RD SHARMA Solutions for Class 12-science Maths Chapter 25 - Vector or Cross Product

Chapter 25 - Vector or Cross Product Exercise Ex. 25.1

Question 1

If $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = -\hat{i} + 3\hat{k}$, find $\left|\vec{a} \times \vec{b}\right|$

Solution 1

If $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and

$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix}$$

$$= \hat{i}\left(9 - 0\right) - \hat{j}\left(3 - 2\right) + \hat{k}\left(0 + 3\right)$$

$$\vec{a} \times \vec{b} = 9\hat{i} - \hat{j} + 3\hat{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{\left(9\right)^2 + \left(-1\right)^2 + \left(3\right)^2}$$
$$= \sqrt{81 + 1 + 9}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{91}$$

Question 2(i)

If $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$, find the value of $\left|\vec{a} \times \vec{b}\right|$.

Solution 2(i)

If $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and
$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}\,(4 - 0) - \hat{j}\,(3 - 0) + \hat{k}\,(3 - 4)$$
$$= 4\hat{i} - 3\hat{j} - \hat{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{(4)^2 + (-3)^2 + (-1)^2}$$
$$= \sqrt{16 + 9 + 1}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{26}$$

Question 2(ii)
If $\vec{a} = 2\hat{i} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$, find the magnitude of $\vec{a} \times \vec{b}$.

Solution 2(ii)
If $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and
$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}\,(0 - 1) - \hat{j}\,(2 - 1) + \hat{k}\,(2 - 0)$$
$$= -\hat{i} - \hat{j} + 2\hat{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{(-1)^2 + (-1)^2 + (2)^2}$$
$$= \sqrt{1 + 1 + 4}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{6}$$

Magnitude of $\vec{a} \times \vec{b} = \sqrt{6}$.

Question 3(i)

Find a unit vector perpendicular to both the vectors $4\hat{i} - \hat{j} + 3\hat{k}$, and $-2\hat{i} + \hat{j} - 2\hat{k}$.

Solution 3(i)

A vector perpendicular to both $\vec{a}$ and $\vec{b} = \vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$\vec{c}\,(\text{say}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix}$$

$$\vec{c} = \hat{i}\,(2 - 3) - \hat{j}\,(-8 + 6) + \hat{k}\,(4 - 2)$$
$$\vec{c} = -\hat{i} + 2\hat{j} + 2\hat{k}$$

$\vec{c}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|}$$

$$= \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{(-1)^2 + (2)^2 + (2)^2}}$$
$$= \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1 + 4 + 4}}$$

$$= \frac{1}{3}\left(-\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

So, unit vector perpendicular to both $\vec{a}$ and $\vec{b} = \frac{1}{3}\left(-\hat{i} + 2\hat{j} + 2\hat{k}\right)$.

Question 3(ii)

Find a unit vector perpendicular to the plane containing the vectors
$\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$, and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

Solution 3(ii)

A vector perpendicular to the plane containing the vector $\vec{a}$ and $\vec{b}$ is given by

$\vec{a} \times \vec{b} = \pm\vec{c}$ (Say)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$\vec{c} = \hat{i}\,(1-2) - \hat{j}\,(2-1) + \hat{k}\,(4-1)$$
$$\vec{c} = -\hat{i} - \hat{j} + 3\hat{k}$$

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|}$$

$$= \frac{-\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{(-1)^2 + (-1)^2 + (3)^2}}$$
$$= \frac{-\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{1+1+9}}$$

$$= \frac{1}{\sqrt{11}}\left(-\hat{i} - \hat{j} + 3\hat{k}\right)$$

Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b} = \pm\frac{1}{\sqrt{11}}\left(-\hat{i} - \hat{j} + 3\hat{k}\right)$.

Question 4

Find the magnitude of $\vec{a} = \left(3\hat{k} + 4\hat{j}\right) \times \left(\hat{i} + \hat{j} - \hat{k}\right)$

Solution 4

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 3 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}\left(-4-3\right) - \hat{j}\left(0-3\right) + \hat{k}\left(0-4\right)$$
$$= -7\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{\left(-7\right)^2 + \left(3\right)^2 + \left(-4\right)^2}$$
$$= \sqrt{49 + 9 + 16}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{74}$$

Question 5

If $\vec{a} = 4\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{k}$, then find $\left|2\vec{b} \times \vec{a}\right|$.

Solution 5

$$\vec{b} = \hat{i} - 2\hat{k}$$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|}$$

$$= \frac{\hat{i} - 2\hat{k}}{\sqrt{(1)^2 + (-2)^2}}$$

$$= \frac{\hat{i} - 2\hat{k}}{\sqrt{1+4}}$$

$$= \frac{\hat{i} - 2\hat{k}}{\sqrt{5}}$$

$$2\hat{b} = \frac{2}{\sqrt{5}}\hat{i} - \frac{4}{\sqrt{5}}\hat{k}$$

And, $\vec{a} = 4\hat{i} + 3\hat{j} + \hat{k}$

If $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and

$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$2\hat{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{4}{\sqrt{5}} \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \hat{i}\left(0 + \frac{12}{\sqrt{5}}\right) - \hat{j}\left(\frac{2}{\sqrt{5}} + \frac{16}{\sqrt{5}}\right) + \hat{k}\left(\frac{6}{\sqrt{5}} - 0\right)$$

$$2\hat{b} \times \vec{a} = \frac{12}{\sqrt{5}}\hat{i} - \frac{18}{\sqrt{5}}\hat{j} + \frac{6}{\sqrt{5}}\hat{k}$$

$$\left|2\hat{b} \times \vec{a}\right| = \sqrt{\left(\frac{12}{\sqrt{5}}\right)^2 + \left(-\frac{18}{\sqrt{5}}\right)^2 + \left(\frac{6}{\sqrt{5}}\right)^2}$$

$$= \sqrt{\frac{144}{5} + \frac{324}{5} + \frac{36}{5}}$$

$$\left|2\hat{b} \times \vec{a}\right| = \sqrt{\frac{504}{5}}$$

Question 6

If $\vec{a} = 3\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$, find $(\vec{a} \times 2\vec{b}) \times (2\vec{a} - \vec{b})$.

Solution 6

$$\vec{a}+2\vec{b}=\left(3\hat{i}-\hat{j}-2\hat{k}\right)+2\left(2\hat{i}+3\hat{j}+\hat{k}\right)$$
$$=3\hat{i}-\hat{j}-2\hat{k}+4\hat{i}+6\hat{j}+2\hat{k}$$

$$\vec{a}+2\vec{b}=7\hat{i}+5\hat{j}$$

$$2\vec{a}-\vec{b}=2\left(3\hat{i}-\hat{j}-2\hat{k}\right)-\left(2\hat{i}+3\hat{j}+\hat{k}\right)$$
$$=6\hat{i}-2\hat{j}-4\hat{k}-2\hat{i}-3\hat{j}-\hat{k}$$

$$=4\hat{i}-5\hat{j}-5\hat{k}$$

We know that if $\vec{a}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $\vec{b}=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$ then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$$

Therefore,

$$\left(\vec{a}+2\vec{b}\right)\times\left(2\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 7 & 5 & 0\\ 4 & -5 & -5\end{vmatrix}$$

$$=\hat{i}\left(-25-0\right)-\hat{j}\left(-35-0\right)+\hat{k}\left(-35-20\right)$$
$$=-25\hat{i}+35\hat{j}-55\hat{k}$$

$$\left(\vec{a}+2\vec{b}\right)\times\left(2\vec{a}-\vec{b}\right)=-25\hat{i}+35\hat{j}-55\hat{k}$$

Question 7(i)

Find a vector of magnitude 49, which is perpendicular to both the vectors $2\hat{i}+3\hat{j}+6\hat{k}$ and $3\hat{i}-6\hat{j}+2\hat{k}$.

Solution 7(i)

Let, $\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, $\vec{b} = 3\hat{i} - 6\hat{j} + 2\hat{k}$

If $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and
$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{vmatrix}$$

$$= \hat{i}\left(6 + 36\right) - \hat{j}\left(4 - 18\right) + \hat{k}\left(-12 - 9\right)$$
$$= 42\hat{i} + 14\hat{j} - 21\hat{k}$$

$$= 7\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right)$$

$$\left|\vec{a} \times \vec{b}\right| = 7\sqrt{\left(6\right)^2 + \left(2\right)^2 + \left(-3\right)^2}$$
$$= 7\sqrt{36 + 4 + 9}$$

$$\left|\vec{a} \times \vec{b}\right| = 7\sqrt{49}$$
$$\left|\vec{a} \times \vec{b}\right| = 7 \times 7$$

$$\left|\vec{a} \times \vec{b}\right| = 49$$

Vector perpendicular to $\vec{a}$ and $\vec{b}$

with magnitude 1 = $\dfrac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}$

$$= \frac{1}{49}\left(7\left(6\hat{i}+2\hat{j}-3\hat{k}\right)\right)$$

$$= \frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)$$

vector of magnitude 49, which is perpendicular to $\vec{a}$ and $\vec{b}$

$$= 49\left(\frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}\right)$$

$$= 49\left[\frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)\right]$$

$$= 42\hat{i}+14\hat{j}-21\hat{k}$$

The required vector = $42\hat{i}+14\hat{j}-21\hat{k}$

Question 7(ii)

Find a vector whose length is 3 and which is perpendicular to the vectors $\vec{a} = 3\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b} = 6\hat{i}+5\hat{j}-2\hat{k}$.

Solution 7(ii)

If $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and
$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -4 \\ 6 & 5 & -2 \end{vmatrix}$$

$$= \hat{i}\left(-2 + 20\right) - \hat{j}\left(-6 + 24\right) + \hat{k}\left(15 - 6\right)$$
$$= 18\hat{i} - 18\hat{j} + 9\hat{k}$$

$$= 9\left(2\hat{i} - 2\hat{j} + \hat{k}\right)$$

$$\left|\vec{a} \times \vec{b}\right| = 9\sqrt{2^2 + \left(-2\right)^2 + \left(1\right)^2}$$
$$= 9\sqrt{4 + 4 + 1}$$

$$\left|\vec{a} \times \vec{b}\right| = 9\sqrt{9}$$
$$\left|\vec{a} \times \vec{b}\right| = 9 \times 3$$

$$\left|\vec{a} \times \vec{b}\right| = 27$$

Unit vector perpendicular to the vector

$\vec{a}$ and $\vec{b} = \dfrac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}$

$$= \frac{1}{27}\left(9\left(2\hat{i}-2\hat{j}+\hat{k}\right)\right)$$

$$= \frac{1}{3}\left(2\hat{i}-2\hat{j}+\hat{k}\right)$$

vector with length 3 and which is perpendicular to both $\vec{a}$ and $\vec{b}$

$$= 3\left(\frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}\right)$$

$$= 3\left[\frac{1}{3}\left(2\hat{i}-2\hat{j}+\hat{k}\right)\right]$$

$$= 2\hat{i}-2\hat{j}+\hat{k}$$

Required vector $= 2\hat{i}-2\hat{j}+\hat{k}$

Question 8(i)

Find the area of the parallelogram determined by the vectors:

$2\hat{i}$ and $3\hat{j}$

Solution 8(i)

Here, $\vec{a} = 2\hat{i} + 0.\hat{j} + 0.\hat{k}$

$\vec{b} = 0.\hat{i} + 3\hat{j} + 0.\hat{k}$,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{vmatrix}$$

$$= \hat{i}\,(0 - 0) - \hat{j}\,(0 - 0) + \hat{k}\,(6 - 0)$$
$$= 6\hat{k}$$

Area of parallelogram $= \left|\vec{a} \times \vec{b}\right|$

$$= \left|0\hat{i} + 0.\hat{j} + 6\hat{k}\right|$$

$$= \sqrt{(0)^2 + (0)^2 + (6)^2}$$

Area of parallelogram = 6 sq.unit

Question 8(ii)

Find the area of the parallelogram determined by the vectors:

$2\hat{i} + \hat{j} + 3\hat{k}$ and $\hat{i} - \hat{j}$

Solution 8(ii)

Let, $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$

$\vec{b} = \hat{i} - \hat{j}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \hat{i}\left(0+3\right) - \hat{j}\left(0-3\right) + \hat{k}\left(-2-1\right)$$
$$= 3\hat{i} + 3\hat{j} - 3\hat{k}$$
$$= 3\left(\hat{i} + \hat{j} - \hat{k}\right)$$

Area of parallelogram $= \left|\vec{a} \times \vec{b}\right|$

$$= 3\sqrt{\left(1\right)^2 + \left(1\right)^2 + \left(-1\right)^2}$$

$$= 3\sqrt{3}$$

Area of parallelogram $= 3\sqrt{3}$ sq.unit

Question 8(iii)

Find the area of the parallelogram determined by the vectors:

$3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$

Solution 8(iii)

