Or

[NCERT Solutions for Class 12-science Maths Chapter 9 - Differential Equations]

Chapter 9 - Differential Equations Exercise Ex. 9.1

Solution 1

$$\frac{d^4y}{dx^4}+\sin(y''')=0$$
$$\Rightarrow y''''+\sin(y''')=0$$

The highest order derivative present in the differential equation is y''''. Therefore, its order is four.

The given differential equation is not a polynomial equation in its derivatives. Hence, its degree is not defined.

Solution 2

The given differential equation is:

$$y'+5y=0$$

The highest order derivative present in the differential equation is y'. Therefore, its order is one.

It is a polynomial equation in y'. The highest power raised to y' is 1. Hence, its degree is one.

Solution 3

$$\left(\frac{ds}{dt}\right)^4+3\frac{d^2s}{dt^2}=0$$

The highest order derivative present in the given differential equation is $\frac{d^2s}{dt^2}$. Therefore, its order is two.

It is a polynomial equation in $\frac{d^2s}{dt^2}$ and $\frac{ds}{dt}$. The power raised to $\frac{d^2s}{dt^2}$ is 1.

Hence, its degree is one.

Solution 4

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is 2.

The given differential equation is not a polynomial equation in its derivatives. Hence, its degree is not defined.

Solution 5

$$\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$$

$$\Rightarrow \frac{d^2y}{dx^2} - \cos 3x - \sin 3x = 0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is two.

It is a polynomial equation in $\frac{d^2y}{dx^2}$ and the power raised to $\frac{d^2y}{dx^2}$ is 1.

Hence, its degree is one.

Solution 6

$$(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$$

The highest order derivative present in the differential equation is y'''. Therefore, its order is three.

The given differential equation is a polynomial equation in y''', y'', and y'.

The highest power raised to y''' is 2. Hence, its degree is 2.

Solution 7

$y''' + 2y'' + y' = 0$

The highest order derivative present in the differential equation is y'''. Therefore, its order is three.

It is a polynomial equation in y''', y'' and y'. The highest power raised to y''' is 1. Hence, its degree is 1.

Solution 8

$y' + y = e^x$

$\Rightarrow y' + y - e^x = 0$

The highest order derivative present in the differential equation is y'. Therefore, its order is one.

The given differential equation is a polynomial equation in y' and the highest power raised to y' is one. Hence, its degree is one.

Solution 9

$y'' + (y')^2 + 2y = 0$

The highest order derivative present in the differential equation is y''. Therefore, its order is two.

The given differential equation is a polynomial equation in y'' and y' and the highest power raised to y'' is one.

Hence, its degree is one.

Solution 10

$y'' + 2y' + \sin y = 0$

The highest order derivative present in the differential equation is y''. Therefore, its order is two.

This is a polynomial equation in y'' and y' and the highest power raised to y'' is one. Hence, its degree is one.

Solution 11

$$\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0$$

The given differential equation is not a polynomial equation in its derivatives. Therefore, its degree is not defined.

Hence, the correct answer is D.

Solution 12

$$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is two.

Hence, the correct answer is A.

Chapter 9 - Differential Equations Exercise Ex. 9.2

Solution 1

$$y=e^x+1$$

Differentiating both sides of this equation with respect to x, we get:

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^x+1\right)$$
$$\Rightarrow y'=e^x \qquad \text{...(1)}$$

Now, differentiating equation (1) with respect to x, we get:

$$\frac{d}{dx}(y')=\frac{d}{dx}\left(e^x\right)$$
$$\Rightarrow y''=e^x$$

Substituting the values of y' and y'' in the given differential equation, we get the L.H.S. as:

$$y''-y'=e^x-e^x=0=\text{R.H.S.}$$

Thus, the given function is the solution of the corresponding differential equation.

Solution 2

$y = x^2 + 2x + C$

Differentiating both sides of this equation with respect to x, we get:

$$y' = \frac{d}{dx}\left(x^2 + 2x + C\right)$$
$$\Rightarrow y' = 2x + 2$$

Substituting the value of y' in the given differential equation, we get:

L.H.S. $= y' - 2x - 2 = 2x + 2 - 2x - 2 = 0 =$ R.H.S.

Hence, the given function is the solution of the corresponding differential equation.

Solution 3

$y = \cos x + C$

Differentiating both sides of this equation with respect to x, we get:

$$y' = \frac{d}{dx}\left(\cos x + C\right)$$
$$\Rightarrow y' = -\sin x$$

Substituting the value of y' in the given differential equation, we get:

L.H.S. $= y' + \sin x = -\sin x + \sin x = 0 =$ R.H.S.

Hence, the given function is the solution of the corresponding differential equation.

Solution 4

$$y=\sqrt{1+x^2}$$

Differentiating both sides of the equation with respect to x, we get:

$$y'=\frac{d}{dx}\left(\sqrt{1+x^2}\right)$$
$$y'=\frac{1}{2\sqrt{1+x^2}}\cdot\frac{d}{dx}\left(1+x^2\right)$$
$$y'=\frac{2x}{2\sqrt{1+x^2}}$$
$$y'=\frac{x}{\sqrt{1+x^2}}$$
$$\Rightarrow y'=\frac{x}{1+x^2}\times\sqrt{1+x^2}$$
$$\Rightarrow y'=\frac{x}{1+x^2}\cdot y$$
$$\Rightarrow y'=\frac{xy}{1+x^2}$$

$\therefore$ L.H.S. = R.H.S.

Hence, the given function is the solution of the corresponding differential equation.

Solution 5

$$y=\text{A}x$$

Differentiating both sides with respect to x, we get:

$$y'=\frac{d}{dx}(\text{A}x)$$
$$\Rightarrow y'=\text{A}$$

Substituting the value of y' in the given differential equation, we get:

L.H.S. $= xy' = x\cdot\text{A} = \text{A}x = y =$ R.H.S.

Hence, the given function is the solution of the corresponding differential equation.

Solution 6

$y = x\sin x$

Differentiating both sides of this equation with respect to x, we get:

$$y' = \frac{d}{dx}(x\sin x)$$
$$\Rightarrow y' = \sin x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sin x)$$
$$\Rightarrow y' = \sin x + x\cos x$$

Substituting the value of y' in the given differential equation, we get:

$$\begin{aligned} \text{L.H.S.} = xy' &= x(\sin x + x\cos x) \\ &= x\sin x + x^2 \cos x \\ &= y + x^2 \cdot \sqrt{1-\sin^2 x} \\ &= y + x^2 \sqrt{1-\left(\frac{y}{x}\right)^2} \\ &= y + x\sqrt{x^2 - y^2} \\ &= \text{R.H.S.} \end{aligned}$$

Hence, the given function is the solution of the corresponding differential equation.

Solution 7

$xy = \log y + \text{C}$

Differentiating both sides of this equation with respect to x, we get:

$$\frac{d}{dx}(xy) = \frac{d}{dx}(\log y)$$
$$\Rightarrow y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$
$$\Rightarrow y + xy' = \frac{1}{y}y'$$
$$\Rightarrow y^2 + xy\,y' = y'$$
$$\Rightarrow (xy-1)\,y' = -y^2$$
$$\Rightarrow y' = \frac{y^2}{1-xy}$$

$\therefore$ L.H.S. = R.H.S.

Hence, the given function is the solution of the corresponding differential equation.

Solution 8

$$y - \cos y = x \qquad \ldots(1)$$

Differentiating both sides of the equation with respect to x, we get:

$$\frac{dy}{dx} - \frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$
$$\Rightarrow y' + \sin y \cdot y' = 1$$
$$\Rightarrow y'(1 + \sin y) = 1$$
$$\Rightarrow y' = \frac{1}{1 + \sin y}$$

Substituting the value of y' in equation (1), we get:

$$\text{L.H.S.} = (y \sin y + \cos y + x)\, y'$$
$$= (y \sin y + \cos y + y - \cos y) \times \frac{1}{1 + \sin y}$$
$$= y(1 + \sin y) \cdot \frac{1}{1 + \sin y}$$
$$= y$$
$$= \text{R.H.S.}$$

Hence, the given function is the solution of the corresponding differential equation.

Solution 9

$x + y = \tan^{-1} y$

Differentiating both sides of this equation with respect to x, we get:

$$\frac{d}{dx}(x+y) = \frac{d}{dx}\left(\tan^{-1} y\right)$$
$$\Rightarrow 1 + y' = \left[\frac{1}{1+y^2}\right] y'$$
$$\Rightarrow y'\left[\frac{1}{1+y^2} - 1\right] = 1$$
$$\Rightarrow y'\left[\frac{1-\left(1+y^2\right)}{1+y^2}\right] = 1$$
$$\Rightarrow y'\left[\frac{-y^2}{1+y^2}\right] = 1$$
$$\Rightarrow y' = \frac{-\left(1+y^2\right)}{y^2}$$

Substituting the value of y' in the given differential equation, we get:

$$\begin{aligned}\text{L.H.S.} = y^2 y' + y^2 + 1 &= y^2\left[\frac{-\left(1+y^2\right)}{y^2}\right] + y^2 + 1\\ &= -1 - y^2 + y^2 + 1\\ &= 0\\ &= \text{R.H.S.}\end{aligned}$$

Hence, the given function is the solution of the corresponding differential equation.

Solution 10

$y=\sqrt{a^2-x^2}$

Differentiating both sides of this equation with respect to x, we get:

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{a^2-x^2}\right)$$
$$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{a^2-x^2}}\cdot\frac{d}{dx}\left(a^2-x^2\right)$$
$$=\frac{1}{2\sqrt{a^2-x^2}}(-2x)$$
$$=\frac{-x}{\sqrt{a^2-x^2}}$$

Substituting the value of $\frac{dy}{dx}$ in the given differential equation, we get:

$$\text{L.H.S.}=x+y\frac{dy}{dx}=x+\sqrt{a^2-x^2}\times\frac{-x}{\sqrt{a^2-x^2}}$$
$$=x-x$$
$$=0$$
$$=\text{R.H.S.}$$

Hence, the given function is the solution of the corresponding differential equation.

Solution 11

We know that the number of constants in the general solution of a differential equation of order n is equal to its order.

Therefore, the number of constants in the general equation of fourth order differential equation is four.

Hence, the correct answer is D.

Solution 12

In a particular solution of a differential equation, there are no arbitrary constants.

Hence, the correct answer is D.

Chapter 9 - Differential Equations Exercise Ex. 9.3

Solution 1

$$\frac{x}{a}+\frac{y}{b}=1$$

Differentiating both sides of the given equation with respect to x, we get:

$$\frac{1}{a}+\frac{1}{b}\frac{dy}{dx}=0$$
$$\Rightarrow \frac{1}{a}+\frac{1}{b}y'=0$$

Again, differentiating both sides with respect to x, we get:

$$0+\frac{1}{b}y''=0$$
$$\Rightarrow \frac{1}{b}y''=0$$
$$\Rightarrow y''=0$$

Hence, the required differential equation of the given curve is $y''=0$.

Solution 2

$$y^2=a\left(b^2-x^2\right)$$

Differentiating both sides with respect to x, we get:

$$2y\frac{dy}{dx}=a(-2x)$$
$$\Rightarrow 2yy'=-2ax$$
$$\Rightarrow yy'=-ax \qquad ...(1)$$

Again, differentiating both sides with respect to x, we get:

$$y'\cdot y'+yy''=-a$$
$$\Rightarrow \left(y'\right)^2+yy''=-a \qquad ...(2)$$

Dividing equation (2) by equation (1), we get:

$$\frac{\left(y'\right)^2+yy''}{yy'}=\frac{-a}{-ax}$$
$$\Rightarrow xyy''+x\left(y'\right)^2-yy''=0$$

This is the required differential equation of the given curve.

Solution 3

$y = ae^{3x} + be^{-2x} \qquad ...(1)$

Differentiating both sides with respect to x, we get:

$y' = 3ae^{3x} - 2be^{-2x} \qquad ...(2)$

Again, differentiating both sides with respect to x, we get:

$y'' = 9ae^{3x} + 4be^{-2x} \qquad ...(3)$

Multiplying equation (1) with 2 and then adding it to equation (2), we get:

$$\left(2ae^{3x} + 2be^{-2x}\right) + \left(3ae^{3x} - 2bc^{-2x}\right) = 2y + y'$$
$$\Rightarrow 5ae^{3x} = 2y + y'$$
$$\Rightarrow ae^{3x} = \frac{2y + y'}{5}$$

Now, multiplying equation (1) with 3 and subtracting equation (2) from it, we get:

$$\left(3ae^{3x} + 3be^{-2x}\right) - \left(3ae^{3x} - 2be^{-2x}\right) = 3y - y'$$
$$\Rightarrow 5be^{-2x} = 3y - y'$$
$$\Rightarrow be^{-2x} = \frac{3y - y'}{5}$$

Substituting the values of ae^{3x} and be^{-2x} in equation (3), we get:

$$y'' = 9 \cdot \frac{(2y + y')}{5} + 4\frac{(3y - y')}{5}$$
$$\Rightarrow y'' = \frac{18y + 9y'}{5} + \frac{12y - 4y'}{5}$$
$$\Rightarrow y'' = \frac{30y + 5y'}{5}$$
$$\Rightarrow y'' = 6y + y'$$
$$\Rightarrow y'' - y' - 6y = 0$$

This is the required differential equation of the given curve.

Solution 4

$$y = e^{2x}\left(a+bx\right) \qquad ...(1)$$

Differentiating both sides with respect to x, we get:

$$y' = 2e^{2x}\left(a+bx\right)+e^{2x}\cdot b$$
$$\Rightarrow y' = e^{2x}\left(2a+2bx+b\right) \qquad ...(2)$$

Multiplying equation (1) with 2 and then subtracting it from equation (2), we get:

$$y'-2y = e^{2x}\left(2a+2bx+b\right)-e^{2x}\left(2a+2bx\right)$$
$$\Rightarrow y'-2y = be^{2x} \qquad ...(3)$$

Differentiating both sides with respect to x, we get:

$$y''-2y' = 2be^{2x} \qquad ...(4)$$

Dividing equation (4) by equation (3), we get:

$$\frac{y''-2y'}{y'-2y} = 2$$
$$\Rightarrow y''-2y' = 2y'-4y$$
$$\Rightarrow y''-4y'+4y = 0$$

This is the required differential equation of the given curve.

Solution 5

$$y = e^x\left(a\cos x + b\sin x\right) \qquad ...(1)$$

Differentiating both sides with respect to x, we get:

$$y' = e^x\left(a\cos x + b\sin x\right) + e^x\left(-a\sin x + b\cos x\right)$$
$$\Rightarrow y' = e^x\left[\left(a+b\right)\cos x - \left(a-b\right)\sin x\right] \qquad ...(2)$$

Again, differentiating with respect to x, we get:

$$y'' = e^x\left[\left(a+b\right)\cos x - \left(a-b\right)\sin x\right] + e^x\left[-\left(a+b\right)\sin x - \left(a-b\right)\cos x\right]$$
$$y'' = e^x\left[2b\cos x - 2a\sin x\right]$$
$$y'' = 2e^x\left(b\cos x - a\sin x\right)$$
$$\Rightarrow \frac{y''}{2} = e^x\left(b\cos x - a\sin x\right) \qquad ...(3)$$

Adding equations (1) and (3), we get:

$$y + \frac{y''}{2} = e^x\left[\left(a+b\right)\cos x - \left(a-b\right)\sin x\right]$$
$$\Rightarrow y + \frac{y''}{2} = y'$$
$$\Rightarrow 2y + y'' = 2y'$$
$$\Rightarrow y'' - 2y' + 2y = 0$$

This is the required differential equation of the given curve.

Solution 6

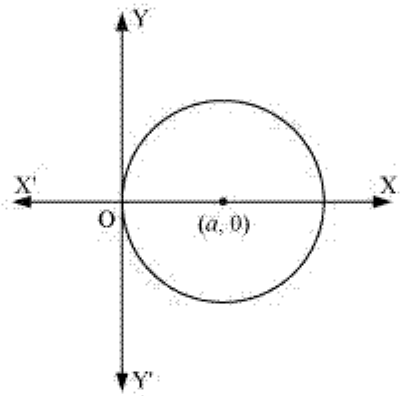
The centre of the circle touching the y-axis at origin lies on the x-axis.

