NCERT Solutions for Class 12- Maths Chapter 5 - Continuity and Differentiability

Chapter 5 - Continuity and Differentiability Exercise Ex. 5.1

Solution 1

The given function is f(x) = 5x - 3

At x = 0, f(0) = 5 × 0 - 3 = -3

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(5x-3) = 5\times 0-3 = -3$$
$$\therefore \lim_{x\to 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

$$\text{At } x=-3,\ f(-3) = 5\times(-3)-3 = -18$$
$$\lim_{x\to -3} f(x) = \lim_{x\to -3}(5x-3) = 5\times(-3)-3 = -18$$
$$\therefore \lim_{x\to -3} f(x) = f(-3)$$

Therefore, f is continuous at $x = -3$

$$\text{At } x=5,\ f(x) = f(5) = 5\times 5-3 = 25-3 = 22$$
$$\lim_{x\to 5} f(x) = \lim_{x\to 5}(5x-3) = 5\times 5-3 = 22$$
$$\therefore \lim_{x\to 5} f(x) = f(5)$$

Therefore, f is continuous at $x = 5$

Solution 2

The given function is $f(x) = 2x^2 - 1$

$$\text{At } x=3,\ f(x) = f(3) = 2\times 3^2 - 1 = 17$$
$$\lim_{x\to 3} f(x) = \lim_{x\to 3}(2x^2-1) = 2\times 3^2 - 1 = 17$$
$$\therefore \lim_{x\to 3} f(x) = f(3)$$

Thus, f is continuous at $x = 3$

Solution 3

(a) The given function is $f(x)=x-5$

It is evident that f is defined at every real number k and its value at k is $k-5$.

It is also observed that, $\lim_{x\to k} f(x)=\lim_{x\to k}(x-5)=k-5=f(k)$

$\therefore \lim_{x\to k} f(x)=f(k)$

Hence, f is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x)=\frac{1}{x-5}, x\neq 5$

For any real number $k\neq 5$, we obtain

$$\lim_{x\to k} f(x)=\lim_{x\to k}\frac{1}{x-5}=\frac{1}{k-5}$$

$$\text{Also, } f(k)=\frac{1}{k-5} \qquad (\text{As } k\neq 5)$$

$$\therefore \lim_{x\to k} f(x)=f(k)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(c) The given function is $f(x)=\dfrac{x^2-25}{x+5}, x \neq -5$

For any real number $c \neq -5$, we obtain

$$\lim_{x\to c} f(x)=\lim_{x\to c}\frac{x^2-25}{x+5}=\lim_{x\to c}\frac{(x+5)(x-5)}{x+5}=\lim_{x\to c}(x-5)=(c-5)$$

$$\text{Also, } f(c)=\frac{(c+5)(c-5)}{c+5}=(c-5) \quad (\text{as } c\neq -5)$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(d) The given function is $f(x)=|x-5|=\begin{cases}5-x, & \text{if } x<5\\ x-5, & \text{if } x\geq 5\end{cases}$

This function f is defined at all points of the real line.

Let c be a point on a real line. Then, $c<5$ or $c=5$ or $c>5$

Case I: $c < 5$

Then, $f(c) = 5 - c$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (5 - x) = 5 - c$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all real numbers less than 5.

Case II : $c = 5$

Then, $f(c) = f(5) = (5 - 5) = 0$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5} (5 - x) = (5 - 5) = 0$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5} (x - 5) = 0$$
$$\therefore \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

Therefore, f is continuous at $x = 5$

Case III: $c > 5$

Then, $f(c) = f(5) = c - 5$
$$\lim_{x \to c} f(x) = \lim_{x \to c} (x - 5) = c - 5$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function

Solution 4

The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

Then, $\lim\limits_{x \to n} f(n) = \lim\limits_{x \to n} (x^n) = n^n$

$\therefore \lim\limits_{x \to n} f(x) = f(n)$

Therefore, f is continuous at n, where n is a positive integer

Solution 5

The given function f is $f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$

At $x = 0$,

It is evident that f is defined at 0 and its value at 0 is 0.

Then, $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x = 0$

$\therefore \lim\limits_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

At $x = 1$,

f is defined at 1 and its value at 1 is 1.

The left hand limit of f at $x = 1$ is,

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1$

The right hand limit of f at $x = 1$ is,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (5) = 5$$
$$\therefore \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

Therefore, f is not continuous at $x = 1$

At $x = 2$,

f is defined at 2 and its value at 2 is 5.

Then, $\lim_{x\to 2} f(x) = \lim_{x\to 2} (5) = 5$

$$\therefore \lim_{x\to 2} f(x) = f(2)$$

Therefore, f is continuous at $x = 2$

Solution 6

The given function f is $f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

It is evident that the given function f is defined at all the points of the real line.

Let c be a point on the real line. Then, three cases arise.

(i) $c < 2$

(ii) $c > 2$

(iii) $c = 2$

Case (i) $c < 2$

Then, $f(c) = 2c + 3$
$$\lim_{x\to c} f(x) = \lim_{x\to c} (2x+3) = 2c + 3$$
$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x < 2$

Case (ii) $c > 2$

Then, $f(c) = 2c - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x - 3) = 2c - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 2$

Case (iii) $c = 2$

Then, the left hand limit of f at $x = 2$ is,

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3) = 2 \times 2 + 3 = 7$

The right hand limit of f at $x = 2$ is,

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3) = 2 \times 2 - 3 = 1$

It is observed that the left and right hand limit of f at $x = 2$ do not coincide.

Therefore, f is not continuous at $x = 2$

Hence, $x = 2$ is the only point of discontinuity of f.

Solution 7

The given function f is $f(x)=\begin{cases}|x|+3=-x+3, \text{ if } x\le -3\\ -2x, \text{ if } -3<x<3\\ 6x+2, \text{ if } x\ge 3\end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$\text{If } c<-3, \text{ then } f(c)=-c+3$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-x+3)=-c+3$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<-3$

Case II:

$\text{If } c=-3, \text{ then } f(-3)=-(-3)+3=6$

$\lim\limits_{x\to -3^-} f(x)=\lim\limits_{x\to -3^-}(-x+3)=-(-3)+3=6$

$\lim\limits_{x\to -3^+} f(x)=\lim\limits_{x\to -3^+}(-2x)=-2\times(-3)=6$

$\therefore \lim\limits_{x\to -3} f(x)=f(-3)$

Therefore, f is continuous at $x=-3$

Case III:

If $-3 < c < 3$, then $f(c) = -2c$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(-2x) = -2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous in $(-3, 3)$.

Case IV:

If $c = 3$, then the left hand limit of f at $x = 3$ is,

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-}(-2x) = -2 \times 3 = -6$

The right hand limit of f at $x = 3$ is,

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+}(6x + 2) = 6 \times 3 + 2 = 20$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

If $c > 3$, then $f(c) = 6c + 2$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(6x + 2) = 6c + 2$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of f.

Solution 8

The given function f is $f(x)=\begin{cases}\frac{|x|}{x} & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$

It is known that, $x<0 \Rightarrow |x|=-x$ and $x>0 \Rightarrow |x|=x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\frac{|x|}{x}=\frac{-x}{x}=-1 \text{ if } x<0\\ 0, \text{ if } x=0\\ \frac{|x|}{x}=\frac{x}{x}=1, \text{ if } x>0\end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<0$, then $f(c)=-1$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-1)=-1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points $x<0$

Case II:

If $c = 0$, then the left hand limit of f at $x = 0$ is,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1) = 1$$

It is observed that the left and right hand limit of f at $x = 0$ do not coincide.

Therefore, f is not continuous at $x = 0$

Case III:

If $c > 0$, then $f(c) = 1$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (1) = 1$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 0$

Hence, $x = 0$ is the only point of discontinuity of f.

Solution 9

The given function f is $f(x)=\begin{cases}\dfrac{x}{|x|}, & \text{if } x<0\\ -1, & \text{if } x\geq 0\end{cases}$

It is known that, $x<0 \Rightarrow |x|=-x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\dfrac{x}{|x|}=\dfrac{x}{-x}=-1, & \text{if } x<0\\ -1, & \text{if } x\geq 0\end{cases}$$

$$\Rightarrow f(x)=-1 \text{ for all } x\in \mathbf{R}$$

Let c be any real number. Then, $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-1)=-1$

Also, $f(c)=-1=\lim\limits_{x\to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Solution 10

The given function f is $f(x)=\begin{cases} x+1, & \text{if } x\geq 1 \\ x^2+1, & \text{if } x<1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c<1, \text{ then } f(c)=c^2+1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2+1\right)=c^2+1$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x < 1$

Case II:

$$\text{If } c=1, \text{ then } f(c)=f(1)=1+1=2$$

The left hand limit of f at $x = 1$ is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^2+1\right)=1^2+1=2$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x+1)=1+1=2$$
$$\therefore \lim_{x\to 1} f(x)=f(1)$$

Therefore, f is continuous at $x = 1$

Case III:

$$\text{If } c>1, \text{ then } f(c)=c+1$$
$$\lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x > 1$

Hence, the given function f has no point of discontinuity.

Solution 11

The given function f is $f(x)=\begin{cases} x^3-3, & \text{if } x \le 2 \\ x^2+1, & \text{if } x>2 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<2$, then $f(c)=c^3-3$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^3-3\right)=c^3-3$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<2$

Case II:

If $c=2$, then $f(c)=f(2)=2^3-3=5$

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}\left(x^3-3\right)=2^3-3=5$

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}\left(x^2+1\right)=2^2+1=5$

$\therefore \lim\limits_{x\to 2} f(x)=f(2)$

Therefore, f is continuous at $x=2$

Case III:

If $c>2$, then $f(c)=c^2+1$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^2+1\right)=c^2+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>2$

Thus, the given function f is continuous at every point on the real line.

Hence, f has no point of discontinuity.

Solution 12

The given function f is $f(x)=\begin{cases} x^{10}-1, & \text{if } x\le 1 \\ x^2, & \text{if } x>1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<1$, then $f(c)=c^{10}-1$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^{10}-1\right)=c^{10}-1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<1$

Case II:

If $c=1$, then the left hand limit of f at $x=1$ is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^{10}-1\right)=1^{10}-1=1-1=0$$

The right hand limit of f at $x=1$ is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(x^2\right)=1^2=1$$

It is observed that the left and right hand limit of f at $x=1$ do not coincide.

Therefore, f is not continuous at $x=1$

Case III:

If $c>1$, then $f(c)=c^2$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^2\right)=c^2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>1$

Thus, from the above observation, it can be concluded that x = 1 is the only point of discontinuity.

Solution 13

The given function is $f(x)=\begin{cases} x+5, & \text{if } x \le 1 \\ x-5, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<1$, then $f(c)=c+5$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x+5)=c+5$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x < 1$

Case II:

If $c=1$, then $f(1)=1+5=6$

The left hand limit of f at $x = 1$ is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(x+5)=1+5=6$

The right hand limit of f at $x = 1$ is,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x-5)=1-5=-4$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III:

If $c > 1$, then $f(c) = c - 5$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(x - 5) = c - 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of f.

Solution 14

The given function is $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$

The given function is defined at all points of the interval [0, 10].

Let c be a point in the interval [0, 10].

Case I:

If $0 \le c < 1$, then $f(c) = 3$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(3) = 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous in the interval [0, 1).

Case II:

If $c = 1$, then $f(3) = 3$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3) = 3$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4) = 4$$

It is observed that the left and right hand limits of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III:

If $1 < c < 3$, then $f(c) = 4$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (4) = 4$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (1, 3).

Case IV:

If $c = 3$, then $f(c) = 5$

The left hand limit of f at $x = 3$ is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4) = 4$$

The right hand limit of f at $x = 3$ is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5) = 5$$

It is observed that the left and right hand limits of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

If $3 < c \leq 10$, then $f(c) = 5$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (5) = 5$

$\lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at $x = 1$ and $x = 3$

Solution 15

The given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = 2c$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (2x) = 2c$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 0$

Case II:

If $c = 0$, then $f(c) = f(0) = 0$

The left hand limit of f at $x = 0$ is,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x) = 2\times 0 = 0$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (0) = 0$$
$$\therefore \lim_{x\to 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

Case III:

If $0 < c < 1$, then $f(x) = 0$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(0) = 0$
$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval (0, 1).

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x) = 4 \times 1 = 4$$

It is observed that the left and right hand limits of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case V:

If $c < 1$, then $f(c) = 4c$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (4x) = 4c$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 1$

Hence, f is not continuous only at $x = 1$

Solution 16

The given function f is $f(x)=\begin{cases}-2, \text{ if } x\le -1\\ 2x, \text{ if } -1<x\le 1\\ 2, \text{ if } x>1\end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c<-1, \text{ then } f(c)=-2 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(-2)=-2$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x<-1$

Case II:

$$\text{If } c=-1, \text{ then } f(c)=f(-1)=-2$$

The left hand limit of f at $x=-1$ is,

$$\lim_{x\to -1^-} f(x)=\lim_{x\to -1^-}(-2)=-2$$

The right hand limit of f at $x=-1$ is,

$$\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+}(2x)=2\times(-1)=-2$$

$$\therefore \lim_{x\to -1} f(x)=f(-1)$$

Therefore, f is continuous at $x=-1$

Case III:

$$\text{If } -1<c<1, \text{ then } f(c)=2c$$
$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x)=2c$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points of the interval $(-1, 1)$.

