RD SHARMA Solutions for Class 12-science Maths Chapter 24 - Scalar or Dot Product

Chapter 24 - Scalar or Dot Product Exercise Ex. 24.1

Question 1

Find $\vec{a}.\vec{b}$, when

$(i)\ \vec{a}=\hat{i}-2\hat{j}+\hat{k}\ and\ \vec{b}=4\hat{i}-4\hat{j}+7\hat{k}$

$(ii)\ \vec{a}=\hat{i}+2\hat{j}\ and\ \vec{b}=2\hat{i}+\hat{k}$

$(iii)\ \vec{a}=\hat{j}-\hat{k}\ and\ \vec{b}=2\hat{i}+3\hat{j}-2\hat{k}$

Solution 1

(i)

$$\begin{aligned}
&\vec{a}\,.\,\vec{b}\\
&=\left(\hat{i}-2\hat{j}+\hat{k}\right).\left(4\hat{i}-4\hat{j}+7\hat{k}\right)\\
&=(1)(4)+(-2).(-4)+(1)(7)\\
&=4+8+7\\
&=19
\end{aligned}$$

$$\vec{a}\,.\,\vec{b}=19$$

(ii)

$$\begin{aligned}
\vec{a}\,.\,\vec{b}&=\left(\hat{j}+2\hat{k}\right).\left(2\hat{i}+\hat{k}\right)\\
&=\left(0\times\hat{i}+\hat{j}+2\hat{k}\right)\left(2\hat{i}+0\times\hat{j}+\hat{k}\right)\\
&=(0)(2)+(1)(0)+(2)(1)\\
&=0+0+2
\end{aligned}$$

$$\vec{a}\,.\,\vec{b}=2$$

(iii)

$$\begin{aligned}
\vec{a}\,.\,\vec{b}&=\left(\hat{j}-\hat{k}\right).\left(2\hat{i}+3\hat{j}-2\hat{k}\right)\\
&=\left(0\times\hat{i}+\hat{j}-\hat{k}\right)\left(2\hat{i}+3\hat{j}-2\hat{k}\right)\\
&=(0)(2)+(1)(3)+(-1)(-2)\\
&=0+3+2
\end{aligned}$$

$$\vec{a}\,.\,\vec{b}=5$$

Question 2

For what value of λ are the vector $\vec{a}$ and $\vec{b}$ perpendicular to each other?

(i) $\vec{a} = \lambda\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 4\hat{i} - 9\hat{j} + 2\hat{k}$

(ii) $\vec{a} = \lambda\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 5\hat{i} - 9\hat{j} + 2\hat{k}$

(iii) $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \lambda\hat{k}$

(iv) $\vec{a} = \lambda\hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 3\hat{k}$

Solution 2

(i)

$\vec{a}$ and $\vec{b}$ are prependicular

$\Rightarrow \vec{a}\vec{b} = 0$

$\Rightarrow \left(\lambda\hat{i} + 2\hat{j} + \hat{k}\right).\left(4\hat{i} - 9\hat{j} + 2\hat{k}\right) = 0$

$\Rightarrow (\lambda)(4) + (2)(-9) + (1)(2) = 0$

$\Rightarrow 4\lambda - 18 + 2 = 0$

$\Rightarrow 4\lambda - 16 = 0$

$\Rightarrow 4\lambda = 16$

$\Rightarrow \lambda = \dfrac{16}{4}$

$\Rightarrow \lambda = 4$

(ii)

$\vec{a}$ and $\vec{b}$ are prependicular

$\Rightarrow \vec{a}\vec{b} = 0$

$\Rightarrow \left(\lambda\hat{i} + 2\hat{j} + \hat{k}\right).\left(5\hat{i} - 9\hat{j} + 2\hat{k}\right) = 0$

$\Rightarrow (\lambda)(5) + (2)(-9) + (1)(2) = 0$

$\Rightarrow 5\lambda - 18 + 2 = 0$

$\Rightarrow 5\lambda - 16 = 0$

$\Rightarrow 5\lambda = 16$

$\Rightarrow \lambda = \dfrac{16}{5}$

(iii)

$\vec{a}$ and $\vec{b}$ are prependicular

$\Rightarrow \vec{a}.\vec{b} = 0$

$\Rightarrow \left(2\hat{i} + 3\hat{j} + 4\hat{k}\right)\left(3\hat{i} + 2\hat{j} - \lambda\hat{k}\right) = 0$

$\Rightarrow (2)(3) + (3)(2) + (4)(-\lambda) = 0$

$\Rightarrow 6 + 6 - 4\lambda = 0$

$\Rightarrow 12 - 4\lambda = 0$

$\Rightarrow -4\lambda = -12$

$\Rightarrow \lambda = \frac{-12}{-4}$

$\Rightarrow \lambda = 3$

(iv)

$\vec{a}$ and $\vec{b}$ are prependicular

$\Rightarrow \vec{a}.\vec{b} = 0$

$\Rightarrow \left(\lambda\hat{i} + 3\hat{j} + 2\hat{k}\right)\left(\hat{i} - \hat{j} + 3\hat{k}\right) = 0$

$\Rightarrow (\lambda)(1) + (3)(-1) + (2)(3) = 0$

$\Rightarrow \lambda - 3 + 6 = 0$

$\Rightarrow \lambda + 3 = 0$

$\Rightarrow \lambda = -3$

Question 3

If $\vec{a}$ and $\vec{b}$ are two vectors such that $\left|\vec{a}\right| = 4, \left|\vec{b}\right| = 3$ and $\vec{a} \times \vec{b} = 6$. Find the angle between $\vec{a}$ and $\vec{b}$.

Solution 3

We know that, if θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{6}{4 \times 3}$$

$$= \frac{6}{12}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{3}$$

Angle between $\vec{a}$ and $\vec{b} = \frac{\pi}{3}$

Question 4

If $\vec{a} = \hat{i} - \hat{j}$ $\vec{b} = -\hat{j} + 2\hat{k}$, Find $\left(\vec{a} - 2\vec{b}\right) \times \left(\vec{a} + \vec{b}\right)$.

Solution 4

$$\left(\vec{a}-2\vec{b}\right)=\left(\hat{i}-\hat{j}\right)-2\left(-\hat{j}+2\hat{k}\right)$$
$$=\left(\hat{i}-\hat{j}\right)+2\hat{j}-4\hat{k}$$
$$=\left(\hat{i}+\hat{j}-4\hat{k}\right)$$

$$\left(\vec{a}+\vec{b}\right)=\left(\hat{i}-\hat{j}\right)+\left(-\hat{j}+2\hat{k}\right)$$
$$=\hat{i}-\hat{j}-\hat{j}+2\hat{k}$$
$$=\left(\hat{i}-2\hat{j}+2\hat{k}\right)$$

Now,

$$\left(\vec{a}-2\vec{b}\right).\left(\vec{a}+\vec{b}\right)$$
$$=\left(\hat{i}+\hat{j}-4\hat{k}\right)\left(\hat{i}-2\hat{j}+2\hat{k}\right)$$
$$=(1)(1)+(1)(-2)+(-4)(2)$$
$$=1-2-8$$
$$=-9$$

$$\left(\vec{a}-2\vec{b}\right).\left(\vec{a}+\vec{b}\right)=-9$$

Question 5 (i)

Find the angle between the vectors $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=\hat{j}+\hat{k}$

Solution 5 (i)

Let θ be the angle between $\vec{a}$ and $\vec{b}$, ther

$$\cos\theta = \frac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|} \qquad ---(i)$$

$$\vec{a}\,.\,\vec{b} = \left(\hat{i}-\hat{j}\right)\left(\hat{j}+\hat{k}\right)$$

$$= \left(\hat{i}-\hat{j}+0\times\hat{k}\right)\left(0\times\hat{i}+\hat{j}+\hat{k}\right)$$

$$= (1)(0)+(-1)(1)+(0)(1)$$

$$= 0-1+0$$

$$\vec{a}\,.\,\vec{b} = -1$$

$$|\vec{a}| = \left|\hat{i}-\hat{j}\right|$$

$$= \left|\hat{i}-\hat{j}+0\times\hat{k}\right|$$

$$= \sqrt{(1)^2+(-1)^2+(0)^2}$$

$$= \sqrt{1+1+0}$$

$$|\vec{a}| = \sqrt{2}$$

$$|\vec{b}| = \left|\hat{j}+\hat{k}\right|$$

$$= \left|0\times\hat{i}+\hat{j}+\hat{k}\right|$$

$$= \sqrt{(0)^2+(1)^2+(1)^2}$$

$$= \sqrt{0+1+1}$$

$$|\vec{b}| = \sqrt{2}$$

Put $\vec{a}\times\vec{b}, |\vec{a}|$ and $|\vec{b}|$ in equation (i)

$$\cos\theta = \frac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{-1}{\sqrt{2}\times\sqrt{2}}$$

$$\cos\theta = \frac{-1}{2}$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = \pi - \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{3}$$

Angle between $\vec{a}$ and $\vec{b}$ = $\frac{2\pi}{3}$

Question 5 (ii)

Find the angle between the vectors $\vec{a} = 3\hat{i} - 2\hat{j} - 6\hat{k}$ and $\vec{b} = 4\hat{i} - \hat{j} + 8\hat{k}$

Solution 5 (ii)

Let θ be the angle between two vactor $\vec{a} = 3\hat{i} - 2\hat{j} - 6\hat{k}$ and $\vec{b} = 4\hat{i} - \hat{j} + 8\hat{k}$

$$\cos\theta = \frac{\overrightarrow{a.b}}{|\vec{a}||\vec{b}|} \cdots (1)$$

$$\begin{aligned}
\vec{a}.\vec{b} &= (3\hat{i} - 2\hat{j} - 6\hat{k})(4\hat{i} - \hat{j} + 8\hat{k}) \\
&= 3 * 4 + (-2)(-1) + (-6)8 \\
&= 12 + 2 - 48 \\
&= -34
\end{aligned}$$

$$\begin{aligned}
|\vec{a}| &= \sqrt{3^2 + (-2)^2 + (-6)^2} \\
&= \sqrt{49} \\
&= 7
\end{aligned}$$

$$\begin{aligned}
|\vec{b}| &= \sqrt{4^2 + (-1)^2 + 8^2} \\
&= \sqrt{81} \\
&= 9
\end{aligned}$$

Putting value of $|\vec{a}|, |\vec{b}|$ and $\vec{a}.\vec{b}$ in equation (1)

$$\begin{aligned}
\cos\theta &= \frac{\overrightarrow{a.b}}{|\vec{a}||\vec{b}|} \\
&= \frac{-34}{7 * 9} \\
&= \frac{-34}{63} \\
\theta &= \cos^{-1}\left(\frac{-34}{63}\right) \\
&= 122.66^{0}
\end{aligned}$$

Question 5 (iii)

Find the angle between the vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 4\hat{i} + 4\hat{j} - 2\hat{k}$

Solution 5 (iii)

Let the angle between $\vec{a}$ and $\vec{b}$ be θ, then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \qquad ---(i)$$

$$\vec{a}.\ \vec{b} = \left(2\hat{i} - \hat{j} + 2\hat{k}\right)\left(4\hat{i} + 4\hat{j} - 2\hat{k}\right)$$
$$= (2)(4) + (-1)(4) + (2)(-2)$$
$$= 8 - 4 - 4$$
$$\vec{a}.\ \vec{b} = 0$$

$$|\vec{a}| = \left|2\hat{i} - \hat{j} + 2\hat{k}\right|$$
$$= \sqrt{(2)^2 + (-1)^2 + (2)^2}$$
$$= \sqrt{4 + 1 + 4}$$
$$= \sqrt{9}$$
$$|\vec{a}| = 3$$

$$|\vec{b}| = \left|4\hat{i} + 4\hat{j} - 2\hat{k}\right|$$
$$= \sqrt{(4)^2 + (4)^2 + (-2)^2}$$
$$= \sqrt{16 + 16 + 4}$$
$$= \sqrt{36}$$
$$|\vec{b}| = 6$$

Put $\vec{a}.\ \vec{b}$, $|\vec{a}|$ and $|\vec{b}|$ in equation (i)

$$\cos\theta = \frac{\vec{a}\ .\ \vec{b}}{|\vec{a}||\vec{b}|}$$
$$= \frac{0}{3\times 6}$$
$$= \frac{0}{18}$$
$$\cos\theta = 0$$
$$\theta = \cos^{-1}(0)$$

Angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$

Question 5 (iv)

Find the angle between the vectors $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$

Solution 5 (iv)

Let θ be the angle between vector $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \qquad ---(i)$$

$$\vec{a} \quad \vec{b} = \left(2\hat{i} - 3\hat{j} + \hat{k}\right)\left(\hat{i} + \hat{j} - 2\hat{k}\right)$$
$$= (2)(1) + (-3)(1) + (1)(-2)$$
$$= 2 - 3 - 2$$
$$\vec{a} \cdot \vec{b} = -3$$

$$|\vec{a}| = \left|2\hat{i} - 3\hat{j} + \hat{k}\right|$$
$$= \sqrt{(2)^2 + (-3)^2 + (-1)^2}$$
$$= \sqrt{4 + 9 + 1}$$
$$= \sqrt{14}$$

$$|\vec{b}| = \left|\hat{i} + \hat{j} - 2\hat{k}\right|$$
$$|\vec{b}| = \sqrt{(1)^2 + (1)^2 + (-2)^2}$$
$$= \sqrt{1 + 1 + 4}$$
$$|\vec{b}| = \sqrt{6}$$

Put $\vec{a}.\ \vec{b}$, $|\vec{a}|$ and $|\vec{b}|$ in equation (i),

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$
$$= \frac{-3}{\sqrt{14} \times \sqrt{6}}$$
$$\cos\theta = \frac{-3}{\sqrt{84}}$$
$$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{84}}\right)$$

Angle between vector $\vec{a}$ and $\vec{b}$ $= \cos^{-1}\left(\frac{-3}{\sqrt{84}}\right)$

Question 5 (v)

Find the angle between the vectors $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

Solution 5 (v)

Let θ be the angle between vector $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \qquad ---(i)$$

$$\vec{a}.\vec{b} = \left(\hat{i} + 2\hat{j} - \hat{k}\right)\left(\hat{i} - \hat{j} + \hat{k}\right)$$
$$= (1)(1) + (2)(-1) + (-1)(1)$$
$$= 1 - 2 - 1$$
$$\vec{a}.\vec{b} = -2$$

$$|\vec{a}| = \left|\hat{i} + 2\hat{j} - \hat{k}\right|$$
$$= \sqrt{(1)^2 + (2)^2 + (-1)^2}$$
$$= \sqrt{1 + 4 + 1}$$
$$= \sqrt{6}$$

$$|\vec{b}| = \left|\hat{i} - \hat{j} + \hat{k}\right|$$
$$|\vec{b}| = \sqrt{(1)^2 + (-1)^2 + (1)^2}$$
$$= \sqrt{1 + 1 + 1}$$
$$|\vec{b}| = \sqrt{3}$$

Put $\vec{a}.\vec{b}$, $|\vec{a}|$, $|\vec{b}|$ in equation (i),

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$
$$= \frac{-2}{\sqrt{6}\sqrt{3}}$$
$$= \frac{-2}{\sqrt{18}}$$
$$= \frac{-2 \times \sqrt{2}}{3\sqrt{2} \times \sqrt{2}}$$
$$= \frac{-2\sqrt{2}}{3 \times 2}$$
$$\cos\theta = \frac{-\sqrt{2}}{3}$$
$$\theta = \cos^{-1}\left(\frac{-\sqrt{2}}{3}\right)$$

Angle between vector $\vec{a}$ and $\vec{b}$ $= \cos^{-1}\left(\frac{-\sqrt{2}}{3}\right)$

Question 6

Find the angle which the vector $\vec{a} = \hat{i} - \hat{j} + \sqrt{2}\hat{k}$ makes with the coordinate axes.

Solution 6

Component along x–, y– and z-axis are $\hat{i}, \hat{j}$ and $\hat{k}$ respectively.

Let θ_1 be the angle between $\vec{a}$ and $\hat{i}$.

$$\cos\theta_1 = \frac{\vec{a}.\hat{i}}{|\vec{a}||\hat{i}|}$$

$$= \frac{\left(\hat{i} - \hat{j} + \sqrt{2}\hat{k}\right)\left(\hat{i} + 0.\hat{j} + 0.\hat{k}\right)}{\left|\hat{i} - \hat{j} + \sqrt{2}\hat{k}\right|\left|\hat{i} + 0.\hat{j} + 0.\hat{k}\right|}$$

$$= \frac{(1)(1) + (-1)(0) + \left(\sqrt{2}\right)(0)}{\sqrt{(1)^2 + (-1)^2 + \left(\sqrt{2}\right)^2}.\sqrt{(1)^2 + (0)^2 + (0)^2}}$$

$$= \frac{1 + 0 + 0}{\sqrt{4}\sqrt{1}}$$

$$\cos\theta_1 = \frac{1}{2}$$

$$\theta_1 = \frac{\pi}{3}$$

Let θ_2 be the angle between $\vec{a}$ and $\hat{j}$.

$$\cos\theta_2 = \frac{\vec{a}.\hat{j}}{|\vec{a}||\hat{j}|}$$

$$= \frac{\left(\hat{i} - \hat{j} + \sqrt{2}\hat{k}\right)\left(0\,\hat{i} + \hat{j} + 0\,\hat{k}\right)}{\sqrt{(1)^2 + (-1)^2 + \left(\sqrt{2}\right)^2}.\sqrt{(0)^2 + (1)^2 + (0)^2}}$$

$$= \frac{(1)(0) + (-1)(1) + \left(\sqrt{2}\right)(0)}{\sqrt{1 + 1 + 2}\ \sqrt{1}}$$

$$= \frac{-1}{\sqrt{4}\sqrt{1}}$$

$$= \frac{-1}{2}$$

$$\cos\theta_2 = -\frac{1}{2}$$

$$\theta_2 = \pi - \frac{\pi}{3}$$

$$\theta_2 = \frac{2\pi}{3}$$

Let θ_3 be the angle between $\vec{a}$ and $\hat{k}$, then

$$\cos\theta_3 = \frac{\vec{a}.\hat{k}}{|\vec{a}||\hat{k}|}$$

$$= \frac{\left(\hat{i} - \hat{j} + \sqrt{2}\hat{k}\right)\left(0.\hat{i} + 0.\hat{j} + \hat{k}\right)}{\sqrt{(1)^2 + (-1)^2 + \left(\sqrt{2}\right)^2}\sqrt{(0)^2 + (0)^2 + (1)^2}}$$

$$= \frac{(1)(0) + (-1)(0) + \left(\sqrt{2}\right)(1)}{\sqrt{1+1+2}\ \sqrt{1}}$$

$$= \frac{\sqrt{2}}{\sqrt{4}\sqrt{1}}$$

$$\cos\theta_3 = \frac{1}{\sqrt{2}}$$

$$\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\theta_3 = \frac{\pi}{4}$$

So, the angle between vector $\vec{a}$ and x-axis is $\frac{\pi}{3}$, vector $\vec{a}$ and y-axis is $\frac{2\pi}{3}$, vector $\vec{a}$ and z-axis is $\frac{\pi}{4}$.

Question 7(i)

Dot product of a vector with $\hat{i} + \hat{j} - 3\hat{k}$, $\hat{i} + 3\hat{j} - 2\hat{k}$ and $2\hat{i} + \hat{j} + 4\hat{k}$ are 0,5 and 8 respectively. Find the vector.

Solution 7(i)

Let the requird vector be $x\hat{i} + y\hat{j} + z\hat{k}$
According to question,

$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\left(\hat{i} + \hat{j} - 3\hat{k}\right) = 0$$
$$(x)(1) + (y)(1) + (z)(-3) = 0$$
$$x + y - 3z = 0 \qquad ---(i)$$

And,
$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\left(\hat{i} + 3\hat{j} - 2\hat{k}\right) = 5$$
$$(x)(1) + (y)(3) + (z)(-2) = 5$$
$$x + 3y - 2z = 5 \qquad ---(ii)$$

And,
$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\left(2\hat{i} + \hat{j} + 4\hat{k}\right) = 8$$
$$(x)(2) + (y)(1) + (z)(4) = 8$$
$$2x + y + 4z = 8 \qquad ---(iii)$$

Subtracting (i) from (ii),
$$\begin{array}{l} x + 3y - 2z = 5 \\ \underset{(-)}{x} \underset{(-)}{+ y} \underset{(+)}{- 3z} = 0 \\ \hline \qquad 2y + z = 5 \end{array} \qquad ---(iv)$$

Subtracting $2 \times$(ii) from (iii),
$$\begin{array}{l} 2x + y + 4z = 8 \\ \underset{(-)}{2x} \underset{(-)}{+ 6y} \underset{(+)}{- 4z} = \underset{(-)}{10} \\ \hline \qquad -5y + 8z = -2 \end{array} \qquad ---(v)$$

Subtracting $8 \times$(iv) from (v),
$$\begin{array}{l} -5y + 8z = -2 \\ \underset{(-)}{16y} \underset{(-)}{+ 8z} = \underset{(-)}{40} \\ \hline -21y \quad = -42 \end{array} \qquad ---(vi)$$
$$y = \frac{-42}{-21}$$
$$y = 2$$

Put $y = 2$ in equation (iv),

$$2y + z = 5$$
$$2(2) + z = 5$$
$$4 + z = 5$$
$$z = 5 - 4$$
$$z = 1$$

Put $y = 2$ and $z = 1$ in equation (i),

$$x + y - 3z = 0$$
$$x + (2) - 3(1) = 0$$
$$x + 2 - 3 = 0$$
$$x - 1 = 0$$
$$x = 1$$

The required vector $= x\hat{i} + y\hat{j} + z\hat{k}$

The required vector $= \hat{i} + 2\hat{j} + \hat{k}$

Question 7(ii)

Dot products of a vector with vectors $\hat{i} - \hat{j} + \hat{k}$, $2\hat{i} + \hat{j} - 3\hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively 4, 0 and 2. Find the vector.

