NCERT Solutions for Class 12- Maths Chapter 6 - Applications of Derivatives

Chapter 6 - Applications of Derivatives Exercise Ex. 6.1

Solution 1

The area of a circle (A) with radius (r) is given by,

$$A = \pi r^2$$

Now, the rate of change of the area with respect to its radius is given by, $\frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r$

1. When $r = 3$ cm,

$$\frac{dA}{dr} = 2\pi(3) = 6\pi$$

Hence, the area of the circle is changing at the rate of 6π cm^2/cm when its radius is 3 cm.

2. When $r = 4$ cm,

$$\frac{dA}{dr} = 2\pi(4) = 8\pi$$

Hence, the area of the circle is changing at the rate of 8π cm^2/cm when its radius is 4 cm.

Solution 2

Let x be the length of a side, V be the volume, and s be the surface area of the cube.

Then, $V = x^3$ and $S = 6x^2$ where x is a function of time t.

It is given that $\frac{dV}{dt} = 8 \text{ cm}^3/\text{s}$.

Then, by using the chain rule, we have:

$$\therefore 8 = \frac{dV}{dt} = \frac{d}{dt}\left(x^3\right) = \frac{d}{dx}\left(x^3\right)\cdot\frac{dx}{dt} = 3x^2 . \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{8}{3x^2} \qquad (1)$$

$$\text{Now, } \frac{dS}{dt} = \frac{d}{dt}\left(6x^2\right) = \frac{d}{dx}\left(6x^2\right)\cdot\frac{dx}{dt} \qquad \text{[By chain rule]}$$

$$= 12x.\frac{dx}{dt} = 12x\cdot\left(\frac{8}{3x^2}\right) = \frac{32}{x}$$

Thus, when $x = 12$ cm, $\frac{dS}{dt} = \frac{32}{12} \text{ cm}^2/\text{s} = \frac{8}{3} \text{ cm}^2/\text{s}$.

Hence, if the length of the edge of the cube is 12 cm, then the surface area is increasing at the rate of $\frac{8}{3}$ cm^2/s

Solution 3

The area of a circle (A) with radius (r) is given by,

$$A = \pi r^2$$

Now, the rate of change of area (A) with respect to time (t) is given by,

$$\frac{dA}{dt} = \frac{d}{dr}\left(\pi r^2\right)\cdot\frac{dr}{dt} = 2\pi r\frac{dr}{dt} \qquad \text{[By chain rule]}$$

It is given that,

$$\frac{dr}{dt} = 3 \text{ cm/s}$$

$$\therefore \frac{dA}{dt} = 2\pi r(3) = 6\pi r$$

Thus, when $r = 10$ cm,

$$\frac{dA}{dt} = 6\pi(10) = 60\pi \text{ cm}^2/\text{s}$$

Hence, the rate at which the area of the circle is increasing when the radius is 10 cm is 60π cm^2/s

Solution 4

Let x be the length of a side and V be the volume of the cube. Then,

$V = x^3.$

$$\therefore \frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} \text{ (By chain rule)}$$

It is given that,

$$\frac{dx}{dt} = 3 \text{ cm/s}$$

$$\therefore \frac{dV}{dt} = 3x^2(3) = 9x^2$$

Thus, when $x = 10$ cm,

$$\frac{dV}{dt} = 9(10)^2 = 900 \text{ cm}^3/\text{s}$$

Hence, the volume of the cube is increasing at the rate of 900 cm^3/s when the edge is 10 cm long.

Solution 5

The area of a circle (A) with radius (r) is given by $A = \pi r^2$.

Therefore, the rate of change of area (A) with respect to time (t) is given by,

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2)\frac{dr}{dt} = 2\pi r\frac{dr}{dt} \text{ [By chain rule]}$$

It is given that $\frac{dr}{dt} = 5 \text{ cm/s}$.

Thus, when $r = 8$ cm,

$$\frac{dA}{dt} = 2\pi(8)(5) = 80\pi$$

Hence, when the radius of the circular wave is 8 cm, the enclosed area is increasing at the rate of $80\pi \text{ cm}^2/\text{s}$

Solution 6

The circumference of a circle (C) with radius (r) is given by

$C = 2\pi r$.

Therefore, the rate of change of circumference (C) with respect to time (t) is given by,

$\frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt}$ (By chain rule)

$= \frac{d}{dr}(2\pi r) \frac{dr}{dt}$

$= 2\pi \cdot \frac{dr}{dt}$

It is given that $\frac{dr}{dt} = 0.7$ cm/s.

Hence, the rate of increase of the circumference is $2\pi(0.7) = 1.4\pi$ cm/s.

Solution 7

Since the length (x) is decreasing at the rate of 5 cm/minute and the width (y) is increasing at the rate of 4 cm/minute, we have:

$$\frac{dx}{dt} = -5 \text{ cm/min and } \frac{dy}{dt} = 4 \text{ cm/min}$$

(a) The perimeter (P) of a rectangle is given by,

$P = 2(x + y)$

$$\therefore \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5+4) = -2 \text{ cm/min}$$

Hence, the perimeter is decreasing at the rate of 2 cm/min.

(b) The area (A) of a rectangle is given by,

$A = x \times y$

$$\therefore \frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt} = -5y + 4x$$

When $x = 8$ cm and $y = 6$ cm, $\frac{dA}{dt} = (-5 \times 6 + 4 \times 8) \text{ cm}^2/\text{min} = 2 \text{ cm}^2/\text{min}$

Hence, the area of the rectangle is increasing at the rate of 2 cm^2/min.

Solution 8

The volume of a sphere (*V*) with radius (*r*) is given by,

$$V = \frac{4}{3}\pi r^3$$

$\therefore$Rate of change of volume (*V*) with respect to time (*t*) is given by,

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} \text{ [By chain rule]}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\cdot\frac{dr}{dt}$$

$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

It is given that $\frac{dV}{dt} = 900 \text{ cm}^3/\text{s}$.

$$\therefore 900 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

Therefore, when radius = 15 cm,

$$\frac{dr}{dt} = \frac{225}{\pi(15)^2} = \frac{1}{\pi}$$

Hence, the rate at which the radius of the balloon increases when the radius is 15 cm is $\frac{1}{\pi}$ cm/s.

Solution 9

The volume of a sphere (V) with radius (r) is given by $V = \frac{4}{3}\pi r^3$.

Rate of change of volume (V) with respect to its radius (r) is given by,

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2$$

Therefore, when radius = 10 cm,

$$\frac{dV}{dr} = 4\pi\left(10\right)^2 = 400\pi$$

Hence, the volume of the balloon is increasing at the rate of 400π cm^3/cm.

Solution 10

Let y m be the height of the wall at which the ladder touches. Also, let the foot of the ladder be x m away from the wall.

Then, by Pythagoras theorem, we have:

$x^2 + y^2 = 25$ [Length of the ladder = 5 m]

$$\Rightarrow y = \sqrt{25 - x^2}$$

Then, the rate of change of height (y) with respect to time (t) is given by,

$$\frac{dy}{dt} = \frac{-x}{\sqrt{25 - x^2}} \cdot \frac{dx}{dt}$$

It is given that $\frac{dx}{dt} = 2$ cm/s.

$$\therefore \frac{dy}{dt} = \frac{-2x}{\sqrt{25 - x^2}}$$

Now, when $x = 4$, we have:

$$\frac{dy}{dt} = \frac{-2 \times 4}{\sqrt{25 - 4^2}} = -\frac{8}{3}$$

Hence, the height of the ladder on the wall is decreasing at the rate of $\frac{8}{3}$ cm/s.

Solution 11

The equation of the curve is given as:

$$6y = x^3 + 2$$

The rate of change of the position of the particle with respect to time (t) is given by,

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 0$$
$$\Rightarrow 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

When the y-coordinate of the particle changes 8 times as fast as the

x-coordinate i.e., $\left(\frac{dy}{dt} = 8\frac{dx}{dt}\right)$, we have:

$$2\left(8\frac{dx}{dt}\right) = x^2\frac{dx}{dt}$$
$$\Rightarrow 16\frac{dx}{dt} = x^2\frac{dx}{dt}$$
$$\Rightarrow \left(x^2 - 16\right)\frac{dx}{dt} = 0$$
$$\Rightarrow x^2 = 16$$
$$\Rightarrow x = \pm 4$$

When $x = 4, y = \frac{4^3 + 2}{6} = \frac{66}{6} = 11$.

When $x = -4, y = \frac{(-4)^3 + 2}{6} = -\frac{62}{6} = -\frac{31}{3}$.

Hence, the points required on the curve are (4, 11) and $\left(-4, \frac{-31}{3}\right)$.

Solution 12

The air bubble is in the shape of a sphere.

Now, the volume of an air bubble (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

The rate of change of volume (V) with respect to time (t) is given by,

$$\frac{dV}{dt} = \frac{4}{3}\pi\frac{d}{dr}\left(r^3\right)\cdot\frac{dr}{dt} \qquad \text{[By chain rule]}$$
$$= \frac{4}{3}\pi\left(3r^2\right)\frac{dr}{dt}$$
$$= 4\pi r^2\frac{dr}{dt}$$

It is given that $\frac{dr}{dt} = \frac{1}{2}$ cm/s.

Therefore, when $r = 1$ cm,

$$\frac{dV}{dt} = 4\pi(1)^2\left(\frac{1}{2}\right) = 2\pi \text{ cm}^3/\text{s}$$

Hence, the rate at which the volume of the bubble increases is 2π cm^3/s

Solution 13

The volume of a sphere (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

It is given that:

$$\text{Diameter} = \frac{3}{2}(2x+1)$$

$$\Rightarrow r = \frac{3}{4}(2x+1)$$

$$\therefore V = \frac{4}{3}\pi\left(\frac{3}{4}\right)^3(2x+1)^3 = \frac{9}{16}\pi(2x+1)^3$$

Hence, the rate of change of volume with respect to x is as

$$\frac{dV}{dx} = \frac{9}{16}\pi\frac{d}{dx}(2x+1)^3 = \frac{9}{16}\pi \times 3(2x+1)^2 \times 2 = \frac{27}{8}\pi(2x+1)^2.$$

Solution 14

The volume of a cone (V) with radius (r) and height (h) is given by,

$$V = \frac{1}{3}\pi r^2 h$$

It is given that,

$$h = \frac{1}{6}r \Rightarrow r = 6h$$

$$\therefore V = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3$$

The rate of change of volume with respect to time (t) is given by,

$$\frac{dV}{dt} = 12\pi\frac{d}{dh}\left(h^3\right)\cdot\frac{dh}{dt} \text{ [By chain rule]}$$

$$= 12\pi\left(3h^2\right)\frac{dh}{dt}$$

$$= 36\pi h^2\frac{dh}{dt}$$

It is also given that $\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$.

Therefore, when $h = 4$ cm, we have:

$$12 = 36\pi(4)^2 \frac{dh}{dt}$$
$$\Rightarrow \frac{dh}{dt} = \frac{12}{36\pi(16)} = \frac{1}{48\pi}$$

Hence, when the height of the sand cone is 4 cm, its height is increasing at the rate of $\frac{1}{48\pi}$ cm/s.

Solution 15

Marginal cost is the rate of change of total cost with respect to output.

$$\therefore \text{Marginal cost (MC)} = \frac{dC}{dx} = 0.007(3x^2) - 0.003(2x) + 15$$

$$= 0.021x^2 - 0.006x + 15$$

When $x = 17$, MC $= 0.021\ (17^2) - 0.006\ (17) + 15$

$= 0.021(289) - 0.006(17) + 15$

$= 6.069 - 0.102 + 15$

$= 20.967$

Hence, when 17 units are produced, the marginal cost is Rs. 20.967

Solution 16

Marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$$\therefore \text{Marginal Revenue (MR)} = \frac{dR}{dx} = 13(2x) + 26 = 26x + 26$$

When $x = 7$,

MR $= 26(7) + 26 = 182 + 26 = 208$

Hence, the required marginal revenue is Rs 208.

Solution 17

The area of a circle (A) with radius (r) is given by,

$$A = \pi r^2$$

Therefore, the rate of change of the area with respect to its radius r is

$$\frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r.$$

$\therefore$When r = 6 cm,

$$\frac{dA}{dr} = 2\pi \times 6 = 12\pi \text{ cm}^2/\text{s}$$

Hence, the required rate of change of the area of a circle is 12π cm^2/s.

The correct answer is B.

Solution 18

Marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$$\therefore \text{Marginal Revenue (MR)} = \frac{dR}{dx} = 3(2x) + 36 = 6x + 36$$

$\therefore$When x = 15,

$$\text{MR} = 6(15) + 36 = 90 + 36 = 126$$

Hence, the required marginal revenue is Rs 126.

The correct answer is D.

Chapter 6 - Applications of Derivatives Exercise Ex. 6.2

Solution 1

Let x_1 and x_2 be any two numbers in **R**.

Then, we have:

$$x_1 < x_2 \Rightarrow 3x_1 < 3x_2 \Rightarrow 3x_1 + 17 < 3x_2 + 17 \Rightarrow f(x_1) < f(x_2)$$

Hence, f is strictly increasing on **R.**

Alternate method:

$f'(x) = 3 > 0$, in every interval of **R**.

Thus, the function is strictly increasing on **R**.

Solution 2

Let x_1 and x_2 be any two numbers in **R**.

Then, we have:

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow e^{2x_1} < e^{2x_2} \Rightarrow f(x_1) < f(x_2)$$

Hence, f is strictly increasing on **R**.

Solution 3

The given function is $f(x) = \sin x$.

$$\therefore f'(x) = \cos x$$

(a) Since for each $x \in \left(0, \frac{\pi}{2}\right)$, $\cos x > 0$, we have $f'(x) > 0$.

Hence, f is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

(b) Since for each $x \in \left(\frac{\pi}{2}, \pi\right)$, $\cos x < 0$, we have $f'(x) < 0$.

Hence, f is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

(c) From the results obtained in (a) and (b), it is clear that f is neither increasing nor decreasing in $(0, \pi)$.

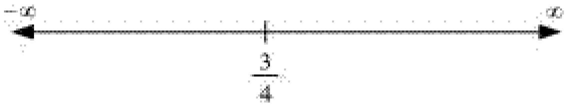
Solution 4

The given function is $f(x) = 2x^2 - 3x$.

$$f'(x) = 4x - 3$$

$$\therefore f'(x) = 0 \Rightarrow x = \frac{3}{4}$$

Now, the point $\frac{3}{4}$ divides the real line into two disjoint intervals i.e., $\left(-\infty, \frac{3}{4}\right)$ and $\left(\frac{3}{4}, \infty\right)$.



In interval $\left(-\infty, \frac{3}{4}\right)$, $f'(x) = 4x - 3 < 0$.

Hence, the given function (f) is strictly decreasing in interval $\left(-\infty, \frac{3}{4}\right)$.

In interval $\left(\frac{3}{4}, \infty\right)$, $f'(x) = 4x - 3 > 0$.

Hence, the given function (f) is strictly increasing in interval $\left(\frac{3}{4}, \infty\right)$.

Solution 5

The given function is $f(x) = 2x^3 - 3x^2 - 36x + 7$.

$$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x+2)(x-3)$$

$$\therefore f'(x) = 0 \Rightarrow x = -2, 3$$

The points $x = -2$ and $x = 3$ divide the real line into three disjoint intervals i.e.,

$(-\infty, -2), (-2, 3)$, and $(3, \infty)$.

In intervals $(-\infty, -2)$ and $(3, \infty)$, $f'(x)$ is positive while in interval

$(-2, 3)$, $f'(x)$ is negative.

Hence, the given function (f) is strictly increasing in intervals

$(-\infty, -2)$ and $(3, \infty)$, while function (f) is strictly decreasing in interval

$(-2, 3)$.

Solution 6

(a) We have,

$f(x) = x^2 + 2x - 5$

$\therefore f'(x) = 2x + 2$

Now,

$f'(x) = 0 \Rightarrow x = -1$

Point $x = -1$ divides the real line into two disjoint intervals i.e., $(-\infty, -1)$ and $(-1, \infty)$.

In interval $(-\infty, -1)$, $f'(x) = 2x + 2 < 0$.

$\therefore f$ is strictly decreasing in interval $(-\infty, -1)$.

Thus, f is strictly decreasing for $x < -1$.

In interval $(-1, \infty)$, $f'(x) = 2x + 2 > 0$.

$\therefore f$ is strictly increasing in interval $(-1, \infty)$.

Thus, f is strictly increasing for $x > -1$.

(b) We have,

$f(x) = 10 - 6x - 2x^2$

$\therefore f'(x) = -6 - 4x$

Now,

$f'(x) = 0 \Rightarrow x = -\frac{3}{2}$

The point $x = -\frac{3}{2}$ divides the real line into two disjoint intervals i.e., $\left(-\infty, -\frac{3}{2}\right)$ and $\left(-\frac{3}{2}, \infty\right)$.

In interval $\left(-\infty, -\frac{3}{2}\right)$ i.e., when $x < -\frac{3}{2}$, $f'(x) = -6 - 4x > 0.$

$\therefore f$ is strictly increasing for $x < -\frac{3}{2}$.

In interval $\left(-\frac{3}{2}, \infty\right)$ i.e., when $x > -\frac{3}{2}$, $f'(x) = -6 - 4x < 0.$

$\therefore f$ is strictly decreasing for $x > -\frac{3}{2}$.

(c) We have,

$f(x) = -2x^3 - 9x^2 - 12x + 1$

$\therefore f'(x) = -6x^2 - 18x - 12 = -6(x^2 + 3x + 2) = -6(x+1)(x+2)$

Now,

$f'(x) = 0 \Rightarrow x = -1$ and $x = -2$

Points $x=-1$ and $x=-2$ divide the real line into three disjoint intervals i.e., $(-\infty,-2),(-2,-1)$, and $(-1,\infty)$.

In intervals $(-\infty,-2)$ and $(-1,\infty)$ i.e., when $x<-2$ and $x>-1$,

$$f'(x)=-6(x+1)(x+2)<0$$

$\therefore f$ is strictly decreasing for $x<-2$ and $x>-1$.

Now, in interval $(-2,-1)$ i.e., when $-2<x<-1$, $f'(x)=-6(x+1)(x+2)>0$.

$\therefore f$ is strictly increasing for $-2<x<-1$.

(d) We have,

$$f(x)=6-9x-x^2$$

$$\therefore f'(x)=-9-2x$$

Now, $f'(x)=0$ gives $x=-\dfrac{9}{2}$

The point $x=-\dfrac{9}{2}$ divides the real line into two disjoint intervals i.e., $\left(-\infty,-\dfrac{9}{2}\right)$ and $\left(-\dfrac{9}{2},\infty\right)$.

In interval $\left(-\infty,-\dfrac{9}{2}\right)$ i.e., for $x<-\dfrac{9}{2}$, $f'(x)=-9-2x>0$.

$\therefore f$ is strictly increasing for $x < -\frac{9}{2}$.

In interval $\left(-\frac{9}{2}, \infty\right)$ i.e., for $x > -\frac{9}{2}$, $f'(x) = -9 - 2x < 0$.

$\therefore f$ is strictly decreasing for $x > -\frac{9}{2}$.

(e) We have,

$f(x) = (x + 1)^3 (x - 3)^3$

$$\begin{aligned} f'(x) &= 3(x+1)^2(x-3)^3 + 3(x-3)^2(x+1)^3 \\ &= 3(x+1)^2(x-3)^2[x-3+x+1] \\ &= 3(x+1)^2(x-3)^2(2x-2) \\ &= 6(x+1)^2(x-3)^2(x-1) \end{aligned}$$

Now,

$f'(x) = 0 \Rightarrow x = -1, 3, 1$

The points $x = -1$, $x = 1$, and $x = 3$ divide the real line into four disjoint intervals i.e., $(-\infty, -1)$, $(-1, 1)$, $(1, 3)$, and $(3, \infty)$.

In intervals $(-\infty, -1)$ and $(-1, 1)$, $f'(x) = 6(x+1)^2(x-3)^2(x-1) < 0$.

$\therefore f$ is strictly decreasing in intervals $(-\infty, -1)$ and $(-1, 1)$.

