NCERT Solutions for Class 12- Maths Chapter 2 - Inverse Trigonometric Functions

Chapter 2 - Inverse Trigonometric Functions Exercise Ex. 2.1

Solution 1

Let $\sin^{-1}(-1/2) = y$. Then $\sin y = -1/2 = -\sin(\pi/6) = \sin(-\pi/6)$.

We know that the range of the principal value branch of $\sin^{-1}$ is $[-\pi/2, \pi/2]$ and $\sin(-\pi/6) = -1/2$. Therefore, the principal value of $\sin^{-1}(-1/2)$ is $-\pi/6$.

Concept Insight:

The principal value of $\sin^{-1}$ is $[-\pi/2, \pi/2]$. This means the first and fourth quadrant in the co-ordinate system. For $\sin^{-1} x = y$, if x is positive then the angle y is in the first quadrant i.e $(0, \pi/2)$ and if x is negative then the angle y is in the fourth quadrant i.e $(-\pi/2, 0)$. Also note that $\sin(-x) = -\sin x$.

In this problem $x = -(1/2)$, so the angle is in the fourth quadrant. Also $\sin(\pi/6) = 1/2$, so the angle is $-\pi/6$.

Solution 2

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$. Then, $\cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

Therefore, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$

Concept Insight:

Note that any inverse trigonometric function gives an angle. Thus $\cos^{-1}x$ is an angle. Once again the principal values are in the interval $[0, \pi]$. For x positive, the angle is always in the first quadrant $[0, \pi/2)$. For x negative, the angle y is in the second quadrant $(\pi/2, \pi]$.

Solution 3

Let $\operatorname{cosec}^{-1}(2) = y$.

"Then, $\operatorname{cosec} y = 2 = \operatorname{cosec}\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

Concept Insight:

Note that the principal range for $\sin^{-1}$ and $\operatorname{cosec}^{-1}$ is similar [$-\pi/2$, $\pi/2$] except the central value '0' as cosec 0 is not defined. So the logic is same as given in the first problem

Solution 4

Let $\tan^{-1}\left(-\sqrt{3}\right) = y$. Then, $\tan y = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right)$ is $-\sqrt{3}$.

Therefore, the principal value of $\tan^{-1}\left(\sqrt{3}\right)$ is $-\frac{\pi}{3}$.

Concept Insight:

The range for $\tan^{-1}$ is same as $\sin^{-1}$ except that it is an open interval, as $\tan(-\pi/2)$ and $\tan(\pi/2)$ are not defined. So the method of finding principal value is same as $\sin^{-1}$ given in the first problem. Also note that $\tan(-x) = -\tan x$.

Solution 5

Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$. Then, $\cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$.

We know that the range of the principal value branch of $\cos^{-1}$ is $[0,\pi]$ and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

Solution 6

Let $\tan^{-1}(-1) = y$. Then, $\tan y = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{4}\right) = -1$.

Therefore, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

Concept Insight:

$\tan(\pi/4) = 1$, so $\tan(-\pi/4) = -1$

Solution 7

Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$. Then, $\sec y = \frac{2}{\sqrt{3}} = \sec\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value of $\sec^{-1}$ is

$[0,\pi] - \left\{\frac{\pi}{2}\right\}$ and $\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$.

Therefore, the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$.

Concept insight:

Note that the principal range for $\cos^{-1}$ and $\sec^{-1}$ is similar $[0, \pi]$ except the central value '$\pi/2$' as $\sec \pi/2$ is not defined. So the logic is same as given in the second problem.

Solution 8

Let $\cot^{-1}\left(\sqrt{3}\right) = y$. Then, $\cot y = \sqrt{3} = \cot\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\cot^{-1}$ is $(0,\pi)$ and $\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$. Therefore, the principal value of $\cot^{-1}\left(\sqrt{3}\right)$ is $\frac{\pi}{6}$.

Concept Insight:

The range for $\cot^{-1}$ is similar to $\cos^{-1}$ except that it is an open interval $(0, \pi)$. So the principal values can be obtained similar to $\cos^{-1}$.

Solution 9

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=y$. Then, $\cos y=-\frac{1}{\sqrt{2}}=-\cos\left(\frac{\pi}{4}\right)=\cos\left(\pi-\frac{\pi}{4}\right)=\cos\left(\frac{3\pi}{4}\right)$.