Case IV:

If $c = 1$, then $f(c) = f(1) = 2 \times 1 = 2$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x) = 2 \times 1 = 2$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2$$

$$\therefore \lim_{x \to 1} f(x) = f(c)$$

Therefore, f is continuous at $x = 2$

Case V:

If $c > 1$, then $f(c) = 2$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (2) = 2$

$$\lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line

Solution 17

The given function f is $f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$

If f is continuous at $x = 3$, then

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) \qquad \ldots(1)$$

Also,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax+1) = 3a+1$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (bx+3) = 3b+3$$
$$f(3) = 3a+1$$

Therefore, from (1), we obtain

$$3a+1 = 3b+3$$
$$\Rightarrow 3a+1 = 3b+3$$
$$\Rightarrow 3a = 3b+2$$
$$\Rightarrow a = b+\frac{2}{3}$$

Therefore, the required relationship is given by, $a = b + \frac{2}{3}$

Solution 18

The given function f is $f(x)=\begin{cases}\lambda\left(x^2-2x\right), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$

If f is continuous at $x=0$, then

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$
$$\Rightarrow \lim_{x\to 0^-} \lambda\left(x^2-2x\right)=\lim_{x\to 0^+}(4x+1)=\lambda\left(0^2-2\times 0\right)$$
$$\Rightarrow \lambda\left(0^2-2\times 0\right)=4\times 0+1=0$$
$$\Rightarrow 0=1=0,\ \text{which is not possible}$$

Therefore, there is no value of λ for which f is continuous at $x=0$

At $x=1$,

$f(1)=4x+1=4\times 1+1=5$

$$\lim_{x\to 1}(4x+1)=4\times 1+1=5$$
$$\therefore \lim_{x\to 1} f(x)=f(1)$$

Therefore, for any values of λ, f is continuous at $x=1$

Solution 19

The given function is $g(x) = x - [x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$$g(n) = n - [n] = n - n = 0$$

The left hand limit of f at $x = n$ is,

$$\lim_{x \to n^-} g(x) = \lim_{x \to n^-} (x - [x]) = \lim_{x \to n^-} (x) - \lim_{x \to n^-} [x] = n - (n-1) = 1$$

The right hand limit of f at $x = n$ is,

$$\lim_{x \to n^+} g(x) = \lim_{x \to n^+} (x - [x]) = \lim_{x \to n^+} (x) - \lim_{x \to n^+} [x] = n - n = 0$$

It is observed that the left and right hand limits of f at $x = n$ do not coincide.

Therefore, f is not continuous at $x = n$

Hence, g is discontinuous at all integral points

Solution 20

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that f is defined at x = ∏

At $x = \pi$, $f(x) = f(\pi) = \pi^2 - \sin\pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Consider $\lim_{x\to\pi} f(x) = \lim_{x\to\pi}\left(x^2 - \sin x + 5\right)$

Put $x = \pi + h$

If $x \to \pi$, *then it is evident that* $h \to 0$

$$\begin{aligned}
\therefore \lim_{x\to\pi} f(x) &= \lim_{x\to\pi}\left(x^2 - \sin x + 5\right)\\
&= \lim_{h\to 0}\left[(\pi+h)^2 - \sin(\pi+h) + 5\right]\\
&= \lim_{h\to 0}(\pi+h)^2 - \lim_{h\to 0}\sin(\pi+h) + \lim_{h\to 0} 5\\
&= (\pi+0)^2 - \lim_{h\to 0}\left[\sin\pi\cos h + \cos\pi\sin h\right] + 5\\
&= \pi^2 - \lim_{h\to 0}\sin\pi\cos h - \lim_{h\to 0}\cos\pi\sin h + 5\\
&= \pi^2 - \sin\pi\cos 0 - \cos\pi\sin 0 + 5\\
&= \pi^2 - 0\times 1 - (-1)\times 0 + 5\\
&= \pi^2 + 5
\end{aligned}$$

$\therefore \lim_{x\to\pi} f(x) = f(\pi)$

Therefore, the given function f is continuous at $x = \pi$

Solution 21

It is known that if g and h are two continuous functions, then

$g+h,\ g-h,$ and $g.h$ are also continuous.

It has to proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$$g(c) = \sin c$$

$$\begin{aligned}\lim_{x\to c} g(x) &= \lim_{x\to c} \sin x\\ &= \lim_{h\to 0} \sin(c+h)\\ &= \lim_{h\to 0}\left[\sin c\cos h + \cos c\sin h\right]\\ &= \lim_{h\to 0}(\sin c\cos h) + \lim_{h\to 0}(\cos c\sin h)\\ &= \sin c\cos 0 + \cos c\sin 0\\ &= \sin c + 0\\ &= \sin c\end{aligned}$$

$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, g is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\begin{aligned}
\lim_{x\to c} h(x) &= \lim_{x\to c} \cos x \\
&= \lim_{h\to 0} \cos(c+h) \\
&= \lim_{h\to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c
\end{aligned}$$

$\therefore \lim_{x\to c} h(x) = h(c)$

Therefore, h is a continuous function.

Therefore, it can be concluded that

(a) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function

(b) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function

(c) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function

Solution 22

It is known that if g and h are two continuous functions, then

(i) $\dfrac{h(x)}{g(x)}$, $g(x) \neq 0$ is continuous

(ii) $\dfrac{1}{g(x)}$, $g(x) \neq 0$ is continuous

(iii) $\dfrac{1}{h(x)}$, $h(x) \neq 0$ is continuous

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$$
\begin{aligned}
g(c) &= \sin c \\
\lim_{x\to c} g(x) &= \lim_{x\to c} \sin x \\
&= \lim_{h\to 0} \sin(c+h) \\
&= \lim_{h\to 0} [\sin c\cos h + \cos c\sin h] \\
&= \lim_{h\to 0} (\sin c\cos h) + \lim_{h\to 0} (\cos c\sin h) \\
&= \sin c\cos 0 + \cos c\sin 0 \\
&= \sin c + 0 \\
&= \sin c \\
\therefore\ \lim_{x\to c} g(x) &= g(c)
\end{aligned}
$$

Therefore, g is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$
\begin{aligned}
\lim_{x\to c} h(x) &= \lim_{x\to c} \cos x \\
&= \lim_{h\to 0} \cos(c+h) \\
&= \lim_{h\to 0} [\cos c\cos h - \sin c\sin h] \\
&= \lim_{h\to 0} \cos c\cos h - \lim_{h\to 0} \sin c\sin h \\
&= \cos c\cos 0 - \sin c\sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c \\
\therefore \lim_{x\to c} h(x) &= h(c)
\end{aligned}
$$

Therefore, $h(x) = \cos x$ is continuous function.

It can be concluded that,

$$
\begin{aligned}
&\operatorname{cosec} x = \frac{1}{\sin x},\ \sin x \neq 0 \text{ is continuous} \\
&\Rightarrow \operatorname{cosec} x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}
\end{aligned}
$$

Therefore, cosecant is continuous except at x = np, n ∈Z

$$\sec x = \frac{1}{\cos x},\ \cos x \neq 0 \text{ is continuous}$$
$$\Rightarrow \sec x,\ x \neq (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z}) \text{ is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z})$

$$\cot x = \frac{\cos x}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \cot x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cotangent is continuous except at x = np, n ∈Z

Solution 23

The given function f is $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c < 0, \text{ then } f(c) = \frac{\sin c}{c} \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c}\left(\frac{\sin x}{x}\right) = \frac{\sin c}{c}$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x < 0$

Case II:

$$\text{If } c > 0, \text{ then } f(c) = c+1 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c}(x+1) = c+1$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 0$

Case III:

If $c = 0$, then $f(c) = f(0) = 0 + 1 = 1$

The left hand limit of f at $x = 0$ is,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1) = 1$$
$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Solution 24

He given function f is $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \neq 0$, then $f(c) = c^2 \sin \frac{1}{c}$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(x^2 \sin \frac{1}{x} \right) = \left(\lim_{x \to c} x^2 \right) \left(\lim_{x \to c} \sin \frac{1}{x} \right) = c^2 \sin \frac{1}{c}$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points $x \neq 0$

Case II:

If $c = 0$, then $f(0) = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \le \sin\frac{1}{x} \le 1,\ x \ne 0$

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-x^2\right) \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$$

$$\Rightarrow \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = 0$$

Similarly, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Solution 25

The given function f is $f(x) = \begin{cases} \sin x - \cos x, \text{if } x \ne 0 \\ -1 \quad\quad\quad\quad \text{if } x = 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \ne 0$, then $f(c) = \sin c - \cos c$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(\sin x - \cos x) = \sin c - \cos c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x \ne 0$

Case II:

If $c = 0$, then $f(0) = -1$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$
$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Solution 26

The given function f is $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

The given function f is continuous at $x = \frac{\pi}{2}$, if f is defined at $x = \frac{\pi}{2}$ and if the value of the f at $x = \frac{\pi}{2}$ equals the limit of f at $x = \frac{\pi}{2}$.

It is evident that f is defined at $x = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 3$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{k\cos x}{\pi - 2x}$$

Put $x = \frac{\pi}{2} + h$

Then, $x \to \frac{\pi}{2} \Rightarrow h \to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)}$$

$$= k\lim_{h\to 0} \frac{-\sin h}{-2h} = \frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h} = \frac{k}{2}.1 = \frac{k}{2}$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2} = 3$$

$$\Rightarrow k = 6$$

Therefore, the required value of k is 6.

Solution 27

The given function is $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$

The given function f is continuous at $x = 2$, if f is defined at $x = 2$ and if the value of f at $x = 2$ equals the limit of f at $x = 2$

It is evident that f is defined at $x = 2$ and $f(2) = k(2)^2 = 4k$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x\to 2^-}(kx^2) = \lim_{x\to 2^+}(3) = 4k$$

$$\Rightarrow k\times 2^2 = 3 = 4k$$

$$\Rightarrow 4k = 3$$

$$\Rightarrow k = \frac{3}{4}$$

Therefore, the required value of k is $\frac{3}{4}$.

Solution 28

$$\text{The given function is } f(x)=\begin{cases} kx+1, & \text{if } x\le\pi \\ \cos x, & \text{if } x>\pi \end{cases}$$

The given function f is continuous at x = π, if f is defined at x = π and if the value of f at x = π equals the limit of f at x = π

$$\lim_{x\to\pi^-} f(x)=\lim_{x\to\pi^+} f(x)=f(\pi)$$
$$\Rightarrow \lim_{x\to\pi^-}(kx+1)=\lim_{x\to\pi^+}\cos x=k\pi+1$$
$$\Rightarrow k\pi+1=\cos\pi=k\pi+1$$
$$\Rightarrow k\pi+1=-1=k\pi+1$$
$$\Rightarrow k=-\frac{2}{\pi}$$

Therefore, the required value of k is $-\frac{2}{\pi}$.

Solution 29

$$\text{The given function } f \text{ is } f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases}$$

The given function f is continuous at $x=5$, if f is defined at $x=5$ and if the value of f at $x=5$ equals the limit of f at $x=5$

It is evident that f is defined at $x=5$ and $f(5)=kx+1=5k+1$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$
$$\Rightarrow \lim_{x\to 5^-}(kx+1)=\lim_{x\to 5^+}(3x-5)=5k+1$$
$$\Rightarrow 5k+1=15-5=5k+1$$
$$\Rightarrow 5k+1=10$$
$$\Rightarrow 5k=9$$
$$\Rightarrow k=\frac{9}{5}$$

Therefore, the required value of k is $\frac{9}{5}$.

Solution 30

The given function f is $f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$

It is evident that the given function f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at $x=2$ and $x=10$

Since f is continuous at $x=2$, we obtain

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$
$$\Rightarrow \lim_{x\to 2^-}(5)=\lim_{x\to 2^+}(ax+b)=5$$
$$\Rightarrow 5=2a+b=5$$
$$\Rightarrow 2a+b=5 \qquad \ldots(1)$$

Since f is continuous at $x=10$, we obtain

$$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$$
$$\Rightarrow \lim_{x\to 10^-}(ax+b)=\lim_{x\to 10^+}(21)=21$$
$$\Rightarrow 10a+b=21=21$$
$$\Rightarrow 10a+b=21 \qquad \ldots(2)$$

On subtracting equation (1) from equation (2), we obtain

$8a=16$

$a=2$

By putting $a=2$ in equation (1), we obtain

$2\times 2+b=5$

$4+b=5$

$b=1$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

Solution 31

The given function is $f(x) = \cos(x^2)$

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = \cos x$ and $h(x) = x^2$

$\left[\because (goh)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x)\right]$

It has to be first proved that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that g is defined for every real number.

Let c be a real number.

Then, $g(c) = \cos c$

Put $x = c + h$

If $x \to c$, then $h \to 0$

$$\begin{aligned}
\lim_{x \to c} g(x) &= \lim_{x \to c} \cos x \\
&= \lim_{h \to 0} \cos(c + h) \\
&= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c
\end{aligned}$$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, $g(x) = \cos x$ is continuous function.

$h(x) = x^2$

Clearly, h is defined for every real number.

Let k be a real number, then $h(k) = k^2$

$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$

$\therefore \lim_{x \to k} h(x) = h(k)$

Therefore, h is a continuous function.

It is known that for real valued functions g and h,such that $(g \circ h)$ is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then $(f \circ g)$ is continuous at c.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Solution 32

The given function is $f(x)=|\cos x|$

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = g\ o\ h$, where $g(x)=|x|$ and $h(x)=\cos x$

$\left[\because (goh)(x)=g(h(x))=g(\cos x)=|\cos x|=f(x)\right]$

It has to be first proved that $g(x)=|x|$ and $h(x)=\cos x$ are continuous functions.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that $x < 0$

Case II:

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that $x > 0$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} (-x) = 0$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (x) = 0$$
$$\therefore \lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} (x) = g(0)$$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\begin{aligned}
\lim_{x\to c} h(x) &= \lim_{x\to c} \cos x \\
&= \lim_{h\to 0} \cos(c+h) \\
&= \lim_{h\to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c
\end{aligned}$$
$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions g and h, such that $(g \text{ o } h)$ is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then $(f \text{ o } g)$ is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is a continuous function

Solution 33

Let $f(x) = \sin|x|$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = |x|$ and $h(x) = \sin x$

$\left[\because (goh)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x)\right]$

It has to be proved first that $g(x) = |x|$ and $h(x) = \sin x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c}(-x) = -c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x < 0$

Case II:

If $c > 0$, then $g(c) = c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} x = c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x > 0$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$$
$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$$
$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = \sin x$

It is evident that $h(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + k$

If $x \to c$, then $k \to 0$

$h(c) = \sin c$

$$\begin{aligned}
h(c) &= \sin c \\
\lim_{x \to c} h(x) &= \lim_{x \to c} \sin x \\
&= \lim_{k \to 0} \sin(c+k) \\
&= \lim_{k \to 0} [\sin c \cos k + \cos c \sin k] \\
&= \lim_{k \to 0} (\sin c \cos k) + \lim_{h \to 0} (\cos c \sin k) \\
&= \sin c \cos 0 + \cos c \sin 0 \\
&= \sin c + 0 \\
&= \sin c \\
\therefore \lim_{x \to c} h(x) &= g(c)
\end{aligned}$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h, such that $(g \circ h)$ is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then $(f \circ g)$ is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Solution 34

The given function is $f(x)=|x|-|x+1|$

The two functions, g and h, are defined as

$g(x)=|x|$ and $h(x)=|x+1|$

Then, $f=g-h$

The continuity of g and h is examined first.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that $x<0$

Case II:

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that $x>0$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$$
$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$$
$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = |x+1|$ can be written as

$$h(x) = \begin{cases} -(x+1), & \text{if, } x < -1 \\ x+1, & \text{if } x \geq -1 \end{cases}$$

Clearly, h is defined for every real number.