Let $(a, 0)$ be the centre of the circle.

Since it touches the y-axis at origin, its radius is a.

Now, the equation of the circle with centre $(a, 0)$ and radius (a) is

$$(x-a)^2 + y^2 = a^2.$$
$$\Rightarrow x^2 + y^2 = 2ax \qquad ...(1)$$



Differentiating equation (1) with respect to x, we get:

$$2x + 2yy' = 2a$$
$$\Rightarrow x + yy' = a$$

Now, on substituting the value of a in equation (1), we get:

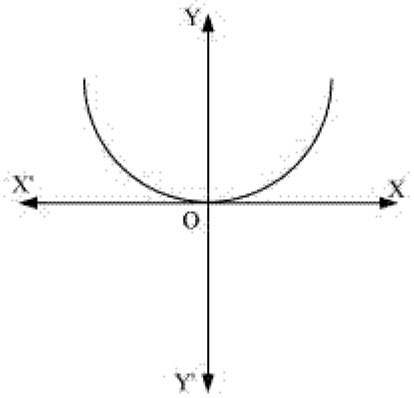
$$x^2 + y^2 = 2(x + yy')x$$
$$\Rightarrow x^2 + y^2 = 2x^2 + 2xyy'$$
$$\Rightarrow 2xyy' + x^2 = y^2$$

This is the required differential equation.

Solution 7

The equation of the parabola having the vertex at origin and the axis along the positive y axis is:

$$x^2 = 4ay \qquad ...(1)$$



Differentiating equation (1) with respect to x, we get:

$$2x = 4ay' \qquad ...(2)$$

Dividing equation (2) by equation (1), we get:

$$\frac{2x}{x^2} = \frac{4ay'}{4ay}$$

$$\Rightarrow \frac{2}{x} = \frac{y'}{y}$$

$$\Rightarrow xy' = 2y$$

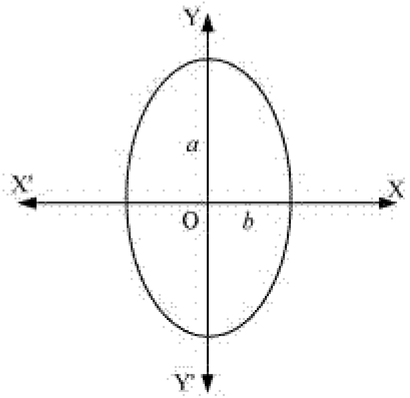
$$\Rightarrow xy' - 2y = 0$$

This is the required differential equation.

Solution 8

The equation of the family of ellipses having foci on the y-axis and the centre at origin is as follows:

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad \text{..(1)}$$



Differentiating equation (1) with respect to x, we get:

$$\frac{2x}{b^2}+\frac{2yy'}{b^2}=0$$

$$\Rightarrow \frac{x}{b^2}+\frac{yy'}{a^2}=0 \qquad \text{...(2)}$$

Again, differentiating with respect to x, we get:

$$\frac{1}{b^2}+\frac{y'.y'+y.y''}{a^2}=0$$

$$\Rightarrow \frac{1}{b^2}+\frac{1}{a^2}\left(y'^2+yy''\right)=0$$

$$\Rightarrow \frac{1}{b^2}=-\frac{1}{a^2}\left(y'^2+yy''\right)$$

Substituting this value in equation (2), we get:

$$x\left[-\frac{1}{a^2}\left(\left(y'\right)^2+yy''\right)\right]+\frac{yy'}{a^2}=0$$

$$\Rightarrow -x\left(y'\right)^2-xyy''+yy'=0$$

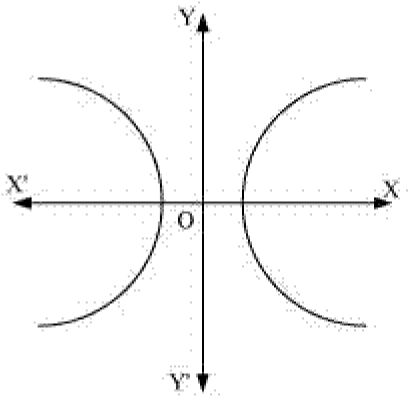
$$\Rightarrow xyy''+x\left(y'\right)^2-yy'=0$$

This is the required differential equation.

Solution 9

The equation of the family of hyperbolas with the centre at origin and foci along the x-axis is:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{...(1)}$$



Differentiating both sides of equation (1) with respect to x, we get:

$$\frac{2x}{a^2}-\frac{2yy'}{b^2}=0$$
$$\Rightarrow \frac{x}{a^2}-\frac{yy'}{b^2}=0 \qquad \text{...(2)}$$

Again, differentiating both sides with respect to x, we get:

$$\frac{1}{a^2}-\frac{y'\cdot y'+yy''}{b^2}=0$$
$$\Rightarrow \frac{1}{a^2}=\frac{1}{b^2}\left((y')^2+yy''\right)$$

Substituting the value of $\frac{1}{a^2}$ in equation (2), we get:

$$\frac{x}{b^2}\left((y')^2+yy''\right)-\frac{yy'}{b^2}=0$$
$$\Rightarrow x(y')^2+xyy''-yy'=0$$
$$\Rightarrow xyy''+x(y')^2-yy'=0$$

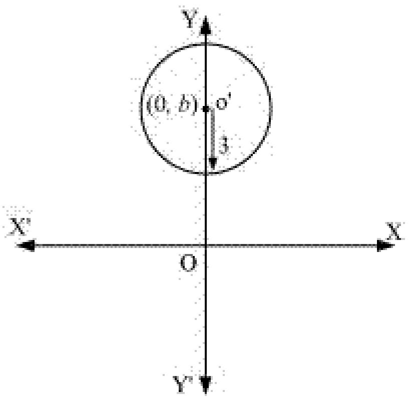
This is the required differential equation.

Solution 10

Let the centre of the circle on y-axis be $(0, b)$.

The differential equation of the family of circles with centre at $(0, b)$ and radius 3 is as follows:

$$x^2 + (y-b)^2 = 3^2$$
$$\Rightarrow x^2 + (y-b)^2 = 9 \qquad \ldots(1)$$



Differentiating equation (1) with respect to x, we get:

$$2x + 2(y-b)\cdot y' = 0$$
$$\Rightarrow (y-b)\cdot y' = -x$$
$$\Rightarrow y-b = \frac{-x}{y'}$$

Substituting the value of $(y-b)$ in equation (1), we get:

$$x^2 + \left(\frac{-x}{y'}\right)^2 = 9$$
$$\Rightarrow x^2\left[1 + \frac{1}{(y')^2}\right] = 9$$
$$\Rightarrow x^2\left((y')^2 + 1\right) = 9(y')^2$$
$$\Rightarrow (x^2 - 9)(y')^2 + x^2 = 0$$

This is the required differential equation.

Solution 11

The given equation is:

$$y = c_1 e^x + c_2 e^{-x} \qquad ...(1)$$

Differentiating with respect to x, we get:

$$\frac{dy}{dx} = c_1 e^x - c_2 e^{-x}$$

Again, differentiating with respect to x, we get:

$$\frac{d^2y}{dx^2} = c_1 e^x + c_2 e^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = y$$

$$\Rightarrow \frac{d^2y}{dx^2} - y = 0$$

This is the required differential equation of the given equation of curve.

Hence, the correct answer is B.

Solution 12

The given equation of curve is $y = x$.

Differentiating with respect to x, we get:

$$\frac{dy}{dx} = 1 \qquad ...(1)$$

Again, differentiating with respect to x, we get:

$$\frac{d^2y}{dx^2} = 0 \qquad ...(2)$$

Now, on substituting the values of y, $\frac{d^2y}{dx^2}$, and $\frac{dy}{dx}$ from equation (1) and (2) in each of the given alternatives, we find that only the differential equation given in alternative C is correct.

$$\begin{aligned}\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy &= 0 - x^2 \cdot 1 + x \cdot x \\ &= -x^2 + x^2 \\ &= 0\end{aligned}$$

Hence, the correct answer is C.

Chapter 9 - Differential Equations Exercise Ex. 9.4

Solution 1

The given differential equation is:

$$\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} = \tan^2 \frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \left(\sec^2 \frac{x}{2} - 1\right)$$

Separating the variables,we get:

$$dy = \left(\sec^2 \frac{x}{2} - 1\right) dx$$

Now, integrating both sides of this equation, we get:

$$\int dy = \int\left(\sec^2 \frac{x}{2} - 1\right) dx = \int \sec^2 \frac{x}{2} dx - \int dx$$

$$\Rightarrow y = 2\tan \frac{x}{2} - x + C$$

This is the required general solution of the given differential equation.

Solution 2

The given differential equation is:

$$\frac{dy}{dx}=\sqrt{4-y^2}$$

Separating the variables, we get:

$$\Rightarrow \frac{dy}{\sqrt{4-y^2}}=dx$$

Now, integrating both sides of this equation, we get:

$$\int\frac{dy}{\sqrt{4-y^2}}=\int dx$$

$$\Rightarrow \sin^{-1}\frac{y}{2}=x+\text{C}$$

$$\Rightarrow \frac{y}{2}=\sin\left(x+\text{C}\right)$$

$$\Rightarrow y=2\sin\left(x+\text{C}\right)$$

This is the required general solution of the given differential equation.

Solution 3

The given differential equation is:

$$\frac{dy}{dx}+y=1$$
$$\Rightarrow dy+y\ dx=dx$$
$$\Rightarrow dy=(1-y)dx$$

Separating the variables, we get:

$$\Rightarrow \frac{dy}{1-y}=dx$$

Now, integrating both sides, we get:

$$\int\frac{dy}{1-y}=\int dx$$
$$\Rightarrow -\log(1-y)=x+\log C$$
$$\Rightarrow -\log C-\log(1-y)=x$$
$$\Rightarrow \log C(1-y)=-x$$
$$\Rightarrow C(1-y)=e^{-x}$$
$$\Rightarrow 1-y=\frac{1}{C}e^{-x}$$
$$\Rightarrow y=1-\frac{1}{C}e^{-x}$$
$$\Rightarrow y=1+Ae^{-x}\ \text{(where } A=-\frac{1}{C}\text{)}$$

This is the required general solution of the given differential equation.

Solution 4

The given differential equation is:

$$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$$
$$\Rightarrow \frac{\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy}{\tan x \tan y} = 0$$
$$\Rightarrow \frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$
$$\Rightarrow \frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy$$

Integrating both sides of this equation, we get

$$\int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy \qquad \ldots(1)$$

Let $\tan x = t$.

$$\therefore \frac{d}{dx}(\tan x) = \frac{dt}{dx}$$
$$\Rightarrow \sec^2 x = \frac{dt}{dx}$$
$$\Rightarrow \sec^2 x \, dx = dt$$

Now, $\int \frac{\sec^2 x}{\tan x} dx = \int \frac{1}{t} dt.$

$$= \log t$$
$$= \log(\tan x)$$

Similarly, $\int \frac{\sec^2 y}{\tan y} dy = \log(\tan y).$

Substituting these values in equation (1), we get:

$$\log(\tan x) = -\log(\tan y) + \log C$$
$$\Rightarrow \log(\tan x) = \log\left(\frac{C}{\tan y}\right)$$
$$\Rightarrow \tan x = \frac{C}{\tan y}$$
$$\Rightarrow \tan x \tan y = C$$

This is the required general solution of the given differential equation.

Solution 5

The given differential equation is:

$$\left(e^{x}+e^{-x}\right)dy-\left(e^{x}-e^{-x}\right)dx=0$$
$$\Rightarrow\left(e^{x}+e^{-x}\right)dy=\left(e^{x}-e^{-x}\right)dx$$
$$\Rightarrow dy=\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]dx$$

Integrating both sides of this equation, we get:

$$\int dy=\int\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]dx+\mathrm{C}$$
$$\Rightarrow y=\int\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]dx+\mathrm{C} \qquad \ldots(1)$$

Let $(e^{x}+e^{-x})=t$.

Differentiating both sides with respect to x, we get:

$$\frac{d}{dx}\left(e^{x}+e^{-x}\right)=\frac{dt}{dx}$$
$$\Rightarrow e^{x}-e^{-x}=\frac{dt}{dx}$$
$$\Rightarrow\left(e^{x}-e^{-x}\right)dx=dt$$

Substituting this value in equation (1), we get:

$$y=\int\frac{1}{t}dt+\mathrm{C}$$
$$\Rightarrow y=\log(t)+\mathrm{C}$$
$$\Rightarrow y=\log\left(e^{x}+e^{-x}\right)+\mathrm{C}$$

This is the required general solution of the given differential equation.

Solution 6

The given differential equation is:

$$\frac{dy}{dx} = \left(1+x^2\right)\left(1+y^2\right)$$
$$\Rightarrow \frac{dy}{1+y^2} = \left(1+x^2\right)dx$$

Integrating both sides of this equation, we get:

$$\int \frac{dy}{1+y^2} = \int \left(1+x^2\right)dx$$
$$\Rightarrow \tan^{-1} y = \int dx + \int x^2 dx$$
$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + C$$

This is the required general solution of the given differential equation.

Solution 7

The given differential equation is:

$$y \log y \, dx - x \, dy = 0$$
$$\Rightarrow y \log y \, dx = x \, dy$$
$$\Rightarrow \frac{dy}{y \log y} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{dy}{y \log y} = \int \frac{dx}{x} \qquad \text{...(1)}$$

Let $\log y = t$.

$$\therefore \frac{d}{dy}(\log y) = \frac{dt}{dy}$$
$$\Rightarrow \frac{1}{y} = \frac{dt}{dy}$$
$$\Rightarrow \frac{1}{y} dy = dt$$

Substituting this value in equation (1), we get:

$$\int \frac{dt}{t} = \int \frac{dx}{x}$$
$$\Rightarrow \log t = \log x + \log C$$
$$\Rightarrow \log(\log y) = \log Cx$$
$$\Rightarrow \log y = Cx$$
$$\Rightarrow y = e^{Cx}$$

This is the required general solution of the given differential equation.

Solution 8

The given differential equation is:

$$x^5 \frac{dy}{dx} = -y^5$$
$$\Rightarrow \frac{dy}{y^5} = -\frac{dx}{x^5}$$
$$\Rightarrow \frac{dx}{x^5} + \frac{dy}{y^5} = 0$$

Integrating both sides, we get:

$$\int \frac{dx}{x^5} + \int \frac{dy}{y^5} = k \quad \text{(where } k \text{ is any constant)}$$
$$\Rightarrow \int x^{-5} dx + \int y^{-5} dy = k$$
$$\Rightarrow \frac{x^{-4}}{-4} + \frac{y^{-4}}{-4} = k$$
$$\Rightarrow x^{-4} + y^{-4} = -4k$$
$$\Rightarrow x^{-4} + y^{-4} = C \qquad (C = -4k)$$

This is the required general solution of the given differential equation.

Solution 9

The given differential equation is:

$$\frac{dy}{dx} = \sin^{-1} x$$
$$\Rightarrow dy = \sin^{-1} x\ dx$$

Integrating both sides, we get:

$$\int dy = \int \sin^{-1} x\, dx$$
$$\Rightarrow y = \int \left(\sin^{-1} x \cdot 1\right) dx$$
$$\Rightarrow y = \sin^{-1} x \cdot \int (1)\, dx - \int \left[\left(\frac{d}{dx}\left(\sin^{-1} x\right) \cdot \int (1)\, dx\right)\right] dx$$

$$\Rightarrow y = \sin^{-1} x \cdot x - \int \left(\frac{1}{\sqrt{1-x^2}} \cdot x\right) dx$$
$$\Rightarrow y = x \sin^{-1} x + \int \frac{-x}{\sqrt{1-x^2}}\, dx \qquad \ldots(1)$$

Let $1 - x^2 = t$.

$$\Rightarrow \frac{d}{dx}\left(1-x^2\right) = \frac{dt}{dx}$$
$$\Rightarrow -2x = \frac{dt}{dx}$$
$$\Rightarrow x\, dx = -\frac{1}{2} dt$$

Substituting this value in equation (1), we get:

$$y = x \sin^{-1} x + \int \frac{1}{2\sqrt{t}}\, dt$$
$$\Rightarrow y = x \sin^{-1} x + \frac{1}{2} \cdot \int (t)^{-\frac{1}{2}}\, dt$$
$$\Rightarrow y = x \sin^{-1} x + \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + \mathrm{C}$$
$$\Rightarrow y = x \sin^{-1} x + \sqrt{t} + \mathrm{C}$$
$$\Rightarrow y = x \sin^{-1} x + \sqrt{1-x^2} + \mathrm{C}$$

This is the required general solution of the given differential equation.