Let, $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$

$\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= \hat{i}\left(4-6\right) - \hat{j}\left(12+2\right) + \hat{k}\left(-9-1\right)$$

$$= -2\hat{i} - 14\hat{j} - 10\hat{k}$$

$$= -2\left(\hat{i} + 7\hat{j} + 5\hat{k}\right)$$

Area of parallelogram $= \left|\vec{a} \times \vec{b}\right|$

$$= 2\sqrt{(1)^2 + (7)^2 + (5)^2}$$

$$= 2\sqrt{1 + 49 + 25}$$

$$= 2\sqrt{75}$$

$$= 10\sqrt{3}$$

Area of parallelogram $= 10\sqrt{3}$ sq.unit

Question 8(iv)

Find the area of the parallelogram determined by the vectors:

$\hat{i} - 3\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$

Solution 8(iv)

Let, $\vec{a} = \hat{i} - 3\hat{j} + \hat{k}$

$\vec{b} = \hat{i} + \hat{j} + \hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}\left(-3-1\right) - \hat{j}\left(1-1\right) + \hat{k}\left(1+3\right)$$
$$= -4\hat{i} - 0.\hat{j} + 4\hat{k}$$

Area of parallelogram $= \left|\vec{a} \times \vec{b}\right|$

$$= \sqrt{\left(-4\right)^2 + \left(0\right)^2 + \left(4\right)^2}$$

$$= \sqrt{16 + 0 + 16}$$
$$= \sqrt{32}$$

$$= 4\sqrt{2}$$

Area of parallelogram $= 4\sqrt{2}$ sq.unit

Question 9(i)

Find the area of the parallelogram whose diagonals are:

$4\hat{i} - \hat{j} - 3\hat{k}$ and $-2\hat{j} + \hat{j} - 2\hat{k}$

Solution 9(i)

Area of parallelogram = $\frac{1}{2}\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right|$

Here, $d_1 = 4\hat{i} - \hat{j} - 3\hat{k}$

$d_2 = -2\hat{i} + \hat{j} - 2\hat{k}$

$$\overrightarrow{d_1} \times \overrightarrow{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & -3 \\ -2 & 1 & -2 \end{vmatrix}$$

$$= \hat{i}\left(2+3\right) - \hat{j}\left(-8-6\right) + \hat{k}\left(4-2\right)$$

$$= 5\hat{i} + 14\hat{j} + 2\hat{k}$$

$$\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right| = \sqrt{\left(5\right)^2 + \left(14\right)^2 + \left(2\right)^2}$$

$$= \sqrt{25 + 196 + 4}$$

$$= \sqrt{225}$$

$$= 15$$

Area of parallelogram = $\frac{1}{2}\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right|$

Area of parallelogram = $\frac{15}{2}$ sq.unit

Question 9(ii)

Find the area of the parallelogram whose diagonals are:

$2\hat{i} + \hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$

Solution 9(ii)

Given, $d_1 = 2\hat{i} + \hat{k}$

$d_2 = \hat{i} + \hat{j} + \hat{k}$

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}(0 - 1) - \hat{j}(2 - 1) + \hat{k}(2 - 0)$$

$$= -\hat{i} - \hat{j} + 2\hat{k}$$

$$\left|\vec{d_1} \times \vec{d_2}\right| = \sqrt{(-1)^2 + (-1)^2 + (2)^2}$$

$$= \sqrt{1 + 1 + 4}$$

$$= \sqrt{6}$$

Area of parallelogram $= \frac{1}{2}\left|\vec{d_1} \times \vec{d_2}\right|$

Area of parallelogram $= \frac{1}{2}\sqrt{6}$ sq.unit

Question 9(iii)

Find the area of the parallelogram whose diagonals are:

$3\hat{i} + 4\hat{j}$ and $\hat{i} + \hat{j} + \hat{k}$

Solution 9(iii)

Given, $d_1 = 3\hat{i} + 4\hat{j}$

$d_2 = \hat{i} + \hat{j} + \hat{k}$

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}\,(4-0) - \hat{j}\,(3-0) + \hat{k}\,(3-4)$$

$$= 4\hat{i} - 3\hat{j} - \hat{k}$$

$$\left|\vec{d_1} \times \vec{d_2}\right| = \sqrt{(4)^2 + (-3)^2 + (-1)^2}$$

$$= \sqrt{16 + 9 + 1}$$

$$= \sqrt{26}$$

Area of parallelogram $= \frac{1}{2}\left|\vec{d_1} \times \vec{d_2}\right|$

Area of parallelogram $= \frac{\sqrt{26}}{2}$ sq.unit

Question 9(iv)

Find the area of the parallelogram whose diagonals are:

$2\hat{i} + 3\hat{j} + 6\hat{k}$ and $3\hat{i} - 6\hat{j} + 2\hat{k}$

Solution 9(iv)

Here, $d_1 = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$d_2 = 3\hat{i} - 6\hat{j} + 2\hat{k}$

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{vmatrix}$$

$$= \hat{i}\left(6 + 36\right) - \hat{j}\left(4 - 18\right) + \hat{k}\left(-12 - 9\right)$$
$$= 42\hat{i} + 14\hat{j} - 21\hat{k}$$

$$= 7\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right)$$

$$\left|\vec{d_1} \times \vec{d_2}\right| = 7\sqrt{(6)^2 + (2)^2 + (-3)^2}$$
$$= 7\sqrt{36 + 4 + 9}$$
$$= 7\sqrt{49}$$
$$= 7 \times 7$$

$$= 49$$

Area of parallelogram $= \dfrac{1}{2}\left|\vec{d_1} \times \vec{d_2}\right|$

Area of parallelogram $= \dfrac{49}{2}$ sq.unit

Question 10

If $\vec{a} = 2\hat{i} + 5\hat{j} - 7\hat{k}$, $\vec{b} = -3\hat{i} + 4\hat{j} + \hat{k}$, and $\vec{c} = \hat{i} - 2\hat{j} - 3\hat{k}$ compute $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ and $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$ and verify that these are not equal.

Solution 10

Given, $\vec{a} = 2\hat{i} + 5\hat{j} - 7\hat{k}$,
$\vec{b} = -3\hat{i} + 4\hat{j} + \hat{k}$,
$\vec{c} = \hat{i} - 2\hat{j} - 3\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 5 & -7 \\ -3 & 4 & 1 \end{vmatrix}$$

$$= \hat{i}(5+28) - \hat{j}(2-21) + \hat{k}(8+15)$$
$$= 33\hat{i} + 19\hat{j} + 23\hat{k}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 33 & 19 & 23 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= \hat{i}(-57+46) - \hat{j}(-99-23) + \hat{k}(-66-19)$$
$$(\vec{a} \times \vec{b}) \times \vec{c} = -11\hat{i} + 122\hat{j} - 85\hat{k} \qquad ---(i)$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 4 & 1 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= \hat{i}(-12+2) - \hat{j}(9-1) + \hat{k}(6-4)$$
$$= -10\hat{i} - 8\hat{j} + 2\hat{k}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 5 & -7 \\ -10 & -8 & 2 \end{vmatrix}$$

$$= \hat{i}(10+56) - \hat{j}(4-70) + \hat{k}(-16+50)$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = 66\hat{i} + 66\hat{j} + 36\hat{k} \qquad ---(ii)$$

From equation (i) and (ii)

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

Question 11

If $|\vec{a}| = 2, |\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, find $\vec{a} \bullet \vec{b}$.

Solution 11

We know that, if θ be the angle between $\vec{a}$ and $\vec{b}$, then,

$$\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}|.\sin\theta.\hat{n}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|.|\sin\theta|.|\hat{n}|$$

$$8 = 2.5.\sin\theta.1 \qquad [\text{As } \hat{n} \text{ is a unit vector}]$$

$$\sin\theta = \frac{8}{10}$$

$$\sin\theta = \frac{4}{5}$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$= 1 - \left(\frac{4}{5}\right)^2$$

$$= 1 - \frac{16}{25}$$

$$= \frac{25 - 16}{25}$$

$$= \frac{9}{25}$$

$$\cos\ \theta = \frac{3}{5}$$

$$\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|.\cos\theta$$

$$= 2.5.\frac{3}{5}$$

$$\vec{a}.\vec{b} = 6$$

Question 12

Given, $\vec{a} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$, $\vec{b} = \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right)$, $\vec{c} = \frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right)$, $\hat{i}, \hat{j}, \hat{k}$ being a right handed orthogonal system of unit vectors in space, show that $\vec{a}, \vec{b}, \vec{c}$ is also another system.

Solution 12

Given, $\vec{a} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$

$\vec{b} = \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right)$

$\vec{c} = \frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right)$,

$$\vec{a} \times \vec{b} = \frac{1}{7} \times \frac{1}{7}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{vmatrix}$$

$$= \frac{1}{49}\left[\hat{i}\left(6 + 36\right) - \hat{j}\left(4 - 18\right) + \hat{k}\left(-12 - 9\right)\right]$$

$$= \frac{1}{49}\left[42\hat{i} + 14\hat{j} - 21\hat{k}\right]$$

$$= \frac{7\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right)}{49}$$

$$= \frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right)$$

$$\vec{a} \times \vec{b} = \vec{c} \qquad ---(\text{i})$$

$$\vec{b} \times \vec{c} = \frac{1}{7} \times \frac{1}{7}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 6 & 2 & -3 \end{vmatrix}$$

$$= \frac{1}{49}\left[\hat{i}\left(18 - 4\right) - \hat{j}\left(-9 - 12\right) + \hat{k}\left(6 + 36\right)\right]$$

$$= \frac{1}{49}\left[14\hat{i} + 21\hat{j} + 42\hat{k}\right]$$

$$= \frac{7\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)}{49}$$

$$\vec{c} \times \vec{a} = \frac{1}{7} \times \frac{1}{7}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 2 & -3 \\ 2 & 3 & 6 \end{vmatrix}$$

$$\vec{c} \times \vec{a} = \frac{1}{49}\left[\hat{i}\left(12 + 9\right) - \hat{j}\left(36 + 6\right) + \hat{k}\left(18 - 4\right)\right]$$

$$= \frac{1}{49}\left[21\hat{i} - 42\hat{j} + 14\hat{k}\right]$$

$$= \frac{7\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right)}{49}$$

$$= \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right)$$

Question 13

If $\left|\vec{a}\right| = 13, \left|\vec{b}\right| = 5$ and $\vec{a}.\vec{b} = 60$, then find $\left|\vec{a} \times \vec{b}\right|$.

Solution 13

We know that, if θ is angle between $\vec{a}$ and $\vec{b}$,

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$60 = 13.5.\cos\theta$$

$$\cos\theta = \frac{60}{65}$$

$$\cos\theta = \frac{12}{13}$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$= 1 - \left(\frac{12}{13}\right)^2$$

$$= 1 - \frac{144}{169}$$

$$= \frac{169 - 144}{169}$$

$$= \frac{25}{169}$$

$$\sin\ \theta = \pm\sqrt{\frac{25}{169}}$$

$$= \pm\frac{5}{13}$$

$$|\sin\theta| = \frac{5}{13}$$

We know that,

$$\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}|.\sin\theta.\hat{n}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}|.|\sin\theta|.|\hat{n}|$$

$$= 13.5.\frac{5}{13}.1 \qquad \left[\text{Since, } \hat{n} \text{ is a unit vector}\right]$$

$$|\vec{a} \times \vec{b}| = 25$$

Question 14

Find the angle between two vectors $\vec{a}$ and $\vec{b}$ if $\left|\vec{a}\times\vec{b}\right| = \vec{a}.\vec{b}$.

Solution 14

We know that, if θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\vec{a}.\vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \cos\theta \qquad ---(i)$$

$$\text{And, } \vec{a}\times\vec{b} = \left|\vec{a}\right| \left|\vec{b}\right|.\sin\theta.\hat{n}$$

$$\left|\vec{a}\times\vec{b}\right| = \left|\vec{a}\right|.\left|\vec{b}\right|.\left|\sin\theta\right|.\left|\hat{n}\right|$$

$$= \left|\vec{a}\right| \left|\vec{b}\right| \left|\sin\theta\right|.1 \qquad \left[\text{Since, } \hat{n} \text{ is a unit vector}\right]$$

$$\left|\vec{a}\times\vec{b}\right| = \left|\vec{a}\right| \left|\vec{b}\right| \sin\theta \qquad ---(ii)$$

Given that, $\left|\vec{a}\times\vec{b}\right| = \vec{a}.\vec{b}$

$$\left|\vec{a}\right| \left|\vec{b}\right| \sin\theta = \left|\vec{a}\right|.\left|\vec{b}\right| \cos\theta$$

$$\sin\theta = \cos\theta$$

$$\theta = \frac{\pi}{4}$$

Angle between $\vec{a}$ and $\vec{b} = \frac{\pi}{4}$

Question 15

If $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} \neq 0$, then show that $\vec{a}+\vec{c} = \overrightarrow{mb}$, where m is any scalar.

