Access answers to RD Sharma Solutions for Class 11 Maths Chapter 10 – Sine and Cosine Formulae and their Applications

EXERCISE 10.1 PAGE NO: 10.12

1. If in a ∆ABC, ∠A = 45°, ∠B = 60°, and ∠C = 75°; find the ratio of its sides.

Solution:

Given: In ∆ABC, ∠A = 45°, ∠B = 60°, and ∠C = 75°

By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Now by substituting the values we get,

$$\frac{a}{\sin 45°} = \frac{b}{\sin 60°} = \frac{c}{\sin 75°}$$

$$\frac{a}{\sin 45°} = \frac{b}{\sin 60°} = \frac{c}{\sin(30° + 45°)}$$

$$\frac{a}{\sin 45°} = \frac{b}{\sin 60°} = \frac{c}{\sin 30° \cos 45° + \sin 45° \cos 30°}$$

We know, sin (a + b) = sin a cos b + sin b cos a

Now by substituting the corresponding values, we get,

$$\frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}}$$

$$\frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{1 + \sqrt{3}}{2\sqrt{2}}}$$

$$a:b:c = \frac{1}{\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

Multiply the above expression by 2√2, we get

a: b: c = 2: √6: (1+√3)

Hence the ratio of the sides of the given triangle is a: b: c = 2: √6: (1+√3)

2. If in any ∆ABC, ∠C = 105°, ∠B = 45°, a = 2, then find b.

Solution:

Given: In ΔABC, $\angle C = 105^o$, $\angle B = 45^o$, $a = 2$

We know in a triangle,

$\angle A + \angle B + \angle C = 180°$

$\angle A = 180° - \angle B - \angle C$

Substituting the given values, we get

$\angle A = 180° - 45° - 105°$

$\angle A = 30°$

By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Now by substituting the corresponding values we get,

$$\frac{2}{\sin 30°} = \frac{b}{\sin 45°}$$

Substitute the equivalent values of the sine, we get

$$\frac{2}{\frac{1}{2}} = \frac{b}{\frac{1}{\sqrt{2}}}$$

$4 = b\sqrt{2}$

$b = 4/\sqrt{2}$

$= 2\sqrt{2}$

Hence the value of b is $2\sqrt{2}$ units.

3. In ΔABC, if a = 18, b = 24 and c = 30 and $\angle C = 90^o$, find sin A, sin B and sin C.

Solution:

Given: In ΔABC, $a = 18$, $b = 24$ and $c = 30$ and $\angle C = 90^o$

By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Now by substituting the given values we get,

$$\frac{18}{\sin A} = \frac{30}{\sin 90°}$$

$$\sin A = \frac{18 \times \sin 90°}{30}$$

$$\sin A = \frac{18 \times 1}{30}$$

$$\sin A = \frac{3}{5}$$

Similarly,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Substitute the given values, we get

$$\frac{24}{\sin B} = \frac{30}{\sin 90°}$$

$$\sin B = \frac{24 \times \sin 90°}{30}$$

$$\sin B = \frac{24 \times 1}{30}$$

$$\sin B = \frac{4}{5}$$

And given, $\angle C = 90°$, so $\sin C = \sin 90° = 1$.

Hence the values of $\sin A = 3/5$, $\sin B = 4/5$ and $\sin C = 1$ respectively.

In any triangle ABC, prove the following:

4. $$\frac{a - b}{a + b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, a = k sin A

Similarly, b = k sin B

And c = k sin C

We know,

a - b = k (sin A - sin B)

a + b = k (sin A + sin B)

Now let us consider LHS:

$$\frac{a-b}{a+b} = \frac{k(\sin A - \sin B)}{k(\sin A + \sin B)}$$

$$= \frac{(\sin A - \sin B)}{(\sin A + \sin B)} \ldots \text{(i)}$$

We know,

Sin A – sin B = 2 sin (A-B)/2 cos (A+B)/2

Sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2

Substituting the above formulas in equation (i), we get

$$\frac{a-b}{a+b} = \frac{\left(2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\right)}{\left(2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right)}$$