Let c be a real number.

Case I:

If $c < -1$, then $h(c) = -(c+1)$ and $\lim_{x \to c} h(x) = \lim_{x \to c} [-(x+1)] = -(c+1)$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, h is continuous at all points x, such that $x < -1$

Case II:

If $c > -1$, then $h(c) = c+1$ and $\lim_{x \to c} h(x) = \lim_{x \to c} (x+1) = c+1$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, h is continuous at all points x, such that $x > -1$

Case III:

If $c=-1$, then $h(c)=h(-1)=-1+1=0$

$$\lim_{x\to -1^-} h(x)=\lim_{x\to -1^-}\left[-(x+1)\right]=-(-1+1)=0$$

$$\lim_{x\to -1^+} h(x)=\lim_{x\to -1^+}(x+1)=(-1+1)=0$$

$$\therefore \lim_{x\to -1^-} h(x)=\lim_{h\to -1^+} h(x)=h(-1)$$

Therefore, h is continuous at $x=-1$

From the above three observations, it can be concluded that h is continuous at all points of the real line.

g and h are continuous functions. Therefore, $f=g-h$ is also a continuous function.

Therefore, f has no point of discontinuity.

Chapter 5 - Continuity and Differentiability Exercise Ex. 5.2

Solution 1

Let $f(x)=\sin\left(x^2+5\right)$, $u(x)=x^2+5$, and $v(t)=\sin t$

Then, (v o u)(x) = v(u(x)) = v(x^2 + 5) = sin (x^2 + 5) = f(x)

Thus, f is a composite of two functions.

Put $t = u(x) = x^2 + 5$

Then, we obtain

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(x^2 + 5)$$

$$\frac{dt}{dx} = \frac{d}{dx}(x^2 + 5) = \frac{d}{dx}(x^2) + \frac{d}{dx}(5) = 2x + 0 = 2x$$

Therefore, by chain rule, $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(x^2 + 5) \times 2x = 2x\cos(x^2 + 5)$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sin(x^2 + 5)\right] &= \cos(x^2 + 5) \cdot \frac{d}{dx}(x^2 + 5) \\ &= \cos(x^2 + 5) \cdot \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(5)\right] \\ &= \cos(x^2 + 5) \cdot [2x + 0] \\ &= 2x\cos(x^2 + 5)\end{aligned}$$

Solution 2

Let $f(x) = \cos(\sin x), u(x) = \sin x,$ and $v(t) = \cos t$

Then, $(vou)(x) = v(u(x)) = v(\sin x) = \cos(\sin x) = f(x)$

Thus, f is a composite function of two functions.

Put $t = u(x) = \sin x$

$$\therefore \frac{dv}{dt} = \frac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x)$$

$$\frac{dt}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

By chain rule,

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$$

Alternate method

$$\frac{d}{dx}\left[\cos(\sin x)\right] = -\sin(\sin x) \cdot \frac{d}{dx}(\sin x) = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$$

Solution 3

Let $f(x) = \sin(ax+b)$, $u(x) = ax+b$, and $v(t) = \sin t$

Then, $(vou)(x) = v(u(x)) = v(ax+b) = \sin(ax+b) = f(x)$

Thus, f is a composite function of two functions, u and v.

Put $t = u(x) = ax + b$

Therefore,

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Hence, by chain rule, we obtain

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax+b) \cdot a = a\cos(ax+b)$$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sin(ax+b)\right] &= \cos(ax+b).\frac{d}{dx}(ax+b)\\ &= \cos(ax+b)\cdot\left[\frac{d}{dx}(ax) + \frac{d}{dx}(b)\right]\\ &= \cos(ax+b).(a+0)\\ &= a\cos(ax+b)\end{aligned}$$

Solution 4

Let $f(x)=\sec\left(\tan\sqrt{x}\right), u(x)=\sqrt{x}, v(t)=\tan t$, and $w(s)=\sec s$

Then, $(wovou)(x)=w\left[v(u(x))\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$

Thus, f is a composite function of three functions, u, v, and w.

Put $s=v(t)=\tan t$ and $t=u(x)=\sqrt{x}$

$$\text{Then, }\frac{dw}{ds}=\frac{d}{ds}(\sec s)=\sec s\tan s=\sec(\tan t).\tan(\tan t) \qquad [s=\tan t]$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right) \qquad \left[t=\sqrt{x}\right]$$

$$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}\cdot x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$

$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Solution 5

The given function is $f(x)=\dfrac{\sin(ax+b)}{\cos(cx+d)}=\dfrac{g(x)}{h(x)}$, where $g(x)=\sin(ax+b)$ and

$h(x)=\cos(cx+d)$

$\therefore f'=\dfrac{g'h-gh'}{h^2}$

Consider $g(x)=\sin(ax+b)$

Let $u(x)=ax+b, v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=g(x)$

$\therefore g$ is a composite function of two functions, u and v.

Put $t=u(x)=ax+b$

$\dfrac{dv}{dt}=\dfrac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$

$\dfrac{dt}{dx}=\dfrac{d}{dx}(ax+b)=\dfrac{d}{dx}(ax)+\dfrac{d}{dx}(b)=a+0=a$

Therefore, by chain rule, we obtain

$g'=\dfrac{dg}{dx}=\dfrac{dv}{dt}\cdot\dfrac{dt}{dx}=\cos(ax+b)\cdot a=a\cos(ax+b)$

Consider $h(x)=\cos(cx+d)$

Let $p(x)=cx+d,\ q(y)=\cos y$

Then, $(qop)(x)=q(p(x))=q(cx+d)=\cos(cx+d)=h(x)$

h is a composite function of two functions, p and q.

Put $y=p(x)=cx+d$

$\dfrac{dq}{dy}=\dfrac{d}{dy}(\cos y)=-\sin y=-\sin(cx+d)$

$\dfrac{dy}{dx}=\dfrac{d}{dx}(cx+d)=\dfrac{d}{dx}(cx)+\dfrac{d}{dx}(d)=c$

Therefore, by chain rule, we obtain

$$h' = \frac{dh}{dx} = \frac{dq}{dy} \cdot \frac{dy}{dx} = -\sin(cx+d) \times c = -c\sin(cx+d)$$

$$\therefore f' = \frac{a\cos(ax+b)\cdot\cos(cx+d) - \sin(ax+b)\{-c\sin(cx+d)\}}{[\cos(cx+d)]^2}$$
$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + c\sin(ax+b)\cdot\frac{\sin(cx+d)}{\cos(cx+d)} \times \frac{1}{\cos(cx+d)}$$
$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Solution 6

The given function is $\cos x^3 . \sin^2(x^5)$.

$$\frac{d}{dx}\left[\cos x^3 \cdot \sin^2(x^5)\right] = \sin^2(x^5) \times \frac{d}{dx}(\cos x^3) + \cos x^3 \times \frac{d}{dx}\left[\sin^2(x^5)\right]$$
$$= \sin^2(x^5) \times (-\sin x^3) \times \frac{d}{dx}(x^3) + \cos x^3 \times 2\sin(x^5) \cdot \frac{d}{dx}\left[\sin x^5\right]$$
$$= -\sin x^3 \sin^2(x^5) \times 3x^2 + 2\sin x^5 \cos x^3 \cdot \cos x^5 \times \frac{d}{dx}(x^5)$$
$$= -3x^2 \sin x^3 \cdot \sin^2(x^5) + 2\sin x^5 \cos x^5 \cos x^3 \cdot \times 5x^4$$
$$= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2(x^5)$$

Solution 7

$$\frac{d}{dx}\left[2\sqrt{\cot\left(x^2\right)}\right]$$

$$= 2\cdot\frac{1}{2\sqrt{\cot\left(x^2\right)}}\times\frac{d}{dx}\left[\cot\left(x^2\right)\right]$$

$$= \sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times -\operatorname{cosec}^2\left(x^2\right)\times\frac{d}{dx}\left(x^2\right)$$

$$= -\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times\frac{1}{\sin^2\left(x^2\right)}\times\left(2x\right)$$

$$= \frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\,\sin x^2}$$

$$= \frac{-2\sqrt{2}\,x}{\sqrt{2\sin x^2\cos x^2}\,\sin x^2}$$

$$= \frac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$$

Solution 8

Let $f(x) = \cos\left(\sqrt{x}\right)$

Also, let $u(x) = \sqrt{x}$

And, $v(t) = \cos t$

Then, $(vou)(x) = v\left(u(x)\right)$

$$= v\left(\sqrt{x}\right)$$
$$= \cos\sqrt{x}$$
$$= f(x)$$

Clearly, f is a composite function of two functions, u and v, such that

$t = u(x) = \sqrt{x}$

Then, $\dfrac{dt}{dx} = \dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{d}{dx}\left(x^{\frac{1}{2}}\right) = \dfrac{1}{2}x^{-\frac{1}{2}}$

$$= \frac{1}{2\sqrt{x}}$$

And, $\dfrac{dv}{dt} = \dfrac{d}{dt}(\cos t) = -\sin t$

$$= -\sin\left(\sqrt{x}\right)$$

By using chain rule, we obtain

$$\frac{dt}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}$$
$$=-\sin\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$
$$=-\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)$$
$$=-\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right]=-\sin\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$
$$=-\sin\left(\sqrt{x}\right)\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$
$$=-\sin\sqrt{x}\times\frac{1}{2}x^{-\frac{1}{2}}$$
$$=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Solution 9

The given function is $f\left(x\right)=\left|x-1\right|,\ x\in\mathbf{R}$

It is known that a function f is differentiable at a point $x=c$ in its domain if both

$\lim\limits_{h\to 0^-}\frac{f\left(c+h\right)-f\left(c\right)}{h}$ and $\lim\limits_{h\to 0^+}\frac{f\left(c+h\right)-f\left(c\right)}{h}$ are finite and equal.

To check the differentiability of the given function at $x=1$,

consider the left hand limit of f at $x=1$

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{|1+h-1|-|1-1|}{h}$$
$$=\lim_{h\to 0^-}\frac{|h|-0}{h}=\lim_{h\to 0^-}\frac{-h}{h} \qquad (h<0\Rightarrow |h|=-h)$$
$$=-1$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{|1+h-1|-|1-1|}{h}$$
$$=\lim_{h\to 0^+}\frac{|h|-0}{h}=\lim_{h\to 0^+}\frac{h}{h} \qquad (h>0\Rightarrow |h|=h)$$
$$=1$$

Since the left and right hand limits of f at $x=1$ are not equal, f is not differentiable at $x=1$

Solution 10

The given function f is $f(x)=[x], 0<x<3$

It is known that a function f is differentiable at a point $x=c$ in its domain if both

$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at $x=1$, consider the left hand limit of f at $x=1$

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[1+h]-[1]}{h}$$
$$=\lim_{h\to 0^-}\frac{0-1}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{[1+h]-[1]}{h}$$
$$=\lim_{h\to 0^+}\frac{1-1}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at $x=1$ are not equal, f is not differentiable at

$x=1$

To check the differentiability of the given function at $x = 2$, consider the left hand limit of f at $x = 2$

$$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \frac{[2+h]-[2]}{h}$$
$$= \lim_{h\to 0^-} \frac{1-2}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

Consider the right hand limit of f at $x = 2$

$$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{[2+h]-[2]}{h}$$
$$= \lim_{h\to 0^+} \frac{2-2}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = 2$ are not equal, f is not differentiable at $x = 2$

Chapter 5 - Continuity and Differentiability Exercise Ex. 5.3

Solution 1

The given relationship is $2x + 3y = \sin x$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(2x+3y) = \frac{d}{dx}(\sin x)$$
$$\Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \cos x$$
$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x$$
$$\Rightarrow 3\frac{dy}{dx} = \cos x - 2$$
$$\therefore \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

Solution 2

The given relationship is $2x+3y=\sin y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \qquad \text{[By using chain rule]}$$

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{2}{\cos y-3}$$

Solution 3

The given relationship is $ax+by^2=\cos y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}(\cos y)$$

$$\Rightarrow a+b\frac{d}{dx}(y^2)=\frac{d}{dx}(\cos y) \qquad ...(1)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y)=-\sin y\frac{dy}{dx}$ ...(2)

From (1) and (2), we obtain

$$a+b\times 2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by+\sin y)\frac{dy}{dx}=-a$$

$$\therefore \frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

Solution 4

The given relationship is $xy+y^2=\tan x+y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(xy+y^2\right)=\frac{d}{dx}\left(\tan x+y\right)$$

$$\Rightarrow \frac{d}{dx}(xy)+\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}(\tan x)+\frac{dy}{dx}$$

$$\Rightarrow \left[y\cdot\frac{d}{dx}(x)+x.\frac{dy}{dx}\right]+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx} \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow y\cdot 1+x.\frac{dy}{dx}+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$

$$\Rightarrow (x+2y-1)\frac{dy}{dx}=\sec^2 x-y$$

$$\therefore \frac{dy}{dx}=\frac{\sec^2 x-y}{(x+2y-1)}$$

Solution 5

The given relationship is $x^2+xy+y^2=100$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(x^2+xy+y^2\right)=\frac{d}{dx}(100)$$

$$\Rightarrow \frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(xy)+\frac{d}{dx}\left(y^2\right)=0 \text{ [Derivative of constant function is 0]}$$

$$\Rightarrow 2x+\left[y\cdot\frac{d}{dx}(x)+x\cdot\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0 \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow 2x+y\cdot 1+x\cdot\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\Rightarrow 2x+y+(x+2y)\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{2x+y}{x+2y}$$

