Access answers to RD Sharma Solutions for Class 11 Maths Chapter 18 – Binomial Theorem

EXERCISE 18.1 PAGE NO: 18.11

1. Using binomial theorem, write down the expressions of the following:

(i) $(2x + 3y)^5$

(ii) $(2x - 3y)^4$

(iii) $\left(x - \frac{1}{x}\right)^6$

(iv) $(1 - 3x)^7$

(v) $\left(ax - \frac{b}{x}\right)^6$

(vi) $\left(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}}\right)^6$

(vii) $\left(\sqrt[3]{x} - \sqrt[3]{a}\right)^6$

(viii) $(1 + 2x - 3x^2)^5$

(ix) $\left(x + 1 - \frac{1}{x}\right)^3$

(x) $(1 - 2x + 3x^2)^3$

Solution:

(i) $(2x + 3y)^5$

Let us solve the given expression:

$(2x + 3y)^5 = {}^5C_0 (2x)^5 (3y)^0 + {}^5C_1 (2x)^4 (3y)^1 + {}^5C_2 (2x)^3 (3y)^2 + {}^5C_3 (2x)^2 (3y)^3 + {}^5C_4 (2x)^1 (3y)^4 + {}^5C_5 (2x)^0 (3y)^5$

$= 32x^5 + 5 (16x^4) (3y) + 10 (8x^3) (9y)^2 + 10 (4x)^2 (27y)^3 + 5 (2x) (81y^4) + 243y^5$

$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$

(ii) $(2x - 3y)^4$

Let us solve the given expression:

$(2x - 3y)^4 = {}^4C_0 (2x)^4 (3y)^0 - {}^4C_1 (2x)^3 (3y)^1 + {}^4C_2 (2x)^2 (3y)^2 - {}^4C_3 (2x)^1 (3y)^3 + {}^4C_4 (2x)^0 (3y)^4$

$= 16x^4 - 4 (8x^3) (3y) + 6 (4x^2) (9y^2) - 4 (2x) (27y^3) + 81y^4$

$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

(iii) $\left(\mathbf{x} - \frac{\mathbf{1}}{\mathbf{x}}\right)^{\mathbf{6}}$

Let us solve the given expression:

$\left(x - \frac{1}{x}\right)^6$

$$= {}^6C_0 x^6 \left(\frac{1}{x}\right)^0 - {}^6C_1 x^5 \left(\frac{1}{x}\right)^1 + {}^6C_2 x^4 \left(\frac{1}{x}\right)^2 - {}^6C_3 x^3 \left(\frac{1}{x}\right)^3$$

$$+ {}^6C_4 x^2 \left(\frac{1}{x}\right)^4 - {}^6C_5 x^1 \left(\frac{1}{x}\right)^5 + {}^6C_6 x^0 \left(\frac{1}{x}\right)^6$$

$$= x^6 - 6x^5 \times \frac{1}{x} + 15x^4 \times \frac{1}{x^2} - 20x^3 \times \frac{1}{x^3} + 15x^2 \times \frac{1}{x^4} - 6x \times \frac{1}{x^5} + \frac{1}{x^6}$$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

(iv) $(1 - 3x)^7$

Let us solve the given expression:

$(1 - 3x)^7 = {}^7C_0 (3x)^0 - {}^7C_1 (3x)^1 + {}^7C_2 (3x)^2 - {}^7C_3 (3x)^3 + {}^7C_4 (3x)^4 - {}^7C_6 (3x)^6 - {}^7C_7 (3x)^7$

$= 1 - 7 (3x) + 21 (9x)^2 - 35 (27x^3) + 35 (81x^4) - 21 (243x^5) + 7 (729x^6) - 2187(x^7)$

$= 1 - 21x + 189x^2 - 945x^3 + 2835x^4 - 5103x^5 + 5103x^6 - 2187x^7$

(v) $\left(ax - \frac{b}{x}\right)^6$

Let us solve the given expression:

$$= {}^6C_0(ax)^6\left(\frac{b}{x}\right)^0 - {}^6C_1(ax)^5\left(\frac{b}{x}\right)^1 + {}^6C_2(ax)^4\left(\frac{b}{x}\right)^2 - {}^6C_3(ax)^3\left(\frac{b}{x}\right)^3$$

$$+ {}^6C_4(ax)^2\left(\frac{b}{x}\right)^4 - {}^6C_5(ax)^1\left(\frac{b}{x}\right)^5 + {}^6C_6(ax)^0\left(\frac{b}{x}\right)^6$$

$$= a^6x^6 - 6a^5x^5 \times \frac{b}{x} + 15a^4x^4 \times \frac{b^2}{x^2} - 20a^3b^3 \times \frac{b^3}{x^3} + 15a^2x^2 \times \frac{b^4}{x^4} - 6ax \times \frac{b^5}{x^5} + \frac{b^6}{x^6}$$

$$= a^6x^6 - 6a^5x^4b + 15a^4x^2b^2 - 20a^3b^3 + 15\frac{a^2b^4}{x^2} - 6\frac{ab^5}{x^4} + \frac{b^6}{x^6}$$

(vi) $\left(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}}\right)^6$

Let us solve the given expression:

$$= {}^6C_0\left(\sqrt{\frac{x}{a}}\right)^6\left(\sqrt{\frac{a}{x}}\right)^0 - {}^6C_1\left(\sqrt{\frac{x}{a}}\right)^5\left(\sqrt{\frac{a}{x}}\right)^1 + {}^6C_2\left(\sqrt{\frac{x}{a}}\right)^4\left(\sqrt{\frac{a}{x}}\right)^2 - {}^6C_3\left(\sqrt{\frac{x}{a}}\right)^3\left(\sqrt{\frac{a}{x}}\right)^3$$

$$+ {}^6C_4\left(\sqrt{\frac{x}{a}}\right)^2\left(\sqrt{\frac{a}{x}}\right)^4 - {}^6C_5\left(\sqrt{\frac{x}{a}}\right)^1\left(\sqrt{\frac{a}{x}}\right)^5 + {}^6C_6\left(\sqrt{\frac{x}{a}}\right)^0\left(\sqrt{\frac{a}{x}}\right)^6$$

$$= \frac{x^3}{a^3} - 6\frac{x^2}{a^2} + 15\frac{x}{a} - 20 + 15\frac{a}{x} - 6\frac{a^2}{x^2} + \frac{a^3}{x^3}$$

(vii) $\left(\sqrt[3]{x} - \sqrt[3]{a}\right)^6$

Let us solve the given expression:

$$= {}^6C_0(\sqrt[3]{x})^6(\sqrt[3]{a})^0 - {}^6C_1(\sqrt[3]{x})^5(\sqrt[3]{a})^1 + {}^6C_2(\sqrt[3]{x})^4(\sqrt[3]{a})^2 - {}^6C_3(\sqrt[3]{x})^3(\sqrt[3]{a})^3$$

$$+ {}^6C_4(\sqrt[3]{x})^2(\sqrt[3]{a})^4 - {}^6C_5(\sqrt[3]{x})^1(\sqrt[3]{a})^5 + {}^6C_6(\sqrt[3]{x})^0(\sqrt[3]{a})^6$$

$$= x^2 - 6x^{5/3}a^{1/3} + 15x^{4/3}a^{2/3} - 20xa + 15x^{2/3}a^{4/3} - 6x^{1/3}a^{5/3} + a^2$$

(viii) $(1 + 2x - 3x^2)^5$

Let us solve the given expression:

Let us consider $(1 + 2x)$ and $3x^2$ as two different entities and apply the binomial theorem.

