

$$\therefore f(x) = |x-1| = \begin{cases} 1-x, x < 1 \\ x-1, x \geq 1 \end{cases}$$

Hence, f(x) is defined for all real numbers x.

Domain (f) = R

When, x < 1, we have x – 1 < 0 or 1 – x > 0.

|x – 1| > 0 ⇒ f(x) > 0

When, x ≥ 1, we have x – 1 ≥ 0.

|x – 1| ≥ 0 ⇒ f(x) ≥ 0

∴ f(x) ≥ 0 or f(x) ∈ [0, ∞)

Range (f) = [0, ∞)

(vii) f (x) = -|x|

$$we\ know\ |x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$$

Now we have,

$$-|x| = \begin{cases} -(-x), x < 0 \\ -x, x \geq 0 \end{cases}$$

$$\therefore f(x) = -|x| = \begin{cases} x, x < 0 \\ -x, x \geq 0 \end{cases}$$

Hence, f(x) is defined for all real numbers x.

Domain (f) = R

When, x < 0, we have –|x| < 0

f (x) < 0

When, x ≥ 0, we have –x ≤ 0.

–|x| ≤ 0 ⇒ f (x) ≤ 0

$\therefore$ f (x) ≤ 0 or f (x) ∈ (–∞, 0]

Range (f) = (–∞, 0]

(viii) f (x) = $\sqrt{(9-x^2)}$

We know the square of a real number is never negative.

f(x) takes real values only when $9 – x^2 ≥ 0$

$9 ≥ x^2$

$x^2 ≤ 9$

$x^2 – 9 ≤ 0$

$x^2 – 3^2 ≤ 0$

(x + 3)(x – 3) ≤ 0

x ≥ –3 and x ≤ 3

$\therefore$ x ∈ [–3, 3]

Domain (f) = [–3, 3]

When, x ∈ [–3, 3], we have $0 ≤ 9 – x^2 ≤ 9$

$0 ≤ \sqrt{(9-x^2)} ≤ 3 \Rightarrow 0 ≤ f(x) ≤ 3$

$\therefore$ f(x) ∈ [0, 3]

Range (f) = [0, 3]

EXERCISE 3.4 PAGE NO: 3.38

1. Find f + g, f – g, cf (c ∈ R, c ≠ 0), fg, 1/f and f/g in each of the following:
(i) f (x) = x^3 + 1 and g (x) = x + 1

(ii) f (x) = $\sqrt{(x-1)}$ and g (x) = $\sqrt{(x+1)}$

Solution:

(i) f (x) = x^3 + 1 and g(x) = x + 1

We have f(x): R → R and g(x): R → R

(a) f + g

We know, (f + g) (x) = f(x) + g(x)

(f + g) (x) = x^3 + 1 + x + 1

= x^3 + x + 2

So, (f + g) (x): R → R

$\therefore$ f + g: R → R is given by (f + g) (x) = x^3 + x + 2

(b) f – g

We know, (f – g) (x) = f(x) – g(x)

(f – g) (x) = x^3 + 1 – (x + 1)

= x^3 + 1 – x – 1

= x^3 – x

So, (f – g) (x): R → R

$\therefore$ f – g: R → R is given by (f – g) (x) = x^3 – x

(c) cf (c ∈ R, c ≠ 0)

We know, $(cf)(x) = c \times f(x)$

$(cf)(x) = c(x^3 + 1)$

$= cx^3 + c$

So, $(cf)(x) : R \to R$

$\therefore$ cf: $R \to R$ is given by $(cf)(x) = cx^3 + c$

(d) fg

We know, $(fg)(x) = f(x)\, g(x)$

$(fg)(x) = (x^3 + 1)(x + 1)$

$= (x + 1)(x^2 - x + 1)(x + 1)$

$= (x + 1)^2(x^2 - x + 1)$

So, $(fg)(x): R \to R$

$\therefore$ fg: $R \to R$ is given by $(fg)(x) = (x + 1)^2(x^2 - x + 1)$

(e) 1/f

We know, $(1/f)(x) = 1/f(x)$

$1/f(x) = 1 / (x^3 + 1)$

Observe that $1/f(x)$ is undefined when $f(x) = 0$ or when $x = -1$.