Upon rearranging we get,

$$= \frac{\left(\sin\left(\frac{A-B}{2}\right)\right)}{\left(\cos\left(\frac{A-B}{2}\right)\right)} \times \frac{\cos\left(\frac{A+B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)}$$

$$= \frac{\left(\tan\left(\frac{A-B}{2}\right)\right)}{1} \times \frac{1}{\tan\left(\frac{A+B}{2}\right)}$$

$$= \frac{\left(\tan\left(\frac{A-B}{2}\right)\right)}{\left(\tan\left(\frac{A+B}{2}\right)\right)}$$

= RHS

Hence proved.

5. (a – b) cos C/2 = C sin (A – B)/2

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, a = k sin A

Similarly, b = k sin B

We know,

a - b = k (sin A - sin B) …. (i)

Now let us consider LHS:

$$(a - b)\cos\frac{C}{2}$$

Substituting equation (i) in above equation, we get

$$(k(\sin A - \sin B))\cos\frac{C}{2} \ldots (ii)$$

We know,

Sin A – sin B = 2 sin (A-B)/2 cos (A+B)/2

Substituting the above formulas in equation (ii), we get

$$(a - b)\cos\frac{C}{2} = \left(k\left(2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\right)\right)\cos\frac{C}{2}$$

$$= \left(k\left(2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\right)\right)\cos\frac{(\pi - (A+B))}{2}$$

$$= \left(2k\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\right)\sin\left(\frac{A+B}{2}\right)$$ [since, cos(π/2 - A) = sin A]

Upon rearranging we get,

$$= k\sin\left(\frac{A-B}{2}\right)\left(2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A+B}{2}\right)\right)$$

We know, sin A = 2 cos (A/2) sin (A/2)

So the above equation becomes,

$$= k\sin\left(\frac{A-B}{2}\right)(\sin(A + B))$$

$$= k\sin\left(\frac{A-B}{2}\right)(\sin(\pi - C))$$ [since, π = A+B+C, where, A+B = π-C]

$$= k\sin(C)\sin\left(\frac{A-B}{2}\right)$$ [since, sin (π - A) = sin A]

From the sine rule,

$$\frac{c}{\sin C} = k \Rightarrow c = k\sin C$$

So the above equation becomes,

$$= c\sin\left(\frac{A-B}{2}\right)$$

= RHS

Hence proved.

6. $\frac{\mathbf{c}}{\mathbf{a-b}} = \frac{\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right)}{\tan\left(\frac{A}{2}\right) - \tan\left(\frac{B}{2}\right)}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, a = k sin A

Similarly, b = k sin B

And c = k sin C … (i)

We know,

a - b = k (sin A - sin B) …. (ii)

Now let us consider LHS:

$$\frac{c}{a - b}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k \sin C}{k(\sin A - \sin B)} = \frac{\sin C}{(\sin A - \sin B)} \quad \text{.... (iii)}$$

By applying half angle rule,

$$\sin C = 2 \sin\frac{C}{2}\cos\frac{C}{2} \quad \text{… (iv)}$$

And we know,

Sin A – sin B = 2 sin (A-B)/2 cos (A+B)/2 … (v)

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\frac{c}{(a - b)} = \frac{2 \sin\frac{C}{2}\cos\frac{C}{2}}{2 \sin\left(\frac{A - B}{2}\right)\cos\left(\frac{A + B}{2}\right)}$$

$$= \frac{\sin\left(\frac{\pi - (A + B)}{2}\right)\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A - B}{2}\right)\cos\left(\frac{A + B}{2}\right)}$$ [since, π = A+B+C, where, C = π – (A+B)]

$$= \frac{\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}$$ [since, sin (π/2 - A) = cos A]

Upon simplification we get,

$$= \frac{\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}$$

$$= \frac{\cos\left(\frac{\pi-(A+B)}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}$$ [since, π = A+B+C, where, C = π – (A+B)]

$$= \frac{\sin\left(\frac{(A+B)}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}$$ … (vi) [since, cos (π/2 - A) = sin A]