Solution 15

We have,

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$$
$$\left(\vec{a} \times \vec{b}\right) - \left(\vec{b} \times \vec{c}\right) = \vec{0}$$
$$\left(\vec{a} \times \vec{b}\right) + \left(\vec{c} \times \vec{b}\right) = \vec{0} \qquad \left[\text{Since, } \left(\vec{b} \times \vec{c}\right) = -\left(\vec{c} \times \vec{b}\right)\right]$$
$$\left(\vec{a} + \vec{c}\right) \times \vec{b} = \vec{0} \qquad \left[\text{Using distributive property}\right]$$

We know that, if $\vec{a} \times \vec{b} = \vec{0}$, then vector $\vec{a}$ is parallel to vector $\vec{b}$.

Thus, $\left(\vec{a} + \vec{c}\right)$ is parallel to $\vec{b}$

$$\left(\vec{a} + \vec{c}\right) = \overrightarrow{mb}$$

Where m is any scalar

Question 16

If $\left|\vec{a}\right| = 2$, $\left|\vec{b}\right| = 7$ and $\vec{a} \times \vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution 16

We know that,

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\;\hat{n}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin\theta|\,|\hat{n}|$$

$$= |\vec{a}|\,|\vec{b}|\,|\sin\theta|.1 \qquad [\text{as } \hat{n} \text{ is a unit vector}]$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin\theta|$$

$$\sqrt{(3)^2 + (2)^2 + (6)^2} = 2.7.|\sin\theta|$$

$$\sqrt{9 + 4 + 36} = 14.|\sin\theta|$$

$$\sqrt{49} = 14|\sin\theta|$$

$$\sin\theta = \frac{7}{14}$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{6}$$

Angle between $\vec{a}$ and $\vec{b}$ = $\frac{\pi}{6}$

Question 17

What inference can you draw if $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a}.\vec{b} = 0$.

Solution 17

Given that $\vec{a} \times \vec{b} = \vec{0}$

This gives us four conclusions about $\vec{a}$ and $\vec{b}$

(i) $\vec{a} = \vec{0}$ or

(ii) $\vec{b} = \vec{0}$ or

(iii) $\vec{a} = \vec{b} = \vec{0}$ or

(iv) $\vec{a}$ is parallel to $\vec{b}$.

Also, it is given that $\vec{a}.\vec{b} = 0$

This also gives us four canclusions about $\vec{a}$ and $\vec{b}$.

(i) $\vec{a} = \vec{0}$ or

(ii) $\vec{b} = \vec{0}$ or

(iii) $\vec{a} = \vec{b} = \vec{0}$ or

(iv) $\vec{a}$ is perpendicular to $\vec{b}$.

Now,

$\vec{a}$ parallel $\vec{b}$ and $\vec{a}$ is perpendicular to $\vec{b}$ are not possible simultaneously.

So,

$\vec{a} = 0$ or $\vec{b} = 0$ or $\vec{a} = \vec{b} = \vec{0}$

Question 18

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three unit vectors such that $\vec{a} \times \vec{b} = \vec{c}$, $\vec{b} \times \vec{c} = \vec{a}$, $\vec{c} \times \vec{a} = \vec{b}$, show that $\vec{a}$, $\vec{b}$, $\vec{c}$ form an orthonormal right handed triad of unit vectors.

Solution 18

Given that $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors such that

$\vec{a} \times \vec{b} = \vec{c}, \ \vec{b} \times \vec{c} = \vec{a}, \ \vec{c} \times \vec{a} = \vec{b},$

$\vec{a} \times \vec{b} = \vec{c}$

$\Rightarrow \quad \vec{c}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ $\quad ---(i)$

$\vec{b} \times \vec{c} = \vec{a}$

$\Rightarrow \quad \vec{a}$ is a vector perpendicular to $\vec{b}$ and $\vec{c}$ $\quad ---(ii)$

$\vec{c} \times \vec{a} = \vec{b}$

$\Rightarrow \quad \vec{b}$ is a vector perpendicular to $\vec{a}$ and $\vec{c}$ $\quad ---(iii)$

Using $(i), (ii)$ and (iii), we can see that $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular unit vectors.

Since, $\vec{a} \times \vec{b} = \vec{c}$
$\vec{b} \times \vec{c} = \vec{a}$
$\vec{c} \times \vec{a} = \vec{b}$

Therefore,
$\vec{a}, \vec{b}, \vec{c}$ form an orthonormal right handed traid of unit vectors.

Question 19

Find the unit vector perpendicular to the plane ABC, where the coordinates of A, B and C are $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$.

Solution 19

Here, Position vector of $A = \left(3\hat{i} - \hat{j} + 2\hat{k}\right)$

Position vector of $B = \left(\hat{i} - \hat{j} - 3\hat{k}\right)$

Position vector of $C = \left(4\hat{i} - 3\hat{j} + \hat{k}\right)$

$$\overrightarrow{AB} = \vec{B} - \vec{A}$$
$$= \left(\hat{i} - \hat{j} - 3\hat{k}\right) - \left(3\hat{i} - \hat{j} + 2\hat{k}\right)$$
$$= \hat{i} - \hat{j} - 3\hat{k} - 3\hat{i} + \hat{j} - 2\hat{k}$$
$$= -2\hat{i} - 5\hat{k}$$

$$\overrightarrow{AC} = \vec{C} - \vec{A}$$
$$= \left(4\hat{i} - 3\hat{j} + \hat{k}\right) - \left(3\hat{i} - \hat{j} + 2\hat{k}\right)$$
$$= 4\hat{i} - 3\hat{j} + \hat{k} - 3\hat{i} + \hat{j} - 2\hat{k}$$
$$= \hat{i} - 2\hat{j} - \hat{k}$$

Vector perpendicular to the plane ABC

$$= \overrightarrow{AC} \times \overrightarrow{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -1 \\ -2 & 0 & -5 \end{vmatrix}$$

$$\overrightarrow{AC} \times \overrightarrow{AB} = \hat{i}\left(10 - 0\right) - \hat{j}\left(-5 - 2\right) + \hat{k}\left(0 - 4\right)$$
$$= 10\hat{i} + 7\hat{j} - 4\hat{k}$$

$$\left|\overrightarrow{AC} \times \overrightarrow{AB}\right| = \sqrt{\left(10\right)^2 + \left(7\right)^2 + \left(-4\right)^2}$$
$$= \sqrt{100 + 49 + 16}$$
$$= \sqrt{165}$$

Therefore, unit vector perpendicular to the plane $ABC = \dfrac{\overrightarrow{AC} \times \overrightarrow{AB}}{\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|}$

$$= \frac{1}{\sqrt{165}}\left(10\hat{i} + 7\hat{j} - 4\hat{k}\right)$$

Unit vector perpendicular to the plane $ABC = \dfrac{1}{\sqrt{165}}\left(10\hat{i} + 7\hat{j} - 4\hat{k}\right)$

Question 20

If a, b, c are the length of sides, BC, CA and AB of a triangle ABC, prove that $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$ and deduce that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Solution 20

Here, It is given that

In $\triangle ABC$

$\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}$

$= \overrightarrow{BA} + \overrightarrow{AB}$

$= \overrightarrow{BA} - \overrightarrow{BA}$ $\qquad \left[\text{Since, } \overrightarrow{BA} = -\overrightarrow{AB}\right]$

$= \vec{0}$

Given that, $\left|\overrightarrow{BC}\right| = a$

$\left|\overrightarrow{CA}\right| = b$

$\left|\overrightarrow{AB}\right| = c$

Let, $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$

We have,

$\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$

$\Rightarrow \quad \overrightarrow{BC} + \overrightarrow{CA} = -\overrightarrow{AB}$

$\Rightarrow \quad \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$

$\Rightarrow \quad \vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \quad \vec{a} \times \left(\vec{a} + \vec{b} + \vec{c}\right) = \vec{a} \times \vec{0}$

$\Rightarrow \quad \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$

$\Rightarrow \quad \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$ $\qquad \left[\text{Since, } \vec{a} \times \vec{a} = \vec{0}\right]$

$\Rightarrow \quad \vec{a} \times \vec{b} = -\left(\vec{a} \times \vec{c}\right)$

$\Rightarrow \quad \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ $\qquad ---(\text{i})$

Again, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$$\Rightarrow \quad \vec{b}\times\left(\vec{a}+\vec{b}+\vec{c}\right)=\vec{b}\times\vec{0}$$
$$\Rightarrow \quad \vec{b}\times\vec{a}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c}=\vec{0}$$
$$\Rightarrow \quad \vec{b}\times\vec{a}+\vec{0}+\vec{b}\times\vec{c}=\vec{0} \qquad \left[\text{Since, } \vec{b}\times\vec{b}=\vec{0}\right]$$
$$\Rightarrow \quad \vec{b}\times\vec{a}+\vec{b}\times\vec{c}=\vec{0}$$
$$\Rightarrow \quad \vec{b}\times\vec{c}=-\left(\vec{b}\times\vec{a}\right)$$

$$\Rightarrow \quad \vec{b}\times\vec{c}=\vec{a}\times\vec{b} \qquad ---(\text{ii})$$

From equation (i) and (ii), we get

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$$

$$\Rightarrow \quad \left|\vec{a}\times\vec{b}\right|=\left|\vec{b}\times\vec{c}\right|=\left|\vec{c}\times\vec{a}\right|$$
$$\Rightarrow \quad \left|\vec{a}\right|\left|\vec{b}\right|\sin\left(\pi-C\right)=\left|\vec{b}\right|\left|\vec{c}\right|\sin\left(\pi-A\right)=\left|\vec{c}\right|\left|\vec{a}\right|\sin\left(\pi-B\right)$$
$$\Rightarrow \quad ab\sin C=bc\sin A=ca\sin B$$

Dividing by abc

$$\Rightarrow \quad \frac{ab\sin C}{abc}=\frac{bc\sin A}{abc}=\frac{ca\sin B}{abc}$$
$$\Rightarrow \quad \frac{\sin C}{c}=\frac{\sin A}{a}=\frac{\sin B}{b}$$

$$\Rightarrow \quad \frac{c}{\sin C}=\frac{a}{\sin A}=\frac{b}{\sin B}$$

Question 21

If $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}-5\hat{k}$, then find $\vec{a}\times\vec{b}$. Verify that $\vec{a}$ and $\vec{a}\times\vec{b}$ are perpendicular to each other.

Solution 21

Here, $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 3 & -5 \end{vmatrix}$$

$$= \hat{i}\left(10 - 9\right) - \hat{j}\left(-5 - 6\right) + \hat{k}\left(3 + 4\right)$$

$$= \hat{i} + 11\hat{j} + 7\hat{k}$$

Now, $\vec{a}.\left(\vec{a} \times \vec{b}\right) = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right).\left(\hat{i} + 11\hat{j} + 7\hat{k}\right)$

$$\vec{a}.\left(\vec{a} \times \vec{b}\right) = (1)(1) + (-2)(11) + (3)(7)$$
$$= 1 - 22 + 21$$
$$= 22 - 22$$

$$\vec{a} \times \left(\vec{a} \times \vec{b}\right) = 0$$

Dot product of $\vec{a}$ and $\vec{a} \times \vec{b}$ is zero, then,

$\vec{a}$ is perpendicular to $\left(\vec{a} \times \vec{b}\right)$

Question 22

If $\vec{p}$ and $\vec{q}$ are unit vectorsforming an angle of 30°; find the area of the parallelogram having $\vec{a} = \vec{p} + 2\vec{q}$ and $\vec{b} = 2\vec{p} + \vec{q}$ as its diagonals.