$(1 + 2x - 3x^2)^5 = {}^5C_0\,(1 + 2x)^5\,(3x^2)^0 - {}^5C_1\,(1 + 2x)^4\,(3x^2)^1 + {}^5C_2\,(1 + 2x)^3\,(3x^2)^2 - {}^5C_3\,(1 + 2x)^2\,(3x^2)^3 + {}^5C_4\,(1 + 2x)^1\,(3x^2)^4 - {}^5C_5\,(1 + 2x)^0\,(3x^2)^5$

$= (1 + 2x)^5 - 5(1 + 2x)^4\,(3x^2) + 10\,(1 + 2x)^3\,(9x^4) - 10\,(1 + 2x)^2\,(27x^6) + 5\,(1 + 2x)\,(81x^8) - 243x^{10}$

$= {}^5C_0\,(2x)^0 + {}^5C_1\,(2x)^1 + {}^5C_2\,(2x)^2 + {}^5C_3\,(2x)^3 + {}^5C_4\,(2x)^4 + {}^5C_5\,(2x)^5 -$
$15x^2\,[{}^4C_0\,(2x)^0 + {}^4C_1\,(2x)^1 + {}^4C_2\,(2x)^2 + {}^4C_3\,(2x)^3 + {}^4C_4\,(2x)^4] + 90x^4\,[1 + 8x^3 + 6x + 12x^2] - 270x^6(1 + 4x^2 + 4x) + 405x^8 + 810x^9 - 243x^{10}$

$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5 - 15x^2 - 120x^3 - 360^4 - 480x^5 - 240x^6 + 90x^4 + 720x^7 + 540x^5 + 1080x^6 - 270x^6 - 1080x^8 - 1080x^7 + 405x^8 + 810x^9 - 243x^{10}$

$= 1 + 10x + 25x^2 - 40x^3 - 190x^4 + 92x^5 + 570x^6 - 360x^7 - 675x^8 + 810x^9 - 243x^{10}$

(ix) $\left(x + 1 - \frac{1}{x}\right)^3$

Let us solve the given expression:

$$={}^3C_0(x+1)^3(\frac{1}{x})^0 - {}^3C_1(x+1)^2(\frac{1}{x})^1 + {}^3C_2(x+1)^1(\frac{1}{x})^2 - {}^3C_3(x+1)^0(\frac{1}{x})^3$$

$$= (x+1)^3 - 3(x+1)^2 \times \frac{1}{x} + 3\frac{x+1}{x^2} - \frac{1}{x^3}$$

$$= x^3 + 1 + 3x + 3x^2 - \frac{3x^2+3+6x}{x} + 3\frac{x+1}{x^2} - \frac{1}{x^3}$$

$$= x^3 + 1 + 3x + 3x^2 - 3x - \frac{3}{x} - 6 + \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}$$

$$= x^3 + 3x^2 - 5 + \frac{3}{x^2} - \frac{1}{x^3}$$

(x) $(1 - 2x + 3x^2)^3$

Let us solve the given expression:

$$={}^3C_0(1-2x)^3+{}^3C_1(1-2x)^2(3x^2)+{}^3C_2(1-2x)(3x^2)^2+{}^3C_3(3x^2)^3$$
$$=(1-2x)^3+9x^2(1-2x)^2+27x^4(1-2x)+27x^6$$
$$=1-8x^3+12x^2-6x+9x^2(1+4x^2-4x)+27x^4-54x^5+27x^6$$
$$=1-8x^3+12x^2-6x+9x^2+36x^4-36x^3+27x^4-54x^5+27x^6$$
$$=1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$$

2. Evaluate the following:

(i) $\left(\sqrt{x+1}+\sqrt{x-1}\right)^6+\left(\sqrt{x+1}-\sqrt{x-1}\right)^6$

(ii) $\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$

(iii) $\left(1+2\sqrt{x}\right)^5+\left(1-2\sqrt{x}\right)^5$

(iv) $\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6$

(v) $\left(3+\sqrt{2}\right)^5-\left(3-\sqrt{2}\right)^5$

(vi) $\left(2+\sqrt{3}\right)^7+\left(2-\sqrt{3}\right)^7$

(vii) $\left(\sqrt{3}+1\right)^5-\left(\sqrt{3}-1\right)^5$

(viii) $(0.99)^5+(1.01)^5$

(ix) $\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$

(x) $\left\{a^2+\sqrt{a^2-1}\right\}^4+\left\{a^2-\sqrt{a^2-1}\right\}^4$

Solution:

(i) $\left(\sqrt{x+1}+\sqrt{x-1}\right)^{6}+\left(\sqrt{x+1}-\sqrt{x-1}\right)^{6}$

Let us solve the given expression:

$= 2\left[{}^{6}C_{0}\left(\sqrt{x+1}\right)^{6}\left(\sqrt{x-1}\right)^{0}+{}^{6}C_{2}\left(\sqrt{x+1}\right)^{4}\left(\sqrt{x-1}\right)^{2}\right.$

$\left.+\,{}^{6}C_{4}\left(\sqrt{x+1}\right)^{2}\left(\sqrt{x-1}\right)^{4}+{}^{6}C_{6}\left(\sqrt{x+1}\right)^{0}\left(\sqrt{x-1}\right)^{6}\right]$

$= 2\left[(x+1)^{3}+15(x+1)^{2}(x-1)+15(x+1)(x-1)^{2}+(x-1)^{3}\right.$

$= 2\left[x^{3}+1+3x+3x^{2}+15\left(x^{2}+2x+1\right)(x-1)+15(x+1)\left(x^{2}+1-2x\right)+x^{3}-1+3x-3x^{2}\right]$

$= 2\left[2x^{3}+6x+15x^{3}-15x^{2}+30x^{2}-30x+15x-15+15x^{3}+15x^{2}-30x^{2}-30x+15x+15\right]$

$= 2\left[32x^{3}-24x\right]$

$= 16x\left[4x^{2}-3\right]$

(ii) $\left(x+\sqrt{x^{2}-1}\right)^{6}+\left(x-\sqrt{x^{2}-1}\right)^{6}$

Let us solve the given expression:

$= 2\left[{}^{6}C_{0}x^{6}\left(\sqrt{x^{2}-1}\right)^{0}+{}^{6}C_{2}x^{4}\left(\sqrt{x^{2}-1}\right)^{2}+{}^{6}C_{4}x^{2}\left(\sqrt{x^{2}-1}\right)^{4}+\right.$

$\left.{}^{6}C_{6}x^{0}\left(\sqrt{x^{2}-1}\right)^{6}\right]$

$= 2\left[x^{6}+15x^{4}\left(x^{2}-1\right)+15x^{2}\left(x^{2}-1\right)^{2}+\left(x^{2}-1\right)^{3}\right]$

$= 2\left[x^{6}+15x^{6}-15x^{4}+15x^{2}\left(x^{4}-2x^{2}+1\right)+\left(x^{6}-1+3x^{2}-3x^{4}\right)\right]$

$= 2\left[x^{6}+15x^{6}-15x^{4}+15x^{6}-30x^{4}+15x^{2}+x^{6}-1+3x^{2}-3x^{4}\right]$

$= 2\left[x^{6}+15x^{6}-15x^{4}+15x^{2}\left(x^{4}-2x^{2}+1\right)+\left(x^{6}-1+3x^{2}-3x^{4}\right)\right]$

$= 2\left[x^{6}+15x^{6}-15x^{4}+15x^{6}-30x^{4}+15x^{2}+x^{6}-1+3x^{2}-3x^{4}\right]$

$= 64x^{6}-96x^{4}+36x^{2}-2$

(iii) $\left(1+2\sqrt{x}\right)^{5}+\left(1-2\sqrt{x}\right)^{5}$

Let us solve the given expression:

$= 2\,[{}^{5}C_{0}\,(2\sqrt{x})^{0} + {}^{5}C_{2}\,(2\sqrt{x})^{2} + {}^{5}C_{4}\,(2\sqrt{x})^{4}]$

$= 2 [1 + 10 (4x) + 5 (16x^2)]$

$= 2 [1 + 40x + 80x^2]$

(iv) $\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6$

Let us solve the given expression:

$= 2 [{}^6C_0 (\sqrt{2})^6 + {}^6C_2 (\sqrt{2})^4 + {}^6C_4 (\sqrt{2})^2 + {}^6C_6 (\sqrt{2})^0]$

$= 2 [8 + 15 (4) + 15 (2) + 1]$

$= 2 [99]$

$= 198$

(v) $\left(3+\sqrt{2}\right)^5-\left(3-\sqrt{2}\right)^5$

Let us solve the given expression:

$= 2 [{}^5C_1 (3^4) (\sqrt{2})^1 + {}^5C_3 (3^2) (\sqrt{2})^3 + {}^5C_5 (3^0) (\sqrt{2})^5]$

$= 2 [5 (81) (\sqrt{2}) + 10 (9) (2\sqrt{2}) + 4\sqrt{2}]$

$= 2\sqrt{2} (405 + 180 + 4)$

$= 1178\sqrt{2}$

(vi) $\left(2+\sqrt{3}\right)^7+\left(2-\sqrt{3}\right)^7$

Let us solve the given expression:

$= 2 [{}^7C_0 (2^7) (\sqrt{3})^0 + {}^7C_2 (2^5) (\sqrt{3})^2 + {}^7C_4 (2^3) (\sqrt{3})^4 + {}^7C_6 (2^1) (\sqrt{3})^6]$

$= 2 [128 + 21 (32)(3) + 35(8)(9) + 7(2)(27)]$

$= 2 [128 + 2016 + 2520 + 378]$

$= 2 [5042]$

$= 10084$

(vii) $\left(\sqrt{3}+1\right)^5-\left(\sqrt{3}-1\right)^5$

Let us solve the given expression:

$= 2 [{}^5C_1 (\sqrt{3})^4 + {}^5C_3 (\sqrt{3})^2 + {}^5C_5 (\sqrt{3})^0]$

$= 2 [5 (9) + 10 (3) + 1]$

= 2 [76]

= 152

(viii) $(0.99)^5 + (1.01)^5$

Let us solve the given expression:

= $(1 – 0.01)^5 + (1 + 0.01)^5$

= 2 [${}^5C_0\ (0.01)^0 + {}^5C_2\ (0.01)^2 + {}^5C_4\ (0.01)^4$]

= 2 [1 + 10 (0.0001) + 5 (0.00000001)]

= 2 [1.00100005]

= 2.0020001

(ix) $\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$

Let us solve the given expression:

= 2 [${}^6C_1\ (\sqrt{3})^5\ (\sqrt{2})^1 + {}^6C_3\ (\sqrt{3})^3\ (\sqrt{2})^3 + {}^6C_5\ (\sqrt{3})^1\ (\sqrt{2})^5$]

= 2 [6 (9√3) (√2) + 20 (3√3) (2√2) + 6 (√3) (4√2)]

= 2 [√6 (54 + 120 + 24)]

= 396 √6

(x) $\left\{a^2+\sqrt{a^2-1}\right\}^4 + \left\{a^2-\sqrt{a^2-1}\right\}^4$

Let us solve the given expression:

$$= 2\left[{}^4C_0\left(a^2\right)^4\left(\sqrt{a^2-1}\right)^0 + {}^4C_2\left(a^2\right)^2\left(\sqrt{a^2-1}\right)^2 + {}^4C_4\left(a^2\right)^0\left(\sqrt{a^2-1}\right)^4\right]$$

$$= 2\left[a^8 + 6a^4\left(a^2-1\right) + \left(a^2-1\right)^2\right]$$

= 2 [$a^8 + 6a^6 – 6a^4 + a^4 + 1 – 2a^2$]

= $2a^8 + 12a^6 – 10a^4 – 4a^2 + 2$

3. Find $(a + b)^4 – (a – b)^4$. Hence, evaluate $(\sqrt{3} + \sqrt{2})^4 – (\sqrt{3} – \sqrt{2})^4$.

Solution:

Firstly, let us solve the given expression:

$(a + b)^4 – (a – b)^4$

The above expression can be expressed as,

$(a + b)^4 – (a – b)^4 = 2 [{}^4C_1 a^3b^1 + {}^4C_3 a^1b^3]$

$= 2 [4a^3b + 4ab^3]$

$= 8 (a^3b + ab^3)$

Now,

Let us evaluate the expression:

$(\sqrt{3} + \sqrt{2})^4 – (\sqrt{3} -\sqrt{2})^4$

So consider, $a = \sqrt{3}$ and $b = \sqrt{2}$ we get,

$(\sqrt{3} + \sqrt{2})^4 – (\sqrt{3} -\sqrt{2})^4 = 8 (a^3b + ab^3)$

$= 8 [(\sqrt{3})^3 (\sqrt{2}) + (\sqrt{3}) (\sqrt{2})^3]$

$= 8 [(3\sqrt{6}) + (2\sqrt{6})]$

$= 8 (5\sqrt{6})$

$= 40\sqrt{6}$

4. Find $(x + 1)^6 + (x – 1)^6$. Hence, or otherwise evaluate $(\sqrt{2} + 1)^6 + (\sqrt{2} – 1)^6$.

Solution:

Firstly, let us solve the given expression:

$(x + 1)^6 + (x – 1)^6$

The above expression can be expressed as,

$(x + 1)^6 + (x – 1)^6 = 2 [{}^6C_0 x^6 + {}^6C_2 x^4 + {}^6C_4 x^2 + {}^6C_6 x^0]$

$= 2 [x^6 + 15x^4 + 15x^2 + 1]$

Now,

Let us evaluate the expression:

$(\sqrt{2} + 1)^6 + (\sqrt{2} – 1)^6$

So consider, $x = \sqrt{2}$ then we get,

$(\sqrt{2} + 1)^6 + (\sqrt{2} – 1)^6 = 2 [x^6 + 15x^4 + 15x^2 + 1]$

$= 2 [(\sqrt{2})^6 + 15 (\sqrt{2})^4 + 15 (\sqrt{2})^2 + 1]$

$= 2 [8 + 15 (4) + 15 (2) + 1]$

= 2 [8 + 60 + 30 + 1]

= 198

5. Using binomial theorem evaluate each of the following:

(i) $(96)^3$

(ii) $(102)^5$

(iii) $(101)^4$

(iv) $(98)^5$

Solution:

(i) $(96)^3$

We have,

$(96)^3$

Let us express the given expression as two different entities and apply the binomial theorem.

$(96)^3 = (100 - 4)^3$

$= {}^3C_0 (100)^3 (4)^0 - {}^3C_1 (100)^2 (4)^1 + {}^3C_2 (100)^1 (4)^2 - {}^3C_3 (100)^0 (4)^3$

= 1000000 – 120000 + 4800 – 64

= 884736

(ii) $(102)^5$

We have,

$(102)^5$

Let us express the given expression as two different entities and apply the binomial theorem.

$(102)^5 = (100 + 2)^5$

$= {}^5C_0 (100)^5 (2)^0 + {}^5C_1 (100)^4 (2)^1 + {}^5C_2 (100)^3 (2)^2 + {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100)^1 (2)^4 + {}^5C_5 (100)^0 (2)^5$

= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32

= 11040808032

(iii) $(101)^4$

We have,

$(101)^4$

Let us express the given expression as two different entities and apply the binomial theorem.

$(101)^4 = (100 + 1)^4$

$= {}^4C_0 (100)^4 + {}^4C_1 (100)^3 + {}^4C_2 (100)^2 + {}^4C_3 (100)^1 + {}^4C_4 (100)^0$

$= 100000000 + 4000000 + 60000 + 400 + 1$

$= 104060401$

(iv) $(98)^5$

We have,

$(98)^5$

Let us express the given expression as two different entities and apply the binomial theorem.

$(98)^5 = (100 – 2)^5$

$= {}^5C_0 (100)^5 (2)^0 – {}^5C_1 (100)^4 (2)^1 + {}^5C_2 (100)^3 (2)^2 – {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100)^1 (2)^4 – {}^5C_5 (100)^0 (2)^5$

$= 10000000000 – 1000000000 + 40000000 – 800000 + 8000 – 32$

$= 9039207968$

6. Using binomial theorem, prove that $2^{3n} – 7n – 1$ is divisible by 49, where $n \in N$.

Solution:

Given:

$2^{3n} – 7n – 1$

So, $2^{3n} – 7n – 1 = 8^n – 7n – 1$

Now,

$8^n – 7n – 1$

$8^n = 7n + 1$

$= (1 + 7)^n$

$= {}^nC_0 + {}^nC_1 (7)^1 + {}^nC_2 (7)^2 + {}^nC_3 (7)^3 + {}^nC_4 (7)^2 + {}^nC_5 (7)^1 + \ldots + {}^nC_n (7)^n$

$8^n = 1 + 7n + 49 [{}^nC_2 + {}^nC_3 (7^1) + {}^nC_4 (7^2) + \ldots + {}^nC_n (7)^{n-2}]$

$8^n – 1 – 7n = 49$ (integer)

So now,

$8^n – 1 – 7n$ is divisible by 49

Or

$2^{3n} – 1 – 7n$ is divisible by 49.

Hence proved.

EXERCISE 18.2 PAGE NO: 18.37

1. Find the 11th term from the beginning and the 11th term from the end in the expansion of $(2x – 1/x^2)^{25}$.

Solution:

Given:

$(2x – 1/x^2)^{25}$

The given expression contains 26 terms.

So, the 11th term from the end is the (26 − 11 + 1) th term from the beginning.

In other words, the 11th term from the end is the 16th term from the beginning.

Then,

$T_{16} = T_{15+1} = {}^{25}C_{15} (2x)^{25-15} (-1/x^2)^{15}$

$= {}^{25}C_{15} (2^{10}) (x)^{10} (-1/x^{30})$

$= – {}^{25}C_{15} (2^{10} / x^{20})$

Now we shall find the 11th term from the beginning.

$T_{11} = T_{10+1} = {}^{25}C_{10} (2x)^{25-10} (-1/x^2)^{10}$

$= {}^{25}C_{10} (2^{15}) (x)^{15} (1/x^{20})$

$= {}^{25}C_{10} (2^{15} / x^5)$

2. Find the 7^{th} term in the expansion of $(3x^2 – 1/x^3)^{10}$.

Solution:

Given:

$(3x^2 – 1/x^3)^{10}$

Let us consider the 7^{th} term as T_7

So,

$T_7 = T_{6+1}$

$= {}^{10}C_6 (3x^2)^{10-6} (-1/x^3)^6$

$= {}^{10}C_6 (3)^4 (x)^8 (1/x^{18})$

$= [10×9×8×7×81] / [4×3×2×x^{10}]$

$= 17010 / x^{10}$

∴ The 7^{th} term of the expression $(3x^2 – 1/x^3)^{10}$ is $17010 / x^{10}$.

