RD SHARMA Solutions for Class 12-science Maths Chapter 22 - Differential Equations

Chapter 22 - Differential Equations Exercise Ex. 22.1

Question 1

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$\frac{d^3x}{dt^3}+\frac{d^2y}{dt^2}+\left(\frac{dx}{dt}\right)^2=e^t$$

Solution 1

$$\frac{d^3x}{dt^3}+\frac{d^{2x}}{dt^2}+\left(\frac{dx}{dt}\right)^2=e^t$$

The highest order differential coefficient is $\frac{d^3x}{dt^3}$ and its power is 1.

So, it is a non-linear differential equation with order 3 and degree 1.

Question 2

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$\frac{d^2y}{dx^2}+4y=0$$

Solution 2

$$\frac{d^2y}{dx^2}+4y=0$$

It is a linear differential equation.

The highest order differential coefficient is $\frac{d^2y}{dx^2}$ and its power is 1.

So, it is a linear differential equation with order 2 and degree 1.

Question 3

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$\left(\frac{dy}{dx}\right)^2+\frac{1}{\left(\frac{dy}{dx}\right)}=2$$

Solution 3

$$\left(\frac{dy}{dx}\right)^2 + \frac{1}{\left(\frac{dy}{dx}\right)} = 2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 + 1 = 2\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 - 2\left(\frac{dy}{dx}\right) + 1 = 0$$

This is a polynomial in $\frac{dy}{dx}$.

The highest order differential coefficient is $\frac{dy}{dx}$ and its power is 3.

So, it is a non-linear differential equation with order 1 and degree 3.

Question 4

Determine the order and degree of the following differential equation. State also whether it is linear or non – linear.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \left(c\frac{d^2y}{dx^2}\right)^{\frac{1}{3}}$$

Solution 4

Consider the given differential equation, $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \left(c\frac{d^2y}{dx^2}\right)^{\frac{1}{3}}$

Squaring on both the sides, we have

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(c\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}$$

Cubing on both the sides, we have

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left\{\left(c\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}\right\}^3$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^6 + 3\left(\frac{dy}{dx}\right)^2 + 3\left(\frac{dy}{dx}\right)^4 = c^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow c^2\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^6 - 3\left(\frac{dy}{dx}\right)^4 - 3\left(\frac{dy}{dx}\right)^2 - 1 = 0$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$ *and its power is 2.*

Therefore, the given differential equation is a non – linear differential equation of second order and second degree.

Question 5

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+xy=0$$

Solution 5

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+xy=0$$

The highest order differential coefficient is $\frac{d^2y}{dx^2}$ and its power is 1.

So, it is a non-linear differential equation with order 2 and degree 1.

Question 6

Determine the order and degree of the following differential equation. State also whether is linear or non – linear.

$$\sqrt[3]{\frac{d^2y}{dx^2}}=\sqrt{\frac{dy}{dx}}$$

Solution 6

Consider the given differential equation,

$$\sqrt[3]{\frac{d^2y}{dx^2}}=\sqrt{\frac{dy}{dx}}$$

Cubing on both the sides of the above equation, we have

$$\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^{\frac{3}{2}}$$

Squaring on both the sides of the above equation, we have

$$\left(\frac{d^2y}{dx^2}\right)^2=\left[\left(\frac{dy}{dx}\right)^{\frac{3}{2}}\right]^2$$

$$\Rightarrow\left(\frac{d^2y}{dx^2}\right)^2=\left[\left(\frac{dy}{dx}\right)\right]^3$$

$$\Rightarrow\left(\frac{d^2y}{dx^2}\right)^2-\left[\left(\frac{dy}{dx}\right)\right]^3=0$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$

and its power is 2.

Therefore, the given differential equation is a non – linear differential equation of second order and second degree.

Question 7

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$\frac{d^4y}{dx^4}=\left[c+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$$

Solution 7

$$\frac{d^4y}{dx^4}=\left[c+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$$

$$\Rightarrow \quad \left(\frac{d^4y}{dx^4}\right)^2=\left[c+\left(\frac{dy}{dx}\right)^2\right]^3$$

$$\Rightarrow \quad \left(\frac{d^4y}{dx^4}\right)^2=c^3+\left(\frac{dy}{dx}\right)^6+3c\left(\frac{dy}{dx}\right)^2+3c^2\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left(\frac{d^4y}{dx^4}\right)^2-\left(\frac{dy}{dx}\right)^6-3c\left(\frac{dy}{dx}\right)^2-3c^2\left(\frac{dy}{dx}\right)-c^3=0$$

The highest order differential coefficient is $\left(\frac{d^4y}{dx^4}\right)$ and its power is 2.

It is a non-linear differential equation with order 4 and degree 2.

Question 8

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$x+\left(\frac{dy}{dx}\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Solution 8

$$x+\left(\frac{dy}{dx}\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \quad \left(x+\frac{dy}{dx}\right)^2=1+\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad x^2+\left(\frac{dy}{dx}\right)^2+2x\left(\frac{dy}{dx}\right)=1+\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad 2x\left(\frac{dy}{dx}\right)+x^2-1=0$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{x}{2}-\frac{1}{2x}=0$$

The highest order differential coefficient is $\frac{dy}{dx}$ and power is 1.

So, it is a linear differential equation with order 1 and degree 1.

Question 9

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$y\frac{d^2x}{dy^2}=y^2+1$$

Solution 9

$$y\frac{d^2x}{dy^2}=y^2+1$$

$$\frac{d^2x}{dy^2}-y-\frac{1}{y}=0$$

The differential coefficient is $\frac{d^2x}{dy^2}$ and its power is 1.

So, it is a linear differential equation with order 2 and degree 1.

Question 10

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$S^2\frac{d^2t}{ds^2}+st\frac{dt}{ds}=s$$

Solution 10

$$S^2\frac{d^2t}{ds^2}+st\frac{dt}{ds}=s$$

The differential coefficient of highest order is $\frac{d^2t}{ds^2}$ and power is 1.

So, it is a non-linear differnetial equation with order 2 and degree 1.