Solution 6

The given relationship is $x^3 + x^2 y + xy^2 + y^3 = 81$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(x^3 + x^2 y + xy^2 + y^3\right) = \frac{d}{dx}(81)$$
$$\Rightarrow \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^2 y\right) + \frac{d}{dx}\left(xy^2\right) + \frac{d}{dx}\left(y^3\right) = 0$$
$$\Rightarrow 3x^2 + \left[y\frac{d}{dx}\left(x^2\right) + x^2\frac{dy}{dx}\right] + \left[y^2\frac{d}{dx}(x) + x\frac{d}{dx}\left(y^2\right)\right] + 3y^2\frac{dy}{dx} = 0$$
$$\Rightarrow 3x^2 + \left[y \cdot 2x + x^2\frac{dy}{dx}\right] + \left[y^2 \cdot 1 + x \cdot 2y \cdot \frac{dy}{dx}\right] + 3y^2\frac{dy}{dx} = 0$$
$$\Rightarrow \left(x^2 + 2xy + 3y^2\right)\frac{dy}{dx} + \left(3x^2 + 2xy + y^2\right) = 0$$
$$\therefore \frac{dy}{dx} = \frac{-\left(3x^2 + 2xy + y^2\right)}{\left(x^2 + 2xy + 3y^2\right)}$$

Solution 7

The given relationship is $\sin^2 y + \cos xy = \pi$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(\sin^2 y + \cos xy\right) = \frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right) + \frac{d}{dx}\left(\cos xy\right) = 0 \qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right) = 2\sin y \frac{d}{dx}(\sin y) = 2\sin y \cos y \frac{dy}{dx} \qquad ...(2)$$

$$\frac{d}{dx}(\cos xy) = -\sin xy \frac{d}{dx}(xy) = -\sin xy\left[y\frac{d}{dx}(x) + x\frac{dy}{dx}\right]$$

$$= -\sin xy\left[y.1 + x\frac{dy}{dx}\right] = -y\sin xy - x\sin xy\frac{dy}{dx} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$2\sin y \cos y \frac{dy}{dx} - y\sin xy - x\sin xy\frac{dy}{dx} = 0$$

$$\Rightarrow \left(2\sin y\cos y - x\sin xy\right)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \left(\sin 2y - x\sin xy\right)\frac{dy}{dx} = y\sin xy$$

$$\therefore \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Solution 8

The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(\sin^2 x + \cos^2 y\right) = \frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right) + \frac{d}{dx}\left(\cos^2 y\right) = 0$$

$$\Rightarrow 2\sin x \cdot \frac{d}{dx}(\sin x) + 2\cos y \cdot \frac{d}{dx}(\cos y) = 0$$

$$\Rightarrow 2\sin x\cos x + 2\cos y(-\sin y)\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \sin 2x - \sin 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

Solution 9

The given relationship is $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \sin y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \cos y\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad ...(1)$$

The function, $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{\left(1+x^2\right).\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$
$$=\frac{\left(1+x^2\right).2-2x\cdot[0+2x]}{\left(1+x^2\right)^2}=\frac{2+2x^2-4x^2}{\left(1+x^2\right)^2}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2} \qquad ...(2)$$

Also, $\sin y = \frac{2x}{1+x^2}$

$$\Rightarrow \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{2x}{1+x^2}\right)^2} = \sqrt{\frac{\left(1+x^2\right)^2 - 4x^2}{\left(1+x^2\right)^2}}$$

$$= \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}} = \frac{1-x^2}{1+x^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{1-x^2}{1+x^2} \times \frac{dy}{dx} = \frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

Solution 10

The given relationship is $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \tan y = \frac{3x-x^3}{1-3x^2} \qquad ...(1)$$

It is known that, $\tan y = \frac{3\tan\frac{y}{3} - \tan^3\frac{y}{3}}{1-3\tan^2\frac{y}{3}}$ $\qquad ...(2)$

Comparing equations (1) and (2), we obtain

$$x = \tan\frac{y}{3}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3}\cdot\frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3}\cdot\frac{1}{3}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{\sec^2\frac{y}{3}} = \frac{3}{1+\tan^2\frac{y}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{1+x^2}$$

Solution 11

The given relationship is,

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}=\frac{1-x^2}{1+x^2}$$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$$\tan\frac{y}{2}=x$$

Differentiating this relationship with respect to x, we obtain

$$\sec^2\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right)=\frac{d}{dx}(x)$$

$$\Rightarrow \sec^2\frac{y}{2}\times\frac{1}{2}\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sec^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+\tan^2\frac{y}{2}}$$

$$\therefore \frac{dy}{dx}=\frac{2}{1+x^2}$$

Solution 12

The given relationship is $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$= \sqrt{\frac{\left(1+x^2\right)^2 - \left(1-x^2\right)^2}{\left(1+x^2\right)^2}} = \sqrt{\frac{4x^2}{\left(1+x^2\right)^2}} = \frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y) = \frac{2x}{1+x^2}\frac{dy}{dx} \qquad ...(2)$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) = \frac{\left(1+x^2\right)\cdot\left(1-x^2\right)' - \left(1-x^2\right)\cdot\left(1+x^2\right)'}{\left(1+x^2\right)^2} \qquad \text{[Using quotient rule]}$$

$$= \frac{(1+x^2)(-2x)-(1-x^2)\cdot(2x)}{(1+x^2)^2}$$

$$= \frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2}$$

$$= \frac{-4x}{(1+x^2)^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Alternate method

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow (1+x^2)\sin y = 1-x^2$$

$$\Rightarrow (1+\sin y)x^2 = 1-\sin y$$

$$\Rightarrow x^2 = \frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2 = \frac{\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2} + \sin\frac{y}{2}\right)^2}$$

$$\Rightarrow x = \frac{\cos\frac{y}{2} - \sin\frac{y}{2}}{\cos\frac{y}{2} + \sin\frac{y}{2}}$$

$$\Rightarrow x = \frac{1 - \tan\frac{y}{2}}{1 + \tan\frac{y}{2}}$$

$$\Rightarrow x = \tan\left(\frac{\pi}{4} - \frac{y}{2}\right)$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx} \cdot \left[\tan\left(\frac{\pi}{4} - \frac{y}{2}\right)\right]$$

$$\Rightarrow 1 = \sec^2\left(\frac{\pi}{4} - \frac{y}{2}\right) \cdot \frac{d}{dx}\left(\frac{\pi}{4} - \frac{y}{2}\right)$$

$$\Rightarrow 1 = \left[1 + \tan^2\left(\frac{\pi}{4} - \frac{y}{2}\right)\right] \cdot \left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow 1 = \left(1 + x^2\right)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1 + x^2}$$

Solution 13

The given relationship is $y=\cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \cos y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}\cdot\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow -\sin y\cdot\frac{dy}{dx}=\frac{(1+x^2)\cdot\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\,\frac{dy}{dx}=\frac{(1+x^2)\times 2-2x\cdot 2x}{(1+x^2)^2}$$
$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx}=-\left[\frac{2(1-x^2)}{(1+x^2)^2}\right]$$
$$\Rightarrow \sqrt{\frac{(1+x^2)^2-4x^2}{(1+x^2)^2}}\,\frac{dy}{dx}=\frac{-2(1-x^2)}{(1+x^2)^2}$$
$$\Rightarrow \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}}\,\frac{dy}{dx}=\frac{-2(1-x^2)}{(1+x^2)^2}$$
$$\Rightarrow \frac{1-x^2}{1+x^2}\cdot\frac{dy}{dx}=\frac{-2(1-x^2)}{(1+x^2)^2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Solution 14

The given relationship is $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$$
$$\Rightarrow \sin y = 2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\cos y\frac{dy}{dx}=2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right)+\sqrt{1-x^2}\frac{dx}{dx}\right]$$
$$\Rightarrow \sqrt{1-\sin^2 y}\frac{dy}{dx}=2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right]$$
$$\Rightarrow \sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\frac{dy}{dx}=2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \sqrt{\left(1-2x^2\right)^2}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \left(1-2x^2\right)\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$

Solution 15

The given relationship is $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y=\frac{1}{2x^2-1}$$

$$\Rightarrow \cos y=2x^2-1$$

$$\Rightarrow 2x^2=1+\cos y$$

$$\Rightarrow 2x^2=2\cos^2\frac{y}{2}$$

$$\Rightarrow x=\cos\frac{y}{2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\cos\frac{y}{2}\right)$$

$$\Rightarrow 1=-\sin\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin\frac{y}{2}}=\frac{1}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{\sin\frac{y}{2}}=\frac{-2}{\sqrt{1-\cos^2\frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{\sqrt{1-x^2}}$$

Chapter 5 - Continuity and Differentiability Exercise Ex. 5.4

Solution 1

Let $y=\frac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\sin x\frac{d}{dx}\left(e^x\right)-e^x\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$=\frac{\sin x.\left(e^x\right)-e^x\cdot(\cos x)}{\sin^2 x}$$

$$=\frac{e^x(\sin x-\cos x)}{\sin^2 x}, x\neq n\pi, n\in\mathbf{Z}$$

Solution 2

Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1} x}\right)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} x} \cdot \frac{d}{dx}\left(\sin^{-1} x\right)$$

$$= e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in (-1,1)$$

Solution 3

Let $y = e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^3}\right) = e^{x^3} \cdot \frac{d}{dx}\left(x^3\right) = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

Solution 4

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right]$$
$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$
$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{1}{1+\left(e^{-x}\right)^2} \cdot \frac{d}{dx}\left(e^{-x}\right)$$
$$= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \cdot e^{-x} \cdot \frac{d}{dx}(-x)$$
$$= \frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \times (-1)$$
$$= \frac{-e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

Solution 5

Let $y = \log\left(\cos e^{x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\cos e^{x}\right)\right]$$
$$= \frac{1}{\cos e^{x}} \cdot \frac{d}{dx}\left(\cos e^{x}\right)$$
$$= \frac{1}{\cos e^{x}} \cdot \left(-\sin e^{x}\right) \cdot \frac{d}{dx}\left(e^{x}\right)$$
$$= \frac{-\sin e^{x}}{\cos e^{x}} \cdot e^{x}$$
$$= -e^{x} \tan e^{x}, e^{x} \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{N}$$

Solution 6

$$\frac{d}{dx}\left(e^{x}+e^{x^{2}}+\ldots+e^{x^{5}}\right)$$
$$=\frac{d}{dx}\left(e^{x}\right)+\frac{d}{dx}\left(e^{x^{2}}\right)+\frac{d}{dx}\left(e^{x^{3}}\right)+\frac{d}{dx}\left(e^{x^{4}}\right)+\frac{d}{dx}\left(e^{x^{5}}\right)$$
$$=e^{x}+\left[e^{x^{2}}\times\frac{d}{dx}\left(x^{2}\right)\right]+\left[e^{x^{3}}\cdot\frac{d}{dx}\left(x^{3}\right)\right]+\left[e^{x^{4}}\cdot\frac{d}{dx}\left(x^{4}\right)\right]+\left[e^{x^{5}}\cdot\frac{d}{dx}\left(x^{5}\right)\right]$$
$$=e^{x}+\left(e^{x^{2}}\times 2x\right)+\left(e^{x^{3}}\times 3x^{2}\right)+\left(e^{x^{4}}\times 4x^{3}\right)+\left(e^{x^{5}}\times 5x^{4}\right)$$
$$=e^{x}+2xe^{x^{2}}+3x^{2}e^{x^{3}}+4x^{3}e^{x^{4}}+5x^{4}e^{x^{5}}$$

Solution 7

Let $y=\sqrt{e^{\sqrt{x}}}$

Then, $y^{2}=e^{\sqrt{x}}$

By differentiating this relationship with respect to x, we obtain

$$y^{2}=e^{\sqrt{x}}$$
$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right) \qquad \text{[By applying the chain rule]}$$
$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{1}{2}\cdot\frac{1}{\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}},\ x>0$$

Solution 8

Let $y = \log(\log x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right]$$
$$= \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$
$$= \frac{1}{\log x} \cdot \frac{1}{x}$$

$$= \frac{1}{x \log x}, x > 1$$

Solution 9

Let $y = \frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x) \times \log x - \cos x \times \frac{d}{dx}(\log x)}{(\log x)^2}$$
$$= \frac{-\sin x \log x - \cos x \times \frac{1}{x}}{(\log x)^2}$$
$$= \frac{-[x \log x . \sin x + \cos x]}{x(\log x)^2}, x > 0$$

Solution 10

Let $y = \cos\left(\log x + e^x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = -\sin\left(\log x + e^x\right) \cdot \frac{d}{dx}\left(\log x + e^x\right)$$
$$= -\sin\left(\log x + e^x\right) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}\left(e^x\right)\right]$$
$$= -\sin\left(\log x + e^x\right) \cdot \left(\frac{1}{x} + e^x\right)$$
$$= -\left(\frac{1}{x} + e^x\right)\sin\left(\log x + e^x\right), x > 0$$

Chapter 5 - Continuity and Differentiability Exercise Ex. 5.5

Solution 1

Let $y = \cos x . \cos 2x . \cos 3x$

Taking logarithm on both the sides, we obtain

$$\log y = \log\left(\cos x . \cos 2x . \cos 3x\right)$$
$$\Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) + \frac{1}{\cos 2x} \cdot \frac{d}{dx}(\cos 2x) + \frac{1}{\cos 3x} \cdot \frac{d}{dx}(\cos 3x)$$
$$\Rightarrow \frac{dy}{dx} = y\left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x} \cdot \frac{d}{dx}(2x) - \frac{\sin 3x}{\cos 3x} \cdot \frac{d}{dx}(3x)\right]$$
$$\therefore \frac{dy}{dx} = -\cos x . \cos 2x . \cos 3x\left[\tan x + 2\tan 2x + 3\tan 3x\right]$$

Solution 2

$$\text{Let } y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Taking logarithm on both the sides, we obtain

$$\log y=\log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$
$$\Rightarrow \log y=\frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$
$$\Rightarrow \log y=\frac{1}{2}\left[\log\{(x-1)(x-2)\}-\log\{(x-3)(x-4)(x-5)\}\right]$$
$$\Rightarrow \log y=\frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\begin{array}{l}\frac{1}{x-1}.\frac{d}{dx}(x-1)+\frac{1}{x-2}.\frac{d}{dx}(x-2)-\frac{1}{x-3}.\frac{d}{dx}(x-3)\\ -\frac{1}{x-4}.\frac{d}{dx}(x-4)-\frac{1}{x-5}.\frac{d}{dx}(x-5)\end{array}\right]$$
$$\Rightarrow \frac{dy}{dx}=\frac{y}{2}\left(\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right)$$
$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

Solution 3

Let $y = (\log x)^{\cos x}$

Taking logarithm on both the sides, we obtain

$\log y = \cos x \cdot \log(\log x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(\cos x) \times \log(\log x) + \cos x \times \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{1}{y} . \frac{dy}{dx} = -\sin x \log(\log x) + \cos x \times \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[-\sin x \log(\log x) + \frac{\cos x}{\log x} \times \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x \log(\log x)\right]$$