We know that the range of the principal value branch of $\cos^{-1}$ is $[0,\pi]$ and $\cos\left(\frac{3\pi}{4}\right)=-\frac{1}{\sqrt{2}}$. Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

Concept Insight:

Here x is negative, so the angle in in the second quadrant $(\pi/2, \pi)$.

Solution 10

Let $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)=y$. Then, $\operatorname{cosec} y=-\sqrt{2}=-\operatorname{cosec}\left(\frac{\pi}{4}\right)=\operatorname{cosec}\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ and $\operatorname{cosec}\left(-\frac{\pi}{4}\right)=-\sqrt{2}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ is $-\frac{\pi}{4}$.

Solution 11

Let $\tan^{-1}(1)=x$. Then, $\tan x=1=\tan\frac{\pi}{4}$.

$\therefore \tan^{-1}(1)=\frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right)=y$. Then, $\cos y=-\frac{1}{2}=-\cos\left(\frac{\pi}{3}\right)=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right)$.

$\therefore \cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right)=z$. Then, $\sin z=-\frac{1}{2}=-\sin\left(\frac{\pi}{6}\right)=\sin\left(-\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

$\therefore \tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

$=\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}$

$=\frac{3\pi+8\pi-2\pi}{12}=\frac{9\pi}{12}=\frac{3\pi}{4}$

Solution 12

Let $\cos^{-1}\left(\frac{1}{2}\right)=x$. Then, $\cos x=\frac{1}{2}=\cos\left(\frac{\pi}{3}\right)$.

$\therefore \cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$

Let $\sin^{-1}\left(\frac{1}{2}\right)=y$. Then, $\sin y=\frac{1}{2}=\sin\left(\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

$\therefore \cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+\frac{2\pi}{6}=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$

Solution 13

It is given that $\sin^{-1} x = y$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Therefore, $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$.

Solution 14

Let $\tan^{-1}\sqrt{3}=x$. Then, $\tan x=\sqrt{3}=\tan\frac{\pi}{3}$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3}=\frac{\pi}{3}$

Let $\sec^{-1}(-2)=y$. Then, $\sec y=-2=-\sec\left(\frac{\pi}{3}\right)=\sec\left(\pi-\frac{\pi}{3}\right)=\sec\frac{2\pi}{3}$.

We know that the range of the principal value branch of $\sec^{-1}$ is $[0,\pi]-\left\{\frac{\pi}{2}\right\}$.

$\therefore \sec^{-1}(-2)=\frac{2\pi}{3}$

Hence, $\tan^{-1}\left(\sqrt{3}\right)-\sec^{-1}(-2)=\frac{\pi}{3}-\frac{2\pi}{3}=-\frac{\pi}{3}$

Concept Insight:

Some points which help to find the principal values of inverse trigonometric functions.

1. The images of inverse trigonometric functions are angles. The principal ranges are either [-π/2, π/2] or [0, π] with minor variations2

2. It is sufficient to know sin, cos and tan of angles in the first quadrant i.e [0, π/2].
For example the students should know the trigonometric ratios of 0, π/6, π/4, π/3, π/2.

3. For x > 0, the angle y is always in the first quadrant i.e (0, π/2) for <u>all</u> inverse trigonometric functions.

4. For x < 0,
$\sin^{-1}$ and $\tan^{-1}$ gives angle in the interval (-π/2, 0) and this can be easily obtained using $\sin^{-1}(-x) = -\sin^{-1} x$ and $\tan^{-1}(-x) = -\tan^{-1} x$. for example $\sin^{-1}(-1/2) = -\sin^{-1}(1/2) = -\pi/6$
$\cos^{-1}$ and $\cot^{-1}$ gives angle in the interval (π/2, π). for example $\cos^{-1}(-1/2) = \pi - \cos^{-1}(1/2) = \pi - \pi/3 = 2\pi/3$

Chapter 2 - Inverse Trigonometric Functions Exercise Ex. 2.2

Solution 1

To prove: $3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Let $x = \sin\theta$. Then, $\sin^{-1} x = \theta$.

We have,

$\text{R.H.S.} = \sin^{-1}\left(3x - 4x^3\right) = \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right)$

$= \sin^{-1}\left(\sin 3\theta\right)$

$= 3\theta$

$= 3\sin^{-1} x$

$= \text{L.H.S.}$

Concept Insight:

This problem is based on the formula of sin3θ.