Question 11

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$x^2\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^4+y^4=0$$

Solution 11

$$x^2\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^4+y^4=0$$

The highest order differential coefficient is $\frac{d^2y}{dx^2}$ and its power is 3.

So, it is a non-linear differnetial equation with order 2 and degree 3.

Question 12

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$\frac{d^3y}{dx^3}+\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)+4y=\sin x$$

Solution 12

$$\frac{d^3y}{dx^3}+\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)+4y=\sin x$$

The highest order differential coefficient is $\frac{d^3y}{dx^3}$ and its power is 1.

So, it is a non-linear differnetial equation with order 3 and degree 1.

Question 13

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$\left(xy^2+x\right)dx+\left(y-x^2y\right)dy=0$

Solution 13

$$\left(xy^2+x\right)dx+\left(y-x^2y\right)dy=0$$

$$\left(y-x^2y\right)\frac{dy}{dx}+xy^2+x=0$$

$$y\left(1-x^2\right)\frac{dy}{dx}+x\left(y^2+1\right)=0$$

The highest order differential coefficient is $\frac{dy}{dx}$ and its power is 1.

So, it is a non-linear differnetial equation with order 1 and degree 1.

Question 14

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$\sqrt{1-y^2}\,dx+\sqrt{1-x^2}\,dy=0$

Solution 14

$$\sqrt{1-y^2}\,dx+\sqrt{1-x^2}\,dy=0$$

$$\sqrt{1-x^2}\,\frac{dy}{dx}+\sqrt{1-y^2}=0$$

$$\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$$

The highest order differential coefficient is $\frac{dy}{dx}$ and its power is 1.

So, it is a non-linear differnetial equation with order 1 and degree 1.

Question 15

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^{\frac{2}{3}}$

Solution 15

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^{\frac{2}{3}}$$

$$\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx}\right)^2$$

$$\left(\frac{d^2y}{dx^2}\right)^3 - \left(\frac{dy}{dx}\right)^2 = 0$$

The highest order differential coefficient is $\frac{d^2y}{dx^2}$ and its power is 3.

So, it is a non-linear differnetial equation with order 2 and degree 3.

Question 16

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$2\frac{d^2y}{dx^2} + 3\sqrt{1 - \left(\frac{dy}{dx}\right)^2 - y} = 0$$

Solution 16

$$2\frac{d^2y}{dx^2} + 3\sqrt{1 - \left(\frac{dy}{dx}\right)^2 - y} = 0$$

$$2\frac{d^2y}{dx^2} = -3\sqrt{1 - \left(\frac{dy}{dx}\right)^2 - y}$$

Squaring both the sides,

$$4\left(\frac{d^2y}{dx^2}\right)^2 = 9\left(1 - \left(\frac{dy}{dx}\right)^2 - y\right)$$

$$4\left(\frac{d^2y}{dx^2}\right)^2 + 9\left(\frac{dy}{dx}\right)^2 + 9y - 9 = 0$$

The highest order differential coefficient is $\frac{d^2y}{dx^2}$ and its power is 2.

So, it is a non-linear differential equation with order 2 and degree 2.

Question 17

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$5\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$$

Solution 17

$$5\ \frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$$

$$\left\{5\ \left(\frac{d^2y}{dx^2}\right)^2\right\} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3$$

$$25\left(\frac{d^2y}{dx^2}\right)^2 = 1+\left(\frac{dy}{dx}\right)^6 + 3\left(\frac{dy}{dx}\right)^2 + 3\left(\frac{dy}{dx}\right)^4$$

$$25\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^6 - 3\left(\frac{dy}{dx}\right)^4 - 3\left(\frac{dy}{dx}\right)^2 - 1 = 0$$

The highest order differential coefficient is $\frac{d^2y}{dx^2}$ and its power is 2.

So, it is a non-linear differential equation with order 2 and degree 2

Question 18

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Solution 18

$$y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(y - x\frac{dy}{dx}\right)^2 = \left(a\sqrt{1-\left(\frac{dy}{dx}\right)^2}\right)^2$$

$$y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} = a\left(1-\left(\frac{dy}{dx}\right)^2\right)$$

$$x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y + a\left(\frac{dy}{dx}\right)^2 - a = 0$$

$$\left(x^2+a\right)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y - a = 0$$

The highest order differential coefficient is $\frac{dy}{dx}$ and poer is 2.

So, it is a non-linear differential equation with order 1 and degree 2.

Question 19

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$y = px + \sqrt{a^2p^2 + b^2}, p = \frac{dy}{dx}$

Solution 19

$$y = px + \sqrt{a^2p^2 + b^2}, p = \frac{dy}{dx}$$
$$y - px = \sqrt{a^2p^2 + b^2}$$
$$(y - px)^2 = \left(a^2p^2 + b^2\right)$$
$$y^2 + p^2x^2 - 2xyp = a^2p^2 + b^2$$
$$x^2p^2 - a^2p^2 - 2xyp + y^2 - b^2 = 0$$
$$\left(x^2 - a^2\right)p^2 - 2xyp + \left(y^2 - b^2\right) = 0$$
$$\left(x^2 - a^2\right)\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + \left(y^2 - b^2\right) = 0$$

The highest order differential coefficient is $\frac{dy}{dx}$ and its power is 2.

So, it is a non-linear differential equation of order 1 and degree 2

Question 20

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$\frac{dy}{dx} + e^y = 0$

Solution 20

$$\frac{dy}{dx} + e^y = 0$$

The highest order differential coefficient is $\frac{dy}{dx}$ and its power is 1.

So, it is a non-linear differential equation of order 1 and degree 1.

Question 21

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x \sin\left(\frac{d^2y}{dx^2}\right)$

Solution 21

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$$

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 - x\sin\left(\frac{d^2y}{dx^2}\right) = 0$$

The highest order differential coefficient is $\left(\frac{d^2y}{dx^2}\right)$ and it is not a polynomial of derviative,

So, it is a non-linear differential equation of order 2 but degree is not defined.

Question 22

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$(y'')^2 + (y')^3 + \sin y = 0$

Solution 22

$$(y'')^2 + (y')^3 + \sin y = 0$$

The highest order of differential coefficient is y'' and its power is 2,

So, it is a non-linear differential equation of order 2 and degree 2.