Solution 4

Let $y = x^x - 2^{\sin x}$

Also, let $x^x = u$ and $2^{\sin x} = v$

$\therefore y = u - v$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} - \dfrac{dv}{dx}$

$u = x^x$

Taking logarithm on both the sides, we obtain

$\log u = x \log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \left[\frac{d}{dx}(x) \times \log x + x \times \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log x + x \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x(\log x + 1)$$

$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$$

$v = 2^{\sin x}$

Taking logarithm on both the sides with respect to x, we obtain

$\log v = \sin x \cdot \log 2$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dy}{dx} = x^x(1 + \log x) - 2^{\sin x} \cos x \log 2$$

Solution 5

Let $y = (x+3)^2 . (x+4)^3 . (x+5)^4$

Taking logarithm on both the sides, we obtain

$$\log y = \log (x+3)^2 + \log (x+4)^3 + \log (x+5)^4$$
$$\Rightarrow \log y = 2 \log (x+3) + 3 \log (x+4) + 4 \log (x+5)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y} . \frac{dy}{dx} = 2 \cdot \frac{1}{x+3} \cdot \frac{d}{dx}(x+3) + 3 \cdot \frac{1}{x+4} \cdot \frac{d}{dx}(x+4) + 4 \cdot \frac{1}{x+5} \cdot \frac{d}{dx}(x+5)$$
$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$
$$\Rightarrow \frac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$
$$\Rightarrow \frac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\frac{2(x+4)(x+5) + 3(x+3)(x+5) + 4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$
$$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2 (x+5)^3 \cdot \left[2(x^2+9x+20) + 3(x^2+8x+15) + 4(x^2+7x+12)\right]$$
$$\therefore \frac{dy}{dx} = (x+3)(x+4)^2 (x+5)^3 (9x^2 + 70x + 133)$$

Solution 6

Let $y = \left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Also, let $u = \left(x+\frac{1}{x}\right)^x$ and $v = x^{\left(1+\frac{1}{x}\right)}$

$\therefore y = u + v$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$

Then, $u = \left(x+\frac{1}{x}\right)^x$

$\Rightarrow \log u = \log\left(x+\frac{1}{x}\right)^x$

$\Rightarrow \log u = x \log\left(x+\frac{1}{x}\right)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx} = \frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right) + x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = 1\times\log\left(x+\frac{1}{x}\right) + x\times\frac{1}{\left(x+\frac{1}{x}\right)}\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log\left(x+\frac{1}{x}\right) + \frac{x}{\left(x+\frac{1}{x}\right)}\times\left(1-\frac{1}{x^2}\right)\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right) + \frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right) + \frac{x^2-1}{x^2+1}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right]$$

$$v = x^{\left(1+\frac{1}{x}\right)}$$

$$\Rightarrow \log v = \log\left[x^{\left(1+\frac{1}{x}\right)}\right]$$

$$\Rightarrow \log v = \left(1+\frac{1}{x}\right)\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx} = \left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x + \left(1+\frac{1}{x}\right)\cdot\frac{d}{dx}\log x$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left(-\frac{1}{x^2}\right)\log x + \left(1+\frac{1}{x}\right)\cdot\frac{1}{x}$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\frac{\log x}{x^2} + \frac{1}{x} + \frac{1}{x^2}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{-\log x + x + 1}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right) \qquad \ldots(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)$$

Solution 7

Let $y = (\log x)^x + x^{\log x}$

Also, let $u = (\log x)^x$ and $v = x^{\log x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

$u = (\log x)^x$

$$\Rightarrow \log u = \log\left[(\log x)^x\right]$$
$$\Rightarrow \log u = x \log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x) \times \log(\log x) + x \cdot \frac{d}{dx}\left[\log(\log x)\right]$$
$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log(\log x) + x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)\right]$$
$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{x}{\log x} \cdot \frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right]$$
$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\frac{\log(\log x) \cdot \log x + 1}{\log x}\right]$$
$$\Rightarrow \frac{du}{dx} = (\log x)^{x-1}\left[1 + \log x . \log(\log x)\right] \qquad ...(2)$$

$v = x^{\log x}$

$$\Rightarrow \log v = \log\left(x^{\log x}\right)$$
$$\Rightarrow \log v = \log x \log x = (\log x)^2$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\frac{d}{dx}\left[(\log x)^2\right]$$

$$\Rightarrow \frac{1}{v}.\frac{dv}{dx}=2(\log x)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dv}{dx}=2v(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow \frac{dv}{dx}=2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow \frac{dv}{dx}=2x^{\log x-1}\cdot\log x \qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]+2x^{\log x-1}\cdot\log x$$

Solution 8

Let $y=(\sin x)^x+\sin^{-1}\sqrt{x}$

Also, let $u=(\sin x)^x$ and $v=\sin^{-1}\sqrt{x}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=(\sin x)^x$

$\Rightarrow \log u=\log(\sin x)^x$

$\Rightarrow \log u=x\log(\sin x)$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\sin x)+x\times\frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx}=u\left[1\cdot\log(\sin x)+x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx}=(\sin x)^x\left[\log(\sin x)+\frac{x}{\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{du}{dx}=(\sin x)^x(x\cot x+\log\sin x) \qquad ...(2)$$

$v=\sin^{-1}\sqrt{x}$

Differentiating both sides with respect to x, we obtain

$$\frac{dv}{dx}=\frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\Rightarrow\frac{dv}{dx}=\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$$

$$\Rightarrow\frac{dv}{dx}=\frac{1}{2\sqrt{x-x^2}}\qquad\qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=(\sin x)^x\left(x\cot x+\log\sin x\right)+\frac{1}{2\sqrt{x-x^2}}$$

Solution 9

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Also, let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

$u = x^{\sin x}$

$\Rightarrow \log u = \log\left(x^{\sin x}\right)$

$\Rightarrow \log u = \sin x \log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(\sin x)\cdot \log x + \sin x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\cos x \log x + \sin x \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right] \qquad ...(2)$$

$v = (\sin x)^{\cos x}$

$\Rightarrow \log v = \log(\sin x)^{\cos x}$

$\Rightarrow \log v = \cos x \log(\sin x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}(\cos x)\times\log(\sin x)+\cos x\times\frac{d}{dx}[\log(\sin x)]$$

$$\Rightarrow\frac{dv}{dx}=v\left[-\sin x.\log(\sin x)+\cos x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\frac{\cos x}{\sin x}\cos x\right]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}[-\sin x\log\sin x+\cot x\cos x]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}[\cot x\cos x-\sin x\log\sin x]\qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{\sin x}\left(\cos x\log x+\frac{\sin x}{x}\right)+(\sin x)^{\cos x}[\cos x\cot x-\sin x\log\sin x]$$

Solution 10

Let $y=x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Also, let $u=x^{x\cos x}$ and $v=\frac{x^2+1}{x^2-1}$

$\therefore y=u+v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\qquad ...(1)$$

$u=x^{x\cos x}$

$\Rightarrow\log u=\log\left(x^{x\cos x}\right)$

$\Rightarrow\log u=x\cos x\log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\cdot\cos x\cdot\log x+x\cdot\frac{d}{dx}(\cos x)\cdot\log x+x\cos x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{du}{dx}=u\left[1\cdot\cos x\cdot\log x+x\cdot(-\sin x)\log x+x\cos x\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}(\cos x\log x-x\sin x\log x+\cos x)$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}[\cos x(1+\log x)-x\sin x\log x]\qquad ...(2)$$

$$v=\frac{x^2+1}{x^2-1}$$
$$\Rightarrow \log v=\log\left(x^2+1\right)-\log\left(x^2-1\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{2x}{x^2+1}-\frac{2x}{x^2-1}$$
$$\Rightarrow \frac{dv}{dx}=v\left[\frac{2x\left(x^2-1\right)-2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$
$$\Rightarrow \frac{dv}{dx}=\frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$
$$\Rightarrow \frac{dv}{dx}=\frac{-4x}{\left(x^2-1\right)^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{x\cos x}\left[\cos x\left(1+\log x\right)-x\sin x\log x\right]-\frac{4x}{\left(x^2-1\right)^2}$$

Solution 11

$$\text{Let } y = (x\cos x)^x + (x\sin x)^{\frac{1}{x}}$$

$$\text{Also, let } u = (x\cos x)^x \text{ and } v = (x\sin x)^{\frac{1}{x}}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(1)$$

$$u = (x\cos x)^x$$

$$\Rightarrow \log u = \log (x\cos x)^x$$

$$\Rightarrow \log u = x\log (x\cos x)$$

$$\Rightarrow \log u = x[\log x + \log\cos x]$$

$$\Rightarrow \log u = x\log x + x\log\cos x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x\log x) + \frac{d}{dx}(x\log\cos x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\left\{\log x\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}(\log x)\right\} + \left\{\log\cos x\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}(\log\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[\left(\log x\cdot 1 + x\cdot\frac{1}{x}\right) + \left\{\log\cos x\cdot 1 + x\cdot\frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[(\log x + 1) + \left\{\log\cos x + \frac{x}{\cos x}\cdot(-\sin x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x \left[(1+\log x) + (\log\cos x - x\tan x)\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x \left[1 - x\tan x + (\log x + \log\cos x)\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x \left[1 - x\tan x + \log(x\cos x)\right] \qquad ...(2)$$

$$v = (x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x}\log(x\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}(\log x + \log\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}\log x + \frac{1}{x}\log\sin x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{x}\log x\right) + \frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x \cdot \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\cdot\frac{d}{dx}(\log x)\right] + \left[\log(\sin x)\cdot\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\cdot\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x \cdot\left(-\frac{1}{x^2}\right) + \frac{1}{x}\cdot\frac{1}{x}\right] + \left[\log(\sin x)\cdot\left(-\frac{1}{x^2}\right) + \frac{1}{x}\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(1-\log x) + \left[-\frac{\log(\sin x)}{x^2} + \frac{1}{x\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x}{x^2} + \frac{-\log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x - \log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x) + x\cot x}{x^2}\right] \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (x\cos x)^x\left[1 - x\tan x + \log(x\cos x)\right] + (x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

Solution 12

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad \ldots(1)$$

$$u = x^y$$
$$\Rightarrow \log u = \log\left(x^y\right)$$
$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log x)$$
$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y \cdot \frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx} = x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) \qquad \ldots(2)$$

$$v = y^x$$
$$\Rightarrow \log v = \log\left(y^x\right)$$
$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$
$$\Rightarrow \frac{dv}{dx} = v\left(\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right)$$
$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right)+y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right)=0$$

$$\Rightarrow\left(x^y\log x+xy^{x-1}\right)\frac{dy}{dx}=-\left(yx^{y-1}+y^x\log y\right)$$

$$\therefore\frac{dy}{dx}=-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$$

Solution 13

The given function is $y^x = x^y$

Taking logarithm on both the sides, we obtain

$$x\log y = y\log x$$

Differentiating both sides with respect to x, we obtain

$$\log y\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log y)=\log x\cdot\frac{d}{dx}(y)+y\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\log y\cdot 1+x\cdot\frac{1}{y}\cdot\frac{dy}{dx}=\log x\cdot\frac{dy}{dx}+y\cdot\frac{1}{x}$$

$$\Rightarrow\log y+\frac{x}{y}\frac{dy}{dx}=\log x\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow\left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\therefore\frac{dy}{dx}=\frac{y}{x}\left(\frac{y-x\log y}{x-y\log x}\right)$$

Solution 14

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$$y \log \cos x = x \log \cos y$$

Differentiating both sides, we obtain

$$\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) = \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) = \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx}(\cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} \cdot (-\sin x) = \log \cos y + \frac{x}{\cos y}(-\sin y) \cdot \frac{dy}{dx}$$

$$\Rightarrow \log \cos x \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \frac{dy}{dx}$$

$$\Rightarrow (\log \cos x + x \tan y)\frac{dy}{dx} = y \tan x + \log \cos y$$

$$\therefore \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$$

Solution 15

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\log(xy) = \log\left(e^{x-y}\right)$$
$$\Rightarrow \log x + \log y = (x-y)\log e$$
$$\Rightarrow \log x + \log y = (x-y)\times 1$$
$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) = \frac{d}{dx}(x) - \frac{dy}{dx}$$
$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$$
$$\Rightarrow \left(1 + \frac{1}{y}\right)\frac{dy}{dx} = 1 - \frac{1}{x}$$
$$\Rightarrow \left(\frac{y+1}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$
$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

Solution 16

The given relationship is $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x)=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{f(x)}\cdot\frac{d}{dx}[f(x)]=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log(1+x^2)+\frac{d}{dx}\log(1+x^4)+\frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow\frac{1}{f(x)}\cdot f'(x)=\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)+\frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)+\frac{1}{1+x^4}\cdot\frac{d}{dx}(1+x^4)+\frac{1}{1+x^8}\cdot\frac{d}{dx}(1+x^8)$$

$$\Rightarrow f'(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}\cdot 4x^3+\frac{1}{1+x^8}\cdot 8x^7\right]$$

$$\therefore f'(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

Hence, $f'(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2\times1}{1+1^2}+\frac{4\times1^3}{1+1^4}+\frac{8\times1^7}{1+1^8}\right]$

$$=2\times2\times2\times2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$=16\times\left(\frac{1+2+4+8}{2}\right)$$

$$=16\times\frac{15}{2}=120$$

Solution 17

(i)

Let $x^2-5x+8=u$ and $x^3+7x+9=v$

$\therefore y=uv$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}\cdot v+u\cdot\frac{dv}{dx} \qquad \text{(By using product rule)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(x^2-5x+8\right)\cdot\left(x^3+7x+9\right)+\left(x^2-5x+8\right)\cdot\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\Rightarrow \frac{dy}{dx}=(2x-5)\left(x^3+7x+9\right)+\left(x^2-5x+8\right)\left(3x^2+7\right)$$

$$\Rightarrow \frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+x^2\left(3x^2+7\right)-5x\left(3x^2+7\right)+8\left(3x^2+7\right)$$

$$\Rightarrow \frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4+7x^2\right)-15x^3-35x+24x^2+56$$

$$\therefore \frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

(ii)

$$\begin{aligned} y&=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\\ &=x^2\left(x^3+7x+9\right)-5x\left(x^3+7x+9\right)+8\left(x^3+7x+9\right)\\ &=x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72\\ &=x^5-5x^4+15x^3-26x^2+11x+72 \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx}&=\frac{d}{dx}\left(x^5-5x^4+15x^3-26x^2+11x+72\right)\\ &=\frac{d}{dx}\left(x^5\right)-5\frac{d}{dx}\left(x^4\right)+15\frac{d}{dx}\left(x^3\right)-26\frac{d}{dx}\left(x^2\right)+11\frac{d}{dx}(x)+\frac{d}{dx}(72)\\ &=5x^4-5\times4x^3+15\times3x^2-26\times2x+11\times1+0\\ &=5x^4-20x^3+45x^2-52x+11 \end{aligned}$$