We know,

Sin (A + B)/2 = sin (A/2 + B/2) = sin A/2 cos B/2 + cos A/2 sin B/2

Sin (A - B)/2 = sin (A/2 - B/2) = sin A/2 cos B/2 - cos A/2 sin B/2

Substituting the above equations in equation (vi) we get,

$$= \frac{\sin\frac{A}{2}\cos\frac{B}{2} + \cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\sin\frac{A}{2}\cos\frac{B}{2} - \cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}$$

Let us divide the numerator and denominator by cos A/2 cos B/2, we get

$$= \frac{\frac{\sin\frac{A}{2}\cos\frac{B}{2} + \cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}{\frac{\sin\frac{A}{2}\cos\frac{B}{2} - \cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}$$

$$= \frac{\frac{\sin\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} + \frac{\cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}{\frac{\sin\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} - \frac{\cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}$$

Upon simplification we get,

$$= \frac{\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} + \frac{\sin\left(\frac{B}{2}\right)}{\cos\frac{B}{2}}}{\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} - \frac{\sin\left(\frac{B}{2}\right)}{\cos\frac{B}{2}}}$$

$$= \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}}$$

= RHS

Hence proved.

7. $\frac{c}{a+b} = \frac{1-\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}{1+\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, a = k sin A

Similarly, b = k sin B

And c = k sin C … (i)

We know,

a + b = k (sin A + sin B) …. (ii)

Now let us consider LHS:

$$\frac{c}{a+b}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k\sin C}{k(\sin A + \sin B)} = \frac{\sin C}{(\sin A + \sin B)} \text{ .... (iii)}$$

By applying half angle rule,

$$\sin C = 2\sin\frac{C}{2}\cos\frac{C}{2} \quad \ldots \text{(iv)}$$

And we know,

Sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2 … (v)

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\frac{c}{(a + b)} = \frac{2\sin\frac{C}{2}\cos\frac{C}{2}}{2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)}$$

$$= \frac{\sin\left(\frac{\pi - (A + B)}{2}\right)\cos\left(\frac{\pi - (A + B)}{2}\right)}{\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)}$$ [Since, π = A+B+C, where, C= π – (A+B)]

$$= \frac{\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A+B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$ [Since, sin (π/2 - A) = cos A, cos (π/2 - A) = sin A]

Upon simplification we get,

$$= \frac{\cos\left(\frac{(A + B)}{2}\right)}{\cos\left(\frac{A - B}{2}\right)} \quad \ldots \text{(vi)}$$

We know,

cos (A + B)/2 = cos (A/2 + B/2) = cos A/2 cos B/2 + sin A/2 sin B/2

cos (A – B)/2 = cos (A/2 – B/2) = cos A/2 cos B/2 – sin A/2 sin B/2

Substituting the above equations in equation (vi) we get,

$$= \frac{\cos\frac{A}{2}\cos\frac{B}{2} + \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2} - \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}$$

Let us divide the numerator and denominator by cos A/2 cos B/2, we get

$$= \frac{\dfrac{\cos\frac{A}{2}\cos\frac{B}{2} + \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}{\dfrac{\cos\frac{A}{2}\cos\frac{B}{2} - \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}$$

$$= \frac{\dfrac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} + \dfrac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}{\dfrac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} - \dfrac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}$$

Upon simplification we get,

$$= \frac{1 + \dfrac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}{1 - \dfrac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}}$$

$$= \frac{1 + \tan\frac{A}{2}\tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}}$$

= RHS

Hence proved.