Question 23

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$$

Solution 23

$$\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$$

The highest order differential coefficient is $\frac{d^2y}{dx^2}$ and its power is 1.

So, it is a non-linear differential equation with order 2 and degree 1.

Question 24

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y\sin y = 0$$

Solution 24

$$\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\frac{dy}{dx}+y\sin y=0$$

The highest order differential coefficient is $\frac{d^3y}{dx^3}$ and its power is 1.

So, it is a linear differential equation of order 3 and degree 1.

Question 25

Determine the order and degree of the following differential equation. State also whether it is linear or non-linear.

$$\frac{d^2y}{dx^3}+3\left(\frac{dy}{dx}\right)^2=x^2\log\left(\frac{d^2y}{dx^2}\right)$$

Solution 25

$$\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2=x^2\log\left(\frac{d^2y}{dx^2}\right)$$

$$\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2-x^2\log\left(\frac{d^2y}{dx^2}\right)=0$$

The highest order derivative is $\frac{d^2y}{dx^2}$ but it is not a polynomial in $\frac{dy}{dx}$.

So, it is a non-linear differential equation of order 2 but degree is not defined.

Question 26

Determine the order and degree of the following differential equations. State also whether they are linear or non-linear.

$$\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$$

Solution 26

The order of a differential equation is the order of the highest order derivative appearing in the equation.
The degree of a differential equation is the degree of the highest order derivative.
Consider the given differential equation

$$\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$$

In the above equation, the order of the highest order derivative is 1.
So the differential equation is of order 1.
In the above differential equation, the power of the highest order derivative is 3.
Hence, it is a differential equation of degree 3.
Since the degree of the above differential equation is 3, more than one, it is a non-linear differential equation.

Chapter 22 - Differential Equations Exercise Ex. 22.2

Question 1

Form the differential equation of the family of curves represented by $y^2=(x-c)^3$.

Solution 1

$$y^2 = (x-c)^3 \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$2y\frac{dy}{dx} = 3(x-c)^2$$

$$(x-c)^2 = \frac{2y}{3}\frac{dy}{dx}$$

$$(x-c)^2 = \left(\frac{2y}{3}\frac{dy}{dx}\right)^{\frac{1}{2}}$$

Put the value of $(x-c)$ in equation (i),

$$y^2 = \left\{\left(\frac{2y}{3}\frac{dy}{dx}\right)^{\frac{1}{2}}\right\}^3$$

$$y^2 = \left(\frac{2y}{3}\frac{dy}{dx}\right)^{\frac{3}{2}}$$

Squaring both the sides,

$$y^4 = \left(\frac{2y}{3}\frac{dy}{dx}\right)^3$$

$$y^4 = \frac{8y^3}{27}\left(\frac{dy}{dx}\right)^3$$

$$27y = 8\left(\frac{dy}{dx}\right)^3.$$

Question 2

Form the differential equation corresponding to $y = e^{mx}$ by eliminating m.

Solution 2

$$y = e^{mx} \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = me^{mx} \qquad \text{—(ii)}$$

From equation (i),

$$y = e^{mx}$$

$$\log y = mx$$

$$m = \frac{\log y}{x}$$

Put the value of m and e^{mx} in equation (i),

$$\frac{dy}{dx} = \frac{\log y}{x}y$$

$$x\frac{dy}{dx} = y\log y$$

Question 3(i)

Form the differential equation from the following primitive where constants are arbitrary:

$y^2 = 4ax$

Solution 3(i)

$$y^2 = 4ax \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$2y\frac{dy}{dx} = 4a \qquad \text{—(ii)}$$

Put the value of a from equation (i) in (ii),

$$2y\frac{dy}{dx} = 4\left(\frac{y^2}{4x}\right)$$

$$2y\frac{dy}{dx} = \frac{y^2}{x}$$

$$2x\frac{dy}{dx} = y$$

Question 3(ii)

Form the differential equation from the following primitive where constants are arbitrary:

$y = cx + 2c^2 + c^3$

Solution 3(ii)

$$y = cx + 2c^2 + c^3 \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = c \qquad \text{—(ii)}$$

Put the value of c from equation (ii) in (i),

$$y = \left(\frac{dy}{dx}\right)x + 2\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^3$$

Question 3(iii)

Form the differential equation from the following primitive where constants are arbitrary:

$xy = a^2$

Solution 3(iii)

$$xy = a^2 \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$x\frac{dy}{dx} + y(1) = 0$$

$$x\frac{dy}{dx} + y = 0$$

Question 3(iv)

Form the differential equation from the following primitive where constants are arbitrary:

$y = ax^2 + bx + c$

Solution 3(iv)

$$y = ax^2 + bx + c$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = 2ax + b$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = 2a$$

Again, differentiating it with respect to x,

$$\frac{d^3y}{dx^3} = 0$$

Question 4

Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$, where A and B are arbitrary constants.

Solution 4

$$y = Ae^{2x} + Be^{-2x} \quad \text{---(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x}$$

$$= 4\left(Ae^{2x} + Be^{-2x}\right)$$

$$\frac{d^2y}{dx^2} = 4y \qquad \left[\text{Using equation (i)}\right]$$

Question 5

Find the differential equation of the family of curves, $x = A\cos nt + B\sin nt$, where A and B are arbitrary constants.

Solution 5

$$x = A\cos nt + B\sin nt$$

Differentiating with respect to t,

$$\frac{dx}{dt} = -An\sin nt + nB\cos nt$$

Again, differentiating with respect to t,

$$\frac{d^2x}{dt^2} = -An^2\cos nt - n^2B\sin t$$

$$= -n^2\left(A\cos nt + B\sin nt\right)$$

$$\frac{d^2x}{dt^2} = -n^2x \qquad \left[\text{Using equation (i)}\right]$$

$$\frac{d^2x}{dt^2} + n^2x = 0$$

Question 6

Form the differential equation corresponding to $y^2 = a\left(b - x^2\right)$ by eliminating a and b.