(iii) $y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$

Taking logarithm on both the sides, we obtain

$$\log y=\log\left(x^2-5x+8\right)+\log\left(x^3+7x+9\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\log\left(x^2-5x+8\right)+\frac{d}{dx}\log\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}\cdot\frac{d}{dx}\left(x^2-5x+8\right)+\frac{1}{x^3+7x+9}\cdot\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{dy}{dx}=y\left[\frac{1}{x^2-5x+8}\times(2x-5)+\frac{1}{x^3+7x+9}\times\left(3x^2+7\right)\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{(2x-5)\left(x^3+7x+9\right)+\left(3x^2+7\right)\left(x^2-5x+8\right)}{\left(x^2-5x+8\right)\left(x^3+7x+9\right)}\right]$$

$$\Rightarrow\frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+3x^2\left(x^2-5x+8\right)+7\left(x^2-5x+8\right)$$

$$\Rightarrow\frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4-15x^3+24x^2\right)+\left(7x^2-35x+56\right)$$

$$\Rightarrow\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

Solution 18

Let $y = u.v.w = u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx} = \frac{du}{dx}.(v.w) + u.\frac{d}{dx}(v \cdot w)$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} v \cdot w + u\left[\frac{dv}{dx} \cdot w + v \cdot \frac{dw}{dx}\right] \qquad \text{(Again applying product rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y = u.v.w$, we obtain

$$\log y = \log u + \log v + \log w$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w.\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

Chapter 5 - Continuity and Differentiability Exercise Ex. 5.6

Solution 1

The given equations are $x = 2at^2$ and $y = at^4$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}(2at^2) = 2a \cdot \frac{d}{dt}(t^2) = 2a \cdot 2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}(at^4) = a \cdot \frac{d}{dt}(t^4) = a \cdot 4 \cdot t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

Solution 2

The given equations are $x = a \cos\theta$ and $y = b \cos\theta$

Then, $\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta) = a(-\sin\theta) = -a\sin\theta$

$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\cos\theta) = b(-\sin\theta) = -b\sin\theta$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$

Solution 3

The given equations are $x = \sin t$ and $y = \cos 2t$

Then, $\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$

$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t \cdot \frac{d}{dt}(2t) = -2\sin 2t$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2\cdot 2\sin t\cos t}{\cos t} = -4\sin t$

Solution 4

The given equations are $x = 4t$ and $y = \frac{4}{t}$

$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4$

$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4\cdot\frac{d}{dt}\left(\frac{1}{t}\right) = 4\cdot\left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$

Solution 5

The given equations are $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta - \cos 2\theta) = \frac{d}{d\theta}(\cos\theta) - \frac{d}{d\theta}(\cos 2\theta)$$
$$= -\sin\theta - (-2\sin 2\theta) = 2\sin 2\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta - \sin 2\theta) = \frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\sin 2\theta)$$
$$= \cos\theta - 2\cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

Solution 6

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta)\right] = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a\left[0 + (-\sin\theta)\right] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1 - \cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Solution 7

The given equations are $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$

Then, $\dfrac{dx}{dt} = \dfrac{d}{dt}\left[\dfrac{\sin^3 t}{\sqrt{\cos 2t}}\right]$

$$= \frac{\sqrt{\cos 2t}\cdot\frac{d}{dt}\left(\sin^3 t\right) - \sin^3 t\cdot\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\cdot 3\sin^2 t\cdot\frac{d}{dt}(\sin t) - \sin^3 t\times\frac{1}{2\sqrt{\cos 2t}}.\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t}\cdot\sin^2 t\cos t - \frac{\sin^3 t}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$= \frac{3\cos 2t\sin^2 t\cos t + \sin^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\left[\dfrac{\cos^3 t}{\sqrt{\cos 2t}}\right]$

$$= \frac{\sqrt{\cos 2t}.\frac{d}{dt}\left(\cos^3 t\right) - \cos^3 t\cdot\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}.3\cos^2 t\cdot\frac{d}{dt}(\cos t) - \cos^3 t.\frac{1}{2\sqrt{\cos 2t}}\cdot\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t}.\cos^2 t(-\sin t) - \cos^3 t\cdot\frac{1}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t\sin 2t}{\cos 2t\cdot\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \sin 2t}{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t\left(2\sin t \cos t\right)}{3\cos 2t \sin^2 t \cos t + \sin^3 t\left(2\sin t \cos t\right)}$$

$$= \frac{\sin t \cos t\left[-3\cos 2t.\cos t + 2\cos^3 t\right]}{\sin t \cos t\left[3\cos 2t \sin t + 2\sin^3 t\right]}$$

$$= \frac{\left[-3\left(2\cos^2 t - 1\right)\cos t + 2\cos^3 t\right]}{\left[3\left(1 - 2\sin^2 t\right)\sin t + 2\sin^3 t\right]} \qquad \left[\begin{array}{l}\cos 2t = \left(2\cos^2 t - 1\right), \\ \cos 2t = \left(1 - 2\sin^2 t\right)\end{array}\right]$$

$$= \frac{-4\cos^3 t + 3\cos t}{3\sin t - 4\sin^3 t}$$

$$= \frac{-\cos 3t}{\sin 3t} \qquad \left[\begin{array}{l}\cos 3t = 4\cos^3 t - 3\cos t, \\ \sin 3t = 3\sin t - 4\sin^3 t\end{array}\right]$$

$$= -\cot 3t$$

Solution 8

The given equations are $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$ and $y = a \sin t$

$$\text{Then, } \frac{dx}{dt} = a \cdot \left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log \tan \frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt}\left(\tan \frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \cot \frac{t}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right)$$

$$= a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin t) = a \cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a \cos t}{\left(a\frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

Solution 9

The given equations are $x = a\sec\theta$ and $y = b\tan\theta$

Then, $\dfrac{dx}{d\theta} = a \cdot \dfrac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$

$\dfrac{dy}{d\theta} = b \cdot \dfrac{d}{d\theta}(\tan\theta) = b\sec^2\theta$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\sec\theta\cot\theta = \frac{b\cos\theta}{a\cos\theta\sin\theta} = \frac{b}{a} \times \frac{1}{\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta$$

Solution 10

The given equations are $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$$
$$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta \cdot \frac{d}{d\theta}(\theta)\right\}\right]$$
$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$
$$= a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Solution 11

The given equations are $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$\Rightarrow x=\left(a^{\sin^{-1}t}\right)^{\frac{1}{2}}$ and $y=\left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}$

$\Rightarrow x=a^{\frac{1}{2}\sin^{-1}t}$ and $y=a^{\frac{1}{2}\cos^{-1}t}$

Consider $x=a^{\frac{1}{2}\sin^{-1}t}$

Taking logarithm on both the sides, we obtain

$\log x=\frac{1}{2}\sin^{-1}t\log a$

$\therefore \frac{1}{x}\cdot\frac{dx}{dt}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\sin^{-1}t\right)$

$\Rightarrow \frac{dx}{dt}=\frac{x}{2}\log a\cdot\frac{1}{\sqrt{1-t^2}}$

$\Rightarrow \frac{dx}{dt}=\frac{x\log a}{2\sqrt{1-t^2}}$

Then, consider $y=a^{\frac{1}{2}\cos^{-1}t}$

Taking logarithm on both the sides, we obtain

$\log y=\frac{1}{2}\cos^{-1}t\log a$

$\therefore \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\cos^{-1}t\right)$

$\Rightarrow \frac{dy}{dt}=\frac{y\log a}{2}\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$

$\Rightarrow \frac{dy}{dt}=\frac{-y\log a}{2\sqrt{1-t^2}}$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)}=-\frac{y}{x}.$$

Hence, proved.

Chapter 5 - Continuity and Differentiability Exercise Ex. 5.7

Solution 1

Let $y = x^2 + 3x + 2$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0 = 2x + 3$$
$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(2x+3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2$$

Solution 2

Let $y = x^{20}$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{20}\right) = 20x^{19}$$
$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(20x^{19}\right) = 20\frac{d}{dx}\left(x^{19}\right) = 20 \cdot 19 \cdot x^{18} = 380x^{18}$$

Solution 3

Let $y = x \cdot \cos x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot \cos x) = \cos x \cdot \frac{d}{dx}(x) + x\frac{d}{dx}(\cos x) = \cos x \cdot 1 + x(-\sin x) = \cos x - x\sin x$$
$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[\cos x - x\sin x] = \frac{d}{dx}(\cos x) - \frac{d}{dx}(x\sin x)$$
$$= -\sin x - \left[\sin x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sin x)\right]$$
$$= -\sin x - (\sin x + x\cos x)$$
$$= -(x\cos x + 2\sin x)$$

Solution 4

Let $y = \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$
$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

Solution 5

Let $y = x^3 \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^3 \log x\right] = \log x \cdot \frac{d}{dx}\left(x^3\right) + x^3 \cdot \frac{d}{dx}\left(\log x\right)$$

$$= \log x \cdot 3x^2 + x^3 \cdot \frac{1}{x} = \log x \cdot 3x^2 + x^2$$

$$= x^2\left(1 + 3\log x\right)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[x^2\left(1 + 3\log x\right)\right]$$

$$= \left(1 + 3\log x\right) \cdot \frac{d}{dx}\left(x^2\right) + x^2 \frac{d}{dx}\left(1 + 3\log x\right)$$

$$= \left(1 + 3\log x\right) \cdot 2x + x^2 \cdot \frac{3}{x}$$

$$= 2x + 6x\log x + 3x$$

$$= 5x + 6x\log x$$

$$= x\left(5 + 6\log x\right)$$

Solution 6

Let $y = e^x \sin 5x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x \sin 5x\right) = \sin 5x \cdot \frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}\left(\sin 5x\right)$$

$$= \sin 5x \cdot e^x + e^x \cdot \cos 5x \cdot \frac{d}{dx}\left(5x\right) = e^x \sin 5x + e^x \cos 5x \cdot 5$$

$$= e^x\left(\sin 5x + 5\cos 5x\right)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^x\left(\sin 5x + 5\cos 5x\right)\right]$$

$$= \left(\sin 5x + 5\cos 5x\right) \cdot \frac{d}{dx}\left(e^x\right) + e^x \cdot \frac{d}{dx}\left(\sin 5x + 5\cos 5x\right)$$

$$= \left(\sin 5x + 5\cos 5x\right)e^x + e^x\left[\cos 5x \cdot \frac{d}{dx}\left(5x\right) + 5\left(-\sin 5x\right) \cdot \frac{d}{dx}\left(5x\right)\right]$$

$$= e^x\left(\sin 5x + 5\cos 5x\right) + e^x\left(5\cos 5x - 25\sin 5x\right)$$

$$= e^x\left(10\cos 5x - 24\sin 5x\right) = 2e^x\left(5\cos 5x - 12\sin 5x\right)$$

Solution 7

Let $y = e^{6x} \cos 3x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x} \cdot \cos 3x\right) = \cos 3x \cdot \frac{d}{dx}\left(e^{6x}\right) + e^{6x} \cdot \frac{d}{dx}\left(\cos 3x\right)$$
$$= \cos 3x \cdot e^{6x} \cdot \frac{d}{dx}(6x) + e^{6x} \cdot (-\sin 3x) \cdot \frac{d}{dx}(3x)$$
$$= 6e^{6x} \cos 3x - 3e^{6x} \sin 3x \qquad \ldots(1)$$
$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(6e^{6x} \cos 3x - 3e^{6x} \sin 3x\right) = 6 \cdot \frac{d}{dx}\left(e^{6x} \cos 3x\right) - 3 \cdot \frac{d}{dx}\left(e^{6x} \sin 3x\right)$$
$$= 6 \cdot \left[6e^{6x} \cos 3x - 3e^{6x} \sin 3x\right] - 3 \cdot \left[\sin 3x \cdot \frac{d}{dx}\left(e^{6x}\right) + e^{6x} \cdot \frac{d}{dx}(\sin 3x)\right] \qquad [\text{Using } (1)]$$
$$= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 3\left[\sin 3x \cdot e^{6x} \cdot 6 + e^{6x} \cdot \cos 3x \cdot 3\right]$$
$$= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 18e^{6x} \sin 3x - 9e^{6x} \cos 3x$$
$$= 27e^{6x} \cos 3x - 36e^{6x} \sin 3x$$
$$= 9e^{6x}\left(3\cos 3x - 4\sin 3x\right)$$

Solution 8

Let $y = \tan^{-1} x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$
$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{d}{dx}\left(1+x^2\right)^{-1} = (-1) \cdot \left(1+x^2\right)^{-2} \cdot \frac{d}{dx}\left(1+x^2\right)$$
$$= \frac{-1}{\left(1+x^2\right)^2} \times 2x = \frac{-2x}{\left(1+x^2\right)^2}$$

Solution 9

Let $y = \log(\log x)$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right] = \frac{1}{\log x}\cdot\frac{d}{dx}(\log x) = \frac{1}{x\log x} = (x\log x)^{-1}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[(x\log x)^{-1}\right] = (-1)\cdot(x\log x)^{-2}\cdot\frac{d}{dx}(x\log x)$$

$$= \frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}(\log x)\right]$$

$$= \frac{-1}{(x\log x)^2}.\left[\log x\cdot 1 + x\cdot\frac{1}{x}\right] = \frac{-(1+\log x)}{(x\log x)^2}$$

Solution 10

Let $y = \sin(\log x)$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin(\log x)\right] = \cos(\log x)\cdot\frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$= \frac{x\cdot\frac{d}{dx}\left[\cos(\log x)\right] - \cos(\log x)\cdot\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\cdot\left[-\sin(\log x)\cdot\frac{d}{dx}(\log x)\right] - \cos(\log x).1}{x^2}$$

$$= \frac{-x\sin(\log x).\frac{1}{x} - \cos(\log x)}{x^2}$$

$$= \frac{-\left[\sin(\log x) + \cos(\log x)\right]}{x^2}$$

Solution 11

It is given that, $y = 5\cos x - 3\sin x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x) - \frac{d}{dx}(3\sin x) = 5\frac{d}{dx}(\cos x) - 3\frac{d}{dx}(\sin x)$$
$$= 5(-\sin x) - 3\cos x = -(5\sin x + 3\cos x)$$
$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(5\sin x + 3\cos x)\right]$$
$$= -\left[5\cdot\frac{d}{dx}(\sin x) + 3\cdot\frac{d}{dx}(\cos x)\right]$$
$$= -\left[5\cos x + 3(-\sin x)\right]$$
$$= -\left[5\cos x - 3\sin x\right]$$
$$= -y$$
$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Hence, proved.