In intervals $(1, 3)$ and $(3, \infty)$, $f'(x) = 6(x+1)^2(x-3)^2(x-1) > 0$

$\therefore f$ is strictly increasing in intervals $(1, 3)$ and $(3, \infty)$.

Solution 7

Consider $y = \log(1 + x) - \frac{2x}{2 + x}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1 + x} - \frac{(2 + x).2 - 2x(1)}{(2 + x)^2}$$

$$= \frac{1}{1 + x} - \frac{4 + \cancel{2x} - \cancel{2x}}{(2 + x)^2}$$

$$= \frac{1}{1 + x} - \frac{4}{(2 + x)^2}$$

$$= \frac{(2 + x)^2 - 4(1 + x)}{(1 + x)(2 + x)^2}$$

$$= \frac{\left(4 + x^2 + 4x\right) - (4 + 4x)}{(1 + x)(2 + x)^2}$$

$$= \frac{x^2}{(1 + x)(2 + x)^2}$$

Thus, $\frac{dy}{dx} = \frac{1}{1 + x}\left(\frac{x}{2 + x}\right)^2$

Now, $x > -1 \quad \Rightarrow 1 + x > 0$

Also, for all $x > -1, \left(\frac{x}{2 + x}\right)^2 > 0$

$\therefore \frac{dy}{dx} = \frac{1}{1 + x}\left(\frac{x}{2 + x}\right)^2 > 0$ for $x > -1$

Hence, f is a increasing function throughout its domain.

Solution 8

We have,

$$y = [x(x-2)]^2 = [x^2 - 2x]^2$$

$$\therefore \frac{dy}{dx} = y' = 2(x^2 - 2x)(2x-2) = 4x(x-2)(x-1)$$

$$\therefore \frac{dy}{dx} = 0 \Rightarrow x = 0, x = 2, x = 1.$$

The points $x = 0$, $x = 1$, and $x = 2$ divide the real line into four disjoint intervals i.e., $(-\infty, 0)$, $(0,1)$ $(1,2)$, and $(2,\infty)$.

In intervals $(-\infty, 0)$ and $(1,2)$, $\frac{dy}{dx} < 0$.

$\therefore y$ is strictly decreasing in intervals $(-\infty, 0)$ and $(1,2)$.

However, in intervals (0, 1) and (2, ∞), $\frac{dy}{dx} > 0$.

$\therefore y$ is strictly increasing in intervals (0, 1) and (2, ∞).

$\therefore$ y is strictly increasing for $0 < x < 1$ and $x > 2$.

Solution 9

We have,

$$y=\frac{4\sin\theta}{(2+\cos\theta)}-\theta$$

$$\therefore \frac{dy}{d\theta}=\frac{(2+\cos\theta)(4\cos\theta)-4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2}-1$$

$$=\frac{8\cos\theta+4\cos^2\theta+4\sin^2\theta}{(2+\cos\theta)^2}-1$$

$$=\frac{8\cos\theta+4}{(2+\cos\theta)^2}-1$$

Now, $\frac{dy}{d\theta}=0.$

$$\Rightarrow \frac{8\cos\theta+4}{(2+\cos\theta)^2}=1$$

$$\Rightarrow 8\cos\theta+4=4+\cos^2\theta+4\cos\theta$$

$$\Rightarrow \cos^2\theta-4\cos\theta=0$$

$$\Rightarrow \cos\theta(\cos\theta-4)=0$$

$$\Rightarrow \cos\theta=0 \text{ or } \cos\theta=4$$

Since $\cos\theta\neq 4$, $\cos\theta=0$.

$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

Now,

$$\frac{dy}{dx} = \frac{8\cos\theta + 4 - \left(4 + \cos^2\theta + 4\cos\theta\right)}{\left(2+\cos\theta\right)^2} = \frac{4\cos\theta - \cos^2\theta}{\left(2+\cos\theta\right)^2} = \frac{\cos\theta\left(4-\cos\theta\right)}{\left(2+\cos\theta\right)^2}$$

In interval $\left(0, \frac{\pi}{2}\right)$, we have $\cos\theta > 0$. Also, $4 > \cos\theta \Rightarrow 4 - \cos\theta > 0$.

$\therefore \cos\theta\left(4-\cos\theta\right) > 0$ and also $\left(2+\cos\theta\right)^2 > 0$

$\Rightarrow \frac{\cos\theta\left(4-\cos\theta\right)}{\left(2+\cos\theta\right)^2} > 0$

$\Rightarrow \frac{dy}{dx} > 0$

Therefore, y is strictly increasing in interval $(0,\pi/2)$.

Also, the given function is continuous at x=0 and x= $\pi/2$

Hence, y is increasing in interval $[0, \pi/2]$.

Solution 10

The given function is $f(x) = \log x$.

$\therefore f'(x) = \frac{1}{x}$

It is clear that for $x > 0$, $f'(x) = \frac{1}{x} > 0$.

Hence, $f(x) = \log x$ is strictly increasing in interval $(0, \infty)$

Solution 11

The given function is $f(x) = x^2 - x + 1$.

$\therefore f'(x) = 2x - 1$

Now, $f'(x) = 0 \Rightarrow x = \frac{1}{2}$.

The point $\frac{1}{2}$ divides the interval $(-1, 1)$ into two disjoint intervals i.e., $\left(-1, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right)$.

Now, in interval $\left(-1, \frac{1}{2}\right)$, $f'(x) = 2x - 1 < 0$.

Therefore, f is strictly decreasing in interval $\left(-1, \frac{1}{2}\right)$.

However, in interval $\left(\frac{1}{2}, 1\right)$, $f'(x) = 2x - 1 > 0$.

Therefore, f is strictly increasing in interval $\left(\frac{1}{2}, 1\right)$.

Hence, f is neither strictly increasing nor decreasing in interval $(-1, 1)$.

Solution 12

(A) Let $f_1(x) = \cos x$.

$\therefore f_1'(x) = -\sin x$

In interval $\left(0, \frac{\pi}{2}\right)$, $f_1'(x) = -\sin x < 0$.

$\therefore f_1(x) = \cos x$ is strictly decreasing in interval $\left(0, \frac{\pi}{2}\right)$.

(B) Let $f_2(x) = \cos 2x$.

$\therefore f_2'(x) = -2\sin 2x$

Now, $0 < x < \frac{\pi}{2} \Rightarrow 0 < 2x < \pi \Rightarrow \sin 2x > 0 \Rightarrow -2\sin 2x < 0$

$\therefore f_2'(x) = -2\sin 2x < 0$ on $\left(0, \frac{\pi}{2}\right)$

$\therefore f_2(x) = \cos 2x$ is strictly decreasing in interval $\left(0, \frac{\pi}{2}\right)$.

(C) Let $f_3(x) = \cos 3x$.

$\therefore f_3'(x) = -3\sin 3x$

Now, $f_3'(x) = 0$.

$\Rightarrow \ \sin 3x = 0 \Rightarrow 3x = \pi, as\ x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow x = \frac{\pi}{3}$

The point $x = \frac{\pi}{3}$ divides the interval $\left(0, \ \frac{\pi}{2}\right)$ into two disjoint intervals

i.e., 0 $\left(0, \ \frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

Now, in interval $\left(0, \frac{\pi}{3}\right)$, $f_3(x) = -3\sin 3x < 0$ $\left[as\ 0 < x < \frac{\pi}{3} \Rightarrow 0 < 3x < \pi\right]$.

$\therefore f_3$ is strictly decreasing in interval $\left(0, \ \frac{\pi}{3}\right)$.

However, in interval $\left(\frac{\pi}{3}, \ \frac{\pi}{2}\right)$, $f_3(x) = -3\sin 3x > 0$ $\left[as\ \frac{\pi}{3} < x < \frac{\pi}{2} \Rightarrow \pi < 3x < \frac{3\pi}{2}\right]$.

$\therefore f_3$ is strictly increasing in interval $\left(\frac{\pi}{3}, \ \frac{\pi}{2}\right)$.

Hence, f_3 is neither increasing nor decreasing in interval $\left(0, \frac{\pi}{2}\right)$.

(D) Let $f_4(x) = \tan x$.

$\therefore f_4'(x) = \sec^2 x$

In interval $\left(0, \frac{\pi}{2}\right)$, $f_4'(x) = \sec^2 x > 0$.

$\therefore f_4$ is strictly increasing in interval $\left(0, \frac{\pi}{2}\right)$.

Therefore, functions $\cos x$ and $\cos 2x$ are strictly decreasing in $\left(0, \frac{\pi}{2}\right)$.

Hence, the correct answers are A and B.

Solution 13

We have,

$f(x) = x^{100} + \sin x - 1$

$\therefore f'(x) = 100x^{99} + \cos x$

In interval $(0,\ 1)$, $\cos x > 0$ and $100x^{99} > 0$.

$\therefore f'(x) > 0$.

Thus, function f is strictly increasing in interval (0, 1).

In interval $\left(\frac{\pi}{2}, \pi\right)$, $\cos x < 0$ and $100x^{99} > 0$. Also, $100x^{99} > \cos x$.

$\therefore f'(x) > 0$ in $\left(\frac{\pi}{2}, \pi\right)$.

Thus, function f is strictly increasing in interval $\left(\frac{\pi}{2},\ \pi\right)$.

In interval $\left(0,\ \frac{\pi}{2}\right)$, $\cos x > 0$ and $100x^{99} > 0$.

$\therefore 100x^{99} + \cos x > 0$

$\Rightarrow f'(x) > 0$ on $\left(0,\ \frac{\pi}{2}\right)$

$\therefore f$ is strictly increasing in interval $\left(0,\ \frac{\pi}{2}\right)$.

Hence, function f is strictly decreasing in none of the intervals.

The correct answer is D.

Solution 14

We have,

$$f(x) = x^2 + ax + 1$$

$$\therefore f'(x) = 2x + a$$

Now, function f will be increasing in (1, 2), if $f'(x) > 0$ in (1, 2).

$$f'(x) > 0$$

$$\Rightarrow 2x + a > 0$$

$$\Rightarrow 2x > -a$$

$$\Rightarrow x > \frac{-a}{2}$$

Therefore, we have to find the least value of a such that

$$x > \frac{-a}{2}, \text{ when } x \in (1, 2).$$

$$\Rightarrow x > \frac{-a}{2} \quad (\text{when } 1 < x < 2)$$

Thus, the least value of a for f to be increasing on (1, 2) is given by,

$$\frac{-a}{2} = 1$$

$$\frac{-a}{2} = 1 \Rightarrow a = -2$$

Hence, the required value of a is -2.

Solution 15

We have,

$$f(x) = x + \frac{1}{x}$$

$$\therefore f'(x) = 1 - \frac{1}{x^2}$$

Now,

$$f'(x) = 0 \Rightarrow \frac{1}{x^2} = 1 \Rightarrow x = \pm 1$$

The points $x = 1$ and $x = -1$ divide the real line in three disjoint intervals i.e., $(-\infty, -1), (-1, 1)$, and $(1, \infty)$.

In interval $(-1, 1)$, it is observed that:

$$-1 < x < 1$$
$$\Rightarrow x^2 < 1$$
$$\Rightarrow 1 < \frac{1}{x^2},\ x \neq 0$$
$$\Rightarrow 1 - \frac{1}{x^2} < 0,\ x \neq 0$$

$$\therefore f'(x) = 1 - \frac{1}{x^2} < 0 \text{ on } (-1, 1) \sim \{0\}.$$

$\therefore f$ is strictly decreasing on $(-1,\ 1)\sim\{0\}$.

In intervals $(-\infty,-1)$ and $(1,\ \infty)$, it is observed that:

$$x<-1 \text{ or } 1<x$$
$$\Rightarrow x^2>1$$
$$\Rightarrow 1>\frac{1}{x^2}$$
$$\Rightarrow 1-\frac{1}{x^2}>0$$

$\therefore f'(x)=1-\dfrac{1}{x^2}>0$ on $(-\infty,-1)$ and $(1,\ \infty)$.

$\therefore f$ is strictly increasing on $(-\infty,\ 1)$ and $(1,\ \infty)$.

Hence, function f is strictly increasing in interval **I** disjoint from $(-1,\ 1)$.

Hence, the given result is proved.

Solution 16

We have,

$$f(x)=\log\sin x$$

$$\therefore f'(x)=\frac{1}{\sin x}\cos x=\cot x$$

In interval $\left(0,\ \dfrac{\pi}{2}\right)$, $f'(x)=\cot x>0$.

$\therefore f$ is strictly increasing in $\left(0,\ \dfrac{\pi}{2}\right)$.

In interval $\left(\dfrac{\pi}{2},\ \pi\right)$, $f'(x)=\cot x<0$.

$\therefore f$ is strictly decreasing in $\left(\dfrac{\pi}{2},\pi\right)$.

Solution 17

We have,

$f(x) = \log \cos x$

$\therefore f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$

In interval $\left(0, \frac{\pi}{2}\right)$, $\tan x > 0 \Rightarrow -\tan x < 0.$

$\therefore f'(x) < 0$ on $\left(0, \frac{\pi}{2}\right)$

$\therefore f$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

In interval $\left(\frac{\pi}{2}, \pi\right)$, $\tan x < 0 \Rightarrow -\tan x > 0.$

$\therefore f'(x) > 0$ on $\left(\frac{3\pi}{2}, 2\pi\right)$

$\therefore f$ is strictly increasing on $\left(\frac{3\pi}{2}, 2\pi\right)$

Solution 18

We have,

$f(x) = x^3 - 3x^2 + 3x - 100$

$$\begin{aligned} f'(x) &= 3x^2 - 6x + 3 \\ &= 3(x^2 - 2x + 1) \\ &= 3(x-1)^2 \end{aligned}$$

For any $x \in \mathbf{R}$, $(x - 1)^2 > 0$.

Thus, $f'(x)$ is always positive in **R**.

Hence, the given function (*f*) is increasing in **R**

Solution 19

We have,

$y = x^2 e^{-x}$

$\therefore \frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x} = xe^{-x}(2-x)$

Now, $\frac{dy}{dx} = 0.$

$\Rightarrow x = 0$ and $x = 2$

The points $x = 0$ and $x = 2$ divide the real line into three disjoint intervals i.e., $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$.

In intervals $(-\infty, 0)$ and $(2, \infty)$, $f'(x) < 0$ as e^{-x} is always positive.

$\therefore f$ is decreasing on $(-\infty, 0)$ and $(2, \infty)$.

In interval $(0, 2)$, $f'(x) > 0$.

$\therefore f$ is strictly increasing on $(0, 2)$.

Hence, f is strictly increasing in interval $(0, 2)$.

The correct answer is D.

Chapter 6 - Applications of Derivatives Exercise Ex. 6.3

Solution 1

The given curve is $y = 3x^4 - 4x$.

Then, the slope of the tangent to the given curve at $x = 4$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=4} = 12x^3 - 4\Big]_{x=4} = 12(4)^3 - 4 = 12(64) - 4 = 764$$

Solution 2

The given curve is $y = \frac{x-1}{x-2}$.

$$\therefore \frac{dy}{dx} = \frac{(x-2)(1)-(x-1)(1)}{(x-2)^2}$$

$$= \frac{x-2-x+1}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

Thus, the slope of the tangent at $x = 10$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=10} = \left.\frac{-1}{(x-2)^2}\right]_{x=10} = \frac{-1}{(10-2)^2} = \frac{-1}{64}$$

Hence, the slope of the tangent at $x = 10$ is $\frac{-1}{64}$.

Solution 3

The given curve is $y = x^3 - x + 1$.

$$\therefore \frac{dy}{dx} = 3x^2 - 1$$

The slope of the tangent to a curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

It is given that $x_0 = 2$.

Hence, the slope of the tangent at the point where the x-coordinate is 2 is given by,

$$\left.\frac{dy}{dx}\right]_{x=2} = 3x^2 - 1\Big]_{x=2} = 3(2)^2 - 1 = 12 - 1 = 11$$

Solution 4

The given curve is $y = x^3 - 3x + 2$.

$$\therefore \frac{dy}{dx} = 3x^2 - 3$$

The slope of the tangent to a curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

Hence, the slope of the tangent at the point where the x-coordinate is 3 is given by,

$$\left.\frac{dy}{dx}\right]_{x=3} = 3x^2 - 3\Big]_{x=3} = 3(3)^2 - 3 = 27 - 3 = 24$$

Solution 5

It is given that $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

$$\therefore \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) = -3a\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{4}$ is given by,

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = -\tan\theta\Big]_{\theta=\frac{\pi}{4}} = -\tan\frac{\pi}{4} = -1$$

Hence, the slope of the normal at $\theta = \frac{\pi}{4}$ is given by,

$$\frac{-1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{-1} = 1$$

Solution 6

It is given that $x = 1 - a\sin\theta$ and $y = b\cos^2\theta$.

$$\therefore \frac{dx}{d\theta} = -a\cos\theta \text{ and } \frac{dy}{d\theta} = 2b\cos\theta(-\sin\theta) = -2b\sin\theta\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2b\sin\theta\cos\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{2}$ is given by,

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{2}} = \left.\frac{2b}{a}\sin\theta\right]_{\theta=\frac{\pi}{2}} = \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}$$

Hence, the slope of the normal at $\theta = \frac{\pi}{2}$ is given by,

$$\frac{-1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{\left(\frac{2b}{a}\right)} = -\frac{a}{2b}$$

Solution 7

The equation of the given curve is $y = x^3 - 3x^2 - 9x + 7$.

$$\therefore \frac{dy}{dx} = 3x^2 - 6x - 9$$

Now, the tangent is parallel to the x-axis if the slope of the tangent is zero.

$$\begin{aligned} \therefore 3x^2 - 6x - 9 = 0 &\Rightarrow x^2 - 2x - 3 = 0 \\ &\Rightarrow (x-3)(x+1) = 0 \\ &\Rightarrow x = 3 \text{ or } x = -1 \end{aligned}$$

When $x = 3$, $y = (3)^3 - 3(3)^2 - 9(3) + 7 = 27 - 27 - 27 + 7 = -20$.

When $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 7 = -1 - 3 + 9 + 7 = 12$.

Hence, the points at which the tangent is parallel to the x-axis are $(3, -20)$ and

$(-1, 12)$.

Solution 8

If a tangent is parallel to the chord joining the points (2, 0) and (4, 4), then the slope of the tangent = the slope of the chord.

The slope of the chord is $\frac{4-0}{4-2}=\frac{4}{2}=2.$

Now, the slope of the tangent to the given curve at a point (x, y) is given by,

$$\frac{dy}{dx}=2(x-2)$$

Since the slope of the tangent = slope of the chord, we have:

$$2(x-2)=2$$
$$\Rightarrow x-2=1 \Rightarrow x=3$$
When $x=3,\ y=(3-2)^2=1.$

Hence, the required point is (3, 1).

Solution 9

The equation of the given curve is $y = x^3 - 11x + 5$.

The equation of the tangent to the given curve is given as $y = x - 11$ (which is of the form $y = mx + c$).

$\therefore$Slope of the tangent = 1

Now, the slope of the tangent to the given curve at the point (x, y) is given by, $\frac{dy}{dx} = 3x^2 - 11$

Then, we have:

$$3x^2 - 11 = 1$$
$$\Rightarrow 3x^2 = 12$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2$$

When $x = 2$, $y = (2)^3 - 11\,(2) + 5 = 8 - 22 + 5 = -9$.

When $x = -2$, $y = (-2)^3 - 11\,(-2) + 5 = -8 + 22 + 5 = 19$.

Hence, the required points are $(2, -9)$ and $(-2, 19)$.

Solution 10

The equation of the given curve is $y=\frac{1}{x-1},\ x\neq 1$.

The slope of the tangents to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx}=\frac{-1}{(x-1)^2}$$

If the slope of the tangent is -1, then we have:

$$\frac{-1}{(x-1)^2}=-1$$
$$\Rightarrow (x-1)^2=1$$
$$\Rightarrow x-1=\pm 1$$
$$\Rightarrow x=2,\ 0$$

When $x = 0$, $y = -1$ and when $x = 2$, $y = 1$.

Thus, there are two tangents to the given curve having slope -1. These are passing through the points $(0, -1)$ and $(2, 1)$.