Solution 22

Given $\vec{p}$ and $\vec{q}$ be unit vector with angle 30° between then

$$\left|\vec{p}\right| = \left|\vec{q}\right| = 1$$

$$\vec{p} \times \vec{q} = \left|\vec{p}\right| \left|\vec{q}\right| \sin 30° \hat{n}$$
$$= 1.1.\left(\frac{1}{2}\right)\hat{n}$$

$$\left|\vec{p} \times \vec{q}\right| = \left|\frac{\hat{n}}{2}\right|$$
$$\left|\vec{p} \times \vec{q}\right| = \frac{1}{2} \qquad ---(i) \qquad \left[\text{Since, } \hat{n} \text{ is a unit vector}\right]$$

$$\text{Area of parallelogram} = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$$
$$= \frac{1}{2}\left|\left(\vec{p} + 2\vec{q}\right) \times \left(2\vec{p} + \vec{q}\right)\right|$$
$$= \frac{1}{2}\left|\vec{p} \times 2\vec{p} + \vec{p} \times \vec{q} + 2\vec{q} \times 2\vec{p} + 2\vec{q} \times \vec{q}\right|$$
$$= \frac{1}{2}\left|\vec{0} + \vec{p} \times \vec{q} + 2\vec{q} \times 2\vec{p} + \vec{0}\right| \qquad \left[\text{Since, } \vec{p} \times 2\vec{q} = \vec{0} \text{ and } 2\vec{q} \times \vec{q} = \vec{0}\right]$$
$$= \frac{1}{2}\left|\vec{p} \times \vec{q} + 4\left(\vec{q} \times \vec{p}\right)\right|$$
$$= \frac{1}{2}\left|\left(\vec{p} \times \vec{q}\right) - 4\left(\vec{p} \times \vec{q}\right)\right| \qquad \left[\text{Since, } \vec{q} \times \vec{p} = -\vec{p} \times \vec{q}\right]$$
$$= \frac{1}{2}\left|-3\left(\vec{p} \times \vec{q}\right)\right|$$
$$= \frac{3}{2}\left|\vec{p} \times \vec{q}\right|$$
$$= \frac{3}{2} \times \frac{1}{2} \qquad \left[\text{Using (i)}\right]$$

$$= \frac{3}{4}$$

$$\text{Area of parallelogram} = \frac{3}{4} \text{ sq. unit}$$

Question 23

For any two vectors $\vec{a}$ and $\vec{b}$, prove that

$$\left|\vec{a} \times \vec{b}\right|^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} \end{vmatrix}$$

Solution 23

We know that

$$\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}|.\sin\theta.\hat{n}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta.|\hat{n}|$$

$$= |\vec{a}||\vec{b}|\sin\theta.1 \qquad [\text{Since, } \hat{n} \text{ is unit vector}]$$

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

Squaring both the sides,

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

$$= |\vec{a}|^2 |\vec{b}|^2 \left(1 - \cos^2\theta\right)$$

$$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$$

$$= |\vec{a}|^2 |\vec{b}|^2 - \left(|\vec{a}||\vec{b}|\cos\theta\right)^2$$

$$= \left(\vec{a}\right)^2 \left(\vec{b}\right)^2 - \left(\vec{a}.\vec{b}\right)^2 \qquad \left[\text{Since, } |\vec{a}||\vec{b}|\cos\theta = \vec{a}.\vec{b}\right]$$

$$|\vec{a} \times \vec{b}|^2 = \left(\vec{a}.\vec{a}\right)\left(\vec{b}.\vec{b}\right) - \left(\vec{a}.\vec{b}\right)\left(\vec{b}.\vec{a}\right) \qquad \left[\text{Since, } \left(\vec{a}.\vec{b}\right) = \left(\vec{b}.\vec{a}\right)\right]$$

$$|\vec{a} \times \vec{b}|^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} \end{vmatrix}$$

Question 24

Define $\vec{a} \times \vec{b}$ and prove that $|\vec{a} \times \vec{b}| = \left(\vec{a}.\vec{b}\right)\tan\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Solution 24

Define of $\vec{a}\times\vec{b}$: – Let $\vec{a}$, $\vec{b}$ be two non-zero, non-parallel vectors. Then $\vec{a}\times\vec{b}$, in that order, is defined as a vector whose magnitude is $|\vec{a}|\,|\vec{b}|\sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ and whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ and this constitute a right handed system.

$$\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$$

Where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a},\vec{b},\hat{n}$ form a right handed system.

Now,

$$\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\ \hat{n}$$

$$|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin\theta|\,|\hat{n}|$$

$$= |\vec{a}|\,|\vec{b}|\,|\sin\theta|.1$$

$$|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin\theta|$$

$$= \frac{\vec{a}.\vec{b}}{\cos\theta}.\sin\theta \qquad \left[\text{Since, } \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right]$$

$$|\vec{a}\times\vec{b}| = \vec{a}.\vec{b}.\tan\theta$$

Question 25

If $|\vec{a}| = \sqrt{26}$, $|\vec{b}| = 7$ and $|\vec{a}\times\vec{b}| = 35$, find $\vec{a}.\vec{b}$.

Solution 25

$\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}| \sin\theta . \hat{n}$

$|\vec{a} \times \vec{b}| = |\vec{a}|\ |\vec{b}|\, |\sin\theta|\ |\hat{n}|$

$35 = \sqrt{26}.7\ |\sin\theta|.1$

$\sin\theta = \dfrac{35}{\sqrt{26}.7}$

$\sin\theta = \dfrac{5}{\sqrt{26}}$

$\cos^2\theta = 1 - \sin^2\theta$

$= 1 - \left(\dfrac{5}{\sqrt{26}}\right)^2$

$= \dfrac{1}{1} - \dfrac{25}{26}$

$= \dfrac{26 - 25}{26}$

$= \dfrac{1}{26}$

$\cos\theta = \dfrac{1}{\sqrt{26}}$

$\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}| \cos\theta$

$= \sqrt{26}.7.\dfrac{1}{\sqrt{26}}$

$\vec{a}.\vec{b} = 7$

Question 26

Find the area of the triangle formed by O, A, B when $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\overrightarrow{OB} = -3\hat{i} - 2\hat{j} + \hat{k}$.

Solution 26

Area of triangle = $\frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$

$$\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix}$$

$$= \hat{i}\left(2+6\right) - \hat{j}\left(1+9\right) + \hat{k}\left(-2+6\right)$$
$$= 8\hat{i} - 10\hat{j} + 4\hat{k}$$

$$= 2\left(4\hat{i} - 5\hat{j} + 2\hat{k}\right)$$

Area of triangle = $\frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$

$$= \frac{1}{2}\left[2\sqrt{\left(4\right)^2 + \left(-5\right)^2 + \left(2\right)^2}\right]$$
$$= \frac{1}{2}\left[2\sqrt{16+25+4}\right]$$
$$= \sqrt{45}$$

$$= 3\sqrt{5}$$

Area of triangle = $3\sqrt{5}$ Sq.unit

Question 27

Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}.\vec{d} = 15$.

Solution 27

Let $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$.

Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, we have:

$$\vec{d} \cdot \vec{a} = 0$$
$$\Rightarrow d_1 + 4d_2 + 2d_3 = 0 \qquad \text{...(i)}$$
And,
$$\vec{d} \cdot \vec{b} = 0$$
$$\Rightarrow 3d_1 - 2d_2 + 7d_3 = 0 \qquad \text{...(ii)}$$

Also, it is given that:

$$\vec{c} \cdot \vec{d} = 15$$
$$\Rightarrow 2d_1 - d_2 + 4d_3 = 15 \qquad \text{...(iii)}$$

On solving (i), (ii), and (iii), we get:

$$d_1 = \frac{160}{3}, d_2 = -\frac{5}{3} \text{ and } d_3 = -\frac{70}{3}$$
$$\therefore \vec{d} = \frac{160}{3}\hat{i} - \frac{5}{3}\hat{j} - \frac{70}{3}\hat{k} = \frac{1}{3}\left(160\hat{i} - 5\hat{j} - 70\hat{k}\right)$$

Hence, the required vector is $\frac{1}{3}\left(160\hat{i} - 5\hat{j} - 70\hat{k}\right)$.

Question 28

Find a unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Solution 28

Given, $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

Let, $\vec{d} = \vec{a} + \vec{b}$

$= \left(3\hat{i} + 2\hat{j} + 2\hat{k}\right) + \left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$

$\vec{d} = 4\hat{i} + 4\hat{j} - 0\,\hat{k}$

And, $\vec{e} = \vec{a} - \vec{b}$

$= \left(3\hat{i} + 2\hat{j} + 2\hat{k}\right) - \left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$

$\vec{e} = 2\hat{i} + 4\hat{k}$

Let, $\vec{f}$ be any vector perpendicular to both $\vec{d}$ and $\vec{e}$

$$\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$= \hat{i}\left(16 - 0\right) - \hat{j}\left(16 - 0\right) + \hat{k}\left(0 - 8\right)$

$\vec{f} = 16\hat{i} - 16\hat{j} - 8\hat{k}$

$= 8\left(2\hat{i} - 2\hat{j} - \hat{k}\right)$

Let $\vec{g}$ be the required vector, then

$\vec{g} = \lambda\vec{f}$ and $\left|\vec{g}\right| = 1$

$\vec{g} = 8\lambda\left(2\hat{i} - 2\hat{j} - \hat{k}\right)$ --- (i)

$\left|\vec{g}\right| = 1$

$8\lambda\sqrt{\left(2\right)^2 + \left(-2\right)^2 + \left(-1\right)^2} = 1$

$8\lambda\sqrt{4 + 4 + 1} = 1$

$8\lambda\sqrt{9} = 1$

$24\lambda = 1$

$\lambda = \dfrac{1}{24}$

Put λ in (i)

$$\vec{g} = 8\left(\frac{1}{24}\right)\left(2\hat{i} - 2\hat{j} - \hat{k}\right)$$
$$\vec{g} = \frac{1}{3}\left(2\hat{i} - 2\hat{j} - \hat{k}\right)$$

Thus,

Unit vector perpendicular to $\left(\vec{a} + \vec{b}\right)$ and $\left(\vec{a} - \vec{b}\right) = \frac{1}{3}\left(2\hat{i} - 2\hat{j} - \hat{k}\right)$

Question 29

Using vectors find the area of the triangle with vertices, $A(2,3,5)$, $B(3,5,8)$ and $C(2,7,8)$.

Solution 29

Given, $A = (2,3,5)$

$B = (3,5,8)$

$C = (2,7,8)$

Position vector of $A = 2\hat{i} + 3\hat{j} + 5\hat{k}$

Position vector of $B = 3\hat{i} + 5\hat{j} + 8\hat{k}$

Position vector of $C = 2\hat{i} + 7\hat{j} + 8\hat{k}$

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= \left(3\hat{i} + 5\hat{j} + 8\hat{k}\right) - \left(2\hat{i} + 3\hat{j} + 5\hat{k}\right)$

$= 3\hat{i} + 5\hat{j} + 8\hat{k} - 2\hat{i} - 3\hat{j} - 5\hat{k}$

$\overrightarrow{AB} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\overrightarrow{AC}$ = Position vector of C – Position vector of A

$= \left(2\hat{i} + 7\hat{j} + 8\hat{k}\right) - \left(2\hat{i} + 3\hat{j} + 5\hat{k}\right)$

$= 2\hat{i} + 7\hat{j} + 8\hat{k} - 2\hat{i} - 3\hat{j} - 5\hat{k}$

$\overrightarrow{AC} = 4\hat{j} + 3\hat{k}$

Area of triangle $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$

$= \hat{i}\,(6 - 12) - \hat{j}\,(3 - 0) + \hat{k}\,(4 - 0)$

$\overrightarrow{AB} \times \overrightarrow{AC} = -6\hat{i} - 3\hat{j} + 4\hat{k}$

$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-6)^2 + (-3)^2 + (4)^2}$

$= \sqrt{36 + 9 + 16}$

$= \sqrt{61}$

Area of triangle $= \frac{1}{2}\sqrt{61}$ Sq. unit

Question 30

If $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{k}$, and $\vec{c} = 2\hat{j} - \hat{k}$ are three vectors, find the area of a parallelogram having diagonals $\left(\vec{a} + \vec{b}\right)$ and $\left(\vec{b} + \vec{c}\right)$.

Solution 30

$\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}, \vec{b} = -\hat{i} + \hat{k}, \vec{c} = 2\hat{j} - \hat{k}$

$(\vec{a} + \vec{b}) = 2\hat{i} - 3\hat{j} + \hat{k} - \hat{i} + \hat{k}$

$= \hat{i} - 3\hat{j} + 2\hat{k}$

$(\vec{b} + \vec{c}) = -\hat{i} + \hat{k} + 2\hat{j} - \hat{k} = -\hat{i} + 2\hat{j}$

$$\text{Area of a parallelogram} = \frac{1}{2}\left|(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})\right|$$

$$= \frac{1}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left|\vec{i}(0-4) - \vec{j}(0+2) + \vec{k}(2-3)\right|$$

$$= \frac{1}{2}\left|-4\vec{i} - 2\vec{j} - \vec{k}\right|$$

$$= \frac{1}{2}\left|\sqrt{(-4)^2 + (-2)^2 + (-1)^2}\right|$$

$$= \frac{1}{2}\left|\sqrt{21}\right| \text{sq.units}$$

Question 31

The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to one of its diagonals. Also, find its area.

Solution 31

let the adjacent sides be $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$

$\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$

$\vec{a} + \vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k} + \hat{i} - 2\hat{j} - 3\hat{k}$

$= 3\hat{i} - 6\hat{j} + 2\hat{k}$

$\vec{a} - \vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k} - \hat{i} + 2\hat{j} + 3\hat{k}$

$= \hat{i} - 2\hat{j} + 8\hat{k}$

Unit vector parallel to $3\hat{i} - 6\hat{j} + 2\hat{k}$

$$\frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{3^2 + (-6)^2 + 2^2}}$$

$$= \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{49}}$$

$$= \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{7}$$

$$= \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$$

Area of the parallelogram,

$$= \frac{1}{2}\left|\left(\vec{a} + \vec{b}\right) \times \left(\vec{a} - \vec{b}\right)\right|$$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 1 & -2 & 8 \end{vmatrix}$$

$$= \frac{1}{2}\left|\hat{i}(-48 + 4) - \hat{j}(24 - 2) + \hat{k}(-6 + 6)\right|$$

$$= \frac{1}{2}\left|-44\hat{i} - 22\hat{j}\right|$$

$$= \frac{1}{2}\left|\sqrt{(-44)^2 + (-22)^2}\right|$$

$$= \frac{1}{2}\left|11\sqrt{(-4)^2 + (-2)^2}\right|$$

$$= \frac{1}{2}\left|11\sqrt{20}\right|$$

$$= \frac{1}{2}\left|22\sqrt{5}\right|$$

$= 11\sqrt{5}$ sq.units

Question 32

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.