Solution 6

$$y^2 = a\left(b - x^2\right)$$

Differentiating it with respect to x,

$$2y\frac{dy}{dx} = a(-2x) \qquad \text{—(i)}$$

Again, differentiating it with respect to x,

$$2\left[y\frac{d^2y}{dx^2} + \frac{dy}{dx} \times \frac{dy}{dx}\right] = -2a$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\left(\frac{2y}{-2x}\frac{dy}{dx}\right)$$

Using equatoin (i)

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{y}{x}\frac{dy}{dx}$$

$$x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$$

Question 7

Form the differential equation corresponding to $y^2 - 2ay + x^2 = a^2$ by eliminating a.

Solution 7

$$y^2 - 2ay + x^2 = a^2 \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$2y\frac{dy}{dx} - 2a\frac{dy}{dx} + 2x = 0$$

$$y\frac{dy}{dx} + x = a\frac{dy}{dx}$$

$$a = \frac{y\frac{dy}{dx} + x}{\frac{dy}{dx}}$$

Put the value of a in equation (i),

$$y^2 - 2\left[\frac{y\frac{dy}{dx} + x}{\frac{dy}{dx}}\right]y + x^2 = \left[\frac{y\frac{dy}{dx} + x}{\frac{dy}{dx}}\right]^2$$

Put $\frac{dy}{dx} = y'$

$$y^2 - 2\left(\frac{yy' + x}{y'}\right)y + x^2 = \left(\frac{yy' + x}{y'}\right)^2$$

$$\frac{y'y^2 - 2y'y^2 - 2xy + y'x^2}{y'} = \frac{y'^2y^2 + x^2 + 2xyy'}{y'^2}$$

$$y'^2y^2 - 2y'^2y^2 - 2xyy' + y'^2x^2 - y'^2y^2 - x^2 - 2xyy' = 0$$

$$-4xyy' + y'^2x^2 - x^2 - 2y'^2y^2 = 0$$

$$y'^2\left(x^2 - 2y^2\right) - 4xyy' - x^2 = 0$$

Question 8

Form the differential equation corresponging to $(x-a)^2+(y-b)^2=r^2$ by eliminating a and b.

Solution 8

$$(x-a)^2+(y-b)^2=r^2 \qquad \text{—(i)}$$

Differentiating with respect to x,

$$2(x-a)+2(y-b)\frac{dy}{dx}=0$$

$$(x-a)+(y-b)\frac{dy}{dx}=0 \qquad \text{—(ii)}$$

Differentiating with respect to x,

$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)=0$$

$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

$$(y-b)=-\left\{\frac{\left(\frac{dy}{dx}\right)^2+1}{\frac{d^2y}{dx^2}}\right\} \qquad \text{—(iii)}$$

Put $(y-b)$ in equation (ii),

$$(x-a)-\left\{\frac{\left(\frac{dy}{dx}\right)^2+1}{\frac{d^2y}{dx^2}}\right\}\frac{dy}{dx}=$$

$$(x-a)\left(\frac{d^2y}{dx^2}\right)-\left(\frac{dy}{dx}\right)^3-\left(\frac{dy}{dx}\right)=0$$

$$(x-a)\frac{d^2y}{dx^2}=\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^3$$

$$(x-a)=\frac{\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^3}{\frac{d^2y}{dx^2}} \qquad \text{—(iv)}$$

Put the value of $(x-a)$ and $(y-b)$ from equation (iii) and (iv) in equation (i),

$$\left\{\frac{\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^3}{\frac{d^2y}{dx^2}}\right\}^2+\left\{\frac{\left(\frac{dy}{dx}\right)^2+1}{\frac{d^2y}{dx^2}}\right\}^2=r^2$$

Put $\frac{dy}{dx}=y'$ and $\frac{d^2y}{dx^2}=y''$

$$\left(y'+y'^3\right)^2+\left(y'^2+1\right)^2=r^2y''^2$$

$$y'^2\left(1+y'^2\right)^2+\left(1+y'^2\right)^2=r^2y''^2$$

$$\left(1+y'^2\right)^2\left(y'^2+1\right)=r^2y''^2$$

$$\left(1+y'^2\right)^3=r^2y''^2$$

Question 9

Find the differential equation of all the circles which pass through the origin and whose centres lie on y-axis.

Solution 9

We know that, equation of a circle with centre at (h, k) and radius r is given by,

$$\left(x-h^2\right)+(y-k)^2=r^2 \qquad \text{—(i)}$$

Here, centre lies, on y-axis, so $h=0$

$$\Rightarrow \quad x^2+(y-k)^2=r^2 \qquad \text{—(ii)}$$

Also, given that, circle is passing through origin, so

$$0+k^2=r^2$$
$$k^2=r^2$$

So, equation (ii) becomes,

$$x^2+(y-k)^2=k^2$$
$$x^2+y^2-2yk=0$$
$$2yk=x^2+y^2$$
$$k=\frac{x^2+y^2}{2y}$$

Differentiating with respect to x,

$$0=\frac{2y\left(2x+2y\frac{dy}{dx}\right)-\left(x^2+y^2\right)2\frac{dy}{dx}}{(2y)^2}$$
$$0=4xy+4y^2\frac{dy}{dx}-2x^2\frac{dy}{dx}-2y^2\frac{dy}{dx}$$
$$0=2y^2\frac{dy}{dx}-2x^2\frac{dy}{dx}+4xy$$
$$x^2\frac{dy}{dx}-y^2\frac{dy}{dx}=2xy$$
$$\left(x^2-y^2\right)\frac{dy}{dx}=2xy$$

Question 10

Find the differential equation of all the circles which pass through the origin and whose centres lie on x-axis.