$\therefore$The equation of the tangent through $(0, -1)$ is given by,

$$y-(-1)=-1(x-0)$$
$$\Rightarrow y+1=-x$$
$$\Rightarrow y+x+1=0$$

$\therefore$The equation of the tangent through $(2, 1)$ is given by,

$$y - 1 = -1\ (x - 2)$$

$$\Rightarrow y - 1 = - x + 2$$

$$\Rightarrow y + x - 3 = 0$$

Hence, the equations of the required lines are $y + x + 1 = 0$ and $y + x - 3 = 0$.

Solution 11

The equation of the given curve is $y = \frac{1}{x-3}$, $x \neq 3$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{-1}{(x-3)^2}$$

If the slope of the tangent is 2, then we have:

$$\frac{-1}{(x-3)^2} = 2$$

$$\Rightarrow 2(x-3)^2 = -1$$

$$\Rightarrow (x-3)^2 = \frac{-1}{2}$$

This is not possible since the L.H.S. is positive while the R.H.S. is negative.

Hence, there is no tangent to the given curve having slope 2.

Solution 12

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx}=\frac{-(2x-2)}{\left(x^2-2x+3\right)^2}=\frac{-2(x-1)}{\left(x^2-2x+3\right)^2}$$

If the slope of the tangent is 0, then we have:

$$\frac{-2(x-1)}{\left(x^2-2x+3\right)^2}=0$$

$$\Rightarrow -2(x-1)=0$$
$$\Rightarrow x=1$$

When $x = 1$, $y=\frac{1}{1-2+3}=\frac{1}{2}$.

$\therefore$The equation of the tangent through $\left(1,\ \frac{1}{2}\right)$ is given by,

$$y-\frac{1}{2}=0(x-1)$$
$$\Rightarrow y-\frac{1}{2}=0$$
$$\Rightarrow y=\frac{1}{2}$$

Hence, the equation of the required line is $y = \frac{1}{2}$.

Solution 13

The equation of the given curve is $\frac{x^2}{9}+\frac{y^2}{16}=1$.

On differentiating both sides with respect to x, we have:

$$\frac{2x}{9}+\frac{2y}{16}\cdot\frac{dy}{dx}=0$$
$$\Rightarrow \frac{dy}{dx}=\frac{-16x}{9y}$$

(i) The tangent is parallel to the x-axis if the slope of the tangent is 0 i.e., $\frac{-16x}{9y}=0$, which is possible if $x = 0$.

Then, $\frac{x^2}{9}+\frac{y^2}{16}=1$ for $x = 0$

$$\Rightarrow y^2=16 \Rightarrow y=\pm 4$$

Hence, the points at which the tangents are parallel to the x-axis are

(0, 4) and (0, − 4).

(ii) The tangent is parallel to the y-axis if the slope of the normal is 0, which gives $\frac{-1}{\left(\frac{-16x}{9y}\right)}=\frac{9y}{16x}=0 \Rightarrow y=0$.

Then, $\frac{x^2}{9}+\frac{y^2}{16}=1$ for $y = 0$.

$$\Rightarrow x=\pm 3$$

Hence, the points at which the tangents are parallel to the y-axis are

(3, 0) and (− 3, 0).

Solution 14

(i) The equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$
$$\left.\frac{dy}{dx}\right]_{(0,\ 5)} = -10$$

Thus, the slope of the tangent at (0, 5) is −10. The equation of the tangent is given as:

$$y - 5 = -10(x - 0)$$

$$\Rightarrow y - 5 = -10x$$

$$\Rightarrow 10x + y = 5$$

The slope of the normal at (0, 5) is $\dfrac{-1}{\text{Slope of the tangent at }(0,\ 5)} = \dfrac{1}{10}$.

Therefore, the equation of the normal at (0, 5) is given as:

$$y - 5 = \frac{1}{10}(x - 0)$$
$$\Rightarrow 10y - 50 = x$$
$$\Rightarrow x - 10y + 50 = 0$$

(ii) The equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$
$$\left.\frac{dy}{dx}\right]_{(1,\,3)} = 4 - 18 + 26 - 10 = 2$$

Thus, the slope of the tangent at (1, 3) is 2. The equation of the tangent is given as:

$$y - 3 = 2(x - 1)$$
$$\Rightarrow y - 3 = 2x - 2$$
$$\Rightarrow y = 2x + 1$$

The slope of the normal at (1, 3) is $\dfrac{-1}{\text{Slope of the tangent at }(1,\,3)} = \dfrac{-1}{2}$.

Therefore, the equation of the normal at (1, 3) is given as:

$$y - 3 = -\frac{1}{2}(x - 1)$$
$$\Rightarrow 2y - 6 = -x + 1$$
$$\Rightarrow x + 2y - 7 = 0$$

(iii) The equation of the curve is $y = x^3$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 3x^2$$
$$\left.\frac{dy}{dx}\right]_{(1,\ 1)} = 3(1)^2 = 3$$

Thus, the slope of the tangent at (1, 1) is 3 and the equation of the tangent is given as:

$$y - 1 = 3(x - 1)$$
$$\Rightarrow y = 3x - 2$$

The slope of the normal at (1, 1) is $\dfrac{-1}{\text{Slope of the tangent at } (1,\ 1)} = \dfrac{-1}{3}$.

Therefore, the equation of the normal at (1, 1) is given as:

$$y - 1 = \frac{-1}{3}(x - 1)$$
$$\Rightarrow 3y - 3 = -x + 1$$
$$\Rightarrow x + 3y - 4 = 0$$

(iv) The equation of the curve is $y = x^2$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 2x$$

$$\left.\frac{dy}{dx}\right]_{(0,\ 0)} = 0$$

Thus, the slope of the tangent at (0, 0) is 0 and the equation of the tangent is given as:

$y - 0 = 0\ (x - 0)$

$\Rightarrow y = 0$

The slope of the normal at (0, 0) is $\dfrac{-1}{\text{Slope of the tangent at } (0,\ 0)} = -\dfrac{1}{0}$, which is not defined.

Therefore, the equation of the normal at $(x_0, y_0) = (0, 0)$ is given by

$x = x_0 = 0.$

(v) The equation of the curve is $x = \cos t$, $y = \sin t$.

$$x = \cos t \text{ and } y = \sin t$$

$$\therefore \frac{dx}{dt} = -\sin t,\ \ \frac{dy}{dt} = \cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos t}{-\sin t} = -\cot t$$

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = -\cot t = -1$$

$\therefore$The slope of the tangent at $t = \frac{\pi}{4}$ is -1.

When $t = \frac{\pi}{4}$, $x = \frac{1}{\sqrt{2}}$ and $y = \frac{1}{\sqrt{2}}$.

Thus, the equation of the tangent to the given curve at $t = \frac{\pi}{4}$ i.e., at $\left[\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]$ is

$$y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right).$$
$$\Rightarrow x + y - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$
$$\Rightarrow x + y - \sqrt{2} = 0$$

The slope of the normal at $t = \frac{\pi}{4}$ is $\dfrac{-1}{\text{Slope of the tangent at } t = \frac{\pi}{4}} = 1$.

Therefore, the equation of the normal to the given curve at $t = \frac{\pi}{4}$ i.e., at $\left[\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]$ is

$$y - \frac{1}{\sqrt{2}} = 1\left(x - \frac{1}{\sqrt{2}}\right).$$
$$\Rightarrow x = y$$

Solution 15

The equation of the given curve is $y = x^2 - 2x + 7$.

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 2x - 2$$

(a) The equation of the line is $2x - y + 9 = 0$.

$2x - y + 9 = 0 \Rightarrow y = 2x + 9$

This is of the form $y = mx + c$.

$\therefore$Slope of the line $= 2$

If a tangent is parallel to the line $2x - y + 9 = 0$, then the slope of the tangent is equal to the slope of the line.

Therefore, we have:

$2 = 2x - 2$

$\Rightarrow 2x = 4$
$\Rightarrow x = 2$

Now, $x = 2$

$\Rightarrow y = 4 - 4 + 7 = 7$

Thus, the equation of the tangent passing through (2, 7) is given by,

$$y - 7 = 2(x - 2)$$
$$\Rightarrow y - 2x - 3 = 0$$

Hence, the equation of the tangent line to the given curve (which is parallel to line $2x - y + 9 = 0$) is $y - 2x - 3 = 0$.

(b) The equation of the line is $5y - 15x = 13$.

$$5y - 15x = 13 \Rightarrow y = 3x + \frac{13}{5}$$

This is of the form $y = mx + c$.

$\therefore$Slope of the line = 3

If a tangent is perpendicular to the line $5y - 15x = 13$, then the slope of the tangent is $\frac{-1}{\text{slope of the line}} = \frac{-1}{3}$.

$$\Rightarrow 2x - 2 = \frac{-1}{3}$$

$$\Rightarrow 2x = \frac{-1}{3} + 2$$

$$\Rightarrow 2x = \frac{5}{3}$$

$$\Rightarrow x = \frac{5}{6}$$

Now, $x = \frac{5}{6}$

$$\Rightarrow y = \frac{25}{36} - \frac{10}{6} + 7 = \frac{25 - 60 + 252}{36} = \frac{217}{36}$$

Thus, the equation of the tangent passing through $\left(\frac{5}{6}, \frac{217}{36}\right)$ is given by,

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right)$$

$$\Rightarrow \frac{36y - 217}{36} = \frac{-1}{18}(6x - 5)$$

$$\Rightarrow 36y - 217 = -2(6x - 5)$$

$$\Rightarrow 36y - 217 = -12x + 10$$

$$\Rightarrow 36y + 12x - 227 = 0$$

Hence, the equation of the tangent line to the given curve (which is perpendicular to line $5y - 15x = 13$) is $36y + 12x - 227 = 0$.

Solution 16

The equation of the given curve is $y = 7x^3 + 11$.

$$\therefore \frac{dy}{dx} = 21x^2$$

The slope of the tangent to a curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

Therefore, the slope of the tangent at the point where $x = 2$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=2} = 21(2)^2 = 84$$

And, the slope of the tangent at the point where $x = -2$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=-2} = 21(-2)^2 = 84$$

It is observed that the slopes of the tangents at the points where $x = 2$ and $x = -2$ are equal.

Hence, the two tangents are parallel.

Solution 17

The equation of the given curve is $y = x^3$.

$$\therefore \frac{dy}{dx} = 3x^2$$

The slope of the tangent at the point (x, y) is given by,

$$\left.\frac{dy}{dx}\right]_{(x,y)} = 3x^2$$

When the slope of the tangent is equal to the y-coordinate of the point, then $y = 3x^2$.

Also, we have $y = x^3$.

$\therefore 3x^2 = x^3$

$\Rightarrow x^2 (x - 3) = 0$

$\Rightarrow x = 0, x = 3$

When $x = 0$, then $y = 0$ and when $x = 3$, then $y = 3(3)^2 = 27$.

Hence, the required points are $(0, 0)$ and $(3, 27)$.

Solution 18

The equation of the given curve is $y = 4x^3 - 2x^5$.

$$\therefore \frac{dy}{dx} = 12x^2 - 10x^4$$

Therefore, the slope of the tangent at a point (x, y) is $12x^2 - 10x^4$.

The equation of the tangent at (x, y) is given by,

$$Y - y = \left(12x^2 - 10x^4\right)\left(X - x\right) \qquad ...(1)$$

When the tangent passes through the origin (0, 0), then $X = Y = 0$.

Therefore, equation (1) reduces to:

$$-y = \left(12x^2 - 10x^4\right)\left(-x\right)$$
$$y = 12x^3 - 10x^5$$

Also, we have $y = 4x^3 - 2x^5$.

$$\therefore 12x^3 - 10x^5 = 4x^3 - 2x^5$$
$$\Rightarrow 8x^5 - 8x^3 = 0$$
$$\Rightarrow x^5 - x^3 = 0$$
$$\Rightarrow x^3\left(x^2 - 1\right) = 0$$
$$\Rightarrow x = 0, \ \pm 1$$

When $x = 0$, $y = 4\left(0\right)^3 - 2\left(0\right)^5 = 0$.

When $x = 1$, $y = 4\ (1)^3 - 2\ (1)^5 = 2$.

When $x = -1$, $y = 4\ (-1)^3 - 2\ (-1)^5 = -2$.

Hence, the required points are (0, 0), (1, 2), and (−1, −2).

Solution 19

The equation of the given curve is $x^2 + y^2 - 2x - 3 = 0$.

On differentiating with respect to x, we have:

$$2x + 2y\frac{dy}{dx} - 2 = 0$$
$$\Rightarrow y\frac{dy}{dx} = 1 - x$$
$$\Rightarrow \frac{dy}{dx} = \frac{1-x}{y}$$

Now, the tangents are parallel to the x-axis if the slope of the tangent is 0.

$$\therefore \frac{1-x}{y} = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1$$

But, $x^2 + y^2 - 2x - 3 = 0$ for $x = 1$.

$$\Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

Hence, the points at which the tangents are parallel to the x-axis are (1, 2) and (1, −2)

Solution 20

The equation of the given curve is $ay^2 = x^3$.

On differentiating with respect to x, we have:

$$2ay\frac{dy}{dx} = 3x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay}$$

The slope of a tangent to the curve at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$.

$\Rightarrow$ The slope of the tangent to the given curve at (am^2, am^3) is

$$\left.\frac{dy}{dx}\right]_{(am^2,\ am^3)} = \frac{3(am^2)^2}{2a(am^3)} = \frac{3a^2m^4}{2a^2m^3} = \frac{3m}{2}.$$

$\therefore$ Slope of normal at (am^2, am^3)

$$= \frac{-1}{\text{slope of the tangent at } (am^2,\ am^3)} = \frac{-2}{3m}$$

Hence, the equation of the normal at (am^2, am^3) is given by,

$$y - am^3 = \frac{-2}{3m}\left(x - am^2\right)$$

$$\Rightarrow 3my - 3am^4 = -2x + 2am^2$$
$$\Rightarrow 2x + 3my - am^2\left(2 + 3m^2\right) = 0$$

Solution 21

The equation of the given curve is $y = x^3 + 2x + 6$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = 3x^2 + 2$$

$\therefore$ Slope of the normal to the given curve at any point (x, y)

$$= \frac{-1}{\text{Slope of the tangent at the point } (x, y)}$$

$$= \frac{-1}{3x^2 + 2}$$

The equation of the given line is $x + 14y + 4 = 0$.

$x + 14y + 4 = 0 \Rightarrow y = -\frac{1}{14}x - \frac{4}{14}$ (which is of the form $y = mx + c$)

$\therefore$Slope of the given line $= \frac{-1}{14}$

If the normal is parallel to the line, then we must have the slope of the normal being equal to the slope of the line.

$$\therefore \frac{-1}{3x^2 + 2} = \frac{-1}{14}$$

$$\Rightarrow 3x^2 + 2 = 14$$
$$\Rightarrow 3x^2 = 12$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2$$

When $x = 2$, $y = 8 + 4 + 6 = 18$.

When $x = -2$, $y = -8 - 4 + 6 = -6$.

Therefore, there are two normals to the given curve with slope $\frac{-1}{14}$ and passing through the points (2, 18) and (−2, −6).

Thus, the equation of the normal through (2, 18) is given by,

$$y - 18 = \frac{-1}{14}(x - 2)$$
$$\Rightarrow 14y - 252 = -x + 2$$
$$\Rightarrow x + 14y - 254 = 0$$

And, the equation of the normal through (−2, −6) is given by,

$$y - (-6) = \frac{-1}{14}[x - (-2)]$$
$$\Rightarrow y + 6 = \frac{-1}{14}(x + 2)$$
$$\Rightarrow 14y + 84 = -x - 2$$
$$\Rightarrow x + 14y + 86 = 0$$

Hence, the equations of the normals to the given curve (which are parallel to the given line) are $x + 14y - 254 = 0$ and $x + 14y + 86 = 0$.

Solution 22

The equation of the given parabola is $y^2 = 4ax$.

On differentiating $y^2 = 4ax$ with respect to x, we have:

$$2y\frac{dy}{dx} = 4a$$
$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$\therefore$The slope of the tangent at $\left(at^2,\ 2at\right)$ is $\left.\frac{dy}{dx}\right]_{\left(at^2,\ 2at\right)} = \frac{2a}{2at} = \frac{1}{t}$.

Then, the equation of the tangent at $\left(at^2,\ 2at\right)$ is given by,

$$y - 2at = \frac{1}{t}\left(x - at^2\right)$$

$$\Rightarrow ty - 2at^2 = x - at^2$$
$$\Rightarrow ty = x + at^2$$

Now, the slope of the normal at $\left(at^2,\ 2at\right)$ is given by,

$$\frac{-1}{\text{Slope of the tangent at } \left(at^2, 2at\right)} = -t$$

Thus, the equation of the normal at $(at^2,\ 2at)$ is given as:

$$y - 2at = -t\left(x - at^2\right)$$
$$\Rightarrow y - 2at = -tx + at^3$$
$$\Rightarrow y = -tx + 2at + at^3$$

Solution 23

The equations of the given curves are given as $x = y^2$ and $xy = k$.

Putting $x = y^2$ in $xy = k$, we get:

$$y^3 = k \Rightarrow y = k^{\frac{1}{3}}$$
$$\therefore x = k^{\frac{2}{3}}$$

Thus, the point of intersection of the given curves is $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$.

Differentiating $x = y^2$ with respect to x, we have:

$$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

Therefore, the slope of the tangent to the curve $x = y^2$ at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is $\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = \frac{1}{2k^{\frac{1}{3}}}$.

On differentiating $xy = k$ with respect to x, we have:

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$\therefore$ Slope of the tangent to the curve $xy = k$ at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is given by,

$$\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}},\,k^{\frac{1}{3}}\right)}=\left.\frac{-y}{x}\right]_{\left(k^{\frac{2}{3}},\,k^{\frac{1}{3}}\right)}=-\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}}=\frac{-1}{k^{\frac{1}{3}}}$$

We know that two curves intersect at right angles if the tangents to the curves at the point of intersection i.e., at $\left(k^{\frac{2}{3}},\, k^{\frac{1}{3}}\right)$ are perpendicular to each other.

This implies that we should have the product of the tangents as − 1.

Thus, the given two curves cut at right angles if the product of the slopes of their respective tangents at $\left(k^{\frac{2}{3}},\, k^{\frac{1}{3}}\right)$ is −1.

i.e., $\left(\frac{1}{2k^{\frac{1}{3}}}\right)\left(\frac{-1}{k^{\frac{1}{3}}}\right)=-1$

$\Rightarrow 2k^{\frac{2}{3}}=1$

$\Rightarrow \left(2k^{\frac{2}{3}}\right)^{3}=(1)^{3}$

$\Rightarrow 8k^{2}=1$

Hence, the given two curves cut at right angels if $8k^2 = 1$.

Solution 24

Differentiating $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with respect to x, we have:

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$
$$\Rightarrow \frac{2y}{b^2}\frac{dy}{dx}=\frac{2x}{a^2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

Therefore, the slope of the tangent at $(x_0,\ y_0)$ is $\left.\frac{dy}{dx}\right]_{(x_0,\ y_0)}=\frac{b^2x_0}{a^2y_0}$.

Then, the equation of the tangent at $(x_0,\ y_0)$ is given by,

$$y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0)$$
$$\Rightarrow a^2yy_0-a^2y_0^2=b^2xx_0-b^2x_0^2$$
$$\Rightarrow b^2xx_0-a^2yy_0-b^2x_0^2+a^2y_0^2=0$$
$$\Rightarrow \frac{xx_0}{a^2}-\frac{yy_0}{b^2}-\left(\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}\right)=0 \qquad \left[\text{On dividing both sides by } a^2b^2\right]$$
$$\Rightarrow \frac{xx_0}{a^2}-\frac{yy_0}{b^2}-1=0 \qquad \left[(x_0,y_0) \text{ lies on the hyperbola } \frac{x^2}{a^2}-\frac{y^2}{b^2}=1\right]$$
$$\Rightarrow \frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$$

Now, the slope of the normal at $(x_0,\ y_0)$ is given by,

$$\frac{-1}{\text{Slope of the tangent at } (x_0,\ y_0)}=\frac{-a^2y_0}{b^2x_0}$$

Hence, the equation of the normal at $(x_0,\ y_0)$ is given by,

$$y-y_0=\frac{-a^2y_0}{b^2x_0}(x-x_0)$$
$$\Rightarrow \frac{y-y_0}{a^2y_0}=\frac{-(x-x_0)}{b^2x_0}$$
$$\Rightarrow \frac{y-y_0}{a^2y_0}+\frac{(x-x_0)}{b^2x_0}=0$$

Solution 25

The equation of the given curve is $y = \sqrt{3x-2}$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}$$

The equation of the given line is $4x - 2y + 5 = 0$.