8. $\frac{a + b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, a = k sin A

Similarly, b = k sin B

And c = k sin C … (i)

We know,

a + b = k (sin A + sin B) …. (ii)

Now let us consider LHS:

$$\frac{a + b}{c}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k(\sin A + \sin B)}{k(\sin C)} = \frac{(\sin A + \sin B)}{(\sin C)} \quad .... \text{(iii)}$$

By applying half angle rule,

$$\sin C = 2 \sin\frac{C}{2}\cos\frac{C}{2} \quad \ldots \text{(iv)}$$

And we know,

Sin A + sin B = 2 sin (A+B)/2 cos (A-B)/2 … (v)

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\frac{a + b}{c} = \frac{2 \sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)}{2 \sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)}$$

$$= \frac{\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)}{\sin\left(\frac{C}{2}\right)\cos\left(\frac{\pi - (A + B)}{2}\right)}$$ [Since, π = A+B+C, where, C= π – (A+B)]

$$= \frac{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)\sin\left(\frac{A+B}{2}\right)}$$ [Since, sin (π/2 - A) = cos A, cos (π/2 - A) = sin A]

Upon simplification we get,

$$= \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

= RHS

Hence proved.

9. $\sin\left(\frac{B-C}{2}\right) = \frac{b-c}{a}\cos\frac{A}{2}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, c = k sin C

Similarly, b = k sin B

We know,

b – c = k (sin B - sin C) …. (i)

Now let us consider RHS:

$$\frac{b-c}{a}\cos\frac{A}{2}$$

Substituting equation (i) in above equation, we get

$$\frac{\left(k(\sin B - \sin C)\right)}{k\sin A}\cos\frac{A}{2} = \frac{(\sin B - \sin C)}{\sin A}\cos\frac{A}{2}$$ …. (ii)

And we know,

Sin B - sin C = 2 sin (B-C)/2 cos (B+C)/2 … (iii)

Substituting the above equation (iii) in equation (ii), we get

$$\frac{b-c}{a}\cos\frac{A}{2} = \frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{\sin A}\cos\left(\frac{\pi-(B+C)}{2}\right)$$ [Since, π = A+B+C, where, C= π – (A+B)]

$$= \frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{\sin A}\sin\left(\frac{(B+C)}{2}\right)$$ [Since, cos (π/2 - A) = sin A]

Upon rearranging the above equation we get,

$$= \frac{\sin\left(\frac{B-C}{2}\right)\left(2\sin\left(\frac{(B+C)}{2}\right)\cos\left(\frac{B+C}{2}\right)\right)}{\sin A}$$

We know sin A = 2 cos (A/2) sin (A/2)

So,

$$= \frac{\sin\left(\frac{B-C}{2}\right)(\sin(B+C))}{\sin A}$$

$$= \frac{\sin\left(\frac{B-C}{2}\right)(\sin(\pi - A))}{\sin A}$$ [Since, π = A+B+C, where, A+B = π-C]

$$= \frac{\sin\left(\frac{B-C}{2}\right)\sin A}{\sin A}$$ [Since, sin (π-A) = sin A]

Upon simplification we get,

$$= \sin\left(\frac{B-C}{2}\right)$$

= LHS

Hence proved.

10. $\frac{a^2 - c^2}{b^2} = \frac{\sin(A - C)}{\sin(A + C)}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, c = k sin C

Similarly, a = k sin A

And b = k sin B

So, a – c = k (sin A – sin C) … (i)

We know,

Now let us consider LHS:

$$\frac{a^2 - c^2}{b^2}$$

Substituting the values in the above equation, we get

$$\frac{(k \sin A)^2 - (k \sin C)^2}{(k \sin B)^2} = \frac{k^2(\sin^2 A - \sin^2 C)}{k^2 \sin^2 B} \quad \text{.... (ii)}$$

And we know,

$\text{Sin}^2 A - \sin^2 C = \sin(A + C) \sin(A - C)$… (iii)

Substituting the above equation (iii) in equation (ii), we get

$$\frac{a^2 - c^2}{b^2} = \frac{\sin(A + C)\sin(A - C)}{\sin^2(\pi - (A + C))} \quad \text{[Since, } \pi = A+B+C\text{, where, } C= \pi - (A+B)\text{]}$$

$$= \frac{\sin(A + C)\sin(A - C)}{\sin^2((A + C))} \quad \text{[Since, } \sin(\pi - A) = \sin A\text{]}$$

$$= \frac{\sin(A - C)}{\sin(A + C)}$$

= RHS

Hence proved.