Solution 7(ii)

Let the unknown vector be '$\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$'

$\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = 2\hat{i} + \hat{j} - 3\hat{k}$, $\vec{d} = \hat{i} + \hat{j} + \hat{k}$

It is given that $\vec{a}.\vec{b} = 4$

$a_1 - b_1 + c_1 = 4$.....(i)

$\vec{a}.\vec{c} = 0$

$2a_1 + b_1 - 3c_1 = 0$.......(ii)

$\vec{a}.\vec{d} = 2$

$a_1 + b_1 + c_1 = 2$........(iii)

Solving (i), (ii) and (iii),

$a_1 = 2, b_1 = -1, c_1 = 1$

$\therefore$ the vector $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$

Question 8 (i)

If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that

$$\cos\frac{\theta}{2} = \frac{1}{2}\left|\hat{a} + \hat{b}\right|$$

Solution 8 (i)

Here, $\hat{a}$ and $\hat{b}$ are unit vectors, then

$$\left|\hat{a}\right| = \left|\hat{b}\right| = 1$$

$$\left|\hat{a}+\hat{b}\right|^2 = \left(\hat{a}+\hat{b}\right)^2$$
$$= \left(\hat{a}\right)^2 + \left(\hat{b}\right)^2 + 2\hat{a}.\hat{b}$$
$$= \left|\hat{a}\right|^2 + \left|\hat{b}\right|^2 + 2\hat{a}.\hat{b}$$
$$= (1)^2 + (1)^2 + 2\hat{a}.\hat{b}$$
$$\left|\hat{a}+\hat{b}\right|^2 = 2 + 2\hat{a}\times\hat{b}$$

$$\left|\hat{a}+\hat{b}\right|^2 = 2 + 2\times\left|\hat{a}\right|\left|\hat{b}\right|\cos\theta \qquad \left[\text{Since } \vec{a}.\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|\cos\theta\right]$$
$$\left|\hat{a}+\hat{b}\right|^2 = 2 + 2\times 1\times 1\times\cos\theta$$
$$= 2 + 2\cos\theta$$
$$\left|\hat{a}+\hat{b}\right|^2 = 2\left(1+\cos\theta\right)$$
$$\left|\hat{a}+\hat{b}\right|^2 = 2\left(2\cos^2\frac{\theta}{2}\right) \qquad \left[\text{Since } 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right]$$

$$\left|\hat{a}+\hat{b}\right|^2 = 4\cos^2\frac{\theta}{2}$$
$$\left|\hat{a}+\hat{b}\right| = \sqrt{4\cos^2\frac{\theta}{2}}$$
$$\left|\hat{a}+\hat{b}\right| = 2\cos\frac{\theta}{2}$$

$$\cos\frac{\theta}{2} = \frac{1}{2}\left|\hat{a}+\hat{b}\right|$$

Question 8 (ii)

If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that

$$\tan\frac{\theta}{2} = \frac{\left|\hat{a}-\hat{b}\right|}{\left|\hat{a}+\hat{b}\right|}$$

Solution 8 (ii)

Here, $\hat{a}$ and $\hat{b}$ are unit vectors

$$|\hat{a}| = |\hat{b}| = 1$$

$$\frac{|\hat{a}-\hat{b}|^2}{|\hat{a}+\hat{b}|^2} = \frac{(\hat{a}-\hat{b})^2}{(\hat{a}+\hat{b})^2}$$

$$= \frac{(\hat{a})^2 + (\hat{b})^2 - 2\hat{a}.\ \hat{b}}{(\hat{a})^2 + (\hat{b})^2 + 2\hat{a}.\ \hat{b}}$$

$$= \frac{|\hat{a}|^2 + |\hat{b}|^2 - 2\hat{a}\ .\ \hat{b}}{|\hat{a}|^2 + |\hat{b}|^2 + 2\hat{a}\ .\ \hat{b}}$$

$$\frac{|\hat{a}-\hat{b}|^2}{|\hat{a}+\hat{b}|^2} = \frac{(1)^2 + (1)^2 - 2|\vec{a}||\vec{b}|\cos\theta}{(1)^2 + (1)^2 + 2|\vec{a}||\vec{b}|\cos\theta} \qquad \left[\text{Since } \vec{a}.\ \vec{b} = |\vec{a}||\vec{b}|\cos\theta\right]$$

$$\frac{|\hat{a}-\hat{b}|^2}{|\hat{a}+\hat{b}|^2} = \frac{1+1-2(1)(1)\cos\theta}{1+1+2(1)(1)\cos\theta}$$

$$= \frac{2-2\cos\theta}{2+2\cos\theta}$$

$$= \frac{2(1-\cos\theta)}{2(1+\cos\theta)}$$

$$= \frac{2\times\sin^2\frac{\theta}{2}}{2\times\cos^2\frac{\theta}{2}} \qquad \left[\text{Since } 1-\cos\theta = 2\sin^2\frac{\theta}{2},\ 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right]$$

$$\frac{|\hat{a}-\hat{b}|^2}{|\hat{a}+\hat{b}|^2} = \tan^2\frac{\theta}{2}$$

$$\tan\frac{\theta}{2} = \frac{|\hat{a}-\hat{b}|}{|\hat{a}+\hat{b}|}$$

Question 9

If the sum of two unit vectors is a unit vector prove that the magnitude of their difference is $\sqrt{3}$.

Solution 9

Let $\hat{a}$ and $\hat{b}$ are two unit vectors

Then, $|\hat{a}| = |\hat{b}| = 1$

And sum of $\hat{a}$ and $\hat{b}$ is a unit vector, then

$$|\hat{a} + \hat{b}| = 1$$

Taking square of both the sides,

$$|\hat{a} + \hat{b}|^2 = (1)^2$$

$$(\hat{a} + \hat{b})^2 = 1$$

$$(\hat{a})^2 + (\hat{b})^2 + 2\hat{a}.\ \hat{b} = 1$$

$$|\hat{a}|^2 + |\hat{b}|^2 + 2\hat{a}.\ \hat{b} = 1$$

$$(1)^2 + (1)^2 + 2\hat{a}.\ \hat{b} = 1$$

$$2 + 2\hat{a}\quad \hat{b} = 1$$

$$2\hat{a}.\ \hat{b} = 1 - 2$$

$$2\hat{a}.:\hat{b} = -1$$

$$\hat{a}.\ \hat{b} = \frac{-1}{2} \qquad ---(i)$$

$$|\hat{a} - \hat{b}|^2 = (\hat{a} - \hat{b})^2$$

$$= (\hat{a})^2 + (\hat{b})^2 - 2\hat{a}.\ \hat{b}$$

$$= |\hat{a}|^2 + |\hat{b}|^2 - 2 \times \hat{a}.\ \hat{b}$$

$$= (1)^2 + (1)^2 - 2 \times \left(\frac{-1}{2}\right)$$

Using equation (i)

$$= 1 + 1 + \frac{2}{2}$$

$$= 1 + 1 + 1$$

$$|\hat{a} - \hat{b}|^2 = 3$$

$$|\hat{a} - \hat{b}| = \sqrt{3}$$

Question 10

If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular unit vectors, then prove that $|\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}$.

Solution 10

Given that $\vec{a}, \vec{b}, \vec{c}$ are mutually prependicular, so,

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$$

and $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors, so

$$|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$$

Now,

$$\begin{aligned}
\left|\vec{a}+\vec{b}+\vec{c}\right|^2 &= \left(\vec{a}+\vec{b}+\vec{c}\right)^2 \\
&= \left(\vec{a}\right)^2 + \left(\vec{b}\right)^2 + \left(\vec{c}\right)^2 + 2\vec{a}\vec{b} + 2\vec{b}\vec{c} + 2\vec{c}\vec{a} \\
&= \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + \left|\vec{c}\right|^2 + 2(0) + 2(0) + 2(0) \\
&= (1)^2 + (1)^2 + (1)^2 + 0 \\
\left|\vec{a}+\vec{b}+\vec{c}\right|^2 &= 1+1+1 \\
\left|\vec{a}+\vec{b}+\vec{c}\right|^2 &= 3 \\
\left|\vec{a}+\vec{b}+\vec{c}\right| &= \sqrt{3}
\end{aligned}$$

Question 11

If $\left|\vec{a}+\vec{b}\right| = 60$, $\left|\vec{a}-\vec{b}\right| = 40$ and $\left|\vec{b}\right| = 46$, find $\left|\vec{a}\right|$

Solution 11

Here, $\left|\vec{a}+\vec{b}\right| = 60$

Squaring both the sides,

$$\left|\vec{a}+\vec{b}\right|^2 = (60)^2$$

$$\left(\vec{a}+\vec{b}\right) = (60)^2$$

$$\left(\vec{a}\right)^2 + \left(\vec{b}\right)^2 + 2\vec{a}\vec{b} = 3600$$

$$\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\vec{a}\vec{b} = 3600 \qquad ---(i)$$

Now, $\left|\vec{a}-\vec{b}\right| = 40$

Squaring both the sides,

$$\left|\vec{a}-\vec{b}\right|^2 = (40)^2$$

$$\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 - 2\vec{a}\vec{b} = 1600 \qquad ---(ii)$$

Adding (i) and (ii),

$$2\left|\vec{a}\right|^2 + 2\left|\vec{b}\right|^2 + 2\vec{a}\vec{b} - 2\vec{a}\vec{b} = 3600 - 1600$$

$$2\left|\vec{a}\right|^2 + 2(46)^2 = 5200$$

$$2\left|\vec{a}\right|^2 = 5200 - 4232$$

$$2\left|\vec{a}\right|^2 = 968$$

$$\left|\vec{a}\right|^2 = \frac{968}{2}$$

$$\left|\vec{a}\right|^2 = 484$$

$$\left|\vec{a}\right| = \sqrt{484}$$

$$\left|\vec{a}\right| = 22$$

Question 12

Show that the verctor $\hat{i} + \hat{j} + \hat{k}$ is equally inclined with the coordinate axes.

Solution 12

Let θ be the angle between $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}$

Then,

$$\cos\theta = \frac{\left(\hat{i}+\hat{j}+\hat{k}\right).\left(\hat{i}\right)}{\left|\hat{i}+\hat{j}+\hat{k}\right|\left|\hat{i}\right|}$$

$$= \frac{1}{\frac{1}{\sqrt{3}}}$$

$$= \sqrt{3}$$

Similarly, if α and γ are angles that $\hat{i}+\hat{j}+\hat{k}$ make with $\hat{j}$ and $\hat{k}$

Then,

$$\cos\alpha = \sqrt{3}$$

and $$\cos\gamma = \sqrt{3}$$

Therefore, $\hat{i}+\hat{j}+\hat{k}$ is equally inclined the three axes.

Question 13

Show that the vectors $\vec{a} = \frac{1}{7}\left(2\hat{i}+3\hat{j}+6\hat{k}\right), \vec{b} = \frac{1}{7}\left(3\hat{i}-6\hat{j}+2\hat{k}\right), \vec{c} = \frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)$ are mutually perpendicular unit vectors.

Solution 13

We have,

$$\vec{a} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$$

$$\vec{b} = \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right)$$

$$\vec{c} = \frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right)$$

Then,

$$\vec{a}\,.\,\vec{b} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right) \times \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right)$$

$$= \frac{1}{49}\left(6 - 18 + 12\right) = 0$$

Similarly,

$$\vec{b}\,.\,\vec{c} = \vec{a}\,.\,\vec{c} = 0$$

$\therefore$ $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular

Question 14

for any two vectors $\vec{a}$ and $\vec{b}$,

Show that $\left(\vec{a} + \vec{b}\right) \times \left(\vec{a} - \vec{b}\right) = 0 \Leftrightarrow \left|\vec{a}\right| = \left|\vec{b}\right|$.

Solution 14

Let $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=0$

$$|\vec{a}|^2-|\vec{b}|^2=0$$

$$|\vec{a}|^2=|\vec{b}|^2$$

$$|\vec{a}|=|\vec{b}|$$

Let $|\vec{a}|=|\vec{b}|$

Squaring both the sides.

$$|\vec{a}|^2=|\vec{b}|^2$$

$$|\vec{a}|^2-|\vec{b}|^2=0$$

$$(\vec{a})^2-(\vec{b})^2=0$$

$$(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=0$$

Thus,

$$(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=0 \Leftrightarrow |\vec{a}|=|\vec{b}|$$

Question 15

If $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+\hat{j}-2\hat{k}$ and $\vec{c}=\hat{i}+3\hat{j}-\hat{k}$,

find λ such that $\vec{a}$ is perpendicular to $\lambda\vec{b}+\vec{c}$

Solution 15

If $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+\hat{j}-2\hat{k}$ and $\vec{c}=\hat{i}+3\hat{j}-\hat{k}$,

find λ

Given that $\vec{a}$ is perpendicular to $\lambda\vec{b}+\vec{c}$

$$\therefore\ \vec{a}\bullet(\lambda\vec{b}+\vec{c})=0$$

$$\lambda\vec{a}\bullet\vec{b}+\vec{a}\bullet\vec{c}=0$$

$$\lambda(2\hat{i}-\hat{j}+\hat{k})\bullet(\hat{i}+\hat{j}-2\hat{k})+(2\hat{i}-\hat{j}+\hat{k})\bullet(\hat{i}+3\hat{j}-\hat{k})=0$$

$$\lambda(2-1-2)+(2-3-1)=0$$

$$-\lambda-2=0$$

$$\lambda=2$$

Question 16

If $\vec{p}=5\hat{i}+\lambda\hat{j}-3\hat{k}$ and $\vec{q}=\hat{i}+3\hat{j}-5\hat{k}$, then find the value of λ,

so that $\vec{p}+\vec{q}$ and $\vec{p}-\vec{q}$ are perpendicular vectors.

Solution 16

$\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$ and $\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$

$\vec{p} + \vec{q}$

$= 5\hat{i} + \lambda\hat{j} - 3\hat{k} + \hat{i} + 3\hat{j} - 5\hat{k}$

$= 6\hat{i} + (\lambda + 3)\hat{j} - 8\hat{k}$

$\vec{p} - \vec{q}$

$= 5\hat{i} + \lambda\hat{j} - 3\hat{k} - \hat{i} - 3\hat{j} + 5\hat{k}$

$= 4\hat{i} + (\lambda - 3)\hat{j} + 2\hat{k}$

$(\vec{p} + \vec{q}).(\vec{p} - \vec{q}) = 0$

$\Rightarrow [6\hat{i} + (\lambda + 3)\hat{j} - 8\hat{k}].[4\hat{i} + (\lambda - 3)\hat{j} + 2\hat{k}] = 0$

$\Rightarrow 24 + (\lambda^2 - 9) - 16 = 0$

$\Rightarrow \lambda^2 - 9 + 8 = 0$

$\Rightarrow \lambda^2 - 1 = 0$

$\therefore \lambda = \pm 1$

Question 17

If $\overline{\alpha} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and $\overline{\beta} = 2\hat{i} + \hat{j} - 4\hat{k}$, then express $\overline{\beta}$ in the form of $\overline{\beta} = \overline{\beta_1} + \overline{\beta_2}$, where $\overline{\beta_1}$ is parallel to $\overline{\alpha}$ and $\overline{\beta_2}$ is perpendicular to $\overline{\alpha}$.

Solution 17

According to question $\overline{\beta_1}$ is parallel to $\overline{\alpha}$. So

$$\overline{\beta_1} = \gamma\overline{\alpha}$$

$$= \gamma\left(3\hat{i} + 4\hat{j} + 5\hat{k}\right)$$

$$\overline{\beta} = \overline{\beta_1} + \overline{\beta_2}$$

$$2\hat{i} + \hat{j} - 4\hat{k} = \gamma\left(3\hat{i} + 4\hat{j} + 5\hat{k}\right) + \overline{\beta_2} \qquad \left(\textit{putting } \overline{\beta} \textit{ and } \overline{\beta_1}\right)$$

$$\overline{\beta_2} = (2 - 3\gamma)\hat{i} + (1 - 4\gamma)\hat{j} - (4 + 5\gamma)\hat{k}$$

Again $\overline{\beta_2}$ is perpendicular to $\overline{\alpha}$. So

$$\overline{\beta_2}.\overline{\alpha} = 0$$

$$\left[(2 - 3\gamma)\hat{i} + (1 - 4\gamma)\hat{j} - (4 + 5\gamma)\hat{k}\right].\left(3\hat{i} + 4\hat{j} + 5\hat{k}\right) = 0$$

$$6 - 9\gamma + 4 - 16\gamma - 20 - 25\gamma = 0$$

$$-50\gamma = 10$$

$$\gamma = -\frac{1}{5}$$

$$\overline{\beta}_1 = -\frac{1}{5}\left(3\hat{i}+4\hat{j}+5\hat{k}\right)$$

$$\overline{\beta} = \overline{\beta_1} + \overline{\beta_2}$$

$$2\hat{i}+\hat{j}-4\hat{k} = -\frac{1}{5}\left(3\hat{i}+4\hat{j}+5\hat{k}\right)+\overline{\beta_2} \qquad \left(\text{putting } \overline{\beta} \text{ and } \overline{\beta_1}\right)$$

$$\overline{\beta_2} = \frac{1}{5}\left(13\hat{i}+9\hat{j}-15\hat{k}\right)$$

$$\overline{\beta} = -\frac{1}{5}\left(3\hat{i}+4\hat{j}+5\hat{k}\right)+\frac{1}{5}\left(13\hat{i}+9\hat{j}-15\hat{k}\right)$$

Question 18

If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

Solution 18

Consider $\vec{a}=2\hat{i}+4\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+3\hat{j}-6\hat{k}$.

Then,

$\vec{a}\cdot\vec{b}=2.3+4.3+3(-6)=6+12-18=0$

We now observe that:

$|\vec{a}|=\sqrt{2^2+4^2+3^2}=\sqrt{29}$

$\therefore \vec{a}\neq\vec{0}$

$|\vec{b}|=\sqrt{3^2+3^2+(-6)^2}=\sqrt{54}$

$\therefore \vec{b}\neq\vec{0}$

Hence, the converse of the given statement need not be true.

Question 19

Show that the vectors $\vec{a}=3\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-3\hat{j}+5\hat{k}$, $\vec{c}=2\hat{i}+\hat{j}-4\hat{k}$ form a right angled triangle.

Solution 19

Here,

$$\vec{b}+\vec{c}=\left(\hat{i}-3\hat{j}+5\hat{k}\right)\left(2\hat{i}+\hat{j}-4\hat{k}\right)$$

$$=3\hat{i}+2\hat{j}+\hat{k}$$

$$\vec{b}+\vec{c}=\vec{a}$$

$\therefore$ $\vec{a},\vec{b},\vec{c}$ are represents the sides of a triangle.

$$\left|\vec{a}\right|=\sqrt{(3)^2+(-2)^2+(1)^2}$$

$$=\sqrt{9+4+1}$$

$$=\sqrt{14}$$

$$\left|\vec{b}\right|=\sqrt{(1)^2+(-3)^2+(5)^2}$$

$$=\sqrt{1+9+25}$$

$$\left|\vec{b}\right|=\sqrt{35}$$

$$\left|\vec{c}\right|=\sqrt{(2)^2+(1)^2+(-4)^2}$$

$$=\sqrt{4+1+16}$$

$$=\sqrt{21}$$

$$\left(\sqrt{21}\right)^2+\left(\sqrt{14}\right)^2=\left(\sqrt{35}\right)^2$$

$$21+14=35$$

$$35=35$$

$$\left|\vec{c}\right|^2+\left|\vec{a}\right|^2=\left|\vec{b}\right|^2$$

$\therefore$ By the pythagorous theorem,

Triangle formed by $\vec{a},\vec{b},\vec{c}$ is a right angled triangled.

Question 20

If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k},\ \vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Solution 20

The given vectors are $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$, and $\vec{c} = 3\hat{i} + \hat{j}$.

Now,

$$\vec{a} + \lambda\vec{b} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(-\hat{i} + 2\hat{j} + \hat{k}\right) = (2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}$$

If $\left(\vec{a} + \lambda\vec{b}\right)$ is perpendicular to $\vec{c}$, then

$$\left(\vec{a} + \lambda\vec{b}\right)\cdot\vec{c} = 0.$$

$$\Rightarrow \left[(2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}\right]\cdot\left(3\hat{i} + \hat{j}\right) = 0$$

$$\Rightarrow (2-\lambda)3 + (2+2\lambda)1 + (3+\lambda)0 = 0$$

$$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0$$

$$\Rightarrow -\lambda + 8 = 0$$

$$\Rightarrow \lambda = 8$$

Hence, the required value of λ is 8.

Question 21

Find the angles of a triangle whose vertices are $A(0,-1,-2)$, $B(3,1,4)$ and $C(5,7,1)$.