3. Find the 5^{th} term in the expansion of $(3x – 1/x^2)^{10}$.

Solution:

Given:

$(3x – 1/x^2)^{10}$

The 5^{th} term from the end is the (11 – 5 + 1)th, is., 7^{th} term from the beginning.

So,

$T_7 = T_{6+1}$

$= {}^{10}C_6 (3x)^{10-6} (-1/x^2)^6$

$= {}^{10}C_6 (3)^4 (x)^4 (1/x^{12})$

$= [10×9×8×7×81] / [4×3×2×x^8]$

$= 17010 / x^8$

∴ The 5^{th} term of the expression $(3x – 1/x^2)^{10}$ is $17010 / x^8$.

4. Find the 8^{th} term in the expansion of $(x^{3/2} y^{1/2} – x^{1/2} y^{3/2})^{10}$.

Solution:

Given:

$(x^{3/2} y^{1/2} – x^{1/2} y^{3/2})^{10}$

Let us consider the 8th term as T_8

So,

$T_8 = T_{7+1}$

$= {}^{10}C_7 (x^{3/2} y^{1/2})^{10-7} (-x^{1/2} y^{3/2})^7$

$= -[10×9×8]/[3×2]\ x^{9/2} y^{3/2} (x^{7/2} y^{21/2})$

$= -120\ x^8y^{12}$

∴ The 8th term of the expression $(x^{3/2} y^{1/2} – x^{1/2} y^{3/2})^{10}$ is $-120\ x^8y^{12}$.

5. Find the 7th term in the expansion of $(4x/5 + 5/2x)^8$.

Solution:

Given:

$(4x/5 + 5/2x)^8$

Let us consider the 7th term as T_7

So,

$T_7 = T_{6+1}$

$$= {}^8C_6\left(\frac{4x}{5}\right)^{8-6}\left(\frac{5}{2x}\right)^6$$

$$= \frac{8\times7\times4\times4\times125\times125}{2\times1\times25\times64}\ x^2\ \left(\frac{1}{x^6}\right)$$

$$= \frac{4375}{x^4}$$

∴ The 7th term of the expression $(4x/5 + 5/2x)^8$ is $4375/x^4$.

6. Find the 4th term from the beginning and 4th term from the end in the expansion of $(x + 2/x)^9$.

Solution:

Given:

$(x + 2/x)^9$

Let T_{r+1} be the 4th term from the end.

Then, T_{r+1} is (10 − 4 + 1)th, i.e., 7th, term from the beginning.

$T_7 = T_{6+1}$

$$= {}^9C_6\ (x^{9-6})\left(\frac{2}{x}\right)^6$$

$$= \frac{9\times8\times7}{3\times2}\ (x^3)\left(\frac{64}{x^6}\right)$$

$$= \frac{5376}{x^3}$$

4th term from the beginning = $T_4 = T_{3+1}$

$$T_4 = {}^9C_3\ (\ x^{9-3})\ \left(\frac{2}{x}\right)^3$$

$$= \frac{9\times8\times7}{3\times2}\ (x^6)\left(\frac{8}{x^3}\right)$$

$= 672\ x^3$

7. Find the 4^{th} term from the end in the expansion of $(4x/5 – 5/2x)^8$.

Solution:

Given:

$(4x/5 – 5/2x)^8$

Let T_{r+1} be the 4th term from the end of the given expression.

Then, T_{r+1} is (10 − 4 + 1)th term, i.e., 7th term, from the beginning.

$T_7 = T_{6+1}$

$$= {}^9C_6\ \left(\frac{4x}{5}\right)^{9-6}\left(\frac{5}{2x}\right)^6$$

$$= \frac{9\times8\times7}{3\times2}\ \left(\frac{64}{125}x^3\right)\left(\frac{125\times125}{64x^6}\right)$$

$$= \frac{10500}{x^3}$$

∴ The 4^{th} term from the end is $10500/x^3$.

8. Find the 7th term from the end in the expansion of $(2x^2 – 3/2x)^8$.

Solution:

Given:

$(2x^2 – 3/2x)^8$

Let T_{r+1} be the 4th term from the end of the given expression.

Then, T_{r+1} is (9 − 7 + 1)th term, i.e., 3rd term, from the beginning.

$T_3 = T_{2+1}$

$$= {}^8C_2 \left(2x^2\right)^{8-2} \left(-\frac{3}{2x}\right)^2$$

$$= \frac{8\times7}{2\times1}\left(64x^{12}\right)\frac{9}{4x^2}$$

$$= 4032\,x^{10}$$

∴ The 7^{th} term from the end is $4032\ x^{10}$.

9. Find the coefficient of:

(i) x^{10} in the expansion of $(2x^2 – 1/x)^{20}$

(ii) x^7 in the expansion of $(x – 1/x^2)^{40}$

(iii) x^{-15} in the expansion of $(3x^2 – a/3x^3)^{10}$

(iv) x^9 in the expansion of $(x^2 – 1/3x)^9$

(v) x^m in the expansion of $(x + 1/x)^n$

(vi) x in the expansion of $(1 – 2x^3 + 3x^5)(1 + 1/x)^8$

(vii) a^5b^7 in the expansion of $(a – 2b)^{12}$

(viii) x in the expansion of $(1 – 3x + 7x^2)(1 – x)^{16}$

Solution:

(i) x^{10} in the expansion of $(2x^2 – 1/x)^{20}$

Given:

$(2x^2 – 1/x)^{20}$

If x^{10} occurs in the (r + 1)th term in the given expression.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{20}C_r \left(2x^2\right)^{20-r} \left(\frac{-1}{x}\right)^r$$
$$= \left(-1\right)^r \, {}^{20}C_r \, (2^{20-r}) \, (x^{40-2r-r})$$

For this term to contain x^{10}, we must have:

40 − 3r =10

3r = 30

r = 10

$$\therefore \text{ Coefficient of } x^{10} = \left(-1\right)^{10} \, {}^{20}C_{10} \, (2^{20-10}) = {}^{20}C_{10} \, (2^{10})$$

(ii) x^7 in the expansion of $(x – 1/x^2)^{40}$

Given:

$(x – 1/x^2)^{40}$

If x^7 occurs at the (r + 1) th term in the given expression.

Then, we have:

$T_{r+1} = {}^nC_r \, x^{n-r} \, a^r$

$$T_{r+1} = {}^{40}C_r \, x^{40-r} \left(\frac{-1}{x^2}\right)^r$$
$$= \left(-1\right)^r \, {}^{40}C_r \, x^{40-r-2r}$$

For this term to contain x^7, we must have:

40 − 3r =7

3r = 40 – 7

3r = 33

r = 13/3

= 11

$$\therefore \text{Coefficient of } x^7 = \left(-1\right)^{11} \, {}^{40}C_{11} = -{}^{40}C_{11}$$

40 − 3r =7

3r = 40 – 7

3r = 33

r = 33/3

= 11

$$\therefore \text{Coefficient of } x^7 = \left(-1\right)^{11} \, {}^{40}C_{11} = -{}^{40}C_{11}$$

(iii) x^{-15} in the expansion of $(3x^2 – a/3x^3)^{10}$

Given:

$(3x^2 – a/3x^3)^{10}$

If x^{-15} occurs at the (r + 1)th term in the given expression.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{10}C_r \left(3x^2\right)^{10-r}\left(\frac{-a}{3x^3}\right)^r$$
$$= \left(-1\right)^r\, {}^{10}C_r\ \left(3^{10-r-r}\right)\left(x^{20-2r-3r}\right)\left(a^r\right)$$

For this term to contain x^{-15}, we must have:

$20 - 5r = -15$

$5r = 20 + 15$

$5r = 35$

$r = 35/5$

$= 7$

$$\therefore \text{Coefficient of } x^{-15} = \left(-1\right)^7\ {}^{10}C_7\ 3^{10-14}\ \left(a^7\right) = -\frac{10\times9\times8}{3\times2\times9\times9}a^7 = -\frac{40}{27}a^7$$

(iv) x^9 in the expansion of $(x^2 – 1/3x)^9$

Given:

$(x^2 – 1/3x)^9$

If x^9 occurs at the (r + 1)th term in the above expression.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^9C_r\left(x^2\right)^{9-r}\left(\frac{-1}{3x}\right)^r$$
$$= \left(-1\right)^r\, {}^9C_r\ \left(x^{18-2r-r}\right)\left(\frac{1}{3^r}\right)$$