Solution 2

To prove: $3\cos^{-1}x = \cos^{-1}\left(4x^3 - 3x\right),\ x \in \left[\frac{1}{2},\ 1\right]$

Let $x = \cos\theta$. Then, $\cos^{-1}x = \theta$.

We have,

$$\begin{aligned}
\text{R.H.S.} &= \cos^{-1}\left(4x^3 - 3x\right) \\
&= \cos^{-1}\left(4\cos^3\theta - 3\cos\theta\right) \\
&= \cos^{-1}\left(\cos 3\theta\right) \\
&= 3\theta \\
&= 3\cos^{-1}x \\
&= \text{L.H.S.}
\end{aligned}$$

Concept Insight:

This problem is based on the formula of $\cos 3\theta$.

Solution 3

To prove: $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

$$\begin{aligned}
\text{L.H.S.} &= \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} \\
&= \tan^{-1}\frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11}\cdot\frac{7}{24}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right] \\
&= \tan^{-1}\frac{\frac{48+77}{11\times 24}}{\frac{11\times 24 - 14}{11\times 24}} \\
&= \tan^{-1}\frac{48+77}{264-14} = \tan^{-1}\frac{125}{250} = \tan^{-1}\frac{1}{2} = \text{R.H.S.}
\end{aligned}$$

Solution 4

To prove: $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

$$\text{L.H.S.}=2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{2\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}+\tan^{-1}\frac{1}{7}\qquad\left[2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$=\tan^{-1}\frac{1}{\left(\frac{3}{4}\right)}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}}\qquad\left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$=\tan^{-1}\frac{\left(\frac{28+3}{21}\right)}{\left(\frac{21-4}{21}\right)}$$

$$=\tan^{-1}\frac{31}{17}=\text{R.H.S.}$$

Solution 5

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$$

Put $x=\tan\theta\Rightarrow\theta=\tan^{-1}x$

$$\therefore\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$$

Concept Insight:

This substitution is based on the fundamental identity $1+\tan^2\theta=\sec^2\theta$.
Thus for $1+x^2$, we substitute $x=\tan\theta$ or $x=\cot\theta$.

Solution 6

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}},\ |x|>1$$

Put $x = \operatorname{cosec}\theta \Rightarrow \theta = \operatorname{cosec}^{-1} x$

$$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}} = \tan^{-1}\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$$

$$= \tan^{-1}\left(\frac{1}{\cot\theta}\right) = \tan^{-1}(\tan\theta)$$

$$= \theta = \operatorname{cosec}^{-1}x = \frac{\pi}{2} - \sec^{-1}x \qquad \left[\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}\right]$$

Concept Insight:

This substitution is based on the fundamental identity $\sec^2\theta - 1 = \tan^2\theta$.

Thus for $x^2 - 1$, we substitute $x = \sec\theta$ or $x = \operatorname{cosec}\theta$.

Solution 7

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),\ x<\pi$$

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right) = \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right) = \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

Concept Insight:

This problem is based on the trigonometric formulae.

Given $\tan^{-1}$, so we express the given function in terms of tan using trigonometric formulae.

Solution 8

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$$

$$=\tan^{-1}\left(\frac{1-\dfrac{\sin x}{\cos x}}{1+\dfrac{\sin x}{\cos x}}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$=\tan^{-1}(1)-\tan^{-1}(\tan x) \qquad \left[\tan^{-1}\frac{x-y}{1-xy}=\tan^{-1}x-\tan^{-1}y\right]$$

$$=\frac{\pi}{4}-x$$

Note: $\tan^{-1}\left(\frac{x-y}{1+xy}\right)=\tan^{-1}x-\tan^{-1}y$

Concept Insight:

This problem is based on the trigonometric formulae.

Given $\tan^{-1}$, so we express the given function in terms of tan using trigonometric formulae.

Solution 9

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$$

Put $x=a\sin\theta \Rightarrow \frac{x}{a}=\sin\theta \Rightarrow \theta=\sin^{-1}\left(\frac{x}{a}\right)$

$$\therefore \tan^{-1}\frac{x}{\sqrt{a^2-x^2}}=\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)=\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

$$=\tan^{-1}(\tan\theta)=\theta=\sin^{-1}\frac{x}{a}$$

Concept Insight:

This substitution is based on the fundamental identity $1-\sin^2\theta=\cos^2\theta$.