Solution 21

$$\vec{A} = 0.\hat{i} - \hat{j} - 2\hat{k}$$
$$\vec{B} = 3\hat{i} + \hat{j} + 4\hat{k}$$
$$\vec{C} = 5\hat{i} + 7\hat{j} + \hat{k}$$

$$\overrightarrow{AB} = \vec{B} - \vec{A}$$
$$= \left(3\hat{i} + \hat{j} + 4\hat{k}\right) - \left(0.\hat{i} - \hat{j} - 2\hat{k}\right)$$
$$= 3\hat{i} + \hat{j} + 4\hat{k} - 0.\hat{i} + \hat{j} + 2\hat{k}$$
$$\overrightarrow{AB} = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\overrightarrow{BC} = \vec{C} - \vec{B}$$
$$= \left(5\hat{i} + 7\hat{j} + \hat{k}\right) - \left(3\hat{i} + \hat{j} + 4\hat{k}\right)$$
$$= 5\hat{i} + 7\hat{j} + \hat{k} - 3\hat{i} - \hat{j} - 4\hat{k}$$
$$\overrightarrow{BC} = 2\hat{i} + 6\hat{j} - 3\hat{k}$$

$$\overrightarrow{AC} = \vec{C} - \vec{A}$$
$$= \left(5\hat{i} + 7\hat{j} + \hat{k}\right) - \left(-\hat{j} - 2\hat{k}\right)$$
$$= 5\hat{i} + 7\hat{j} + \hat{k} + \hat{j} + 2\hat{k}$$
$$\overrightarrow{AC} = 5\hat{i} + 8\hat{j} + 3\hat{k}$$

Angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$,

$$\cos A = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$$
$$= \frac{\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right)\left(5\hat{i} + 8\hat{j} + 3\hat{k}\right)}{\sqrt{(3)^2 + (2)^2 + (6)^2}\ \sqrt{(5)^2 + (8)^2 + (3)^2}}$$
$$= \frac{(3)(5) + (2)(8) + (6)(3)}{\sqrt{9 + 4 + 36}\ \sqrt{25 + 64 + 9}}$$
$$= \frac{15 + 16 + 18}{\sqrt{49}\ \sqrt{98}}$$
$$= \frac{49}{\sqrt{49}\ \sqrt{49 \times 2}}$$
$$\cos A = \frac{49}{49\sqrt{2}}$$
$$\cos A = \frac{1}{\sqrt{2}}$$
$$A = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{4}$$

$$\angle A = \frac{\pi}{4}$$

Angle between $\overrightarrow{BC}$ and $\overrightarrow{BA}$

$$\cos B = \frac{\overrightarrow{BC}.\overrightarrow{BA}}{\left|\overrightarrow{BC}\right|\left|\overrightarrow{BA}\right|}$$

$$= \frac{\left(2\hat{i} + 6\hat{j} - 3\hat{k}\right)\left(-3\hat{i} - 2\hat{j} - 6\hat{k}\right)}{\sqrt{(2)^2 + (6)^2 + (-3)^2}\sqrt{(-3)^2 + (-2)^2 + (-6)^2}}$$

$$= \frac{(2)(-3) + (6)(-2) + (-3)(-6)}{\sqrt{4 + 36 + 9}\sqrt{9 + 4 + 36}}$$

$$= \frac{-6 - 12 + 18}{\sqrt{49}\sqrt{98}}$$

$$\cos B = \frac{-18 + 18}{49}$$

$$= \frac{0}{49}$$

$$\cos B = 0$$

$$B = \cos^{-1}(0)$$

$$\angle B = \frac{\pi}{2}$$

We know that,

$$\angle A + \angle B + \angle C = \pi$$

$$\frac{\pi}{4} + \frac{\pi}{2} + \angle C = \pi$$

$$\frac{3\pi}{4} + \angle C = \pi$$

$$\angle C = \frac{\pi}{1} - \frac{3\pi}{4}$$

$$\angle C = \frac{4\pi - 3\pi}{4}$$

$$\angle C = \frac{\pi}{4}$$

$$\angle A = \frac{\pi}{4}$$

$$\angle B = \frac{\pi}{2}$$

Question 22

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.

Solution 22

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

It is given that $|\vec{a}| = |\vec{b}|$, $\vec{a} \cdot \vec{b} = \frac{1}{2}$, and $\theta = 60°$. ...(1)

We know that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \frac{1}{2} = |\vec{a}||\vec{a}|\cos 60° \qquad [\text{Using (1)}]$$
$$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$$
$$\Rightarrow |\vec{a}|^2 = 1$$
$$\Rightarrow |\vec{a}| = |\vec{b}| = 1$$

Question 23

Show that the points whose position vectors are $\vec{a} = 4\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$, $\vec{c} = \hat{i} - \hat{j}$ form a right triangle.

Solution 23

Given

$$\vec{a} = 4\hat{i} - 3\hat{j} + \hat{k}$$
$$\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$$
$$\vec{c} = \hat{i} - \hat{j}$$

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$$= \left(2\hat{i} - 4\hat{j} + 5\hat{k}\right) - \left(4\hat{i} - 3\hat{j} + \hat{k}\right)$$
$$= 2\hat{i} - 4\hat{j} + 5\hat{k} - 4\hat{i} + 3\hat{j} - \hat{k}$$
$$\overrightarrow{AB} = -2\hat{i} - \hat{j} + 4\hat{k}$$

$\overrightarrow{BC}$ = Position vector of C – Position vector of B

$$= \left(\hat{i} - \hat{j}\right) - \left(2\hat{i} - 4\hat{j} + 5\hat{k}\right)$$
$$= \hat{i} - \hat{j} - 2\hat{i} + 4\hat{j} - 5\hat{k}$$
$$= -\hat{i} + 3\hat{j} - 5\hat{k}$$

$\overrightarrow{CA}$ = Position vector of A – Position vector of C

$$= \left(4\hat{i} - 3\hat{j} + \hat{k}\right) - \left(\hat{i} - \hat{j}\right)$$
$$= 4\hat{i} - 3\hat{j} + \hat{k} - \hat{i} + \hat{j}$$
$$= 3\hat{i} - 2\hat{j} + \hat{k}$$

Now, $\overrightarrow{AB}.\overrightarrow{CA}$

$$= \left(-2\hat{i} - \hat{j} + 4\hat{k}\right).\left(3\hat{i} - 2\hat{j} + \hat{k}\right)$$
$$= (-2)(3) + (-1)(-2) + (4)(1)$$
$$= -6 + 2 + 4$$
$$= -6 + 6$$
$$= 0$$

So, $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CA}$

$\angle A$ is right angle.

Hence, ABC is a right triangle

Question 24

If the vertices A, B, C of $\triangle ABC$ have position vectors $(1,2,3), (-1,0,0), (0,1,2)$ respectively, what is the magnitude of $\angle ABC$?

Solution 24

Given,

$$A=(1,2,3)$$
$$B=(-1,0,0)$$
$$C=(0,1,2)$$

Position vector of $A=\hat{i}+2\hat{j}+3\hat{k}$

Position vector of $B=-\hat{i}+0\hat{j}+0\hat{k}$

Position vector of $C=0\hat{i}+\hat{j}+2\hat{k}$

$\overline{AB}$ = Position vector of B - Position vector of A

$$=\left(-\hat{i}+0\hat{j}+0\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$
$$=-2\hat{i}-2\hat{j}-3\hat{k}$$

$\overline{BC}$ = Position vector of C - Position vector of B

$$=\left(0\hat{i}+\hat{j}+2\hat{k}\right)-\left(\widehat{-i}+0\hat{j}+0\hat{k}\right)$$
$$=\hat{i}+\hat{j}+2\hat{k}$$

$\overline{AC}$ = Position vector of C - Position vector of A

$$=\left(0\hat{i}+\hat{j}+2\hat{k}\right)-\left(1\hat{i}+2\hat{j}+3\hat{k}\right)$$
$$=-\hat{i}-\hat{j}-\hat{k}$$

$$\overline{AB}.\overline{BC}=\left(-2\hat{i}-2\hat{j}-3\hat{k}\right).\left(\hat{i}+\hat{j}+2\hat{k}\right)$$
$$=-2-2-6$$
$$=-10$$

$$\angle ABC=\frac{\overrightarrow{AB}.\overrightarrow{BC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|}$$

$$=\frac{-10}{\sqrt{(-2)^2+(-2)^2+(-3)^2}\sqrt{1^2+1^2+2^2}}$$

$$=\frac{-10}{\sqrt{17}\sqrt{6}}$$

$$=\frac{-10}{\sqrt{102}}$$

$$\angle ABC=\cos^{-1}\left(\frac{-10}{\sqrt{102}}\right)$$

Question 25

If A, B, C have position vectors $(0,1,1)$, $(3,1,5)$, $(0,3,3)$ respectively, show that ΔABC is right angled at C.

Solution 25

Given

$A = (0,1,1)$

$B = (\ 3,1,5)$

$C = (0,3,3)$

Position vector of $A = 0\hat{i} + \hat{j} + \hat{k}$

Position vector of $B = 3\hat{i} + \hat{j} + 5\hat{k}$

Position vector of $C = 0.\hat{i} + 3\hat{j} + 3\hat{k}$

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= (3\hat{i} + \hat{j} + 5\hat{k}) - (0.\hat{i} + \hat{j} + \hat{k})$

$= 3\hat{i} + \hat{j} + 5\hat{k} - \hat{j} - \hat{k}$

$\overrightarrow{AB} = 3\hat{i} + 4\hat{k}$

$\overrightarrow{BC}$ = Position vector of C – Position vector of B

$= (0.\hat{i} + 3\hat{j} + 3\hat{k}) - (3\hat{i} + \hat{j} + 5\hat{k})$

$\overrightarrow{BC} = 3\hat{j} + 3\hat{k} - 3\hat{i} - \hat{j} - 5\hat{k}$

$= -3\hat{i} + 2\hat{j} - 2\hat{k}$

$\overrightarrow{AC}$ = Position vector of C – Position vector of A

$= (-3\hat{j} + 3\hat{k}) - (\hat{j} + \hat{k})$

$= 3\hat{j} + 3\hat{k} - \hat{j} - \hat{k}$

$= 2\hat{j} + 2\hat{k}$

$\overrightarrow{BC}.\overrightarrow{AC}$

$= (-3\hat{i} + 2\hat{j} - 2\hat{k})(2\hat{j} + 2\hat{k})$

$= (-3)(0) + (2)(2) + (-2)(+2)$

$= 0 + 4 - 4$

$= 0$

So, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ is perpendicular

$\Rightarrow$ $\angle C$ is right angle.

Question 26

Find the projection of $\vec{b} + \vec{c}$ on $\vec{a}$, where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

Solution 26

Projection of $\left(\vec{b}+\vec{c}\right)$ on $\vec{a}$

$$= \frac{\left(\vec{b}+\vec{c}\right).\vec{a}}{\left|\vec{a}\right|}$$

$$= \frac{\vec{b}.\vec{a}+\vec{c}.\vec{a}}{\sqrt{(2)^2+(-2)^2+(1)^2}}$$

$$= \frac{\left(\hat{i}+2\hat{j}-2\hat{k}\right)\left(2\hat{i}-2\hat{j}+\hat{k}\right)\left(2\hat{i}-\hat{j}+4\hat{k}\right)\left(2\hat{i}-2\hat{j}+\hat{k}\right)}{\sqrt{4+4+1}}$$

$$= \frac{(1)(2)+(2)(-2)+(-2)(1)+(2)(2)+(-1)(-2)+(4)(1)}{\sqrt{9}}$$

$$= \frac{2-4-2+4+2+4}{3}$$

$$= \frac{12-6}{3} = \frac{6}{3} = 2$$

Projection of $\left(\vec{b}+\vec{c}\right) = 2$

Question 27

If $\vec{a} = 5\hat{i}-\hat{j}-3\hat{k}$ and $\vec{b} = \hat{i}+3\hat{j}-5\hat{k}$, then show that the vectors $\vec{a}+\vec{b}$ and $\left(\vec{a}-\vec{b}\right)$ are orthogonal.

Solution 27

$$\vec{a}+\vec{b} = \left(5\hat{i}-\hat{j}-3\hat{k}\right)+\left(\hat{i}+3\hat{j}-5\hat{k}\right)$$
$$= 5\hat{i}-\hat{j}-3\hat{k}+\hat{i}+3\hat{j}-5\hat{k}$$
$$\vec{a}+\vec{b} = 6\hat{i}+2\hat{j}-8\hat{k} \qquad ---(i)$$

$$\vec{a}-\vec{b} = \left(5\hat{i}-\hat{j}-3\hat{k}\right)-\left(\hat{i}+3\hat{j}-5\hat{k}\right)$$
$$= 5\hat{i}-\hat{j}-3\hat{k}-\hat{i}-3\hat{j}+5\hat{k}$$
$$\vec{a}-\vec{b} = 4\hat{i}-4\hat{j}+2\hat{k} \qquad ---(ii)$$

Now, $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right)$

$$= \left(6\hat{i}+2\hat{j}-8\hat{k}\right).\left(4\hat{i}-4\hat{j}+2\hat{k}\right)$$
$$= (6)(4)+(2)(-4)+(-8)(2)$$
$$= 24-8-16$$
$$= 0$$

So, $\left(\vec{a}+\vec{b}\right)$ and $\left(\vec{a}-\vec{b}\right)$ are perpendicular.

Question 28

A unit vector $\vec{a}$ makes an angle $\frac{\pi}{4}$ and $\frac{\pi}{3}$ with $\hat{i}$ and $\hat{j}$ respectively and an acute angle θ with $\hat{k}$. Find the angle θ and components of $\vec{a}$.

Solution 28

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

$$\Rightarrow \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$.

Also, it is given that $\vec{a}$ makes angles $\frac{\pi}{4}$ with $\hat{i}$, $\frac{\pi}{3}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$.

Then, we have:

$$\cos\frac{\pi}{4} = \frac{a_1}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = a_1 \qquad [|\vec{a}| = 1]$$

$$\cos\frac{\pi}{3} = \frac{a_2}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{2} = a_2 \qquad [|\vec{a}| = 1]$$

Also, $\cos\theta = \frac{a_3}{|\vec{a}|}$.

$$\Rightarrow a_3 = \cos\theta$$

Now,

$$|a| = 1$$

$$\Rightarrow \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$$

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\theta = 1$$

$$\Rightarrow \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1$$

$$\Rightarrow \frac{3}{4} + \cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\therefore a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$$

Hence, $\theta = \frac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right)$.

Question 29

If two vector $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a}\vec{b} = 1$, then find the value of $(2\vec{a} - 5\vec{b}).(3\vec{a} + \vec{b})$.

Solution 29

$$\left(3\vec{a} - 5\vec{b}\right)\cdot\left(2\vec{a} + 7\vec{b}\right)$$

$$= 3\vec{a}\cdot 2\vec{a} + 3\vec{a}\cdot 7\vec{b} - 5\vec{b}\cdot 2\vec{a} - 5\vec{b}\cdot 7\vec{b}$$

$$= 6\vec{a}\cdot\vec{a} + 21\vec{a}\cdot\vec{b} - 10\vec{a}\cdot\vec{b} - 35\vec{b}\cdot\vec{b}$$

$$= 6|\vec{a}|^2 + 11\vec{a}\cdot\vec{b} - 35|\vec{b}|^2$$

$$= \mathbf{6*2^2 + 11*1 - 35*1^2}$$

$$= \mathbf{35 - 35}$$

$$= \mathbf{0}$$

Question 30(i)

If $\bar{a}$ is a unit vector, then find $|\bar{x}|$ in each of the following:

$$(\bar{x} - \bar{a}).(\bar{x} + \bar{a}) = 8$$

Solution 30(i)

We have,

$$\left(\bar{x}-\bar{a}\right).\left(\bar{x}+\bar{a}\right)=8$$

$$\Rightarrow \left|\bar{x}\right|^2-\left|\bar{a}\right|^2=8$$

$$\Rightarrow \left|\bar{x}\right|^2-1^2=8 \qquad \sin ce\left|\bar{a}\right|=1$$

$$\Rightarrow \left|\bar{x}\right|^2=8+1$$

$$\Rightarrow \left|\bar{x}\right|^2=9$$

$$\Rightarrow \left|\bar{x}\right|=3$$

Question 30(ii)

If $\bar{a}$ is a unit vector, then find $\left|\bar{x}\right|$ in each of the following:

$$\left(\bar{x}-\bar{a}\right).\left(\bar{x}+\bar{a}\right)=12$$

Solution 30(ii)

We have,

$$\left(\bar{x}-\bar{a}\right).\left(\bar{x}+\bar{a}\right)=12$$

$$\Rightarrow \left|\bar{x}\right|^2-\left|\bar{a}\right|^2=12$$

$$\Rightarrow \left|\bar{x}\right|^2-1^2=12 \qquad \sin ce\left|\bar{a}\right|=1$$

$$\Rightarrow \left|\bar{x}\right|^2=12+1$$

$$\Rightarrow \left|\bar{x}\right|^2=13$$

$$\Rightarrow \left|\bar{x}\right|=\sqrt{13}$$

Question 31(i)

Find $\left|\vec{a}\right|$ and $\left|\vec{b}\right|$, if

$$\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right)=12 \text{ and } \left|\vec{a}\right|=2\left|\vec{b}\right|$$

Solution 31(i)

Here, $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 12$

$|\vec{a}|^2 - |\vec{b}|^2 = 12$

$(2|\vec{b}|)^2 - |\vec{b}|^2 = 12$ $\quad\left[\text{Using } |\vec{a}| = 2|\vec{b}|\right]$

$4|\vec{b}|^2 - |\vec{b}|^2 = 12$

$3|\vec{b}|^2 = 12$

$|\vec{b}|^2 = \frac{12}{3}$

$|\vec{b}|^2 = 4$

$|\vec{b}| = 2$

$|\vec{a}| = 2|\vec{b}| = 2(2)$

$|\vec{a}| = 4$

$|\vec{b}| = 2$

Question 31(ii)

Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.

Solution 31(ii)

$(\vec{a}\cdot\vec{b}).(\vec{a}-\vec{b}) = 8$

$\Rightarrow \vec{a}\cdot\vec{a} - \vec{a}.\vec{b} + \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{b} = 8$

$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8$

$\Rightarrow (8|\vec{b}|)^2 - |\vec{b}|^2 = 8$ $\quad\left[|\vec{a}| = 8|\vec{b}|\right]$

$\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8$

$\Rightarrow 63|\vec{b}|^2 = 8$

$\Rightarrow |\vec{b}|^2 = \frac{8}{63}$

$\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}}$ $\quad$ [Magnitude of a vector is non-negative]

$\Rightarrow |\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}}$

$|\vec{a}| = 8|\vec{b}| = \frac{8\times 2\sqrt{2}}{3\sqrt{7}} = \frac{16\sqrt{2}}{3\sqrt{7}}$

Question 31(iii)

Find $\left|\vec{a}\right|$ and $\left|\vec{b}\right|$, if

$$\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right) = 3 \text{ and } \left|\vec{a}\right| = 2\left|\vec{b}\right|$$

Solution 31(iii)

Here, $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right) = 3$

$$\left|\vec{a}\right|^2 - \left|\vec{b}\right|^2 = 3$$

$$\left(2\left|\vec{b}\right|\right)^2 - \left|\vec{b}\right|^2 = 3 \qquad \left[\text{Using } \left|\vec{a}\right| = 2\left|\vec{b}\right|\right]$$

$$4\left|\vec{b}\right|^2 - \left|\vec{b}\right|^2 = 3$$

$$3\left|\vec{b}\right|^2 = 3$$

$$\left|\vec{b}\right|^2 = \frac{3}{3}$$

$$\left|\vec{b}\right|^2 = 1$$

$$\left|\vec{b}\right| = 1$$

$$\left|\vec{a}\right| = 2\left|\vec{b}\right|$$
$$= 2(1)$$

$$\left|\vec{a}\right| = 2$$
$$\left|\vec{b}\right| = 1$$

Question 32(i)

Find $\left|\vec{a}-\vec{b}\right|$, if

$$\left|\vec{a}\right| = 2,\ \left|\vec{b}\right| = 5 \text{ and } \vec{a}.\vec{b} = 8$$

Solution 32(i)

$$\left|\vec{a}-\vec{b}\right|^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 - 2\vec{a}.\vec{b}$$
$$= (2)^2 + (5)^2 - 2(8)$$
$$= 4 + 25 - 16$$
$$\left|\vec{a}-\vec{b}\right|^2 = 13$$

$$\left|\vec{a}-\vec{b}\right| = \sqrt{13}$$

Question 32(ii)

Find $\left|\vec{a}-\vec{b}\right|$, if

$$\left|\vec{a}\right| = 3,\ \left|\vec{b}\right| = 4 \text{ and } \vec{a}.\vec{b} = 1$$

Solution 32(ii)

$$\left|\vec{a}-\vec{b}\right|^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 - 2\vec{a}.\vec{b}$$
$$= (3)^2 + (4)^2 - 2.(1)$$
$$= 9 + 16 - 2$$
$$\left|\vec{a}-\vec{b}\right|^2 = 23$$
$$\left|\vec{a}-\vec{b}\right| = \sqrt{23}$$

Question 32(iii)

Find $\left|\vec{a}-\vec{b}\right|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $\left|\vec{a}\right| = 2$, $\left|\vec{b}\right| = 3$ and $\vec{a}.\vec{b} = 4$

Solution 32(iii)

We have

$$\left|\vec{a}-\vec{b}\right|^2 = \left(\vec{a}-\vec{b}\right).\left(\vec{a}-\vec{b}\right)$$
$$= \vec{a}.\vec{a} - \vec{a}.\vec{b} - \vec{b}.\vec{a} + \vec{b}.\vec{b}$$
$$= \left|\vec{a}\right|^2 - 2\left(\vec{a}.\vec{b}\right) + \left|\vec{b}\right|^2 = (2)^2 - 2(4) + (3)^2 = 5$$
$$\therefore \quad \left|\vec{a}-\vec{b}\right| = \sqrt{5}$$

Question 33(i)

Find the angle between two vectors $\vec{a}$ and $\vec{b}$

$\left|\vec{a}\right| = \sqrt{3}, \left|\vec{b}\right| = 2$ and $\vec{a}.\vec{b} = \sqrt{6}$

Solution 33(i)

We have,

$$\left|\bar{a}\right| = \sqrt{3}, \left|\bar{b}\right| = 2 \quad \text{and} \quad \bar{a}.\bar{b} = \sqrt{6}$$

Let θ be the angle between $\bar{a}$ and $\bar{b}$. Then

$$\cos\theta = \frac{\bar{a}.\bar{b}}{\left|\bar{a}\right|\left|\bar{b}\right|}$$

$$= \frac{\sqrt{6}}{\sqrt{3} \times 2}$$

$$= \frac{1}{\sqrt{2}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{4}$$

Question 33(ii)

Find the angle between two vectors $\vec{a}$ and $\vec{b}$, if

$$\left|\vec{a}\right| = 3, \ \left|\vec{b}\right| = 3 \text{ and } \vec{a}.\vec{b} = 1$$

Solution 33(ii)

Let the angle between $\vec{a}$ and $\vec{b}$ is θ, then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}$$

$$= \frac{1}{3.3}$$

$$\cos\theta = \frac{1}{9}$$

$$\theta = \cos^{-1}\left(\frac{1}{9}\right)$$

Question 34

Express the vector $\vec{a} = 5\hat{i} + 2\hat{j} + 5\hat{k}$ as the sum of two vectors such that one is parallel to the vector $\vec{b} = 3\hat{i} + \hat{k}$ and other is perpendicular to $\vec{b}$.