11. b sin B – c sin C = a sin (B – C)

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, c = k sin C

Similarly, a = k sin A

And b = k sin B

We know,

Now let us consider LHS:

b sin B – c sin C

Substituting the values of b and c in the above equation, we get

k sin B sin B – k sin C sin C = k ($\sin^2$ B – $\sin^2$ C) ……….(i)

We know,

Sin^2 B – $\sin^2$ C = sin (B + C) sin (B – C),

Substituting the above values in equation (i), we get

k ($\sin^2$ B – $\sin^2$ C) = k (sin (B + C) sin (B – C)) [since, π = A + B + C ⇒ B + C = π –A]

The above equation becomes,

= k (sin (π –A) sin (B – C)) [since, sin (π – θ) = sin θ]

= k (sin (A) sin (B – C))

From sine rule, a = k sin A, so the above equation becomes,

= a sin (B – C)

= RHS

Hence proved.

12. a^2 sin (B – C) = (b^2 – c^2) sin A

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, c = k sin C

Similarly, a = k sin A

And b = k sin B

We know,

Now let us consider RHS:

$(b^2 – c^2)$ sin A …

Substituting the values of b and c in the above equation, we get

$(b^2 – c^2)$ sin A = $[(k \sin B)^2 – (k \sin C)^2] \sin A$

= $k^2 (\sin^2 B – \sin^2 C) \sin A$………. (i)

We know,

$\sin^2 B – \sin^2 C = \sin (B + C) \sin (B – C)$,

Substituting the above values in equation (i), we get

= $k^2 (\sin (B + C) \sin (B – C)) \sin A$ [since, $\pi = A + B + C \Rightarrow B + C = \pi – A$]

= $k^2 (\sin (\pi – A) \sin (B – C)) \sin A$

= $k^2 (\sin (A) \sin (B – C)) \sin A$ [since, $\sin (\pi – \theta) = \sin \theta$]

Rearranging the above equation we get

= $(k \sin (A))(\sin (B – C)) (k \sin A)$

From sine rule, a = k sin A, so the above equation becomes,

= $a^2 \sin (B – C)$

= RHS

Hence proved.

13. $\dfrac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}+\sqrt{\sin B}}=\dfrac{a+b-2\sqrt{ab}}{a-b}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

$$\sin A=\frac{a}{k}, \sin B=\frac{b}{k}, \sin C=\frac{c}{k}$$

Let us consider LHS,

$$\frac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}+\sqrt{\sin B}}$$

Let us multiply and divide the above expression by $\dfrac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}-\sqrt{\sin B}}$ we get,

$$\frac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}+\sqrt{\sin B}}\times\frac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}-\sqrt{\sin B}}=\frac{(\sqrt{\sin A}-\sqrt{\sin B})^2}{(\sqrt{\sin A})^2-(\sqrt{\sin B})^2}$$

$$=\frac{(\sqrt{\sin A})^2+(\sqrt{\sin B})^2-(2\sqrt{\sin A}\times\sqrt{\sin B})}{\sin A-\sin B}$$

$$=\frac{\sin A+\sin B-(2\sqrt{\sin A\times\sin B})}{\sin A-\sin B}$$

Substituting the values of a and b from sine rule in the above equation, we get

$$=\frac{\frac{a}{k}+\frac{b}{k}-\left(2\sqrt{\frac{a}{k}\times\frac{b}{k}}\right)}{\frac{a}{k}-\frac{b}{k}}$$

$$=\frac{\frac{1}{k}(a+b-2\sqrt{ab})}{\frac{1}{k}(a-b)}$$

$$=\frac{a+b-2\sqrt{ab}}{a-b}$$

= RHS

Hence proved.

14. a (sin B – sin C) + b (sin C – sin A) + c (sin A – sin B) = 0

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

a = k sin A, b = k sin B, c = k sin C

Let us consider LHS:

a (sin B – sin C) + b (sin C – sin A) + c (sin A – sin B)

Substituting the values of a, b, c from sine rule in above equation, we get

a (sin B – sin C) + b (sin C – sin A) + c (sin A – sin B) = k sin A (sin B – sin C) + k sin B (sin C – sin A) + k sin C (sin A – sin B)

= k sin A sin B – k sin A sin C + k sin B sin C – k sin B sin A + k sin C sin A – k sin C sin B

Upon simplification, we get

= 0

= RHS

Hence proved.