Thus for a^2-x^2, we substitute $x=a\sin\theta$ or $x=a\cos\theta$.

Solution 10

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$$

Put $x = a\tan\theta \Rightarrow \frac{x}{a} = \tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = \tan^{-1}\left(\frac{3a^2\cdot a\tan\theta - a^3\tan^3\theta}{a^3-3a\cdot a^2\tan^2\theta}\right)$$

$$= \tan^{-1}\left(\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right)$$

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta$$

$$= 3\tan^{-1}\frac{x}{a}$$

Concept Insight:

This problem is based on the formula of tan3θ.

Solution 11

$$\therefore \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

$$\therefore \tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right] = \tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[2\cos\frac{\pi}{3}\right] = \tan^{-1}\left[2\times\frac{1}{2}\right]$$

$$= \tan^{-1}1 = \frac{\pi}{4}$$

Concept Insight:

Solve the innermost bracket first, so first find the principal value of $\sin^{-1}(1/2)$

Solution 12

$$\cot\left(\tan^{-1}a + \cot^{-1}a\right)$$

$$= \cot\left(\frac{\pi}{2}\right) \qquad \left[\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$= 0$$

Solution 13

Let $x = \tan\theta$. Then, $\theta = \tan^{-1} x$.

$$\therefore \sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1} x$$

Let $y = \tan\Phi$. Then, $\Phi = \tan^{-1} y$.

$$\therefore \cos^{-1}\frac{1-y^2}{1+y^2} = \cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right) = \cos^{-1}(\cos 2\phi) = 2\phi = 2\tan^{-1} y$$

$$\therefore \tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$= \tan\frac{1}{2}\left[2\tan^{-1} x + 2\tan^{-1} y\right]$$

$$= \tan\left[\tan^{-1} x + \tan^{-1} y\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \frac{x+y}{1-xy}$$

Concept Insight:

This problem is based on the formula of $\sin 2\theta$ and $\cos 2\theta$ in terms of $\tan\theta$.

Solution 14

$\sin(\sin^{-1}(1/5) + \cos^{-1} x) = 1$

$\therefore \sin^{-1}(1/5) + \cos^{-1} x = \sin^{-1} 1$

$\sin^{-1}(1/5) + \cos^{-1} x = \pi/2$

$\sin^{-1}(1/5) = \pi/2 - \cos^{-1} x$

$\sin^{-1}(1/5) = \sin^{-1} x$

$\therefore x = 1/5$

We use the results: $\sin^{-1} 1 = \pi/2$ and $\sin^{-1} x + \cos^{-1} x = \pi/2$

Concept Insight:

As R.H.S is '1', it is easier to take $\sin^{-1} 1 = \pi/2$

Solution 15

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4} \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+2)(x-2)-(x-1)(x+1)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan\left[\tan^{-1}\frac{4-2x^2}{3}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{4-2x^2}{3}=1$$

$$\Rightarrow 4-2x^2=3$$

$$\Rightarrow 2x^2=4-3=1$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

Hence, the value of x is $\pm\frac{1}{\sqrt{2}}$.

Solution 16

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

We know that $\sin^{-1}(\sin x) = x$ if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1}x$.

Here, $\frac{2\pi}{3} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ can be written as:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{3}\right)\right] = \sin^{-1}\left(\sin\frac{\pi}{3}\right) \text{ where } \frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Concept Insight:

For problems of the form $\sin^{-1}(\sin x)$, always check whether the angle is in the principal range. If not use allied angle formulae to find the equivalent angle having same trigonometric ratio. This angle must be in the principal range $[-\pi/2, \pi/2]$.

Solution 17

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$$

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{3\pi}{4} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\tan^{-1}(\tan(3\pi/4)) = \tan^{-1}(\tan(\pi - \pi/4))$

$= \tan^{-1}(-\tan(\pi/4))$ as $\tan(\pi-x) = -\tan x$

$= -\tan^{-1}(\tan(\pi/4))$ as $\tan^{-1}(-x) = -\tan^{-1}x$

$= -\pi/4$

Concept Insight:

For problems of the form $\tan^{-1}(\tan x)$, always check whether the angle is in the principal range. If not use allied angle formulae to find the equivalent angle having same trigonometric ratio. This angle must be in the principal range $(-\pi/2, \pi/2)$.