Solution 12

It is given that, $y = \cos^{-1} x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} = -(1-x^2)^{\frac{-1}{2}}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(1-x^2)^{\frac{-1}{2}}\right]$$
$$= -\left(-\frac{1}{2}\right)\cdot(1-x^2)^{\frac{-3}{2}}\cdot\frac{d}{dx}(1-x^2)$$
$$= \frac{1}{2\sqrt{(1-x^2)^3}} \times (-2x)$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x}{\sqrt{(1-x^2)^3}} \qquad \ldots(i)$$
$$y = \cos^{-1} x \Rightarrow x = \cos y$$

Putting $x = \cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$= \frac{-\cos y}{\sin^3 y}$$

$$= \frac{-\cos y}{\sin y} \times \frac{1}{\sin^2 y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\cot y \cdot \operatorname{cosec}^2 y$$

Solution 13

It is given that, $y = 3\cos(\log x) + 4\sin(\log x)$

Then,

$$y_1 = 3 \cdot \frac{d}{dx}\left[\cos(\log x)\right] + 4 \cdot \frac{d}{dx}\left[\sin(\log x)\right]$$

$$= 3 \cdot \left[-\sin(\log x) \cdot \frac{d}{dx}(\log x)\right] + 4 \cdot \left[\cos(\log x) \cdot \frac{d}{dx}(\log x)\right]$$

$$\therefore y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

$$\therefore y_2 = \frac{d}{dx}\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)$$

$$= \frac{x\left\{4\cos(\log x) - 3\sin(\log x)\right\}' - \left\{4\cos(\log x) - 3\sin(\log x)\right\}(x)'}{x^2}$$

$$= \frac{x\left[4\left\{\cos(\log x)\right\}' - 3\left\{\sin(\log x)\right\}'\right] - \left\{4\cos(\log x) - 3\sin(\log x)\right\}.1}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).(\log x)' - 3\cos(\log x).(\log x)'\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).\frac{1}{x} - 3\cos(\log x).\frac{1}{x}\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

$$\therefore x^2 y_2 + xy_1 + y$$
$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) + 3\cos(\log x) + 4\sin(\log x)$$
$$= -\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 3\cos(\log x) + 4\sin(\log x)$$
$$= 0$$

Hence, proved.

Solution 14

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A\cdot\frac{d}{dx}\left(e^{mx}\right) + B\cdot\frac{d}{dx}\left(e^{nx}\right) = A\cdot e^{mx}.\frac{d}{dx}(mx) + B\cdot e^{nx}\cdot\frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right) = Am\cdot\frac{d}{dx}\left(e^{mx}\right) + Bn\cdot\frac{d}{dx}\left(e^{nx}\right)$$
$$= Am\cdot e^{mx}\cdot\frac{d}{dx}(mx) + Bn\cdot e^{nx}\cdot\frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx}$$
$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$
$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)\cdot\left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$
$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amn\,e^{mx} + Bmn\,e^{nx}$$
$$= 0$$

Hence, proved.

Solution 15

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\begin{aligned}
\frac{dy}{dx} &= 500.\frac{d}{dx}\left(e^{7x}\right) + 600.\frac{d}{dx}\left(e^{-7x}\right)\\
&= 500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)\\
&= 3500e^{7x} - 4200e^{-7x}
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{d^2y}{dx^2} &= 3500 \cdot \frac{d}{dx}\left(e^{7x}\right) - 4200.\frac{d}{dx}\left(e^{-7x}\right)\\
&= 3500 \cdot e^{7x}.\frac{d}{dx}(7x) - 4200 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)\\
&= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x}\\
&= 49 \times 500e^{7x} + 49 \times 600e^{-7x}\\
&= 49\left(500e^{7x} + 600e^{-7x}\right)\\
&= 49y
\end{aligned}$$

Hence, proved

Solution 16

The given relationship is $e^{y}(x+1)=1$

$$e^{y}(x+1)=1$$
$$\Rightarrow e^{y}=\frac{1}{x+1}$$

Taking logarithm on both the sides, we obtain

$$y=\log\frac{1}{(x+1)}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right)=(x+1).\frac{-1}{(x+1)^2}=\frac{-1}{x+1}$$
$$\therefore \frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\frac{1}{x+1}\right)=-\left(\frac{-1}{(x+1)^2}\right)=\frac{1}{(x+1)^2}$$
$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{-1}{x+1}\right)^2$$
$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Hence, proved.

Solution 17

The given relationship is $y = \left(\tan^{-1} x\right)^2$

Then,

$$y_1 = 2\tan^{-1} x \frac{d}{dx}\left(\tan^{-1} x\right)$$
$$\Rightarrow y_1 = 2\tan^{-1} x . \frac{1}{1+x^2}$$
$$\Rightarrow \left(1+x^2\right) y_1 = 2\tan^{-1} x$$

Again differentiating with respect to x on both the sides, we obtain

$$\left(1+x^2\right) y_2 + 2xy_1 = 2\left(\frac{1}{1+x^2}\right)$$
$$\Rightarrow \left(1+x^2\right)^2 y_2 + 2x\left(1+x^2\right) y_1 = 2$$

Hence, proved.

Chapter 5 - Continuity and Differentiability Exercise Ex. 5.8

Solution 1

The given function, $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in $[-4, 2]$ and is differentiable in $(-4, 2)$.

$$f(-4) = (-4)^2 + 2\times(-4) - 8 = 16 - 8 - 8 = 0$$
$$f(2) = (2)^2 + 2\times 2 - 8 = 4 + 4 - 8 = 0$$

$$f(-4) = f(2) = 0$$

The value of $f(x)$ at -4 and 2 coincides.

Rolle's theorem states that there is a point c $\in$ (-4, -2) such that f'(c) = 0

$$f(x) = x^2 + 2x - 8$$
$$\Rightarrow f'(x) = 2x + 2$$
$$\therefore f'(c) = 0$$
$$\Rightarrow 2c + 2 = 0$$
$$\Rightarrow c = -1, \text{ where } c = -1 \in (-4, 2)$$

Hence, Rolle's Theorem is verified for the given function.

Solution 2

By Rolle's Theorem, for a function $f:[a, b] \to \mathbf{R}$, if

(a) f is continuous on $[a, b]$

(b) f is differentiable on (a, b)

(c) $f(a) = f(b)$

then there exists some c $\in$(a, b) such that f'(c) = 0

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5, 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$f(x)$ is not continuous in $[5, 9]$.

Also, $f(5) = [5] = 5$ and $f(9) = [9] = 9$
$\therefore f(5) \neq f(9)$

The differentiability of f in $(5, 9)$ is checked as follows.

Let n be an integer such that n $\in$ (5, 9).

The left hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

$\therefore f(-2) \neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that n $\in$ (-2, 2).

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

f is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

$$f(1) = (1)^2 - 1 = 0$$
$$f(2) = (2)^2 - 1 = 3$$

$f(1) \neq f(2)$

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Solution 3

It is given that $f : [-5, 5] \to \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on $[-5, 5]$.

(b) f is differentiable on $(-5, 5)$.

Therefore, by the Mean Value Theorem, there exists c ϵ (-5, 5) such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$
$$\Rightarrow 10f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c) \neq 0$$
$$\Rightarrow 10f'(c) \neq 0$$
$$\Rightarrow f(5) - f(-5) \neq 0$$
$$\Rightarrow f(5) \neq f(-5)$$

Hence, proved.

Solution 4

The given function is $f(x) = x^2 - 4x - 3$

f, being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$ whose derivative is $2x - 4$.

$$f(1) = 1^2 - 4 \times 1 - 3 = -6, \; f(4) = 4^2 - 4 \times 4 - 3 = -3$$
$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1$$

Mean Value Theorem states that there is a point c $\in$(1, 4) such that f'(c) = 1

$$f'(c) = 1$$
$$\Rightarrow 2c - 4 = 1$$
$$\Rightarrow c = \frac{5}{2}, \text{ where } c = \frac{5}{2} \in (1,\ 4)$$

Hence, Mean Value Theorem is verified for the given function

Solution 5

The given function f is $f(x) = x^3 - 5x^2 - 3x$

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2 - 10x - 3$.

$$f(1) = 1^3 - 5\times1^2 - 3\times1 = -7,\ \ f(3) = 3^3 - 5\times3^2 - 3\times3 = -27$$
$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1} = \frac{-27-(-7)}{3-1} = -10$$

Mean Value Theorem states that there exist a point c □ (1, 3) such that $f'(c) = -10$

$$\begin{aligned}
&f'(c) = -10 \\
&\Rightarrow 3c^2 - 10c - 3 = 10 \\
&\Rightarrow 3c^2 - 10c + 7 = 0 \\
&\Rightarrow 3c^2 - 3c - 7c + 7 = 0 \\
&\Rightarrow 3c(c-1) - 7(c-1) = 0 \\
&\Rightarrow (c-1)(3c-7) = 0 \\
&\Rightarrow c = 1,\ \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1,\ 3)
\end{aligned}$$

Hence, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1,\ 3)$ is the only point for which $f'(c) = 0$

Solution 6

Mean Value Theorem states that for a function $f:[a,\ b]\to\mathbf{R}$, if

(a) f is continuous on $[a, b]$

(b) f is differentiable on (a, b)

then, there exists some $c \in (a, b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$\Rightarrow f(x)$ is not continuous in $[5, 9]$.

The differentiability of f in $(5, 9)$ is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x=n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(5, 9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x=-2$ and $x=2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n\in(-2, 2)$.

The left hand limit of f at $x=n$ is,

$$\lim_{h\to0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to0^-}\frac{n-1-n}{h}=\lim_{h\to0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to0^+}\frac{n-n}{h}=\lim_{h\to0^+}0=0$$

Since the left and right hand limits of f at $x=n$ are not equal, f is not differentiable at $x=n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

It is evident that f, being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x)=x^2-1$ for $x\in[1,\ 2]$.

It can be proved as follows.

$$f(1)=1^2-1=0,\ f(2)=2^2-1=3$$
$$\therefore\frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(1)}{2-1}=\frac{3-0}{1}=3$$

$$f'(x)=2x$$
$$\therefore f'(c)=3$$
$$\Rightarrow 2c=3$$
$$\Rightarrow c=\frac{3}{2}=1.5,\ \text{where } 1.5\in[1, 2]$$

Chapter 5 - Continuity and Differentiability Exercise Misc. Ex.

Solution 1

Let $y=\left(3x^2-9x+5\right)^9$

Using chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(3x^2-9x+5\right)^9\\&=9\left(3x^2-9x+5\right)^8\cdot\frac{d}{dx}\left(3x^2-9x+5\right)\\&=9\left(3x^2-9x+5\right)^8\cdot(6x-9)\\&=9\left(3x^2-9x+5\right)^8\cdot3(2x-3)\\&=27\left(3x^2-9x+5\right)^8(2x-3)\end{aligned}$$

Solution 2

Let $y=\sin^3 x+\cos^6 x$

$$\begin{aligned}\therefore\frac{dy}{dx}&=\frac{d}{dx}\left(\sin^3 x\right)+\frac{d}{dx}\left(\cos^6 x\right)\\&=3\sin^2 x\cdot\frac{d}{dx}(\sin x)+6\cos^5 x\cdot\frac{d}{dx}(\cos x)\\&=3\sin^2 x\cdot\cos x+6\cos^5 x\cdot(-\sin x)\\&=3\sin x\cos x\left(\sin x-2\cos^4 x\right)\end{aligned}$$

Solution 3

Let $y = (5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$\log y = 3\cos 2x \log 5x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\log 5x \cdot \frac{d}{dx}(\cos 2x) + \cos 2x \cdot \frac{d}{dx}(\log 5x)\right]$$
$$\Rightarrow \frac{dy}{dx} = 3y\left[\log 5x(-\sin 2x)\cdot \frac{d}{dx}(2x) + \cos 2x \cdot \frac{1}{5x}\cdot\frac{d}{dx}(5x)\right]$$
$$\Rightarrow \frac{dy}{dx} = 3y\left[-2\sin 2x \log 5x + \frac{\cos 2x}{x}\right]$$
$$\Rightarrow \frac{dy}{dx} = y\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$
$$\therefore \frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

Solution 4

Let $y = \sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$
$$= \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}} \times \frac{d}{dx}\left(x\sqrt{x}\right)$$
$$= \frac{1}{\sqrt{1-x^3}}\cdot\frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$
$$= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2}\cdot x^{\frac{1}{2}}$$
$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$
$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Solution 5

Let $y=\dfrac{\cos^{-1}\dfrac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\sqrt{2x+7}\,\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right)-\left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$=\frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{x}{2}\right)\right]-\left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2\sqrt{2x+7}}\cdot\frac{d}{dx}(2x+7)}{2x+7}$$

$$=\frac{\sqrt{2x+7}\,\frac{-1}{\sqrt{4-x^2}}-\left(\cos^{-1}\frac{x}{2}\right)\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$=\frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)}-\frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$=-\left[\frac{1}{\sqrt{4-x^2}\,\sqrt{2x+7}}+\frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Solution 6

$$\text{Let } y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right] \qquad ...(1)$$

$$\text{Then, } \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$=\frac{1+\cos x}{\sin x}$$

$$=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

Therefore, equation (1) becomes

$$y=\cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y=\frac{x}{2}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}$$

Solution 7

Let $y = (\log x)^{\log x}$

Taking logarithm on both the sides, we obtain

$$\log y = \log x \cdot \log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log x \cdot \log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x).\frac{1}{x} + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