For this term to contain x^9, we must have:

$18 - 3r = 9$

$3r = 18 - 9$

$3r = 9$

$r = 9/3$

$= 3$

$$\therefore \text{Coefficient of } x^9 = \left(-1\right)^3\, {}^9C_3\ \frac{1}{3^3} = -\frac{9\times8\times7}{2\times9\times9} = \frac{-28}{9}$$

For this term to contain x^9, we must have:

$18 - 3r = 9$

$3r = 18 - 9$

$3r = 9$

$r = 9/3$

$= 3$

$$\therefore \text{Coefficient of } x^9 = \left(-1\right)^3 {}^9C_3 \frac{1}{3^3} = -\frac{9\times8\times7}{2\times9\times9} = \frac{-28}{9}$$

(v) x^m in the expansion of $(x + 1/x)^n$

Given:

$(x + 1/x)^n$

If x^m occurs at the $(r + 1)$th term in the given expression.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^nC_r\, x^{n-r}\, \frac{1}{x^r}$$

$$= {}^nC_r\, x^{n-2r}$$

For this term to contain x^m, we must have:

$n - 2r = m$

$2r = n - m$

$r = (n - m)/2$

$$\therefore \text{Coefficient of } x^m = {}^nC_{(n-m)/2} = \frac{n!}{\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}$$

(vi) x in the expansion of $(1 - 2x^3 + 3x^5)(1 + 1/x)^8$

Given:

$(1 - 2x^3 + 3x^5)(1 + 1/x)^8$

If x occurs at the $(r + 1)$th term in the given expression.

Then, we have:

$(1 - 2x^3 + 3x^5)(1 + 1/x)^8$ **=** $(1 - 2x^3 + 3x^5)({}^8C_0 + {}^8C_1 (1/x) + {}^8C_2 (1/x)^2 + {}^8C_3 (1/x)^3 + {}^8C_4 (1/x)^4 + {}^8C_5 (1/x)^5 + {}^8C_6 (1/x)^6 + {}^8C_7 (1/x)^7 + {}^8C_8 (1/x)^8)$

So, ‘x’ occurs in the above expression at $-2x^3.{}^8C_2\ (1/x^2) + 3x^5.{}^8C_4\ (1/x^4)$

∴ Coefficient of x = -2 (8!/(2!6!)) + 3 (8!/(4! 4!))

= -56 + 210

= 154

(vii) a^5b^7 in the expansion of $(a – 2b)^{12}$

Given:

$(a – 2b)^{12}$

If a^5b^7 occurs at the (r + 1)th term in the given expression.

Then, we have:

$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$T_{r+1} = {}^{12}C_r\ a^{12-r}\ (-2b)^r$$
$$= (-1)^r\ {}^{12}C_r\ (a^{12-r})\ (b^r)\ (2^r)$$

For this term to contain a^5b^7, we must have:

12 – r = 5

r = 12 – 5

= 7

$$\therefore \text{ Required coefficient } = (-1)^7\ {}^{12}C_7\ (2^7)$$
$$= -\frac{12\times11\times10\times9\times8\times128}{5\times4\times3\times2}$$
$$= -101376$$

(viii) x in the expansion of $(1 – 3x + 7x^2)(1 – x)^{16}$

Given:

$(1 – 3x + 7x^2)(1 – x)^{16}$

If x occurs at the (r + 1)th term in the given expression.

Then, we have:

$(1 – 3x + 7x^2)(1 – x)^{16} = (1 – 3x + 7x^2)({}^{16}C_0 + {}^{16}C_1(-x) + {}^{16}C_2(-x)^2 + {}^{16}C_3(-x)^3 + {}^{16}C_4(-x)^4 + {}^{16}C_5(-x)^5 + {}^{16}C_6(-x)^6 + {}^{16}C_7(-x)^7 + {}^{16}C_8(-x)^8 + {}^{16}C_9(-x)^9 + {}^{16}C_{10}(-x)^{10} + {}^{16}C_{11}(-x)^{11} + {}^{16}C_{12}(-x)^{12} + {}^{16}C_{13}(-x)^{13} + {}^{16}C_{14}(-x)^{14} + {}^{16}C_{15}(-x)^{15} + {}^{16}C_{16}(-x)^{16})$

So, ‘x’ occurs in the above expression at ${}^{16}C_1(-x) – 3x\,{}^{16}C_0$

∴ Coefficient of x = -(16!/(1! 15!)) – 3(16!/(0! 16!))

= -16 – 3

= -19

10. Which term in the expansion of $\left\{\left(\frac{x}{\sqrt{y}}\right)^{1/3}+\left(\frac{y}{x^{1/3}}\right)^{1/2}\right\}^{21}$ **contains x and y to one and the same power.**

Solution:

Let us consider T_{r+1} th term in the given expansion contains x and y to one and the same power.

Then we have,

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{21}C_r \left[\left(\frac{x}{\sqrt{y}}\right)^{1/3}\right]^{21-r} \left[\left(\frac{y}{x^{1/3}}\right)^{1/2}\right]^{r}$$

$$= {}^{21}C_r \left(\frac{x^{(21-r)/3}}{x^{r/6}}\right)\left(\frac{y^{r/2}}{y^{(21-r)/6}}\right)$$

$$= {}^{21}C_r\ (x)^{7-r/2}(y)^{2r/3-7/2}$$

$$= {}^{21}C_r\ (x)^{7-r/2}(y)^{2r/3-7/2}$$

If x and y have the same power, then

7 – r/2 = 2r/3 – 7/2

2r/3 + r/2 = 7 + 7/2

(4r + 3r)/6 = (14+7)/2

7r/6 = 21/2

r = (21 × 6)/(2 × 7)

= 3×3

= 9

Hence, the required term is the 10^{th} term.

11. Does the expansion of (2x^2 – 1/x) contain any term involving x^9?

Solution:

Given:

($2x^2$ – 1/x)

If x^9 occurs at the (r + 1)th term in the given expression.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{20}C_r \left(2x^2\right)^{20-r} \left(\frac{-1}{x}\right)^r$$
$$= \left(-1\right)^r\ {}^{20}C_r (2)^{20-r}\ (x)^{40-2r-r}$$

For this term to contain x^9, we must have

40 – 3r = 9

3r = 40 – 9

3r = 31

r = 31/3

It is not possible, since r is not an integer.

Hence, there is no term with x^9 in the given expansion.

12. Show that the expansion of $(x^2 + 1/x)^{12}$ does not contain any term involving x^{-1}.

Solution:

Given:

$(x^2 + 1/x)^{12}$

If x^{-1} occurs at the (r + 1)th term in the given expression.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{12}C_r \left(x^2\right)^{12-r} \left(\frac{1}{x}\right)^r$$
$$= {}^{12}C_r\ x^{24-2r-r}$$

For this term to contain x^{-1}, we must have

24 – 3r = -1

3r = 24 + 1

3r = 25

$r = 25/3$

It is not possible, since r is not an integer.

Hence, there is no term with x^{-1} in the given expansion.

13. Find the middle term in the expansion of:

(i) $(2/3x – 3/2x)^{20}$

(ii) $(a/x + bx)^{12}$

(iii) $(x^2 – 2/x)^{10}$

(iv) $(x/a – a/x)^{10}$

Solution:

(i) $(2/3x – 3/2x)^{20}$

We have,

$(2/3x – 3/2x)^{20}$ where, $n = 20$ (even number)

So the middle term is $(n/2 + 1) = (20/2 + 1) = (10 + 1) = 11$. ie., 11^{th} term

Now,

$T_{11} = T_{10+1}$

$= {}^{20}C_{10}\ (2/3x)^{20-10}\ (3/2x)^{10}$

$= {}^{20}C_{10}\ 2^{10}/3^{10} \times 3^{10}/2^{10}\ x^{10-10}$

$= {}^{20}C_{10}$

Hence, the middle term is ${}^{20}C_{10}$.

(ii) $(a/x + bx)^{12}$

We have,

$(a/x + bx)^{12}$ where, $n = 12$ (even number)

So the middle term is $(n/2 + 1) = (12/2 + 1) = (6 + 1) = 7$. ie., 7^{th} term

Now,

$T_7 = T_{6+1}$

$$= {}^{12}C_6 \left(\frac{a}{x}\right)^{12-6} \left(bx\right)^6$$
$$= {}^{12}C_6\, a^6\, b^6$$
$$= \frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2} a^6\, b^6$$
$$= 924\ a^6b^6$$

= 924 a^6b^6

Hence, the middle term is 924 a^6b^6.