Solution 10

The given differential equation is:

$$e^x \tan y\, dx + \left(1-e^x\right)\sec^2 y\, dy = 0$$

$$\left(1-e^x\right)\sec^2 y\, dy = -e^x \tan y\, dx$$

Separating the variables, we get:

$$\frac{\sec^2 y}{\tan y}dy = \frac{-e^x}{1-e^x}dx$$

Integrating both sides, we get:

$$\int\frac{\sec^2 y}{\tan y}dy = \int\frac{-e^x}{1-e^x}dx \qquad ...(1)$$

Let $\tan y = u$.

$$\Rightarrow \frac{d}{dy}\left(\tan y\right) = \frac{du}{dy}$$

$$\Rightarrow \sec^2 y = \frac{du}{dy}$$

$$\Rightarrow \sec^2 y dy = du$$

$$\therefore \int\frac{\sec^2 y}{\tan y}dy = \int\frac{du}{u} = \log u = \log\left(\tan y\right)$$

Now, let $1-e^x = t$.

$$\therefore \frac{d}{dx}\left(1-e^x\right) = \frac{dt}{dx}$$

$$\Rightarrow -e^x = \frac{dt}{dx}$$

$$\Rightarrow -e^x dx = dt$$

$$\Rightarrow \int\frac{-e^x}{1-e^x}dx = \int\frac{dt}{t} = \log t = \log\left(1-e^x\right)$$

Substituting the values of $\int\frac{\sec^2 y}{\tan y}dy$ and $\int\frac{-e^x}{1-e^x}dx$ in equation (1), we get:

$$\Rightarrow \log\left(\tan y\right) = \log\left(1-e^x\right) + \log \mathrm{C}$$

$$\Rightarrow \log\left(\tan y\right) = \log\left[\mathrm{C}\left(1-e^x\right)\right]$$

$$\Rightarrow \tan y = \mathrm{C}\left(1-e^x\right)$$

This is the required general solution of the given differential equation.

Solution 11

The given differential equation is:

$$\left(x^3+x^2+x+1\right)\frac{dy}{dx}=2x^2+x$$

$$\Rightarrow \frac{dy}{dx}=\frac{2x^2+x}{\left(x^3+x^2+x+1\right)}$$

$$\Rightarrow dy=\frac{2x^2+x}{(x+1)\left(x^2+1\right)}dx$$

Integrating both sides, we get:

$$\int dy=\int\frac{2x^2+x}{(x+1)\left(x^2+1\right)}dx \qquad ...(1)$$

$$\text{Let } \frac{2x^2+x}{(x+1)\left(x^2+1\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}. \qquad ...(2)$$

$$\Rightarrow \frac{2x^2+x}{(x+1)\left(x^2+1\right)}=\frac{Ax^2+A+(Bx+C)(x+1)}{(x+1)\left(x^2+1\right)}$$

$$\Rightarrow 2x^2+x=Ax^2+A+Bx^2+Bx+Cx+C$$

$$\Rightarrow 2x^2+x=(A+B)x^2+(B+C)x+(A+C)$$

Comparing the coefficients of x^2 and x, we get:

$A+B=2$

$B+C=1$

$A+C=0$

Solving these equations, we get:

$$A=\frac{1}{2},\ B=\frac{3}{2} \text{ and } C=\frac{-1}{2}$$

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{2x^2+x}{(x+1)\left(x^2+1\right)}=\frac{1}{2}\cdot\frac{1}{(x+1)}+\frac{1}{2}\frac{(3x-1)}{\left(x^2+1\right)}$$

Therefore, equation (1) becomes:

$$\int dy = \frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{3x-1}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{2}\int \frac{x}{x^2+1}dx - \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\cdot\int \frac{2x}{x^2+1}dx - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}\left[2\log(x+1) + 3\log(x^2+1)\right] - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}\left[(x+1)^2(x^2+1)^3\right] - \frac{1}{2}\tan^{-1}x + C \qquad ...(3)$$

Now, $y = 1$ when $x = 0$.

$$\Rightarrow 1 = \frac{1}{4}\log(1) - \frac{1}{2}\tan^{-1}0 + C$$

$$\Rightarrow 1 = \frac{1}{4}\times 0 - \frac{1}{2}\times 0 + C$$

$$\Rightarrow C = 1$$

Substituting $C = 1$ in equation (3), we get:

$$y = \frac{1}{4}\left[\log(x+1)^2(x^2+1)^3\right] - \frac{1}{2}\tan^{-1}x + 1$$

Solution 12

$$x\left(x^2-1\right)\frac{dy}{dx}=1$$
$$\Rightarrow dy=\frac{dx}{x\left(x^2-1\right)}$$
$$\Rightarrow dy=\frac{1}{x(x-1)(x+1)}dx$$

Integrating both sides, we get:

$$\int dy=\int\frac{1}{x(x-1)(x+1)}dx \qquad ...(1)$$

$$\text{Let } \frac{1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}. \qquad ...(2)$$
$$\Rightarrow \frac{1}{x(x-1)(x+1)}=\frac{A(x-1)(x+1)+Bx(x+1)+Cx(x-1)}{x(x-1)(x+1)}$$
$$=\frac{(A+B+C)x^2+(B-C)x-A}{x(x-1)(x+1)}$$

Comparing the coefficients of x^2, x, and constant, we get:

$$A=-1$$
$$B-C=0$$
$$A+B+C=0$$

Solving these equations, we get $B=\frac{1}{2}$ and $C=\frac{1}{2}$.

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{1}{x(x-1)(x+1)}=\frac{-1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$$

Therefore, equation (1) becomes:

$$\int dy=-\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{x-1}dx+\frac{1}{2}\int\frac{1}{x+1}dx$$
$$\Rightarrow y=-\log x+\frac{1}{2}\log(x-1)+\frac{1}{2}\log(x+1)+\log k$$
$$\Rightarrow y=\frac{1}{2}\log\left[\frac{k^2(x-1)(x+1)}{x^2}\right] \qquad ...(3)$$

Now, $y = 0$ when $x = 2$.

$$\Rightarrow 0 = \frac{1}{2}\log\left[\frac{k^2(2-1)(2+1)}{4}\right]$$

$$\Rightarrow \log\left(\frac{3k^2}{4}\right) = 0$$

$$\Rightarrow \frac{3k^2}{4} = 1$$

$$\Rightarrow 3k^2 = 4$$

$$\Rightarrow k^2 = \frac{4}{3}$$

Substituting the value of k^2 in equation (3), we get:

$$y = \frac{1}{2}\log\left[\frac{4(x-1)(x+1)}{3x^2}\right]$$

$$y = \frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$$

$$y = \frac{1}{2}\log\left[\frac{x^2-1}{x^2}\right] - \frac{1}{2}\log\frac{3}{4}$$

Solution 13

$$\cos\left(\frac{dy}{dx}\right) = a$$
$$\Rightarrow \frac{dy}{dx} = \cos^{-1} a$$
$$\Rightarrow dy = \cos^{-1} a \, dx$$

Integrating both sides, we get:

$$\int dy = \cos^{-1} a \int dx$$
$$\Rightarrow y = \cos^{-1} a \cdot x + C$$
$$\Rightarrow y = x\cos^{-1} a + C \qquad \ldots(1)$$

Now, $y = 1$ when $x = 0$.

$$\Rightarrow 1 = 0 \cdot \cos^{-1} a + C$$
$$\Rightarrow C = 1$$

Substituting $C = 1$ in equation (1), we get:

$$y = x\cos^{-1} a + 1$$
$$\Rightarrow \frac{y-1}{x} = \cos^{-1} a$$
$$\Rightarrow \cos\left(\frac{y-1}{x}\right) = a$$

Solution 14

$$\frac{dy}{dx} = y \tan x$$

$$\Rightarrow \frac{dy}{y} = \tan x \, dx$$

Integrating both sides, we get:

$$\int \frac{dy}{y} = \int \tan x \, dx$$

$$\Rightarrow \log y = \log(\sec x) + \log C$$

$$\Rightarrow \log y = \log(C \sec x)$$

$$\Rightarrow y = C \sec x \qquad \ldots(1)$$

Now, $y = 1$ when $x = 0$.

$$\Rightarrow 1 = C \times \sec 0$$

$$\Rightarrow 1 = C \times 1$$

$$\Rightarrow C = 1$$

Substituting $C = 1$ in equation (1), we get:

$$y = \sec x$$

Solution 15

The differential equation of the curve is:

$$y' = e^x \sin x$$
$$\Rightarrow \frac{dy}{dx} = e^x \sin x$$
$$\Rightarrow dy = e^x \sin x \, dx$$

Integrating both sides, we get:

$$\int dy = \int e^x \sin x \, dx \qquad \ldots(1)$$

Let $I = \int e^x \sin x \, dx$.

$$\Rightarrow I = \sin x \int e^x dx - \int\left(\frac{d}{dx}(\sin x) \cdot \int e^x dx\right) dx$$

$$\Rightarrow I = \sin x \cdot e^x - \int \cos x \cdot e^x dx$$
$$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot \int e^x dx - \int\left(\frac{d}{dx}(\cos x) \cdot \int e^x dx\right) dx\right]$$
$$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot e^x - \int(-\sin x) \cdot e^x dx\right]$$
$$\Rightarrow I = e^x \sin x - e^x \cos x - I$$
$$\Rightarrow 2I = e^x (\sin x - \cos x)$$
$$\Rightarrow I = \frac{e^x (\sin x - \cos x)}{2}$$

Substituting this value in equation (1), we get:

$$y = \frac{e^x(\sin x - \cos x)}{2} + C \qquad ...(2)$$

Now, the curve passes through point (0, 0).

$$\therefore 0 = \frac{e^0(\sin 0 - \cos 0)}{2} + C$$
$$\Rightarrow 0 = \frac{1(0-1)}{2} + C$$
$$\Rightarrow C = \frac{1}{2}$$

Substituting $C = \frac{1}{2}$ in equation (2), we get:

$$y = \frac{e^x(\sin x - \cos x)}{2} + \frac{1}{2}$$
$$\Rightarrow 2y = e^x(\sin x - \cos x) + 1$$
$$\Rightarrow 2y - 1 = e^x(\sin x - \cos x)$$

Hence, the required equation of the curve is $2y - 1 = e^x(\sin x - \cos x)$.

Solution 16

The differential equation of the given curve is:

$$xy\frac{dy}{dx}=(x+2)(y+2)$$
$$\Rightarrow\left(\frac{y}{y+2}\right)dy=\left(\frac{x+2}{x}\right)dx$$
$$\Rightarrow\left(1-\frac{2}{y+2}\right)dy=\left(1+\frac{2}{x}\right)dx$$

Integrating both sides, we get:

$$\int\left(1-\frac{2}{y+2}\right)dy=\int\left(1+\frac{2}{x}\right)dx$$
$$\Rightarrow \int dy-2\int\frac{1}{y+2}dy=\int dx+2\int\frac{1}{x}dx$$
$$\Rightarrow y-2\log(y+2)=x+2\log x+\mathrm{C}$$
$$\Rightarrow y-x-\mathrm{C}=\log x^2+\log(y+2)^2$$
$$\Rightarrow y-x-\mathrm{C}=\log\left[x^2(y+2)^2\right] \qquad ...(1)$$

Now, the curve passes through point (1, −1).

$$\Rightarrow -1-1-\mathrm{C}=\log\left[(1)^2(-1+2)^2\right]$$
$$\Rightarrow -2-\mathrm{C}=\log 1=0$$
$$\Rightarrow \mathrm{C}=-2$$

Substituting $\mathrm{C}=-2$ in equation (1), we get:

$$y-x+2=\log\left[x^2(y+2)^2\right]$$

This is the required solution of the given curve.

Solution 17

Let x and y be the x-coordinate and y-coordinate of the point on the curve respectively.

We know that the slope of a tangent to the curve in the coordinate axes is given by the relation,

$$\frac{dy}{dx}$$

According to the given information, we get:

$$y \cdot \frac{dy}{dx} = x$$
$$\Rightarrow y\,dy = x\,dx$$

Integrating both sides, we get:

$$\int y\,dy = \int x\,dx$$
$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$
$$\Rightarrow y^2 - x^2 = 2C \qquad ...(1)$$

Now, the curve passes through point (0, –2).

$$\therefore\ (-2)^2 - 0^2 = 2C$$

$$\Rightarrow 2C = 4$$

Substituting $2C = 4$ in equation (1), we get:

$$y^2 - x^2 = 4$$

This is the required equation of the curve.

Solution 18

It is given that (x, y) is the point of contact of the curve and its tangent.

The slope (m_1) of the line segment joining (x, y) and $(-4, -3)$ is $\frac{y+3}{x+4}$.

We know that the slope of the tangent to the curve is given by the relation,

$$\frac{dy}{dx}$$

$$\therefore \text{Slope } (m_2) \text{ of the tangent } = \frac{dy}{dx}$$

According to the given information:

$$m_2 = 2m_1$$
$$\Rightarrow \frac{dy}{dx} = \frac{2(y+3)}{x+4}$$
$$\Rightarrow \frac{dy}{y+3} = \frac{2dx}{x+4}$$

Integrating both sides, we get:

$$\int \frac{dy}{y+3} = 2\int \frac{dx}{x+4}$$
$$\Rightarrow \log(y+3) = 2\log(x+4) + \log C$$
$$\Rightarrow \log(y+3) = \log C(x+4)^2$$
$$\Rightarrow y+3 = C(x+4)^2 \qquad ...(1)$$

This is the general equation of the curve.

It is given that it passes through point $(-2, 1)$.

$$\Rightarrow 1+3 = C(-2+4)^2$$
$$\Rightarrow 4 = 4C$$
$$\Rightarrow C = 1$$

Substituting $C = 1$ in equation (1), we get:

$$y + 3 = (x + 4)^2$$

This is the required equation of the curve.

Solution 19

Let the rate of change of the volume of the balloon be k (where k is a constant)

$$\Rightarrow \frac{dv}{dt} = k$$

$$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \qquad \left[\text{Volume of sphere} = \frac{4}{3}\pi r^3\right]$$

$$\Rightarrow \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} = k$$

$$\Rightarrow 4\pi r^2\, dr = k\, dt$$

Integrating both sides, we get:

$$4\pi \int r^2 dr = k \int dt$$

$$\Rightarrow 4\pi \cdot \frac{r^3}{3} = kt + C$$

$$\Rightarrow 4\pi r^3 = 3(kt + C) \qquad ...(1)$$

Now, at $t = 0$, $r = 3$:

$4\pi \times 3^3 = 3\ (k \times 0 + C)$

$108\pi = 3C$

$C = 36\pi$

At $t = 3$, $r = 6$:

$4\pi \times 6^3 = 3\ (k \times 3 + C)$

$864\pi = 3\ (3k + 36\pi)$

$3k = 288\pi - 36\pi = 252\pi$

$k = 84\pi$

Substituting the values of k and C in equation (1), we get:

$$4\pi r^3 = 3\left[84\pi t + 36\pi\right]$$

$$\Rightarrow 4\pi r^3 = 4\pi(63t + 27)$$

$$\Rightarrow r^3 = 63t + 27$$

$$\Rightarrow r = (63t + 27)^{\frac{1}{3}}$$

Thus, the radius of the balloon after t seconds is $(63t + 27)^{\frac{1}{3}}$.

Solution 20

Let p, t, and r represent the principal, time, and rate of interest respectively.