So, $1/f: R - \{-1\} \to R$ is given by $1/f(x) = 1 / (x^3 + 1)$

(f) f/g

We know, $(f/g)(x) = f(x)/g(x)$

$(f/g)(x) = (x^3 + 1) / (x + 1)$

Observe that $(x^3 + 1) / (x + 1)$ is undefined when $g(x) = 0$ or when $x = -1$.

Using $x^3 + 1 = (x + 1)(x^2 - x + 1)$, we have

$(f/g)(x) = [(x+1)(x^2 - x+1)/(x+1)]$

$= x^2 - x + 1$

$\therefore$ $f/g: R - \{-1\} \to R$ is given by $(f/g)(x) = x^2 - x + 1$

(ii) $f(x) = \sqrt{(x-1)}$ and $g(x) = \sqrt{(x+1)}$

We have $f(x): [1, \infty) \to R^+$ and $g(x): [-1, \infty) \to R^+$ as real square root is defined only for non-negative numbers.

(a) f + g

We know, $(f + g)(x) = f(x) + g(x)$

$(f+g)(x) = \sqrt{(x-1)} + \sqrt{(x+1)}$

Domain of $(f + g)$ = Domain of f $\cap$ Domain of g

Domain of $(f + g) = [1, \infty) \cap [-1, \infty)$

Domain of $(f + g) = [1, \infty)$

$\therefore$ $f + g: [1, \infty) \to R$ is given by $(f+g)(x) = \sqrt{(x-1)} + \sqrt{(x+1)}$

(b) f – g

We know, $(f - g)(x) = f(x) - g(x)$

$(f-g)(x) = \sqrt{(x-1)} - \sqrt{(x+1)}$

Domain of $(f - g)$ = Domain of f $\cap$ Domain of g

Domain of $(f - g) = [1, \infty) \cap [-1, \infty)$

Domain of (f – g) = $[1, \infty)$

$\therefore$ f – g: $[1, \infty) \rightarrow R$ is given by (f-g) (x) = $\sqrt{(x-1)} - \sqrt{(x+1)}$

(c) cf ($c \in R$, $c \neq 0$)

We know, (cf) (x) = c × f(x)

(cf) (x) = $c\sqrt{(x-1)}$

Domain of (cf) = Domain of f

Domain of (cf) = $[1, \infty)$

$\therefore$ cf: $[1, \infty) \rightarrow R$ is given by (cf) (x) = $c\sqrt{(x-1)}$

(d) fg

We know, (fg) (x) = f(x) g(x)

(fg) (x) = $\sqrt{(x-1)}\ \sqrt{(x+1)}$

= $\sqrt{(x^2 -1)}$

Domain of (fg) = Domain of f ∩ Domain of g

Domain of (fg) = $[1, \infty) \cap [-1, \infty)$

Domain of (fg) = $[1, \infty)$

$\therefore$ fg: $[1, \infty) \rightarrow R$ is given by (fg) (x) = $\sqrt{(x^2 -1)}$

(e) 1/f

We know, (1/f) (x) = 1/f(x)

(1/f) (x) = $1/\sqrt{(x-1)}$

Domain of (1/f) = Domain of f

Domain of (1/f) = $[1, \infty)$

Observe that $1/\sqrt{(x-1)}$ is also undefined when x – 1 = 0 or x = 1.

$\therefore$ 1/f: $(1, \infty) \rightarrow R$ is given by (1/f) (x) = $1/\sqrt{(x-1)}$

(f) f/g

We know, (f/g) (x) = f(x)/g(x)

(f/g) (x) = $\sqrt{(x-1)}/\sqrt{(x+1)}$

(f/g) (x) = $\sqrt{[(x-1)/(x+1)]}$

Domain of (f/g) = Domain of f ∩ Domain of g

Domain of (f/g) = $[1, \infty) \cap [-1, \infty)$

Domain of (f/g) = $[1, \infty)$

$\therefore$ f/g: $[1, \infty) \rightarrow R$ is given by (f/g) (x) = $\sqrt{[(x-1)/(x+1)]}$

2. Let f(x) = 2x + 5 and g(x) = x^2 + x. Describe

(i) f + g
(ii) f – g
(iii) fg
(iv) f/g
Find the domain in each case.