(iii) $(x^2 – 2/x)^{10}$

We have,

$(x^2 – 2/x)^{10}$ where, n = 10 (even number)

So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6^{th} term

Now,

$T_6 = T_{5+1}$

$$= {}^{10}C_5 \left(x^2\right)^{10-5} \left(\frac{-2}{x}\right)^5$$
$$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2} \times 32x^5$$
$$= -\ 8064\ x^5$$

Hence, the middle term is $-8064x^5$.

(iv) $(x/a – a/x)^{10}$

We have,

$(x/a – a/x)^{10}$ where, n = 10 (even number)

So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6^{th} term

Now,

$T_6 = T_{5+1}$

$$= {}^{10}C_5 \left(\frac{x}{a}\right)^{10-5} \left(\frac{-a}{x}\right)^5$$
$$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2}$$
$$= -\ 252$$

Hence, the middle term is -252.

14. Find the middle terms in the expansion of:

(i) $(3x - x^3/6)^9$

(ii) $(2x^2 - 1/x)^7$

(iii) $(3x - 2/x^2)^{15}$

(iv) $(x^4 - 1/x^3)^{11}$

Solution:

(i) $(3x - x^3/6)^9$

We have,

$(3x - x^3/6)^9$ where, n = 9 (odd number)

So the middle terms are ((n+1)/2) = ((9+1)/2) = 10/2 = 5 and

((n+1)/2 + 1) = ((9+1)/2 + 1) = (10/2 + 1) = (5 + 1) = 6

The terms are 5th and 6th.

Now,

$T_5 = T_{4+1}$

$$= {}^9C_4 \left(3x\right)^{9-4} \left(\frac{-x^3}{6}\right)^4$$

$$= \frac{9\times8\times7\times6}{4\times3\times2} \times 27 \times 9 \times \frac{1}{36\times36} x^{17}$$

$$= \frac{189}{8} x^{17}$$

And,

$T_6 = T_{5+1}$

$$= {}^9C_5 \left(3x\right)^{9-5} \left(\frac{-x^3}{6}\right)^5$$

$$= -\frac{9\times8\times7\times6}{4\times3\times2} \times 81 \times \frac{1}{216\times36} x^{19}$$

$$= -\frac{21}{16} x^{19}$$

Hence, the middle term are $189/8\ x^{17}$ and $-21/16\ x^{19}$.

(ii) $(2x^2 - 1/x)^7$

We have,

$(2x^2 - 1/x)^7$ where, n = 7 (odd number)

So the middle terms are ((n+1)/2) = ((7+1)/2) = 8/2 = 4 and

((n+1)/2 + 1) = ((7+1)/2 + 1) = (8/2 + 1) = (4 + 1) = 5

The terms are 4th and 5th.

Now,

$T_4 = T_{3+1}$

$$= {}^7C_3 \left(2x^2\right)^{7-3} \left(\frac{-1}{x}\right)^3$$

$$= -\frac{7\times6\times5}{3\times2} \times 16\, x^8 \times \frac{1}{x^3}$$

$$= -560\, x^5$$

And,

$T_5 = T_{4+1}$

$$= {}^7C_4 \left(2x^2\right)^{7-4} \left(\frac{-1}{x}\right)^4$$

$$= 35 \times 8 \;\times x^6 \times \frac{1}{x^4}$$

$$= 280\, x^2$$

Hence, the middle term are $-560x^5$ and $280x^2$.

(iii) $(3x – 2/x^2)^{15}$

We have,

$(3x – 2/x^2)^{15}$ where, n = 15 (odd number)

So the middle terms are ((n+1)/2) = ((15+1)/2) = 16/2 = 8 and ((n+1)/2 + 1) = ((15+1)/2 + 1) = (16/2 + 1) = (8 + 1) = 9

The terms are 8th and 9th.

Now,

$T_8 = T_{7+1}$

$$= {}^{15}C_7 \left(3x\right)^{15-7} \left(\frac{-2}{x^2}\right)^7$$

$$= -\frac{15\times14\times13\times12\times11\times10\times9}{7\times6\times5\times4\times3\times2} \times 3^8 \times 2^7\, x^{8-14}$$

$$= \frac{-6435\times3^8\times2^7}{x^6}$$

And,

$T_9 = T_{8+1}$

$$= {}^{15}C_8 \left(3x\right)^{15-8} \left(\frac{-2}{x^2}\right)^8$$

$$= \frac{15\times14\times13\times12\times11\times10\times9}{7\times6\times5\times4\times3\times2} \times 3^7 \times 2^8 \;\times x^{7-16}$$

$$= \frac{6435\times3^7\times2^8}{x^9}$$

Hence, the middle term are $(-6435\times3^8\times2^7)/x^6$ and $(6435\times3^7\times2^8)/x^9$.

(iv) $(x^4 – 1/x^3)^{11}$

We have,

$(x^4 – 1/x^3)^{11}$

where, n = 11 (odd number)

So the middle terms are ((n+1)/2) = ((11+1)/2) = 12/2 = 6 and

((n+1)/2 + 1) = ((11+1)/2 + 1) = (12/2 + 1) = (6 + 1) = 7

The terms are 6^{th} and 7^{th}.

Now,

$$T_6 = T_{5+1}$$
$$= {}^{11}C_5\left(x^4\right)^{11-5}\left(\frac{-1}{x^3}\right)^5$$
$$= -\frac{11\times10\times9\times8\times7}{5\times4\times3\times2}\times(x)^{24-15}$$
$$= -462\,x^9$$

And,

$$T_7 = T_{6+1}$$
$$= {}^{11}C_6\left(x^4\right)^{11-6}\left(\frac{-1}{x^3}\right)^6$$
$$= \frac{11\times10\times9\times8\times7}{5\times4\times3\times2}(x)^{20-18}$$
$$= 462\,x^2$$

Hence, the middle term are $-462x^9$ and $462x^2$.

15. Find the middle terms in the expansion of:

(i) $(x – 1/x)^{10}$

(ii) $(1 – 2x + x^2)^n$

(iii) $(1 + 3x + 3x^2 + x^3)^{2n}$

(iv) $(2x – x^2/4)^9$

(v) $(x – 1/x)^{2n+1}$

(vi) $(x/3 + 9y)^{10}$

(vii) $(3 – x^3/6)^7$

(viii) $(2ax – b/x^2)^{12}$

(ix) $(p/x + x/p)^9$

(x) $(x/a – a/x)^{10}$

Solution:

(i) $(x – 1/x)^{10}$

We have,

$(x – 1/x)^{10}$ where, n = 10 (even number)

So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6^{th} term

Now,

$T_6 = T_{5+1}$

$$= {}^{10}C_5\, x^{10-5} \left(\frac{-1}{x}\right)^5$$
$$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2}$$
$$= -252$$

Hence, the middle term is -252.

(ii) $(1 – 2x + x^2)^n$

We have,

$(1 – 2x + x^2)^n = (1 – x)^{2n}$ where, n is an even number.

So the middle term is (2n/2 + 1) = (n + 1)th term.

Now,

$T_n = T_{n+1}$

$= {}^{2n}C_n (-1)^n (x)^n$

$= (2n)!/(n!)^2 (-1)^n x^n$

Hence, the middle term is $(2n)!/(n!)^2 (-1)^n x^n$.

(iii) $(1 + 3x + 3x^2 + x^3)^{2n}$

We have,

$(1 + 3x + 3x^2 + x^3)^{2n} = (1 + x)^{6n}$ where, n is an even number.

So the middle term is (n/2 + 1) = (6n/2 + 1) = (3n + 1)th term.

Now,

$T_{2n} = T_{3n+1}$

$= {}^{6n}C_{3n}\, x^{3n}$

$= (6n)!/(3n!)^2\, x^{3n}$

Hence, the middle term is $(6n)!/(3n!)^2\, x^{3n}$.

(iv) $(2x – x^2/4)^9$

We have,

$(2x – x^2/4)^9$ where, n = 9 (odd number)

So the middle terms are ((n+1)/2) = ((9+1)/2) = 10/2 = 5 and

((n+1)/2 + 1) = ((9+1)/2 + 1) = (10/2 + 1) = (5 + 1) = 6

The terms are 5th and 6th.

Now,

$T_5 = T_{4+1}$

$$= {}^9C_4 \left(2x\right)^{9-4} \left(\frac{-x^2}{4}\right)^4$$

$$= \frac{9\times8\times7\times6}{4\times3\times2} \times 2^5\ \frac{1}{4^4} x^{5+8}$$

$$= \frac{63}{4} x^{13}$$

And,

$T_6 = T_{5+1}$

$$= {}^9C_5 \left(2x\right)^{9-5} \left(\frac{-x^2}{4}\right)^5$$

$$= -\frac{9\times8\times7\times6}{4\times3\times2} \times 2^4\ \frac{1}{4^5} x^{4+10}$$

$$= -\frac{63}{32} x^{14}$$

Hence, the middle term is $63/4\ x^{13}$ and $-63/32\ x^{14}$.