It is given that the principal increases continuously at the rate of $r\%$ per year.

$$\Rightarrow \frac{dp}{dt}=\left(\frac{r}{100}\right)p$$

$$\Rightarrow \frac{dp}{p}=\left(\frac{r}{100}\right)dt$$

Integrating both sides, we get:

$$\int\frac{dp}{p}=\frac{r}{100}\int dt$$

$$\Rightarrow \log p=\frac{rt}{100}+k$$

$$\Rightarrow p=e^{\frac{rt}{100}+k} \qquad ...(1)$$

It is given that when $t = 0$, $p = 100$.

$$\Rightarrow 100 = e^k \; \ldots \; (2)$$

Now, if $t = 10$, then $p = 2 \times 100 = 200$.

$$200=e^{\frac{r}{10}+k}$$

$$\Rightarrow 200=e^{\frac{r}{10}}\cdot e^{k}$$

$$\Rightarrow 200=e^{\frac{r}{10}}\cdot 100 \qquad (\text{From } (2))$$

$$\Rightarrow e^{\frac{r}{10}}=2$$

$$\Rightarrow \frac{r}{10}=\log_e 2$$

$$\Rightarrow \frac{r}{10}=0.6931$$

$$\Rightarrow r=6.931$$

Hence, the value of r is 6.93%.

Solution 21

Let p and t be the principal and time respectively.

It is given that the principal increases continuously at the rate of 5% per year.

$$\Rightarrow \frac{dp}{dt} = \left(\frac{5}{100}\right)p$$
$$\Rightarrow \frac{dp}{dt} = \frac{p}{20}$$
$$\Rightarrow \frac{dp}{p} = \frac{dt}{20}$$

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{1}{20}\int dt$$
$$\Rightarrow \log p = \frac{t}{20} + C$$
$$\Rightarrow p = e^{\frac{t}{20}+C} \quad ...(1)$$

Now, when $t = 0$, $p = 1000$.

$1000 = e^{C}$ ... (2)

At $t = 10$, equation (1) becomes:

$$p = e^{\frac{1}{2}+C}$$
$$\Rightarrow p = e^{0.5} \times e^{C}$$
$$\Rightarrow p = 1.648 \times 1000$$
$$\Rightarrow p = 1648$$

Hence, after 10 years the amount will worth Rs 1648.

Solution 22

Let y be the number of bacteria at any instant t.

It is given that the rate of growth of the bacteria is proportional to the number present.

$$\therefore \frac{dy}{dt} \propto y$$
$$\Rightarrow \frac{dy}{dt} = ky \text{ (where } k \text{ is a constant)}$$
$$\Rightarrow \frac{dy}{y} = kdt$$

Integrating both sides, we get:

$$\int \frac{dy}{y} = k \int dt$$
$$\Rightarrow \log y = kt + \text{C} \qquad \ldots(1)$$

Let y_0 be the number of bacteria at $t = 0$.

$$\log y_0 = \text{C}$$

Substituting the value of C in equation (1), we get:

$$\log y = kt + \log y_0$$
$$\Rightarrow \log y - \log y_0 = kt$$
$$\Rightarrow \log\left(\frac{y}{y_0}\right) = kt$$
$$\Rightarrow kt = \log\left(\frac{y}{y_0}\right) \qquad \ldots(2)$$

Also, it is given that the number of bacteria increases by 10% in 2 hours.

$$\Rightarrow y = \frac{110}{100} y_0$$
$$\Rightarrow \frac{y}{y_0} = \frac{11}{10} \qquad \ldots(3)$$

Substituting this value in equation (2), we get:

$$k \cdot 2 = \log\left(\frac{11}{10}\right)$$
$$\Rightarrow k = \frac{1}{2}\log\left(\frac{11}{10}\right)$$

Therefore, equation (2) becomes:

$$\frac{1}{2}\log\left(\frac{11}{10}\right)\cdot t = \log\left(\frac{y}{y_0}\right)$$
$$\Rightarrow t = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} \qquad ...(4)$$

Now, let the time when the number of bacteria increases from 100000 to 200000 be t_1.

$y = 2y_0$ at $t = t_1$

From equation (4), we get:

$$t_1 = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} = \frac{2\log 2}{\log\left(\frac{11}{10}\right)}$$

Hence, in $\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$ hours the number of bacteria increases from 100000 to 200000.

Solution 23

$$\frac{dy}{dx} = e^{x+y} = e^x \cdot e^y$$
$$\Rightarrow \frac{dy}{e^y} = e^x dx$$
$$\Rightarrow e^{-y} dy = e^x dx$$

Integrating both sides, we get:

$$\int e^{-y} dy = \int e^x dx$$
$$\Rightarrow -e^{-y} = e^x + k$$
$$\Rightarrow e^x + e^{-y} = -k$$
$$\Rightarrow e^x + e^{-y} = c \qquad (c = -k)$$

Hence, the correct answer is A.

Chapter 9 - Differential Equations Exercise Ex. 9.5

Solution 1

The given differential equation i.e., $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$ can be written as:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} \qquad ...(1)$$

Let $F(x, y) = \frac{x^2 + y^2}{x^2 + xy}$.

Now, $F(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{(\lambda x)^2 + (\lambda x)(\lambda y)} = \frac{x^2 + y^2}{x^2 + xy} = \lambda^0 \cdot F(x, y)$

This shows that equation (1) is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of v and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2}{x^2 + x(vx)}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{1+v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v = \frac{(1+v^2) - v(1+v)}{1+v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v}$$
$$\Rightarrow \left(\frac{1+v}{1-v}\right)dv = \frac{dx}{x}$$
$$\Rightarrow \left(\frac{2-1+v}{1-v}\right)dv = \frac{dx}{x}$$
$$\Rightarrow \left(\frac{2}{1-v} - 1\right)dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$-2\log(1-v) - v = \log x - \log k$$
$$\Rightarrow v = -2\log(1-v) - \log x + \log k$$
$$\Rightarrow v = \log\left[\frac{k}{x(1-v)^2}\right]$$
$$\Rightarrow \frac{y}{x} = \log\left[\frac{k}{x\left(1-\frac{y}{x}\right)^2}\right]$$
$$\Rightarrow \frac{y}{x} = \log\left[\frac{kx}{(x-y)^2}\right]$$
$$\Rightarrow \frac{kx}{(x-y)^2} = e^{\frac{y}{x}}$$
$$\Rightarrow (x-y)^2 = kxe^{-\frac{y}{x}}$$

This is the required solution of the given differential equation.

Solution 2

The given differential equation is:

$$y' = \frac{x+y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x} \qquad \ldots(1)$$

Let $F(x, y) = \frac{x+y}{x}$.

Now, $F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x} = \frac{x+y}{x} = \lambda^0 F(x, y)$

Thus, the given equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{x+vx}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 + v$$

$$x\frac{dv}{dx} = 1$$

$$\Rightarrow dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$v = \log x + \mathrm{C}$$

$$\Rightarrow \frac{y}{x} = \log x + \mathrm{C}$$

$$\Rightarrow y = x\log x + \mathrm{C}x$$

This is the required solution of the given differential equation.

Solution 3

The given differential equation is:

$$(x-y)dy-(x+y)dx=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{x+y}{x-y} \qquad ...(1)$$

Let $F(x,y)=\dfrac{x+y}{x-y}$.

$$\therefore\ F(\lambda x,\lambda y)=\frac{\lambda x+\lambda y}{\lambda x-\lambda y}=\frac{x+y}{x-y}=\lambda^0\cdot F(x,y)$$

Thus, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y=vx$$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v+x\frac{dv}{dx}=\frac{x+vx}{x-vx}=\frac{1+v}{1-v}$$

$$x\frac{dv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v-v(1-v)}{1-v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v^2}{1-v}$$

$$\Rightarrow \frac{1-v}{\left(1+v^2\right)}dv=\frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{1+v^2}-\frac{v}{1+v^2}\right)dv=\frac{dx}{x}$$

Integrating both sides, we get:

$$\tan^{-1}v-\frac{1}{2}\log\left(1+v^2\right)=\log x+\mathrm{C}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\log\left[1+\left(\frac{y}{x}\right)^2\right]=\log x+\mathrm{C}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right)=\log x+\mathrm{C}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\left[\log\left(x^2+y^2\right)-\log x^2\right]=\log x+\mathrm{C}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log\left(x^2+y^2\right)+\mathrm{C}$$

This is the required solution of the given differential equation.

Solution 4

The given differential equation is:

$$\left(x^2-y^2\right)dx+2xy\ dy=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\left(x^2-y^2\right)}{2xy} \qquad \ldots(1)$$

Let $F(x,y)=\dfrac{-\left(x^2-y^2\right)}{2xy}$.

$$\therefore\ F(\lambda x,\lambda y)=-\left[\frac{(\lambda x)^2-(\lambda y)^2}{2(\lambda x)(\lambda y)}\right]=\frac{-\left(x^2-y^2\right)}{2xy}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y=vx$$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v+x\frac{dv}{dx}=-\left[\frac{x^2-(vx)^2}{2x\cdot(vx)}\right]$$

$$v+x\frac{dv}{dx}=\frac{v^2-1}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-1}{2v}-v=\frac{v^2-1-2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=-\frac{(1+v^2)}{2v}$$

$$\Rightarrow \frac{2v}{1+v^2}dv=-\frac{dx}{x}$$

Integrating both sides, we get:

$$\log(1+v^2)=-\log x+\log C=\log\frac{C}{x}$$

$$\Rightarrow 1+v^2=\frac{C}{x}$$

$$\Rightarrow \left[1+\frac{y^2}{x^2}\right]=\frac{C}{x}$$

$$\Rightarrow x^2+y^2=Cx$$

This is the required solution of the given differential equation.

Solution 5

The given differential equation is:

$$x^2\frac{dy}{dx}=x^2-2y^2+xy$$

$$\frac{dy}{dx}=\frac{x^2-2y^2+xy}{x^2} \qquad ...(1)$$

Let $F(x,y)=\dfrac{x^2-2y^2+xy}{x^2}$.

$$\therefore\ F(\lambda x,\lambda y)=\frac{(\lambda x)^2-2(\lambda y)^2+(\lambda x)(\lambda y)}{(\lambda x)^2}=\frac{x^2-2y^2+xy}{x^2}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$$y=vx$$

$$\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\dfrac{dy}{dx}$ in equation (1), we get:

$$v+x\frac{dv}{dx}=\frac{x^2-2(vx)^2+x\cdot(vx)}{x^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=1-2v^2+v$$

$$\Rightarrow x\frac{dv}{dx}=1-2v^2$$

$$\Rightarrow\frac{dv}{1-2v^2}=\frac{dx}{x}$$

$$\Rightarrow\frac{1}{2}\cdot\frac{dv}{\frac{1}{2}-v^2}=\frac{dx}{x}$$

$$\Rightarrow\frac{1}{2}\cdot\left[\frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2-v^2}\right]=\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\cdot\frac{1}{2\times\frac{1}{\sqrt{2}}}\log\left|\frac{\frac{1}{\sqrt{2}}+v}{\frac{1}{\sqrt{2}}-v}\right|=\log|x|+C$$

$$\Rightarrow\frac{1}{2\sqrt{2}}\log\left|\frac{\frac{1}{\sqrt{2}}+\frac{y}{x}}{\frac{1}{\sqrt{2}}-\frac{y}{x}}\right|=\log|x|+C$$

$$\Rightarrow\frac{1}{2\sqrt{2}}\log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right|=\log|x|+C$$

This is the required solution for the given differential equation.

Solution 6

$$xdy-ydx=\sqrt{x^2+y^2}\,dx$$

$$\Rightarrow xdy=\left[y+\sqrt{x^2+y^2}\right]dx$$

$$\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x^2} \qquad ...(1)$$

Let $F(x,y)=\frac{y+\sqrt{x^2+y^2}}{x^2}$.

$$\therefore\ F(\lambda x,\lambda y)=\frac{\lambda x+\sqrt{(\lambda x)^2+(\lambda y)^2}}{\lambda x}=\frac{y+\sqrt{x^2+y^2}}{x}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y=vx$

$$\Rightarrow\frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of v and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + (vx)^2}}{x}$$
$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$
$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left|v + \sqrt{1+v^2}\right| = \log|x| + \log C$$
$$\Rightarrow \log\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right| = \log|Cx|$$
$$\Rightarrow \log\left|\frac{y + \sqrt{x^2+y^2}}{x}\right| = \log|Cx|$$
$$\Rightarrow y + \sqrt{x^2+y^2} = Cx^2$$

This is the required solution of the given differential equation.

Solution 7

The given differential equation is:

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdy$$

$$\frac{dy}{dx}=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x} \qquad ...(1)$$

Let $F(x,y)=\dfrac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$.

$$\therefore\ F(\lambda x,\lambda y)=\frac{\left\{\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)+\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda y}{\left\{\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)-\lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda x}$$

$$=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x = \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{(x\cos v + vx\sin v)\cdot vx}{(vx\sin v - x\cos v)\cdot x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \left[\frac{v\sin v - \cos v}{v\cos v}\right]dv = \frac{2dx}{x}$$

$$\Rightarrow \left(\tan v - \frac{1}{v}\right)dv = \frac{2dx}{x}$$

Integrating both sides, we get:

$$\log(\sec v) - \log v = 2\log x + \log C$$
$$\Rightarrow \log\left(\frac{\sec v}{v}\right) = \log\left(Cx^2\right)$$
$$\Rightarrow \left(\frac{\sec v}{v}\right) = Cx^2$$
$$\Rightarrow \sec v = Cx^2 v$$
$$\Rightarrow \sec\left(\frac{y}{x}\right) = C \cdot x^2 \cdot \frac{y}{x}$$
$$\Rightarrow \sec\left(\frac{y}{x}\right) = Cxy$$
$$\Rightarrow \cos\left(\frac{y}{x}\right) = \frac{1}{Cxy} = \frac{1}{C} \cdot \frac{1}{xy}$$
$$\Rightarrow xy\cos\left(\frac{y}{x}\right) = k \qquad \left(k = \frac{1}{C}\right)$$

This is the required solution of the given differential equation.

Solution 8

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$
$$\Rightarrow x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$
$$\Rightarrow \frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x} \qquad \ldots(1)$$

Let $F(x, y) = \dfrac{y - x\sin\left(\frac{y}{x}\right)}{x}$.

$$\therefore\ F(\lambda x, \lambda y) = \frac{\lambda y - \lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$
$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{vx - x\sin v}{x}$$
$$\Rightarrow v + x\frac{dv}{dx} = v - \sin v$$
$$\Rightarrow -\frac{dv}{\sin v} = \frac{dx}{x}$$
$$\Rightarrow \operatorname{cosec} v\,dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left|\operatorname{cosec} v - \cot v\right| = -\log x + \log C = \log\frac{C}{x}$$
$$\Rightarrow \operatorname{cosec}\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right) = \frac{C}{x}$$
$$\Rightarrow \frac{1}{\sin\left(\frac{y}{x}\right)} - \frac{\cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)} = \frac{C}{x}$$
$$\Rightarrow x\left[1 - \cos\left(\frac{y}{x}\right)\right] = C\sin\left(\frac{y}{x}\right)$$

This is the required solution of the given differential equation.

Solution 9

$$y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$$

$$\Rightarrow y\,dx = \left[2x - x\log\left(\frac{y}{x}\right)\right]dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \qquad \ldots(1)$$

$$\text{Let } F(x, y) = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}.$$

$$\therefore\ F(\lambda x, \lambda y) = \frac{\lambda y}{2(\lambda x) - (\lambda x)\log\left(\frac{\lambda y}{\lambda x}\right)} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx)$$
$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log v}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{2 - \log v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$
$$\Rightarrow x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{v\log v - v}{2 - \log v}$$
$$\Rightarrow \frac{2 - \log v}{v(\log v - 1)}dv = \frac{dx}{x}$$
$$\Rightarrow \left[\frac{1 + (1 - \log v)}{v(\log v - 1)}\right]dv = \frac{dx}{x}$$

$$\Rightarrow\left[\frac{1}{v(\log v-1)}-\frac{1}{v}\right]dv=\frac{dx}{x}$$

Integrating both sides, we get:

$$\int\frac{1}{v(\log v-1)}\,dv-\int\frac{1}{v}\,dv=\int\frac{1}{x}\,dx$$

$$\Rightarrow\int\frac{dv}{v(\log v-1)}-\log v=\log x+\log \mathrm{C} \qquad ...(2)$$

$$\Rightarrow \text{Let } \log v-1=t$$

$$\Rightarrow\frac{d}{dv}(\log v-1)=\frac{dt}{dv}$$

$$\Rightarrow\frac{1}{v}=\frac{dt}{dv}$$

$$\Rightarrow\frac{dv}{v}=dt$$

Therefore, equation (1) becomes:

$$\Rightarrow\int\frac{dt}{t}-\log v=\log x+\log \mathrm{C}$$

$$\Rightarrow\log t-\log\left(\frac{y}{x}\right)=\log(\mathrm{C}x)$$

$$\Rightarrow\log\left[\log\left(\frac{y}{x}\right)-1\right]-\log\left(\frac{y}{x}\right)=\log(\mathrm{C}x)$$

$$\Rightarrow\log\left[\frac{\log\left(\frac{y}{x}\right)-1}{\frac{y}{x}}\right]=\log(\mathrm{C}x)$$

$$\Rightarrow\frac{x}{y}\left[\log\left(\frac{y}{x}\right)-1\right]=\mathrm{C}x$$

$$\Rightarrow\log\left(\frac{y}{x}\right)-1=\mathrm{C}y$$

This is the required solution of the given differential equation.