Solution:

Given:

f(x) = 2x + 5 and g(x) = x^2 + x

Both f(x) and g(x) are defined for all $x \in R$.

So, domain of f = domain of g = R

(i) f + g

We know, (f + g)(x) = f(x) + g(x)

$(f + g)(x) = 2x + 5 + x^2 + x$

$= x^2 + 3x + 5$

(f + g)(x) Is defined for all real numbers x.

∴ The domain of (f + g) is R

(ii) f – g

We know, (f – g)(x) = f(x) – g(x)

$(f – g)(x) = 2x + 5 – (x^2 + x)$

$= 2x + 5 – x^2 – x$

$= 5 + x – x^2$

(f – g)(x) is defined for all real numbers x.

∴ The domain of (f – g) is R

(iii) fg

We know, (fg)(x) = f(x)g(x)

$(fg)(x) = (2x + 5)(x^2 + x)$

$= 2x(x^2 + x) + 5(x^2 + x)$

$= 2x^3 + 2x^2 + 5x^2 + 5x$

$= 2x^3 + 7x^2 + 5x$

(fg)(x) is defined for all real numbers x.

∴ The domain of fg is R

(iv) f/g

We know, (f/g) (x) = f(x)/g(x)

$(f/g) (x) = (2x+5)/(x^2+x)$

(f/g) (x) is defined for all real values of x, except for the case when $x^2 + x = 0$.

$x^2 + x = 0$

$x(x + 1) = 0$

x = 0 or x + 1 = 0

x = 0 or –1

When x = 0 or –1, (f/g) (x) will be undefined as the division result will be indeterminate.

∴ The domain of f/g = R – {–1, 0}

3. If f(x) be defined on [–2, 2] and is given by

$$f(x) = \begin{cases} -1, -2 \leq x \leq 0 \\ x - 1, 0 < x \leq 2 \end{cases}$$ **and g(x) = f(|x|) + |f(x)|. Find g(x).**

Solution:

Given:

$f(x) = \begin{cases} -1, -2 \le x \le 0 \\ x-1, 0 < x \le 2 \end{cases}$ and

g(x) = f(|x|) + |f(x)|

Now we have,

$f(|x|) = \begin{cases} -1, -2 \le |x| \le 0 \\ |x|-1, 0 < |x| \le 2 \end{cases}$

However, |x| ≥ 0 ⇒ f(|x|) = |x| – 1 when 0 < |x| ≤ 2

We also have,

$|f(x)| = \begin{cases} |-1|, -2 \le x \le 0 \\ |x-1|, 0 < x \le 2 \end{cases}$

$= \begin{cases} 1, -2 \le x \le 0 \\ |x-1|, 0 < x \le 2 \end{cases}$

We also know,

$|x-1| = \begin{cases} -(x-1), x-1 < 0 \\ x-1, x-1 \ge 0 \end{cases}$

$= \begin{cases} -(x-1), x < 1 \\ x-1, x \ge 1 \end{cases}$

Here, we shall only the range between [0, 2].

$|x-1| = \begin{cases} -(x-1), 0 < x < 1 \\ x-1, 1 \le x \le 2 \end{cases}$

Substituting this value of |x – 1| in |f(x)|, we get

$|f(x)| = \begin{cases} 1, -2 \le x \le 0 \\ -(x-1), 0 < x < 1 \\ x-1, 1 \le x \le 2 \end{cases}$

$= \begin{cases} 1, -2 \le x \le 0 \\ 1-x, 0 < x < 1 \\ x-1, 1 \le x \le 2 \end{cases}$