$4x - 2y + 5 = 0 \Rightarrow y = 2x + \frac{5}{2}$ (which is of the form $y = mx + c$)

$\therefore$Slope of the line = 2

Now, the tangent to the given curve is parallel to the line $4x - 2y - 5 = 0$ if the slope of the tangent is equal to the slope of the line.

$$\frac{3}{2\sqrt{3x-2}} = 2$$

$$\Rightarrow \sqrt{3x-2} = \frac{3}{4}$$

$$\Rightarrow 3x-2=\frac{9}{16}$$
$$\Rightarrow 3x=\frac{9}{16}+2=\frac{41}{16}$$
$$\Rightarrow x=\frac{41}{48}$$

When $x=\frac{41}{48}$, $y=\sqrt{3\left(\frac{41}{48}\right)-2}=\sqrt{\frac{41}{16}-2}=\sqrt{\frac{41-32}{16}}=\sqrt{\frac{9}{16}}=\frac{3}{4}$.

$\therefore$Equation of the tangent passing through the point $\left(\frac{41}{48},\ \frac{3}{4}\right)$ is given by,

$$y-\frac{3}{4}=2\left(x-\frac{41}{48}\right)$$

$$\Rightarrow \frac{4y-3}{4}=2\left(\frac{48x-41}{48}\right)$$
$$\Rightarrow 4y-3=\frac{48x-41}{6}$$
$$\Rightarrow 24y-18=48x-41$$
$$\Rightarrow 48x-24y=23$$

Hence, the equation of the required tangent is $48x-24y=23$

Solution 26

The equation of the given curve is $y=2x^2+3\sin x$.

Slope of the tangent to the given curve at $x=0$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=0}=4x+3\cos x\Big]_{x=0}=0+3\cos 0=3$$

Hence, the slope of the normal to the given curve at $x=0$ is

$$\frac{-1}{\text{Slope of the tangent at } x=0}=\frac{-1}{3}.$$

The correct answer is D.

Solution 27

The equation of the given curve is $y^2 = 4x$.

Differentiating with respect to x, we have:

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

Therefore, the slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{2}{y}$$

The given line is $y = x + 1$ (which is of the form $y = mx + c$)

$\therefore$ Slope of the line = 1

The line $y = x + 1$ is a tangent to the given curve if the slope of the line is equal to the slope of the tangent. Also, the line must intersect the curve.

Thus, we must have:

$$\frac{2}{y} = 1$$

$$\Rightarrow y = 2$$

Now, $y = x + 1 \Rightarrow x = y - 1 \Rightarrow x = 2 - 1 = 1$

Hence, the line $y = x + 1$ is a tangent to the given curve at the point (1, 2).

The correct answer is A.

Chapter 6 - Applications of Derivatives Exercise Ex. 6.4

Solution 1

(i) $\sqrt{25.3}$

Consider $y = \sqrt{x}$. Let $x = 25$ and $\Delta x = 0.3$.

Then,

$$\Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{25.3} - \sqrt{25} = \sqrt{25.3} - 5$$
$$\Rightarrow \sqrt{25.3} = \Delta y + 5$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.3) \qquad \left[\text{as } y = \sqrt{x}\right]$$

$$= \frac{1}{2\sqrt{25}}(0.3) = 0.03$$

Hence, the approximate value of $\sqrt{25.3}$ is 0.03 + 5 = 5.03.

(ii) $\sqrt{49.5}$

Consider $y = \sqrt{x}$. Let $x = 49$ and $\Delta x = 0.5$.

Then,

$$\Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{49.5} - \sqrt{49} = \sqrt{49.5} - 7$$

$$\Rightarrow \sqrt{49.5} = 7 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.5) \qquad \left[\text{as } y = \sqrt{x}\right]$$

$$= \frac{1}{2\sqrt{49}}(0.5) = \frac{1}{14}(0.5) = 0.035$$

Hence, the approximate value of $\sqrt{49.5}$ is 7 + 0.035 = 7.035.

(iii) $\sqrt{0.6}$

Consider $y = \sqrt{x}$. Let $x = 1$ and $\Delta x = -\,0.4$.

Then,

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{0.6} - 1$$
$$\Rightarrow \sqrt{0.6} = 1 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[\text{as } y = \sqrt{x}\right]$$

$$= \frac{1}{2}(-0.4) = -0.2$$

Hence, the approximate value of $\sqrt{0.6}$ is 1 + (−0.2) = 1 − 0.2 = 0.8.

(iv) $(0.009)^{\frac{1}{3}}$

Consider $y = x^{\frac{1}{3}}$. Let $x = 0.008$ and $\Delta x = 0.001$.

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}} = (0.009)^{\frac{1}{3}} - (0.008)^{\frac{1}{3}} = (0.009)^{\frac{1}{3}} - 0.2$$
$$\Rightarrow (0.009)^{\frac{1}{3}} = 0.2 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{3}}\right]$$
$$= \frac{1}{3\times 0.04}(0.001) = \frac{0.001}{0.12} = 0.008$$

Hence, the approximate value of $(0.009)^{\frac{1}{3}}$ is 0.2 + 0.008 = 0.208.

(v) $(0.999)^{\frac{1}{10}}$

Consider $y = (x)^{\frac{1}{10}}$. Let $x = 1$ and $\Delta x = -0.001$.

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{10}} - (x)^{\frac{1}{10}} = (0.999)^{\frac{1}{10}} - 1$$
$$\Rightarrow (0.999)^{\frac{1}{10}} = 1 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{10(x)^{\frac{9}{10}}}(\Delta x) \qquad \left[\text{as } y = (x)^{\frac{1}{10}}\right]$$
$$= \frac{1}{10}(-0.001) = -0.0001$$

Hence, the approximate value of $(0.999)^{\frac{1}{10}}$ is 1 + (−0.0001) = 0.9999.

(vi) $(15)^{\frac{1}{4}}$

Consider $y = x^{\frac{1}{4}}$. Let x = 16 and Δx = −1.

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}} = (15)^{\frac{1}{4}} - (16)^{\frac{1}{4}} = (15)^{\frac{1}{4}} - 2$$
$$\Rightarrow (15)^{\frac{1}{4}} = 2 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{4}}\right]$$

$$= \frac{1}{4(16)^{\frac{3}{4}}}(-1) = \frac{-1}{4\times 8} = \frac{-1}{32} = -0.03125$$

Hence, the approximate value of $(15)^{\frac{1}{4}}$ is 2 + (−0.03125) = 1.96875.

(vii) $(26)^{\frac{1}{3}}$

Consider $y = (x)^{\frac{1}{3}}$. Let $x = 27$ and $\Delta x = -1$.

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}} = (26)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (26)^{\frac{1}{3}} - 3$$
$$\Rightarrow (26)^{\frac{1}{3}} = 3 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\text{as } y = (x)^{\frac{1}{3}}\right]$$

$$= \frac{1}{3(27)^{\frac{2}{3}}}(-1) = \frac{-1}{27} = -0.0\overline{370}$$

Hence, the approximate value of $(26)^{\frac{1}{3}}$ is 3 + (−0.0370) = 2.9629.

(viii) $(255)^{\frac{1}{4}}$

Consider $y = (x)^{\frac{1}{4}}$. Let $x = 256$ and $\Delta x = -1$.

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} = (255)^{\frac{1}{4}} - (256)^{\frac{1}{4}} = (255)^{\frac{1}{4}} - 4$$
$$\Rightarrow (255)^{\frac{1}{4}} = 4 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{4}}\right]$$
$$= \frac{1}{4(256)^{\frac{3}{4}}}(-1) = \frac{-1}{4\times 4^3} = -0.0039$$

Hence, the approximate value of $(255)^{\frac{1}{4}}$ is 4 + (−0.0039) = 3.9961.

(ix) $(82)^{\frac{1}{4}}$

Consider $y = x^{\frac{1}{4}}$. Let $x = 81$ and $\Delta x = 1$.

Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} = (82)^{\frac{1}{4}} - (81)^{\frac{1}{4}} = (82)^{\frac{1}{4}} - 3$$
$$\Rightarrow (82)^{\frac{1}{4}} = \Delta y + 3$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{4}}\right]$$
$$= \frac{1}{4(81)^{\frac{3}{4}}}(1) = \frac{1}{4(3)^3} = \frac{1}{108} = 0.009$$

Hence, the approximate value of $(82)^{\frac{1}{4}}$ is 3 + 0.009 = 3.009.

(x) $(401)^{\frac{1}{2}}$

Consider $y = x^{\frac{1}{2}}$. Let $x = 400$ and $\Delta x = 1$.

Then,

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{401} - \sqrt{400} = \sqrt{401} - 20$$
$$\Rightarrow \sqrt{401} = 20 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{2}}\right]$$
$$= \frac{1}{2 \times 20}(1) = \frac{1}{40} = 0.025$$

Hence, the approximate value of $\sqrt{401}$ is 20 + 0.025 = 20.025.

(xi) $(0.0037)^{\frac{1}{2}}$

Consider $y = x^{\frac{1}{2}}$. Let $x = 0.0036$ and $\Delta x = 0.0001$.

Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{2}} - (x)^{\frac{1}{2}} = (0.0037)^{\frac{1}{2}} - (0.0036)^{\frac{1}{2}} = (0.0037)^{\frac{1}{2}} - 0.06$$
$$\Rightarrow (0.0037)^{\frac{1}{2}} = 0.06 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{2}}\right]$$

$$= \frac{1}{2 \times 0.06}(0.0001)$$

$$= \frac{0.0001}{0.12} = 0.00083$$

Thus, the approximate value of $(0.0037)^{\frac{1}{2}}$ is 0.06 + 0.00083 = 0.06083.

(xii) $(26.57)^{\frac{1}{3}}$

Consider $y = x^{\frac{1}{3}}$. Let $x = 27$ and $\Delta x = -0.43$.

Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}} = (26.57)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (26.57)^{\frac{1}{3}} - 3$$

$$\Rightarrow (26.57)^{\frac{1}{3}} = 3 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{3}}\right]$$

$$= \frac{1}{3(9)}(-0.43)$$

$$= \frac{-0.43}{27} = -0.015$$

Hence, the approximate value of $(26.57)^{\frac{1}{3}}$ is 3 + (−0.015) = 2.984.

(xiii) $(81.5)^{\frac{1}{4}}$

Consider $y = x^{\frac{1}{4}}$. Let $x = 81$ and $\Delta x = 0.5$.

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} = (81.5)^{\frac{1}{4}} - (81)^{\frac{1}{4}} = (81.5)^{\frac{1}{4}} - 3$$

$$\Rightarrow (81.5)^{\frac{1}{4}} = 3 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{4}}\right]$$

$$= \frac{1}{4(3)^3}(0.5) = \frac{0.5}{108} = 0.0046$$

Hence, the approximate value of $(81.5)^{\frac{1}{4}}$ is 3 + 0.0046 = 3.0046.

(xiv) $(3.968)^{\frac{3}{2}}$

Consider $y = x^{\frac{3}{2}}$. Let $x = 4$ and $\Delta x = -0.032$.

Then,

$$\Delta y = (x+\Delta x)^{\frac{3}{2}} - x^{\frac{3}{2}} = (3.968)^{\frac{3}{2}} - (4)^{\frac{3}{2}} = (3.968)^{\frac{3}{2}} - 8$$
$$\Rightarrow (3.968)^{\frac{3}{2}} = 8 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{3}{2}(x)^{\frac{1}{2}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{3}{2}}\right]$$
$$= \frac{3}{2}(2)(-0.032)$$
$$= -0.096$$

Hence, the approximate value of $(3.968)^{\frac{3}{2}}$ is 8 + (−0.096) = 7.904.

(xv) $(32.15)^{\frac{1}{5}}$

Consider $y = x^{\frac{1}{5}}$. Let $x = 32$ and $\Delta x = 0.15$.

Then,

$$\Delta y=(x+\Delta x)^{\frac{1}{5}}-x^{\frac{1}{5}}=(32.15)^{\frac{1}{5}}-(32)^{\frac{1}{5}}=(32.15)^{\frac{1}{5}}-2$$

$$\Rightarrow (32.15)^{\frac{1}{5}}=2+\Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{5(x)^{\frac{4}{5}}}\cdot(\Delta x) \qquad \left[\text{as } y=x^{\frac{1}{5}}\right]$$

$$=\frac{1}{5\times(2)^4}(0.15)$$

$$=\frac{0.15}{80}=0.00187$$

Hence, the approximate value of $(32.15)^{\frac{1}{5}}$ is $2 + 0.00187 = 2.00187$.

Solution 2

Let $x = 2$ and $\Delta x = 0.01$. Then, we have:

$$f(2.01) = f(x + \Delta x) = 4(x + \Delta x)^2 + 5(x + \Delta x) + 2$$

Now, $\Delta y = f(x + \Delta x) - f(x)$

$$\therefore f(x + \Delta x) = f(x) + \Delta y$$

$$\approx f(x)+f'(x)\cdot\Delta x \qquad (\text{as } dx=\Delta x)$$

$$\Rightarrow f(2.01)\approx(4x^2+5x+2)+(8x+5)\Delta x$$

$$=\left[4(2)^2+5(2)+2\right]+\left[8(2)+5\right](0.01) \qquad [\text{as } x=2,\ \Delta x=0.01]$$

$$=(16+10+2)+(16+5)(0.01)$$

$$=28+(21)(0.01)$$

$$=28+0.21$$

$$=28.21$$

Hence, the approximate value of $f(2.01)$ is 28.21.

Solution 3

Let $x = 5$ and $\Delta x = 0.001$. Then, we have:

$$\begin{aligned}
&f(5.001) = f(x+\Delta x) = (x+\Delta x)^3 - 7(x+\Delta x)^2 + 15 \\
&\text{Now, } \Delta y = f(x+\Delta x) - f(x) \\
&\therefore f(x+\Delta x) = f(x) + \Delta y \\
&\qquad\qquad \approx f(x) + f'(x)\cdot\Delta x \qquad\qquad (\text{as } dx = \Delta x) \\
&\Rightarrow f(5.001) \approx (x^3 - 7x^2 + 15) + (3x^2 - 14x)\Delta x \\
&\qquad\qquad = \left[(5)^3 - 7(5)^2 + 15\right] + \left[3(5)^2 - 14(5)\right](0.001) \qquad\qquad [x = 5, \Delta x = 0.001] \\
&\qquad\qquad = (125 - 175 + 15) + (75 - 70)(0.001) \\
&\qquad\qquad = -35 + (5)(0.001) \\
&\qquad\qquad = -35 + 0.005 \\
&\qquad\qquad = -34.995
\end{aligned}$$

Hence, the approximate value of $f(5.001)$ is -34.995.

Solution 4

The volume of a cube (V) of side x is given by $V = x^3$.

$$\begin{aligned}
\therefore dV &= \left(\frac{dV}{dx}\right)\Delta x \\
&= (3x^2)\Delta x \\
&= (3x^2)(0.01x) \qquad\qquad [\text{as } 1\% \text{ of } x \text{ is } 0.01x] \\
&= 0.03x^3
\end{aligned}$$

Hence, the approximate change in the volume of the cube is $0.03x^3$ m³.

Solution 5

The surface area of a cube (S) of side x is given by $S = 6x^2$.

$$\begin{aligned}
\therefore dS &= \left(\frac{dS}{dx}\right)\Delta x \\
&= (12x)\Delta x \\
&= (12x)(-0.01x) \qquad\qquad [\text{as } 1\% \text{ of } x \text{ is } 0.01x] \\
&= -0.12x^2
\end{aligned}$$

Hence, the approximate change in the surface area of the cube is $-0.12x^2$ m^2.

Solution 6

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then,

$r = 7$ m and $\Delta r = 0.02$ m

Now, the volume V of the sphere is given by,

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

$$\begin{aligned}\therefore dV &= \left(\frac{dV}{dr}\right)\Delta r \\ &= \left(4\pi r^2\right)\Delta r \\ &= 4\pi(7)^2(0.02)\ \text{m}^3 = 3.92\pi\ \text{m}^3\end{aligned}$$

Hence, the approximate error in calculating the volume is $3.92\ \pi\ \text{m}^3$.

Solution 7

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then,

$r = 9$ m and $\Delta r = 0.03$ m

Now, the surface area of the sphere (S) is given by,

$S = 4\pi r^2$

$$\therefore \frac{dS}{dr} = 8\pi r$$

$$\begin{aligned}\therefore dS &= \left(\frac{dS}{dr}\right)\Delta r \\ &= (8\pi r)\Delta r \\ &= 8\pi(9)(0.03)\ \text{m}^2 \\ &= 2.16\pi\ \text{m}^2\end{aligned}$$

Hence, the approximate error in calculating the surface area is $2.16\pi\ \text{m}^2$.

Solution 8

Let $x = 3$ and $\Delta x = 0.02$. Then, we have:

$$
\begin{aligned}
&f(3.02) = f(x+\Delta x) = 3(x+\Delta x)^2 + 15(x+\Delta x) + 5 \\
&\text{Now, } \Delta y = f(x+\Delta x) - f(x) \\
&\Rightarrow f(x+\Delta x) = f(x) + \Delta y \\
&\qquad\qquad \approx f(x) + f'(x)\Delta x \qquad\qquad (\text{As } dx = \Delta x) \\
&\Rightarrow f(3.02) \approx (3x^2 + 15x + 5) + (6x + 15)\Delta x \\
&\qquad\qquad = \left[3(3)^2 + 15(3) + 5\right] + \left[6(3) + 15\right](0.02) \qquad [\text{As } x = 3,\ \Delta x = 0.02] \\
&\qquad\qquad = (27 + 45 + 5) + (18 + 15)(0.02) \\
&\qquad\qquad = 77 + (33)(0.02) \\
&\qquad\qquad = 77 + 0.66 \\
&\qquad\qquad = 77.66
\end{aligned}
$$

Hence, the approximate value of $f(3.02)$ is 77.66.

The correct answer is D.

Solution 9

The volume of a cube (V) of side x is given by $V = x^3$.

$$
\begin{aligned}
\therefore dV &= \left(\frac{dV}{dx}\right)\Delta x \\
&= (3x^2)\Delta x \\
&= (3x^2)(0.03x) \qquad\qquad [\text{As } 3\% \text{ of } x \text{ is } 0.03x] \\
&= 0.09x^3 \text{ m}^3
\end{aligned}
$$

Hence, the approximate change in the volume of the cube is $0.09x^3$ m^3.

The correct answer is C.

Chapter 6 - Applications of Derivatives Exercise Ex. 6.5

Solution 1

(i) The given function is $f(x) = (2x - 1)^2 + 3$.

It can be observed that $(2x - 1)^2 \geq 0$ for every $x \in \mathbf{R}$.

Therefore, $f(x) = (2x - 1)^2 + 3 \geq 3$ for every $x \in \mathbf{R}$.

The minimum value of f is attained when $2x - 1 = 0$.

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$\therefore$Minimum value of $f = f\left(\frac{1}{2}\right) = \left(2 \cdot \frac{1}{2} - 1\right)^2 + 3 = 3$

Hence, function f does not have a maximum value.

(ii) The given function is $f(x) = 9x^2 + 12x + 2 = (3x + 2)^2 - 2$.

It can be observed that $(3x + 2)^2 \geq 0$ for every $x \in \mathbf{R}$.

Therefore, $f(x) = (3x + 2)^2 - 2 \geq -2$ for every $x \in \mathbf{R}$.

The minimum value of f is attained when $3x + 2 = 0$.

$3x + 2 = 0 \Rightarrow x = \frac{-2}{3}$

$\therefore$Minimum value of $f = f\left(-\frac{2}{3}\right) = \left(3\left(\frac{-2}{3}\right)+2\right)^2 - 2 = -2$

Hence, function f does not have a maximum value.

(iii) The given function is $f(x) = -(x-1)^2 + 10$.