Solution 32

Take any parallel non-zero vectors so that $\vec{a}\times\vec{b}=\vec{0}$.

Let $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k},\ \vec{b}=4\hat{i}+6\hat{j}+8\hat{k}$.

Then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 4 & 6 & 8\end{vmatrix}=\hat{i}\left(24-24\right)-\hat{j}\left(16-16\right)+\hat{k}\left(12-12\right)=0\hat{i}+0\hat{j}+0\hat{k}=\vec{0}$$

It can now be observed that:

$$|\vec{a}|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$$
$$\therefore \vec{a}\neq\vec{0}$$
$$|\vec{b}|=\sqrt{4^2+6^2+8^2}=\sqrt{116}$$
$$\therefore \vec{b}\neq\vec{0}$$

Hence, the converse of the given statement need not be true.

Question 33

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$,
then verify that $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.

Solution 33

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$
$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\vec{a} \times (\vec{b} + \vec{c})$$
$$= \left[a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right] \times \left[(b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}\right]$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ (b_1 + c_1) & (b_2 + c_2) & (b_3 + c_3) \end{vmatrix}$$
$$= \hat{i}\left[a_2(b_3 + c_3) - a_3(b_2 + c_2)\right] - \hat{j}\left[a_1(b_3 + c_3) - a_3(b_1 + c_1)\right] + \hat{k}\left[a_1(b_2 + c_2) - a_2(b_1 + c_1)\right]$$

$$\vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$
$$= \left[a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right] \times \left[b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right] + \left[a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right] \times \left[c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\right]$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= \hat{i}\left[a_2b_3 - a_3b_2\right] - \hat{j}\left[a_1b_3 - a_3b_1\right] + \hat{k}\left[a_1b_2 - a_2b_1\right] + \hat{i}\left[a_2c_3 - a_3c_2\right] - \hat{j}\left[a_1c_3 - a_3c_1\right] + \hat{k}\left[a_1c_2 - a_2c_1\right]$$
$$= \hat{i}\left[a_2(b_3 + c_3) - a_3(b_2 + c_2)\right] - \hat{j}\left[a_1(b_3 + c_3) - a_3(b_1 + c_1)\right] + \hat{k}\left[a_1(b_2 + c_2) - a_2(b_1 + c_1)\right]$$
$$= \vec{a} \times (\vec{b} + \vec{c})$$

Question 34

Using Vectors, find the area of the triangle with vertices: (i) A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5) (ii) A (1, 2, 3), B(2, -1, 4) and C (4, 5, -1).

Solution 34

Given that

$$A = (1, 1, 2)$$
$$B = (2, 3, 5)$$
$$C = (1, 5, 5)$$

Position vector of $A = \hat{i} + \hat{j} + 2\hat{k}$

Position vector of $B = 2\hat{i} + 3\hat{j} + 5\hat{k}$

Position vector of $C = \hat{i} + 5\hat{j} + 5\hat{k}$

$\overline{AB}$ = Position vector of B − Position vector of A

$$= 2\hat{i} + 3\hat{j} + 5\hat{k} - (\hat{i} + \hat{j} + 2\hat{k})$$
$$= \hat{i} + 2\hat{j} + 3\hat{k}$$

$\overline{AC}$ − Position vector of C − Position vector of A

$$= \hat{i} + 5\hat{j} + 5\hat{k} - (\hat{i} + \hat{j} + 2\hat{k})$$
$$= 4\hat{j} + 3\hat{k}$$

Area of triangle $= \frac{1}{2}\left|\overline{AB} \times \overline{AC}\right|$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$
$$= \hat{i}(6-12) - \hat{j}(3-0) + \hat{k}(4-0)$$
$$= -6\hat{i} - 3\hat{j} + 4\hat{k}$$
$$\left|\overline{AB} \times \overline{AC}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2}$$
$$- \sqrt{36 + 9 + 16}$$
$$= \sqrt{61}$$

Area of the triangle $= \frac{1}{2}\sqrt{61}$ Sq.unit

Question 35

Find all vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane of $\hat{i} + 2\hat{j} + \hat{k}$ and $-\hat{i} + 3\hat{j} + 4\hat{k}$.

Solution 35

$\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$

$\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{a} \times \vec{b}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{vmatrix}$

$= 5\hat{i} - 5\hat{j} + 5\hat{k}$

$\left|\vec{a} \times \vec{b}\right| = \sqrt{25 + 25 + 25} = \pm 5\sqrt{3}$

$\therefore \text{Required vector} = 10\sqrt{3}\left[\frac{\vec{a} \times \vec{b}}{\left|\vec{a} \times \vec{b}\right|}\right] = 10\sqrt{3}\left[\frac{5\hat{i} - 5\hat{j} + 5\hat{k}}{\pm 5\sqrt{3}}\right] = \pm 10\left(\hat{i} - \hat{j} + \hat{k}\right)$

Chapter 25 - Vector or Cross Product Exercise MCQ

Question 1

If $\vec{a}$ is any vector, then $\left(\vec{a} \times \hat{i}\right)^2 + \left(\vec{a} \times \hat{j}\right)^2 + \left(\vec{a} \times \hat{k}\right)^2 =$

(a) $\vec{a}^2$

(b) $2\vec{a}^2$

(c) $3\vec{a}^2$

(d) $4\vec{a}^2$

Solution 1

Correct option: (b)

To find $\left(\vec{a}\times\hat{i}\right)^2+\left(\vec{a}\times\hat{j}\right)^2+\left(\vec{a}\times\hat{k}\right)^2$

consider, $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$\vec{a}\times\hat{i}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ 1 & 0 & 0\end{vmatrix}$$

$\vec{a}\times\hat{i}=a_3\hat{j}-a_2\hat{k}$

Similarly you can find

$\vec{a}\times\hat{j}=-a_3\hat{i}+a_1\hat{k}$

$\vec{a}\times\hat{k}=a_2\hat{i}-a_1\hat{j}$

$\left(\vec{a}\times\hat{i}\right)^2+\left(\vec{a}\times\hat{j}\right)^2+\left(\vec{a}\times\hat{k}\right)^2=2\left(a_1^2+a_2^2+a_3^2\right)$

$\left(\vec{a}\times\hat{i}\right)^2+\left(\vec{a}\times\hat{j}\right)^2+\left(\vec{a}\times\hat{k}\right)^2=2\vec{a}^2$

As $\sqrt{a_1^2+a_2^2+a_3^2}=\vec{a}$

Question 2

If $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$ and $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, $\vec{a}\neq 0$, then

(a) $\vec{b}=\vec{c}$

(b) $\vec{b}=\vec{0}$

(c) $\vec{b}+\vec{c}=\vec{0}$

(d) none of these

Solution 2

Correct option: (a)

$\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$ and $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$

$\Rightarrow\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}=0$ and $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$

$\Rightarrow\vec{a}\cdot\left(\vec{b}-\vec{c}\right)=0$ and $\vec{a}\times\left(\vec{b}-\vec{c}\right)=0$

Obviously $\vec{b}-\vec{c}=0\Rightarrow\vec{b}=\vec{c}$

Also,

$\Rightarrow\left|\vec{a}\right|\left|\left(\vec{b}-\vec{c}\right)\right|\cos\theta=0$ and $\left|\vec{a}\right|\left|\left(\vec{b}-\vec{c}\right)\right|\sin\theta=0$

$\Rightarrow$ If $\theta=\dfrac{\pi}{2}\Rightarrow\sin\theta=1\Rightarrow\vec{b}-\vec{c}=0\Rightarrow\vec{b}=\vec{c}$

Question 3

The vector $\vec{b} = 3\hat{i} + 4\hat{k}$ is to be written as the sum of a vector $\vec{\alpha}$ parallel to $\vec{a} = \hat{i} + \hat{j}$ and a vector $\vec{\beta}$ perpendicular to $\vec{a}$. Then $\vec{\alpha} =$

(a) $\frac{3}{2}\left(\hat{i} + \hat{j}\right)$

(b) $\frac{2}{3}\left(\hat{i} + \hat{j}\right)$

(c) $\frac{1}{2}\left(\hat{i} + \hat{j}\right)$

(d) $\frac{1}{3}\left(\hat{i} + \hat{j}\right)$

Solution 3

Correct option: (a)

Let $\vec{\alpha} = \alpha_1\hat{i} + \alpha_2\hat{j} + \alpha_3\hat{k}$, $\vec{\beta} = \beta_1\hat{i} + \beta_2\hat{j} + \beta_3\hat{k}$

$\vec{b} = 3\hat{i} + 4\hat{k}$

$\vec{\alpha} + \vec{\beta} = 3\hat{i} + 4\hat{k}$

$(\alpha_1 + \beta_1)\hat{i} + (\alpha_2 + \beta_2)\hat{j} + (\alpha_3 + \beta_3)\hat{k} = 3\hat{i} + 4\hat{k}$

$\Rightarrow \alpha_1 + \beta_1 = 3$

$\alpha_2 + \beta_2 = 0$

$\alpha_3 + \beta_3 = 4$

Given that $\vec{\alpha}$ is parallel to $\vec{a}$.

$\vec{\alpha} \times \vec{a} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha_1 & \alpha_2 & \alpha_3 \\ 1 & 1 & 0 \end{vmatrix} = 0 \qquad \left(\text{Given } \vec{a} = \hat{i} + \hat{j}\right)$$

$-\alpha_3\hat{i} + \alpha_3\hat{j} + (\alpha_1 - \alpha_2)\hat{k} = 0$

$\alpha_3 = 0,\ \alpha_1 - \alpha_2 = 0$

$\alpha_3 = 0,\ \alpha_1 = \alpha_2$

Given $\vec{\beta}$ is perpendicular to $\vec{a}$.

$\vec{\beta} \cdot \vec{a} = 0$

$(\beta_1\hat{i} + \beta_2\hat{j} + \beta_3\hat{k}) \cdot (\hat{i} + \hat{j}) = 0$

$\beta_1 + \beta_2 = 0$

$\beta_1 = -\beta_2$

Solving $\alpha_3 = 0,\ \alpha_1 = \alpha_2, \alpha_1 + \beta_1 = 3$

$\alpha_2 + \beta_2 = 0, \alpha_3 + \beta_3 = 4,\ \beta_1 = -\beta_2$

$\Rightarrow \alpha_1 = \alpha_2 = \dfrac{3}{2},\ \alpha_3 = 0$

$\vec{\alpha} = \alpha_1\hat{i} + \alpha_2\hat{j} + \alpha_3\hat{k}$

$\vec{\alpha} = \dfrac{3}{2}(\hat{i} + \hat{j})$

Question 4

The unit vector perpendicular to the plane passing through points $P(\hat{i} - \hat{j} + 2\hat{k})$, $Q(2\hat{i} - \hat{k})$ and $R\left(2\hat{j} + \hat{k}\right)$ is

(a) $2\hat{i} + \hat{j} + \hat{k}$

(b) $\sqrt{6}\left(2\hat{i} + \hat{j} + \hat{k}\right)$

(c) $\frac{1}{\sqrt{6}}\left(2\hat{i} + \hat{j} + \hat{k}\right)$

(d) $\frac{1}{6}\left(2\hat{i} + \hat{j} + \hat{k}\right)$

Solution 4

Correct option: (c)

$P(\hat{i} - \hat{j} + 2\hat{k})$, $Q(2\hat{i} - \hat{k})$ and $R\left(2\hat{j} + \hat{k}\right)$

$\overrightarrow{PQ} = \hat{i} + \hat{j} - 3\hat{k}$

$\overrightarrow{PR} = -\hat{i} + 3\hat{j} - \hat{k}$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

$\overrightarrow{PQ} \times \overrightarrow{PR} = 8\hat{i} + 4\hat{j} + 4\hat{k}$

$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = 4\sqrt{6}$

$$\text{unit vector} = \frac{8\hat{i} + 4\hat{j} + 4\hat{k}}{4\sqrt{6}} = \frac{1}{\sqrt{6}}\left(2\hat{i} + \hat{j} + \hat{k}\right)$$

Question 5

If $\vec{a}$, $\vec{b}$ represent the diagonals of a rhombus, then

(a) $\vec{a} \times \vec{b} = \vec{0}$

(b) $\vec{a} \cdot \vec{b} = 0$

(c) $\vec{a} \cdot \vec{b} = 1$

(d) $\vec{a} \times \vec{b} = \vec{a}$

Solution 5

Correct option: (b)

Diagonals of a rhombus are perpendicular to each other.