15. $$\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C} = 0$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

a = k sin A, b = k sin B, c = k sin C

Let us consider LHS:

$$\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C}$$

Substituting the values of a, b and c from sine rule in the above equation, we get

$$= \frac{(k \sin A)^2 \sin(B-C)}{\sin A} + \frac{(k \sin B)^2 \sin(C-A)}{\sin B} + \frac{(k \sin C)^2 \sin(A-B)}{\sin C}$$

$$= \frac{k^2 \sin^2 A \sin(B-C)}{\sin A} + \frac{k^2 \sin^2 B \sin(C-A)}{\sin B} + \frac{k^2 \sin^2 C \sin(A-B)}{\sin C}$$

Upon simplification we get,

$= k^2$ [sin A sin (B – C) + sin B sin (C – A) + sin C sin (A – B)]

We know, sin (A – B) = sin A cos B – cos A sin B

Sin (B – C) = sin B cos C – cos B sin C

Sin (C – A) = sin C cos A – cos C sin A

So the above equation becomes,

$= k^2$ [sin A (sin B cos C – cos B sin C) + sin B (sin C cos A – cos C sin A) + sin C (sin A cos B – cos A sin B)]

$= k^2$ [sin A sin B cos C – sin A cos B sin C + sin B sin C cos A – sin B cos C sin A + sin C sin A cos B – sin C cos A sin B)]

Upon simplification we get,

= 0

= RHS

Hence proved.

EXERCISE 10.2 PAGE NO: 10.25

In any ∆ABC, prove the following:

1. In a ∆ABC, if a = 5, b = 6 and C = 60°, show that its area is (15√3)/2 sq. units.

Solution:

Given:

In a ∆ABC, a = 5, b = 6 and C = 60°

By using the formula,

Area of ∆ABC = 1/2 ab sin θ where, a and b are the lengths of the sides of a triangle and θ is the angle between sides.

So,

Area of ∆ABC = 1/2 ab sin θ

= 1/2 × 5 × 6 × sin 60°

= 30/2 × √3/2

= (15√3)/2 sq. units

Hence proved.

2. In a ΔABC, if a = √2, b = √3 and c = √5 show that its area is1/2 √6 sq. units.

Solution:

Given:

In a ΔABC, a = √2, b = √3 and c = √5

By using the formulas,

We know, cos A = $(b^2 + c^2 – a^2)/2bc$

By substituting the values we get,

= $[(\sqrt{3})^2 + (\sqrt{5})^2 – (\sqrt{2})^2] / [2 \times \sqrt{3} \times \sqrt{5}]$

= 3/√15

We know, Area of ΔABC = 1/2 bc sin A

To find sin A:

Sin A = $\sqrt{(1 – \cos^2 A)}$ [by using trigonometric identity]

= $\sqrt{(1 – (3/\sqrt{15})^2)}$

= √(1- (9/15))

= √(6/15)

Now,

Area of ΔABC = 1/2 bc sin A

= 1/2 × √3 × √5 × √(6/15)

= 1/2 √6 sq. units

Hence proved.

3. The sides of a triangle are a = 4, b = 6 and c = 8, show that: 8 cos A + 16 cos B + 4 cos C = 17.

Solution:

Given:

Sides of a triangle are a = 4, b = 6 and c = 8

By using the formulas,

Cos A = $(b^2 + c^2 – a^2)/2bc$

Cos B = $(a^2 + c^2 - b^2)/2ac$

Cos C = $(a^2 + b^2 - c^2)/2ab$

So now let us substitute the values of a, b and c we get,

Cos A = $(b^2 + c^2 - a^2)/2bc$

$= (6^2 + 8^2 - 4^2)/2\times6\times8$

$= (36 + 64 - 16)/96$

$= 84/96$

$= 7/8$

Cos B = $(a^2 + c^2 - b^2)/2ac$

$= (4^2 + 8^2 - 6^2)/2\times4\times8$

$= (16 + 64 - 36)/64$

$= 44/64$

Cos C = $(a^2 + b^2 - c^2)/2ab$

$= (4^2 + 6^2 - 8^2)/2\times4\times6$

$= (16 + 36 - 64)/48$

$= -12/48$

$= -1/4$

Now considering LHS:

$8 \cos A + 16 \cos B + 4 \cos C = 8 \times 7/8 + 16 \times 44/64 + 4 \times (-1/4)$

$= 7 + 11 - 1$

$= 17$

Hence proved.

4. In a △ABC, if a = 18, b = 24, c = 30, find cos A, cos B and cos C

Solution:

Given:

Sides of a triangle are a = 18, b = 24 and c = 30

By using the formulas,

Cos A = $(b^2 + c^2 – a^2)/2bc$

Cos B = $(a^2 + c^2 – b^2)/2ac$

Cos C = $(a^2 + b^2 – c^2)/2ab$

So now let us substitute the values of a, b and c we get,

Cos A = $(b^2 + c^2 – a^2)/2bc$

= $(24^2 + 30^2 – 18^2)/2×24×30$

= 1152/1440

= 4/5

Cos B = $(a^2 + c^2 – b^2)/2ac$

= $(18^2 + 30^2 – 24^2)/2×18×30$

= 648/1080

= 3/5

Cos C = $(a^2 + b^2 – c^2)/2ab$

= $(18^2 + 24^2 – 30^2)/2×18×24$

= 0/864

= 0

∴ cos A = 4/5, cos B = 3/5, cos C = 0

5. For any ΔABC, show that b (c cos A – a cos C) = $c^2 – a^2$

Solution:

Let us consider LHS:

b (c cos A – a cos C)

As LHS contain bc cos A and ab cos C which can be obtained from cosine formulae.

From cosine formula we have:

Cos A = $(b^2 + c^2 – a^2)/2bc$

bc cos A = $(b^2 + c^2 – a^2)/2$ … (i)

Cos C = $(a^2 + b^2 – c^2)/2ab$

ab cos C = $(a^2 + b^2 – c^2)/2$ … (ii)

Now let us subtract equation (i) and (ii) we get,

$bc \cos A - ab \cos C = (b^2 + c^2 - a^2)/2 - (a^2 + b^2 - c^2)/2$

$= c^2 - a^2$

$\therefore b (c \cos A - a \cos C) = c^2 - a^2$

Hence proved.

6. For any Δ ABC show that $c (a \cos B - b \cos A) = a^2 - b^2$

Solution:

Let us consider LHS:

$c (a \cos B - b \cos A)$

As LHS contain ca cos B and cb cos A which can be obtained from cosine formulae.

From cosine formula we have:

$\text{Cos } A = (b^2 + c^2 - a^2)/2bc$

$bc \cos A = (b^2 + c^2 - a^2)/2 \ldots \text{(i)}$

$\text{Cos } B = (a^2 + c^2 - b^2)/2ac$

$ac \cos B = (a^2 + c^2 - b^2)/2 \ldots \text{(ii)}$

Now let us subtract equation (ii) from (i) we get,

$ac \cos B - bc \cos A = (a^2 + c^2 - b^2)/2 - (b^2 + c^2 - a^2)/2$

$= a^2 - b^2$

$\therefore c (a \cos B - b \cos A) = a^2 - b^2$

Hence proved.

7. For any Δ ABC show that
$2 (bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$

Solution:

Let us consider LHS:

$2 (bc \cos A + ca \cos B + ab \cos C)$

As LHS contain 2ca cos B, 2ab cos C and 2cb cos A, which can be obtained from cosine formulae.