Now, we need to find g(x)

g(x) = f(|x|) + |f(x)|

= |x| – 1 when 0 < |x| ≤ 2 + $\begin{cases} 1, -2 \le x \le 0 \\ 1-x, 0 < x < 1 \\ x-1, 1 \le x \le 2 \end{cases}$

$$g(x) = \begin{cases} -x-1, -2 \le x \le 0 \\ x-1, 0 < x < 1 \\ x-1, 1 \le x \le 2 \end{cases} + \begin{cases} 1, -2 \le x \le 0 \\ 1-x, 0 < x < 1 \\ x-1, 1 \le x \le 2 \end{cases}$$

$$= \begin{cases} -x-1+1, -2 \le x \le 0 \\ x-1+1-x, 0 < x < 1 \\ x-1+x-1, 1 \le x \le 2 \end{cases}$$

$$= \begin{cases} -x, -2 \le x \le 0 \\ 0, 0 < x < 1 \\ 2(x-1), 1 \le x \le 2 \end{cases}$$

$$\therefore g(x) = f(|x|) + |f(x)|$$

$$= \begin{cases} -x, -2 \le x \le 0 \\ 0, 0 < x < 1 \\ 2(x-1), 1 \le x \le 2 \end{cases}$$

4. Let f, g be two real functions defined by f(x) = √(x+1) and g(x) = √(9-x²). Then, describe each of the following functions.
(i) f + g
(ii) g – f
(iii) fg
(iv) f/g
(v) g/f
(vi) 2f – √5g
(vii) f² + 7f
(viii) 5/g

Solution:

Given:

$f(x) = \sqrt{(x+1)}$ and $g(x) = \sqrt{(9-x^2)}$

We know the square of a real number is never negative.

So, f(x) takes real values only when $x + 1 \ge 0$

$x \ge -1$, $x \in [-1, \infty)$

Domain of f = $[-1, \infty)$

Similarly, g(x) takes real values only when $9 - x^2 \ge 0$

$9 \ge x^2$

$x^2 \le 9$

$x^2 - 9 \le 0$

$x^2 - 3^2 \le 0$

$(x + 3)(x - 3) \le 0$

$x \ge -3$ and $x \le 3$

$\therefore x \in [-3, 3]$

Domain of g = $[-3, 3]$

(i) f + g

We know, $(f + g)(x) = f(x) + g(x)$

$(f + g)(x) = \sqrt{(x+1)} + \sqrt{(9-x^2)}$

Domain of f + g = Domain of f ∩ Domain of g

= [–1, ∞) ∩ [–3, 3]

= [–1, 3]

∴ f + g: [–1, 3] → R is given by $(f + g)(x) = f(x) + g(x) = \sqrt{(x+1)} + \sqrt{(9-x^2)}$

(ii) g – f

We know, $(g – f)(x) = g(x) – f(x)$

$(g – f)(x) = \sqrt{(9-x^2)} – \sqrt{(x+1)}$

Domain of g – f = Domain of g ∩ Domain of f

= [–3, 3] ∩ [–1, ∞)

= [–1, 3]

∴ g – f: [–1, 3] → R is given by $(g – f)(x) = g(x) – f(x) = \sqrt{(9-x^2)} – \sqrt{(x+1)}$

(iii) fg

We know, $(fg)(x) = f(x)g(x)$

$(fg)(x) = \sqrt{(x+1)}\sqrt{(9-x^2)}$

$= \sqrt{[(x+1)(9-x^2)]}$

$= \sqrt{[x(9-x^2) + (9-x^2)]}$

$= \sqrt{(9x-x^3+9-x^2)}$

$= \sqrt{(9+9x-x^2-x^3)}$

Domain of fg = Domain of f ∩ Domain of g

= [–1, ∞) ∩ [–3, 3]

= [–1, 3]

∴ fg: [–1, 3] → R is given by $(fg)(x) = f(x)g(x) = \sqrt{(x+1)}\sqrt{(9-x^2)} = \sqrt{(9+9x-x^2-x^3)}$

(iv) f/g

We know, $(f/g)(x) = f(x)/g(x)$

$(f/g)(x) = \sqrt{(x+1)} / \sqrt{(9-x^2)}$

$= \sqrt{[(x+1) / (9-x^2)]}$

Domain of f/g = Domain of f ∩ Domain of g

= [–1, ∞) ∩ [–3, 3]

= [–1, 3]

However, $(f/g)(x)$ is defined for all real values of x ∈ [–1, 3], except for the case when $9 – x^2 = 0$ or $x = \pm 3$

When $x = \pm 3$, $(f/g)(x)$ will be undefined as the division result will be indeterminate.