(v) $(x – 1/x)^{2n+1}$

We have,

$(x – 1/x)^{2n+1}$ where, n = (2n + 1) is an (odd number)

So the middle terms are ((n+1)/2) = ((2n+1+1)/2) = (2n+2)/2 = (n + 1) and ((n+1)/2 + 1) = ((2n+1+1)/2 + 1) = ((2n+2)/2 + 1) = (n + 1 + 1) = (n + 2)

The terms are $(n + 1)^{th}$ and $(n + 2)^{th}$.

Now,

$T_n = T_{n+1}$

$$= {}^{2n+1}C_n\, x^{2n+1-n} \times \frac{(-1)^n}{x^n}$$
$$= \left(-1\right)^n\, {}^{2n+1}C_n\, x$$

And,

$T_{n+2} = T_{n+1+1}$

$$= {}^{2n+1}C_n x^{2n+1-n-1}\ \frac{(-1)^{n+1}}{x^{n+1}}$$
$$= \left(-1\right)^{n+1}\, {}^{2n+1}C_n \times \frac{1}{x}$$

Hence, the middle term is $(-1)^n.{}^{2n+1}C_n$ x and $(-1)^{n+1}.{}^{2n+1}C_n$ (1/x).

(vi) $(x/3 + 9y)^{10}$

We have,

$(x/3 + 9y)^{10}$ where, n = 10 is an even number.

So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. i.e., 6th term.

Now,

$T_6 = T_{5+1}$

$$= {}^{10}C_5\, \left(\tfrac{x}{3}\right)^{10-5} \left(9y\right)^5$$
$$= \frac{10\times9\times8\times7\times6}{5\times4\times3\times2} \times \frac{1}{3^5} \times 9^5 \times x^5\, y^5$$
$$= 61236\, x^5\, y^5$$

Hence, the middle term is $61236x^5y^5$.

(vii) $(3 – x^3/6)^7$

We have,

$(3 – x^3/6)^7$ where, n = 7 (odd number).

So the middle terms are ((n+1)/2) = ((7+1)/2) = 8/2 = 4 and

$((n+1)/2 + 1) = ((7+1)/2 + 1) = (8/2 + 1) = (4 + 1) = 5$

The terms are 4^{th} and 5^{th}.

Now,

$T_4 = T_{3+1}$

$= {}^7C_3 (3)^{7-3} (-x^3/6)^3$

$= -105/8\ x^9$

And,

$T_5 = T_{4+1}$

$= {}^9C_4 (3)^{9-4} (-x^3/6)^4$

$$= \frac{7\times6\times5}{3\times2} \times 3^5 \times \frac{1}{6^4}\ x^{12}$$

$$= \frac{35}{48} x^{12}$$

Hence, the middle terms are $-105/8\ x^9$ and $35/48\ x^{12}$.

(viii) $(2ax – b/x^2)^{12}$

We have,

$(2ax – b/x^2)^{12}$ where, n = 12 is an even number.

So the middle term is $(n/2 + 1) = (12/2 + 1) = (6 + 1) = 7$. i.e., 7th term.

Now,

$T_7 = T_{6+1}$

$$= {}^{12}C_6\ (2ax)^{12-6} \left(\frac{-b}{x^2}\right)^6$$

$$= \frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2\times1} \times \left(\frac{2ab}{x}\right)^6$$

$$= \frac{59136\ a^6b^6}{x^6}$$

Hence, the middle term is $(59136a^6b^6)/x^6$.

(ix) $(p/x + x/p)^9$

We have,

$(p/x + x/p)^9$ where, n = 9 (odd number).

So the middle terms are $((n+1)/2) = ((9+1)/2) = 10/2 = 5$ and ((n+1)/2 + 1) = ((9+1)/2 + 1) = (10/2 + 1) = (5 + 1) = 6

The terms are 5^{th} and 6^{th}.

Now,

$T_5 = T_{4+1}$

$$= {}^9C_4 \left(\frac{p}{x}\right)^{9-4} \left(\frac{x}{p}\right)^4$$
$$= \frac{9\times8\times7\times6}{4\times3\times2\times1} \times \left(\frac{p}{x}\right)$$
$$= \frac{126\,p}{x}$$

And,

$T_6 = T_{5+1}$

$= {}^9C_5\ (p/x)^{9-5}\ (x/p)^5$

$$= \frac{9\times8\times7\times6}{4\times3\times2\times1} \times \left(\frac{x}{p}\right)$$
$$= \frac{126\,x}{p}$$

Hence, the middle terms are 126p/x and 126x/p.

(x) $(x/a – a/x)^{10}$

We have,

$(x/a – a/x)^{10}$ where, n = 10 (even number)

So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6^{th} term

Now,

$T_6 = T_{5+1}$

$$= {}^{10}C_5 \left(\frac{x}{a}\right)^{10-5} \left(\frac{-a}{x}\right)^5$$
$$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2}$$
$$= -252$$

Hence, the middle term is -252.

16. Find the term independent of x in the expansion of the following expressions:

(i) $(3/2\ x^2 – 1/3x)^9$

(ii) $(2x + 1/3x^2)^9$

(iii) $(2x^2 – 3/x^3)^{25}$

(iv) $(3x – 2/x^2)^{15}$

(v) $((\sqrt{x}/3) + \sqrt{3}/2x^2)^{10}$

(vi) $(x – 1/x^2)^{3n}$

(vii) $(1/2\ x^{1/3} + x^{-1/5})^8$

(viii) $(1 + x + 2x^3)\ (3/2x^2 – 3/3x)^9$

(ix) $(\sqrt[3]{x} + 1/2\sqrt[3]{x})^{18}$, x > 0

(x) $(3/2x^2 – 1/3x)^6$

Solution:

(i) $(3/2\ x^2 – 1/3x)^9$

Given:

$(3/2\ x^2 – 1/3x)^9$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$= {}^9C_r\ \left(\frac{3}{2}x^2\right)^{9-r}\ \left(\frac{-1}{3x}\right)^r$$

$$= \left(-1\right)^r\ {}^9C_r\ .\ \frac{3^{9-2r}}{2^{9-r}}\ \times\ x^{18-2r-r}$$

For this term to be independent of x, we must have

18 – 3r = 0

3r = 18

r = 18/3

= 6

So, the required term is 7^{th} term.

We have,

$T_7 = T_{6+1}$

$= {}^9C_6 \times (3^{9-12})/(2^{9-6})$

$= (9×8×7)/(3×2) \times 3^{-3} \times 2^{-3}$

$= 7/18$

Hence, the term independent of x is 7/18.

(ii) $(2x + 1/3x^2)^9$

Given:

$(2x + 1/3x^2)^9$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$= {}^9C_r \left(2x\right)^{9-r} \left(\frac{1}{3x^2}\right)^r$$

$$= {}^9C_r . \frac{2^{9-r}}{3^r}\, x^{9-r-2r}$$

For this term to be independent of x, we must have

$9 – 3r = 0$

$3r = 9$

$r = 9/3$

$= 3$

So, the required term is 4^{th} term.

We have,

$T_4 = T_{3+1}$

$= {}^9C_3 \times (2^6)/(3^3)$

$= {}^9C_3 \times 64/27$

Hence, the term independent of x is ${}^9C_3 \times 64/27$.

(iii) $(2x^2 – 3/x^3)^{25}$

Given:

$(2x^2 – 3/x^3)^{25}$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$= {}^{25}C_r\, (2x^2)^{25-r}\, (-3/x^3)^r$

$= (-1)^r\, {}^{25}C_r \times 2^{25-r} \times 3^r\, x^{50-2r-3r}$

For this term to be independent of x, we must have

$50 – 5r = 0$

$5r = 50$

$r = 50/5$

$= 10$

So, the required term is 11^{th} term.

We have,

$T_{11} = T_{10+1}$

$= (-1)^{10}\, {}^{25}C_{10} \times 2^{25-10} \times 3^{10}$

$= {}^{25}C_{10}\, (2^{15} \times 3^{10})$

Hence, the term independent of x is ${}^{25}C_{10}\, (2^{15} \times 3^{10})$.

(iv) $(3x – 2/x^2)^{15}$

Given:

$(3x – 2/x^2)^{15}$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$= {}^{15}C_r\, (3x)^{15-r}\, (-2/x^2)^r$

$= (-1)^r\, {}^{15}C_r \times 3^{15-r} \times 2^r\, x^{15-r-2r}$

For this term to be independent of x, we must have

$15 – 3r = 0$

$3r = 15$

$r = 15/3$

$= 5$

So, the required term is 6^{th} term.