Solution 10

$$\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$$

$$\Rightarrow\left(1+e^{\frac{x}{y}}\right)dx=-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy$$

$$\Rightarrow\frac{dx}{dy}=\frac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}\qquad\ldots(1)$$

Let $F(x,y)=\dfrac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}$.

$$\therefore\ F(\lambda x,\lambda y)=\frac{-e^{\frac{\lambda x}{\lambda y}}\left(1-\frac{\lambda x}{\lambda y}\right)}{1+e^{\frac{\lambda x}{\lambda y}}}=\frac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$x = vy$

$$\Rightarrow \frac{d}{dy}(x) = \frac{d}{dy}(vy)$$

$$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the values of x and $\frac{dx}{dy}$ in equation (1), we get:

$$v + y\frac{dv}{dy} = \frac{-e^{v}(1-v)}{1+e^{v}}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^{v} + ve^{v}}{1+e^{v}} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^{v} + ve^{v} - v - ve^{v}}{1+e^{v}}$$

$$\Rightarrow y\frac{dv}{dy} = -\left[\frac{v+e^{v}}{1+e^{v}}\right]$$

$$\Rightarrow \left[\frac{1+e^{v}}{v+e^{v}}\right]dv = -\frac{dy}{y}$$

Integrating both sides, we get:

$$\Rightarrow \log\left(v+e^{v}\right) = -\log y + \log C = \log\left(\frac{C}{y}\right)$$

$$\Rightarrow \left[\frac{x}{y} + e^{\frac{x}{y}}\right] = \frac{C}{y}$$

$$\Rightarrow x + ye^{\frac{x}{y}} = C$$

This is the required solution of the given differential equation.

Solution 11

$$(x+y)dy+(x-y)dx=0$$
$$\Rightarrow (x+y)dy=-(x-y)dx$$
$$\Rightarrow \frac{dy}{dx}=\frac{-(x-y)}{x+y} \qquad ...(1)$$
Let $F(x,y)=\dfrac{-(x-y)}{x+y}$.
$$\therefore\ F(\lambda x,\lambda y)=\frac{-(\lambda x-\lambda y)}{\lambda x-\lambda y}=\frac{-(x-y)}{x+y}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y=vx$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$
$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\dfrac{dy}{dx}$ in equation (1), we get:

$$v+x\frac{dv}{dx}=\frac{-(x-vx)}{x+vx}$$
$$\Rightarrow v+x\frac{dv}{dx}=\frac{v-1}{v+1}$$
$$\Rightarrow x\frac{dv}{dx}=\frac{v-1}{v+1}-v=\frac{v-1-v(v+1)}{v+1}$$
$$\Rightarrow x\frac{dv}{dx}=\frac{v-1-v^2-v}{v+1}=\frac{-(1+v^2)}{v+1}$$
$$\Rightarrow \frac{(v+1)}{1+v^2}dv=-\frac{dx}{x}$$
$$\Rightarrow \left[\frac{v}{1+v^2}+\frac{1}{1+v^2}\right]dv=-\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\log\left(1+v^2\right)+\tan^{-1}v=-\log x+k$$
$$\Rightarrow \log\left(1+v^2\right)+2\tan^{-1}v=-2\log x+2k$$
$$\Rightarrow \log\left[\left(1+v^2\right)\cdot x^2\right]+2\tan^{-1}v=2k$$
$$\Rightarrow \log\left[\left(1+\frac{y^2}{x^2}\right)\cdot x^2\right]+2\tan^{-1}\frac{y}{x}=2k$$
$$\Rightarrow \log\left(x^2+y^2\right)+2\tan^{-1}\frac{y}{x}=2k \qquad ...(2)$$

Now, $y=1$ at $x=1$.

$$\Rightarrow \log 2+2\tan^{-1}1=2k$$
$$\Rightarrow \log 2+2\times\frac{\pi}{4}=2k$$
$$\Rightarrow \frac{\pi}{2}+\log 2=2k$$

Substituting the value of $2k$ in equation (2), we get:

$$\log\left(x^2+y^2\right)+2\tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{2}+\log 2$$

This is the required solution of the given differential equation.

Solution 12

$$x^2 dy + \left(xy + y^2\right) dx = 0$$
$$\Rightarrow x^2 dy = -\left(xy + y^2\right) dx$$
$$\Rightarrow \frac{dy}{dx} = \frac{-\left(xy + y^2\right)}{x^2} \qquad \ldots(1)$$
Let $F(x, y) = \dfrac{-\left(xy + y^2\right)}{x^2}$.
$$\therefore\ F(\lambda x, \lambda y) = \frac{\left[\lambda x \cdot \lambda y + (\lambda y)^2\right]}{(\lambda x)^2} = \frac{-\left(xy + y^2\right)}{x^2} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$
$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{-\left[x \cdot vx + (vx)^2\right]}{x^2} = -v - v^2$$
$$\Rightarrow x\frac{dv}{dx} = -v^2 - 2v = -v(v+2)$$
$$\Rightarrow \frac{dv}{v(v+2)} = -\frac{dx}{x}$$
$$\Rightarrow \frac{1}{2}\left[\frac{(v+2)-v}{v(v+2)}\right]dv = -\frac{dx}{x}$$
$$\Rightarrow \frac{1}{2}\left[\frac{1}{v} - \frac{1}{v+2}\right]dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\left[\log v - \log(v+2)\right] = -\log x + \log \mathrm{C}$$
$$\Rightarrow \frac{1}{2}\log\left(\frac{v}{v+2}\right) = \log\frac{\mathrm{C}}{x}$$
$$\Rightarrow \frac{v}{v+2} = \left(\frac{\mathrm{C}}{x}\right)^2$$

$$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+2} = \left(\frac{C}{x}\right)^2$$

$$\Rightarrow \frac{y}{y+2x} = \frac{C^2}{x^2}$$

$$\Rightarrow \frac{x^2 y}{y+2x} = C^2 \qquad \ldots(2)$$

Now, $y = 1$ at $x = 1$.

$$\Rightarrow \frac{1}{1+2} = C^2$$

$$\Rightarrow C^2 = \frac{1}{3}$$

Substituting $C^2 = \frac{1}{3}$ in equation (2), we get:

$$\frac{x^2 y}{y+2x} = \frac{1}{3}$$

$$\Rightarrow y + 2x = 3x^2 y$$

This is the required solution of the given differential equation.

Solution 13

$$\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx+xdy=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\left[x\sin^2\left(\frac{y}{x}\right)-y\right]}{x} \qquad \ldots(1)$$

Let $F(x,y)=\dfrac{-\left[x\sin^2\left(\frac{y}{x}\right)-y\right]}{x}$.

$$\therefore\ F(\lambda x,\lambda y)=\frac{-\left[\lambda x\cdot\sin^2\left(\frac{\lambda x}{\lambda y}\right)-\lambda y\right]}{\lambda x}=\frac{-\left[x\sin^2\left(\frac{y}{x}\right)-y\right]}{x}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve this differential equation, we make the substitution as:

$y=vx$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v+x\frac{dv}{dx}=\frac{-\left[x\sin^2 v-vx\right]}{x}$$
$$\Rightarrow v+x\frac{dv}{dx}=-\left[\sin^2 v-v\right]=v-\sin^2 v$$
$$\Rightarrow x\frac{dv}{dx}=-\sin^2 v$$
$$\Rightarrow \frac{dv}{\sin^2 v}=-\frac{dx}{x}$$
$$\Rightarrow \operatorname{cosec}^2 v\,dv=-\frac{dx}{x}$$

Integrating both sides, we get:

$$-\cot v=-\log|x|-C$$
$$\Rightarrow \cot v=\log|x|+C$$
$$\Rightarrow \cot\left(\frac{y}{x}\right)=\log|x|+\log C$$
$$\Rightarrow \cot\left(\frac{y}{x}\right)=\log|Cx| \qquad ...(2)$$

Now, $y=\frac{\pi}{4}$ at $x=1$.

$$\Rightarrow \cot\left(\frac{\pi}{4}\right)=\log|C|$$
$$\Rightarrow 1=\log C$$
$$\Rightarrow C=e^1=e$$

Substituting $C=e$ in equation (2), we get:

$$\cot\left(\frac{y}{x}\right)=\log|ex|$$

This is the required solution of the given differential equation.

Solution 14

$$\frac{dy}{dx}-\frac{y}{x}+\operatorname{cosec}\left(\frac{y}{x}\right)=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}-\operatorname{cosec}\left(\frac{y}{x}\right) \qquad \ldots(1)$$

$$\text{Let } F(x,y)=\frac{y}{x}-\operatorname{cosec}\left(\frac{y}{x}\right).$$

$$\therefore\ F(\lambda x,\lambda y)=\frac{\lambda y}{\lambda x}-\operatorname{cosec}\left(\frac{\lambda y}{\lambda x}\right)$$

$$\Rightarrow\ F(\lambda x,\lambda y)=\frac{y}{x}-\operatorname{cosec}\left(\frac{y}{x}\right)=F(x,y)=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y=vx$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = v - \text{cosec}\, v$$
$$\Rightarrow -\frac{dv}{\text{cosec}\, v} = -\frac{dx}{x}$$
$$\Rightarrow -\sin v dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$\cos v = \log x + \log \text{C} = \log|\text{C}x|$$
$$\Rightarrow \cos\left(\frac{y}{x}\right) = \log|\text{C}x| \qquad ...(2)$$

This is the required solution of the given differential equation.

Now, $y = 0$ at $x = 1$.

$$\Rightarrow \cos(0) = \log \text{C}$$
$$\Rightarrow 1 = \log \text{C}$$
$$\Rightarrow \text{C} = e^1 = e$$

Substituting $\text{C} = e$ in equation (2), we get:

$$\cos\left(\frac{y}{x}\right) = \log|(ex)|$$

This is the required solution of the given differential equation

Solution 15

$$2xy+y^2-2x^2\frac{dy}{dx}=0$$

$$\Rightarrow 2x^2\frac{dy}{dx}=2xy+y^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{2xy+y^2}{2x^2} \qquad \ldots(1)$$

Let $F(x,y)=\dfrac{2xy+y^2}{2x^2}$.

$$\therefore\ F(\lambda x,\lambda y)=\frac{2(\lambda x)(\lambda y)+(\lambda y)^2}{2(\lambda x)^2}=\frac{2xy+y^2}{2x^2}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y=vx$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{2x(vx)+(vx)^2}{2x^2}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{2v+v^2}{2}$$
$$\Rightarrow v + x\frac{dv}{dx} = v + \frac{v^2}{2}$$
$$\Rightarrow \frac{2}{v^2}dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$2\cdot\frac{v^{-2+1}}{-2+1} = \log|x| + C$$
$$\Rightarrow -\frac{2}{v} = \log|x| + C$$
$$\Rightarrow -\frac{2}{\frac{y}{x}} = \log|x| + C \quad .... (2)$$

Now, $y = 2$ at $x = 1$.

$$\Rightarrow -1 = \log(1) + C$$
$$\Rightarrow C = -1$$

Substituting $C = -1$ in equation (2), we get:

$$-\frac{2x}{y} = \log|x| - 1$$
$$\Rightarrow \frac{2x}{y} = 1 - \log|x|$$
$$\Rightarrow y = \frac{2x}{1-\log|x|},\ (x \neq 0,\ x \neq e)$$

This is the required solution of the given differential equation.

Solution 16

For solving the homogeneous equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$, we need to make the substitution as $x = vy$.

Hence, the correct answer is C.

Solution 17

Function F(x, y) is said to be the homogenous function of degree n, if

$F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any non-zero constant (λ).

Consider the equation given in alternativeD:

$$y^2dx + \left(x^2 - xy - y^2\right)dy = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{-y^2}{x^2 - xy - y^2} = \frac{y^2}{y^2 + xy - x^2}$$
$$\text{Let } F(x, y) = \frac{y^2}{y^2 + xy - x^2}.$$
$$\Rightarrow F(\lambda x, \lambda y) = \frac{(\lambda y)^2}{(\lambda y)^2 + (\lambda x)(\lambda y) - (\lambda x)^2}$$
$$= \frac{\lambda^2 y^2}{\lambda^2\left(y^2 + xy - x^2\right)}$$
$$= \lambda^0\left(\frac{y^2}{y^2 + xy - x^2}\right)$$
$$= \lambda^0 \cdot F(x, y)$$

Hence, the differential equation given in alternative D is a homogenous equation.

Chapter 9 - Differential Equations Exercise Ex. 9.6
Solution 1

The given differential equation is $\frac{dy}{dx}+2y=\sin x$.

This is in the form of $\frac{dy}{dx}+py=Q$ (where $p=2$ and $Q=\sin x$).

Now, I.F $=e^{\int p\,dx}=e^{\int 2\,dx}=e^{2x}$.

The solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})dx+\text{C}$$
$$\Rightarrow ye^{2x}=\int\sin x\cdot e^{2x}dx+\text{C} \qquad ...(1)$$

Let $I=\int\sin x\cdot e^{2x}$.

$$\Rightarrow I=\sin x\cdot\int e^{2x}dx-\int\left(\frac{d}{dx}(\sin x)\cdot\int e^{2x}dx\right)dx$$
$$\Rightarrow I=\sin x\cdot\frac{e^{2x}}{2}-\int\left(\cos x\cdot\frac{e^{2x}}{2}\right)dx$$
$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x\cdot\int e^{2x}-\int\left(\frac{d}{dx}(\cos x)\cdot\int e^{2x}dx\right)dx\right]$$
$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x\cdot\frac{e^{2x}}{2}-\int\left[(-\sin x)\cdot\frac{e^{2x}}{2}\right]dx\right]$$
$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}-\frac{1}{4}\int\left(\sin x.e^{2x}\right)dx$$
$$\Rightarrow I=\frac{e^{2x}}{4}(2\sin x-\cos x)-\frac{1}{4}I$$
$$\Rightarrow\frac{5}{4}I=\frac{e^{2x}}{4}(2\sin x-\cos x)$$
$$\Rightarrow I=\frac{e^{2x}}{5}(2\sin x-\cos x)$$

Therefore, equation (1) becomes:

$$ye^{2x}=\frac{e^{2x}}{5}(2\sin x-\cos x)+\text{C}$$
$$\Rightarrow y=\frac{1}{5}(2\sin x-\cos x)+\text{C}e^{-2x}$$

This is the required general solution of the given differential equation.

Solution 2

The given differential equation is $\frac{dy}{dx}+py=Q$ (where $p=3$ and $Q=e^{-2x}$).

Now, I.F $=e^{\int p\,dx}=e^{\int 3\,dx}=e^{3x}$.

The solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})\,dx+\text{C}$$
$$\Rightarrow ye^{3x}=\int\left(e^{-2x}\times e^{3x}\right)+\text{C}$$
$$\Rightarrow ye^{3x}=\int e^{x}\,dx+\text{C}$$
$$\Rightarrow ye^{3x}=e^{x}+\text{C}$$
$$\Rightarrow y=e^{-2x}+\text{C}e^{-3x}$$

This is the required general solution of the given differential equation.

Solution 3

The given differential equation is:

$$\frac{dy}{dx}+py=Q \text{ (where } p=\frac{1}{x} \text{ and } Q=x^2\text{)}$$

Now, I.F $=e^{\int p\,dx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$.

The solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})\,dx+\text{C}$$
$$\Rightarrow y(x)=\int\left(x^2\cdot x\right)dx+\text{C}$$
$$\Rightarrow xy=\int x^3\,dx+\text{C}$$
$$\Rightarrow xy=\frac{x^4}{4}+\text{C}$$

This is the required general solution of the given differential equation.

Solution 4

The given differential equation is:

$$\frac{dy}{dx}+py=Q \text{ (where } p=\sec x \text{ and } Q=\tan x)$$

Now, I.F $=e^{\int p\,dx}=e^{\int \sec x\,dx}=e^{\log(\sec x+\tan x)}=\sec x+\tan x.$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})dx+\text{C}$$

$$\Rightarrow y(\sec x+\tan x)=\int\tan x(\sec x+\tan x)dx+\text{C}$$

$$\Rightarrow y(\sec x+\tan x)=\int\sec x\tan x dx+\int\tan^2 x dx+\text{C}$$

$$\Rightarrow y(\sec x+\tan x)=\sec x+\int(\sec^2 x-1)dx+\text{C}$$

$$\Rightarrow y(\sec x+\tan x)=\sec x+\tan x-x+\text{C}$$

Solution 5

The given differential equation is:

$$\cos^2 x\frac{dy}{dx}+y=\tan x$$

$$\Rightarrow \frac{dy}{dx}+\sec^2 x\cdot y=\sec^2 x\tan x$$

This equation is in the form of:

$$\frac{dy}{dx}+py=Q \text{ (where } p=\sec^2 x \text{ and } Q=\sec^2 x\tan x)$$

Now, I.F $=e^{\int p\,dx}=e^{\int \sec^2 x\,dx}=e^{\tan x}.$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})dx+\text{C}$$

$$\Rightarrow y\cdot e^{\tan x}=\int e^{\tan x}\cdot\sec^2 x\tan x\,dx+\text{C} \qquad ...(1)$$

Let $\tan x=t.$

$$\Rightarrow \frac{d}{dx}(\tan x)=\frac{dt}{dx}$$

$$\Rightarrow \sec^2 x=\frac{dt}{dx}$$

$$\Rightarrow \sec^2 x\,dx=dt$$

Therefore, equation (1) becomes:

$$y \cdot e^{\tan x} = \int \left(e^t \cdot t\right) dt + \text{C}$$
$$\Rightarrow y \cdot e^{\tan x} = \int \left(t \cdot e^t\right) dt + \text{C}$$
$$\Rightarrow y \cdot e^{\tan x} = t \cdot \int e^t dt - \int \left(\frac{d}{dt}(t) \cdot \int e^t dt\right) dt + \text{C}$$
$$\Rightarrow y \cdot e^{\tan x} = t \cdot e^t - \int e^t dt + \text{C}$$
$$\Rightarrow ye^{\tan x} = (t-1)e^t + \text{C}$$
$$\Rightarrow ye^{\tan x} = (\tan x - 1)e^{\tan x} + \text{C}$$
$$\Rightarrow y = (\tan x - 1) + \text{C}e^{-\tan x}$$

Solution 6

The given differential equation is:

$$x\frac{dy}{dx} + 2y = x^2 \log x$$
$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = x \log x$$

This equation is in the form of a linear differential equation as:

$$\frac{dy}{dx} + py = Q \text{ (where } p = \frac{2}{x} \text{ and } Q = x \log x)$$

Now, I.F $= e^{\int p\,dx} = e^{\int \frac{2}{x} dx} = e^{2\log x} = e^{\log x^2} = x^2$.

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\, dx + \text{C}$$

$$\Rightarrow y \cdot x^2 = \int \left(x \log x \cdot x^2 \right) dx + \text{C}$$

$$\Rightarrow x^2 y = \int \left(x^3 \log x \right) dx + \text{C}$$

$$\Rightarrow x^2 y = \log x \cdot \int x^3 dx - \int \left[\frac{d}{dx} (\log x) \cdot \int x^3 dx \right] dx + \text{C}$$

$$\Rightarrow x^2 y = \log x \cdot \frac{x^4}{4} - \int \left(\frac{1}{x} \cdot \frac{x^4}{4} \right) dx + \text{C}$$

$$\Rightarrow x^2 y = \frac{x^4 \log x}{4} - \frac{1}{4} \int x^3 dx + \text{C}$$

$$\Rightarrow x^2 y = \frac{x^4 \log x}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + \text{C}$$

$$\Rightarrow x^2 y = \frac{1}{16} x^4 \left(4 \log x - 1 \right) + \text{C}$$

$$\Rightarrow y = \frac{1}{16} x^2 \left(4 \log x - 1 \right) + \text{C} x^{-2}$$

Solution 7

The given differential equation is:

$$x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x^2}$$

This equation is the form of a linear differential equation as:

$$\frac{dy}{dx} + py = Q \text{ (where } p = \frac{1}{x \log x} \text{ and } Q = \frac{2}{x^2})$$

$$\text{Now, I.F } = e^{\int p dx} = e^{\int \frac{1}{x \log} dx} = e^{\log(\log x)} = \log x.$$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\, dx + \text{C}$$

$$\Rightarrow y \log x = \int \left(\frac{2}{x^2} \log x \right) dx + \text{C} \qquad \ldots(1)$$

$$\text{Now, } \int\left(\frac{2}{x^2}\log x\right)dx = 2\int\left(\log x \cdot \frac{1}{x^2}\right)dx.$$

$$= 2\left[\log x \cdot \int\frac{1}{x^2}dx - \int\left\{\frac{d}{dx}(\log x)\cdot\int\frac{1}{x^2}dx\right\}dx\right]$$

$$= 2\left[\log x\left(-\frac{1}{x}\right) - \int\left(\frac{1}{x}\cdot\left(-\frac{1}{x}\right)\right)dx\right]$$

$$= 2\left[-\frac{\log x}{x} + \int\frac{1}{x^2}dx\right]$$

$$= 2\left[-\frac{\log x}{x} - \frac{1}{x}\right]$$

$$= -\frac{2}{x}(1+\log x)$$

Substituting the value of $\int\left(\frac{2}{x^2}\log x\right)dx$ in equation (1), we get:

$$y\log x = -\frac{2}{x}(1+\log x) + \text{C}$$

This is the required general solution of the given differential equation.

Solution 8

$$\left(1+x^2\right)dy + 2xy\ dx = \cot x dx$$

$$\Rightarrow \frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{\cot x}{1+x^2}$$

This equation is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \text{ (where } p = \frac{2x}{1+x^2} \text{ and } Q = \frac{\cot x}{1+x^2}\text{)}$$

$$\text{Now, I.F } = e^{\int p\,dx} = e^{\int\frac{2x}{1+x^2}dx} = e^{\log\left(1+x^2\right)} = 1+x^2.$$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int(\text{Q}\times\text{I.F.})dx + \text{C}$$

$$\Rightarrow y\left(1+x^2\right) = \int\left[\frac{\cot x}{1+x^2}\times\left(1+x^2\right)\right]dx + \text{C}$$

$$\Rightarrow y\left(1+x^2\right) = \int\cot x dx + \text{C}$$

$$\Rightarrow y\left(1+x^2\right) = \log|\sin x| + \text{C}$$

Solution 9

$$x\frac{dy}{dx}+y-x+xy\cot x=0$$

$$\Rightarrow x\frac{dy}{dx}+y\left(1+x\cot x\right)=x$$

$$\Rightarrow \frac{dy}{dx}+\left(\frac{1}{x}+\cot x\right)y=1$$

This equation is a linear differential equation of the form:

$$\frac{dy}{dx}+py=Q \text{ (where } p=\frac{1}{x}+\cot x \text{ and } Q=1)$$

Now, I.F $=e^{\int pdx}=e^{\int\left(\frac{1}{x}+\cot x\right)dx}=e^{\log x+\log(\sin x)}=e^{\log(x\sin x)}=x\sin x.$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})dx+\text{C}$$

$$\Rightarrow y\left(x\sin x\right)=\int\left(1\times x\sin x\right)dx+\text{C}$$

$$\Rightarrow y\left(x\sin x\right)=\int\left(x\sin x\right)dx+\text{C}$$

$$\Rightarrow y\left(x\sin x\right)=x\int\sin x\,dx-\int\left[\frac{d}{dx}(x)\cdot\int\sin x\,dx\right]+\text{C}$$

$$\Rightarrow y\left(x\sin x\right)=x\left(-\cos x\right)-\int 1\cdot\left(-\cos x\right)dx+\text{C}$$

$$\Rightarrow y\left(x\sin x\right)=-x\cos x+\sin x+\text{C}$$

$$\Rightarrow y=\frac{-x\cos x}{x\sin x}+\frac{\sin x}{x\sin x}+\frac{\text{C}}{x\sin x}$$

$$\Rightarrow y=-\cot\cdot x+\frac{1}{x}+\frac{\text{C}}{x\sin x}$$

Solution 10

$$(x+y)\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x+y}$$

$$\Rightarrow \frac{dx}{dy}=x+y$$

$$\Rightarrow \frac{dx}{dy}-x=y$$

This is a linear differential equation of the form:

$\frac{dy}{dx}+px=Q$ (where $p=-1$ and $Q=y$)

Now, I.F $=e^{\int p\,dy}=e^{\int -dy}=e^{-y}$.

The general solution of the given differential equation is given by the relation,

$$x(\text{I.F.})=\int(\text{Q}\times\text{I.F.})dy+\text{C}$$

$$\Rightarrow xe^{-y}=\int\left(y\cdot e^{-y}\right)dy+\text{C}$$

$$\Rightarrow xe^{-y}=y\cdot\int e^{-y}dy-\int\left[\frac{d}{dy}(y)\int e^{-y}dy\right]dy+\text{C}$$

$$\Rightarrow xe^{-y}=y\left(-e^{-y}\right)-\int\left(-e^{-y}\right)dy+\text{C}$$

$$\Rightarrow xe^{-y}=-ye^{-y}+\int e^{-y}dy+\text{C}$$

$$\Rightarrow xe^{-y}=-ye^{-y}-e^{-y}+\text{C}$$

$$\Rightarrow x=-y-1+\text{C}e^{y}$$

$$\Rightarrow x+y+1=\text{C}e^{y}$$

Solution 11

$$y\ dx+\left(x-y^2\right)dy=0$$
$$\Rightarrow ydx=\left(y^2-x\right)dy$$
$$\Rightarrow \frac{dx}{dy}=\frac{y^2-x}{y}=y-\frac{x}{y}$$
$$\Rightarrow \frac{dx}{dy}+\frac{x}{y}=y$$

This is a linear differential equation of the form:

$$\frac{dy}{dx}+px=Q \text{ (where } p=\frac{1}{y} \text{ and } Q=y)$$

Now, I.F $=e^{\int p\,dy}=e^{\int \frac{1}{y}dy}=e^{\log y}=y.$

The general solution of the given differential equation is given by the relation,

$$x(\text{I.F.})=\int(\text{Q}\times\text{I.F.})dy+\text{C}$$
$$\Rightarrow xy=\int(y\cdot y)dy+\text{C}$$
$$\Rightarrow xy=\int y^2dy+\text{C}$$
$$\Rightarrow xy=\frac{y^3}{3}+\text{C}$$
$$\Rightarrow x=\frac{y^2}{3}+\frac{\text{C}}{y}$$

Solution 12

$$\left(x+3y^2\right)\frac{dy}{dx}=y$$
$$\Rightarrow \frac{dy}{dx}=\frac{y}{x+3y^2}$$
$$\Rightarrow \frac{dx}{dy}=\frac{x+3y^2}{y}=\frac{x}{y}+3y$$
$$\Rightarrow \frac{dx}{dy}-\frac{x}{y}=3y$$

This is a linear differential equation of the form:

$$\frac{dx}{dy}+px=Q \text{ (where } p=-\frac{1}{y} \text{ and } Q=3y)$$
$$\text{Now, I.F } =e^{\int p\,dy}=e^{-\int\frac{dy}{y}}=e^{-\log y}=e^{\log\left(\frac{1}{y}\right)}=\frac{1}{y}.$$

The general solution of the given differential equation is given by the relation,

$$x\left(\text{I.F.}\right)=\int\left(\text{Q}\times\text{I.F.}\right)dy+\text{C}$$
$$\Rightarrow x\times\frac{1}{y}=\int\left(3y\times\frac{1}{y}\right)dy+\text{C}$$
$$\Rightarrow \frac{x}{y}=3y+\text{C}$$
$$\Rightarrow x=3y^2+\text{C}y$$

Solution 13

The given differential equation is $\frac{dy}{dx} + 2y\tan x = \sin x$.

This is a linear equation of the form:

$$\frac{dy}{dx} + py = Q \text{ (where } p = 2\tan x \text{ and } Q = \sin x\text{)}$$

$$\text{Now, I.F } = e^{\int p\,dx} = e^{\int 2\tan x\,dx} = e^{2\log|\sec x|} = e^{\log(\sec^2 x)} = \sec^2 x.$$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\,dx + \text{C}$$
$$\Rightarrow y(\sec^2 x) = \int (\sin x \cdot \sec^2 x)\,dx + \text{C}$$
$$\Rightarrow y\sec^2 x = \int (\sec x \cdot \tan x)\,dx + \text{C}$$
$$\Rightarrow y\sec^2 x = \sec x + \text{C} \qquad ...(1)$$

Now, $y = 0$ at $x = \frac{\pi}{3}$.

Therefore,

$$0 \times \sec^2\frac{\pi}{3} = \sec\frac{\pi}{3} + \text{C}$$
$$\Rightarrow 0 = 2 + \text{C}$$
$$\Rightarrow \text{C} = -2$$

Substituting $\text{C} = -2$ in equation (1), we get:

$$y\sec^2 x = \sec x - 2$$
$$\Rightarrow y = \cos x - 2\cos^2 x$$

Hence, the required solution of the given differential equation is $y = \cos x - 2\cos^2 x$.

Solution 14

$$\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$$
$$\Rightarrow \frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{1}{\left(1+x^2\right)^2}$$

This is a linear differential equation of the form:

$$\frac{dy}{dx}+py=Q \text{ (where } p=\frac{2x}{1+x^2} \text{ and } Q=\frac{1}{\left(1+x^2\right)^2}\text{)}$$

Now, I.F $=e^{\int p\,dx}=e^{\int\frac{2x\,dx}{1+x^2}}=e^{\log\left(1+x^2\right)}=1+x^2.$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})\,dx+\text{C}$$
$$\Rightarrow y\left(1+x^2\right)=\int\left[\frac{1}{\left(1+x^2\right)^2}\cdot\left(1+x^2\right)\right]dx+\text{C}$$
$$\Rightarrow y\left(1+x^2\right)=\int\frac{1}{1+x^2}\,dx+\text{C}$$
$$\Rightarrow y\left(1+x^2\right)=\tan^{-1}x+\text{C} \qquad ...(1)$$

Now, $y=0$ at $x=1$.

Therefore,

$$0=\tan^{-1}1+\text{C}$$
$$\Rightarrow \text{C}=-\frac{\pi}{4}$$

Substituting $\text{C}=-\frac{\pi}{4}$ in equation (1), we get:

$$y\left(1+x^2\right)=\tan^{-1}x-\frac{\pi}{4}$$

This is the required general solution of the given differential equation.

Solution 15

The given differential equation is $\frac{dy}{dx} - 3y\cot x = \sin 2x.$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \text{ (where } p = -3\cot x \text{ and } Q = \sin 2x\text{)}$$

$$\text{Now, I.F } = e^{\int p\,dx} = e^{-3\int \cot x\,dx} = e^{-3\log|\sin x|} = e^{\log\left|\frac{1}{\sin^3 x}\right|} = \frac{1}{\sin^3 x}.$$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\,dx + \text{C}$$
$$\Rightarrow y \cdot \frac{1}{\sin^3 x} = \int \left[\sin 2x . \frac{1}{\sin^3 x}\right] dx + \text{C}$$
$$\Rightarrow y\,\text{cosec}^3 x = 2\int (\cot x \text{cosec} x)\,dx + \text{C}$$
$$\Rightarrow y\,\text{cosec}^3 x = 2\text{cosec}\, x + \text{C}$$
$$\Rightarrow y = -\frac{2}{\text{cosec}^2 x} + \frac{3}{\text{cosec}^3 x}$$
$$\Rightarrow y = -2\sin^2 x + \text{C}\sin^3 x \qquad ...(1)$$

Now, $y = 2$ at $x = \frac{\pi}{2}.$

Therefore, we get:

$$2 = -2 + \text{C}$$
$$\Rightarrow \text{C} = 4$$

Substituting $\text{C} = 4$ in equation (1), we get:

$$y = -2\sin^2 x + 4\sin^3 x$$
$$\Rightarrow y = 4\sin^3 x - 2\sin^2 x$$

This is the required particular solution of the given differential equation.