Domain of f/g = [–1, 3] – {–3, 3}

Domain of f/g = [–1, 3)

∴ f/g: [–1, 3) → R is given by $(f/g)(x) = f(x)/g(x) = \sqrt{(x+1)} / \sqrt{(9-x^2)}$

(v) g/f

We know, $(g/f)(x) = g(x)/f(x)$

$(g/f)(x) = \sqrt{(9-x^2)} / \sqrt{(x+1)}$

$= \sqrt{[(9-x^2) / (x+1)]}$

Domain of g/f = Domain of f ∩ Domain of g

= [–1, ∞) ∩ [–3, 3]

= [–1, 3]

However, (g/f) (x) is defined for all real values of x ∈ [–1, 3], except for the case when x + 1 = 0 or x = –1

When x = –1, (g/f) (x) will be undefined as the division result will be indeterminate.

Domain of g/f = [–1, 3] – {–1}

Domain of g/f = (–1, 3]

∴ g/f: (–1, 3] → R is given by (g/f) (x) = g(x)/f(x) = $\sqrt{(9-x^2)}$ / $\sqrt{(x+1)}$

(vi) 2f – √5g

We know, (2f – √5g) (x) = 2f(x) – √5g(x)

(2f – √5g) (x) = 2f (x) – √5g (x)

= 2√(x+1) – √5√($9-x^2$)

= 2√(x+1) – √($45- 5x^2$)

Domain of 2f – √5g = Domain of f ∩ Domain of g

= [–1, ∞) ∩ [–3, 3]

= [–1, 3]

∴ 2f – √5g: [–1, 3] → R is given by (2f – √5g) (x) = 2f (x) – √5g (x) = 2√(x+1) – √($45- 5x^2$)

(vii) f^2 + 7f

We know, (f^2 + 7f) (x) = f^2(x) + (7f)(x)

(f^2 + 7f) (x) = f(x) f(x) + 7f(x)

= √(x+1) √(x+1) + 7√(x+1)

= x + 1 + 7√(x+1)

Domain of f^2 + 7f is same as domain of f.

Domain of f^2 + 7f = [–1, ∞)

∴ f^2 + 7f: [–1, ∞) → R is given by (f^2 + 7f) (x) = f(x) f(x) + 7f(x) = x + 1 + 7√(x+1)

(viii) 5/g

We know, (5/g) (x) = 5/g(x)

(5/g) (x) = 5/√($9-x^2$)

Domain of 5/g = Domain of g = [–3, 3]

However, (5/g) (x) is defined for all real values of x ∈ [–3, 3], except for the case when 9 – x^2 = 0 or x = ± 3

When x = ±3, (5/g) (x) will be undefined as the division result will be indeterminate.

Domain of 5/g = [–3, 3] – {–3, 3}

= (–3, 3)

∴ 5/g: (–3, 3) → R is given by (5/g) (x) = 5/g(x) = 5/√($9-x^2$)

5. If f(x) = $\log_e$(1 – x) and g(x) = [x], then determine each of the following functions:
(i) f + g
(ii) fg

(iii) f/g
(iv) g/f

Also, find (f + g) (–1), (fg) (0), (f/g) (1/2) and (g/f) (1/2).

Solution:

Given:

$f(x) = \log_e(1 - x)$ and $g(x) = [x]$

We know, $f(x)$ takes real values only when $1 - x > 0$

$1 > x$

$x < 1$, $\therefore x \in (-\infty, 1)$

Domain of $f = (-\infty, 1)$

Similarly, $g(x)$ is defined for all real numbers x.