$\Rightarrow \vec{a} \cdot \vec{b} = 0$

Question 6

Vectors $\vec{a}$ and $\vec{b}$ are inclined at angle $\theta = 120°$. If $|\vec{a}| = 1$, $|\vec{b}| = 2$, then $\left[\left(\vec{a} + 3\vec{b}\right) \times \left(3\vec{a} - \vec{b}\right)\right]^2$ is equal to

a. 300
b. 325
c. 275

d. 225

Solution 6

Correct option: (a)

$$\left(\vec{a} + 3\vec{b}\right) \times \left(3\vec{a} - \vec{b}\right)$$

$$= 3\left(\vec{a} \times \vec{a}\right) - \left(\vec{a} \times \vec{b}\right) + 9\left(\vec{b} \times \vec{a}\right) - 3\left(\vec{b} \times \vec{b}\right)$$

$$= -10\left(\vec{a} \times \vec{b}\right)$$

$$\left|\left(\vec{a} + 3\vec{b}\right) \times \left(3\vec{a} - \vec{b}\right)\right|^2$$

$$= 100\left(\vec{a} \times \vec{b}\right)^2$$

$$= 100 \times 4 \times \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= 300$$

Question 7

If $\vec{a} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - \hat{k}$,

then a unit vector normal to the vectors $\vec{a} + \vec{b}$ and $\vec{b} - \vec{c}$ is

(a) $\hat{i}$

(b) $\hat{j}$

(c) $\hat{k}$

(d) none of these

Solution 7

Correct option: (a)

$\vec{a} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - \hat{k}$

$\vec{a} + \vec{b} = 3\hat{j} + \hat{k}$, $\vec{b} - \vec{c} = 3\hat{k}$

$$\left(\vec{a} + \vec{b}\right) \times \left(\vec{b} - \vec{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{vmatrix}$$

$$\left(\vec{a} + \vec{b}\right) \times \left(\vec{b} - \vec{c}\right) = 9\hat{i}$$

$$\left(\vec{a} + \vec{b}\right) \times \left(\vec{b} - \vec{c}\right) = 9$$

Unit vector perpendicular to both $\left(\vec{a} + \vec{b}\right)$ and $\left(\vec{b} - \vec{c}\right)$

$$= \frac{9\hat{i}}{9} = \hat{i}$$

Question 8

A unit vector perpendicular to both $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$ is

(a) $\hat{i}-\hat{j}+\hat{k}$

(b) $\hat{i}+\hat{j}+\hat{k}$

(c) $\frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right)$

(d) $\frac{1}{\sqrt{3}}\left(\hat{i}-\hat{j}+\hat{k}\right)$

Solution 8

vector perpendicular to both $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 0\\ 0 & 1 & 1\end{vmatrix}$$

$$\vec{a}\times\vec{b}=\hat{i}-\hat{j}+\hat{k}$$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{3}$$

Unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$$=\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$$

NOTE: Answer not matching with back answer.

Question 9

If $\vec{a}=2\hat{i}-3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+4\hat{j}-2\hat{k}$, then $\vec{a}\times\vec{b}$ is

(a) $10\hat{i}+2\hat{j}+11\hat{k}$

(b) $10\hat{i}+3\hat{j}+11\hat{k}$

(c) $10\hat{i}-3\hat{j}+11\hat{k}$

(d) $10\hat{i}-2\hat{j}-10\hat{k}$

Solution 9

Correct option: (b)

Given $\vec{a}=2\hat{i}-3\hat{j}-\hat{k}$ and $\hat{i}+4\hat{j}-2\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & -1\\ 1 & 4 & -2\end{vmatrix}$$

$$\vec{a}\times\vec{b}=10\hat{i}+3\hat{j}+11\hat{k}$$

Question 10

If $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors, then

(a) $\hat{i} \cdot \hat{j} = 1$

(b) $\hat{i} \cdot \hat{i} = 1$

(c) $\hat{i} \times \hat{j} = 1$

(d) $\hat{i} \times \left(\hat{j} \times \hat{k}\right) = 1$

Solution 10

Correct option: (b)

Using property of dot and cross product.

Question 11

If θ is the angle between the vectors $2\hat{i} - 2\hat{j} + 4\hat{k}$ and $3\hat{i} + \hat{j} + 2\hat{k}$, then $\sin\theta =$

(a) $\frac{2}{3}$

(b) $\frac{2}{\sqrt{7}}$

(c) $\frac{\sqrt{2}}{7}$

(d) $\sqrt{\frac{2}{7}}$

Solution 11

Correct option: (b)

Let, $\vec{p} = 2\hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{q} = 3\hat{i} + \hat{j} + 2\hat{k}$,

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 4 \\ 3 & 1 & 2 \end{vmatrix} = -8\hat{i} + 8\hat{j} + 8\hat{k}$$

$$\left|\vec{p} \times \vec{q}\right| = \left|\vec{p}\right|\left|\vec{q}\right| \sin\theta$$

$$\sqrt{64 + 64 + 64} = \sqrt{4 + 4 + 16}\sqrt{9 + 1 + 4} \sin\theta$$

$$8\sqrt{3} = 2\sqrt{6}\sqrt{14} \sin\theta$$

$$\sin\theta = \frac{8\sqrt{3}}{2\sqrt{6}\sqrt{14}}$$

$$\sin\theta = \frac{2}{\sqrt{7}}$$

Question 12

If $|\vec{a}\times\vec{b}| = 4$, $|\vec{a}\cdot\vec{b}| = 2$, then $|\vec{a}|^2|\vec{b}|^2 =$

(a) 6

(b) 2

(c) 20

(d) 8

Solution 12

Correct option: (c)

We know that

$$\left(\vec{a}\cdot\vec{b}\right)^2 + \left(\vec{a}\times\vec{b}\right)^2 = |\vec{a}|^2 \cdot |\vec{b}|^2$$

$$|\vec{a}|^2 \cdot |\vec{b}|^2 = 2^2 + 4^2$$

$$|\vec{a}|^2 \cdot |\vec{b}|^2 = 20$$

Question 13

The value of $\left(\vec{a}\times\vec{b}\right)^2$ is

(a) $|\vec{a}|^2 + |\vec{b}|^2 - \left(\vec{a}\cdot\vec{b}\right)^2$

(b) $|\vec{a}|^2 |\vec{b}|^2 - \left(\vec{a}\cdot\vec{b}\right)^2$

(c) $|\vec{a}|^2 + |\vec{b}|^2 - 2\left(\vec{a}\cdot\vec{b}\right)$

(d) $|\vec{a}|^2 + |\vec{b}|^2 - \vec{a}\cdot\vec{b}$

Solution 13

Correct option: (b)

$$\left(\vec{a}\cdot\vec{b}\right)^2 + \left(\vec{a}\times\vec{b}\right)^2 = \left(|\vec{a}|\cdot|\vec{b}|\cos\theta\right)^2 + \left(|\vec{a}|\cdot|\vec{b}|\sin\theta\right)^2$$

$$\left(\vec{a}\cdot\vec{b}\right)^2 + \left(\vec{a}\times\vec{b}\right)^2 = |\vec{a}|^2\cdot|\vec{b}|^2\cos^2\theta + |\vec{a}|^2\cdot|\vec{b}|^2\sin^2\theta$$

$$\left(\vec{a}\cdot\vec{b}\right)^2 + \left(\vec{a}\times\vec{b}\right)^2 = |\vec{a}|^2\cdot|\vec{b}|^2\left(\cos^2\theta + \sin^2\theta\right)$$

$$\left(\vec{a}\cdot\vec{b}\right)^2 + \left(\vec{a}\times\vec{b}\right)^2 = |\vec{a}|^2\cdot|\vec{b}|^2$$

$$\left(\vec{a}\times\vec{b}\right)^2 = |\vec{a}|^2\cdot|\vec{b}|^2 - \left(\vec{a}\cdot\vec{b}\right)^2$$

Question 14

The value of $\hat{i}\cdot\left(\hat{j}\times\hat{k}\right)+\hat{j}\cdot\left(\hat{i}\times\hat{k}\right)+\hat{k}\cdot\left(\hat{i}\times\hat{j}\right)$, is

(a) 0

(b) -1

(c) 1

(d) 3

Solution 14

Correct option: (c)

$$\hat{i}\cdot\left(\hat{j}\times\hat{k}\right)+\hat{j}\cdot\left(\hat{i}\times\hat{k}\right)+\hat{k}\cdot\left(\hat{i}\times\hat{j}\right)$$
$$=\hat{i}\cdot\hat{i}+\hat{j}\cdot\left(-\hat{j}\right)+\hat{k}\cdot\hat{k}$$
$$=1-1+1$$
$$=1$$

Question 15

If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}\cdot\vec{b}|=|\vec{a}\times\vec{b}|$ when θ is equal to

(a) 0

(b) $\pi/4$

(c) $\pi/2$

(d) π

Solution 15

Correct option: (b)

$$|\vec{a}\cdot\vec{b}|=|\vec{a}\times\vec{b}|$$
$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta=|\vec{a}||\vec{b}|\sin\theta$$
$$\Rightarrow \cos\theta=\sin\theta$$
$$\Rightarrow \theta=\frac{\pi}{4}$$

Chapter 25 - Vector or Cross Product Exercise Ex. 25VSAQ

Question 1

Define vector product of two vectors.

Solution 1

Let, $\vec{a}.\vec{b}$ be two non-zero, non-parallel vectors. Then the vector product $\vec{a}\times\vec{b}$, in that order, is defined as a vector whose magnitude is $|\vec{a}|\,|\vec{b}|\sin\theta$, where θ is the angle between $\vec{a}\times\vec{b}$. And whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in such a way $\vec{a}$, $\vec{b}$ and this direction constitute a right handed system.

$\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta.\hat{n}$

Where θ is angle between $\vec{a}$ and $\vec{b}\,\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

Now, $\vec{a}\times\vec{b}$ gives a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

Question 2

Write the value $\left(\hat{i}\times\hat{j}\right).\hat{k}+\hat{i}.\hat{j}$.

Solution 2

Here, $\left(\hat{i}\times\hat{j}\right).\hat{k}+\hat{i}.\hat{j}$

$= \hat{k}.\hat{k}+\hat{i}.\hat{j}$ $\quad\left[\text{Since, } \hat{i}\times\hat{j}=\hat{k}\right]$

$= 1+0$ $\quad\left[\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1,\ \hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0\right]$

$= 1$

$\left(\hat{i}\times\hat{j}\right).\hat{k}+\hat{i}.\hat{j} = 1$

Question 3

Write the value of $\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{k}\times\hat{i}\right)+\hat{k}.\left(\hat{j}\times\hat{i}\right)$.

Solution 3

$\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{k}\times\hat{i}\right)+\hat{k}.\left(\hat{j}\times\hat{i}\right)$

$= \hat{i}.\left(\hat{i}\right)+\hat{j}.\left(\hat{j}\right)+\hat{k}.\left(-\hat{k}\right)$ $\quad\left[\text{Since, } \hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i}, \hat{k}\times\hat{i}=\hat{j},\ \hat{j}\times\hat{i}=-\hat{k}\right]$

$= 1+1-1$ $\quad\left[\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1\right]$

$= 2-1$

$= 1$

$\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{k}\times\hat{i}\right)+\hat{k}.\left(\hat{j}\times\hat{i}\right) = 1$

Question 4

Write the value of $\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{k}\times\hat{i}\right)+\hat{k}.\left(\hat{i}\times\hat{j}\right)$.

Solution 4

$\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{k}\times\hat{i}\right)+\hat{k}.\left(\hat{i}\times\hat{j}\right)$

$=\hat{i}.\left(\hat{i}\right)+\hat{j}.\left(\hat{j}\right)+\hat{k}.\left(\hat{k}\right)$ $\left[\text{Since, } \hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}\right]$

$=1+1+1$ $\left[\hat{i}\hat{i}=\hat{j}.\hat{j}=\hat{k}\hat{k}=1\right]$

$=3$

$\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{k}\times\hat{i}\right)+\hat{k}.\left(\hat{i}\times\hat{j}\right)=3$

Question 5

Write the value of $\hat{i}\times\left(\hat{j}+\hat{k}\right)+\hat{j}\times\left(\hat{k}+\hat{i}\right)+\hat{k}\times\left(\hat{i}+\hat{j}\right)$.