Solution 18

Let $\sin^{-1}\frac{3}{5}=x$.

Then, $\sin x=\frac{3}{5}\Rightarrow \cos x=\sqrt{1-\sin^2 x}=\frac{4}{5}\Rightarrow \sec x=\frac{5}{4}$.

$\therefore \tan x=\sqrt{\sec^2 x-1}=\sqrt{\frac{25}{16}-1}=\frac{3}{4}$

$\therefore x=\tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$ ...(i)

Now, $\cot^{-1}\frac{3}{2}=\tan^{-1}\frac{2}{3}$ ...(ii) $\left[\tan^{-1}\frac{1}{x}=\cot^{-1}x\right]$

Hence, $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

$=\tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$ [Using (i) and (ii)]

$=\tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)$ $\left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$

$=\tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$

$=\tan\left(\tan^{-1}\frac{17}{6}\right)=\frac{17}{6}$

Alternatively:

Let $\sin^{-1}(3/5)=y$ so $\sin y=3/5$ and $y \in (0,\pi/2)$ so all ratios of y are positive.

Hence $\cos y=4/5$ and $\tan y=3/4$

So $\tan^{-1}(3/4)=y$

Also $\cot^{-1}(3/2)=\tan^{-1}(2/3)$ as $\cot^{-1}x=\tan^{-1}(1/x)$

$\tan(\sin^{-1}(3/5)+\cot^{-1}(3/2))=\tan(\tan^{-1}(3/4)+\tan^{-1}(2/3))$

$$=\tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)=\tan\left(\tan^{-1}\left(\frac{17}{6}\right)\right)=\frac{17}{6}$$

Concept Insight:

1.Every angle θ can be expressed in 6 ways, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, $\sec^{-1}$, cosec^{-1}, $\cot^{-1}$.
For example, $\pi/6=\sin^{-1}(1/2)=\cos^{-1}(\sqrt{3}/2)=\tan^{-1}(1/\sqrt{3})=\sec^{-1}(2/\sqrt{3})=\mathrm{cosec}^{-1}(2)=$
$=\cot^{-1}(\sqrt{3})$

2. in the above example we need tan so we express both angles as $\tan^{-1}$ and then use the formula.

Solution 19

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{7\pi}{6} \notin x \in [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as:

$$\begin{aligned}\cos^{-1}\cos(7\pi/6) &= \cos^{-1}\cos(\pi + \pi/6)\\ &= \cos^{-1}(-\cos(\pi/6)) \quad \text{as } \cos(\pi + x) = -\cos x\\ &= \pi - \cos^{-1}\cos(\pi/6) \quad \text{as } \cos^{-1}(-x) = \pi - \cos^{-1} x\\ &= \pi - \pi/6 = 5\pi/6\end{aligned}$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$$

The correct answer is B.

Concept insight:

For problems of the form $\tan^{-1}(\tan x)$, always check whether the angle is in the principal range. If not use allied angle formulae to find the equivalent angle having same trigonometric ratio. This angle must be in the principal range $(-\pi/2, \pi/2)$.

Solution 20

$$\begin{aligned}\sin^{-1}(-1/2) &= -\sin^{-1}(1/2) \text{ as } \sin^{-1}(-x) = -\sin^{-1} x\\ &= -\pi/6 \text{ as } \mathrm{Sin}(\pi/6) = ½\end{aligned}$$

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$$

$$\therefore \sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

The correct answer is D.

Solution 21

$$\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})$$
$$=\tan^{-1}\sqrt{3}-\left[\pi-\cot^{-1}\sqrt{3}\right]$$
$$=\tan^{-1}\sqrt{3}+\cot^{-1}\sqrt{3}-\pi$$
$$=\pi/2-\pi$$
$$=-(\pi/2)$$

Using results : $\cot^{-1}(-x) = \pi - \cot^{-1} x$ and $\tan^{-1} x + \cot^{-1} x = \pi/2$

The correct answer is B.

Concept Insight:

Use the result $\tan^{-1} x + \cot^{-1} x = \pi/2$.

Chapter 2 - Inverse Trigonometric Functions Exercise Misc. Ex.