Solution 34

Let $\vec{a} = \vec{u} + \vec{v}$

$5\hat{i} - 2\hat{j} + 5\hat{k} = \vec{u} + \vec{v}$ ---(i)

Such that $\vec{u}$ is parallel to $\vec{b}$ and $\vec{v}$ is perpendicular to $\vec{b}$.

Now, $\vec{u}$ is parallel to $\vec{b}$

$$\vec{u} = \lambda\vec{b}$$
$$= \lambda\left(3\hat{i} + \hat{k}\right)$$
$$\vec{u} = 3\lambda\hat{i} + \lambda\hat{k} \qquad ---(ii)$$

Put value of $\vec{u}$ in equation (i),

$$5\hat{i} - 2\hat{j} + 5\hat{k} = \left(3\lambda\hat{i} + \lambda\hat{k}\right) + \vec{v}$$
$$\vec{v} = 5\hat{i} - 2\hat{j} + 5\hat{k} - 3\lambda\hat{i} - \lambda\hat{k}$$
$$\vec{v} = (5 - 3\lambda)\hat{i} + (-2)\hat{j} + (5 - \lambda)\hat{k}$$

$\vec{v}$ is perpendicular to $\vec{b}$

Then, $\vec{v}.\vec{b} = 0$

$$\left[(5 - 3\lambda)\hat{i} + (-2)\hat{j} + (5 - \lambda)\hat{k}\right].\left(3\hat{i} + 0 \times \hat{j} + \hat{k}\right) = 0$$
$$(5 - 3\lambda)(3) + (-2)(0) + (5 - \lambda)(1) = 0$$
$$15 - 9\lambda + 0 + 5 - \lambda = 0$$
$$20 - 10\lambda = 0$$
$$-10\lambda = -20$$
$$\lambda = \frac{-20}{-10}$$
$$\lambda = 2$$

Put λ in equation (ii)

$$\vec{u} = 3\lambda\hat{i} + \lambda\hat{k}$$
$$= 3(2)\hat{i} + (2)\hat{k}$$
$$\vec{u} = 6\hat{i} + 2\hat{k}$$

Put the value of $\vec{u}$ in equation (i)

$$5\hat{i} - 2\hat{j} + 5\hat{k} = \vec{u} + \vec{v}$$
$$5\hat{i} - 2\hat{j} + 5\hat{k} = \left(6\hat{i} + 2\hat{k}\right) + \vec{v}$$
$$\vec{v} = 5\hat{i} - 2\hat{j} + 5\hat{k} - 6\hat{i} - 2\hat{k}$$
$$\vec{v} = -\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{a} = \left(6\hat{i} + 2\hat{k}\right) + \left(-\hat{i} - 2\hat{j} + 3\hat{k}\right)$$

Question 35

If $\vec{a}$ and $\vec{b}$ are two vectors of the same magnitude inclined at an angle of $30°$ such that $\vec{a}\vec{b} = 3$, Find $|\vec{a}|, |\vec{b}|$.

Solution 35

Vectors $\vec{a}$ and $\vec{b}$ have same magnitude, then

$$|\vec{a}| = |\vec{b}| = x \quad \text{(Say)}$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos 30° = \frac{3}{x.x}$$

$$\frac{\sqrt{3}}{2} = \frac{3}{x^2}$$

$$\sqrt{3}x^2 = 6$$

$$x^2 = \frac{6}{\sqrt{3}}$$

Rationalizing the denominator,

$$x^2 = \frac{6 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$

$$x^2 = \frac{6\sqrt{3}}{3}$$

$$x^2 = 2\sqrt{3}$$

$$x = \sqrt{2\sqrt{3}}$$

$$|\vec{a}| = |\vec{b}| = \sqrt{2\sqrt{3}}$$

Question 36

Express $2\hat{i} - \hat{j} + 3\hat{k}$ as the sum of a vector parallel and a vector perpendicular to $2\hat{i} + 4\hat{j} - 2\hat{k}$.

Solution 36

Let $\left(2\hat{i} - \hat{j} + 3\hat{k}\right) = \vec{a} + \vec{b}$ --- (i)

Such that $\vec{a}$ is a vector parallel to vector $\left(2\hat{i} + 4\hat{j} - 2\hat{k}\right)$ and $\vec{b}$ is a vector perpendicular to the vector $\left(2\hat{i} + 4\hat{j} - 2\hat{k}\right)$.

Since, $\vec{a}$ is parallel to $\left(2\hat{i} + 4\hat{j} - 2\hat{k}\right)$

$$\vec{a} = \lambda\left(2\hat{i} + 4\hat{j} - 2\hat{k}\right)$$
$$\vec{a} = 2\lambda\hat{i} + 4\lambda\hat{j} - 2\lambda\hat{k} \qquad \text{--- (ii)}$$

Put value of $\vec{a}$ in equation (i),

$$\left(2\hat{i} - \hat{j} + 3\hat{k}\right) = \left(2\lambda\hat{i} + 4\lambda\hat{j} - 2\lambda\hat{k}\right) + \vec{b}$$
$$\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k} - 2\lambda\hat{i} - 4\lambda\hat{j} + 2\lambda\hat{k}$$
$$\vec{b} = (2 - 2\lambda)\hat{i} + (-1 - 4\lambda)\hat{j} + (3 + 2\lambda)\hat{k}$$

$\vec{b}$ is a vector perpendicular to the vector $\left(2\hat{i} + 4\hat{j} - 2\hat{k}\right)$, then

$$\vec{b}.\left(2\hat{i} + 4\hat{j} - 2\hat{k}\right) = 0$$
$$\left[(2 - 2\lambda)\hat{i} + (-1 - 4\lambda)\hat{j} + (3 + 2\lambda)\hat{k}\right]\left(2\hat{i} + 4\hat{j} - 2\hat{k}\right) = 0$$
$$(2 - 2\lambda)(2) + (-1 - 4\lambda)(4) + (3 + 2\lambda)(-2) = 0$$
$$4 - 4\lambda - 4 - 16\lambda - 6 - 4\lambda = 0$$
$$-6 - 24\lambda = 0$$
$$-24\lambda = 6$$
$$\lambda = \frac{6}{-24}$$
$$\lambda = -\frac{1}{4}$$

Put λ in equation (ii),

$$\vec{a} = 2\lambda\hat{i} + 4\lambda\hat{j} - 2\lambda\hat{k}$$
$$= 2\left(-\frac{1}{4}\right)\hat{i} + 4\left(-\frac{1}{4}\right)\hat{j} - 2\left(-\frac{1}{4}\right)\hat{k}$$
$$\vec{a} = -\frac{1}{2}\hat{i} - \hat{j} + \frac{1}{2}\hat{k}$$

Put the value of $\vec{a}$ in equation (i),

$$\left(2\hat{i} - \hat{j} + 3\hat{k}\right) = \left(-\frac{1}{2}\hat{i} - \hat{j} + \frac{1}{2}\hat{k}\right) + \vec{b}$$

$$\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k} + \frac{1}{2}\hat{i} + \hat{j} - \frac{1}{2}\hat{k}$$

$$= \frac{4\hat{i} - 2\hat{j} + 6\hat{k} + \hat{i} + 2\hat{j} - \hat{k}}{2}$$

$$= \frac{5\hat{i} + 5\hat{k}}{2}$$

$$\vec{b} = \frac{5}{2}\left(\hat{i} + \hat{k}\right)$$

$$\left(2\hat{i} - \hat{j} + 3\hat{k}\right) = \left(-\frac{1}{2}\hat{i} - \hat{j} + \frac{1}{2}\hat{k}\right) + \frac{5}{2}\left(\hat{i} + \hat{k}\right)$$

Question 37

Decompose the vector $6\hat{i} - 3\hat{j} - 6\hat{k}$ into vectors which are parallel and perpendicular to the vector $\hat{i} + \hat{j} + \hat{k}$.

Solution 37

Let $\left(6\hat{i} - 3\hat{j} - 6\hat{k}\right) = \vec{a} + \vec{b}$ --- (i)

Such that $\vec{a}$ is parallel to $\left(\hat{i} + \hat{j} + \hat{k}\right)$ and $\vec{b}$ is perpendicular to $\left(\hat{i} + \hat{j} + \hat{k}\right)$.

Since, $\vec{a}$ is parallel to $\left(\hat{i} + \hat{j} + \hat{k}\right)$

$$\vec{a} = \lambda\left(\hat{i} + \hat{j} + \hat{k}\right) \qquad \text{--- (ii)}$$

Put $\vec{a}$ in equation (i),

$$\left(6\hat{i} - 3\hat{j} - 6\hat{k}\right) = \left(\lambda\hat{i} + \lambda\hat{j} + \lambda\hat{k}\right) + \vec{b}$$
$$\vec{b} = 6\hat{i} - \lambda\hat{i} - 3\hat{j} - \lambda\hat{j} - 6\hat{k} - \lambda\hat{k}$$
$$\vec{b} = (6 - \lambda)\hat{i} + (-3 - \lambda)\hat{j} + (-6 - \lambda)\hat{k}$$

$\vec{b}$ is a vector perpendicular to the vector $\left(\hat{i} + \hat{j} + \hat{k}\right)$, then

$$\vec{b}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 0$$
$$\left[(6 - \lambda)\hat{i} + (-3 - \lambda)\hat{j} + (-6 - \lambda)\hat{k}\right]\left(\hat{i} + \hat{j} + \hat{k}\right) = 0$$
$$(6 - \lambda)(1) + (-3 - \lambda)(1) + (-6 - \lambda)(1) = 0$$
$$6 - \lambda - 3 - \lambda - 6 - \lambda = 0$$
$$-3 - 3\lambda = 0$$
$$-3 = 3\lambda$$
$$\lambda = \frac{-3}{3}$$
$$\lambda = -1$$

Put value of λ in (ii),

$$\vec{a} = -1.\left(\hat{i} + \hat{j} + \hat{k}\right)$$
$$\vec{a} = -\hat{i} - \hat{j} - \hat{k}$$

Using $\vec{a}$ in equation (i),

$$\left(6\hat{i} - 3\hat{j} - 6\hat{k}\right) = \left(-\hat{i} - \hat{j} - \hat{k}\right) + \vec{b}$$
$$\vec{b} = 6\hat{i} + \hat{i} - 3\hat{j} + \hat{j} - 6\hat{k} + \hat{k}$$
$$\vec{b} = 7\hat{i} - 2\hat{j} - 5\hat{k}$$

Thus,

Vector $\vec{a} = -\hat{i} - \hat{j} - \hat{k}$ and

$\vec{b} = 7\hat{i} - 2\hat{j} - 5\hat{k}$

are required vectors.

Question 38

Let $\vec{a} = 5\hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \lambda\hat{k}$. Find λ such that $\vec{a} + \vec{b}$ is orthogonal to $\vec{a} - \vec{b}$.

Solution 38

Here, $\left(\vec{a}+\vec{b}\right)$ is orthogonal to $\left(\vec{a}-\vec{b}\right)$

Then, $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right) = 0$

$$\left|\vec{a}\right|^2 = \left|\vec{b}\right|^2 = 0$$

$$\left\{\sqrt{(5)^2+(-1)^2+(7)^2}\right\}^2 - \left\{\sqrt{(1)^2+(-1)^2+(\lambda)^2}\right\}^2 = 0$$

$$(25+1+49)-\left(1+1+\lambda^2\right) = 0$$

$$75-\left(2+\lambda^2\right) = 0$$

$$75-2-\lambda^2 = 0$$

$$-\lambda^2 = -73$$

$$\lambda = \sqrt{73}$$

Question 39

If $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$, then what can be concluded about the vector $\vec{b}$?

Solution 39

It is given that $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$.

Now,

$\vec{a}\cdot\vec{a}=0 \Rightarrow \left|\vec{a}\right|^2=0 \Rightarrow \left|\vec{a}\right|=0$

$\therefore \vec{a}$ is a zero vector.

Hence, vector $\vec{b}$ satisfying $\vec{a}\cdot\vec{b}=0$ can be any vector

Question 40

If $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, then prove that it is perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

Solution 40

Given that $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, so,

$$\vec{a}\cdot\vec{c} = 0 \text{ and } \vec{b}\cdot\vec{c} = 0$$

Now, $\vec{c}\cdot\left(\vec{a}+\vec{b}\right)$

$= \vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}$

$= 0 + 0$

$= 0$

$\therefore \vec{c}$ is perpendicular to $\left(\vec{a}+\vec{b}\right)$

$\vec{c}\cdot\left(\vec{a}-\vec{b}\right)$

$= \vec{c}\cdot\vec{a}-\vec{c}\cdot\vec{b}$

$= 0 - 0$

$= 0$

$\therefore \vec{c}$ is perpendicular to $\left(\vec{a}-\vec{b}\right)$

Question 41

If $\left|\vec{a}\right| = a$ and $\left|\vec{b}\right| = b$, prove that $\left(\frac{\vec{a}}{a^2} - \frac{\vec{b}}{b^2}\right)^2 = \left(\frac{\vec{a}-\vec{b}}{ab}\right)^2$.

Solution 41

Here $|\vec{a}| = a,\ |\vec{b}| = b$

$$\text{LHS} = \left(\frac{\vec{a}}{a^2} - \frac{\vec{b}}{h^2}\right)^2$$

$$= \left(\frac{\vec{a}}{a^2}\right)^2 + \left(\frac{\vec{b}}{b^2}\right)^2 - 2\frac{\vec{a}}{a^2}\cdot\frac{\vec{b}}{b^2}$$

$$= \frac{|\vec{a}|^2}{a^4} + \frac{|\vec{b}|^2}{b^4} - \frac{2\vec{a}\vec{b}}{a^2b^2}$$

$$= \frac{a^2}{a^4} + \frac{b^2}{b^4} - \frac{2\vec{a}\vec{b}}{a^2b^2} \qquad \left[\text{Since } |\vec{a}| = a,\ |\vec{b}| = b\right]$$

$$= \frac{1}{a^2} + \frac{1}{b^2} - \frac{2\vec{a}\vec{b}}{a^2b^2}$$

$$= \frac{b^2 + a^2 - 2\vec{a}\vec{b}}{a^2b^2}$$

$$= \frac{|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\vec{b}}{a^2b^2}$$

$$= \frac{(\vec{a} - \vec{b})^2}{a^2b^2}$$

$$= \left(\frac{\vec{a} - \vec{b}}{ab}\right)^2$$

= RHS

Hence proved

$$\therefore \quad \left(\frac{\vec{a}}{a^2} - \frac{\vec{b}}{b^2}\right)^2 = \left(\frac{\vec{a} - \vec{b}}{ab}\right)^2$$

Question 42

If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{d}.\vec{a} = \vec{d}.\vec{b} = \vec{d}\vec{c}$ then show that $\vec{d}$ is the null vector.

Solution 42

Given that

$\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that

$\vec{d}.\vec{a} = \vec{d}.\vec{b} = \vec{d}.\vec{c} = 0$

Given that

$\vec{d}.\vec{a} = 0$

$\Rightarrow$ $\vec{d}$ perpendicular to $\vec{a}$

or $\vec{d} = 0$ $\quad ---(i)$

$\vec{d}.\vec{b} = 0$

$\Rightarrow$ $\vec{d}$ is perpendicular to $\vec{b}$ or $\vec{d} = 0$ $\quad ---(ii)$

$\vec{d}.\vec{c} = 0$

$\Rightarrow$ $\vec{d}$ is perpendicular to $\vec{c}$ or $\vec{d} = 0$ $\quad ---(iii)$

From (i), (ii), (iii), we get

$\vec{d}$ is perpendicular to $\vec{a}, \vec{b}, \vec{c}$ or $\vec{d} = 0$, but $\vec{d}$ can not be perpendicular to $\vec{a}, \vec{b}$ and $\vec{c}$ because $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors, so

$\vec{d} = 0$

Question 43

If a vector $\vec{a}$ is perpendicular to two non-collinear vectors $\vec{b}$ and $\vec{c}$, then $\vec{a}$ is perpendicular to every vector in the plane of $\vec{b}$ and $\vec{c}$.

Solution 43

Given that

$\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$

It means,

$\vec{a}.\vec{b} = 0$ and $\vec{a}.\vec{c} = 0$ $\quad --- (i)$

Let $\vec{r}$ be some vector in the plane of $\vec{b}$ and $\vec{c}$

Then, $\vec{r}, \vec{b}, \vec{c}$ are coplanar

We know that,

Three vectors are coplanar if one of them is expressible as linear combination of other two vectors.

Let $\vec{r} = x\vec{b} + y\vec{c}$

where x and y are same scalar

$\vec{r}.\vec{a} = \left(x\vec{b} + y\vec{c}\right).\vec{a}$ $\quad$ [Taking dot product with $\vec{a}$ on both the side]

$\vec{r}.\vec{a} = x\vec{b}.\vec{a} + y\vec{c}.\vec{a}$

$= x.0 + y.0$ $\quad$ [Using (i)]

$\vec{r}.\vec{a} = 0 + 0$

$\vec{r}.\vec{a} = 0$

So, $\vec{r}$ is perpendicular to $\vec{a}$

Thus,

$\vec{a}$ is perpendicular to every vector in the plane of $\vec{b}$ and $\vec{c}$

Question 44

If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, show that the angle θ between the vectors $\vec{b}$ and $\vec{c}$ is given by

$$\cos\theta = \frac{\left|\vec{a}\right|^2 - \left|\vec{b}\right|^2 - \left|\vec{c}\right|^2}{2\left|\vec{b}\right|\left|\vec{c}\right|}.$$

Solution 44

We have,

$$\vec{a}+\vec{b}+\vec{c}=\vec{0}$$
$$\vec{b}+\vec{c}=-\vec{a}$$

Squaring both the sides.

$$\left(\vec{b}+\vec{c}\right)^2=\left(-\vec{a}\right)^2$$
$$\left|\vec{b}\right|^2+\left|\vec{c}\right|^2+2\vec{b}.\vec{c}=\left|\vec{a}\right|^2$$
$$2\vec{b}.\vec{c}=\left|\vec{a}\right|^2-\left|\vec{b}\right|^2-\left|\vec{c}\right|^2$$
$$2\left|\vec{b}\right|\left|\vec{c}\right|\cos\theta=\left|\vec{a}\right|^2-\left|\vec{b}\right|^2-\left|\vec{c}\right|^2 \qquad \left[\text{Since } \vec{b}.\vec{c}=\left|\vec{b}\right|\left|\vec{c}\right|\cos\theta\right]$$

$$\cos\theta=\frac{\left|\vec{a}\right|^2-\left|\vec{b}\right|^2-\left|\vec{c}\right|^2}{2\left|\vec{b}\right|\left|\vec{c}\right|}$$

Question 45

Let $\vec{u},\vec{v}$ and $\vec{w}$ be vector such that $\vec{u}+\vec{v}+\vec{w}=0$. If $\left|\vec{u}\right|=3,\left|\vec{v}\right|=4$ and $\left|\vec{w}\right|=5$, then find $\vec{u}.\vec{v}+\vec{v}.\vec{w}+\vec{w}.\vec{u}$.