From cosine formula we have:

Cos A = $(b^2 + c^2 – a^2)/2bc$

2bc cos A = $(b^2 + c^2 – a^2)$ … (i)

Cos B = $(a^2 + c^2 – b^2)/2ac$

2ac cos B = $(a^2 + c^2 – b^2)$… (ii)

Cos C = $(a^2 + b^2 – c^2)/2ab$

2ab cos C = $(a^2 + b^2 – c^2)$ … (iii)

Now let us add equation (i), (ii) and (ii) we get,

2bc cos A + 2ac cos B + 2ab cos C = $(b^2 + c^2 – a^2) + (a^2 + c^2 – b^2) + (a^2 + b^2 – c^2)$

Upon simplification we get,

= $c^2 + b^2 + a^2$

2 (bc cos A + ac cos B + ab cos C) = $a^2 + b^2 + c^2$

Hence proved.

8. For any Δ ABC show that
$(c^2 – a^2 + b^2)$ tan A = $(a^2 – b^2 + c^2)$ tan B = $(b^2 – c^2 + a^2)$ tan C

Solution:

Let us consider LHS:

$(c^2 – a^2 + b^2)$, $(a^2 – b^2 + c^2)$, $(b^2 – c^2 + a^2)$

We know sine rule in Δ ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

As LHS contain $(c^2 – a^2 + b^2)$, $(a^2 – b^2 + c^2)$ and $(b^2 – c^2 + a^2)$, which can be obtained from cosine formulae.

From cosine formula we have:

Cos A = $(b^2 + c^2 – a^2)/2bc$

2bc cos A = $(b^2 + c^2 – a^2)$

Let us multiply both the sides by tan A we get,

2bc cos A tan A = $(b^2 + c^2 – a^2)$ tan A

2bc cos A (sin A/cos A) = $(b^2 + c^2 – a^2)$ tan A

$2bc \sin A = (b^2 + c^2 – a^2) \tan A$ … (i)

$\cos B = (a^2 + c^2 – b^2)/2ac$

$2ac \cos B = (a^2 + c^2 – b^2)$

Let us multiply both the sides by tan B we get,

$2ac \cos B \tan B = (a^2 + c^2 – b^2) \tan B$

$2ac \cos B (\sin B/\cos B) = (a^2 + c^2 – b^2) \tan B$

$2ac \sin B = (a^2 + c^2 – b^2) \tan B$ … (ii)

$\cos C = (a^2 + b^2 – c^2)/2ab$

$2ab \cos C = (a^2 + b^2 – c^2)$

Let us multiply both the sides by tan C we get,

$2ab \cos C \tan C = (a^2 + b^2 – c^2) \tan C$

$2ab \cos C (\sin C/\cos C) = (a^2 + b^2 – c^2) \tan C$

$2ab \sin C = (a^2 + b^2 – c^2) \tan C$ … (iii)

As we are observing that sin terms are being involved so let's use sine formula.

From sine formula we have,

$$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} \Rightarrow \frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$$

Let us multiply abc to each of the expression we get,

$$\frac{abc\ sinA}{a} = \frac{abc\ sinB}{b} = \frac{abc\ sinC}{c}$$

$bc \sin A = ac \sin B = ab \sin C$

$2bc \sin A = 2ac \sin B = 2ab \sin C$

∴ From equation (i), (ii) and (iii) we have,

$(c^2 – a^2 + b^2) \tan A = (a^2 – b^2 + c^2) \tan B = (b^2 – c^2 + a^2) \tan C$

Hence proved.

9. For any Δ ABC show that:

$$\frac{\mathbf{c - b \cos A}}{\mathbf{b - c \cos A}} = \frac{\mathbf{\cos B}}{\mathbf{\cos C}}$$

Solution:

Let us consider LHS:

$$\frac{c - b\cos A}{b - c\cos A}$$

We can observe that we can get terms c – b cos A and b – c cos A from projection formula

From projection formula we get,

c = a cos B + b cos A

c – b cos A = a cos B …. (i)

And,

b = c cos A + a cos C

b – c cos A = a cos C …. (ii)

Dividing equation (i) by (ii), we get,

$$\frac{c - b\cos A}{b - c\cos A} = \frac{a\cos B}{a\cos C}$$

$$= \frac{\cos B}{\cos C}$$

= RHS

Hence proved.