Solution 16

Let $F(x, y)$ be the curve passing through the origin.

At point (x, y), the slope of the curve will be $\frac{dy}{dx}$.

According to the given information:

$$\frac{dy}{dx} = x + y$$
$$\Rightarrow \frac{dy}{dx} - y = x$$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \text{ (where } p = -1 \text{ and } Q = x\text{)}$$

Now, I.F $= e^{\int p\,dx} = e^{\int(-1)dx} = e^{-x}$.

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int(\text{Q} \times \text{I.F.})\,dx + \text{C}$$
$$\Rightarrow ye^{-x} = \int xe^{-x}dx + \text{C} \qquad \ldots(1)$$

Now, $\int xe^{-x}dx = x\int e^{-x}dx - \int\left[\frac{d}{dx}(x)\cdot\int e^{-x}dx\right]dx.$

$$= -xe^{-x} - \int -e^{-x}dx$$
$$= -xe^{-x} + \left(-e^{-x}\right)$$
$$= -e^{-x}(x+1)$$

Substituting in equation (1), we get:

$$ye^{-x} = -e^{-x}(x+1) + C$$
$$\Rightarrow y = -(x+1) + Ce^{x}$$
$$\Rightarrow x + y + 1 = Ce^{x} \qquad ...(2)$$

The curve passes through the origin.

Therefore, equation (2) becomes:

1 = C

C = 1

Substituting C = 1 in equation (2), we get:

$$x + y + 1 = e^{x}$$

Hence, the required equation of curve passing through the origin is $x + y + 1 = e^{x}$.

Solution 17

Let $F(x, y)$ be the curve and let (x, y) be a point on the curve. The slope of the tangent to the curve at (x, y) is $\frac{dy}{dx}$.

According to the given information:

$$\frac{dy}{dx}+5=x+y$$
$$\Rightarrow \frac{dy}{dx}-y=x-5$$

This is a linear differential equation of the form:

$$\frac{dy}{dx}+py=Q \text{ (where } p=-1 \text{ and } Q=x-5)$$
$$\text{Now, I.F } =e^{\int p dx}=e^{\int(-1)dx}=e^{-x}.$$

The general equation of the curve is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})dx+\text{C}$$
$$\Rightarrow y\cdot e^{-x}=\int(x-5)e^{-x}dx+\text{C} \qquad ...(1)$$

Now, $\int (x-5)e^{-x}dx = (x-5)\int e^{-x}dx - \int\left[\frac{d}{dx}(x-5).\int e^{-x}dx\right]dx.$

$$= (x-5)(-e^{-x}) - \int(-e^{-x})dx$$
$$= (5-x)e^{-x} + (-e^{-x})$$
$$= (4-x)e^{-x}$$

Therefore, equation (1) becomes:

$$ye^{-x} = (4-x)e^{-x} + C$$
$$\Rightarrow y = 4 - x + Ce^{x}$$
$$\Rightarrow x + y - 4 = Ce^{x} \qquad ...(2)$$

The curve passes through point (0, 2).

Therefore, equation (2) becomes:

$$0 + 2 - 4 = Ce^{0}$$
$$\Rightarrow -2 = C$$
$$\Rightarrow C = -2$$

Substituting $C = -2$ in equation (2), we get:

$$x + y - 4 = -2e^{x}$$
$$\Rightarrow y = 4 - x - 2e^{x}$$

This is the required equation of the curve.

Solution 18

The given differential equation is:

$$x\frac{dy}{dx}-y=2x^2$$
$$\Rightarrow \frac{dy}{dx}-\frac{y}{x}=2x$$

This is a linear differential equation of the form:

$$\frac{dy}{dx}+py=Q \text{ (where } p=-\frac{1}{x} \text{ and } Q=2x\text{)}$$

The integrating factor (I.F) is given by the relation,

$$e^{\int p dx}$$

$$\therefore \text{I.F } = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log\left(x^{-1}\right)} = x^{-1} = \frac{1}{x}$$

Hence, the correct answer is C.

Solution 19

The given differential equation is:

$$\left(1-y^2\right)\frac{dx}{dy}+yx=ay$$
$$\Rightarrow \frac{dx}{dy}+\frac{yx}{1-y^2}=\frac{ay}{1-y^2}$$

This is a linear differential equation of the form:

$$\frac{dx}{dy}+py=Q \text{ (where } p=\frac{y}{1-y^2} \text{ and } Q=\frac{ay}{1-y^2}\text{)}$$

The integrating factor (I.F) is given by the relation,

$$e^{\int p dx}$$

$$\therefore \text{I.F } = e^{\int p dy} = e^{\int \frac{y}{1-y^2}dy} = e^{-\frac{1}{2}\log\left(1-y^2\right)} = e^{\log\left[\frac{1}{\sqrt{1-y^2}}\right]} = \frac{1}{\sqrt{1-y^2}}$$

Hence, the correct answer is D.

Chapter 9 - Differential Equations Exercise Misc. Ex.

Solution 1

(i) The differential equation is given as:

$$\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$$

$$\Rightarrow \frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y-\log x=0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$. Thus, its order is two. The highest power raised to $\frac{d^2y}{dx^2}$ is one. Hence, its degree is one.

(ii) The differential equation is given as:

$$\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y-\sin x=0$$

The highest order derivative present in the differential equation is $\frac{dy}{dx}$. Thus, its order is one. The highest power raised to $\frac{dy}{dx}$ is three. Hence, its degree is three.

(iii) The differential equation is given as:

$$\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$$

The highest order derivative present in the differential equation is $\frac{d^4y}{dx^4}$. Thus, its order is four.

However, the given differential equation is not a polynomial equation. Hence, its degree is not defined.

Solution 2

(i)

$xy = ae^x + be^{-x} + x^2$

Differentiating both sides w.r.t x we get

$$\frac{d(xy)}{dx} = a\frac{d}{dx}(e^x) + b\frac{d}{dx}(e^{-x}) + \frac{d}{dx}(2x)$$

$$\Rightarrow x\frac{dy}{dx} + y = ae^x - be^{-x} + 2x$$

Again, differentiating both the sides w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = a\frac{d}{dx}(e^x) - b\frac{d}{dx}(e^{-x}) + 2\frac{d}{dx}(x)$$

$$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = ae^x + be^{-x} + 2$$

$$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - x^2 + 2$$

$$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$$

Hence, the given function is a solution of the corresponding differential equation.

(ii) $y = e^x\left(a\cos x + b\sin x\right) = ae^x\cos x + be^x\sin x$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = a\cdot\frac{d}{dx}\left(e^x\cos x\right) + b\cdot\frac{d}{dx}\left(e^x\sin x\right)$$

$$\Rightarrow \frac{dy}{dx} = a\left(e^x\cos x - e^x\sin x\right) + b\cdot\left(e^x\sin x + e^x\cos x\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(a+b\right)e^x\cos x + \left(b-a\right)e^x\sin x$$

Again, differentiating both sides with respect to x, we get:

$$\frac{d^2y}{dx^2} = \left(a+b\right)\cdot\frac{d}{dx}\left(e^x\cos x\right) + \left(b-a\right)\frac{d}{dx}\left(e^x\sin x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(a+b\right)\cdot\left[e^x\cos x - e^x\sin x\right] + \left(b-a\right)\left[e^x\sin x + e^x\cos x\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x\left[\left(a+b\right)\left(\cos x - \sin x\right) + \left(b-a\right)\left(\sin x + \cos x\right)\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x\left[a\cos x - a\sin x + b\cos x - b\sin x + b\sin x + b\cos x - a\sin x - a\cos x\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left[2e^x\left(b\cos x - a\sin x\right)\right]$$

Now, on substituting the values of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in the L.H.S. of the given differential equation, we get:

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y$$
$$=2e^x(b\cos x-a\sin x)-2e^x\left[(a+b)\cos x+(b-a)\sin x\right]+2e^x(a\cos x+b\sin x)$$
$$=e^x\begin{bmatrix}(2b\cos x-2a\sin x)-(2a\cos x+2b\cos x)\\-(2b\sin x-2a\sin x)+(2a\cos x+2b\sin x)\end{bmatrix}$$
$$=e^x\left[(2b-2a-2b+2a)\cos x\right]+e^x\left[(-2a-2b+2a+2b)\sin x\right]$$
$$=0$$

Hence, the given function is a solution of the corresponding differential equation.

(iii) $y=x\sin 3x$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx}=\frac{d}{dx}(x\sin 3x)=\sin 3x+x\cdot\cos 3x\cdot 3$$
$$\Rightarrow\frac{dy}{dx}=\sin 3x+3x\cos 3x$$

Again, differentiating both sides with respect to x, we get:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(\sin 3x)+3\frac{d}{dx}(x\cos 3x)$$
$$\Rightarrow\frac{d^2y}{dx^2}=3\cos 3x+3\left[\cos 3x+x(-\sin 3x)\cdot 3\right]$$
$$\Rightarrow\frac{d^2y}{dx^2}=6\cos 3x-9x\sin 3x$$

Substituting the value of $\frac{d^2y}{dx^2}$ in the L.H.S. of the given differential equation, we get:

$$\frac{d^2y}{dx^2}+9y-6\cos 3x$$
$$=(6\cdot\cos 3x-9x\sin 3x)+9x\sin 3x-6\cos 3x$$
$$=0$$

Hence, the given function is a solution of the corresponding differential equation.

(iv) $x^2=2y^2\log y$

Differentiating both sides with respect to x, we get:

$$2x=2\cdot\frac{d}{dx}\left[y^2\log y\right]$$
$$\Rightarrow x=\left[2y\cdot\log y\cdot\frac{dy}{dx}+y^2\cdot\frac{1}{y}\cdot\frac{dy}{dx}\right]$$
$$\Rightarrow x=\frac{dy}{dx}(2y\log y+y)$$
$$\Rightarrow \frac{dy}{dx}=\frac{x}{y(1+2\log y)}$$

Substituting the value of $\frac{dy}{dx}$ in the L.H.S. of the given differential equation, we get:

$$\left(x^2+y^2\right)\frac{dy}{dx}-xy$$
$$=\left(2y^2\log y+y^2\right)\cdot\frac{x}{y(1+2\log y)}-xy$$
$$=y^2(1+2\log y)\cdot\frac{x}{y(1+2\log y)}-xy$$
$$=xy-xy$$
$$=0$$

Hence, the given function is a solution of the corresponding differential equation.

Solution 3

$$(x-a)^2+2y^2=a^2$$
$$\Rightarrow x^2+a^2-2ax+2y^2=a^2$$
$$\Rightarrow 2y^2=2ax-x^2 \quad \ldots(1)$$

Differentiating with respect to x, we get:

$$2y\frac{dy}{dx}=\frac{2a-2x}{2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{a-x}{2y}$$
$$\Rightarrow \frac{dy}{dx}=\frac{2ax-2x^2}{4xy} \quad \ldots(2)$$

From equation (1), we get:

$$2ax=2y^2+x^2$$

On substituting this value in equation (2), we get:

$$\frac{dy}{dx}=\frac{2y^2+x^2-2x^2}{4xy}$$
$$\Rightarrow \frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$$

Hence, the differential equation of the family of curves is given as $\frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$.

Solution 4

$$\left(x^3 - 3xy^2\right)dx = \left(y^3 - 3x^2y\right)dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \qquad \ldots(1)$$

This is a homogeneous equation. To simplify it, we need to make the substitution as:

$$y = vx$$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{x^3 - 3x(vx)^2}{(vx)^3 - 3x^2(vx)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - 3v^2 - v\left(v^3 - 3v\right)}{v^3 - 3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - v^4}{v^3 - 3v}$$

$$\Rightarrow \left(\frac{v^3 - 3v}{1 - v^4}\right)dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$\int\left(\frac{v^3-3v}{1-v^4}\right)dv=\log x+\log C' \qquad ...(2)$$

Now, $\int\left(\frac{v^3-3v}{1-v^4}\right)dv=\int\frac{v^3dv}{1-v^4}-3\int\frac{vdv}{1-v^4}$

$$\Rightarrow\int\left(\frac{v^3-3v}{1-v^4}\right)dv=I_1-3I_2,\text{ where } I_1=\int\frac{v^3dv}{1-v^4}\text{ and } I_2=\int\frac{vdv}{1-v^4} \qquad ...(3)$$

Let $1-v^4=t$.

$\therefore \frac{d}{dv}\left(1-v^4\right)=\frac{dt}{dv}$

$\Rightarrow -4v^3=\frac{dt}{dv}$

$\Rightarrow v^3dv=-\frac{dt}{4}$

Now, $I_1=\int\frac{-dt}{4t}=-\frac{1}{4}\log t=-\frac{1}{4}\log\left(1-v^4\right)$

And, $I_2=\int\frac{vdv}{1-v^4}=\int\frac{vdv}{1-\left(v^2\right)^2}$

Let $v^2 = p$.

$$\therefore \frac{d}{dv}\left(v^2\right) = \frac{dp}{dv}$$

$$\Rightarrow 2v = \frac{dp}{dv}$$

$$\Rightarrow vdv = \frac{dp}{2}$$

$$\Rightarrow \; I_2 = \frac{1}{2}\int \frac{dp}{1-p^2} = \frac{1}{2\times 2}\log\left|\frac{1+p}{1-p}\right| = \frac{1}{4}\log\left|\frac{1+v^2}{1-v^2}\right|$$

Substituting the values of I_1 and I_2 in equation (3), we get:

$$\int\left(\frac{v^3-3v}{1-v^4}\right)dv = -\frac{1}{4}\log\left(1-v^4\right) - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right|$$

Therefore, equation (2) becomes:

$$-\frac{1}{4}\log\left(1-v^4\right) - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log x + \log C'$$

$$\Rightarrow -\frac{1}{4}\log\left[\left(1-v^4\right)\left(\frac{1+v^2}{1-v^2}\right)^3\right] = \log C'x$$

$$\Rightarrow \frac{\left(1+v^2\right)^4}{\left(1-v^2\right)^2} = \left(C'x\right)^{-4}$$

$$\Rightarrow \frac{\left(1+\frac{y^2}{x^2}\right)^4}{\left(1-\frac{y^2}{x^2}\right)^2} = \frac{1}{C'^4x^4}$$

$$\Rightarrow \frac{\left(x^2+y^2\right)^4}{x^4\left(x^2-y^2\right)^2} = \frac{1}{C'^4x^4}$$

$$\Rightarrow \left(x^2-y^2\right)^2 = C'^4\left(x^2+y^2\right)^4$$

$$\Rightarrow \left(x^2-y^2\right) = C'^2\left(x^2+y^2\right)^2$$

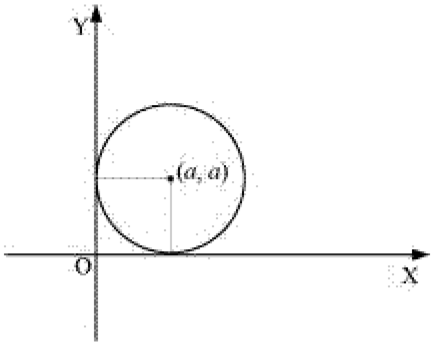
$$\Rightarrow x^2-y^2 = C\left(x^2+y^2\right)^2, \text{ where } C = C'^2$$

Hence, the given result is proved.