Solution 1

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{13\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right)=\cos^{-1}\left[\cos\left(2\pi+\frac{\pi}{6}\right)\right]=\cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right], \text{ where } \frac{\pi}{6}\in[0,\pi].$$

$$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right)=\cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right]=\frac{\pi}{6}$$

Concept insight:

For problems of the form $\cos^{-1}(\cos x)$, always check whether the angle is in the principal range. If not use allied angle formulae to find the equivalent angle having same trigonometric ratio. This angle must be in the principal range $[0, \pi]$.

Solution 2

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{7\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\tan^{-1}\tan(7\pi/6) = \tan^{-1}\tan(\pi + \pi/6)$

$= \tan^{-1}(\tan \pi/6)$ as $\tan(\pi + x) = \tan x$

$= \pi/6 \in (-\pi/2, \pi/2)$

Concept insight:

For problems of the form $\tan^{-1}(\tan x)$, always check whether the angle is in the principal range. If not use allied angle formulae to find the equivalent angle having same trigonometric ratio. This angle must be in the principal range $(-\pi/2, \pi/2)$.

Solution 3

Let $\sin^{-1}\frac{3}{5} = x$. Then, $\sin x = \frac{3}{5}$.

$\Rightarrow \cos x = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}$

$\therefore \tan x = \frac{3}{4}$

$\therefore x = \tan^{-1}\frac{3}{4} \Rightarrow \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}$

Now, we have:

$$\text{L.H.S.} = 2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4}$$

$$= \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right) \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{16-9}{16}}\right) = \tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)$$

$$= \tan^{-1}\frac{24}{7} = \text{R.H.S.}$$

Concept insight:

The R.H.S is $\tan^{-1}$ so we express L.H.S in $\tan^{-1}$

Solution 4

Let $\sin^{-1}\frac{8}{17}=x$. Then, $\sin x=\frac{8}{17}\Rightarrow \cos x=\sqrt{1-\left(\frac{8}{17}\right)^2}=\sqrt{\frac{225}{289}}=\frac{15}{17}$.

$\therefore \tan x=\frac{8}{15}\Rightarrow x=\tan^{-1}\frac{8}{15}$

$\therefore \sin^{-1}\frac{8}{17}=\tan^{-1}\frac{8}{15}$ ...(1)

Now, let $\sin^{-1}\frac{3}{5}=y$. Then, $\sin y=\frac{3}{5}\Rightarrow \cos y=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$.

$\therefore \tan y=\frac{3}{4}\Rightarrow y=\tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$ ...(2)

Now, we have:

$$\text{L.H.S.}=\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}$$

$$=\tan^{-1}\frac{8}{15}+\tan^{-1}\frac{3}{4} \qquad [\text{Using (1) and (2)}]$$

$$=\tan^{-1}\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}$$

$$=\tan^{-1}\left(\frac{32+45}{60-24}\right) \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$=\tan^{-1}\frac{77}{36}=\text{R.H.S.}$$

Concept insight:

The R.H.S is $\tan^{-1}$ so we express L.H.S in $\tan^{-1}$ and then use the formula.

Solution 5

Let $\cos^{-1}\frac{4}{5} = x$. Then, $\cos x = \frac{4}{5} \Rightarrow \sin x = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$.

$\therefore \tan x = \frac{3}{4} \Rightarrow x = \tan^{-1}\frac{3}{4}$

$\therefore \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}$ ...(1)

Now, let $\cos^{-1}\frac{12}{13} = y$. Then, $\cos y = \frac{12}{13} \Rightarrow \sin y = \frac{5}{13}$.

$\therefore \tan y = \frac{5}{12} \Rightarrow y = \tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13} = \tan^{-1}\frac{5}{12}$ ...(2)

Let $\cos^{-1}\frac{33}{65} = z$. Then, $\cos z = \frac{33}{65} \Rightarrow \sin z = \frac{56}{65}$.