Solution 10

Equation of circle with centre (h,k) and radius r is given by

$$(x-h)^2+(y-k)^2=r^2 \qquad \text{---(i)}$$

Here, centre lie on x-axis, so

$$k=0$$

$$\Rightarrow \quad (x-h)^2+y^2=r^2 \qquad \text{---(ii)}$$

Also, given that, circle is passing through $(0,0)$, so,

$$h^2=r^2$$

So, equation (ii) becomes,

$$(x-h)^2+y^2=h^2$$

$$x^2+h^2-2xh+y^2=h^2$$

$$x^2-2xh+y^2=0$$

$$2xh=x^2+y^2$$

$$h=\frac{x^2+y^2}{2x}$$

Differentiating it with respect to x,

$$0=\frac{\left(2x+2y\frac{dy}{dx}\right)2x-\left(x^2+y^2\right)2}{(2x)^2}$$

$$\left(2x+2y\frac{dy}{dx}\right)2x-\left(x^2+y^2\right)^2=0$$

$$2x^2+2yx\frac{dy}{dx}-x^2-y^2=0$$

$$\left(x^2-y^2\right)+2xy\frac{dy}{dx}=0$$

Question 11

Assume that a rain drop evaporates at a rate proportional to its surface area. From a differential equation involving the rate of change of the radius of the rain drop.

Solution 11

Let A be the surface area of rain drain, V be its volume, and r be the radius of rain drop.

Given,

$$\frac{dV}{dt}\propto A$$

$$\frac{dV}{dt}=-kA \qquad [\text{negative because } V \text{ decreases with increase in } t]$$

where k is the constant of proportionality.

So,

$$\frac{d}{dt}\left(\frac{4\pi}{3}r^3\right)=-k\left(4\pi r^2\right)$$

$$4\pi r^2\frac{dr}{dt}=-k\left(4\pi r^2\right)$$

$$\frac{dr}{dt}=-k$$

Question 12

Find the differential equation of all the parabolas with latus rectum '4a' and whose axes are parallel to x-axis.

Solution 12

Equation of parabolas with lotus rectum '(4a)' and whose area is parallel to x axes and vertex at (h,k) is given by,

$$(y-k)^2 = 4a(x-h)$$

Differentiating with respect to x,

$$2(y-k)y_1 = 4a(1)$$

$$(y-k)y_1 = 2a \qquad \text{---(i)}$$

Differentiating with respect to x,

$$(y-k)y_2 + (y_1)(y_1) = 0$$

$$(y-k)y_2 + (y_1)^2 = 0$$

$$\left(\frac{2a}{y_1}\right)^{y_2} + (y_1)^2 = 0$$

Using equation (i)

$$2ay_2 + (y_1)^3 = 0$$

Question 13

Show that the differential equation of which $y = 2(x^2-1) + ce^{-x^2}$ is a solution, is $\frac{dy}{dx} + 2xy = 4x^3$.

Solution 13

$$y = 2(x^2-1) + ce^{-x^2} \qquad \text{---(i)}$$

Differentiating it in equation (i),

$$\frac{dy}{dx} = 4x - 2cxe^{-x^2} \qquad \text{---(ii)}$$

Now,

$$\frac{dy}{dx} + 2xy$$

$$= 4x - 2cxe^{-x^2} + 2x\left[2(x^2-1) + ce^{-x^2}\right]$$

$$= 4x - 2cxe^{-x^2} + 4x^3 - 4x + 2xce^{-x^2}$$

$$= 4x^3$$

So,

$$\frac{dy}{dx} + 2xy = 4x^3$$

Which is given equation, so

$y = 2(x^2+1) + ce^{-x^2}$ **is the solution of the equation.**

Question 14

Form the differential equation having y = $(\sin^{-1}x)^2$ + A $\cos^{-1}$ x + B, where A and B are arbitrary constants, as its general solution.

Solution 14

$y = \left(\sin^{-1}x\right)^2 + A\cos^{-1}x + B$

$\frac{dy}{dx} = 2\sin^{-1}x \times \left(\frac{1}{\sqrt{1-x^2}}\right) + A \times \left(\frac{-1}{\sqrt{1-x^2}}\right) + 0$

$\sqrt{1-x^2}\frac{dy}{dx} = 2\sin^{-1}x - A$

$\sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{1}{\sqrt{1-x^2}}\right)(-2x) = 2 \times \left(\frac{1}{\sqrt{1-x^2}}\right) - 0$

$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2 = 0$

Note: Answer given in the book is incorrect.

Question 15(i)

Form the differential equation of the family of curves represented by the equation ('a' being the parameter).

$(2x+a)^2 + y^2 = a^2$

Solution 15(i)

Consider the given equation.,

$(2x+a)^2+y^2=a^2 \ldots.(1)$

Differentiating the above equation with respect to x, we have,

$$2(2x+a)+2y\frac{dy}{dx}=0$$

$$\Rightarrow (2x+a)+y\frac{dy}{dx}=0$$

$$\Rightarrow 2x+a=-y\frac{dy}{dx}$$

$$\Rightarrow a=-2x-y\frac{dy}{dx}$$

Substituting the value of a in equation (1), we have

$$\left(2x-2x-y\frac{dy}{dx}\right)^2+y^2=\left(-2x-y\frac{dy}{dx}\right)^2$$

$$\Rightarrow \left(y\frac{dy}{dx}\right)^2+y^2=\left(4x^2+y^2\left(\frac{dy}{dx}\right)^2+4xy\frac{dy}{dx}\right)$$

$$\Rightarrow y^2=4x^2+4xy\frac{dy}{dx}$$

$$\Rightarrow y^2-4x^2-4xy\frac{dy}{dx}=0$$

Question 15(ii)

Represent the following family of curves by forming the corresponding differential equation (a,b being parameters):

$(2x-a)^2-y^2=a^2$

Solution 15(ii)

$$(2x-a)^2 - y^2 = a^2$$
$$4x^2 + a^2 - ax - y^2 = a^2$$
$$4x^2 - 4ax - y^2 = 0$$
$$4ax = 4x^2 - y^2$$
$$a = \frac{4x^2 - y^2}{4x}$$

Differentiating it with respect to *x*,

$$0 = \left[\frac{4x\left(8x - 2y\frac{dy}{dx}\right) - 4\left(4x^2 - y^2\right)}{(4x)^2}\right]$$
$$32x^2 - 8xy\frac{dy}{dx} - 16x^2 + 4y^2 = 0$$
$$16x^2 - 8xy\frac{dy}{dx} + 4y^2 = 0$$
$$4x^2 + y^2 = 2xy\frac{dy}{dx}$$

Question 15(iii)

Form the differential equation of the family of curves represented by the equation ('a' being the parameter):