It can be observed that $(x-1)^2 \geq 0$ for every $x \in \mathbf{R}$.

Therefore, $f(x) = -(x-1)^2 + 10 \leq 10$ for every $x \in \mathbf{R}$.

The maximum value of f is attained when $(x-1) = 0$.

$(x-1) = 0 \Rightarrow x = 1$

$\therefore$Maximum value of $f = f(1) = -(1-1)^2 + 10 = 10$

Hence, function f does not have a minimum value.

(iv) The given function is $g(x) = x^3 + 1$.

Hence, function g neither has a maximum value nor a minimum value

Solution 2

(i) $f(x) = |x+2|-1$

We know that $|x+2| \geq 0$ for every $x \in \mathbf{R}$.

Therefore, $f(x) = |x+2|-1 \geq -1$ for every $x \in \mathbf{R}$.

The minimum value of f is attained when $|x+2| = 0$.

$|x+2| = 0$
$\Rightarrow x = -2$

$\therefore$Minimum value of $f = f(-2) = |-2+2|-1 = -1$

Hence, function f does not have a maximum value.

(ii) $g(x) = -|x+1|+3$

We know that $-|x+1| \leq 0$ for every $x \in \mathbf{R}$.

Therefore, $g(x) = -|x+1|+3 \leq 3$ for every $x \in \mathbf{R}$.

The maximum value of g is attained when $|x+1| = 0$.

$|x+1| = 0$

$\Rightarrow x = -1$

$\therefore$Maximum value of $g = g(-1) = -|-1+1| + 3 = 3$

Hence, function g does not have a minimum value.

(iii) $h(x) = \sin 2x + 5$

We know that $-1 \le \sin 2x \le 1$.

$\Rightarrow -1 + 5 \le \sin 2x + 5 \le 1 + 5$

$\Rightarrow 4 \le \sin 2x + 5 \le 6$

Hence, the maximum and minimum values of h are 6 and 4 respectively.

(iv) $f(x) = |\sin 4x + 3|$

We know that $-1 \le \sin 4x \le 1$.

$\Rightarrow 2 \le \sin 4x + 3 \le 4$

$\Rightarrow 2 \le |\sin 4x + 3| \le 4$

Hence, the maximum and minimum values of f are 4 and 2 respectively.

(v) $h(x) = x + 1, x \in (-1, 1)$

Here, if a point x_0 is closest to -1, then we find $\frac{x_0}{2} + 1 < x_0 + 1$ for all $x_0 \in (-1, 1)$.

Also, if x_1 is closest to 1, then $x_1 + 1 < \frac{x_1 + 1}{2} + 1$ for all $x_1 \in (-1, 1)$.

Hence, function $h(x)$ has neither maximum nor minimum value in $(-1, 1)$.

Solution 3

(i) $f(x) = x^2$

$\therefore f'(x) = 2x$

Now,

$f'(x) = 0 \Rightarrow x = 0$

Thus, $x = 0$ is the only critical point which could possibly be the point of local maxima or local minima of f.

We have $f''(0) = 2$, which is positive.

Therefore, by second derivative test, $x = 0$ is a point of local minima and local minimum value of f at $x = 0$ is $f(0) = 0$.

(ii) $g(x) = x^3 - 3x$

$\therefore g'(x) = 3x^2 - 3$

Now,

$g''(x) = 0 \Rightarrow 3x^2 = 3 \Rightarrow x = \pm 1$

$g''(x) = 6x$

$g''(1) = 6 > 0$

$g''(-1) = -6 < 0$

By second derivative test, $x = 1$ is a point of local minima and local minimum value of g at $x = 1$ is $g(1) = 1^3 - 3 = 1 - 3 = -2$. However,

$x = -1$ is a point of local maxima and local maximum value of g at

$x = -1$ is $g(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$.

(iii) $h(x) = \sin x + \cos x,\ 0 < x < \frac{\pi}{2}$

$$\therefore h'(x) = \cos x - \sin x$$

$$h'(x) = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4} \in \left(0,\ \frac{\pi}{2}\right)$$

$$h''(x) = -\sin x - \cos x = -(\sin x + \cos x)$$

$$h''\left(\frac{\pi}{4}\right) = -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0$$

Therefore, by second derivative test, $x = \frac{\pi}{4}$ is a point of local maxima and the local maximum value of h at $x = \frac{\pi}{4}$ is $h\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$.

(iv) $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$

$\therefore f'(x) = \cos x + \sin x$

$f'(x) = 0 \Rightarrow \cos x = -\sin x \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi)$

$f''(x) = -\sin x + \cos x$

$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$

$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$

Therefore, by second derivative test, $x = \frac{3\pi}{4}$ is a point of local maxima and the local maximum value of f at $x = \frac{3\pi}{4}$ is

$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$. However, $x = \frac{7\pi}{4}$ is a point of local minima and the local minimum value of f at $x = \frac{7\pi}{4}$ is $f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$.

(v) $f(x) = x^3 - 6x^2 + 9x + 15$

$\therefore f'(x) = 3x^2 - 12x + 9$

$f'(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0$

$\Rightarrow 3(x-1)(x-3) = 0$

$\Rightarrow x = 1,\ 3$

Now, $f''(x) = 6x - 12 = 6(x-2)$

$f''(1) = 6(1-2) = -6 < 0$

$f''(3) = 6(3-2) = 6 > 0$

Therefore, by second derivative test, $x = 1$ is a point of local maxima and the local maximum value of f at $x = 1$ is $f(1) = 1 - 6 + 9 + 15 = 19$. However, $x = 3$ is a point of local minima and the local minimum value of f at $x = 3$ is $f(3) = 27 - 54 + 27 + 15 = 15$.

(vi) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

$\therefore g'(x) = \frac{1}{2} - \frac{2}{x^2}$

Now,

$g'(x) = 0$ gives $\frac{2}{x^2} = \frac{1}{2} \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

Since $x > 0$, we take $x = 2$.

Now,

$g''(x) = \frac{4}{x^3}$

$$g''(2)=\frac{4}{2^3}=\frac{1}{2}>0$$

Therefore, by second derivative test, $x=2$ is a point of local minima and the local minimum value of g at $x=2$ is $g(2)=\frac{2}{2}+\frac{2}{2}=1+1=2.$

(vii) $g(x)=\frac{1}{x^2+2}$

$$\therefore g'(x)=\frac{-(2x)}{(x^2+2)^2}$$

$$g'(x)=0 \Rightarrow \frac{-2x}{(x^2+2)^2}=0 \Rightarrow x=0$$

Now, for values close to $x=0$ and to the left of 0, $g'(x)>0$. Also, for values close to $x=0$ and to the right of 0, $g'(x)<0$.

Therefore, by first derivative test, $x=0$ is a point of local maxima and the local maximum value of $g(0)$ is $\frac{1}{0+2}=\frac{1}{2}$..

(viii) $f(x)=x\sqrt{1-x},\ x>0$

$$\therefore f'(x)=\sqrt{1-x}+x\cdot\frac{1}{2\sqrt{1-x}}(-1)=\sqrt{1-x}-\frac{x}{2\sqrt{1-x}}$$
$$=\frac{2(1-x)-x}{2\sqrt{1-x}}=\frac{2-3x}{2\sqrt{1-x}}$$
$$f'(x)=0\Rightarrow\frac{2-3x}{2\sqrt{1-x}}=0\Rightarrow 2-3x=0\Rightarrow x=\frac{2}{3}$$
$$f''(x)=\frac{1}{2}\left[\frac{\sqrt{1-x}(-3)-(2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x}\right]$$
$$=\frac{\sqrt{1-x}(-3)+(2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)}{2(1-x)}$$
$$=\frac{-6(1-x)+(2-3x)}{4(1-x)^{\frac{3}{2}}}$$
$$=\frac{3x-4}{4(1-x)^{\frac{3}{2}}}$$

$$f''\left(\frac{2}{3}\right)=\frac{3\left(\frac{2}{3}\right)-4}{4\left(1-\frac{2}{3}\right)^{\frac{3}{2}}}=\frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}}=\frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}}<0$$

Therefore, by second derivative test, $x=\frac{2}{3}$ is a point of local maxima and the local maximum value of f at $x=\frac{2}{3}$ is

$$f\left(\frac{2}{3}\right)=\frac{2}{3}\sqrt{1-\frac{2}{3}}=\frac{2}{3}\sqrt{\frac{1}{3}}=\frac{2}{3\sqrt{3}}=\frac{2\sqrt{3}}{9}.$$

Solution 4

i. We have,

$f(x) = e^x$

$\therefore f'(x) = e^x$

Now, if $f'(x) = 0$, then $e^x = 0$. But, the exponential function can never assume 0 for any value of x.

Therefore, there does not exist $c \in \mathbf{R}$ such that $f'(c) = 0$.

Hence, function f does not have maxima or minima.

ii. We have,

$g(x) = \log x$

$\therefore g'(x) = \frac{1}{x}$

Since $\log x$ is defined for a positive number x, $g'(x) > 0$ for any x.

Therefore, there does not exist $c \in \mathbf{R}$ such that $g'(c) = 0$.

Hence, function g does not have maxima or minima.

iii. We have,

$h(x) = x^3 + x^2 + x + 1$

$\therefore h'(x) = 3x^2 + 2x + 1$

Now,

$h(x) = 0 \Rightarrow 3x^2 + 2x + 1 = 0 \Rightarrow x = \frac{-2 \pm 2\sqrt{2}\, i}{6} = \frac{-1 \pm \sqrt{2}\, i}{3} \notin \mathbf{R}$

Therefore, there does not exist $c \in \mathbf{R}$ such that $h'(c) = 0$.

Hence, function h does not have maxima or minima.

Solution 5

(i) The given function is $f(x) = x^3$.

$\therefore f'(x) = 3x^2$

Now,

$f'(x) = 0 \Rightarrow x = 0$

Then, we evaluate the value of f at critical point $x = 0$ and at end points of the interval $[-2, 2]$.

$f(0) = 0$

$f(-2) = (-2)^3 = -8$

$f(2) = (2)^3 = 8$

Hence, we can conclude that the absolute maximum value of f on $[-2, 2]$ is 8 occurring at $x = 2$. Also, the absolute minimum value of f on $[-2, 2]$ is -8 occurring at $x = -2$.

(ii) The given function is $f(x) = \sin x + \cos x$.

$\therefore f'(x) = \cos x - \sin x$

Now,

$f'(x) = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$

Then, we evaluate the value of f at critical point $x=\frac{\pi}{4}$ and at the end points of the interval $[0, \pi]$.

$$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$
$$f(0)=\sin 0+\cos 0=0+1=1$$
$$f(\pi)=\sin\pi+\cos\pi=0-1=-1$$

Hence, we can conclude that the absolute maximum value of f on $[0, \pi]$ is $\sqrt{2}$ occurring at $x=\frac{\pi}{4}$ and the absolute minimum value of f on $[0, \pi]$ is -1 occurring at $x=\pi$.

(iii) The given function is $f(x)=4x-\frac{1}{2}x^2$.

$$\therefore f'(x)=4-\frac{1}{2}(2x)=4-x$$

Now,

$$f'(x)=0 \Rightarrow x=4$$

Then, we evaluate the value of f at critical point $x=4$ and at the end points of the interval $\left[-2, \frac{9}{2}\right]$.

$$f(4)=16-\frac{1}{2}(16)=16-8=8$$
$$f(-2)=-8-\frac{1}{2}(4)=-8-2=-10$$
$$f\left(\frac{9}{2}\right)=4\left(\frac{9}{2}\right)-\frac{1}{2}\left(\frac{9}{2}\right)^2=18-\frac{81}{8}=18-10.125=7.875$$

Hence, we can conclude that the absolute maximum value of f on $\left[-2, \frac{9}{2}\right]$ is 8 occurring at $x = 4$ and the absolute minimum value of f on $\left[-2, \frac{9}{2}\right]$ is -10 occurring at $x = -2$.

(iv) The given function is $f(x)=(x-1)^2+3$.

$\therefore f'(x)=2(x-1)$

Now,

$f'(x)=0 \Rightarrow 2(x-1)=0 \Rightarrow x=1$

Then, we evaluate the value of f at critical point $x = 1$ and at the end points of the interval $[-3, 1]$.

$$f(1)=(1-1)^2+3=0+3=3$$
$$f(-3)=(-3-1)^2+3=16+3=19$$

Hence, we can conclude that the absolute maximum value of f on $[-3, 1]$ is 19 occurring at $x = -3$ and the minimum value of f on $[-3, 1]$ is 3 occurring at $x = 1$.

Solution 6

The profit function is given as $p(x) = 41 - 72x - 18x^2$.

$\therefore p'(x) = -72 - 36x$

$p''(x) = -36$

Now,

$p'(x) = 0 \Rightarrow x = \frac{-72}{36} = -2$

Also,

$p''(-2) = -36 < 0$

By second derivative test, $x = -2$ is the point of local maxima of p.

$\therefore$ Maximum profit $= p(-2)$

$$= 41 - 72(-2) - 18(-2)^2$$
$$= 41 + 144 - 72$$
$$= 113$$

Hence, the maximum profit that the company can make is 113 units.

Solution 7

Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$.

$$\begin{aligned}
\therefore f'(x) &= 12x^3 - 24x^2 + 24x - 48 \\
&= 12\left(x^3 - 2x^2 + 2x - 4\right) \\
&= 12\left[x^2(x-2) + 2(x-2)\right] \\
&= 12(x-2)\left(x^2+2\right)
\end{aligned}$$

Now, $f'(x) = 0$ gives $x = 2$ or $x^2 + 2 = 0$ for which there are no real roots.

Therefore, we consider only $x = 2 \in [0, 3]$.

Now, we evaluate the value of f at critical point $x = 2$ and at the end points of the interval [0, 3].

$$\begin{aligned}
f(2) &= 3(16) - 8(8) + 12(4) - 48(2) + 25 \\
&= 48 - 64 + 48 - 96 + 25 \\
&= -39 \\
f(0) &= 3(0) - 8(0) + 12(0) - 48(0) + 25 \\
&= 25 \\
f(3) &= 3(81) - 8(27) + 12(9) - 48(3) + 25 \\
&= 243 - 216 + 108 - 144 + 25 = 16
\end{aligned}$$

Hence, we can conclude that the absolute maximum value of f on [0, 3] is 25 occurring at $x = 0$ and the absolute minimum value of f at [0, 3] is -39 occurring at $x = 2$.

Solution 8

Let $f(x) = \sin 2x$.

$\therefore f'(x) = 2\cos 2x$

Now,

$f'(x) = 0 \Rightarrow \cos 2x = 0$

$\Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Then, we evaluate the values of f at critical points $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and at the end points of the interval $[0, 2\pi]$.

$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1, f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{2} = -1$

$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{2} = 1, f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{2} = -1$

$f(0) = \sin 0 = 0, f(2\pi) = \sin 2\pi = 0$

Hence, we can conclude that the absolute maximum value of f on $[0, 2\pi]$ is occurring at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Solution 9

Let $f(x) = \sin x + \cos x$.

$\therefore f'(x) = \cos x - \sin x$

$f'(x) = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots,$

$f''(x) = -\sin x - \cos x = -(\sin x + \cos x)$

Now, $f''(x)$ will be negative when $(\sin x + \cos x)$ is positive i.e., when $\sin x$ and $\cos x$ are both positive. Also, we know that $\sin x$ and $\cos x$ both are positive in the first quadrant.

Then, $f''(x)$ will be negative when $x \in \left(0, \frac{\pi}{2}\right)$.

Thus, we consider $x = \frac{\pi}{4}$.

$$f''\left(\frac{\pi}{4}\right) = -\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = -\left(\frac{2}{\sqrt{2}}\right) = -\sqrt{2} < 0$$

$\therefore$By second derivative test, f will be the maximum at $x = \frac{\pi}{4}$ and the maximum value of f is $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

Solution 10

Let $f(x) = 2x^3 - 24x + 107$.

$\therefore f'(x) = 6x^2 - 24 = 6(x^2 - 4)$

Now,

$f'(x) = 0 \Rightarrow 6(x^2 - 4) = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

We first consider the interval [1, 3].

Then, we evaluate the value of f at the critical point $x = 2 \in [1, 3]$ and at the end points of the interval [1, 3].

$f(2) = 2(8) - 24(2) + 107 = 16 - 48 + 107 = 75$

$f(1) = 2(1) - 24(1) + 107 = 2 - 24 + 107 = 85$

$f(3) = 2(27) - 24(3) + 107 = 54 - 72 + 107 = 89$

Hence, the absolute maximum value of $f(x)$ in the interval [1, 3] is 89 occurring at $x = 3$.

Next, we consider the interval $[-3, -1]$.

Evaluate the value of f at the critical point $x = -2 \in [-3, -1]$ and at the end points of the interval [1, 3].

$f(-3) = 2(-27) - 24(-3) + 107 = -54 + 72 + 107 = 125$

$f(-1) = 2(-1) - 24(-1) + 107 = -2 + 24 + 107 = 129$

$f(-2) = 2(-8) - 24(-2) + 107 = -16 + 48 + 107 = 139$

Hence, the absolute maximum value of $f(x)$ in the interval $[-3, -1]$ is 139 occurring at $x = -2$

Solution 11

Let $f(x) = x^4 - 62x^2 + ax + 9$.

$\therefore f'(x) = 4x^3 - 124x + a$

It is given that function f attains its maximum value on the interval [0, 2] at $x = 1$.

$\therefore f'(1) = 0$
$\Rightarrow 4 - 124 + a = 0$
$\Rightarrow a = 120$

Hence, the value of a is 120.

Solution 12

Let $f(x) = x + \sin 2x$.

$\therefore f'(x) = 1 + 2\cos 2x$

Now, $f'(x) = 0 \Rightarrow \cos 2x = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$2x = 2n\pi \pm \frac{2\pi}{3}, \quad n \in \mathbf{Z}$

$\Rightarrow x = n\pi \pm \frac{\pi}{3}, \; n \in \mathbf{Z}$

$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \in [0, 2\pi]$

Then, we evaluate the value of f at critical points $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ and at the end points of the interval $[0, 2\pi]$.

$$f\left(\frac{\pi}{3}\right)=\frac{\pi}{3}+\sin\frac{2\pi}{3}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$$
$$f\left(\frac{2\pi}{3}\right)=\frac{2\pi}{3}+\sin\frac{4\pi}{3}=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$
$$f\left(\frac{4\pi}{3}\right)=\frac{4\pi}{3}+\sin\frac{8\pi}{3}=\frac{4\pi}{3}+\frac{\sqrt{3}}{2}$$
$$f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}+\sin\frac{10\pi}{3}=\frac{5\pi}{3}-\frac{\sqrt{3}}{2}$$
$$f(0)=0+\sin 0=0$$
$$f(2\pi)=2\pi+\sin 4\pi=2\pi+0=2\pi$$

Hence, we can conclude that the absolute maximum value of $f(x)$ in the interval $[0, 2\pi]$ is 2π occurring at $x = 2\pi$ and the absolute minimum value of $f(x)$ in the interval $[0, 2\pi]$ is 0 occurring at $x = 0$.

Solution 13

Let one number be x. Then, the other number is $(24 - x)$.

Let $P(x)$ denote the product of the two numbers. Thus, we have:

$$P(x)=x(24-x)=24x-x^2$$
$$\therefore P'(x)=24-2x$$
$$P''(x)=-2$$

Now,

$$P'(x)=0 \Rightarrow x=12$$

Also,

$$P''(12)=-2<0$$

$\therefore$By second derivative test, $x = 12$ is the point of local maxima of P. Hence, the product of the numbers is the maximum when the numbers are 12 and $24 - 12 = 12$.

Solution 14

The two numbers are x and y such that $x + y = 60$.

$\Rightarrow y = 60 - x$

Let $f(x) = xy^3$.

$$\Rightarrow f(x) = x(60-x)^3$$

$$\therefore f'(x) = (60-x)^3 - 3x(60-x)^2$$

$$= (60-x)^2[60-x-3x]$$

$$= (60-x)^2(60-4x)$$

$$\text{And, } f''(x) = -2(60-x)(60-4x) - 4(60-x)^2$$

$$= -2(60-x)[60-4x+2(60-x)]$$

$$= -2(60-x)(180-6x)$$

$$= -12(60-x)(30-x)$$

Now, $f'(x) = 0 \Rightarrow x = 60$ or $x = 15$

When $x = 60$, $f''(x) = 0$.