Solution 5

$\hat{i}\times\left(\hat{j}+\hat{k}\right)+\hat{j}\times\left(\hat{k}+\hat{i}\right)+\hat{k}\times\left(\hat{i}+\hat{j}\right)$

$=\hat{i}\times\hat{j}+\hat{i}\times\hat{k}+\hat{j}\times\hat{k}+\hat{j}\times\hat{i}+\hat{k}\times\hat{i}+\hat{k}\times\hat{j}$

$=\hat{i}\times\hat{j}-\hat{k}\times\hat{i}+\hat{j}\times\hat{k}-\hat{i}\times\hat{j}+\hat{k}\times\hat{i}-\hat{j}\times\hat{k}$ $\left[\hat{i}\times\hat{j}=-\hat{j}\times\hat{i},\hat{i}\times\hat{k}=-\hat{k}\times\hat{i},\hat{j}\times\hat{k}=-\hat{k}\times\hat{j}\right]$

$=\left(\hat{i}\times\hat{j}\right)-\left(\hat{i}\times\hat{j}\right)-\left(\hat{k}\times\hat{i}\right)+\left(\hat{k}\times\hat{i}\right)+\left(\hat{j}\times\hat{k}\right)-\left(\hat{j}\times\hat{k}\right)$

$=0$

$\hat{i}\times\left(\hat{j}+\hat{k}\right)+\hat{j}\times\left(\hat{k}+\hat{i}\right)+\hat{k}\times\left(\hat{i}+\hat{j}\right)=0$

Question 6

Write the expression for the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as its diagonals.

Solution 6

Let, $\vec{a}$ and $\vec{b}$ be the two vectors representing the diagonals of a parallelogram

Then,

Area of parallelogram $=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$

Question 7

For any two vectors $\vec{a}$ and $\vec{b}$ write the value of $\left(\vec{a}\vec{b}\right)^2+\left|\vec{a}\times\vec{b}\right|^2$ in terms of their magnitudes.

Solution 7

Let, θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}|.\sin\theta.\hat{n}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}|.|\sin\theta|.|\hat{n}|$$

$$= |\vec{a}|.|\vec{b}|.|\sin\theta|.1 \qquad \left[\text{Since, } \hat{n} \text{ is the unit vector}\right]$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}|.|\sin\theta|$$

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta \qquad ---(i)$$

Also,

$$\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}| \cos\theta$$

$$\left(\vec{a}.\vec{b}\right)^2 = |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta \qquad ---(ii)$$

Adding equation (i) and (ii),

$$|\vec{a} \times \vec{b}|^2 + \left(\vec{a}.\vec{b}\right)^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta + |\vec{a}|^2.|\vec{b}|^2.\cos^2\theta$$

$$= |\vec{a}|^2.|\vec{b}|^2\left(\sin^2\theta + \cos^2\theta\right)$$

$$= |\vec{a}|^2.|\vec{b}|^2.1 \qquad \left[\text{Since, } \sin^2\theta + \cos^2\theta = 1\right]$$

$$|\vec{a} \times \vec{b}|^2 + \left(\vec{a}.\vec{b}\right)^2 = |\vec{a}|^2.|\vec{b}|^2$$

Question 8

If $\vec{a}$ and $\vec{b}$ are two vectors of magnitudes 3 and $\frac{\sqrt{2}}{3}$ respectively such that $\vec{a} \times \vec{b}$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$.

Solution 8

Let, θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}|\sin\theta.\hat{n}$$
$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|.|\sin\theta|.|\hat{n}|$$

$$1 = 3.\frac{\sqrt{2}}{3}.|\sin\theta|.1 \qquad \left[\text{Given, } |\vec{a}| = 3,\ |\vec{b}| = \frac{\sqrt{2}}{3} \text{ and } \vec{a} \times \vec{b} \text{ is a unit vector } \hat{n} \text{ is a unit vector}\right]$$
$$1 = \sqrt{2}\,|\sin\theta|$$
$$|\sin\theta| = \frac{1}{\sqrt{2}}$$
$$\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\theta = \frac{\pi}{4} \quad \text{and} \quad \pi - \frac{\pi}{4}$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\theta = 45^\circ, 135^\circ$$

Question 9

If $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $|\vec{a} \times \vec{b}| = 16$, find $\vec{a}.\vec{b}$.

Solution 9

Let, θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}|.\sin\theta.\hat{n}$$
$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|.|\sin\theta|.|\hat{n}|$$

$$16 = 10.2.|\sin\theta|.1 \qquad \left[\text{Since, } \hat{n} \text{ is a unit vector}\right]$$
$$16 = 20|\sin\theta|$$
$$\sin\theta = \frac{16}{20}$$
$$\sin\theta = \frac{4}{5}$$

$$\cos^2\theta = 1 - \sin^2\theta$$
$$\cos^2\theta = 1 - \left(\frac{4}{5}\right)^2$$
$$\cos^2\theta = \frac{1}{1} - \frac{16}{25}$$
$$\cos^2\theta = \frac{25-16}{25}$$
$$\cos^2\theta = \frac{9}{25}$$

$$\cos\theta = \pm\frac{3}{5}$$

We know,

$$\vec{a} \bullet \vec{b} = |\vec{a}|.|\vec{b}|.\cos\theta$$
$$= 10.2.\pm\frac{3}{5}$$

$$\vec{a} \bullet \vec{b} = \pm 12$$

Question 10

For any two vectors $\vec{a}$ and $\vec{b}$, find $\vec{a}.(\vec{b} \times \vec{a})$.

Solution 10

Let, $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$$= \hat{i}\left(b_2c_1 - b_1c_2\right) - \hat{j}\left(a_2c_1 - a_1c_2\right) + \hat{k}\left(a_2b_1 - a_1b_2\right)$$

$$\vec{a}.\left(\vec{b} \times \vec{a}\right) = \left(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}\right)\left[\left(b_2c_1 - b_1c_2\right)\hat{i} - \hat{j}\left(a_2c_1 - a_1c_2\right) + \hat{k}\left(a_2b_1 - a_1b_2\right)\right]$$

$$= \left(a_1\right)\left(b_2c_1 - b_1c_2\right) + \left(b_1\right)\left[-\left(a_2c_1 - a_1c_2\right)\right] + \left(c_1\right)\left(a_2b_1 - a_1b_2\right)$$

$$= a_1b_2c_1 - a_1b_1c_2 - a_2b_1c_1 + a_1b_1c_2 + a_2b_1c_1 - a_1b_2c_1$$

$$= 0$$

$$\vec{a}.\left(\vec{b} \times \vec{a}\right) = 0$$

Question 11

If $\vec{a}$ and $\vec{b}$ are two vectors such that $\left|\vec{a} \times \vec{b}\right| = \sqrt{3}$ and $\vec{a}.\vec{b} = 1$, find the angle between.

Solution 11

Let, θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = \left|\vec{a}\right| . \left|\vec{b}\right| . \sin\theta . \hat{n}$$
$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right| \left|\vec{b}\right| \left|\sin\theta\right| \left|\hat{n}\right|$$

$$\sqrt{3} = \left|\vec{a}\right| \left|\vec{b}\right| \left|\sin\theta\right| . 1 \qquad \left[\text{Since, } \hat{n} \text{ is a unit vector and } \left|\vec{a} \times \vec{b}\right| = \sqrt{3} \ (\text{given})\right]$$
$$\sqrt{3} = \left|\vec{a}\right| \left|\vec{b}\right| \left|\sin\theta\right| \qquad ---(i)$$

Now,
$$\vec{a} . \vec{b} = \left|\vec{a}\right| . \left|\vec{b}\right| . \cos\theta$$
$$1 = \left|\vec{a}\right| \left|\vec{b}\right| \cos\theta \qquad ---(ii) \qquad \left[\vec{a} . \vec{b} = 1 \text{ given}\right]$$

Dividing equation (i) by (ii),

$$\frac{\sqrt{3}}{1} = \frac{\left|\vec{a}\right| . \left|\vec{b}\right| . \sin\theta}{\left|\vec{a}\right| . \left|\vec{b}\right| . \cos\theta}$$
$$\sqrt{3} = \tan\theta$$
$$\theta = \tan^{-1}\left(\sqrt{3}\right)$$

$$\theta = \frac{\pi}{3}$$

Question 12

For any three vectors write the value of $\vec{a} \times \left(\vec{b} + \vec{c}\right) + \vec{b} \times \left(\vec{c} + \vec{a}\right) + \vec{c} \times \left(\vec{a} + \vec{b}\right)$.

Solution 12

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) + \vec{b} \times \left(\vec{c} + \vec{a}\right) + \vec{c} \times \left(\vec{a} + \vec{b}\right)$$
$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$$

$$= \cancel{\vec{a} \times \vec{b}} + \cancel{\vec{a} \times \vec{c}} + \cancel{\vec{b} \times \vec{c}} - \cancel{\vec{a} \times \vec{b}} - \cancel{\vec{a} \times \vec{c}} - \cancel{\vec{b} \times \vec{c}} \qquad \left[\text{Since, } \vec{b} \times \vec{a} = -\left(\vec{a} \times \vec{b}\right), \left(\vec{c} \times \vec{a}\right) = -\left(\vec{a} \times \vec{c}\right), \left(\vec{c} \times \vec{b}\right) = -\left(\vec{b} \times \vec{c}\right)\right]$$

$$= \vec{0}$$

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) + \vec{b} \times \left(\vec{c} + \vec{a}\right) + \vec{c} \times \left(\vec{a} + \vec{b}\right) = \vec{0}$$

Question 13

For any two vectors $\vec{a}$ and $\vec{b}$, find $\left(\vec{a} \times \vec{b}\right) . \vec{b}$.

Solution 13

Let, $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

$$\left(\vec{a} \times \vec{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \hat{i}\left(b_1c_2 - b_2c_1\right) - \hat{j}\left(a_1c_2 - a_2c_1\right) + \hat{k}\left(a_1b_2 - a_2b_1\right)$$

$$\left(\vec{a} \times \vec{b}\right).\vec{b} = \left[\left(b_1c_2 - b_2c_1\right)\hat{i} - \hat{j}\left(a_1c_2 - a_2c_1\right) + \hat{k}\left(a_1b_2 - a_2b_1\right)\right].\left(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}\right)$$

$$= \left(b_1c_2 - b_2c_1\right)\left(a_1\right) + \left(-\left(a_1c_2 - a_2c_1\right)\right).b_1 + \left(a_1b_2 - a_2b_1\right)\left(c_1\right)$$

$$= \cancel{a_1b_1c_2} - \cancel{a_1b_2c_1} - \cancel{a_1b_1c_2} + \cancel{a_2b_1c_1} + \cancel{a_1b_2c_1} - \cancel{a_2b_1c_1}$$

$$= 0$$

$$\left(\vec{a} \times \vec{b}\right).\vec{b} = 0$$

Question 14

Write the value of $\hat{i} \times \left(\hat{j} \times \hat{k}\right)$.

Solution 14

$\hat{i} \times \left(\hat{j} \times \hat{k}\right) = \hat{i} \times \hat{i}$ $\qquad$ $\left[\text{Since, } \hat{j} \times \hat{k} = \hat{i}\right]$

$= \vec{0}$ $\qquad$ $\left[\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}\right]$

$\hat{i} \times \left(\hat{j} \times \hat{k}\right) = \vec{0}$

Question 15

If $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$, then find $\left(\vec{a} \times \vec{b}\right)\vec{a}$.

Solution 15

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \hat{i}\,(1 - 2) - \hat{j}\,(-3 - 4) + \hat{k}\,(3 + 2)$$
$$\vec{a} \times \vec{b} = -\hat{i} + 7\hat{j} + 5\hat{k}$$

$$\left(\vec{a} \times \vec{b}\right).\vec{a} = \left(-\hat{i} + 7\hat{j} + 5\hat{k}\right)\left(3\hat{i} - \hat{j} + 2\hat{k}\right)$$

In vectors, there are two kinds of prducts, the one a scalar and the other a vector.
But the product given in the problem is not having a dot or a cross
and this product is meaning less.

Question 16

Write a unit vector perpendicular to $\hat{i} + \hat{j}$ and $\hat{j} + \hat{k}$.

Solution 16

$\vec{a} = \hat{i} + \hat{j}$
$\vec{a} = \hat{i} + \hat{j} + 0.\hat{k}$
$\vec{b} = \hat{j} + \hat{k}$
$\vec{b} = 0.\hat{i} + \hat{j} + \hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$\vec{a} \times \vec{b} = \hat{i}\left(1 - 0\right) - \hat{j}\left(1 - 0\right) + \hat{k}\left(1 - 0\right)$
$\vec{a} \times \vec{b} = \hat{i} - \hat{j} + \hat{k}$

Unit vector perpendicular to $\vec{a}$ and $\vec{b} = \dfrac{\vec{a} \times \vec{b}}{\left|\vec{a} \times \vec{b}\right|}$

$$= \frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{\left(1\right)^2 + \left(-1\right)^2 + \left(1\right)^2}}$$

$$= \frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{1 + 1 + 1}}$$

$$= \frac{1}{\sqrt{3}}\left(\hat{i} - \hat{j} + \hat{k}\right)$$

Unit vector perpendicular to $\left(\hat{i} + \hat{j}\right)$ and $\left(\hat{j} + \hat{k}\right) = \dfrac{1}{\sqrt{3}}\left(\hat{i} - \hat{j} + \hat{k}\right)$

Question 17

If $\left|\vec{a} \times \vec{b}\right|^2 + \left(\vec{a}\,\vec{b}\right)^2 = 144$ and $\left|\vec{a}\right| = 4,$ find $\left|\vec{b}\right|$.