$(x-a)^2 + 2y^2 = a^2$

Solution 15(iii)

Consider the given equation,

$(x-a)^2+2y^2=a^2 \ldots.(1)$

Differentiating the above equation with respect to x, we have

$$2(x-a)+4y\frac{dy}{dx}=0$$

$$\Rightarrow (x-a)+2y\frac{dy}{dx}=0$$

$$\Rightarrow (x-a)=-2y\frac{dy}{dx}$$

$$\Rightarrow a=x+2y\frac{dy}{dx}$$

Substituting the value of a in equation (1), we have

$$\left(x-x+2y\frac{dy}{dx}\right)^2+2y^2=\left(x+2y\frac{dy}{dx}\right)^2$$

$$\Rightarrow 4y^2\left(\frac{dy}{dx}\right)^2+2y^2=x^2+4y^2\left(\frac{dy}{dx}\right)^2+4xy\frac{dy}{dx}$$

$$\Rightarrow 2y^2-x^2=4xy\frac{dy}{dx}$$

Question 16(i)

Represent the following family of curves by forming the corresponding differential equation (a,b being parameters):

$x^2+y^2=a^2$

Solution 16(i)

$$x^2+y^2=a^2$$

Differentiating it with respect to x,

$$2x+2y\frac{dy}{dx}=0$$

$$x+y\frac{dy}{dx}=0$$

Question 16(ii)

Represent the following family of curves by forming the corresponding differential equation (a,b being parameters):

$x^2-y^2=a^2$

Solution 16(ii)

$$x^2 - y^2 = a^2$$

Differentiating it with respect to x,

$$2x - 2y\frac{dy}{dx} = 0$$

$$x - y\frac{dy}{dx} = 0$$

Question 16(iii)

Represent the following family of curves by forming the corresponding differential equation (a, b being parameters):

$y^2 = 4ax$

Solution 16(iii)

$$y^2 = 4ax$$

$$\frac{y^2}{x} = 4a$$

Differentiating it with respect to x,

$$\left[\frac{x \times 2y\frac{dy}{dx} - y^2(1)}{x^2}\right] = 0$$

$$2xy\frac{dy}{dx} - y^2 = 0$$

$$2x\frac{dy}{dx} - y = 0$$

Question 16(iv)

Represent the following family of curves by forming the corresponding differential equation (a, b being parameters):

$x^2 + (y - b)^2 = 1$

Solution 16(iv)

$$x^2 + (y-b)^2 = 1 \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$2x + 2(y-b)\frac{dy}{dx} = 0$$

$$x + (y-b)\frac{dy}{dx} = 0$$

$$(y-b)\frac{dy}{dx} = -x$$

$$(y-b) = \frac{-x}{\frac{dy}{dx}}$$

Put the value of $(y-b)$ is equation (i)

$$x^2\left(\frac{-x}{\frac{dy}{dx}}\right)^2 = 1$$

$$x^2\left(\frac{dy}{dx}\right)^2 + x^2 = \left(\frac{dy}{dx}\right)^2$$

$$x^2\left\{\left(\frac{dy}{dx}\right)^2 + 1\right\} = \left(\frac{dy}{dx}\right)^2$$

Question 16(v)

Represent the following family of curves by forming the corresponding differential equation (a, b being parameters):

$(x-a)^2 - y^2 = 1$

Solution 16(v)

$$(x-a)^2 - y^2 = 1 \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$2(x-a) - 2y\frac{dy}{dx} = 0$$

$$(x-a) - y\frac{dy}{dx} = 0$$

$$(x-a) = y\frac{dy}{dx}$$

Put the value of $(x-a)$ is equation (i)

$$\left(y\frac{dy}{dx}\right)^2 - y^2 = 1$$

$$y^2\left(\frac{dy}{dx}\right)^2 - y^2 = 1$$

Question 16(vi)

Represent the following family of curves by forming the corresponding differential equation (a, b being parameters):

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Solution 16(vi)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
$$\frac{b^2x^2-a^2y^2}{a^2b^2}=1$$
$$b^2x^2-a^2y^2=a^2b^2$$

Differentiating it with respect to x,

$$2xb^2-2a^2y\frac{dy}{dx}=0$$
$$xb^2-ya^2\frac{dy}{dx}=0 \qquad \text{—(i)}$$

Again, differentiating it with respect to x,

$$b^2-a^2\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)\right)=0$$
$$b^2=a^2\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)$$

Put the value of b^2 in equation (i)

$$xb^2-ya^2\frac{dy}{dx}=0$$
$$xa^2\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)-ya^2\frac{dy}{dx}=0$$
$$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$
$$x\left\{y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right\}=y\frac{dy}{dx}$$

Question 16(vii)

Represent the following family of curves by forming the corresponding differential equation (a,b being parameters):

$y^2=4a(x-b)$

Solution 16(vii)

$$y^2=4a(x-b)$$

Differentiating it with respect to x,

$$2y\frac{dy}{dx}=4a$$

Again, differentiating it with respect to x,

$$2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)\right]=0$$
$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

Question 16(viii)

Represent the following family of curves by forming the corresponding differential equation $(a,b$ being parameters$)$:

$y = ax^3$

Solution 16(viii)

$$y = ax^3$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = 3ax^2$$

$$= 3\left(\frac{y}{x^3}\right)x^2$$

Using equation (i)

$$\frac{dy}{dx} = \frac{3y}{x}$$

$$x\frac{dy}{dx} = 3y$$

Question 16(ix)

Represent the following family of curves by forming the corresponding differential equation $(a,b$ being parameters$)$:

$x^2 + y^2 = ax^3$

Solution 16(ix)

$$x^2 + y^2 = ax^3$$

$$\frac{x^2 + y^2}{x^3} = a$$

Differentiating it with respect to x,

$$\left[\frac{\left(x^3\right)\left(2x + 2y\frac{dy}{dx}\right) - \left(x^2 + y^2\right)\left(3x^2\right)}{\left(x^3\right)^2}\right] = 0$$

$$2x^4 + 2x^3y\frac{dy}{dx} - 3x^4 - 3x^2y^2 = 0$$

$$2x^3y\frac{dy}{dx} - x^4 - 3x^2y^2 = 0$$

$$2x^3y\frac{dy}{dx} = x^4 + 3x^2y^2$$

$$2x^3y\frac{dy}{dx} = x^2\left(x^2 + 3y^2\right)$$

$$2xy\frac{dy}{dx} = \left(x^2 + 3y^2\right)$$

Question 16(x)