When $x = 15$, $f''(x) = -12(60-15)(30-15) = -12\times 45\times 15 < 0$.

$\therefore$By second derivative test, $x = 15$ is a point of local maxima of f. Thus, function xy^3 is maximum when $x = 15$ and $y = 60 - 15 = 45$.

Hence, the required numbers are 15 and 45.

Solution 15

Let one number be x. Then, the other number is $y = (35 - x)$.

Let $P(x) = x^2y^5$. Then, we have:

$$P(x) = x^2(35-x)^5$$
$$\therefore P'(x) = 2x(35-x)^5 - 5x^2(35-x)^4$$
$$= x(35-x)^4\left[2(35-x)-5x\right]$$
$$= x(35-x)^4(70-7x)$$
$$= 7x(35-x)^4(10-x)$$

$$\text{And, } P''(x) = 7(35-x)^4(10-x) + 7x\left[-(35-x)^4 - 4(35-x)^3(10-x)\right]$$
$$= 7(35-x)^4(10-x) - 7x(35-x)^4 - 28x(35-x)^3(10-x)$$
$$= 7(35-x)^3\left[(35-x)(10-x) - x(35-x) - 4x(10-x)\right]$$
$$= 7(35-x)^3\left[350 - 45x + x^2 - 35x + x^2 - 40x + 4x^2\right]$$
$$= 7(35-x)^3\left(6x^2 - 120x + 350\right)$$

Now, $P'(x) = 0 \Rightarrow x = 0, x = 35, x = 10$

When $x = 35$, $P'(x) = P(x) = 0$ and $y = 35 - 35 = 0$. This will make the product $x^2 y^5$ equal to 0.
When $x = 0$, $y = 35 - 0 = 35$ and the product x^2y^2 will be 0.

$\therefore x = 0$ and $x = 35$ cannot be the possible values of x.

When $x = 10$, we have:

$$P''(x) = 7(35-10)^3(6\times100 - 120\times10 + 350)$$
$$= 7(25)^3(-250) < 0$$

$\therefore$ By second derivative test, $P(x)$ will be the maximum when $x = 10$ and $y = 35 - 10 = 25$.

Hence, the required numbers are 10 and 25.

Solution 16

Let one number be x. Then, the other number is $(16 - x)$.

Let the sum of the cubes of these numbers be denoted by $S(x)$. Then,

$$S(x) = x^3 + (16 - x)^3$$
$$\therefore S'(x) = 3x^2 - 3(16 - x)^2,\ S''(x) = 6x + 6(16 - x)$$
$$\text{Now, } S'(x) = 0 \Rightarrow 3x^2 - 3(16 - x)^2 = 0$$
$$\Rightarrow x^2 - (16 - x)^2 = 0$$
$$\Rightarrow x^2 - 256 - x^2 + 32x = 0$$
$$\Rightarrow x = \frac{256}{32} = 8$$

Now, $S''(8) = 6(8) + 6(16 - 8) = 48 + 48 = 96 > 0$

$\therefore$ By second derivative test, $x = 8$ is the point of local minima of S.

Hence, the sum of the cubes of the numbers is the minimum when the numbers are 8 and $16 - 8 = 8$.

Solution 17

Let the side of the square to be cut off be x cm. Then, the length and the breadth of the box will be $(18 - 2x)$ cm each and the height of the box is x cm.

Therefore, the volume $V(x)$ of the box is given by,

$$V(x) = x(18 - 2x)^2$$

$$\begin{aligned}\therefore V'(x) &= (18 - 2x)^2 - 4x(18 - 2x)\\ &= (18 - 2x)[18 - 2x - 4x]\\ &= (18 - 2x)(18 - 6x)\\ &= 6 \times 2(9 - x)(3 - x)\\ &= 12(9 - x)(3 - x)\end{aligned}$$

$$\begin{aligned}\text{And, } V''(x) &= 12[-(9 - x) - (3 - x)]\\ &= -12(9 - x + 3 - x)\\ &= -12(12 - 2x)\\ &= -24(6 - x)\end{aligned}$$

Now, $V'(x)=0 \Rightarrow x=9$ or $x=3$

If $x=9$, then the length and the breadth will become 0.

$\therefore x \neq 9.$

$\Rightarrow x=3.$

Now, $V''(3)=-24(6-3)=-72<0$

$\therefore$ By second derivative test, $x=3$ is the point of maxima of V.

Hence, if we remove a square of side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

Solution 18

Let the side of the square to be cut off be x cm. Then, the height of the box is x, the length is $45-2x$, and the breadth is $24-2x$.

Therefore, the volume $V(x)$ of the box is given by,

$$\begin{aligned}
V(x) &= x(45-2x)(24-2x) \\
&= x\left(1080-90x-48x+4x^2\right) \\
&= 4x^3-138x^2+1080x \\
\therefore V'(x) &= 12x^2-276x+1080 \\
&= 12\left(x^2-23x+90\right) \\
&= 12(x-18)(x-5) \\
V''(x) &= 24x-276 = 12(2x-23)
\end{aligned}$$

Now, $V'(x)=0 \Rightarrow x=18$ and $x=5$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet. Thus, x cannot be equal to 18.

$\therefore x = 5$

Now, $V''(5) = 12(10-23) = 12(-13) = -156 < 0$

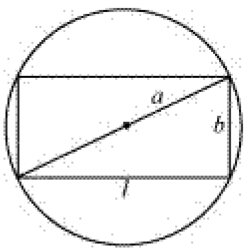
$\therefore$ By second derivative test, $x = 5$ is the point of maxima.

Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5 cm.

Solution 19

Let a rectangle of length l and breadth b be inscribed in the given circle of radius a.

Then, the diagonal passes through the centre and is of length $2a$ cm.



Now, by applying the Pythagoras theorem, we have:

$$(2a)^2 = l^2 + b^2$$
$$\Rightarrow b^2 = 4a^2 - l^2$$
$$\Rightarrow b = \sqrt{4a^2 - l^2}$$

$\therefore$ Area of the rectangle, $A = l\sqrt{4a^2 - l^2}$

$$\therefore \frac{dA}{dl} = \sqrt{4a^2 - l^2} + l\frac{1}{2\sqrt{4a^2 - l^2}}(-2l) = \sqrt{4a^2 - l^2} - \frac{l^2}{\sqrt{4a^2 - l^2}}$$
$$= \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}}$$

$$\frac{d^2A}{dl^2}=\frac{\sqrt{4a^2-l^2}(-4l)-\left(4a^2-2l^2\right)\frac{(-2l)}{2\sqrt{4a^2-l^2}}}{\left(4a^2-l^2\right)}$$

$$=\frac{\left(4a^2-l^2\right)(-4l)+l\left(4a^2-2l^2\right)}{\left(4a^2-l^2\right)^{\frac{3}{2}}}$$

$$=\frac{-12a^2l+2l^3}{\left(4a^2-l^2\right)^{\frac{3}{2}}}=\frac{-2l\left(6a^2-l^2\right)}{\left(4a^2-l^2\right)^{\frac{3}{2}}}$$

Now, $\frac{dA}{dl}=0$ gives $4a^2=2l^2\Rightarrow l=\sqrt{2}a$

$$\Rightarrow b=\sqrt{4a^2-2a^2}=\sqrt{2a^2}=\sqrt{2}a$$

Now, when $l=\sqrt{2}a$,

$$\frac{d^2A}{dl^2}=\frac{-2\left(\sqrt{2}a\right)\left(6a^2-2a^2\right)}{2\sqrt{2}a^3}=\frac{-8\sqrt{2}a^3}{2\sqrt{2}a^3}=-4<0$$

$\therefore$ By the second derivative test, when $l=\sqrt{2}a$, then the area of the rectangle is the maximum.

Since $l=b=\sqrt{2}a$, the rectangle is a square.

Hence, it has been proved that of all the rectangles inscribed in the given fixed circle, the square has the maximum area.

Solution 20

Let r and h be the radius and height of the cylinder respectively.

Then, the surface area (S) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi rh$$
$$\Rightarrow h = \frac{S - 2\pi r^2}{2\pi r}$$
$$= \frac{S}{2\pi}\left(\frac{1}{r}\right) - r$$

Let V be the volume of the cylinder. Then,

$$V = \pi r^2 h = \pi r^2\left[\frac{S}{2\pi}\left(\frac{1}{r}\right) - r\right] = \frac{Sr}{2} - \pi r^3$$

Then, $\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$, $\frac{d^2V}{dr^2} = -6\pi r$

Now, $\frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} = 3\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi}$

When $r^2 = \frac{S}{6\pi}$, then $\frac{d^2V}{dr^2} = -6\pi\left(\sqrt{\frac{S}{6\pi}}\right) < 0.$

$\therefore$ By second derivative test, the volume is the maximum when $r^2 = \frac{S}{6\pi}$.

Now, when $r^2 = \frac{S}{6\pi}$, then $h = \frac{6\pi r^2}{2\pi}\left(\frac{1}{r}\right) - r = 3r - r = 2r.$

Hence, the volume is the maximum when the height is twice the radius i.e., when the height is equal to the diameter.

Solution 21

Let r and h be the radius and height of the cylinder respectively.

Then, volume (V) of the cylinder is given by,

$$V = \pi r^2 h = 100 \qquad \text{(given)}$$

$$\therefore h = \frac{100}{\pi r^2}$$

Surface area (S) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{200}{r}$$

$$\therefore \frac{dS}{dr} = 4\pi r - \frac{200}{r^2}, \quad \frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$

$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Now, it is observed that when $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $\frac{d^2S}{dr^2} > 0$.

$\therefore$By second derivative test, the surface area is the minimum when the radius of the cylinder is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

When $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $h = \dfrac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}} = \dfrac{2\times 50}{(50)^{\frac{2}{3}}(\pi)^{1-\frac{2}{3}}} = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

Hence, the required dimensions of the can which has the minimum surface area is given by radius $= \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height $= 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

Solution 22

Let a piece of length l be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length $(28 - l)$ m.

Now, side of square $= \frac{l}{4}$.

Let r be the radius of the circle. Then, $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28 - l)$.

The combined areas of the square and the circle (A) is given by,

$$A = (\text{side of the square})^2 + \pi r^2$$
$$= \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28 - l)\right]^2$$
$$= \frac{l^2}{16} + \frac{1}{4\pi}(28 - l)^2$$
$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28 - l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28 - l)$$
$$\frac{d^2A}{dl^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$
$$\text{Now, } \frac{dA}{dl} = 0 \Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28 - l) = 0$$
$$\Rightarrow \frac{\pi l - 4(28 - l)}{8\pi} = 0$$
$$\Rightarrow (\pi + 4)l - 112 = 0$$
$$\Rightarrow l = \frac{112}{\pi + 4}$$

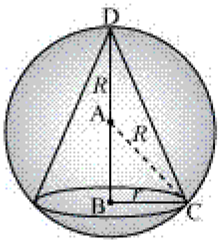
Thus, when $l = \frac{112}{\pi + 4}, \frac{d^2\text{A}}{dl^2} > 0$.

$\therefore$ By second derivative test, the area (A) is the minimum when $l = \frac{112}{\pi + 4}$.

Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi + 4}$ cm while the length of the wire in making the circle is $28 - \frac{112}{\pi + 4} = \frac{28\pi}{\pi + 4}$ cm.

Solution 23

Let r and h be the radius and height of the cone respectively inscribed in a sphere of radius R.



Let V be the volume of the cone.

Then, $V = \frac{1}{3}\pi r^2 h$

Height of the cone is given by,

$h = R + \text{AB} = R + \sqrt{R^2 - r^2}$ [ABC is a right triangle]

$$\therefore V = \frac{1}{3}\pi r^2 \left(R + \sqrt{R^2 - r^2}\right)$$
$$= \frac{1}{3}\pi r^2 R + \frac{1}{3}\pi r^2 \sqrt{R^2 - r^2}$$
$$\therefore \frac{dV}{dr} = \frac{2}{3}\pi r R + \frac{2}{3}\pi r \sqrt{R^2 - r^2} + \frac{1}{3}\pi r^2 \cdot \frac{(-2r)}{2\sqrt{R^2 - r^2}}$$
$$= \frac{2}{3}\pi r R + \frac{2}{3}\pi r \sqrt{R^2 - r^2} - \frac{1}{3}\pi \frac{r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi rR + \frac{2\pi r\left(R^2 - r^2\right) - \pi r^3}{3\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi rR + \frac{2\pi rR^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}}$$

$$\frac{d^2V}{dr^2} = \frac{2\pi R}{3} + \frac{3\sqrt{R^2 - r^2}\left(2\pi R^2 - 9\pi r^2\right) - \left(2\pi rR^2 - 3\pi r^3\right)\cdot\frac{(-2r)}{6\sqrt{R^2 - r^2}}}{9\left(R^2 - r^2\right)}$$

$$= \frac{2}{3}\pi R + \frac{9\left(R^2 - r^2\right)\left(2\pi R^2 - 9\pi r^2\right) + 2\pi r^2 R^2 - 3\pi r^4}{27\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

Now, $\frac{dV}{dr} = 0 \Rightarrow \frac{2}{3}\pi R = \frac{3\pi r^3 - 2\pi rR^2}{3\sqrt{R^2 - r^2}}$

$$\Rightarrow 2R = \frac{3r^2 - 2R^2}{\sqrt{R^2 - r^2}} \Rightarrow 2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2$$

$$\Rightarrow 4R^2\left(R^2 - r^2\right) = \left(3r^2 - 2R^2\right)^2$$

$$\Rightarrow 4R^4 - 4R^2r^2 = 9r^4 + 4R^4 - 12r^2R^2$$

$$\Rightarrow 9r^4 = 8R^2r^2$$

$$\Rightarrow r^2 = \frac{8}{9}R^2$$

When $r^2 = \frac{8}{9}R^2$, then $\frac{d^2V}{dr^2} < 0$.

$\therefore$ By second derivative test, the volume of the cone is the maximum when $r^2 = \frac{8}{9}R^2$.

When $r^2 = \frac{8}{9}R^2$, $h = R + \sqrt{R^2 - \frac{8}{9}R^2} = R + \sqrt{\frac{1}{9}R^2} = R + \frac{R}{3} = \frac{4}{3}R$.

Therefore,

$$= \frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)$$

$$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{8}{27}\times\left(\text{Volume of the sphere}\right)$$

Hence, the volume of the largest cone that can be inscribed in the sphere is $\frac{8}{27}$ the volume of the sphere.

Solution 24

Let r and h be the radius and the height (altitude) of the cone respectively.

Then, the volume (V) of the cone is given as:

$$V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2}$$

The surface area (S) of the cone is given by,

$S = \pi r l$ (where l is the slant height)

$$= \pi r\sqrt{r^2 + h^2}$$
$$= \pi r\sqrt{r^2 + \frac{9V^2}{\pi^2 r^4}}$$
$$= \pi r\sqrt{\frac{\pi^2 r^6 + 9V^2}{\pi^2 r^4}}$$
$$= \frac{\pi r}{\pi r^2}\sqrt{\pi^2 r^6 + 9V^2}$$
$$= \frac{1}{r}\sqrt{\pi^2 r^6 + 9V^2}$$

$$\therefore \frac{dS}{dr} = \frac{r.\frac{1}{2\sqrt{\pi^2 r^6 + 9V^2}}.6\pi^2 r^5 - \sqrt{\pi^2 r^6 + 9V^2}}{r^2}$$
$$= \frac{3\pi^2 r^6 - \pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}} = \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

Now,

$$\frac{dS}{dr} = 0 \Rightarrow 2\pi^2 r^6 = 9V^2 \Rightarrow r^6 = \frac{9V^2}{2\pi^2}$$

Thus, it can be easily verified that when $r^6 = \frac{9V^2}{2\pi^2}, \frac{d^2S}{dr^2} > 0$

$\therefore$ By second derivative test, the surface area of the cone is the least when $r^6 = \frac{9V^2}{2\pi^2}$

When $r^6 = \frac{9V^2}{2\pi^2}$, $h = \frac{3V}{\pi r^2} = \frac{3}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}\right)^{\frac{1}{2}} = \frac{3}{\pi r^2}\cdot\frac{\sqrt{2}\pi r^3}{3} = \sqrt{2}r.$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to $\sqrt{2}$ times the radius of the base.

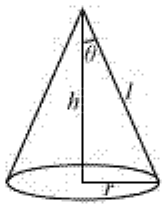
Solution 25

Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left[0, \frac{\pi}{2}\right]$.

Let r, h, and l be the radius, height, and the slant height of the cone respectively.

The slant height of the cone is given as constant.



Now, $r = l \sin \theta$ and $h = l \cos \theta$

The volume (V) of the cone is given by,

$$
\begin{aligned}
V &= \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi\left(l^2 \sin^2\theta\right)\left(l\cos\theta\right) \\
&= \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta
\end{aligned}
$$

$$\therefore \frac{dV}{d\theta} = \frac{l^3\pi}{3}\left[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)\right]$$

$$= \frac{l^3\pi}{3}\left[-\sin^3\theta + 2\sin\theta\cos^2\theta\right]$$

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}\left[-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta\right]$$

$$= \frac{l^3\pi}{3}\left[2\cos^3\theta - 7\sin^2\theta\cos\theta\right]$$

Now, $\frac{dV}{d\theta} = 0$

$\Rightarrow \sin^3\theta = 2\sin\theta\cos^2\theta$

$\Rightarrow \tan^2\theta = 2$

$\Rightarrow \tan\theta = \sqrt{2}$

$\Rightarrow \theta = \tan^{-1}\sqrt{2}$

Now, when $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2$ or $\sin^2\theta = 2\cos^2\theta$.

Then, we have:

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}\left[2\cos^3\theta - 14\cos^3\theta\right] = -4\pi l^3\cos^3\theta < 0 \text{ for } \theta \in \left[0, \frac{\pi}{2}\right]$$

$\therefore$By second derivative test, the volume (V) is the maximum when $\theta = \tan^{-1}\sqrt{2}$.

Hence, for a given slant height, the semi-vertical angle of the cone of the maximum volume is $\tan^{-1}\sqrt{2}$.

Solution 26

Total Surface Area of the cone = S= πr (l+r) .... (1)

[Where r and l are radius and slant height of the cone respectively]

Volume of cone = V= $\frac{1}{3}\pi r^2$ h

$V^2 = \frac{1}{9}\pi^2 r^4 h^2$

$V^2 = \frac{1}{9}\pi^2 r^4(l^2-r^2)$

Using (1), we have

$V^2 = \frac{1}{9}\pi^2 r^4\left(\left(\frac{S}{\pi r} - r\right)^2 - r^2\right)$

Then by solving further we get-

$V^2 = \frac{1}{9}(S(S r^2 - 2\pi r^4))$

P = V^2

differentiating P with respect to r we gwt

$\frac{dP}{dr} = \frac{1}{9}(S(2Sr - 8\pi r^3))$

equate $\frac{dP}{dr}$ =0

S= $4\pi r^2$

Differentiating again with respect to r we find that $\frac{d^2P}{dr^2} < 0$ therefore P is maximum when S= $4\pi r^2$

Again therefore V is maximum when S= $4\pi r^2$

πr(l+r) = $4\pi r^2$

Thus l= 3r

$\sin\theta = \frac{r}{l} = \frac{1}{3}$

$\theta = \sin^{-1}\left(\frac{1}{3}\right)$

Solution 27

The given curve is $x^2 = 2y$.

For each value of x, the position of the point will be $\left(x, \dfrac{x^2}{2}\right)$.