$\therefore \tan z = \frac{56}{33} \Rightarrow z = \tan^{-1}\frac{56}{33}$

$\therefore \cos^{-1}\frac{33}{65} = \tan^{-1}\frac{56}{33}$ ...(3)

Now, we will prove that:

$$\text{L.H.S.} = \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13}$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{5}{12} \qquad [\text{Using (1) and (2)}]$$

$$= \tan^{-1}\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}.\frac{5}{12}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{36+20}{48-15}$$

$$= \tan^{-1}\frac{56}{33}$$

$$= \cos^{-1}\frac{33}{65} \qquad \text{by (3)}$$

$$= \text{RHS}$$

Alternate method:

Let $\cos^{-1}(4/5) = A$ and $\cos^{-1}(12/13) = B$

So $\cos A = 4/5$ and $\cos B = 12/13$

Hence $\sin A = 3/5$ and $\sin B = 5/13$

As R.H.S is $\cos^{-1}$ we use $\cos(A+B)$.

$\cos(A+B) = \cos A \cos B - \sin A \sin B = (4/5)(12/13) - (3/5)(5/13)$

$= 48/65 - 15/65 = 33/65$

Thus $A+B = \cos^{-1}(33/65)$. hence proved.

Concept insight:

If R.H.S is $\cos^{-1}$ or $\sin^{-1}$ then use $\mathrm{Cos}(A+B)$ or $\sin(A+B)$ as the case maybe.

Solution 6

Let $\sin^{-1}(3/5) = A$ and $\cos^{-1}(12/13) = B$

So $\sin A = 3/5$ and $\cos B = 12/13$

Hence $\cos A = 4/5$ and $\sin B = 5/13$

As R.H.S is $\sin^{-1}$ we use $\sin(A + B)$

$\mathrm{Sin}(A + B) = \sin A \cos B + \cos A \sin B = (3/5)(12/13) + (4/5)(5/13)$

$= 36/65 + 20/65 = 56/65$

Thus $A + B = \sin^{-1}(56/65)$ hence proved.

Concept insight:

If R.H.S is cos-1 or $\sin^{-1}$ then use $\mathrm{Cos}(A + B)$ or $\sin(A + B)$ as the case may be.

Solution 7

Let $\sin^{-1}\frac{5}{13}=x$. Then, $\sin x=\frac{5}{13}\Rightarrow\cos x=\frac{12}{13}$.

$\therefore \tan x=\frac{5}{12}\Rightarrow x=\tan^{-1}\frac{5}{12}$

$\therefore \sin^{-1}\frac{5}{13}=\tan^{-1}\frac{5}{12}$ ...(1)

Let $\cos^{-1}\frac{3}{5}=y$. Then, $\cos y=\frac{3}{5}\Rightarrow\sin y=\frac{4}{5}$.

$\therefore \tan y=\frac{4}{3}\Rightarrow y=\tan^{-1}\frac{4}{3}$

$\therefore \cos^{-1}\frac{3}{5}=\tan^{-1}\frac{4}{3}$ ...(2)

Using (1) and (2), we have

$$\begin{aligned}
\text{R.H.S.} &= \sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}\\
&= \tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}\\
&= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right) \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]\\
&= \tan^{-1}\left(\frac{15+48}{36-20}\right)\\
&= \tan^{-1}\frac{63}{16}\\
&= \text{L.H.S.}
\end{aligned}$$

Concept Insight:

As L.H.S is $\tan^{-1}$ express the terms in R.H.S in the form of $\tan^{-1}$

Solution 8

$$\text{L.H.S.} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{7+5}{35-1}\right) + \tan^{-1}\left(\frac{8+3}{24-1}\right)$$

$$= \tan^{-1}\frac{12}{34} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

$$= \tan^{-1}\left(\frac{138+187}{391-66}\right)$$

$$= \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}1$$

$$= \frac{\pi}{4} = \text{R.H.S.}$$

Concept insight:

As L.H.S is $\tan^{-1}$, we express R.H.S in $\tan^{-1}$

Solution 9

Let $x = \tan^2\theta$. Then, $\sqrt{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$.

$$\therefore \frac{1-x}{1+x} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

Now, we have:

$$\text{R.H.S.} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta) = \frac{1}{2}\times 2\theta = \theta = \tan^{-1}\sqrt{x} = \text{L.H.S.}$$

Concept Insight:

Use the formula for $\cos 2\theta$ on the R.H.S, hence put $x = \tan^2\theta$.