Represent the following family of curves by forming the corresponding differential equation $(a,b$ being parameters$)$:

$y = e^{ax}$

Solution 16(x)

$$y = e^{ax} \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = ae^{ax}$$

$$\frac{dy}{dx} = ay \qquad \text{—(ii)}$$

From equation (i),

$$y = e^{ax}$$

$$\log y = ax$$

$$a = \frac{\log y}{x}$$

Put the value of a in equation (ii),

$$\frac{dy}{dx} = \left(\frac{\log y}{x}\right)y$$

$$x\frac{dy}{dx} = y\log y$$

Question 17

Form the differential equation representing the family of ellipses having foci on x-axis and centre at the origin.

Solution 17

We know that the equation of said family of ellipses is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{---------(i)}$$

DIfferentiating (i) w.r.t. x , we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{y}{x}\left(\frac{dy}{dx}\right) = \frac{-b^2}{a^2} \qquad \text{---------(ii)}$$

DIfferentiating (ii) w.r.t. x , we get

$$\frac{y}{x}\left(\frac{d^2y}{dx^2}\right) + \left(\frac{x\frac{dy}{dx} - y}{x^2}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

which is the required differential equation.

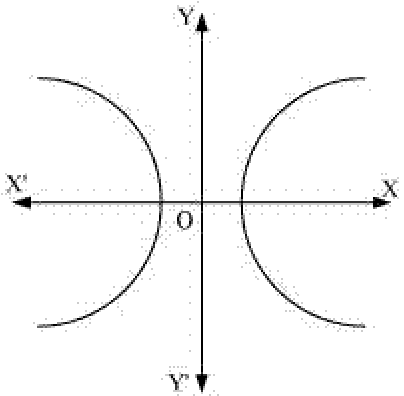
Question 18

Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.

Solution 18

The equation of the family of hyperbolas with the centre at origin and foci along the x-axis is:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad ...(1)$$



Differentiating both sides of equation (1) with respect to x, we get:

$$\frac{2x}{a^2}-\frac{2yy'}{b^2}=0$$

$$\Rightarrow \frac{x}{a^2}-\frac{yy'}{b^2}=0 \qquad ...(2)$$

Again, differentiating both sides with respect to x, we get:

$$\frac{1}{a^2}-\frac{y'\cdot y'+yy''}{b^2}=0$$

$$\Rightarrow \frac{1}{a^2}=\frac{1}{b^2}\left((y')^2+yy''\right)$$

Substituting the value of $\frac{1}{a^2}$ in equation (2), we get:

$$\frac{x}{b^2}\left((y')^2+yy''\right)-\frac{yy'}{b^2}=0$$

$$\Rightarrow x(y')^2+xyy''-yy'=0$$

$$\Rightarrow xyy''+x(y')^2-yy'=0$$

This is the required differential equation.

Question 19

Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Solution 19

Let C denote the family of circles in the second quadrant and touching the coordinate axes.

Let $(-a,a)$ be the coordinate of the centre of any member of this family.

Equation representing the family C is

$$(x+a)^2 + (y-a)^2 = a^2 \qquad ---------(i)$$

or $x^2 + y^2 + 2ax - 2ay + a^2 = 0 \qquad ---------(ii)$

Differentiating eqn (ii) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} + 2a - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow x + y\frac{dy}{dx} = a\left(\frac{dy}{dx} - 1\right)$$

$$\Rightarrow \qquad a = \frac{x + yy'}{y' - 1}$$

Substituting the value of a in (ii), we get

$$\left[x + \frac{x + yy'}{y' - 1}\right]^2 + \left[y - \frac{x + yy'}{y' - 1}\right]^2 = \left[\frac{x + yy'}{y' - 1}\right]^2$$

$$\Rightarrow \left[xy' - x + x + yy'\right]^2 + \left[yy' - y - x - yy'\right]^2 = \left[x + yy'\right]^2$$

$$\Rightarrow (x+y)^2 y'^2 + (x+y)^2 = \left[x + yy'\right]^2$$

$$\Rightarrow (x+y)^2\left[(y')^2 + 1\right] = \left[x + yy'\right]^2$$

which is the differential equation representing the given family of circles.

Chapter 22 - Differential Equations Exercise Ex. 22.3

Question 1

Show that $y = be^x + ce^{2x}$ is solution of differential equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Solution 1

$$y = be^x + ce^{2x} \qquad \text{—(i)}$$

Differentiating both sides with respect to x,

$$\frac{dy}{dx} = be^x + 2ce^{2x} \qquad \text{—(ii)}$$

Differentiating both sides with respect to x,

$$\frac{d^2y}{dx^2} = be^x + 4ce^{2x} \qquad \text{—(iii)}$$

Now,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y$$
$$= be^x + 4ce^{2x} - 3\left(be^x + 2ce^{2x}\right) + 2\left(be^x + ce^{2x}\right)$$
$$= be^x + 4ce^{2x} - 3be^x - 6ce^{2x} + 2be^x + 2ce^{2x}$$
$$= 3be^x - 3be^x + 6ce^{2x} - 6ce^{2x}$$
$$= 0$$

So,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Question 2

Verify that $y = 4\,sin 3x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + 9y = 0$$

Solution 2

$$y = 4\sin 3x \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = 4(3)\cos 3x$$
$$\frac{dy}{dx} = 12\cos 3x \qquad \text{—(ii)}$$

Differentiating it wtih respect to x,

$$\frac{d^2y}{dx^2} = -12(3)\sin 3x$$
$$\frac{d^2y}{dx^2} = -36\sin 3x \qquad \text{—(iii)}$$

Now,

$$\frac{d^2y}{dx^2} + 9y$$
$$= -36\sin 3x + 9(4\sin 3x)$$
$$= -36\sin 3x + 36\sin 3x$$
$$= 0$$