Solution 45

Here, $\vec{u}+\vec{v}+\vec{w}=0$

Squaring both the sides,

$$\left(\vec{u}+\vec{v}+\vec{w}\right)^2=\left(0\right)^2$$
$$\left|\vec{u}\right|^2+\left|\vec{v}\right|^2+\left|\vec{w}\right|^2+2\vec{u}\vec{v}+2\vec{v}\vec{w}+2\vec{w}\vec{u}=0$$
$$\left(3\right)^2+\left(4\right)^2+\left(5\right)^2+2\left(\vec{u}.\vec{v}+\vec{v}.\vec{w}+\vec{w}.\vec{u}\right)=0$$
$$9+16+25+2\left(\vec{u}\vec{v}+\vec{v}\vec{w}+\vec{w}\vec{u}\right)=0$$
$$2\left(\vec{u}\vec{v}+\vec{v}\vec{w}+\vec{w}\vec{u}\right)=-50$$
$$\vec{u}\vec{v}+\vec{v}\vec{w}+\vec{w}\vec{u}=\frac{-50}{2}$$

$$\vec{u}\vec{v}+\vec{v}\vec{w}+\vec{w}\vec{u}=-25$$

Question 46

Let $\vec{a}=x^2\hat{i}+2\hat{j}-2\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$, and $\vec{c}=x^2\hat{i}+5\hat{j}-4k$ be three vectors. Find the value of x for which the angle between $\vec{a}$ and $\vec{b}$ is acute and the angle between $\vec{b}$ and $\vec{c}$ is obtuse.

Solution 46

Given

$$\vec{a} = x^2\hat{i} + 2\hat{j} - 2\hat{k}$$
$$\vec{b} = \hat{i} - \hat{j} + \hat{k}$$
$$\vec{c} = x^2\hat{i} + 5\hat{j} - 4k$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\vec{a}.\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$$
$$\cos\theta = \frac{\vec{a}\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}$$
$$= \frac{\left(x^2\hat{i} + 2\hat{j} - 2\hat{k}\right)\left(\hat{i} - \hat{j} + \hat{k}\right)}{\sqrt{\left(x^2\right)^2 + (2)^2 + (-2)^2}\sqrt{(1)^2 + (-1)^2 + (1)^2}}$$
$$= \frac{\left(x^2\right)(1) + (2)(-1) + (-2)(1)}{\sqrt{x^4 + 4 + 4}\sqrt{1 + 1 + 1}}$$
$$= \frac{x^2 - 2 - 2}{\sqrt{8 + x^2}\sqrt{3}}$$
$$\cos\theta = \frac{x^2 - 4}{\sqrt{3}\sqrt{8 + x^4}}$$

Since θ is an acute angle, so

$$\cos\theta > 0$$
$$\frac{x^2 - 4}{\sqrt{3}\sqrt{8 + x^4}} > 0$$
$$x^2 - 4 > 0$$
$$x^2 > 4$$

$$\Rightarrow \qquad x < -2 \text{ or } x > 2 \qquad\qquad ---(i)$$

Again, let ϕ be the angle between $\vec{b}$ and $\vec{c}$,

$$\cos\phi = \frac{\vec{b}\vec{c}}{\left|\vec{b}\right|\left|\vec{c}\right|}$$
$$= \frac{\left(\hat{i} - \hat{j} + \hat{k}\right)\left(x^2\hat{i} + 5\hat{j} - 4k\right)}{\sqrt{(1)^2 + (-1)^2 + (1)^2}\sqrt{\left(x^2\right)^2 + (5)^2 + (-4)^2}}$$
$$\cos\phi = \frac{(1)\left(x^2\right) + (-1)(5) + (1)(-4)}{\sqrt{3}\sqrt{x^4 + 25 + 16}}$$

$$\cos\phi = \frac{x^2 - 5 - 4}{\sqrt{3}\sqrt{x^2 + 41}}$$

$$\cos\phi = \frac{x^2 - 9}{\sqrt{3}\sqrt{x^2 + 41}}$$

Since ϕ is an obtuse angle, so

$$\cos\phi < 0$$

$$\frac{x^2 - 9}{\sqrt{3}\sqrt{x^2 + 41}} < 0$$

$$x^2 - 9 < 0$$

$$x^2 < 9$$

$\Rightarrow$ $x > -3$ and $x < 3$ --- (ii)

From

$$-3 < x < -2 \text{ and } 2 < x < 3$$

$$x \in (-3, -2) \cup (2, 3)$$

Question 47

Find the value of x and y if the vectors $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ are mutually perpenducular vectors of equal magnitude.

Solution 47

Here, $\vec{a}$ and $\vec{b}$ are mutually perpendicular, then

$$\vec{a}.\vec{b} = 0$$

$$\left(3\hat{i} + x\hat{j} - \hat{k}\right)\left(2\hat{i} + \hat{j} + y\hat{k}\right) = 0$$

$$(3)(2) + (x)(1) + (-1)(y) = 0$$

$$6 + x - y = 0$$

$$x - y = -6 \qquad ---(i)$$

Also, $\vec{a}$ and $\vec{b}$ have equal magnitude,

$$\left|\vec{a}\right| = \left|\vec{b}\right|$$

$$\sqrt{(3)^2 + (x)^2 + (-1)^2} = \sqrt{(2)^2 + (1)^2 + (y)^2}$$

$$9 + x^2 + 1 = 4 + 1 + y^2$$

$$x^2 + 10 = 5y^2$$

$$x^2 - y^2 = 5 - 10$$

$$x^2 - y^2 = -5$$

$$(x + y)(x - y) = -5$$

$$(x + y)(-6) = -5 \qquad [\text{Using } (i)]$$

$$-6x - 6y = -5$$

$$-(6x + 6y) = -5$$

$$6x + 6y = 5 \qquad ---(ii)$$

Solving (i) and (ii),

$$\begin{array}{l} 6x + 6y = 5 \\ \underline{6x - 6y = -36} \qquad [(i) \times 6] \\ 12x \qquad = -31 \end{array}$$

$$x = \frac{-31}{12}$$

Put value of x in equation (i),

$$x - y = -6$$

$$\frac{-31}{12} - y = -6$$

$$-y = \frac{-6}{1} + \frac{31}{12}$$

$$-y = \frac{-72 + 31}{12}$$

$$y = \frac{41}{12}$$

$$y = \frac{41}{12}$$

$$x = \frac{-31}{12}$$

Question 48

If $\vec{a}$ and $\vec{b}$ are two non-coplinear unit vectors such that $\left|\vec{a}+\vec{b}\right| = \sqrt{3}$ find $\left(2\vec{a}-5\vec{b}\right).\left(3\vec{a}+\vec{b}\right)$.

Solution 48

Given

$\vec{a}$ and $\vec{b}$ are unit vectors

Then, $\left|\vec{a}\right| = \left|\vec{b}\right| = 1$

$\left|\vec{a}+\vec{b}\right| = \sqrt{3}$

Squaring both the sides,

$$\left|\vec{a}+\vec{b}\right|^2 = \left(\sqrt{3}\right)^2$$
$$\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\vec{a}.\vec{b} = 3$$
$$1+1+2\vec{a}.\vec{b} = 3$$
$$2+2\vec{a}.\vec{b} = 3$$
$$2\vec{a}.\vec{b} = 3-2$$
$$2\vec{a}.\vec{b} = 1$$
$$\vec{a}.\vec{b} = \frac{1}{2}$$

Now, $\left(2\vec{a}-5\vec{b}\right).\left(3\vec{a}+\vec{b}\right)$

$$= 2\vec{a}.3\vec{a} + 2\vec{a}\vec{b} - 5\vec{b}.3\vec{a} - 5\vec{b}.\vec{b}$$
$$= 6\left(\vec{a}\right)^2 + 2\vec{a}.\vec{b} - 15\vec{a}.\vec{b} - 5\left(\vec{b}\right)^2$$
$$= 6\left|\vec{a}\right|^2 - 13\vec{a}\vec{b} - 5\left|\vec{b}\right|^2$$
$$= 6\left(1\right)^2 - 13\left(\frac{1}{2}\right) - 5\left(1\right)^2$$
$$= \frac{6}{1} - \frac{13}{2} - \frac{5}{1}$$
$$= \frac{12-13-10}{2}$$
$$= \frac{12-23}{2}$$
$$= -\frac{11}{2}$$

$$\left(2\vec{a}-5\vec{b}\right).\left(3\vec{a}+\vec{b}\right) = -\frac{11}{2}$$

Question 49

If $\vec{a}$, $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}| = |\vec{b}|$, then prove that $\vec{a}+2\vec{b}$ is perpendicular to $\vec{a}$.

Solution 49

$|\vec{a}+\vec{b}|^2 = |\vec{b}|^2$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = \vec{b}.\vec{b}$

$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = \vec{b}.\vec{b}$

$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{a}.\vec{b} + \vec{b}.\vec{b} = \vec{b}.\vec{b}$

$\Rightarrow \vec{a}.\vec{a} + 2\,\vec{a}.\vec{b} = 0$

$\Rightarrow \vec{a}.(\vec{a} + 2\vec{b}) = 0$

$\therefore\ \vec{a} + 2\vec{b}$ is perpendicular to $\vec{a}$.

Chapter 24 - Scalar Or Dot Product Exercise Ex. 24.2

Question 1

In a triangle OAB, $\angle AOB = 90°$. If P and Q are points of trisection of AB, prove that $OP^2 + OQ^2 = \frac{5}{9}AB^2$.

Solution 1

Let $\vec{o}$, $\vec{a}$ and $\vec{b}$ be the position vector of the O, A and B.

P and Q are points of trisection of AB.

Position vector of point P = $\dfrac{2\vec{a}+\vec{b}}{3}$

Position vector of point Q = $\dfrac{\vec{a}+2\vec{b}}{3}$

$OP = \dfrac{2\vec{a}+\vec{b}}{3} - \vec{o} = \dfrac{2\vec{a}+\vec{b}-3\vec{o}}{3} = \dfrac{2OA+OB}{3}$

$OQ = \dfrac{\vec{a}+2\vec{b}}{3} - \vec{o} = \dfrac{\vec{a}+2\vec{b}-3\vec{o}}{3} = \dfrac{OA+2OB}{3}$

$OP^2 + OQ^2 = \left(\dfrac{2OA+OB}{3}\right)^2 + \left(\dfrac{OA+2OB}{3}\right)^2$

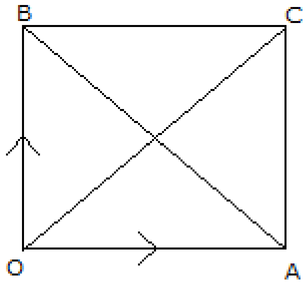
$= \dfrac{5\left(OA^2+OB^2\right)+8(OA)(OB)\cos 90°}{9}$

$= \dfrac{5}{9}AB^2 \ldots\ldots\ldots\left[\because OA^2+OB^2 = AB^2 \text{ and } \cos 90° = 0\right]$

Question 2

Prove that: If the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

Solution 2



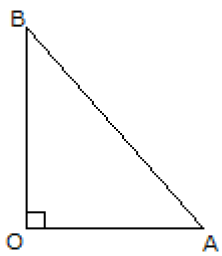
Let OACB be a quadrilateral such that its diagonal bisect each other at right angles.
We know that if the diagonals of a quadrilateral bisect each other then its a parallelogram.
$\therefore$ OACB is a parallelogram.
$\Rightarrow$ OA = BC and OB = AC.

Taking O as origin let $\vec{a}$ and $\vec{b}$ be the position vector of the A and B.

AB and OC be the diagonals of quadrilateral which bisect each other at right angles.
$\therefore \overline{OC} \bullet \overline{AB} = 0$
$\Rightarrow \left(\vec{a}+\vec{b}\right) \bullet \left(\vec{b}-\vec{a}\right) = 0$
$\Rightarrow \left|\vec{b}\right|^2 = \left|\vec{a}\right|^2$
$\Rightarrow$ OB = OA
Simillarly we can show that
OA = OB = BC = CA
Hence OACB is a rhombus.

Question 3
(Pythagoras's Theorem) Prove by vector method that in a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other rum sides.

Solution 3



Let OAC be a right triangle, right angled at O.

Taking O as origin let $\vec{a}$ and $\vec{b}$ be the position vector of the $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$

$\therefore\ \overrightarrow{OA} \bullet \overrightarrow{OB} = 0$

$\vec{a} \bullet \vec{b} = 0$

Now,

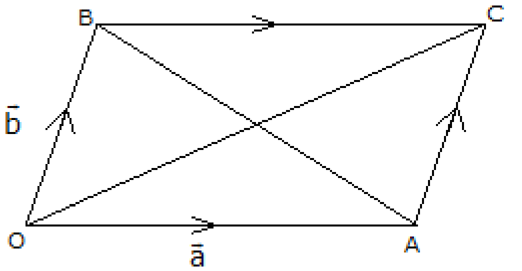
$\overrightarrow{AB}^2 = \left(\vec{b}-\vec{a}\right)^2 = (\vec{a})^2 + \left(\vec{b}\right)^2 - 2\vec{a}\bullet\vec{b} = (\vec{a})^2 + \left(\vec{b}\right)^2 - 0 = \left(\overrightarrow{OA}\right)^2 + \left(\overrightarrow{OB}\right)^2$

Hence proved.

Question 4

Prove by vector method that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

Solution 4



Let OAC be a right triangle, right angled at O.

Taking O as origin let $\vec{a}$ and $\vec{b}$ be the position vector of the $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$

$\therefore\ \overrightarrow{OA} \bullet \overrightarrow{OB} = 0$

$\vec{a} \bullet \vec{b} = 0$

Now,

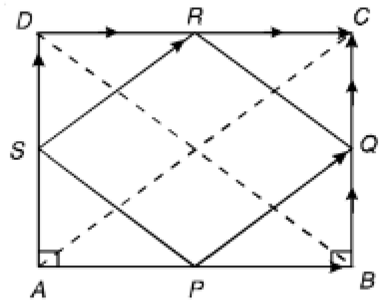
$\overrightarrow{AB}^2 = \left(\vec{b}-\vec{a}\right)^2 = (\vec{a})^2 + \left(\vec{b}\right)^2 - 2\vec{a}\bullet\vec{b} = (\vec{a})^2 + \left(\vec{b}\right)^2 - 0 = \left(\overrightarrow{OA}\right)^2 + \left(\overrightarrow{OB}\right)^2$

Hence proved.

Question 5

Prove using vectors: The quadrilateral obtained by joining mid-points of adjacent sides of a rectangle is a rhombus.

Solution 5



ABCD be a rectangle.

Let P, Q, R and S be the midpoints of the sides AB, BC, CD and DA respectively.

Now,

$\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) = \frac{1}{2}\overrightarrow{AC}$........(i)

$\overrightarrow{SR} = \overrightarrow{SD} + \overrightarrow{DR} = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{DC}\right) = \frac{1}{2}\overrightarrow{AC}$........(ii)

From (i) and (ii), we have

$\overrightarrow{PQ} = \overrightarrow{SR}$ i.e. sides PQ and SR are equal and parallel.

$\therefore$ PQRS is a parallelogram.

$(PQ)^2 = \overrightarrow{PQ} \bullet \overrightarrow{PQ} = \left(\overrightarrow{PB} + \overrightarrow{BQ}\right) \bullet \left(\overrightarrow{PB} + \overrightarrow{BQ}\right) = |PB|^2 + |BQ|^2$........(iii)

$(PS)^2 = \overrightarrow{PS} \bullet \overrightarrow{PS} = \left(\overrightarrow{PA} + \overrightarrow{PS}\right) \bullet \left(\overrightarrow{PA} + \overrightarrow{PS}\right) = |PA|^2 + |AS|^2 = |PB|^2 + |BQ|^2$........(iv)

From (iii) and (iv) we get,

$(PQ)^2 = (PQ)^2$ i. e. PQ = PS

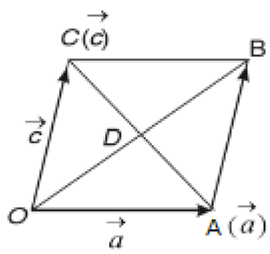
$\Rightarrow$ The adjacent sides of PQRS are equal.

$\therefore$ PQRS is a rhombus.

Question 6

Prove that the diagonals of a rhombus are perpendicular bisectors of each other.

Solution 6



Let OABC be a rhombus, whose diagonals OB and AC intersect at point D.
Let O be the origin.
Let the position vector of A and C be $\vec{a}$ and $\vec{c}$ respectively then,

$\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OC} = \vec{c}$

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{OC} = \vec{a} + \vec{c}$............$\left[\because \overrightarrow{AB} = \overrightarrow{OC}\right]$

Position vector of mid-point of $\overrightarrow{OB} = \frac{1}{2}\left(\vec{a} + \vec{c}\right)$

Position vector of mid-point of $\overrightarrow{AC} = \frac{1}{2}\left(\vec{a} + \vec{c}\right)$

$\therefore$ Midpoints of OB and AC coincide.
$\therefore$ Diagonal OB and AC bisect each other.

$\overrightarrow{OB} \bullet \overrightarrow{AC} = \left(\vec{a} + \vec{c}\right) \bullet \left(\vec{c} - \vec{a}\right) = \left(\vec{c} + \vec{a}\right) \bullet \left(\vec{c} - \vec{a}\right) = \left|\vec{c}\right|^2 - \left|\vec{a}\right|^2 = \overrightarrow{OC} - \overrightarrow{OA} = 0$

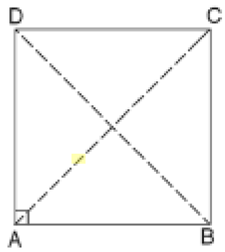
$\left[\because \text{OC and OA are sides of the rhombus}\right]$

$\Rightarrow \overrightarrow{OB} \perp \overrightarrow{AC}$

Question 7

Prove that the diagonals of a rectangle are perpendicular if and only if the rectangle is a square.

Solution 7



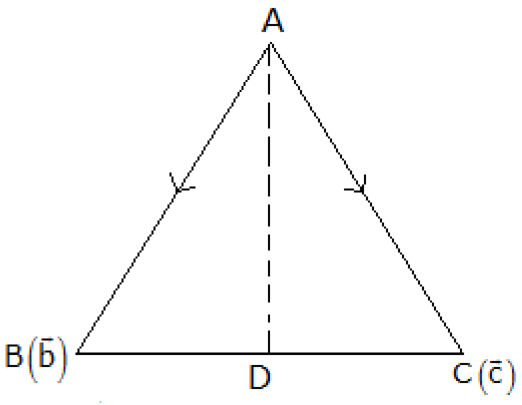
Let ABCD be a rectangle.
Take A as origin.
Let position vectors of point B, D be $\vec{a}$ and $\vec{b}$ respectively.
By parallelogram law,
$\overrightarrow{AC} = \vec{a} + \vec{b}$ and $\overrightarrow{BD} = \vec{a} - \vec{b}$
As ABCD is a rectangle, AB $\perp$ AD
$\Rightarrow \vec{a} \bullet \vec{b} = 0$.........(i)

Now, diagonals AC and BD are perpendicular iff $\overrightarrow{AC} \bullet \overrightarrow{BD} = 0$
$\Rightarrow \left(\vec{a} + \vec{b}\right)\left(\vec{a} - \vec{b}\right) = 0$
$\Rightarrow \left(\vec{a}\right)^2 - \left(\vec{b}\right)^2 = 0$
$\Rightarrow \left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{AD}\right|^2$
$\Rightarrow |AB| = |AD|$
Hence ABCD is a square.

Question 8

If AD is the median of D ABC, using vectors, prove that
$AB^2 + AC^2 = 2\ (AD^2 + CD^2)$.

Solution 8



Take A as origin, let the position vectors of B and C are $\vec{b}$ and $\vec{c}$ respectively.

Position vector of D $= \frac{\vec{b}+\vec{c}}{2}$, $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AC} = \vec{c}$.

$$\overrightarrow{AD} = \frac{\vec{b}+\vec{c}}{2} - \vec{0} = \frac{\vec{b}+\vec{c}}{2}$$

Consider, $2\left(AD^2 + CD^2\right)$

$$= 2\left[\left(\frac{\vec{b}+\vec{c}}{2}\right)^2 + \left(\frac{\vec{b}+\vec{c}}{2} - \vec{c}\right)^2\right]$$

$$= 2\left[\left(\frac{\vec{b}+\vec{c}}{2}\right)^2 + \left(\frac{\vec{b}-\vec{c}}{2}\right)^2\right]$$

$$= \frac{1}{2}\left[\left(\vec{b}+\vec{c}\right)^2 + \left(\vec{b}-\vec{c}\right)^2\right]$$

$$= \left(\vec{b}\right)^2 + \left(\vec{c}\right)^2$$

$$= \left(\overrightarrow{AB}\right)^2 + \left(\overrightarrow{AC}\right)^2$$

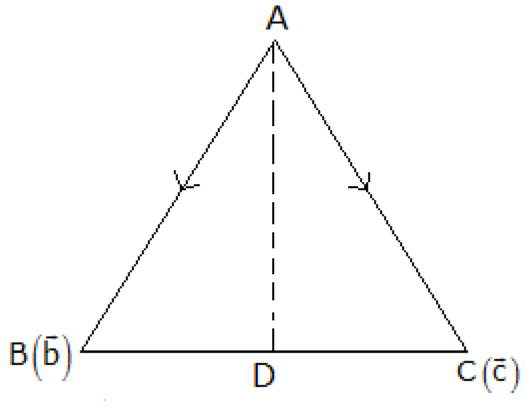
$$= AB^2 + AC^2$$

Hence proved.