Solution 10

Consider $\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2} \qquad \text{(by rationalizing)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x-1+\sin x}$$

$$=\frac{2\left(1+\sqrt{1-\sin^2 x}\right)}{2\sin x}=\frac{1+\cos x}{\sin x}=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

$$\therefore \text{L.H.S.}=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}=\text{R.H.S.}$$

Concept Insight:

Simplify the trigonometric expression. As there is no direct formula for 1 – sinx, we rationalize the denominator.

Solution 11

Put $x=\cos 2\theta$ so that $\theta=\dfrac{1}{2}\cos^{-1}x$. Then, we have:

$$\text{L.H.S.}=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$=\tan^{-1}1-\tan^{-1}(\tan\theta) \qquad \left[\tan^{-1}\left(\frac{x-y}{1+xy}\right)=\tan^{-1}x-\tan^{-1}y\right]$$

$$=\frac{\pi}{4}-\theta=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x=\text{R.H.S.}$$

Concept Insight:

Use trigonometric formulae to simplify the given expression.

Solution 12

$$\text{L.H.S.} = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$

$$= \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right)$$

$$= \frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right) \qquad . \qquad ....(1) \quad \left[\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

Now, let $\cos^{-1}\frac{1}{3} = x$. Then, $\cos x = \frac{1}{3} \Rightarrow \sin x = \sqrt{1-\left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$.

$$\therefore x = \sin^{-1}\frac{2\sqrt{2}}{3} \Rightarrow \cos^{-1}\frac{1}{3} = \sin^{-1}\frac{2\sqrt{2}}{3}$$

$$\therefore \text{L.H.S.} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3} = \text{R.H.S.}$$

Solution 13

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x) \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow \frac{2\cos x}{1-\cos^2 x} = 2\operatorname{cosec} x$$

$$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\therefore x = \frac{\pi}{4}$$

Concept insight:

If $\tan^{-1}x = \tan^{-1}y$, then $x = y$.

Solution 14

$$\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x \qquad \left[\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow x=\tan\frac{\pi}{6}$$

$$\therefore x=\frac{1}{\sqrt{3}}$$

Solution 15

Let $\tan^{-1}x=y$.

Then, $\tan y=x\Rightarrow \sin y=\frac{x}{\sqrt{1+x^2}}$.

$$\therefore y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\Rightarrow \tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$\therefore \sin\left(\tan^{-1}x\right)=\sin\left(\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)=\frac{x}{\sqrt{1+x^2}}$$

The correct answer is D.

Concept insight:

As $\sin^{-1}(\sin x)=x$, we express $\tan^{-1}$ in terms of $\sin^{-1}$.

Solution 16

Given : $\sin^{-1}(1-x) - 2\sin^{-1}x = \pi/2$

We put x = siny in the equation,

$\sin^{-1}(1-\sin y) - 2\sin^{-1}\sin y = \pi/2$

$\sin^{-1}(1-\sin y) - 2y = \pi/2$

$\sin^{-1}(1-\sin y) = \pi/2 + 2y$

$1-\sin y = \sin(\pi/2 + 2y)$

$1-\sin y = \cos 2y$ using $\sin(\pi/2 + x) = \cos x$

$1-\cos 2y = \sin y$

$2\sin^2 y = \sin y$

$2\sin^2 y - \sin y = 0$

$\sin y(2\sin y - 1) = 0$

$\sin y = 0$ or ½

x = 0 or ½

But, when $x = \frac{1}{2}$, it can be observed that:

$$\text{L.H.S.} = \sin^{-1}\left(1-\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2}$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2}$$

$$= -\sin^{-1}\frac{1}{2}$$

$$= -\frac{\pi}{6} \neq \frac{\pi}{2} \neq \text{R.H.S.}$$

$\therefore x = \frac{1}{2}$ is not the **Solution** of the given equation.

Thus, $x = 0$.

Hence, the correct answer is **C**.

Concept Insight:

To solve the equation, we remove inverse function, so we first put x = siny.

We then solve the trigonometric equation.

Solution 17

$$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\frac{x-y}{x+y}$$

$$=\tan^{-1}\left[\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right] \qquad \left[\tan^{-1}y-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}\right]$$

$$=\tan^{-1}\left[\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right]$$

$$=\tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right)$$

$$=\tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)=\tan^{-1}1=\frac{\pi}{4}$$

Hence, the correct answer is **C**.

Concept Insight:

As both are $\tan^{-1}$, the formula for $\tan^{-1}x - \tan^{-1}y$ is used.