The distance $d(x)$ between the points $\left(x, \dfrac{x^2}{2}\right)$ and $(0, 5)$ is given by,

$$d(x)=\sqrt{(x-0)^2+\left(\frac{x^2}{2}-5\right)^2}=\sqrt{x^2+\frac{x^4}{4}+25-5x^2}=\sqrt{\frac{x^4}{4}-4x^2+25}$$

$$\therefore d'(x)=\frac{\left(x^3-8x\right)}{2\sqrt{\frac{x^4}{4}-4x^2+25}}=\frac{\left(x^3-8x\right)}{\sqrt{x^4-16x^2+100}}$$

$$\begin{aligned}\text{Now, } d'(x)=0 &\Rightarrow x^3-8x=0\\ &\Rightarrow x\left(x^2-8\right)=0\\ &\Rightarrow x=0,\pm 2\sqrt{2}\end{aligned}$$

$$\text{And, } d''(x) = \frac{\sqrt{x^4 - 16x^2 + 100}\left(3x^2 - 8\right) - \left(x^3 - 8x\right) \cdot \frac{4x^3 - 32x}{2\sqrt{x^4 - 16x^2 + 100}}}{\left(x^4 - 16x^2 + 100\right)}$$

$$= \frac{\left(x^4 - 16x^2 + 100\right)\left(3x^2 - 8\right) - 2\left(x^3 - 8x\right)\left(x^3 - 8x\right)}{\left(x^4 - 16x^2 + 100\right)^{\frac{3}{2}}}$$

$$= \frac{\left(x^4 - 16x^2 + 100\right)\left(3x^2 - 8\right) - 2\left(x^3 - 8x\right)^2}{\left(x^4 - 16x^2 + 100\right)^{\frac{3}{2}}}$$

When, $x = 0$, then $d''(x) = \frac{36(-8)}{6^3} < 0$.

When, $x = \pm 2\sqrt{2}$, $d''(x) > 0$.

$\therefore$By second derivative test, $d(x)$ is the minimum at $x = \pm 2\sqrt{2}$.

When $x = \pm 2\sqrt{2}$, $y = \frac{\left(2\sqrt{2}\right)^2}{2} = 4$.

Hence, the point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is $\left(\pm 2\sqrt{2},\ 4\right)$.

The correct answer is A.

Solution 28

Let $f(x)=\frac{1-x+x^2}{1+x+x^2}$.

$$\therefore f'(x)=\frac{\left(1+x+x^2\right)(-1+2x)-\left(1-x+x^2\right)(1+2x)}{\left(1+x+x^2\right)^2}$$

$$=\frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{\left(1+x+x^2\right)^2}$$

$$=\frac{2x^2-2}{\left(1+x+x^2\right)^2}=\frac{2\left(x^2-1\right)}{\left(1+x+x^2\right)^2}$$

$$\therefore f'(x)=0 \Rightarrow x^2=1\Rightarrow x=\pm 1$$

$$\text{Now, } f''(x)=\frac{2\left[\left(1+x+x^2\right)^2(2x)-\left(x^2-1\right)(2)\left(1+x+x^2\right)(1+2x)\right]}{\left(1+x+x^2\right)^4}$$

$$=\frac{4\left(1+x+x^2\right)\left[\left(1+x+x^2\right)x-\left(x^2-1\right)(1+2x)\right]}{\left(1+x+x^2\right)^4}$$

$$=\frac{4\left[x+x^2+x^3-x^2-2x^3+1+2x\right]}{\left(1+x+x^2\right)^3}$$

$$=\frac{4\left(1+3x-x^3\right)}{\left(1+x+x^2\right)^3}$$

$$\text{And, } f''(1)=\frac{4(1+3-1)}{(1+1+1)^3}=\frac{4(3)}{(3)^3}=\frac{4}{9}>0$$

$$\text{Also, } f''(-1)=\frac{4(1-3+1)}{(1-1+1)^3}=4(-1)=-4<0$$

$\therefore$By second derivative test, f is the minimum at $x=1$ and the minimum value is given by $f(1)=\frac{1-1+1}{1+1+1}=\frac{1}{3}$.

The correct answer is D.

Solution 29

Let $f(x)=[x(x-1)+1]^{\frac{1}{3}}$.

$$\therefore f'(x)=\frac{2x-1}{3[x(x-1)+1]^{\frac{2}{3}}}$$

Now, $f'(x)=0 \Rightarrow x=\frac{1}{2}$

Then, we evaluate the value of f at critical point $x=\frac{1}{2}$ and at the end points of the interval [0, 1] {i.e., at $x = 0$ and $x = 1$}.

$$f(0)=[0(0-1)+1]^{\frac{1}{3}}=1$$
$$f(1)=[1(1-1)+1]^{\frac{1}{3}}=1$$
$$f\left(\frac{1}{2}\right)=\left[\frac{1}{2}\left(\frac{-1}{2}\right)+1\right]^{\frac{1}{3}}=\left(\frac{3}{4}\right)^{\frac{1}{3}}$$

Hence, we can conclude that the maximum value of f in the interval [0, 1] is 1.

The correct answer is C.

Chapter 6 - Applications of Derivatives Exercise Misc. Ex.

Solution 1

(a) Consider $y = x^{\frac{1}{4}}$. Let $x = \frac{16}{81}$ and $\Delta x = \frac{1}{81}$.

Then, $\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \left(\frac{16}{81}\right)^{\frac{1}{4}}$$
$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \frac{2}{3}$$

$$\therefore \left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left(\text{as } y = x^{\frac{1}{4}}\right)$$
$$= \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right) = \frac{27}{4\times 8}\times\frac{1}{81} = \frac{1}{32\times 3} = \frac{1}{96} = 0.010$$

Hence, the approximate value of $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ is $\frac{2}{3}+0.010 = 0.667 + 0.010$

$= 0.677.$

(b) Consider $y = x^{-\frac{1}{5}}$. Let $x = 32$ and $\Delta x = 1$.

Then, $\Delta y = (x+\Delta x)^{-\frac{1}{5}} - x^{-\frac{1}{5}} = (33)^{-\frac{1}{5}} - (32)^{-\frac{1}{5}} = (33)^{-\frac{1}{5}} - \frac{1}{2}$

$\therefore (33)^{-\frac{1}{5}} = \frac{1}{2} + \Delta y$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)(\Delta x) = \frac{-1}{5(x)^{\frac{6}{5}}}(\Delta x) \qquad \left(\text{as } y = x^{-\frac{1}{5}}\right)$$

$$= -\frac{1}{5(2)^{6}}(1) = -\frac{1}{320} = -0.003$$

Hence, the approximate value of $(33)^{-\frac{1}{5}}$ is $\frac{1}{2} + (-0.003)$

$= 0.5 - 0.003 = 0.497.$

Solution 2

The given function is $f(x)=\frac{\log x}{x}$.

$$f'(x)=\frac{x\left(\frac{1}{x}\right)-\log x}{x^2}=\frac{1-\log x}{x^2}$$

Now, $f'(x)=0$

$\Rightarrow 1-\log x=0$

$\Rightarrow \log x=1$

$\Rightarrow \log x=\log e$

$\Rightarrow x=e$

$$\text{Now, } f''(x)=\frac{x^2\left(-\frac{1}{x}\right)-(1-\log x)(2x)}{x^4}$$

$$=\frac{-x-2x(1-\log x)}{x^4}$$

$$=\frac{-3+2\log x}{x^3}$$

$$\text{Now, } f''(e)=\frac{-3+2\log e}{e^3}=\frac{-3+2}{e^3}=\frac{-1}{e^3}<0$$

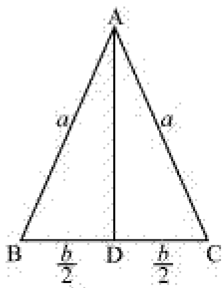
Therefore, by second derivative test, f is the maximum at $x=e$.

Solution 3

Let ΔABC be isosceles where BC is the base of fixed length b.

Let the length of the two equal sides of ΔABC be a.

Draw AD⊥BC.



Now, in ΔADC, by applying the Pythagoras theorem, we have:

$$\text{AD} = \sqrt{a^2 - \frac{b^2}{4}}$$

$$\therefore \text{Area of triangle}\,(A) = \frac{1}{2}b\sqrt{a^2 - \frac{b^2}{4}}$$

The rate of change of the area with respect to time (t) is given by,

$$\frac{dA}{dt} = \frac{1}{2}b \cdot \frac{2a}{2\sqrt{a^2 - \frac{b^2}{4}}}\frac{da}{dt} = \frac{ab}{\sqrt{4a^2 - b^2}}\frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm per second.

$$\therefore \frac{da}{dt} = -3 \text{ cm/s}$$

$$\therefore \frac{dA}{dt} = \frac{-3ab}{\sqrt{4a^2 - b^2}}$$

Then, when $a = b$, we have:

$$\frac{dA}{dt} = \frac{-3b^2}{\sqrt{4b^2 - b^2}} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3}b$$

Hence, if the two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $\sqrt{3}\,b \text{ cm}^2/\text{s}$.

Solution 4

The equation of the given curve is $y^2 = 4x$.

Differentiating with respect to x, we have:

$$2y\frac{dy}{dx} = 4$$

$$\Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

$$\therefore \left.\frac{dy}{dx}\right]_{(1,2)} = \frac{2}{2} = 1$$

Now, the slope of the normal at point (1, 2) is $\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,2)}} = \frac{-1}{1} = -1.$

$\therefore$Equation of the normal at (1, 2) is $y - 2 = -1(x - 1)$.

$$\Rightarrow y - 2 = -x + 1$$

$$\Rightarrow x + y - 3 = 0$$

OR

If we consider the equation of the curve as $x^2 = 4y$

Differentiating with respect to x, we have:

$$2x = 4\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

$$\therefore \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{1}{2}$$

Now, the slope of the normal at point (1, 2) is $\frac{-1}{\left.\frac{dy}{dx}\right|_{(1,2)}} = \frac{-1}{\frac{1}{2}} - 2$

$\therefore$ Equation of the normal at (1, 2) is $y - 2 = -2(x - 1)$

$\Rightarrow y-2=-2x+2$

$\Rightarrow 2x+y-4=0$

Solution 5

We have $x = a\cos\theta + a\theta\sin\theta$.

$$\therefore \frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta = a\theta\cos\theta$$

$$y = a\sin\theta - a\theta\cos\theta$$

$$\therefore \frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta = a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

$\therefore$ Slope of the normal at any point θ is $-\frac{1}{\tan\theta}$.

The equation of the normal at a given point (x, y) is given by,

$$y - a\sin\theta + a\theta\cos\theta = \frac{-1}{\tan\theta}(x - a\cos\theta - a\theta\sin\theta)$$

$$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

$$\Rightarrow x\cos\theta + y\sin\theta - a\left(\sin^2\theta + \cos^2\theta\right) = 0$$

$$\Rightarrow x\cos\theta + y\sin\theta - a = 0$$

Now, the perpendicular distance of the normal from the origin is

$\dfrac{|-a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = \dfrac{|-a|}{\sqrt{1}} = |-a|$, which is independent of θ.

Hence, the perpendicular distance of the normal from the origin is constant.

Solution 6

$$f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$$

$$\therefore f'(x)=\frac{(2+\cos x)(4\cos x-2-\cos x+x\sin x)-(4\sin x-2x-x\cos x)(-\sin x)}{(2+\cos x)^2}$$

$$=\frac{(2+\cos x)(3\cos x-2+x\sin x)+\sin x(4\sin x-2x-x\cos x)}{(2+\cos x)^2}$$

$$=\frac{6\cos x-4+2x\sin x+3\cos^2 x-2\cos x+x\sin x\cos x+4\sin^2 x-2x\sin x-x\sin x\cos x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-4+3\cos^2 x+4\sin^2 x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-4+3\cos^2 x+4-4\cos^2 x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-\cos^2 x}{(2+\cos x)^2}=\frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

N $f'(x)=0$

$\Rightarrow \cos x=0$ or $\cos x=4$

But, $\cos x\neq 4$

$\therefore \cos x=0$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

Now, $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ divides $(0, 2\pi)$ into three disjoint intervals i.e.,

$\left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$.

In intervals $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, $f'(x) > 0$.

Thus, $f(x)$ is increasing for $0 < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$.

In the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $f'(x) < 0$.

Thus, $f(x)$ is decreasing for $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

Solution 7

$f(x) = x^3 + \frac{1}{x^3}$

$\therefore f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$

Then, $f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = 1 \Rightarrow x = \pm 1$

Now, the points $x = 1$ and $x = -1$ divide the real line into three disjoint intervals i.e., $(-\infty, -1), (-1, 1)$, and $(1, \infty)$.

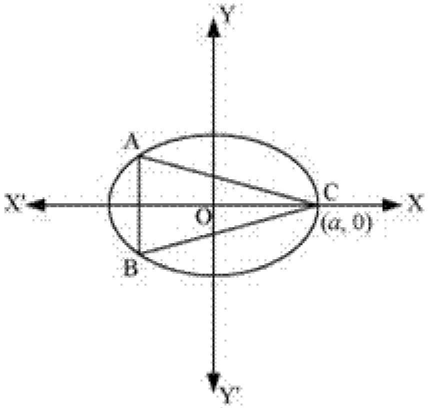
In intervals $(-\infty, -1)$ and $(1, \infty)$ i.e., when $x < -1$ and $x > 1$, $f'(x) > 0$.

Thus, when $x < -1$ and $x > 1$, f is increasing.

In interval $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x) < 0$.

Thus, when $-1 < x < 1$, f is decreasing.

Solution 8



The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let the major axis be along the x −axis.

Let ABC be the triangle inscribed in the ellipse where vertex C is at $(a, 0)$.

Since the ellipse is symmetrical with respect to the x−axis and y −axis, we can assume the coordinates of A to be $(-x_1, y_1)$ and the coordinates of B to be $(-x_1, -y_1)$.

Now, we have $y_1 = \pm\frac{b}{a}\sqrt{a^2-x_1^2}$.

∴Coordinates of A are $\left(-x_1,\ \frac{b}{a}\sqrt{a^2-x_1^2}\right)$ and the coordinates of B are $\left(-x_1,\ -\frac{b}{a}\sqrt{a^2-x_1^2}\right)$.

As the point (x_1, y_1) lies on the ellipse, the area of triangle ABC (A) is given by,

$$A = \frac{1}{2}\left| a\left(\frac{2b}{a}\sqrt{a^2 - x_1^2}\right) + (-x_1)\left(-\frac{b}{a}\sqrt{a^2 - x_1^2}\right) + (-x_1)\left(-\frac{b}{a}\sqrt{a^2 - x_1^2}\right)\right|$$

$$\Rightarrow A = b\sqrt{a^2 - x_1^2} + x_1\frac{b}{a}\sqrt{a^2 - x_1^2} \qquad \ldots(1)$$

$$\therefore \frac{dA}{dx_1} = \frac{-2x_1 b}{2\sqrt{a^2 - x_1^2}} + \frac{b}{a}\sqrt{a^2 - x_1^2} - \frac{2bx_1^2}{a2\sqrt{a^2 - x_1^2}}$$

$$= \frac{b}{a\sqrt{a^2 - x_1^2}}\left[-x_1 a + \left(a^2 - x_1^2\right) - x_1^2\right]$$

$$= \frac{b\left(-2x_1^2 - x_1 a + a^2\right)}{a\sqrt{a^2 - x_1^2}}$$

Now, $\frac{dA}{dx_1} = 0$

$$\Rightarrow -2x_1^2 - x_1 a + a^2 = 0$$

$$\Rightarrow x_1 = \frac{a \pm \sqrt{a^2 - 4(-2)\left(a^2\right)}}{2(-2)}$$

$$= \frac{a \pm \sqrt{9a^2}}{-4}$$

$$= \frac{a \pm 3a}{-4}$$

$$\Rightarrow x_1 = -a, \ \frac{a}{2}$$

But, x_1 cannot be equal to $-a$.

$$\therefore x_1 = \frac{a}{2} \Rightarrow y_1 = \frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}} = \frac{ba}{2a}\sqrt{3} = \frac{\sqrt{3}b}{2}$$

Now, $$\frac{d^2A}{dx_1^2} = \frac{b}{a}\left\{\frac{\sqrt{a^2 - x_1^2}\left(-4x_1 - a\right) - \left(-2x_1^2 - x_1 a + a^2\right)\frac{(-2x_1)}{2\sqrt{a^2 - x_1^2}}}{a^2 - x_1^2}\right\}$$

$$= \frac{b}{a}\left\{\frac{\left(a^2 - x_1^2\right)\left(-4x_1 - a\right) + x_1\left(-2x_1^2 - x_1 a + a^2\right)}{\left(a^2 - x_1^2\right)^{\frac{3}{2}}}\right\}$$

$$= \frac{b}{a}\left\{\frac{2x_1^3 - 3a^2 x_1 - a^3}{\left(a^2 - x_1^2\right)^{\frac{3}{2}}}\right\}$$

Also, when $x_1 = \frac{a}{2}$, then

$$\frac{d^2A}{dx_1^2} = \frac{b}{a}\left\{\frac{2\frac{a^3}{8} - 3\frac{a^3}{2} - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} = \frac{b}{a}\left\{\frac{\frac{a^3}{4} - \frac{3}{2}a^3 - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}$$

$$= -\frac{b}{a}\left\{\frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} < 0$$

Thus, the area is the maximum when $x_1 = \frac{a}{2}$.

$\therefore$ Maximum area of the triangle is given by,

$$A = b\sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}}$$

$$= ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right)\frac{b}{a} \times \frac{a\sqrt{3}}{2}$$

$$= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab$$

Solution 9

Let l, b, and h represent the length, breadth, and height of the tank respectively.

Then, we have height $(h) = 2$ m

Volume of the tank = 8m^3

Volume of the tank = $l \times b \times h$

$\therefore 8 = l \times b \times 2$

$\Rightarrow lb = 4 \Rightarrow b = \dfrac{4}{l}$

Now, area of the base = $lb = 4$

Area of the 4 walls $(A) = 2h\,(l + b)$

$$\therefore A = 4\left(l + \frac{4}{l}\right)$$
$$\Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$$
Now, $\dfrac{dA}{dl} = 0$
$$\Rightarrow 1 - \frac{4}{l^2} = 0$$
$$\Rightarrow l^2 = 4$$

$\Rightarrow l = \pm 2$

However, the length cannot be negative.

Therefore, we have $l = 2$.

$\therefore b = \frac{4}{l} = \frac{4}{2} = 2$

Now, $\frac{d^2A}{dl^2} = \frac{32}{l^3}$

When $l = 2$, $\frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0.$

Thus, by second derivative test, the area is the minimum when $l = 2$.

We have $l = b = h = 2$.

$\therefore$Cost of building the base = Rs 70 × (lb) = Rs 70 (4) = Rs 280

Cost of building the walls = Rs $2h$ $(l + b)$ × 45 = Rs 90 (2) (2 + 2)

= Rs 8 (90) = Rs 720

Required total cost = Rs (280 + 720) = Rs 1000

Hence, the total cost of the tank will be Rs 1000.

Solution 10

Let r be the radius of the circle and a be the side of the square.

Then, we have:

$$2\pi r + 4a = k \text{ (where } k \text{ is constant)}$$
$$\Rightarrow a = \frac{k - 2\pi r}{4}$$

The sum of the areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$
$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k - 2\pi r)}{4}$$

$$\text{Now}, \frac{dA}{dr} = 0$$
$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$
$$8r = k - 2\pi r$$
$$\Rightarrow (8 + 2\pi)r = k$$
$$\Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$
$$\text{Now}, \ \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$
$$\therefore \text{When } r = \frac{k}{2(4 + \pi)}, \ \frac{d^2A}{dr^2} > 0.$$

$\therefore$ The sum of the areas is least when $r = \dfrac{k}{2(4 + \pi)}$.

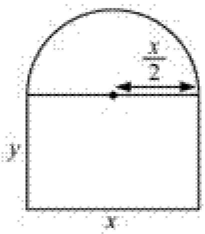
$$\text{When } r = \frac{k}{2(4 + \pi)}, \ a = \frac{k - 2\pi\left[\frac{k}{2(4 + \pi)}\right]}{4} = \frac{k(4 + \pi) - \pi k}{4(4 + \pi)} = \frac{4k}{4(4 + \pi)} = \frac{k}{4 + \pi} = 2r.$$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

Solution 11

Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening $= \frac{x}{2}$



It is given that the perimeter of the window is 10 m.

$$\therefore x + 2y + \frac{\pi x}{2} = 10$$
$$\Rightarrow x\left(1 + \frac{\pi}{2}\right) + 2y = 10$$
$$\Rightarrow 2y = 10 - x\left(1 + \frac{\pi}{2}\right)$$
$$\Rightarrow y = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

$\therefore$Area of the window (A) is given by,

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$
$$= x\left[5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)\right] + \frac{\pi}{8}x^2$$
$$= 5x - x^2\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{8}x^2$$
$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{4}x$$
$$= 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x$$
$$\therefore \frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

Now, $\frac{dA}{dx} = 0$

$$\Rightarrow 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x = 0$$
$$\Rightarrow 5 - x - \frac{\pi}{4}x = 0$$
$$\Rightarrow x\left(1 + \frac{\pi}{4}\right) = 5$$
$$\Rightarrow x = \frac{5}{\left(1 + \frac{\pi}{4}\right)} = \frac{20}{\pi + 4}$$

Thus, when $x = \frac{20}{\pi + 4}$ then $\frac{d^2A}{dx^2} < 0$.