We have,

$T_6 = T_{5+1}$

$= (-1)^5\ {}^{15}C_5 \times 3^{15-5} \times 2^5$

$= -3003 \times 3^{10} \times 2^5$

Hence, the term independent of x is $-3003 \times 3^{10} \times 2^5$.

(v) $((\sqrt{x}/3) + \sqrt{3}/2x^2)^{10}$

Given:

$((\sqrt{x}/3) + \sqrt{3}/2x^2)^{10}$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$= {}^{10}C_r \left(\sqrt{\frac{x}{3}}\right)^{10-r} \left(\frac{3}{2x^2}\right)^r$$

$$= {}^{10}C_r \cdot \frac{3^{r-\frac{10-r}{2}}}{2^r}\, x^{\frac{10-r}{2}-2r}$$

For this term to be independent of x, we must have

$(10-r)/2 – 2r = 0$

$10 – 5r = 0$

$5r = 10$

$r = 10/5$

$= 2$

So, the required term is 3rd term.

We have,

$T_3 = T_{2+1}$

$$= {}^{10}C_2 \times \frac{3^{2-\frac{10-2}{2}}}{2^2}$$

$$= \frac{10\times9}{2\times4\times9}$$

= 90/72

= 15/12

= 5/4

Hence, the term independent of x is 5/4.

(vi) $(x - 1/x^2)^{3n}$

Given:

$(x - 1/x^2)^{3n}$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$= {}^{3n}C_r\, x^{3n-r}\, (-1/x^2)^r$

$= (-1)^r\, {}^{3n}C_r\, x^{3n-r-2r}$

For this term to be independent of x, we must have

$3n - 3r = 0$

$r = n$

So, the required term is (n+1)th term.

We have,

$(-1)^n\, {}^{3n}C_n$

Hence, the term independent of x $(-1)^n\, {}^{3n}C_n$

(vii) $(1/2\, x^{1/3} + x^{-1/5})^8$

Given:

$(1/2\, x^{1/3} + x^{-1/5})^8$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$= {}^8C_r\left(\frac{1}{2}x^{1/3}\right)^{8-r}\left(x^{-1/5}\right)^r$$

$$= {}^8C_r.\ \frac{1}{2^{8-r}}\ x^{\frac{8-r}{3}-\frac{r}{5}}$$

For this term to be independent of x, we must have

$(8-r)/3 - r/5 = 0$

(40 – 5r – 3r)/15 = 0

40 – 5r – 3r = 0

40 – 8r = 0

8r = 40

r = 40/8

= 5

So, the required term is 6th term.

We have,

$T_5 = T_{5+1}$

$= {}^8C_5 \times 1/(2^{8-5})$

= (8×7×6)/(3×2×8)

= 7

Hence, the term independent of x is 7.

(viii) $(1 + x + 2x^3)\,(3/2x^2 – 3/3x)^9$

Given:

$(1 + x + 2x^3)\,(3/2x^2 – 3/3x)^9$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$(1 + x + 2x^3)\,(3/2x^2 – 3/3x)^9$ **=**

$$= (1 + x + 2x^3)\left[\left(\frac{3}{2}x^2\right)^9 - {}^9C_1\left(\frac{3}{2}x^2\right)^8\frac{1}{3x}\ldots + {}^9C_6\left(\frac{3}{2}x^2\right)^3\left(\frac{1}{3x}\right)^6 - {}^9C_7\left(\frac{3}{2}x^2\right)^2\left(\frac{1}{3x}\right)^7\right]$$

By computing we get,

The term independent of x

$$= 1\left[{}^9C_6\frac{3^3}{2^3}\times\frac{1}{3^6}\right] - 2x^3\left[{}^9C_7\frac{3^3}{2^3}\times\frac{1}{3^7}\times\frac{1}{x^3}\right]$$

$$= \left[\frac{9\times8\times7}{1\times2\times3}\times\frac{1}{8\times27}\right] - 2\left[\frac{9\times8}{1\times2} - \frac{1}{4\times243}\right]$$

$$= \frac{7}{18} - \frac{2}{27}$$

$$= \frac{17}{54}$$

$= 7/18 – 2/27$

$= (189 – 36)/486$

$= 153/486$ (divide by 9)

$= 17/54$

Hence, the term independent of x is 17/54.

(ix) $(\sqrt[3]{x} + 1/2\sqrt[3]{x})^{18}, x > 0$

Given:

$(\sqrt[3]{x} + 1/2\sqrt[3]{x})^{18}, x > 0$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$= {}^{18}C_r \left(x^{1/3}\right)^{18-r} \left(\frac{1}{2\,x^{1/3}}\right)^r$$

$$= {}^{18}C_r \times \frac{1}{2^r}\, x^{\frac{18-r}{3}-\frac{r}{3}}$$

For this term to be independent of r, we must have

$(18-r)/3 – r/3 = 0$

$(18 – r – r)/3 = 0$

$18 – 2r = 0$

$2r = 18$

$r = 18/2$

$= 9$

So, the required term is 10th term.

We have,

$T_{10} = T_{9+1}$

$= {}^{18}C_9 \times 1/2^9$

Hence, the term independent of x is ${}^{18}C_9 \times 1/2^9$.

(x) $(3/2x^2 – 1/3x)^6$

Given:

$(3/2x^2 – 1/3x)^6$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$= {}^6C_r \left(\frac{3}{2}x^2\right)^{6-r} \left(\frac{-1}{3x}\right)^r$$
$$= (-1)^r\ {}^6C_r \times \frac{3^{6-r-r}}{2^{6-r}}\ x^{12-2r-r}$$

For this term to be independent of r, we must have

12 – 3r = 0

3r = 12

r = 12/3

= 4

So, the required term is 5th term.

We have,

$T_5 = T_{4+1}$

$$= {}^6C_4 \times \frac{3^{6-4-4}}{2^{6-4}}$$
$$= \frac{6\times5}{2\times1\times4\times9}$$
$$= \frac{5}{12}$$

Hence, the term independent of x is 5/12.

17. If the coefficients of (2r + 4)th and (r – 2)th terms in the expansion of $(1 + x)^{18}$ are equal, find r.

Solution:

Given:

$(1 + x)^{18}$

We know, the coefficient of the r term in the expansion of $(1 + x)^n$ is ${}^nC_{r-1}$

So, the coefficients of the (2r + 4) and (r – 2) terms in the given expansion are ${}^{18}C_{2r+4-1}$ and ${}^{18}C_{r-2-1}$

For these coefficients to be equal, we must have

$^{18}C_{2r+4-1} = {}^{18}C_{r-2-1}$

$^{18}C_{2r+3} = {}^{18}C_{r-3}$

2r + 3 = r – 3 (or) 2r + 3 + r – 3 = 18 [Since, $^nC_r = {}^nC_s$ => r = s (or) r + s = n]

2r – r = -3 – 3 (or) 3r = 18 – 3 + 3

r = -6 (or) 3r = 18

r = -6 (or) r = 18/3

r = -6 (or) r = 6

∴ r = 6 [since, r should be a positive integer.]

18. If the coefficients of (2r + 1)th term and (r + 2)th term in the expansion of $(1 + x)^{43}$ are equal, find r.

Solution:

Given:

$(1 + x)^{43}$

We know, the coefficient of the r term in the expansion of $(1 + x)^n$ is $^nC_{r-1}$

So, the coefficients of the (2r + 1) and (r + 2) terms in the given expansion are $^{43}C_{2r+1-1}$ and $^{43}C_{r+2-1}$

For these coefficients to be equal, we must have

$^{43}C_{2r+1-1} = {}^{43}C_{r+2-1}$

$^{43}C_{2r} = {}^{43}C_{r+1}$

2r = r + 1 (or) 2r + r + 1 = 43 [Since, $^nC_r = {}^nC_s$ => r = s (or) r + s = n]

2r – r = 1 (or) 3r + 1 = 43

r = 1 (or) 3r = 43 – 1

r = 1 (or) 3r = 42

r = 1 (or) r = 42/3

r = 1 (or) r = 14

∴ r = 14 [since, value ‘1’ gives the same term]

19. Prove that the coefficient of (r + 1)th term in the expansion of $(1 + x)^{n+1}$ is equal to the sum of the coefficients of rth and (r + 1)th terms in the expansion of $(1 + x)^n$.

Solution:

We know, the coefficients of (r + 1)th term in $(1 + x)^{n+1}$ is ${}^{n+1}C_r$

So, sum of the coefficients of the rth and (r + 1)th terms in $(1 + x)^n$ is

$(1 + x)^n = {}^nC_{r-1} + {}^nC_r$

$= {}^{n+1}C_r$ [since, ${}^nC_{r+1} + {}^nC_r = {}^{n+1}C_{r+1}$]

Hence proved.