Domain of $g = [x], x \in R$

$= R$

(i) f + g

We know, $(f + g)(x) = f(x) + g(x)$

$(f + g)(x) = \log_e(1 - x) + [x]$

Domain of $f + g$ = Domain of $f \cap$ Domain of g

Domain of $f + g = (-\infty, 1) \cap R$

$= (-\infty, 1)$

$\therefore f + g: (-\infty, 1) \to R$ is given by $(f + g)(x) = \log_e(1 - x) + [x]$

(ii) fg

We know, $(fg)(x) = f(x)\, g(x)$

$(fg)(x) = \log_e(1 - x) \times [x]$

$= [x] \log_e(1 - x)$

Domain of fg = Domain of $f \cap$ Domain of g

$= (-\infty, 1) \cap R$

$= (-\infty, 1)$

$\therefore fg: (-\infty, 1) \to R$ is given by $(fg)(x) = [x] \log_e(1 - x)$

(iii) f/g

We know, $(f/g)(x) = f(x)/g(x)$

$(f/g)(x) = \log_e(1 - x) / [x]$

Domain of f/g = Domain of $f \cap$ Domain of g

$= (-\infty, 1) \cap R$

$= (-\infty, 1)$

However, $(f/g)(x)$ is defined for all real values of $x \in (-\infty, 1)$, except for the case when $[x] = 0$.

We have, $[x] = 0$ when $0 \le x < 1$ or $x \in [0, 1)$

When $0 \le x < 1$, $(f/g)(x)$ will be undefined as the division result will be indeterminate.

Domain of $f/g = (-\infty, 1) - [0, 1)$

$= (-\infty, 0)$

$\therefore f/g: (-\infty, 0) \to R$ is given by $(f/g)(x) = \log_e(1 - x) / [x]$

(iv) g/f

We know, $(g/f)(x) = g(x)/f(x)$

$(g/f)(x) = [x] / \log_e(1 - x)$

However, (g/f) (x) is defined for all real values of $x \in (-\infty, 1)$, except for the case when $\log_e(1 - x) = 0$.

$\log_e(1 - x) = 0 \Rightarrow 1 - x = 1$ or $x = 0$

When x = 0, (g/f) (x) will be undefined as the division result will be indeterminate.

Domain of $g/f = (-\infty, 1) - \{0\}$

$= (-\infty, 0) \cup (0, 1)$

$\therefore$ g/f: $(-\infty, 0) \cup (0, 1) \rightarrow R$ is given by $(g/f)(x) = [x] / \log_e(1 - x)$

(a) We need to find (f + g) (–1).

We have, $(f + g)(x) = \log_e(1 - x) + [x]$, $x \in (-\infty, 1)$

Substituting x = –1 in the above equation, we get

$(f + g)(-1) = \log_e(1 - (-1)) + [-1]$

$= \log_e(1 + 1) + (-1)$

$= \log_e 2 - 1$

$\therefore (f + g)(-1) = \log_e 2 - 1$

(b) We need to find (fg) (0).

We have, $(fg)(x) = [x] \log_e(1 - x)$, $x \in (-\infty, 1)$

Substituting x = 0 in the above equation, we get

$(fg)(0) = [0] \log_e(1 - 0)$

$= 0 \times \log_e 1$

$\therefore (fg)(0) = 0$

(c) We need to find (f/g) (1/2)

We have, $(f/g)(x) = \log_e(1 - x) / [x]$, $x \in (-\infty, 0)$

However, 1/2 is not in the domain of f/g.

$\therefore$ (f/g) (1/2) does not exist.

(d) We need to find (g/f) (1/2)

We have, $(g/f)(x) = [x] / \log_e(1 - x)$, $x \in (-\infty, 0) \cup (0, \infty)$

Substituting x=1/2 in the above equation, we get

$(g/f)(1/2) = [x] / \log_e(1 - x)$

$= (1/2)/ \log_e(1 - 1/2)$

$= 0.5/ \log_e(1/2)$

$= 0 / \log_e(1/2)$

$= 0$

$\therefore (g/f)(1/2) = 0$