Solution 8

Let $y = \cos(a\cos x + b\sin x)$

By using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x) \cdot \frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x) \cdot \left[a(-\sin x) + b\cos x\right]$$

$$= (a\sin x - b\cos x) \cdot \sin(a\cos x + b\sin x)$$

Solution 9

Let $y=(\sin x-\cos x)^{(\sin x-\cos x)}$

Taking logarithm on both the sides, we obtain

$$\log y=\log\left[(\sin x-\cos x)^{(\sin x-\cos x)}\right]$$
$$\Rightarrow \log y=(\sin x-\cos x)\cdot\log(\sin x-\cos x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[(\sin x-\cos x)\log(\sin x-\cos x)\right]$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot\frac{d}{dx}(\sin x-\cos x)+(\sin x-\cos x)\cdot\frac{d}{dx}\log(\sin x-\cos x)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot(\cos x+\sin x)+(\sin x-\cos x)\cdot\frac{1}{(\sin x-\cos x)}\cdot\frac{d}{dx}(\sin x-\cos x)$$
$$\Rightarrow \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}\left[(\cos x+\sin x)\cdot\log(\sin x-\cos x)+(\cos x+\sin x)\right]$$
$$\therefore \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

where sin x > cosx

Solution 10

Let $y = x^x + x^a + a^x + a^a$

Also, let $x^x = u,\ x^a = v,\ a^x = w,$ and $a^a = s$

$\therefore y = u + v + w + s$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \qquad ...(1)$$

$u = x^x$

$\Rightarrow \log u = \log x^x$

$\Rightarrow \log u = x \log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \cdot 1 + x \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x[\log x + 1] = x^x(1 + \log x) \qquad ...(2)$$

$v = x^a$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}(x^a)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1} \qquad ...(3)$$

$w = a^x$

$\Rightarrow \log w = \log a^x$

$\Rightarrow \log w = x \log a$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{w} \cdot \frac{dw}{dx} = \log a \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w \log a$$

$$\Rightarrow \frac{dw}{dx} = a^x \log a \qquad ...(4)$$

$s = a^a$

Since a is constant, a^a is also a constant.

$$\frac{ds}{dx} = 0 \qquad \ldots(5)$$

From (1), (2), (3), (4), and (5), we obtain

$$\begin{aligned}\frac{dy}{dx} &= x^x(1+\log x) + ax^{a-1} + a^x \log a + 0 \\ &= x^x(1+\log x) + ax^{a-1} + a^x \log a\end{aligned}$$

Solution 11

Let $y = x^{x^2-3} + (x-3)^{x^2}$

Also, let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

$\therefore y = u + v$

Differentiating both sides with respect to x, we obtain

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(1)$$

$u = x^{x^2-3}$

$\therefore \log u = \log\left(x^{x^2-3}\right)$

$\log u = (x^2-3)\log x$

Differentiating with respect to x, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx} = \log x \cdot \frac{d}{dx}(x^2-3) + (x^2-3)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log x \cdot 2x + (x^2-3)\cdot\frac{1}{x}$$

$$\Rightarrow \frac{du}{dx} = x^{x^2-3}\cdot\left[\frac{x^2-3}{x} + 2x\log x\right]$$

Also,

$$v = (x-3)^{x^2}$$
$$\therefore \log v = \log (x-3)^{x^2}$$
$$\Rightarrow \log v = x^2 \log(x-3)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx} = \log(x-3)\cdot\frac{d}{dx}(x^2) + x^2\cdot\frac{d}{dx}[\log(x-3)]$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \log(x-3)\cdot 2x + x^2\cdot\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)$$
$$\Rightarrow \frac{dv}{dx} = v\left[2x\log(x-3) + \frac{x^2}{x-3}\cdot 1\right]$$
$$\Rightarrow \frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

Substituting the expressions of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right] + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

Solution 12

It is given that, $y = 12(1-\cos t), x = 10(t-\sin t)$

$$\therefore \frac{dx}{dt} = \frac{d}{dt}[10(t-\sin t)] = 10\cdot\frac{d}{dt}(t-\sin t) = 10(1-\cos t)$$
$$\frac{dy}{dt} = \frac{d}{dt}[12(1-\cos t)] = 12\cdot\frac{d}{dt}(1-\cos t) = 12\cdot[0-(-\sin t)] = 12\sin t$$
$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{12\sin t}{10(1-\cos t)} = \frac{12\cdot 2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{10\cdot 2\sin^2\frac{t}{2}} = \frac{6}{5}\cot\frac{t}{2}$$

Solution 13

It is given that, $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{x}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{2x\sqrt{1-x^2}}(-2x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx}=0$$

Solution 14

It is given that,

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

$$\Rightarrow x\sqrt{1+y}=-y\sqrt{1+x}$$

Squaring both sides, we obtain

$$x^2(1+y)=y^2(1+x)$$

$$\Rightarrow x^2+x^2y=y^2+xy^2$$

$$\Rightarrow x^2-y^2=xy^2-x^2y$$

$$\Rightarrow x^2-y^2=xy(y-x)$$

$$\Rightarrow (x+y)(x-y)=xy(y-x)$$

$$\therefore x+y=-xy$$

$$\Rightarrow (1+x)y=-x$$

$$\Rightarrow y=\frac{-x}{(1+x)}$$

Differentiating both sides with respect to x, we obtain

$$y=\frac{-x}{(1+x)}$$

$$\frac{dy}{dx}=-\frac{(1+x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+x)}{(1+x)^2}=-\frac{(1+x)-x}{(1+x)^2}=-\frac{1}{(1+x)^2}$$

Hence, proved.

Solution 15

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}(c^2)$$

$$\Rightarrow 2(x-a)\cdot\frac{d}{dx}(x-a)+2(y-b)\cdot\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a)\cdot 1+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \qquad \ldots(1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$= -\left[\frac{(y-b)\cdot\frac{d}{dx}(x-a)-(x-a)\cdot\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad [\text{Using } (1)]$$

$$= -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= \frac{\left[\frac{c^2}{(y-b)^2}\right]}{\frac{-c^2}{(y-b)^3}} = \frac{\frac{c^3}{(y-b)^3}}{\frac{-c^2}{(y-b)^3}} = -c \text{ which is constant and is independent of a and b}$$

Hence, proved.

Solution 16

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx}[\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x\cdot[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \qquad ...(1)$$

Since $\cos y = x\cos(a+y)$, $x = \dfrac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to

$$\left[\frac{\cos y}{\cos(a+y)}\cdot\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow [\cos y\cdot\sin(a+y) - \sin y\cdot\cos(a+y)]\cdot\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Solution 17

It is given that, $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\therefore \frac{dx}{dt} = a\cdot\frac{d}{dt}(\cos t + t\sin t)$$

$$= a\left[-\sin t + \sin t\cdot\frac{d}{dx}(t) + t\cdot\frac{d}{dt}(\sin t)\right]$$

$$= a[-\sin t + \sin t + t\cos t] = at\cos t$$

$$\frac{dy}{dt} = a\cdot\frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\cos t - \left\{\cos t\cdot\frac{d}{dt}(t) + t\cdot\frac{d}{dt}(\cos t)\right\}\right]$$

$$= a\left[\cos t - \{\cos t - t\sin t\}\right] = at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

Then, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(\tan t) = \sec^2 t\cdot\dfrac{dt}{dx}$

$$= \sec^2 t\cdot\frac{1}{at\cos t} \qquad \left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

Solution 18

It is known that, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, when $x \geq 0$, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When $x < 0$, $f(x) = |x|^3 = (-x)^3 = -x^3$

In this case, $f'(x) = -3x^2$ and hence, $f''(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

Solution 19

To prove: $\mathrm{P}(n): \frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n

For $n = 1$,

$$\mathrm{P}(1): \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$$

$\mathrm{P}(n)$ is true for $n = 1$

Let $\mathrm{P}(k)$ is true for some positive integer k.

That is, $\mathrm{P}(k): \frac{d}{dx}(x^k) = kx^{k-1}$

It has to be proved that $\mathrm{P}(k + 1)$ is also true.

$$\text{Consider } \frac{d}{dx}\left(x^{k+1}\right)=\frac{d}{dx}\left(x\cdot x^{k}\right)$$

$$=x^{k}\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}\left(x^{k}\right) \qquad \text{[By applying product rule]}$$

$$=x^{k}\cdot 1+x\cdot k\cdot x^{k-1}$$

$$=x^{k}+kx^{k}$$

$$=(k+1)\cdot x^{k}$$

$$=(k+1)\cdot x^{(k+1)-1}$$

Thus, P(k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, the statement P(n) is true for every positive integer n.

Hence, proved.

Solution 20

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[\sin(A+B)\right]=\frac{d}{dx}(\sin A\cos B)+\frac{d}{dx}(\cos A\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B)=\cos B\cdot\frac{d}{dx}(\sin A)+\sin A\cdot\frac{d}{dx}(\cos B)$$
$$+\sin B\cdot\frac{d}{dx}(\cos A)+\cos A\cdot\frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B)=\cos B\cdot\cos A\frac{dA}{dx}+\sin A(-\sin B)\frac{dB}{dx}$$
$$+\sin B(-\sin A)\cdot\frac{dA}{dx}+\cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\cdot\left[\frac{dA}{dx}+\frac{dB}{dx}\right]=(\cos A\cos B-\sin A\sin B)\cdot\left[\frac{dA}{dx}+\frac{dB}{dx}\right]$$

$$\therefore \cos(A+B)=\cos A\cos B-\sin A\sin B$$

Solution 21

Yes.

Consider the function f(x)=|x-1|+|x-2|

Since we know that the modulus function is continuous everywhere, so there sum is also continuous

Therefore, function f is continuous everywhere

Now, let us check the differentiability of f(x) at x=1,2

At x=1

LHD = $\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}$

$$=\lim_{x\to1^-}\frac{|x-1|+|x-2|-|1-1|-|1-2|}{x-1}$$

$$=\lim_{x\to1^-}\frac{|x-1|+|x-2|-1}{x-1}$$

[Take x=1-h, h>0 such that h→0 as $x\to1^-$]

$$=\lim_{h\to0}\frac{|1-h-1|+|1-h-2|-1}{1-h-1}$$

$$=\lim_{h\to0}\frac{|-h|+|-h-1|-1}{-h}$$

$$=\lim_{h\to0}\frac{h+h+1-1}{-h}=\lim_{h\to0}\frac{2h}{-h}$$

$$=-2$$

Now,

RHD = $\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}$

$$=\lim_{x\to1^+}\frac{|x-1|+|x-2|-|1-1|-|1-2|}{x-1}$$

$$=\lim_{x\to1^+}\frac{|x-1|+|x-2|-1}{x-1}$$

[Take x=1+h, h>0 such that h→0 as $x\to1^+$]

$$= \lim_{h\to 0}\frac{|1+h-1|+|1+h-2|-1}{1+h-1}$$

$$= \lim_{h\to 0}\frac{|h|+|h-1|-1}{h}$$

$$= \lim_{h\to 0}\frac{h+|-(1-h)|-1}{-h}$$

$$= \lim_{h\to 0}\frac{h+1-h-1}{-h}$$

$$= 0$$

$\neq$ LHD

Therefore, f is not differentiable at x=1.

At x=2

LHD = $\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$

$$= \lim_{x\to 2^-}\frac{|x-1|+|x-2|-|2-1|-|2-2|}{x-1}$$

$$= \lim_{x\to 2^-}\frac{|x-1|+|x-2|-1}{x-2}$$

[Take x=2-h, h>0 such that h→0 as $x\to 2^-$]

$$= \lim_{h\to 0}\frac{|2-h-1|+|2-h-2|-1}{2-h-2}$$

$$= \lim_{h\to 0}\frac{|1-h|+|-h|-1}{-h}$$

$$= \lim_{h\to 0}\frac{1-h+h-1}{-h}$$

$$= -0$$

Now,

RHD = $\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$

$$= \lim_{x \to 2^+} \frac{|x-1| + |x-2| - |2-1| - |2-2|}{x-2}$$

$$= \lim_{x \to 2^+} \frac{|x-1| + |x-2| - 1}{x-2}$$

[Take x=2+h, h>0 such that h→0 as x→2^+]

$$= \lim_{h \to 0} \frac{|2+h-1| + |2+h-2| - 1}{2+h-2}$$

$$= \lim_{h \to 0} \frac{|1+h| + |h| - 1}{h}$$

$$= \lim_{h \to 0} \frac{1+h \ \ + h - 1}{h}$$

$$= \lim_{h \to 0} \frac{2h}{h}$$

$$= 2$$

$\neq$ LHD

Therefore, f is not differentiable at x=2.

Hence, f is not differentiable at exactly two points.

Solution 22

$$y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\Rightarrow y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$

Then, $\frac{dy}{dx}=\frac{d}{dx}\left[(mc-nb)f(x)\right]-\frac{d}{dx}\left[(lc-na)g(x)\right]+\frac{d}{dx}\left[(lb-ma)h(x)\right]$

$$=(mc-nb)f'(x)-(lc-na)g'(x)+(lb-ma)h'(x)$$

$$=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Thus,

$$\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Solution 23

It is given that, $y=e^{a\cos^{-1}x}$

Taking logarithm on both the sides, we obtain

$\log y=a\cos^{-1}x\log e$

$\log y=a\cos^{-1}x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=a\times\frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow\frac{dy}{dx}=\frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2=\frac{a^2y^2}{1-x^2}$$

$$\Rightarrow(1-x^2)\left(\frac{dy}{dx}\right)^2=a^2y^2$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2=a^2y^2$$

Again differentiating both sides with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}\left(1-x^2\right)+\left(1-x^2\right)\times\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right]=a^2\frac{d}{dx}\left(y^2\right)$$

$$\Rightarrow\left(\frac{dy}{dx}\right)^2(-2x)+\left(1-x^2\right)\times 2\frac{dy}{dx}.\frac{d^2y}{dx^2}=a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow\left(\frac{dy}{dx}\right)^2(-2x)+\left(1-x^2\right)\times 2\frac{dy}{dx}.\frac{d^2y}{dx^2}=a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow -x\frac{dy}{dx}+\left(1-x^2\right)\frac{d^2y}{dx^2}=a^2.y \qquad \left[\frac{dy}{dx}\neq 0\right]$$

$$\Rightarrow\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$$

Hence, proved.