Question 9

If the median to the base of a triangle is perpendicular to the base, then triangle is isosceles.

Solution 9



Take A as origin, let the position vectors of B and C are $\vec{b}$ and $\vec{c}$ respectively.

Position vector of D $= \dfrac{\vec{b}+\vec{c}}{2}$, $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AC} = \vec{c}$.

$$\overrightarrow{AD} = \frac{\vec{b}+\vec{c}}{2} - \vec{0} = \frac{\vec{b}+\vec{c}}{2}$$

AD is perpendicular to BC

$\Rightarrow \overrightarrow{AD} \bullet \overrightarrow{BC} = 0$

$\Rightarrow \left(\dfrac{\vec{b}+\vec{c}}{2}\right) \bullet \left(\vec{c} - \vec{b}\right) = 0$

$\Rightarrow \left(\vec{b}+\vec{c}\right) \bullet \left(\vec{c} - \vec{b}\right) = 0$

$\Rightarrow |\vec{c}|^2 = |\vec{b}|^2$

$\Rightarrow |\vec{c}| = |\vec{b}|$

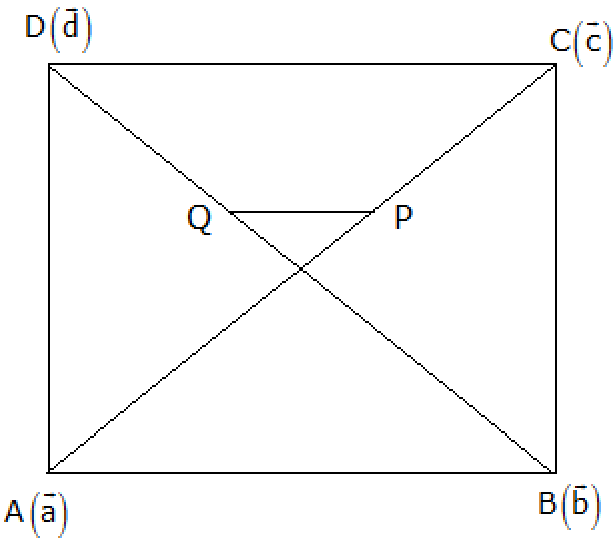
$\Rightarrow AC = AB$

Hence ΔABC is an isoscales triangle.

Question 10

In a quadrilateral ABCD, prove that $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4\,PQ^2$, where P and Q are middle points of diagonals AC and BD.

Solution 10



Take O as origin, let the position vectors of A, B C and D are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively.

Position vector of P = $\dfrac{\vec{a}+\vec{c}}{2}$

Position vector of Q = $\dfrac{\vec{a}+\vec{d}}{2}$

$$LHS = AB^2 + BC^2 + CD^2 + DA^2$$
$$= \left(\vec{b}-\vec{a}\right)^2 + \left(\vec{c}-\vec{b}\right)^2 + \left(\vec{d}-\vec{c}\right)^2 + \left(\vec{d}-\vec{a}\right)^2$$
$$= 2\left[\left(\vec{a}\right)^2 + \left(\vec{b}\right)^2 + \left(\vec{c}\right)^2 + \left(\vec{d}\right)^2 - \vec{a}\vec{b}\cos\theta_1 - \vec{b}\vec{c}\cos\theta_2 - \vec{d}\vec{c}\cos\theta_3 - \vec{c}\vec{a}\cos\theta_4\right]$$

$$RHS = AC^2 + BD^2 + 4PQ^2$$
$$= \left(\vec{c}-\vec{a}\right)^2 + \left(\vec{d}-\vec{b}\right)^2 + 4\left(\frac{\vec{a}+\vec{d}}{2} - \frac{\vec{a}+\vec{c}}{2}\right)^2$$
$$= 2\left[\left(\vec{a}\right)^2 + \left(\vec{b}\right)^2 + \left(\vec{c}\right)^2 + \left(\vec{d}\right)^2 - \vec{a}\vec{b}\cos\theta_1 - \vec{b}\vec{c}\cos\theta_2 - \vec{d}\vec{c}\cos\theta_3 - \vec{c}\vec{a}\cos\theta_4\right]$$
$$= LHS$$

Hence proved.

Chapter 24 - Scalar Or Dot Product Exercise MCQ

Question 1

The vector $\vec{a}$ and $\vec{b}$ satisfy the equation $2\vec{a}+\vec{b}=\vec{p}$ and $\vec{a}+2\vec{b}=\vec{q}$, where $\vec{p}=\hat{i}+\hat{j}$ and $\vec{q}=\hat{i}-\hat{j}$.

If θ is the angle between $\vec{a}$ and $\vec{b}$, then

(a) $\cos\theta = \dfrac{4}{5}$

(b) $\sin\theta = \dfrac{1}{\sqrt{2}}$

(c) $\cos\theta = -\dfrac{4}{5}$

(d) $\cos\theta = -\dfrac{3}{5}$

Solution 1

Correct option: (c)

$2\vec{a}+\vec{b}=\vec{p}$........(i)

$\vec{a}+2\vec{b}=\vec{q}$........(ii)

$\Rightarrow \vec{a}=\dfrac{2\vec{p}-\vec{q}}{3}, \vec{b}=\dfrac{2\vec{q}-\vec{p}}{3}$

putting in above equations $\vec{p}=\hat{i}+\hat{j}, \vec{q}=\hat{i}-\hat{j}$

$\vec{a}=\dfrac{2(\hat{i}+\hat{j})-(\hat{i}-\hat{j})}{3}=\dfrac{\hat{i}+3\hat{j}}{3}$

$\vec{b}=\dfrac{2(\hat{i}-\hat{j})-(\hat{i}+\hat{j})}{3}=\dfrac{\hat{i}-3\hat{j}}{3}$

$\Rightarrow |\vec{a}|=\dfrac{\sqrt{10}}{3}, |\vec{b}|=\dfrac{\sqrt{10}}{3}$

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

$\dfrac{1}{9}(1-9)=\dfrac{\sqrt{10}}{3}\times\dfrac{\sqrt{10}}{3}\cos\theta$

$\dfrac{-8}{9}=\dfrac{10}{9}\cos\theta$

$\cos\theta=\dfrac{-4}{5}$

Question 2

If $\vec{a}\cdot\hat{i}=\vec{a}\cdot(\hat{i}+\hat{j})=\vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})=1$, then $\vec{a}=$

(a) $\vec{0}$

(b) $\hat{i}$

(c) $\hat{j}$

(d) $\hat{i}+\hat{j}+\hat{k}$

Solution 2

Correct option: (b)

$\vec{a}\cdot\hat{i}=\vec{a}\cdot(\hat{i}+\hat{j})=\vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})=1$

$\Rightarrow \vec{a}\cdot i=a_1, \vec{a}\cdot(\hat{i}+\hat{j})=a_1+a_2, \vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})=a_1+a_2+a_3$

$\Rightarrow a_1=a_1+a_2=a_1+a_2+a_3=1$

Consider, $a_1=a_1+a_2 \Rightarrow a_2=0$

$a_1+a_2=a_1+a_2+a_3$

$\Rightarrow a_3=0$

$\Rightarrow a_1=\hat{i}$

Question 3

If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, $|\vec{a}|=3, |\vec{b}|=5, |\vec{c}|=7$, then the angle between them, $\vec{a}$ and $\vec{b}$ is

(a) $\frac{\pi}{6}$

(b) $\frac{2\pi}{3}$

(c) $\frac{5\pi}{3}$

(d) $\frac{\pi}{3}$

Solution 3

Correct option: (d)

Given that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \vec{a}+\vec{b}=-\vec{c}$

squaring on both sides,

$\left(\vec{a}+\vec{b}\right)^2=\left(-\vec{c}\right)^2$

$\vec{a}^2+2\vec{a}\cdot\vec{b}+\vec{b}^2=\vec{c}^2$

Given that $|\vec{a}|=3, |\vec{b}|=5, |\vec{c}|=7$

$\Rightarrow 9+2\times 3\times 7\cos\theta+25=49$

$\cos\theta=\frac{1}{2}$

$\Rightarrow \theta=\frac{\pi}{3}$

Question 4

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and α be the angle between them, then $\vec{a}+\vec{b}$ is a unit vector, if

(a) $\alpha=\frac{\pi}{4}$

(b) $\alpha=\frac{\pi}{3}$

(c) $\alpha=\frac{2\pi}{3}$

(d) $\alpha=\frac{\pi}{2}$

Solution 4

Correct option: (c)

$|\vec{a}| = 1, |\vec{b}| = 1$

$\Rightarrow |\vec{a} + \vec{b}| = 1$

$(\vec{a} + \vec{b})^2 = 1$

$\vec{a}^2 + 2\vec{a} \cdot \vec{b} + \vec{b}^2 = 1$

As both are unit vectors,

$1 + 2\cos\alpha + 1 == 1$

$2\cos\alpha = -1$

$\cos\alpha = \frac{-1}{2}$

$\Rightarrow \alpha = \frac{2\pi}{3}$

Question 5

The vector $(\cos\alpha \cos\beta)\hat{i} + (\cos\alpha \sin\beta)\hat{j} + (\sin\alpha)\hat{k}$ is a

(a) Null vector
(b) Unit vector
(c) Constant vector
(d) None of these

Solution 5

Correct option: (b)

$\cos\alpha \cos\beta\,\hat{i} + \cos\alpha \sin\beta\,\hat{j} + \sin\alpha\hat{k}$

Magnitude of a vector

$= \sqrt{\cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha}$

$= \sqrt{\cos^2\alpha\left(\cos^2\beta + \sin^2\beta\right) + \sin^2\alpha}$

$= \sqrt{\cos^2\alpha + \sin^2\alpha}$

$= 1$

Hence, it is unit vector.

Question 6

If the position vectors of P and Q are $\hat{i} + 3\hat{j} - 7\hat{k}$ and $5\hat{i} - 2\hat{j} + 4\hat{k}$ then the cosine of the angle between $\overrightarrow{PQ}$ and y-axis is

(a) $\frac{5}{\sqrt{162}}$

(b) $\frac{4}{\sqrt{162}}$

(c) $-\frac{5}{\sqrt{162}}$

(d) $\frac{11}{\sqrt{162}}$

Solution 6

Correct option: (c)

Let O be the origin.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$\overrightarrow{PQ} = 5\hat{i} - 2\hat{j} + 4\hat{k} - \left(\hat{i} + 3\hat{j} - 7\hat{k}\right)$

$\overrightarrow{PQ} = 4\hat{i} - 5\hat{j} + 11\hat{k}$

We have to find angle between $\overrightarrow{PQ}$ and y – axis.

Unit vector along y – axis is $\hat{j}$.

$$\cos\theta = \frac{\left(4\hat{i} - 5\hat{j} + 11\hat{k}\right)\cdot\hat{j}}{\sqrt{162}} = \frac{-5}{\sqrt{162}}$$

Question 7

If $\vec{a}$ and $\vec{b}$ are unit vectors, then which of the following values of $\vec{a}.\vec{b}$ is not possible?

(a) $\sqrt{3}$

(b) $\sqrt{3}/2$

(c) $1/\sqrt{2}$

(d) $-1/2$

Solution 7

Correct option: (a)

$\vec{a}$ and $\vec{b}$ are unit vectors.

$\Rightarrow \vec{a}\cdot\vec{b} = \vec{a}\vec{b}\cos\theta$

$\Rightarrow \vec{a}\cdot\vec{b} = \cos\theta$

$-1 < \cos\theta < 1$

Hene, it can not be > 1.

Option (a) is incorrect.

Question 8

If the vectors $\hat{i} - 2x\hat{j} + 3y\hat{k}$ and $\hat{i} + 2x\hat{j} - 3y\hat{k}$ are perpendicular, then the locus of (x, y) is

a. A circle
b. An ellipse
c. A hyperbola
d. None of these

Solution 8

Correct option: (b)

Let, $\vec{a} = \hat{i} - 2x\hat{j} + 3y\hat{k}$ and $\vec{b} = \hat{i} + 2x\hat{j} - 3y\hat{k}$

Given that vectors are perpendicular.

$\Rightarrow \vec{a} \cdot \vec{b} = 0$

$1 - 4x^2 - 9y^2 = 0$

$4x^2 + 9y^2 = 1$

This is equation of ellipse.

Question 9

The vector componet of $\vec{b}$ perpendicular to $\vec{a}$ is

(a) $(\vec{b} \,.\, \vec{c})\,\vec{a}$

(b) $\dfrac{\vec{a} \times (\vec{b} \times \vec{a})}{|\vec{a}|^2}$

(c) $\vec{a} \times (\vec{b} \times \vec{a})$

(d) none of these

Solution 9

Correct option: (b)

Vector $\vec{b}$ perpendicular to $\vec{a}$ is given by

$$\frac{\vec{a} \times (\vec{b} \times \vec{a})}{|\vec{a}|^2}$$

Question 10

The length of the longer diagonal of the parallelogram constructed on $5\vec{a} + 2\vec{b}$ and $\vec{a} - 3\vec{b}$ if it is given that $|\vec{a}| = 2\sqrt{2}, |\vec{b}| = 3$ and angle between $\vec{a}$ and $\vec{b}$ is $\pi/4$, is

(a) 15

(b) $\sqrt{113}$

(c) $\sqrt{593}$

(d) $\sqrt{369}$

Solution 10

Correct option: (c)

Given that $\overrightarrow{AD} = \overrightarrow{BC} = \vec{a} - 3\vec{b}$

$\overrightarrow{AB} = \overrightarrow{DC} = 5\vec{a} + 2\vec{b}$

And diagonals are BD and AC.

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{AC} = 6\vec{a} - \vec{b}$

$\left|\overrightarrow{AC}\right| = \sqrt{36a^2 + b^2 - 12ab\cos\frac{\pi}{4}}$

$\left|\overrightarrow{AC}\right| = \sqrt{36 \times 8 + 9 - 12 \times 6\sqrt{2} \times \frac{1}{\sqrt{2}}}$

$\left|\overrightarrow{AC}\right| = \sqrt{297 - 72}$

$\left|\overrightarrow{AC}\right| = \sqrt{225} = 15$ units

similarly using $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{BD}$

we can find $\overrightarrow{BD} = \sqrt{593}$ units

$\Rightarrow$ The larger diagonal is $\sqrt{593}$ units

Question 11

If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ is a non-zero scalar, then $\lambda\,\vec{a}$ is a unit vector if

(a) $\lambda = 1$

(b) $\lambda = -1$

(c) $a = |\lambda|$

(d) $a = \frac{1}{|\lambda|}$

Solution 11

Correct option: (d)

Given that $\lambda\vec{a}$ is unit vector.

$\Rightarrow \lambda\vec{a} = 1$

$\Rightarrow |\lambda|\left|\vec{a}\right| = 1$

$\Rightarrow \left|\vec{a}\right| = \frac{1}{|\lambda|}$

Question 12

If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a} \cdot \vec{b} \geq 0$ only when

(a) $0 < \theta < \frac{\pi}{2}$

(b) $0 \leq \theta \leq \frac{\pi}{2}$

(c) $0 < \theta < \pi$

(d) $0 \leq \theta \leq \pi$

Solution 12

Correct option: (b)

Given that $\vec{a} \cdot \vec{b} \geq 0$

$\Rightarrow \vec{a}\vec{b}\cos\theta \geq 0$

$\Rightarrow \cos\theta \geq 0$

$0 \leq \theta \leq \frac{\pi}{2}$

Question 13

The values of x for which the angle between $\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$, $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse and the angle between $\vec{b}$ and the z-axis is acute and less than $\frac{\pi}{6}$ are

(a) $x > \frac{1}{2}$ or $x < 0$

(b) $0 < x < \frac{1}{2}$

(c) $\frac{1}{2} < x < 15$

(d) ϕ

Solution 13

Correct option: (b)

$\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k},$

$\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$

Let, $\cos A$ be the angle between vector a and b.

But A is obtuse angle.

$\cos A < 0$

$$\frac{14x^2 - 7x}{\sqrt{\left(4x^4 + 16x^2 + 1\right)\left(53 + x^2\right)}} < 0$$

$\Rightarrow 14x^2 - 7x < 0$

$\Rightarrow x(2x - 1) < 0$

$\Rightarrow x > 0 \text{ or } x < \frac{1}{2}$

The angle between $\vec{b}$ – axis is B.

$0 < \cos B < \cos\frac{\pi}{6}$

$0 < \frac{x}{\sqrt{53 + x^2}} < \frac{\sqrt{3}}{2}$

$\Rightarrow 0 < x < \frac{1}{2}$

Question 14

If $\vec{a}, \vec{b}, \vec{c}$ are any three mutually perpendicular vectors of equal magnitude a, then $|\vec{a} + \vec{b} + \vec{c}|$ is equal to

(a) a

(b) $\sqrt{2}$ a

(c) $\sqrt{3}$ a

(d) 2 a

(e) none of these

Solution 14

Correct option: (c)

Given that $\vec{a}, \vec{b}, \vec{c}$ are any three mutually perpendicular vectors of equal magnitude a.

$\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}| = a$

Also,

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$

Hence,

$(\vec{a} + \vec{b} + \vec{c})^2 = a^2 + b^2 + c^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a}$

$(\vec{a} + \vec{b} + \vec{c})^2 = a^2 + a^2 + a^2$

$(\vec{a} + \vec{b} + \vec{c})^2 = 3a^2$

$|\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}a$

Question 15

If the vectors $3\hat{i} + \lambda\hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} + 8\hat{k}$ are perpendicular, then λ is equal to

(a) -14

(b) 7

(c) 14

(d) $\frac{1}{7}$

Solution 15

Correct option: (c)

The vectors $3\hat{i} + \lambda\hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} + 8\hat{k}$ are perpendicular.

$\Rightarrow 6 - \lambda + 8 = 0$

$\Rightarrow \lambda = 14$

Question 16

The projection of the vector $\hat{i} + \hat{j} + \hat{k}$ along the vector of $\hat{j}$ is

a. 1
b. 0
c. 2
d. -1
e. -2

Solution 16

Correct option: (a)

The projection of the vector $\hat{i}+\hat{j}+\hat{k}$ along the vector of $\hat{j}$ is

$\vec{p}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{q}=\hat{j}$

The projection of $\vec{p}$ on $\vec{q}$

$$\frac{\left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\hat{j}}{\left|\hat{j}\right|}=1$$

Question 17

The vectors $2\hat{i}+3\hat{j}-4\hat{k}$ and $a\hat{i}+b\hat{j}+c\hat{k}$ are perpendicular,if

(a) a = 2, b = 3, c = -4

(b) a = 4, b = 4, c = 5

(c) a = 4, b = 4, c = -5

(d)a = -4, b = 4, c = -5

Solution 17

Correct option: (b)

$2\hat{i}+3\hat{j}-4\hat{k}$ and $a\hat{i}+b\hat{j}+c\hat{k}$ are perpendicular.

$\Rightarrow 2a+3b-4c=0$

For option

$a=4, b=4, c=5$

$2a+3b-4c=0$ satisfies.

Question 18

If $|\vec{a}| = |\vec{b}|$, then $(\vec{a}+\vec{b})(\vec{a}-\vec{b})=$

a. Positive
b. Negative
c. 0
d. None of these

Solution 18

Correct option: (c)

$(\vec{a}+\vec{b})(\vec{a}-\vec{b})=|\vec{a}|^2-|\vec{b}|^2$

Given that $|\vec{a}|=|\vec{b}|$

$\Rightarrow(\vec{a}+\vec{b})(\vec{a}-\vec{b})=0$

Question 19

If $\vec{a}$ and $\vec{b}$ are unit vectors inclined at an angle θ, then the value of $|\vec{a}-\vec{b}|$ is

(a) $2\sin\dfrac{\theta}{2}$

(b) $2\sin\theta$

(c) $2\cos\dfrac{\theta}{2}$

(d) $2\cos\theta$

Solution 19

Correct option: (a)

$\vec{a}$ and $\vec{b}$ are unit vectors.