Solution 17

Let, θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}|.\sin\theta.\hat{n}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta|.|\hat{n}|$$

$$= |\vec{a}|.|\vec{b}|.|\sin\theta|.1 \qquad \left[\text{Since, } \hat{n} \text{ is a unit vector}\right]$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}|.|\sin\theta|$$

Squaring both the sides,

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2.|\vec{b}|^2.\sin^2\theta$$

$$|\vec{a} \times \vec{b}| = (4)^2.|\vec{b}|^2 \sin^2\theta$$

$$|\vec{a} \times \vec{b}| = 16|\vec{b}|\sin^2\theta \qquad ---(i)$$

We have,

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

Squaring both the sides,

$$(\vec{a}.\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$(\vec{a}.\vec{b})^2 = (4)^2|\vec{b}|^2\cos^2\theta$$

$$(\vec{a}.\vec{b})^2 = 16|\vec{b}|^2\cos^2\theta \qquad ---(ii)$$

Adding (i) and (ii),

$$|\vec{a} \times \vec{b}|^2 + (\vec{a}.\vec{b})^2 = 16|\vec{b}|^2\sin^2\theta + 16|\vec{b}|^2\cos^2\theta$$

$$144 = 16.|\vec{b}|^2.(\sin^2\theta + \cos^2\theta)$$

$$144 = 16|\vec{b}|^2(1) \qquad \left[\text{Since, } \sin^2\theta + \cos^2\theta = 1\right]$$

$$16.|\vec{b}|^2 = 144$$

$$|\vec{b}|^2 = \frac{144}{16}$$

$$|\vec{b}|^2 = 9$$

$$|\vec{b}| = 3$$

Question 18

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then write the value of $|\vec{r} \times \hat{i}|^2$.

Solution 18

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$$

$$\vec{r}\times\hat{i}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ 1 & 0 & 0\end{vmatrix}$$

$$=\hat{i}\,(0-0)-\hat{j}\,(0-z)+\hat{k}\,(0-y)$$

$$\vec{r}\times\hat{i}=0\,\hat{i}+z.\hat{j}-y.\hat{k}$$

$$\left|\vec{r}\times\hat{i}\right|=\sqrt{(0)^2+(z)^2+(-y)^2}$$

$$\left|\vec{r}\times\hat{i}\right|=\sqrt{z^2+y^2}$$

Squaring both the sides,

$$\left|\vec{r}\times\hat{i}\right|^2=z^2+y^2$$

Question 19

If $\vec{a}$ and $\vec{b}$ are unit vectors such that $\vec{a}\times\vec{b}$ is also a unit vector, find the angle between $\vec{a}$ and $\vec{b}$.

Solution 19

Let, θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a}\times\vec{b}=\left|\vec{a}\right|.\left|\vec{b}\right|.\sin\theta.\hat{n}$$

$$\left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|.\left|\vec{b}\right|.\left|\sin\theta\right|.\left|\hat{n}\right|$$

$$1=1.1.\left|\sin\theta\right|.1 \qquad \left[\text{Since, } \hat{n} \text{ is a unit vector and given that } \vec{a}, \vec{b} \text{ and } \vec{a}\times\vec{b} \text{ are unit vector}\right]$$

$$1=\left|\sin\theta\right|$$

$$\theta=\sin^{-1}(1)$$

$$\theta=\frac{\pi}{2}$$

Question 20

If $\vec{a}$ and $\vec{b}$ are two vectors such that $\left|\vec{a}.\vec{b}\right|=\left|\vec{a}\times\vec{b}\right|$, write the angle between $\vec{a}$ and $\vec{b}$.

Solution 20

Let, angle between $\vec{a}$ and $\vec{b}$ be θ.

$\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}|\sin\theta.\hat{n}$

$|\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}|.|\sin\theta|.|\hat{n}|$

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta|.1$ $\quad$ [Since, $\hat{n}$ is a unit vector]

$|\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}|.|\sin\theta|$ $\quad$ ---(i)

Now,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$|\vec{a}.\vec{b}| = |\vec{a}|.|\vec{b}|.|\cos\theta|$ $\quad$ ---(ii)

We have,

$|\vec{a} \times \vec{b}| = |\vec{a}.\vec{b}|$

$|\vec{a}||\vec{b}||\sin\theta| = |\vec{a}|.|\vec{b}|.|\cos\theta|$

$|\sin\theta| = |\cos\theta|$

$\theta = \frac{\pi}{4}$

Question 21

If $\vec{a}$ and $\vec{b}$ are unit vectors, then write the value of $|\vec{a} \times \vec{b}|^2 + (\vec{a}.\vec{b})^2$

Solution 21

Here, $\vec{a}$ and $\vec{b}$ are unit vectors

$\left|\vec{a}\right| = 1, \ \left|\vec{b}\right| = 1$

Let, θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = \left|\vec{a}\right| . \left|\vec{b}\right| . \sin\theta . \hat{n}$$
$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right| . \left|\vec{b}\right| . \left|\sin\theta\right| . \left|\hat{n}\right|$$
$$= 1.1.\left|\sin\theta\right|.1 \qquad \left[\text{Since, } \hat{n} \text{ is a unit vector}\right]$$
$$\left|\vec{a} \times \vec{b}\right| = \left|\sin\theta\right|$$

Squaring both the sides,

$$\left|\vec{a} \times \vec{b}\right|^2 = \sin^2\theta \qquad ---(i)$$
$$\vec{a}.\vec{b} = \left|\vec{a}\right| . \left|\vec{b}\right| . \left|\cos\theta\right|$$
$$= 1.1.\cos\theta$$
$$\vec{a}.\vec{b} = \cos\theta$$

Squaring both the sides,

$$\left(\vec{a}.\vec{b}\right)^2 = \cos^2\theta \qquad ---(ii)$$

Adding (i) and (ii),

$$\left|\vec{a} \times \vec{b}\right|^2 + \left(\vec{a}.\vec{b}\right)^2 = \sin^2\theta + \cos^2\theta$$

$$\left|\vec{a} \times \vec{b}\right|^2 + \left(\vec{a}.\vec{b}\right)^2 = 1 \qquad \left[\text{Since, } \sin^2\theta + \cos^2\theta = 1\right]$$

Question 22

If $\vec{a}$ is a unit vector such that $\vec{a} \times \hat{i} = \hat{j}$, find $\vec{a}.\hat{i}$.

Solution 22

Here, $\vec{a}$ is a unit vector, so

$|\vec{a}| = 1$

Let, $a = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$\vec{a} \times \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \hat{i}\,(0 - 0) - \hat{j}\,(0 - c_1) + \hat{k}\,(0 - b_1)$$

$\vec{a} \times \hat{i} = 0.\hat{i} + c_1\hat{j} - b_1\hat{k}$

$\hat{j} = 0\,\hat{i} + c_1\hat{j} - b_1\hat{k}$ $\left[\text{Since, } \vec{a} \times \hat{i} = \hat{j}\right]$

$0.\hat{i} + \hat{j} + 0\,\hat{k} = 0.\hat{i} + c_1\hat{j} - b_1\hat{k}$

$c_1 = 1,\ b_1 = 0$

Now, $|\vec{a}| = 1$

$\left|a_1\hat{i} + b_1\hat{j} + c_1\hat{k}\right| = 1$

$\sqrt{(a_1)^2 + (b_1)^2 + (c_1)^2} = 1$

Put value of $b_1 = 0,\ c_1 = 1$

$\sqrt{(a_1)^2 + (0)^2 + (1)^2} = 1$

Squaring both the sides,

$a_1^2 + 1 = 1$

$a_1^2 = 1 - 1$

$a_1^2 = 0$

$a_1 = 0$

So, $a_1 = 0,\ b_1 = 0,\ c_1 = 1$

$$\therefore \quad \vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$$
$$= 0\,\hat{i} + 0.\hat{j} + 1.\hat{k}$$

$\vec{a} = \hat{k}$

$\vec{a}\hat{i} = \hat{k}\,\hat{i}$

$= 0$

Question 23

If $\vec{c}$ is a unit vector perpendicular to the vectors $\vec{a}$ and $\vec{b}$, write another unit vector perpendicular to $\vec{a}$ and $\vec{b}$.

Solution 23

Here, it is given that $\vec{c}$ is a unit vector perpendicular to the vector $\vec{a}$ and $\vec{b}$. thus

$$\vec{c} = \vec{a} \times \vec{b}$$

Multiplying by (-1) on both the sides,

$$-\vec{c} = -\left(\vec{a} \times \vec{b}\right)$$

$$-\vec{c} = \vec{b} \times \vec{a} \qquad \left[\text{Since, } -\left(\vec{a} \times \vec{b}\right) = \vec{b} \times \vec{a}\right]$$

So,

$\left(-\vec{c}\right)$ is another unit vector which is perpendicular to both $\vec{a}$ and $\vec{b}$.

Question 24

Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and when $\left|\vec{a} \times \vec{b}\right| = \sqrt{3}$.

Solution 24

Here, $\left|\vec{a}\right| = 1$, $\left|\vec{b}\right| = 2$, $\left|\vec{a} \times \vec{b}\right| = \sqrt{3}$

Let, θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = \left|\vec{a}\right|.\left|\vec{b}\right|.\sin\theta.\hat{n}$$

$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right|\left|\sin\theta\right|\left|\hat{n}\right|$$

$$\sqrt{3} = 1.2.\left|\sin\theta\right|.1 \qquad \left[\text{Since, } \hat{n} \text{ is a unit vector}\right]$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\theta = \frac{\pi}{3}$$

Question 25

Vectors $\vec{a}$ and $\vec{b}$ are such that $\left|\vec{a}\right| = \sqrt{3}$, $\left|\vec{b}\right| = \frac{2}{3}$ and $\left(\vec{a} \times \vec{b}\right)$ is a unit vector, write the angle between $\vec{a}$ and $\vec{b}$.

Solution 25

Here, $\vec{a} \times \vec{b}$ is a unit vector

$$\Rightarrow \quad \left|\vec{a} \times \vec{b}\right| = 1$$

$$\vec{a} \times \vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|.\sin\theta.\hat{n}$$

$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right|.\left|\sin\theta\right|.\left|\hat{n}\right|$$

$$1 = \sqrt{3}.\frac{2}{3}.\left|\sin\theta\right|.1 \qquad \left[\text{Since, } \hat{n} \text{ is a unit vector}\right]$$

$$\sin\theta = \frac{3 \times \sqrt{3}}{2\sqrt{3} \times \sqrt{3}}$$

$$= \frac{3\sqrt{3}}{3.2}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\theta = \frac{\pi}{3}$$

Question 26

Find λ, if $\left(2\hat{i} + 6\hat{j} + 14\hat{k}\right) \times \left(\hat{i} - \lambda\hat{j} + 7\hat{k}\right) = \vec{0}$.

Solution 26

If $\vec{a} \times \vec{b} = 0$, Then vector $\vec{a}$ is parallel to vector $\vec{b}$.

$\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and

$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ are parallel, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\frac{2}{1} = \frac{6}{-\lambda} = \frac{14}{7}$$

Cross-multiplying the first two,

$$-2\lambda = 6$$

$$\lambda = \frac{6}{-2}$$

$$\lambda = -3$$

Question 27

Write the value of the area of the parallelogram determined by the vectors $2\vec{i}$ and $3\vec{j}$.

Solution 27

Let

$$\hat{a} = 2\hat{i}$$
$$= 2\hat{i} + 0\hat{j} + 0\hat{k}$$
$$\hat{b} = 3\hat{j}$$
$$= 0\hat{i} + 3\hat{j} + 0\hat{k}$$

The area of the parallelogram is $|\hat{a} \times \hat{b}|$.

Now

$$\hat{a} \times \hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{vmatrix}$$
$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(6-0)$$
$$= 6\hat{k}$$

Therefore,

$$|\hat{a} \times \hat{b}| = 6|\hat{k}|$$
$$= 6 \cdot 1$$
$$= 6 \text{ sq. unit}$$

Question 28

Write the value of $(\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} + \hat{k}) \cdot \hat{j}$.

Solution 28

The given expression is

$$(\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} + \hat{k}) \cdot \hat{j}$$

Now,

$$(\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} + \hat{k}) \cdot \hat{j} = \hat{k} \cdot \hat{k} + \hat{j} \cdot \hat{j} + \hat{k} \cdot \hat{j}$$
$$= 1 + 1 + 0$$
$$= 2$$

Therefore, the value is 2.