Therefore, by second derivative test, the area is the maximum when length $x = \frac{20}{\pi + 4}$ m.

Now,

$$y = 5 - \frac{20}{\pi + 4}\left(\frac{2 + \pi}{4}\right) = 5 - \frac{5(2 + \pi)}{\pi + 4} = \frac{10}{\pi + 4} \text{ m}$$

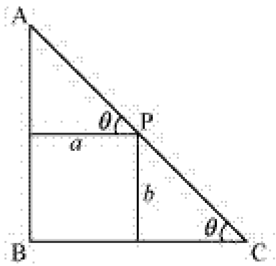
Hence, the required dimensions of the window to admit maximum light is given by length $= \frac{20}{\pi + 4}$ m and breadth $= \frac{10}{\pi + 4}$ m.

Solution 12

Let ΔABC be right-angled at B. Let AB = x and BC = y.

Let P be a point on the hypotenuse of the triangle such that P is at a distance of a and b from the sides AB and BC respectively.

Let $\angle$C = θ.



We have,

$$\text{AC} = \sqrt{x^2 + y^2}$$

Now,

PC = b cosec θ

And, AP = a sec θ

$\therefore$AC = AP + PC

$\Rightarrow$ AC = b cosec $\theta + a$ sec θ … (1)

$$\therefore \frac{d(\text{AC})}{d\theta} = -b\operatorname{cosec}\theta \cot\theta + a\sec\theta\tan\theta$$

$$\therefore \frac{d(\text{AC})}{d\theta} = 0$$

$$\Rightarrow a\sec\theta\tan\theta = b\operatorname{cosec}\theta\cot\theta$$

$$\Rightarrow \frac{a}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{b}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow a\sin^3\theta = b\cos^3\theta$$

$$\Rightarrow (a)^{\frac{1}{3}}\sin\theta = (b)^{\frac{1}{3}}\cos\theta$$

$$\Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\therefore \sin\theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}} \text{ and } \cos\theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}} \qquad \text{...(2)}$$

It can be clearly shown that $\frac{d^2(\text{AC})}{d\theta^2} < 0$ when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Therefore, by second derivative test, the length of the hypotenuse is the maximum when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Now, when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we have:

$$\text{AC} = \frac{b\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{a^{\frac{1}{3}}} \qquad \text{[Using (1) and (2)]}$$

$$= \sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}\left(b^{\frac{2}{3}}+a^{\frac{2}{3}}\right)$$

$$= \left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Hence, the maximum length of the hypotenuses is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Solution 13

The given function is $f(x)=(x-2)^4(x+1)^3$.

$$\therefore f'(x)=4(x-2)^3(x+1)^3+3(x+1)^2(x-2)^4$$
$$=(x-2)^3(x+1)^2\left[4(x+1)+3(x-2)\right]$$
$$=(x-2)^3(x+1)^2(7x-2)$$

Now, $f'(x)=0 \Rightarrow x=-1$ and $x=\frac{2}{7}$ or $x=2$

Now, for values of x close to $\frac{2}{7}$ and to the left of $\frac{2}{7}$, $f'(x)>0$. Also, for values of x close to $\frac{2}{7}$ and to the right of $\frac{2}{7}$, $f'(x)<0$.

Thus, $x=\frac{2}{7}$ is the point of local maxima.

Now, for values of x close to 2 and to the left of 2, $f'(x)<0$. Also, for values of x close to 2 and to the right of 2, $f'(x)>0$.

Thus, $x=2$ is the point of local minima.

Now, as the value of x varies through -1, $f'(x)$ does not changes its sign.

Thus, $x=-1$ is the point of inflexion.

Solution 14

$$f(x)=\cos^2 x+\sin x$$
$$f'(x)=2\cos x(-\sin x)+\cos x$$
$$=-2\sin x\cos x+\cos x$$
Now, $f'(x)=0$
$$\Rightarrow 2\sin x\cos x=\cos x\Rightarrow \cos x(2\sin x-1)=0$$
$$\Rightarrow \sin x=\frac{1}{2} \text{ or } \cos x=0$$
$$\Rightarrow x=\frac{\pi}{6}, \text{ or } \frac{\pi}{2} \text{ as } x\in[0,\pi]$$

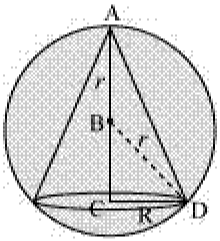
Now, evaluating the value of f at critical points $x=\frac{\pi}{2}$ and $x=\frac{\pi}{6}$ and at the end points of the interval $[0,\pi]$ (i.e., at $x=0$ and $x=\pi$), we have:

$$f\left(\frac{\pi}{6}\right)=\cos^2\frac{\pi}{6}+\sin\frac{\pi}{6}=\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1}{2}=\frac{5}{4}$$
$$f(0)=\cos^2 0+\sin 0=1+0=1$$
$$f(\pi)=\cos^2\pi+\sin\pi=(-1)^2+0=1$$
$$f\left(\frac{\pi}{2}\right)=\cos^2\frac{\pi}{2}+\sin\frac{\pi}{2}=0+1=1$$

Hence, the absolute maximum value of f is $\frac{5}{4}$ occurring at $x=\frac{\pi}{6}$ and the absolute minimum value of f is 1 occurring at $x=0,\frac{\pi}{2}$, and π.

Solution 15

Let R and h be the radius and the height of the cone respectively.



The volume (V) of the cone is given by,

$$V=\frac{1}{3}\pi R^2 h$$

Now, from the right triangle BCD, we have:

$$\text{BC}=\sqrt{r^2-R^2}$$

$$\therefore h=r+\sqrt{r^2-R^2}$$

$$\therefore V=\frac{1}{3}\pi R^2\left(r+\sqrt{r^2-R^2}\right)=\frac{1}{3}\pi R^2 r+\frac{1}{3}\pi R^2\sqrt{r^2-R^2}$$

$$\therefore \frac{dV}{dR}=\frac{2}{3}\pi Rr+\frac{2}{3}\pi R\sqrt{r^2-R^2}+\frac{\pi R^2}{3}\cdot\frac{(-2R)}{2\sqrt{r^2-R^2}}$$

$$=\frac{2}{3}\pi Rr+\frac{2}{3}\pi R\sqrt{r^2-R^2}-\frac{R^3}{3\sqrt{r^2-R^2}}$$
$$=\frac{2}{3}\pi Rr+\frac{2\pi R\left(r^2-R^2\right)-\pi R^3}{3\sqrt{r^2-R^2}}$$
$$=\frac{2}{3}\pi Rr+\frac{2\pi Rr^2-3\pi R^3}{3\sqrt{r^2-R^2}}$$

Now, $\frac{dV}{dR}=0$

$$\Rightarrow \frac{2\pi rR}{3}=\frac{3\pi R^3-2\pi Rr^2}{3\sqrt{r^2-R^2}}$$
$$\Rightarrow 2r\sqrt{r^2-R^2}=3R^2-2r^2$$
$$\Rightarrow 4r^2\left(r^2-R^2\right)=\left(3R^2-2r^2\right)^2$$
$$\Rightarrow 4r^4-4r^2R^2=9R^4+4r^4-12R^2r^2$$
$$\Rightarrow 9R^4-8r^2R^2=0$$
$$\Rightarrow 9R^2=8r^2$$
$$\Rightarrow R^2=\frac{8r^2}{9}$$

Now, $\frac{d^2V}{dR^2}=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)-\left(2\pi Rr^2-3\pi R^3\right)(-6R)\frac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$

$$=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)+\left(2\pi Rr^2-3\pi R^3\right)(3R)\frac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$$

Now, when $R^2=\frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2}<0$.

$\therefore$ The volume is the maximum when $R^2=\frac{8r^2}{9}$.

When $R^2=\frac{8r^2}{9}$, height of the cone $=r+\sqrt{r^2-\frac{8r^2}{9}}=r+\sqrt{\frac{r^2}{9}}=r+\frac{r}{3}=\frac{4r}{3}$.

Hence, it can be seen that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Solution 16

Since $f'(x) > 0$ on (a, b)

$\therefore$ f is a differentiable function (a, b)

Also every differentible function is continuous, therefore f is continous on $[a, b]$

Let $x_1, x_2 \in (a, b)$ and $x_2 > x_1$ then by LMV theorem, there exist $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$\Rightarrow$ $f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$

$\Rightarrow$ $f(x_2) - f(x_1) > 0$ as $x_2 > x_1$ and $f'(x) > 0$

$\Rightarrow$ $f(x_2) > f(x_1)$

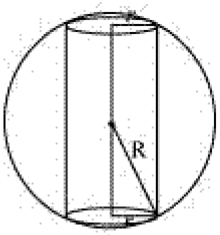
$\therefore$ for $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$\Rightarrow$ f is an increasing funtion.

Solution 17

A sphere of fixed radius *(R)* is given.

Let *r* and *h* be the radius and the height of the cylinder respectively.



From the given figure, we have $h = 2\sqrt{R^2 - r^2}$.

The volume *(V)* of the cylinder is given by,

$$V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$$

$$\therefore \frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r(R^2 - r^2) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

Now, $\frac{dV}{dr} = 0 \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

$$\text{Now, } \frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2}\left(4\pi R^2 - 18\pi r^2\right) - \left(4\pi r R^2 - 6\pi r^3\right)\frac{(-2r)}{2\sqrt{R^2 - r^2}}}{\left(R^2 - r^2\right)}$$

$$= \frac{\left(R^2 - r^2\right)\left(4\pi R^2 - 18\pi r^2\right) + r\left(4\pi r R^2 - 6\pi r^3\right)}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

Now, it can be observed that at $r^2 = \frac{2R^2}{3}, \frac{d^2V}{dr^2} < 0$.

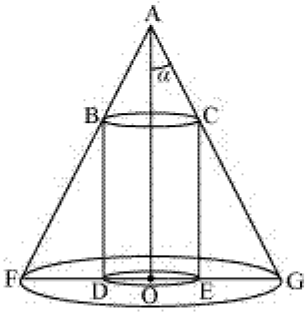
$\therefore$ The volume is the maximum when $r^2 = \frac{2R^2}{3}$.

When $r^2 = \frac{2R^2}{3}$, the height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$.

Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\frac{2R}{\sqrt{3}}$.

Solution 18

The given right circular cone of fixed height (h) and semi-vertical angle (α) can be drawn as:



Here, a cylinder of radius R and height H is inscribed in the cone.

Then, $\angle GAO = \alpha$, $OG = r$, $OA = h$, $OE = R$, and $CE = H$.

We have,

$r = h \tan \alpha$

Now, since ΔAOG is similar to ΔCEG, we have:

$$\frac{AO}{OG} = \frac{CE}{EG}$$
$$\Rightarrow \frac{h}{r} = \frac{H}{r-R} \qquad [EG = OG - OE]$$
$$\Rightarrow H = \frac{h}{r}(r-R) = \frac{h}{h\tan\alpha}(h\tan\alpha - R) = \frac{1}{\tan\alpha}(h\tan\alpha - R)$$

Now, the volume (V) of the cylinder is given by,

$$V = \pi R^2 H = \frac{\pi R^2}{\tan \alpha}\left(h \tan \alpha - R\right) = \pi R^2 h - \frac{\pi R^3}{\tan \alpha}$$

$$\therefore \frac{dV}{dR} = 2\pi R h - \frac{3\pi R^2}{\tan \alpha}$$

Now, $\frac{dV}{dR} = 0$

$$\Rightarrow 2\pi R h = \frac{3\pi R^2}{\tan \alpha}$$

$$\Rightarrow 2h \tan \alpha = 3R$$

$$\Rightarrow R = \frac{2h}{3} \tan \alpha$$

Now, $\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi R}{\tan \alpha}$

And, for $R = \frac{2h}{3} \tan \alpha$, we have:

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan \alpha}\left(\frac{2h}{3} \tan \alpha\right) = 2\pi h - 4\pi h = -2\pi h < 0$$

$\therefore$By second derivative test, the volume of the cylinder is the greatest when $R = \frac{2h}{3} \tan \alpha$.

When $R = \frac{2h}{3} \tan \alpha$, $H = \frac{1}{\tan \alpha}\left(h \tan \alpha - \frac{2h}{3} \tan \alpha\right) = \frac{1}{\tan \alpha}\left(\frac{h \tan \alpha}{3}\right) = \frac{h}{3}$.

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Now, the maximum volume of the cylinder can be obtained as:

$$\pi\left(\frac{2h}{3} \tan \alpha\right)^2\left(\frac{h}{3}\right) = \pi\left(\frac{4h^2}{9} \tan^2 \alpha\right)\left(\frac{h}{3}\right) = \frac{4}{27} \pi h^3 \tan^2 \alpha$$

Hence, the given result is proved.

Solution 19

Then, volume (V) of the cylinder is given by,

$$V = \pi(\text{radius})^2 \times \text{height}$$
$$= \pi(10)^2 h \qquad (\text{radius} = 10 \text{ m})$$
$$= 100\pi h$$

Differentiating with respect to time t, we have:

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

The tank is being filled with wheat at the rate of 314 cubic metres per hour.

$$\therefore \frac{dV}{dt} = 314 \text{ m}^3/\text{h}$$

Thus, we have:

$$314 = 100\pi \frac{dh}{dt}$$
$$\Rightarrow \frac{dh}{dt} = \frac{314}{100(3.14)} = \frac{314}{314} = 1$$

Hence, the depth of wheat is increasing at the rate of 1 m/h.

The correct answer is A.

Solution 20

The given curve is $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$.

$$\therefore \frac{dx}{dt} = 2t + 3 \text{ and } \frac{dy}{dt} = 4t - 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{4t - 2}{2t + 3}$$

The given point is (2, −1).

At $x = 2$, we have:

$$t^2 + 3t - 8 = 2$$
$$\Rightarrow t^2 + 3t - 10 = 0$$
$$\Rightarrow (t - 2)(t + 5) = 0$$
$$\Rightarrow t = 2 \text{ or } t = -5$$

At $y = -1$, we have:

$$2t^2 - 2t - 5 = -1$$
$$\Rightarrow 2t^2 - 2t - 4 = 0$$
$$\Rightarrow 2(t^2 - t - 2) = 0$$
$$\Rightarrow (t - 2)(t + 1) = 0$$
$$\Rightarrow t = 2 \text{ or } t = -1$$

The common value of t is 2.

Hence, the slope of the tangent to the given curve at point (2, −1) is

$$\left.\frac{dy}{dx}\right]_{t=2} = \frac{4(2) - 2}{2(2) + 3} = \frac{8 - 2}{4 + 3} = \frac{6}{7}.$$

The correct answer is B.

Solution 21

The equation of the tangent to the given curve is $y = mx + 1$.

Now, substituting $y = mx + 1$ in $y^2 = 4x$, we get:

$$\Rightarrow (mx+1)^2 = 4x$$
$$\Rightarrow m^2x^2 + 1 + 2mx - 4x = 0$$
$$\Rightarrow m^2x^2 + x(2m-4) + 1 = 0 \qquad \text{...(i)}$$

Since a tangent touches the curve at one point, the roots of equation (i) must be equal.

Therefore, we have:

$$\text{Discriminant} = 0$$
$$(2m-4)^2 - 4(m^2)(1) = 0$$
$$\Rightarrow 4m^2 + 16 - 16m - 4m^2 = 0$$
$$\Rightarrow 16 - 16m = 0$$
$$\Rightarrow m = 1$$

Hence, the required value of m is 1.

The correct answer is A.

Solution 22

The equation of the given curve is $2y + x^2 = 3$.

Differentiating with respect to x, we have:

$$\frac{2dy}{dx} + 2x = 0$$
$$\Rightarrow \frac{dy}{dx} = -x$$
$$\therefore \left.\frac{dy}{dx}\right]_{(1,1)} = -1$$

The slope of the normal to the given curve at point (1, 1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,1)}} = 1.$$

Hence, the equation of the normal to the given curve at (1, 1) is given as:

$$\Rightarrow y - 1 = 1(x - 1)$$
$$\Rightarrow y - 1 = x - 1$$
$$\Rightarrow x - y = 0$$

The correct answer is B.

Solution 23

The equation of the given curve is $x^2 = 4y$.

Differentiating with respect to x, we have:

$$2x = 4 \cdot \frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

The slope of the normal to the given curve at point (h, k) is given by,

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(h,k)}} = -\frac{2}{h}$$

$\therefore$Equation of the normal at point (h, k) is given as:

$$y - k = \frac{-2}{h}(x - h)$$

Now, it is given that the normal passes through the point (1, 2).

Therefore, we have:

$$2 - k = \frac{-2}{h}(1 - h) \text{ or } k = 2 + \frac{2}{h}(1 - h) \qquad \text{... (i)}$$

Since (h, k) lies on the curve $x^2 = 4y$, we have $h^2 = 4k$.

$$\Rightarrow k=\frac{h^2}{4}$$

From equation (i), we have:

$$\frac{h^2}{4}=2+\frac{2}{h}(1-h)$$
$$\Rightarrow \frac{h^3}{4}=2h+2-2h=2$$
$$\Rightarrow h^3=8$$
$$\Rightarrow h=2$$
$$\therefore k=\frac{h^2}{4}\Rightarrow k=1$$

Hence, the equation of the normal is given as:

$$\Rightarrow y-1=\frac{-2}{2}(x-2)$$
$$\Rightarrow y-1=-(x-2)$$
$$\Rightarrow x+y=3$$

The correct answer is A.

Solution 24

The equation of the given curve is $9y^2 = x^3$.

Differentiating with respect to x, we have:

$$9(2y)\frac{dy}{dx} = 3x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

The slope of the normal to the given curve at point (x_1, y_1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(x_1, y_1)}} = -\frac{6y_1}{x_1^2}.$$

$\therefore$ The equation of the normal to the curve at (x_1, y_1) is

$$y - y_1 = \frac{-6y_1}{x_1^2}(x - x_1).$$
$$\Rightarrow x_1^2 y - x_1^2 y_1 = -6xy_1 + 6x_1 y_1$$
$$\Rightarrow 6xy_1 + x_1^2 y = 6x_1 y_1 + x_1^2 y_1$$
$$\Rightarrow \frac{6xy_1}{6x_1 y_1 + x_1^2 y_1} + \frac{x_1^2 y}{6x_1 y_1 + x_1^2 y_1} = 1$$
$$\Rightarrow \frac{x}{\frac{x_1(6 + x_1)}{6}} + \frac{y}{\frac{y_1(6 + x_1)}{x_1}} = 1$$

It is given that the normal makes equal intercepts with the axes.

Therefore, We have:

$$\therefore \frac{x_1(6+x_1)}{6} = \frac{y_1(6+x_1)}{x_1}$$
$$\Rightarrow \frac{x_1}{6} = \frac{y_1}{x_1}$$
$$\Rightarrow x_1^2 = 6y_1 \qquad \text{...(i)}$$

Also, the point (x_1, y_1) lies on the curve, so we have

$$9y_1^2 = x_1^3 \qquad \text{...(ii)}$$

From (i) and (ii), we have:

$$9\left(\frac{x_1^2}{6}\right)^2 = x_1^3 \Rightarrow \frac{x_1^4}{4} = x_1^3 \Rightarrow x_1 = 4$$

From (ii), we have:

$$9y_1^2 = (4)^3 = 64$$
$$\Rightarrow y_1^2 = \frac{64}{9}$$
$$\Rightarrow y_1 = \pm\frac{8}{3}$$

Hence, the required points are $\left(4, \pm\frac{8}{3}\right)$.

The correct answer is A.