$\vec{a}\cdot\vec{b} = \vec{a}\,\vec{b}\cos\theta$

$\vec{a}\cdot\vec{b} = 1\times 1\times\cos\theta$

$\vec{a}\cdot\vec{b} = \cos\theta$

$|\vec{a}-\vec{b}|^2 = \vec{a}^2 - 2\vec{a}\cdot\vec{b} + \vec{b}^2$

$|\vec{a}-\vec{b}|^2 = 1 - 2\cos\theta + 1$

$|\vec{a}-\vec{b}|^2 = 2 - 2\cos\theta$

$|\vec{a}-\vec{b}|^2 = 2(1-\cos\theta)$

$|\vec{a}-\vec{b}|^2 = 2\left(2\sin^2\dfrac{\theta}{2}\right)$

$|\vec{a}-\vec{b}| = 2\sin\dfrac{\theta}{2}$

Question 20

If $\vec{a}$ and $\vec{b}$ are unit vectors, then the greatest value of $\sqrt{3}\,|\vec{a}+\vec{b}| + |\vec{a}-\vec{b}|$ is

(a) 2

(b) $2\sqrt{2}$

(c) 4

(d) none of these

Solution 20

Correct option: (c)

$$\sqrt{3}\,|\vec{a}+\vec{b}\,|+|\,\vec{a}-\vec{b}\,|=\sqrt{3}\sqrt{a^2+b^2+2ab\cos\theta}+\sqrt{a^2+b^2-2ab\cos\theta}$$

But $|\vec{a}\,|=|\,\vec{b}\,|=1$

$$\Rightarrow\sqrt{3}\,|\,\vec{a}+\vec{b}\,|+|\,\vec{a}-\vec{b}\,|=\sqrt{3}\sqrt{1+1+2\cos\theta}+\sqrt{1+1-2\cos\theta}$$

$$\Rightarrow\sqrt{3}\,|\,\vec{a}+\vec{b}\,|+|\,\vec{a}-\vec{b}\,|=\sqrt{3}\sqrt{2+2\cos\theta}+\sqrt{2-2\cos\theta}$$

$$\Rightarrow\sqrt{3}\,|\,\vec{a}+\vec{b}\,|+|\,\vec{a}-\vec{b}\,|=\sqrt{3}\sqrt{2(1+\cos\theta)}+\sqrt{2(1-\cos\theta)}$$

$$\Rightarrow\sqrt{3}\,|\,\vec{a}+\vec{b}\,|+|\,\vec{a}-\vec{b}\,|=\sqrt{3}\sqrt{2\times 2\cos^2\frac{\theta}{2}}+\sqrt{2\times 2\sin^2\frac{\theta}{2}}$$

$$\Rightarrow\sqrt{3}\,|\,\vec{a}+\vec{b}\,|+|\,\vec{a}-\vec{b}\,|=2\sqrt{3}\cos\frac{\theta}{2}+2\sin\frac{\theta}{2}$$

$$\Rightarrow\sqrt{3}\,|\,\vec{a}+\vec{b}\,|+|\,\vec{a}-\vec{b}\,|=2\times 2\left(\frac{\sqrt{3}}{2}\cos\frac{\theta}{2}+\frac{1}{2}\sin\frac{\theta}{2}\right)$$

$$\Rightarrow\sqrt{3}\,|\,\vec{a}+\vec{b}\,|+|\,\vec{a}-\vec{b}\,|=2\times 2\left(\frac{\sqrt{3}}{2}\cos\frac{\theta}{2}+\frac{1}{2}\sin\frac{\theta}{2}\right)$$

$$\Rightarrow\sqrt{3}\,|\,\vec{a}+\vec{b}\,|+|\,\vec{a}-\vec{b}\,|=4\left(\sin\frac{\pi}{3}\cos\frac{\theta}{2}+\cos\frac{\pi}{3}\sin\frac{\theta}{2}\right)$$

$$\Rightarrow\sqrt{3}\,|\,\vec{a}+\vec{b}\,|+|\,\vec{a}-\vec{b}\,|=4\left(\sin\left(\frac{\pi}{3}+\frac{\theta}{2}\right)\right)$$

$$-1\le\left(\sin\left(\frac{\pi}{3}+\frac{\theta}{2}\right)\right)\le 1$$

$$\Rightarrow 4\left(\sin\left(\frac{\pi}{3}+\frac{\theta}{2}\right)\right)\le 4$$

Maximum value of $\sqrt{3}\,|\,\vec{a}+\vec{b}\,|+|\,\vec{a}-\vec{b}\,|$ is 4.

Question 21

If the angle between the vectors $x\hat{i}+3\hat{j}-7\hat{k}$ and $x\hat{i}-x\hat{j}+4\hat{k}$ is acute, then x lies in the interval

a. (-4,7)
b. [-4,7]
c. R - [-4,7]
d. R - (4,7)

Solution 21

Correct option: (c)

The angle between the vectors $x\hat{i} + 3\hat{j} - 7\hat{k}$ and $x\hat{i} - x\hat{j} + 4\hat{k}$

$$\cos\theta = \frac{x^2 - 3x - 28}{\sqrt{x^2 + 53}\sqrt{2x^2 + 16}}$$

$\theta < \frac{\pi}{2}$

$\Rightarrow \cos\theta > 0$

$x^2 - 3x - 28 > 0$

$(x - 7)(x + 4) > 0$

$\Rightarrow x \in R - [-4, 7]$

NOTE: Answer not matching with back answer.

Question 22

If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at an angle θ such that $|\vec{a} + \vec{b}| < 1$, then

(a) $\theta < \frac{\pi}{3}$

(b) $\theta > \frac{2\pi}{3}$

(c) $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$

(d) $\frac{2\pi}{3} < \theta < \pi$

Solution 22

Correct option: (d)

$\vec{a}$ and $\vec{b}$ are two unit vectors indined at an angle θ such that $|\vec{a} + \vec{b}| < 1$

$|\vec{a} + \vec{b}|^2 < 1$

$\Rightarrow 2 + 2\cos\theta < 1$

$\Rightarrow 2(1 + \cos\theta) < 1$

$\Rightarrow 2\cos^2\frac{\theta}{2} < 1$

$\Rightarrow \cos^2\frac{\theta}{2} < \frac{1}{4}$

$\Rightarrow \left|\cos\frac{\theta}{2}\right| < \frac{1}{2}$

$\cos\theta$ is always between $[-\pi, \pi]$

$\Rightarrow \frac{2\pi}{3} < \theta < \pi$

NOTE: Answer not matching with back answer.

Question 23

Let $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors such that $|\vec{a}+\vec{b}+\vec{c}| = 1$ and $\vec{a}$ is perpendicular to $\vec{b}$. If $\vec{c}$ makes angle α and β with $\vec{a}$ and $\vec{b}$ respectively, then $\cos\alpha + \cos\beta =$

(a) $-\frac{3}{2}$

(b) $\frac{3}{2}$

(c) 1

(d) -1

Solution 23

Correct option: (d)

$$\left|\vec{a}+\vec{b}+\vec{c}\right| = 1$$
$$\Rightarrow \left|\vec{a}+\vec{b}+\vec{c}\right|^2 = 1$$
$$\Rightarrow a^2 + b^2 + c^2 + 2a \cdot b + 2b \cdot c + 2c \cdot a = 1$$
$$\Rightarrow 1 + 1 + 1 + 0 + 2\cos\alpha + 2\cos\beta = 1$$
$$\Rightarrow \cos\alpha + \cos\beta = -1$$

Question 24

The orthogonal projection of $\vec{a}$ on $\vec{b}$ is

(a) $\frac{(\vec{a}\cdot\vec{b})\,\vec{a}}{|\vec{a}|^2}$

(b) $\frac{(\vec{a}\cdot\vec{b})\,\vec{b}}{|\vec{b}|^2}$

(c) $\frac{\vec{a}}{|\vec{a}|}$

(d) $\frac{\vec{b}}{|\vec{b}|}$

Solution 24

Correct option: (b)

The orthogonal projection of $\vec{a}$ on $\vec{b}$ is

$$\frac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}$$

Question 25

If θ is an acute angle and the vector $(\sin\theta)\hat{i} + (\cos\theta)\hat{j}$ is perpendicular to the vector $\hat{i} - \sqrt{3}\hat{j}$, then $\theta =$

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{5}$

(c) $\frac{\pi}{4}$

(d) $\frac{\pi}{3}$

Solution 25

Correct option: (d)

The vector $(\sin\theta)\hat{i} + (\cos\theta)\hat{j}$

is perpendicular to the vector $\hat{i} - \sqrt{3}\hat{j}$

$\Rightarrow \sin\theta - \sqrt{3}\cos\theta = 0$

$\Rightarrow \sin\theta = \sqrt{3}\cos\theta$

$\Rightarrow \tan\theta = \sqrt{3}$

$\Rightarrow \theta = \frac{\pi}{3}$

Question 26

If $\vec{a}$ and $\vec{b}$ be two unit vector and θ is the angle between them.

Then $\vec{a} + \vec{b}$ is a unit vector,

(a) $\frac{\pi}{4}$

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$

(d) $\frac{2\pi}{3}$

Solution 26

Correct option: (d)

$\left|\vec{a}\right| = 1,\ \left|\vec{b}\right| = 1$

$\Rightarrow \left|\vec{a}+\vec{b}\right| = 1$

$\left(\vec{a}+\vec{b}\right)^2 = 1$

$\vec{a}^2 + 2\vec{a}\cdot\vec{b} + \vec{b}^2 = 1$

As both are unit vectors,

$1 + 2\cos\theta + 1 == 1$

$2\cos\theta = -1$

$\cos\theta = \frac{-1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}$

Chapter 24 - Scalar Or Dot Product Exercise Ex. 24VSAQ

Question 1

What is the angle between $\vec{a}$ and $\vec{b}$ with magnitudes 2 and $\sqrt{3}$ respectively?

Solution 1

Given

$\vec{a}.\vec{b} = \sqrt{3}$

Here, $\left|\vec{a}\right| = 2,\ \left|\vec{b}\right| = \sqrt{3},\ \vec{a}.\vec{b} = \sqrt{3}$

Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}$$

$$= \frac{\sqrt{3}}{2\sqrt{3}}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{3}$$

Question 2

If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a}.\vec{b} = 6$, $\left|\vec{a}\right| = 3$ and $\left|\vec{b}\right| = 4$. Write the projection of $\vec{a}$ on $\vec{b}$.

Solution 2

Here, $\vec{a}.\vec{b} = 6$, $|\vec{a}| = 3$, $|\vec{b}| = 4$

Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$= \frac{6}{4}$$
$$= \frac{3}{2}$$

Projection of $\vec{a}$ on $\vec{b} = \dfrac{3}{2}$

Question 3

Find the cosine of the angle between the vectors $4\hat{i} - 3\hat{j} + 3\hat{k}$ and $2\hat{i} - \hat{j} - \hat{k}$.

Solution 3

Here, Let angle between vector $\vec{a}$ and $\vec{b}$ is θ, then

$$\cos\theta = \frac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|}$$
$$= \frac{\left(4\hat{i} - 3\hat{j} + 3\hat{k}\right)\left(2\hat{i} - \hat{j} - \hat{k}\right)}{\sqrt{(4)^2 + (-3)^2 + (3)^2}\sqrt{(2)^2 + (-1)^2 + (-1)^2}}$$
$$= \frac{(4)(2) + (-3)(-1) + (3)(-1)}{\sqrt{16 + 9 + 9}\sqrt{4 + 1 + 1}}$$
$$= \frac{8 + 3 - 3}{\sqrt{34}\sqrt{6}}$$
$$= \frac{8}{\sqrt{17 \times 2 \times 6}}$$
$$= \frac{8}{2\sqrt{51}}$$

$$\cos\theta = \frac{4}{\sqrt{51}}$$

Question 4

If the vectors $3\hat{i} + m\hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} - 8\hat{k}$ are orthogonal, find m.

Solution 4

We know that,

Vector $\vec{a}$ and $\vec{b}$ are orthogonal (perpendicular) if

$\vec{a}.\vec{b} = 0$

$\left(3\hat{i} + m\hat{j} + \hat{k}\right).\left(2\hat{i} - \hat{j} - 8\hat{k}\right) = 0$

$(3)(2) + (m)(-1) + (1)(-8) = 0$

$6 - m - 8 = 0$

$-m - 2 = 0$

$-m = 2$

$m = -2$

Question 5

If the vectors $3\hat{i} - 2\hat{j} - 4\hat{k}$ and $18\hat{i} - 12\hat{j} - m\hat{k}$ are parallel, find the value of m.

Solution 5

We know that,

Vector $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and

$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ are parallel when

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\therefore \quad \frac{3}{18} = \frac{(-2)}{(-12)} = \frac{(-4)}{(-m)}$$

$$\frac{1}{6} = \frac{1}{6} = \frac{4}{m}$$

Cross multiplying the last two,

$m = 24$

Question 6

If $\vec{a}$ and $\vec{b}$ are vectors of equal magnitude, with the value of $\left(\vec{a} + \vec{b}\right).\left(\vec{a} - \vec{b}\right)$.

Solution 6

Here, $\left|\vec{a}\right| = \left|\vec{b}\right| = x$ (Say)

$$\begin{aligned}\left(\vec{a} + \vec{b}\right)\left(\vec{a} - \vec{b}\right) &= \left(\vec{a}\right)^2 - \left(\vec{b}\right)^2 \\ &= \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2 \\ &= x^2 - x^2 \\ &= 0\end{aligned}$$

$$\therefore \quad \left(\vec{a} + \vec{b}\right).\left(\vec{a} - \vec{b}\right) = 0$$

Question 7

If $\vec{a}$ and $\vec{b}$ are two vectors such that $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right) = 0$, find the relation between the magnitudes of $\vec{a}$ and $\vec{b}$.

Solution 7

Here, $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right) = 0$

$$\left(\vec{a}\right)^2 - \left(\vec{b}\right)^2 = 0$$

$$\left|\vec{a}\right|^2 - \left|\vec{b}\right|^2 = 0$$

$$\left|\vec{a}\right|^2 = \left|\vec{b}\right|^2$$

$\therefore \quad \left|\vec{a}\right| = \left|\vec{b}\right|$

Question 8

For any two vectors $\vec{a}$ and $\vec{b}$ write when $\left|\vec{a}+\vec{b}\right| = \left|\vec{a}\right| + \left|\vec{b}\right|$ holds.

Solution 8

Here, $\left|\vec{a}+\vec{b}\right| = \left|\vec{a}\right| + \left|\vec{b}\right|$

Squaring both the sides,

$$\left(\left|\vec{a}+\vec{b}\right|\right)^2 = \left(\left|\vec{a}\right| + \left|\vec{b}\right|\right)^2$$

$$\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\vec{a}.\vec{b} = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\left|\vec{a}\right|\left|\vec{b}\right|$$

$$2\vec{a}.\vec{b} = 2\left|\vec{a}\right|\left|\vec{b}\right|$$

$$\vec{a}\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right| \qquad ---(i)$$

Now, let θ be the angle between $\vec{a}$ and $\vec{b}$,

$$\cos\theta = \frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}$$

$$\frac{\left|\vec{a}\right|\left|\vec{b}\right|}{\left|\vec{a}\right|\left|\vec{b}\right|} \qquad \left[\text{Using } (1)\right]$$

$$\cos\theta = 1$$

$$\theta = \cos^{-1}(1)$$

$$\theta = 0°$$

Thus, $\vec{a}$ is parallel to $\vec{b}$.

Question 9

For any two vectors $\vec{a}$ and $\vec{b}$ write when $\left|\vec{a}+\vec{b}\right| = \left|\vec{a}-\vec{b}\right|$ holds.

Solution 9

Here, $\left|\vec{a}+\vec{b}\right| = \left|\vec{a}-\vec{b}\right|$

Squaring both the sides,

$$\left|\vec{a}+\vec{b}\right|^2 = \left|\vec{a}-\vec{b}\right|^2$$
$$\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\vec{a}.\vec{b} = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\vec{a}.\vec{b}$$
$$2\vec{a}\vec{b} = -2\vec{a}\vec{b}$$
$$2\vec{a}\vec{b} + 2\vec{a}\vec{b} = 0$$
$$4\vec{a}.\vec{b} = 0$$
$$\vec{a}.\vec{b} = 0$$

Thus, $\vec{a}$ is perpendicular to $\vec{b}$.

Question 10

If $\vec{a}$ and $\vec{b}$ are two vectors of the same magnitude indined at an angle of 60°
such that $\vec{a}\vec{b} = 8$, write the value of their magnitude.

Solution 10

Here, $\left|\vec{a}\right| = \left|\vec{b}\right| = x$ (Say)

Angle between $\vec{a}$ and $\vec{b} = \theta = 60°$
and, $\vec{a}\vec{b} = 8$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}$$
$$\cos 60° = \frac{8}{x.x}$$
$$\frac{1}{2} = \frac{8}{x^2}$$
$$x^2 = 16$$
$$x = 4$$

$$\therefore \quad \left|\vec{a}\right| = \left|\vec{b}\right| = 4$$

Question 11

If $\vec{a}.\vec{a} = 0$ and $\vec{a}.\vec{b} = 0$, what can you condude about the vector $\vec{b}$?

Solution 11

Here, $\vec{a}.\vec{a} = 0$

$\left(\vec{a}\right)^2 = 0$

$\vec{a} = 0$

and, $\vec{a}.\vec{b} = 0$

Thus, $\vec{b}$ is a non zero vecto

Question 12

If $\vec{b}$ is a unit vector such that $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right) = 8$, find $\left|\vec{a}\right|$.

Solution 12

Here, $\vec{b}$ is a unit vector

$\Rightarrow \quad \left|\vec{b}\right| = 1$

$$\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right) = 8$$

$$\left(\vec{a}\right)^2 - \left(\vec{b}\right)^2 = 8$$

$$\left|\vec{a}\right|^2 - \left|\vec{b}\right|^2 = 8$$

$$\left|\vec{a}\right|^2 - 1 = 8$$

$$\left|\vec{a}\right|^2 = 8+1$$

$$\left|\vec{a}\right|^2 = 9$$

$$\left|\vec{a}\right| = 3$$

Question 13

If $\vec{a}, \vec{b}$ are unit vectors such that $\hat{a}+\hat{b}$ is a unit vector, write the value of $\left|\hat{a}-\hat{b}\right|$.

Solution 13

Here, $\vec{a}, \vec{b}$ and $\left(\hat{a}+\hat{b}\right)$ are unit vectors,

$$\left|\vec{a}\right| = 1 = \left|\vec{b}\right| \quad \text{and} \quad \left|\hat{a}+\hat{b}\right| = 1$$

Now, $\left|\hat{a}+\hat{b}\right| = 1$

Squaring both the sides,

$$\left|\hat{a}+\hat{b}\right|^2 = 1^2$$

$$\left|\hat{a}\right|^2 + \left|\hat{b}\right|^2 + 2\hat{a}\hat{b} = 1$$

$$(1)^2 + (1)^2 + 2\hat{a}\hat{b} = 1$$

$$2\hat{a}\hat{b} = 1 - 2$$

$$2\hat{a}\hat{b} = -1$$

$$\hat{a}\hat{b} = \frac{-1}{2}$$

Now,

$$\left|\hat{a}-\hat{b}\right|^2 = \left|\hat{a}\right|^2 + \left|\hat{b}\right|^2 - 2\hat{a}\hat{b}$$

$$= (1)^2 + (1)^2 - 2\left(-\frac{1}{2}\right)$$

$$\left(\vec{a}-\vec{b}\right)^2 = 3$$

$$\vec{a}-\vec{b} = \sqrt{3}$$

Question 14

If $\left|\vec{a}\right| = 2$, $\left|\vec{b}\right| = 5$ and $\vec{a}.\vec{b} = 2$, find $\left|\vec{a}-\vec{b}\right|$.

Solution 14

Here, $\left|\vec{a}-\vec{b}\right|^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 - 2\vec{a}\vec{b}$

$$\left|\vec{a}-\vec{b}\right|^2 = (2)^2 + (5)^2 - 2.(2)$$

$$\left|\vec{a}-\vec{b}\right|^2 = 4 + 25 - 4$$

$$\left|\vec{a}-\vec{b}\right|^2 = 25$$

$$\left|\vec{a}-\vec{b}\right| = 5$$

Question 15

If $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = -\hat{j} + \hat{k}$, find the projection of $\vec{a}$ on $\vec{b}$.

Solution 15

Now, projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a}.\vec{b}}{\left|\vec{b}\right|}$$

$$= \frac{\left(\hat{i} - \hat{j}\right).\left(-\hat{j} + \hat{k}\right)}{\left|-\hat{j} + \hat{k}\right|}$$

$$= \frac{\left(\hat{i} - \hat{j} + 0 \times \hat{k}\right).\left(0 \times \hat{i} - \hat{j} + \hat{k}\right)}{\sqrt{\left(-1\right)^2 + \left(1\right)^2}}$$

$$= \frac{\left(1\right)\left(0\right) + \left(-1\right)\left(-1\right) + \left(0\right)\left(1\right)}{\sqrt{1+1}}$$

$$= \frac{1}{\sqrt{2}}$$

Projection of $\vec{a}$ on $\vec{b} = \frac{1}{\sqrt{2}}$

Question 16

For any two non-zero vectors, write the value of $\frac{\left|\vec{a} + \vec{b}\right|^2 + \left|\vec{a} - \vec{b}\right|^2}{\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2}$.

Solution 16

$$\frac{\left|\vec{a} + \vec{b}\right|^2 + \left|\vec{a} - \vec{b}\right|^2}{\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2}$$

$$= \frac{\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\vec{a}.\vec{b} + \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 - 2\vec{a}.\vec{b}}{\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2}$$

$$= \frac{2\left|\vec{a}\right|^2 + 2\left|\vec{b}\right|^2}{\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2}$$

$$= \frac{2\left(\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2\right)}{\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2}$$

$$= 2$$

$$\frac{\left|\vec{a} + \vec{b}\right|^2 + \left|\vec{a} - \vec{b}\right|^2}{\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2} = 2$$

Question 17

Write the projections of $\vec{r} = 3\hat{i} - 4\hat{j} + 12\hat{k}$ on the coordinate axes.