Solution 5

The equation of a circle in the first quadrant with centre (a, a) and radius (a) which touches the coordinate axes is:

$$(x-a)^2+(y-a)^2=a^2 \qquad \dots(1)$$



Differentiating equation (1) with respect to x, we get:

$$2(x-a)+2(y-a)\frac{dy}{dx}=0$$
$$\Rightarrow (x-a)+(y-a)y'=0$$
$$\Rightarrow x-a+yy'-ay'=0$$
$$\Rightarrow x+yy'-a(1+y')=0$$
$$\Rightarrow a=\frac{x+yy'}{1+y'}$$

Substituting the value of a in equation (1), we get:

$$\left[x-\left(\frac{x+yy'}{1+y'}\right)\right]^2+\left[y-\left(\frac{x+yy'}{1+y'}\right)\right]^2=\left(\frac{x+yy'}{1+y'}\right)^2$$
$$\Rightarrow \left[\frac{(x-y)y'}{(1+y')}\right]^2+\left[\frac{y-x}{1+y'}\right]^2=\left[\frac{x+yy'}{1+y'}\right]^2$$
$$\Rightarrow (x-y)^2\cdot y'^2+(x-y)^2=(x+yy')^2$$
$$\Rightarrow (x-y)^2\left[1+(y')^2\right]=(x+yy')^2$$

Hence, the required differential equation of the family of circles is $(x-y)^2\left[1+(y')^2\right]=(x+yy')^2$.

Solution 6

$$\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$$
$$\Rightarrow \frac{dy}{dx}=-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$
$$\Rightarrow \frac{dy}{\sqrt{1-y^2}}=\frac{-dx}{\sqrt{1-x^2}}$$

Integrating both sides, we get:

$$\sin^{-1} y=-\sin^{-1} x+\mathrm{C}$$
$$\Rightarrow \sin^{-1}x+\sin^{-1} y=\mathrm{C}$$

Solution 7

$$\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$$
$$\Rightarrow \frac{dy}{dx}=-\frac{\left(y^2+y+1\right)}{x^2+x+1}$$
$$\Rightarrow \frac{dy}{y^2+y+1}=\frac{-dx}{x^2+x+1}$$
$$\Rightarrow \frac{dy}{y^2+y+1}+\frac{dx}{x^2+x+1}=0$$

Integrating both sides, we get:

$$\int\frac{dy}{y^2+y+1}+\int\frac{dx}{x^2+x+1}=\mathrm{C}$$
$$\Rightarrow \int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\mathrm{C}$$
$$\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]+\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]=\mathrm{C}$$

$$\Rightarrow \tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right]+\tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1-\frac{(2y+1)}{\sqrt{3}}\cdot\frac{(2x+1)}{\sqrt{3}}}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2x+2y+2}{\sqrt{3}}}{1-\left(\frac{4xy+2x+2y+1}{3}\right)}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{2\sqrt{3}(x+y+1)}{3-4xy-2x-2y-1}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{2\sqrt{3}(x+y+1)}{2(1-x-y-2xy)}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow \frac{\sqrt{3}(x+y+1)}{(1-x-y-2xy)}=\tan\left(\frac{\sqrt{3}C}{2}\right)=B,\text{ where } B=\tan\left(\frac{\sqrt{3}C}{2}\right)$$

$$\Rightarrow x+y+1=\frac{B}{\sqrt{3}}(1-xy-2xy)$$

$$\Rightarrow x+y+1=A(1-x-y-2xy),\text{ where } A=\frac{B}{\sqrt{3}}$$

Hence, the given result is proved.

Solution 8

The differential equation of the given curve is:

$$\sin x \cos y dx + \cos x \sin y dy = 0$$

$$\Rightarrow \frac{\sin x \cos y dx + \cos x \sin y dy}{\cos x \cos y} = 0$$

$$\Rightarrow \tan x dx + \tan y dy = 0$$

Integrating both sides, we get:

$$\log(\sec x) + \log(\sec y) = \log C$$

$$\log(\sec x \cdot \sec y) = \log C$$

$$\Rightarrow \sec x \cdot \sec y = C \qquad ...(1)$$

The curve passes through point $\left(0, \frac{\pi}{4}\right)$.

$$\therefore 1 \times \sqrt{2} = C$$

$$\Rightarrow C = \sqrt{2}$$

On substituting $C = \sqrt{2}$ in equation (1), we get:

$$\sec x \cdot \sec y = \sqrt{2}$$

$$\Rightarrow \sec x \cdot \frac{1}{\cos y} = \sqrt{2}$$

$$\Rightarrow \cos y = \frac{\sec x}{\sqrt{2}}$$

Hence, the required equation of the curve is $\cos y = \frac{\sec x}{\sqrt{2}}$.

Solution 9

$$\left(1+e^{2x}\right)dy+\left(1+y^2\right)e^x dx=0$$

$$\Rightarrow \frac{dy}{1+y^2}+\frac{e^x dx}{1+e^{2x}}=0$$

Integrating both sides, we get:

$$\tan^{-1} y+\int \frac{e^x dx}{1+e^{2x}}=\mathrm{C} \qquad ...(1)$$

Let $e^x=t \Rightarrow e^{2x}=t^2$.

$$\Rightarrow \frac{d}{dx}\left(e^x\right)=\frac{dt}{dx}$$

$$\Rightarrow e^x=\frac{dt}{dx}$$

$$\Rightarrow e^x dx=dt$$

Substituting these values in equation (1), we get:

$$\tan^{-1} y+\int \frac{dt}{1+t^2}=\mathrm{C}$$

$$\Rightarrow \tan^{-1} y+\tan^{-1} t=\mathrm{C}$$

$$\Rightarrow \tan^{-1} y+\tan^{-1}\left(e^x\right)=\mathrm{C} \qquad ...(2)$$

Now, $y=1$ at $x=0$.

Therefore, equation (2) becomes:

$$\tan^{-1} 1+\tan^{-1} 1=C$$

$$\Rightarrow \frac{\pi}{4}+\frac{\pi}{4}=\mathrm{C}$$

$$\Rightarrow \mathrm{C}=\frac{\pi}{2}$$

Substituting $\mathrm{C}=\frac{\pi}{2}$ in equation (2), we get:

$$\tan^{-1} y+\tan^{-1}\left(e^x\right)=\frac{\pi}{2}$$

This is the required particular solution of the given differential equation.

Solution 10

$$ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y^2\right)dy$$

$$\Rightarrow ye^{\frac{x}{y}}\frac{dx}{dy} = xe^{\frac{x}{y}} + y^2$$

$$\Rightarrow e^{\frac{x}{y}}\left[y \cdot \frac{dx}{dy} - x\right] = y^2$$

$$\Rightarrow e^{\frac{x}{y}} \cdot \frac{\left[y \cdot \frac{dx}{dy} - x\right]}{y^2} = 1 \qquad ...(1)$$

Let $e^{\frac{x}{y}} = z$.

Differentiating it with respect to y, we get:

$$\frac{d}{dy}\left(e^{\frac{x}{y}}\right) = \frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}} \cdot \frac{d}{dy}\left(\frac{x}{y}\right) = \frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}} \cdot \left[\frac{y \cdot \frac{dx}{dy} - x}{y^2}\right] = \frac{dz}{dy} \qquad ...(2)$$

From equation (1) and equation (2), we get:

$$\frac{dz}{dy} = 1$$

$$\Rightarrow dz = dy$$

Integrating both sides, we get:

$$z = y + \mathrm{C}$$

$$\Rightarrow e^{\frac{x}{y}} = y + \mathrm{C}$$

Solution 11

$(x-y)(dx+dy)=dx-dy$

$\Rightarrow (x-y+1)dy=(1-x+y)dx$

$\Rightarrow \frac{dy}{dx}=\frac{1-x+y}{x-y+1}$

$\Rightarrow \frac{dy}{dx}=\frac{1-(x-y)}{1+(x-y)} \qquad ...(1)$

Let $x-y=t$.

$\Rightarrow \frac{d}{dx}(x-y)=\frac{dt}{dx}$

$\Rightarrow 1-\frac{dy}{dx}=\frac{dt}{dx}$

$\Rightarrow 1-\frac{dt}{dx}=\frac{dy}{dx}$

Substituting the values of $x-y$ and $\frac{dy}{dx}$ in equation (1), we get:

$1-\frac{dt}{dx}=\frac{1-t}{1+t}$

$\Rightarrow \frac{dt}{dx}=1-\left(\frac{1-t}{1+t}\right)$

$\Rightarrow \frac{dt}{dx}=\frac{(1+t)-(1-t)}{1+t}$

$\Rightarrow \frac{dt}{dx}=\frac{2t}{1+t}$

$\Rightarrow \left(\frac{1+t}{t}\right)dt=2dx$

$\Rightarrow \left(1+\frac{1}{t}\right)dt=2dx \qquad ...(2)$

Integrating both sides, we get:

$t+\log|t|=2x+C$

$\Rightarrow (x-y)+\log|x-y|=2x+C$

$\Rightarrow \log|x-y|=x+y+C \qquad ...(3)$

Now, $y = -1$ at $x = 0$.

Therefore, equation (3) becomes:

$\log 1 = 0 - 1 + C$

$C = 1$

Substituting $C = 1$ in equation (3) we get:

$\log|x - y| = x + y + 1$

This is the required particular solution of the given differential equation.

Solution 12

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

This equation is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{\sqrt{x}}$ and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

Now, $\text{I.F} = e^{\int P dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$

The general solution of the given differential equation is given by,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\, dx + \text{C}$$

$$\Rightarrow ye^{2\sqrt{x}} = \int\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} \times e^{2\sqrt{x}}\right)dx + \text{C}$$

$$\Rightarrow ye^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\, dx + \text{C}$$

$$\Rightarrow ye^{2\sqrt{x}} = 2\sqrt{x} + \text{C}$$

Solution 13

The given differential equation is:

$$\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x$$

This equation is a linear differential equation of the form

$\frac{dy}{dx} + py = Q$, where $p = \cot x$ and $Q = 4x \operatorname{cosec} x$.

Now, $\text{I.F} = e^{\int p dx} = e^{\int \cot x dx} = e^{\log|\sin x|} = \sin x$

The general solution of the given differential equation is given by,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})dx + \text{C}$$
$$\Rightarrow y\sin x = \int (4x\operatorname{cosec} x \cdot \sin x)dx + \text{C}$$
$$\Rightarrow y\sin x = 4\int x\,dx + \text{C}$$
$$\Rightarrow y\sin x = 4 \cdot \frac{x^2}{2} + \text{C}$$
$$\Rightarrow y\sin x = 2x^2 + \text{C} \qquad ...(1)$$

Now, $y = 0$ at $x = \frac{\pi}{2}$.

Therefore, equation (1) becomes:

$$0 = 2\times\frac{\pi^2}{4} + \text{C}$$
$$\Rightarrow \text{C} = -\frac{\pi^2}{2}$$

Substituting $\text{C} = -\frac{\pi^2}{2}$ in equation (1), we get:

$$y\sin x = 2x^2 - \frac{\pi^2}{2}$$

This is the required particular solution of the given differential equation.

Solution 14

$$(x+1)\frac{dy}{dx}=2e^{-y}-1$$
$$\Rightarrow \frac{dy}{2e^{-y}-1}=\frac{dx}{x+1}$$
$$\Rightarrow \frac{e^{y}dy}{2-e^{y}}=\frac{dx}{x+1}$$

Integrating both sides, we get:

$$\int\frac{e^{y}dy}{2-e^{y}}=\log|x+1|+\log C \qquad ...(1)$$

Let $2-e^{y}=t$.

$$\therefore \frac{d}{dy}\left(2-e^{y}\right)=\frac{dt}{dy}$$
$$\Rightarrow -e^{y}=\frac{dt}{dy}$$
$$\Rightarrow e^{y}dy=-dt$$

Substituting this value in equation (1), we get:

$$\int\frac{-dt}{t}=\log|x+1|+\log C$$
$$\Rightarrow -\log|t|=\log|C(x+1)|$$
$$\Rightarrow -\log|2-e^{y}|=\log|C(x+1)|$$
$$\Rightarrow \frac{1}{2-e^{y}}=C(x+1)$$
$$\Rightarrow 2-e^{y}=\frac{1}{C(x+1)} \qquad ...(2)$$

Now, at $x=0$ and $y=0$, equation (2) becomes:

$$\Rightarrow 2-1=\frac{1}{C}$$
$$\Rightarrow C=1$$

Substituting C = 1 in equation (2), we get:

$$2-e^{y}=\frac{1}{x+1}$$
$$\Rightarrow e^{y}=2-\frac{1}{x+1}$$
$$\Rightarrow e^{y}=\frac{2x+2-1}{x+1}$$
$$\Rightarrow e^{y}=\frac{2x+1}{x+1}$$
$$\Rightarrow y=\log\left|\frac{2x+1}{x+1}\right|,\ (x\neq-1)$$

This is the required particular solution of the given differential equation.

Solution 15

Let the population at any instant (t) be y.

It is given that the rate of increase of population is proportional to the number of inhabitants at any instant.

$$\therefore \frac{dy}{dt}\propto y$$
$$\Rightarrow \frac{dy}{dt}=ky \qquad (k \text{ is a constant})$$
$$\Rightarrow \frac{dy}{y}=kdt$$

Integrating both sides, we get:

$\log y = kt + \text{C} \ldots (1)$

In the year 1999, $t = 0$ and $y = 20000$.

Therefore, we get:

$\log 20000 = \text{C} \ldots (2)$

In the year 2004, $t = 5$ and $y = 25000$.

Therefore, we get:

$$\log 25000 = k \cdot 5 + C$$
$$\Rightarrow \log 25000 = 5k + \log 20000$$
$$\Rightarrow 5k = \log\left(\frac{25000}{20000}\right) = \log\left(\frac{5}{4}\right)$$
$$\Rightarrow k = \frac{1}{5}\log\left(\frac{5}{4}\right) \qquad ...(3)$$

In the year 2009, $t = 10$ years.

Now, on substituting the values of t, k, and C in equation (1), we get:

$$\log y = 10 \times \frac{1}{5}\log\left(\frac{5}{4}\right) + \log(20000)$$
$$\Rightarrow \log y = \log\left[20000 \times \left(\frac{5}{4}\right)^2\right]$$
$$\Rightarrow y = 20000 \times \frac{5}{4} \times \frac{5}{4}$$
$$\Rightarrow y = 31250$$

Hence, the population of the village in 2009 will be 31250.

Solution 16

The given differential equation is:

$$\frac{ydx - xdy}{y} = 0$$
$$\Rightarrow \frac{ydx - xdy}{xy} = 0$$
$$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy = 0$$

Integrating both sides, we get:

$$\log|x| - \log|y| = \log k$$
$$\Rightarrow \log\left|\frac{x}{y}\right| = \log k$$
$$\Rightarrow \frac{x}{y} = k$$
$$\Rightarrow y = \frac{1}{k}x$$
$$\Rightarrow y = Cx \text{ where } C = \frac{1}{k}$$

Hence, the correct answer is C.

Solution 17

The integrating factor of the given differential equation $\frac{dx}{dy} + P_1 x = Q_1$ is $e^{\int P_1 dy}$.

The general solution of the differential equation is given by,

$$x(\text{I.F.}) = \int (Q \times \text{I.F.})dy + C$$
$$\Rightarrow x \cdot e^{\int P_1 dy} = \int\left(Q_1 e^{\int P_1 dy}\right)dy + C$$

Hence, the correct answer is C.

Solution 18

The given differential equation is:

$$e^x dy + \left(ye^x + 2x\right)dx = 0$$

$$\Rightarrow e^x \frac{dy}{dx} + ye^x + 2x = 0$$

$$\Rightarrow \frac{dy}{dx} + y = -2xe^{-x}$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = 1$ and $Q = -2xe^{-x}$.

Now, $\text{I.F} = e^{\int P dx} = e^{\int dx} = e^x$

The general solution of the given differential equation is given by,

$$y\left(\text{I.F.}\right) = \int\left(\text{Q} \times \text{I.F.}\right)dx + \text{C}$$

$$\Rightarrow ye^x = \int\left(-2xe^{-x} \cdot e^x\right)dx + \text{C}$$

$$\Rightarrow ye^x = -\int 2x\,dx + \text{C}$$

$$\Rightarrow ye^x = -x^2 + \text{C}$$

$$\Rightarrow ye^x + x^2 = \text{C}$$

Hence, the correct answer is C.