Solution 17

We know that,

Component along x-axes $= \hat{i}$

Component along y-axes $= \hat{j}$

Component along z-axes $= \hat{k}$

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$$

$$\text{Projection of } \vec{r} \text{ on } \hat{i} = \frac{\vec{r}\,\hat{i}}{|\hat{i}|}$$

$$= \frac{\left(3\hat{i} - 4\hat{j} + 12\hat{k}\right)\left(\hat{i} + 0 \times \hat{j} + 0 \times \hat{k}\right)}{1}$$

$$= \frac{(3)(1) + (-4)(0) + (12)(0)}{1}$$

$$= 3$$

Thus, Projection of $\vec{r}$ on x-axis $= 3$

$$\text{Projection of } \vec{r} \text{ on } \hat{j} = \frac{\vec{r}.\hat{j}}{|\hat{j}|}$$

$$\text{Projection of } \vec{r} \text{ on } \hat{j} = \frac{\left(3\hat{i} - 4\hat{j} + 12\hat{k}\right).\left(0 \times \hat{i} + \hat{j} + 0 \times \hat{k}\right)}{|\hat{j}|}$$

$$= \frac{(3)(0) + (-4)(1) + (12)(0)}{1}$$

$$= -4$$

Projection of $\vec{r}$ on y-axis $= -4$

$$\text{Projection of } \vec{r} \text{ on } \hat{k} = \frac{\vec{r}\,\hat{k}}{|\hat{k}|}$$

$$= \frac{\left(3\hat{i} - 4\hat{j} + 12\hat{k}\right)\left(0 \times \hat{i} + 0 \times \hat{j} + \hat{k}\right)}{1}$$

$$= \frac{(3)(0) + (-4)(0) + (12)(1)}{1}$$

$$= 12$$

Projection of $\vec{r}$ on z-axis $= 12$

Question 18

Write the component of $\vec{b}$ along $\vec{a}$.

Solution 18

The component of $\vec{b}$ along $\vec{a}$

$$= \left\{\frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|^2}\right\}.\vec{a}$$

Question 19

Write the value of $\left(\vec{a}.\hat{i}\right)\hat{i} + \left(\vec{a}.\hat{j}\right)\hat{j} + \left(\vec{a}.\hat{k}\right)\hat{k}$.

Solution 19

Let $\quad \vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\left(\vec{a}.\hat{i}\right)\hat{i} + \left(\vec{a}.\hat{j}\right)\hat{j} + \left(\vec{a}.\hat{k}\right)\hat{k}$$

$$= \left[\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\hat{i}\right]\hat{i} + \left[\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\hat{j}\right]\hat{j} + \left[\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\hat{k}\right]\hat{k}$$

$$= \left[(x)(1) + (y)(0) + (z)(0)\right]\hat{i} + \left[(x)(0) + (y)(1) + (z)(0)\right]\hat{j} + \left[(x)(0) + (y)(0) + (z)(1)\right]\hat{k}$$

$$= [x]\hat{i} + [y]\vec{j} + [z]\hat{k}$$

$$= x\hat{i} + y\vec{j} + z\hat{k}$$

$$= \vec{a}$$

$$\left(\vec{a}.\hat{i}\right)\hat{i} + \left(\vec{a}.\hat{j}\right)\hat{j} + \left(\vec{a}.\hat{k}\right)\hat{k} = \vec{a}$$

Question 20

Find the value of $\theta \in \left(0, \frac{\pi}{2}\right)$ for which vectors $\vec{a} = (\sin\theta)\,i + (\cos\theta)\,j$ and $\vec{b} = \hat{i} - \sqrt{3}\hat{j} + 2\hat{k}$ are perpendicular.

Solution 20

Here, $\vec{a}$ and $\vec{b}$ are perpendicular

$\Rightarrow \quad \vec{a}.\vec{b} = 0$

$$\left[(\sin\theta)i + (\cos\theta)\,j + 0.\hat{k}\right]\left(\hat{i} - \sqrt{3}\hat{j} + 2\hat{k}\right) = 0$$

$$(\sin\theta)(1) + (\cos\theta)\left(-\sqrt{3}\right) + (0)(2) = 0$$

$$\sin\theta - \sqrt{3}\cos\theta + 0 = 0$$

$$\sin\theta - \sqrt{3}\cos\theta = 0$$

Divide the equation by $\cos\theta$,

$$\frac{\sin\theta}{\cos\theta} - \frac{\sqrt{3}\cos\theta}{\cos\theta} = \frac{0}{\cos\theta}$$

$$\tan\theta - \sqrt{3} = 0$$

$$\tan\theta = \sqrt{3}$$

$$\tan\theta = \tan\frac{\pi}{3}$$

$$\theta = \frac{\pi}{3}$$

Question 21

Write the projection of $\hat{i} + \hat{j} + \hat{k}$ along the vector $\hat{j}$.

Solution 21

$$\text{Projection of } \vec{a} \text{ along } \vec{b} = \frac{\vec{a}.\vec{b}}{\left|\vec{b}\right|}$$

$$= \frac{\left(\hat{i} + \hat{j} + \hat{k}\right)\left(\hat{j}\right)}{\left|\hat{j}\right|}$$

$$= \frac{\left(\hat{i} + \hat{j} + \hat{k}\right)\left(0\,\hat{i} + \hat{j} + 0\,\hat{k}\right)}{1}$$

$$= (1)(0) + (1)(1) + (1)(0)$$

$$= 1$$

Projection of $\left(\hat{i} + \hat{j} + \hat{k}\right)$ along $\hat{j} = 1$

Question 22

Write the vector satisfying $\vec{a} \times \hat{i} = \vec{a}.\left(\hat{i} + \hat{j}\right) = \vec{a}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 1$.

Solution 22

Let the required vector $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{a}.\hat{i} = 1$$
$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\hat{i} = 1$$
$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(\hat{i} + 0.\hat{j} + 0\,\hat{k}\right) = 1$$
$$(x)(1) + (y)(0) + (z)(0) = 1$$
$$x + 0 + 0 = 1$$
$$x = 1 \qquad ---(i)$$

Again,

$$\vec{a}.\left(\hat{i} + \hat{j}\right) = 1$$
$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(\hat{i} + \hat{j} + 0.\hat{k}\right) = 1$$
$$(x)(1) + (y)(1) + (z)(0) = 1$$
$$x + y + 0 = 1$$
$$x + y = 1 \qquad ---(ii)$$

Again,

$$\vec{a}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 1$$
$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\left(\hat{i} + \hat{j} + \hat{k}\right) = 1$$
$$(x)(1) + (y)(1) + (z)(1) = 1$$
$$x + y + z = 1 \qquad ---(iii)$$

Subtracting equation (i) from (ii),

$$x + y - x = 1 - 1$$
$$y = 0 \qquad ---(iv)$$

Put value of x from (i) and y from (iv) in equation (iii),

$$x + y + z = 1$$
$$1 + 0 + z = 1$$
$$1 + z = 1$$
$$z = 0 \qquad ---(v)$$

$$\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= (1)\hat{i} + (0)\hat{j} + (0)\hat{k} \qquad \left[\text{Using } (i),(iv),(v)\right]$$

$$\vec{a} = \hat{i}$$

Question 23

If $\vec{a}$ and $\vec{b}$ are unit vectors, find the angle between $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

Solution 23

Here, $\vec{a}$ and $\vec{b}$ are unit vector

$$|\vec{a}| = |\vec{b}| = 1$$

Let angle between $\left(\vec{a}+\vec{b}\right)$ and $\left(\vec{a}-\vec{b}\right)$ is θ

$$\cos\theta = \frac{\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right)}{\left|\vec{a}-\vec{b}\right|.\left|\vec{a}-\vec{b}\right|}$$

$$= \frac{|\vec{a}|^2 - |\vec{b}|^2}{\left|\vec{a}-\vec{b}\right|.\left|\vec{a}-\vec{b}\right|}$$

$$= \frac{(1)^2 - (1)^2}{\left|\vec{a}-\vec{b}\right|.\left|\vec{a}-\vec{b}\right|}$$

$$= \frac{0}{\left|\vec{a}-\vec{b}\right|.\left|\vec{a}-\vec{b}\right|}$$

$$\cos\theta = 0$$

$$\theta = \cos^{-1}(0)$$

$$\theta = \frac{\pi}{2}$$

Question 24

If $\vec{a}$ and $\vec{b}$ are mutually perpendicualr unit vectors, write the value of $\left|\vec{a}+\vec{b}\right|$.

Solution 24

Here, $\vec{a}$ and $\vec{b}$ are mutually perpendicualr, then

$$\vec{a}.\vec{b} = 0$$

and, $\vec{a},\vec{b}$ are unit vectors,

$$|\vec{a}| = |\vec{b}| = 1$$

Now,

$$\left|\vec{a}+\vec{b}\right|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b}$$

$$= (1)^2 + (1)^2 + 2.(0)$$

$$= 1 + 1 + 0$$

$$\left|\vec{a}+\vec{b}\right|^2 = 2$$

$$\left|\vec{a}+\vec{b}\right| = \sqrt{2}$$

Question 25

If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicualr unit vectors, write the value of $\left|\vec{a}+\vec{b}+\vec{c}\right|$.

Solution 25

Here, $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicualr vectors, then

$$\vec{a}.\vec{b} = \vec{b}\vec{c} = \vec{c}.\vec{a} = 0$$

and, $\vec{a}, \vec{b}, \vec{c}$ are unit vectors

$$\left|\vec{a}\right| = \left|\vec{b}\right| = \left|\vec{c}\right| = 1$$

Now,

$$\begin{aligned}\left|\vec{a}+\vec{b}+\vec{c}\right|^2 &= \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + \left|\vec{c}\right|^2 + 2\vec{a}.\vec{b} + 2\vec{b}\vec{c} + 2\vec{c}.\vec{a} \\ &= (1)^2 + (1)^2 + (1)^2 + 2(0) + 2(0) + 2(0) \\ &= 1+1+1+0+0+0 \\ &= 3\end{aligned}$$

$$\left|\vec{a}+\vec{b}+\vec{c}\right| = \sqrt{3}$$

Question 26

Find the angle between the vectors $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$.

Solution 26

Let θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\begin{aligned}\cos\theta &= \frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|} \\ &= \frac{\left(\hat{i}-\hat{j}+\hat{k}\right)\left(\hat{i}+\hat{j}-\hat{k}\right)}{\sqrt{(1)^2+(-1)^2+(1)^2}\sqrt{(1)^2+(1)^2+(-1)^2}} \\ &= \frac{(1)(1)+(-1)(1)+(1)(-1)}{\sqrt{1+1+1}\sqrt{1+1+1}} \\ &= \frac{1-1-1}{\sqrt{3}\sqrt{3}} \\ \cos\theta &= \frac{-1}{3}\end{aligned}$$

$$\theta = \cos^{-1}\left(-\frac{1}{3}\right)$$

Question 27

For what value of λ are the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ perpendicular to each other?

Solution 27

We know that,

$\vec{a}$ and $\vec{b}$ are perpendicular if $\vec{a}\vec{b} = 0$

$\left(2\hat{i} + \lambda\hat{j} + \hat{k}\right)\left(\hat{i} - 2\hat{j} + 3\hat{k}\right) = 0$

$(2)(1) + (\lambda)(-2) + (1)(3) = 0$

$2 - 2\lambda + 3 = 0$

$5 - 2\lambda = 0$

$-2\lambda = -5$

$\lambda = \frac{-5}{-2}$

$\lambda = \frac{5}{2}$

Question 28

Find the projection of $\vec{a}$ on $\vec{b}$ if $\vec{a}.\vec{b} = 8$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$.

Solution 28

Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{\left|\vec{b}\right|}$

$= \frac{8}{\sqrt{(2)^2 + (6)^2 + (3)^2}}$

$= \frac{8}{\sqrt{4 + 36 + 9}}$

$= \frac{8}{\sqrt{49}}$

$= \frac{8}{7}$

Projection of $\vec{a}$ on $\vec{b}$ $= \frac{8}{7}$

Question 29

Write the value of p for which $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel vectors.

Solution 29

Vectors $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and

$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ are parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3}$$

Cross multiplying the first two,

$$3p = 2$$

$$p = \frac{2}{3}$$

Question 30

Find the value of λ if the vectors $2\hat{i} + \lambda\hat{j} + 3\hat{k}$ and $3\hat{i} + 2\hat{j} - 4\hat{k}$ are perpendicular to each other?

Solution 30

Vectors $\vec{a}$ and $\vec{b}$ are perpendicular if $\vec{a}.\vec{b} = 0$

$$\left(2\hat{i} + \lambda\hat{j} + 3\hat{k}\right).\left(3\hat{i} + 2\hat{j} - 4\hat{k}\right) = 0$$

$$(2)(3) + (\lambda)(2) + (3)(-4) = 0$$

$$6 + 2\lambda - 12 = 0$$

$$2\lambda - 6 = 0$$

$$2\lambda = 6$$

$$\lambda = \frac{6}{2}$$

$$\lambda = 3$$

Question 31

If $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}.\vec{b} = 3$, find the projection of $\vec{b}$ on $\vec{a}$.

Solution 31

The projection of $\vec{b}$ on $\vec{a}$

$$= \frac{\vec{a}\vec{b}}{|\vec{a}|}$$

$$= \frac{3}{2}$$

Projection of $\vec{b}$ on $\vec{a}$ $= \frac{3}{2}$

Question 32

Write the angle between two vector $\vec{a}$ *and* $\vec{b}$ with magnitude $\sqrt{3}$ and 2 respectively having $\vec{a}.\vec{b} = \sqrt{6}$

Solution 32

Given

$$|\vec{a}| = \sqrt{3}, |\vec{b}| = 2, \text{ and } \vec{a}.\vec{b} = \sqrt{6}$$

Let θ be the angle between $\overline{a}$ and $\vec{b}$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$
$$= \frac{\sqrt{6}}{\sqrt{3}.2}$$
$$= \frac{1}{\sqrt{2}}$$
$$\theta = \cos^{-1}\frac{1}{\sqrt{2}}$$
$$= \frac{\pi}{4}$$

Question 33

Write the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Solution 33

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$

$\vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$

$$= \vec{a}.\frac{\vec{b}}{|\vec{b}|}$$
$$= (\hat{i} + 3\hat{j} + 7\hat{k}).\frac{(2\hat{i} - 3\hat{j} + 6\hat{k})}{|2\hat{i} - 3\hat{j} + 6\hat{k}|}$$
$$= \frac{(\hat{i} + 3\hat{j} + 7\hat{k}).(2\hat{i} - 3\hat{j} + 6\hat{k})}{|2\hat{i} - 3\hat{j} + 6\hat{k}|}$$
$$= \frac{1 \times 2 + 3 \times (-3) + 7 \times 6}{|\sqrt{2^2 + (-3)^2 + 6^2}|}$$
$$= \frac{33}{\sqrt{49}}$$
$$= \frac{33}{7}$$

Question 34

Find λ, when the projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 unit.

Solution 34

the projection of $\vec{a}=\lambda\hat{i}+\hat{j}+4\hat{k}$ and $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$

$$\frac{(\lambda\hat{i}+\hat{j}+4\hat{k}).(2\hat{i}+6\hat{j}+3\hat{k})}{|2\hat{i}+6\hat{j}+3\hat{k}|}=4$$

$$\Rightarrow 2\lambda+6+12=4.\sqrt{2^2+6^2+3^2}$$
$$\Rightarrow 2\lambda=28-18$$
$$\Rightarrow 2\lambda=10$$
$$\Rightarrow \lambda=5$$

Question 35

For what value of λ are the vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ are perpendicular to each other.

Solution 35

The vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ are perpendicular to each other. So

$$\vec{a}.\vec{b}=0$$
$$(2\hat{i}+\lambda\hat{j}+\hat{k})(\hat{i}-2\hat{j}+3\hat{k})=0$$
$$2-2\lambda+3=0$$
$$\lambda=\frac{5}{2}$$

Question 36

Write the projection of the vector $7\hat{i}+\hat{j}-4\hat{k}$ on the vector $2\hat{i}+6\hat{j}+3\hat{k}$.

Solution 36

Let $\vec{a}=7\hat{i}+\hat{j}-4\hat{k}, \vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$

Projection of $\vec{a}$ on $\vec{b}=\vec{a}.\frac{\vec{b}}{|\vec{b}|}$

$$=(7\hat{i}+\hat{j}-4\hat{k}).\frac{(2\hat{i}+6\hat{j}+3\hat{k})}{|\sqrt{2^2+6^2+3^2}|}$$
$$=\frac{(7\hat{i}+\hat{j}-4\hat{k}).(2\hat{i}+6\hat{j}+3\hat{k})}{|\sqrt{2^2+6^2+3^2}|}$$
$$=\frac{[7\times2+1\times6+(-4)\times3]}{|\sqrt{2^2+6^2+3^2}|}$$
$$=\frac{[14+6-12]}{|\sqrt{2^2+6^2+3^2}|}$$
$$=\frac{8}{|\sqrt{49}|}=\frac{8}{7}$$

Question 37

Write the value of λ so that the vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ are perpendicular to each other.

Solution 37

$\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$
$\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$
Since $\vec{a}$ and $\vec{b}$ are perpendicular to each other
$\vec{a}.\vec{b} = 0$
$\Rightarrow (2\hat{i} + \lambda\hat{j} + \hat{k}).(\hat{i} - 2\hat{j} + 3\hat{k}) = 0$
$\Rightarrow 2 \times 1 + \lambda \times (-2) + 1 \times 3 = 0$
$\Rightarrow 2 - 2\lambda + 3 = 0$
$\Rightarrow 5 = 2\lambda$
$\therefore \lambda = \frac{5}{2}$

Question 38
Write the projection of $\vec{b} + \vec{c}$ on $\vec{a}$, when
$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

Solution 38
$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}, \vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

$\vec{b} + \vec{c} = \hat{i} + 2\hat{j} - 2\hat{k} + 2\hat{i} - \hat{j} + 4\hat{k}$
$\vec{b} + \vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$

$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$
Projection of $\vec{b} + \vec{c}$ on $a = (\vec{b} + \vec{c}).\frac{\vec{a}}{|\vec{a}|}$
$= \frac{(\vec{b} + \vec{c}).\vec{a}}{|\vec{a}|}$
$= \frac{(3\hat{i} + \hat{j} + 2\hat{k}).(2\hat{i} - 2\hat{j} + \hat{k})}{\left|2\hat{i} - 2\hat{j} + \hat{k}\right|}$
$= \frac{3 \times 2 - 1 \times 2 + 2 \times 1}{\left|\sqrt{2^2 + (-2)^2 + 1^2}\right|}$
$= \frac{6}{\sqrt{9}} = \frac{6}{3} = 2$

Question 39
If $\vec{a}$ and $\vec{b}$ are perpendicular vectors,
$|\vec{a} + \vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$.

Solution 39

Here, $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

$\vec{a}.\vec{b} = 0$

$|\vec{a}+\vec{b}|^2 = 13^2$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 169$

$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = 169$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 9$

$\Rightarrow 5^2 + 2 \times 0 + |\vec{b}|^2 = 169$

$\Rightarrow |\vec{b}|^2 = 169 - 25$

$\Rightarrow |\vec{b}|^2 = 144$

$\therefore |\vec{b}| = 12$

Question 40

If vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$.

Solution 40

$|\vec{a}| = 3$, $|\vec{b}| = \frac{2}{3}$

$|\vec{a} \times \vec{b}| = ||\vec{a}||\vec{b}|\sin\theta.|\hat{n}||$

$\Rightarrow 1 = |3 \times \frac{2}{3} \times \sin\theta.1|$

$\Rightarrow 1 = 3 \times \frac{2}{3} \times \sin\theta$

$\Rightarrow \sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}$

Question 41

If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

Solution 41

$|\vec{a}| = 1, |\vec{b}| = 1,$

$|\vec{a} + \vec{b}| = 1$

$\Rightarrow |\vec{a} + \vec{b}|^2 = 1^2$

$\Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = 1$

$\Rightarrow (\vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b}) = 1$

$\Rightarrow |\vec{a}|^2 + \vec{a}.\vec{b} + \vec{a}.\vec{b} + |\vec{b}|^2 = 1$

$\Rightarrow 1 + 2\vec{a}.\vec{b} + 1 = 1$

$\Rightarrow 2\vec{a}.\vec{b} = -1$

$\Rightarrow \vec{a}.\vec{b} = -\frac{1}{2}$

$\Rightarrow |\vec{a}||\vec{b}|.\cos\theta = -\frac{1}{2}$

$\Rightarrow 1 \times 1 \times \cos\theta = -\frac{1}{2}$

$\Rightarrow \cos\theta = \cos\frac{2\pi}{3}$

$\therefore \theta = \frac{2\pi}{3}$

Question 42

If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the angle between $\vec{a}$ and $\vec{b}$, given that $(\sqrt{3}\,\vec{a} - \vec{b})$ is a unit vector.

Solution 42

$|\vec{a}| = 1, |\vec{b}| = 1,$

$|\vec{a}\sqrt{3} - \vec{b}| = 1$

$\Rightarrow |\vec{a}\sqrt{3} - \vec{b}|^2 = 1^2$

$\Rightarrow (\vec{a}\sqrt{3} - \vec{b}).(\vec{a}\sqrt{3} - \vec{b}) = 1$

$\Rightarrow 3\vec{a}.\vec{a} - \sqrt{3}\,\vec{a}.\vec{b} - \sqrt{3}\,\vec{a}.\vec{b} + \vec{b}.\vec{b} = 1$

$\Rightarrow 3 \times 1^2 - 2\sqrt{3}\,\vec{a}.\vec{b} + 1^2 = 1$

$\Rightarrow 3 - 2\sqrt{3}\,\vec{a}.\vec{b} + 1 = 1$

$\Rightarrow -2\sqrt{3}\,\vec{a}.\vec{b} = -3$

$\Rightarrow \vec{a}.\vec{b} = \frac{3}{2\sqrt{3}}$

$\Rightarrow |\vec{a}|.|\vec{b}|\cos\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \cos\theta = \cos\frac{\pi}{6}$

$\therefore \theta = \frac{\pi}{6}$