So, $y = 4\sin 3x$ is a solution of

$$\frac{d^2y}{dx^2} + 9y = 0$$

Question 3

Show that $y = ae^{2x} + be^{-x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$

Solution 3

$$y = ae^{2x} + be^{-x} \qquad \text{---(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = 2ae^{2x} - be^{-x} \qquad \text{---(ii)}$$

Differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x} \qquad \text{---(ii)}$$

Now,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y$$
$$= \left(4ae^{2x} + be^{-x}\right) - \left(2ae^{2x} - be^{-x}\right) - 2\left(ae^{2x} + be^{-x}\right)$$
$$= 4ae^{2x} + be^{-x} - 2ae^{2x} + be^{-x} - 2ae^{2x} - 2be^{-x}$$
$$= 4ae^{2x} - 4ae^{2x} + 2be^{-x} - 2be^{-x}$$
$$= 0$$

So,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

Question 4

Verify that the function y= Acos x + Bsin x, where a,b $\in$ R is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Solution 4

The given function is $y = A\cos x + B\sin x$ ---------(i)

Differentiating both sides of eqn (i) w.r.t x, successively, we get

$\frac{dy}{dx} = -A\sin x + B\cos x$

$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$

Substituting these values of $\frac{d^2y}{dx^2}$ and y in the given differential equation,

L.H.S = $(-A\cos x - B\sin x) + (A\cos x + B\sin x) = 0 = $ R.H.S

Therefore, the given function is a solution of the given differential equation.

Question 5

Show that the function $y = A\cos 2x - B\sin 2x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

Solution 5

$$y = A\cos 2x - B\sin 2x \qquad \text{---(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = -2A\sin 2x - 2B\cos 2x$$

$$\frac{dy}{dx} = -2(A\sin 2x + B\cos 2x) \qquad \text{---(ii)}$$

Differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = -2[2A\cos 2x - 2B\sin 2x]$$

$$= -4[A\cos 2x - B\sin 2x]$$

$$\frac{d^2y}{dx^2} = -4y$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

Question 6

Show that $y = Ae^{Bx}$ is a solution of the differential equation $\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2$

Solution 6

$$y = Ae^{Bx} \qquad \text{---(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = ABe^{Bx} \qquad \text{---(ii)}$$

Differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = AB^2e^{Bx}$$

$$= \frac{\left(ABe^{Bx}\right)^2}{\left(Ae^{Bx}\right)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2$$

Question 7

Verify that $y = \frac{a}{x} + b$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0$

Solution 7

$$y = \frac{a}{x} + b \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = -\frac{a}{x^2} \qquad \text{—(ii)}$$

Differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = \frac{2a}{x^3}$$

$$= -\frac{2}{x}\left(-\frac{a}{x^2}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{2}{x}\left(\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0$$

Question 8

Verify that $y^2 = 4ax$ is a solution of the differential equation $y = x\frac{dy}{dx} + a\frac{dx}{dy}$

Solution 8

$$y^2 = 4ax \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y}$$

$$\frac{dy}{dx} = \frac{2a}{y} \qquad \text{—(ii)}$$

Now,

$$x\frac{dy}{dx} + a\frac{dy}{dx}$$

$$= 2\frac{xa}{y} + a\left(\frac{y}{2a}\right)$$

$$= \frac{4a^2x + ay^2}{2ay}$$

$$= \frac{ay^2 + ay^2}{2ay}$$

$$= y$$

So,

$$x\frac{dy}{dx} + a\frac{dx}{dy} = y$$

Question 9

Show that $Ax^2 + By^2 = 1$ is a solution of the differential equation

$$x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$$

Solution 9

$$Ax^2 + By^2 = 1$$

Differentiating it with respect to x,

$$2Ax + 2By\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = \frac{-2Ax}{2B}$$

$$y\frac{dy}{dx} = -\frac{Ax}{B} \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\frac{A}{B}$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{y}{x}\frac{dy}{dx}$$

Using equation (i)

$$x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$$

Question 10

Show that $y = ax^3 + bx^2 + c$ is a solution of the differential equation $\frac{d^3y}{dx^3} = 6a$

Solution 10

$$y = ax^3 + bx^2 + c$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = 3ax^2 + 2bx$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = 6ax + 2b$$

Differentiating it with respect to x

$$\frac{d^3y}{dx^3} = 6a$$

Question 11

Show that $y = \frac{c-x}{1+cx}$ is a solution of the differential equation

$$\left(1+x^2\right)\frac{dy}{dx}+\left(1+y^2\right)=0$$

Solution 11

$$y = \frac{c-x}{1+cx} \qquad \text{---(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = \left[\frac{(1+cx)(-1)-(c-x)(c)}{(1+cx)}\right]$$

$$\frac{dy}{dx} = \left[\frac{-1-cx-c^2+cx}{(1+cx)^2}\right]$$

$$= \frac{-1-c^2}{(1+cx)^2}$$

$$\frac{dy}{dx} = \frac{-\left(1+c^2\right)}{(1+cx)^2} \qquad \text{---(ii)}$$

Now,

$$\left(1+x^2\right)\frac{dy}{dx}+\left(1+y^2\right)$$

$$=\left(1+x^2\right)\left[\frac{-\left(1+c^2\right)}{(1+cx)^2}\right]+\left[1+\left(\frac{c-x}{1+cx}\right)^2\right]$$

$$=\frac{-\left(1+x^2\right)\left(1+c^2\right)}{(1+cx)^2}+\left[\frac{(1+cx)^2+(c-x)^2}{(1+cx)^2}\right]$$

$$=\frac{-1-x^2-c^2-x^2c^2+1+c^2x^2+2cx+c^2+x^2-2cx}{(1+cx)^2}$$

$$=\frac{0}{(1+cx)^2}$$

$$=0$$

So,

$$\left(1+x^2\right)\frac{dy}{dx}+\left(1+y^2\right)=0$$

Question 12

Show that $y = e^x(A \cos x + B \sin x)$ is the solution of the differential equation

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$$

Solution 12

$$y = e^x(A\cos x + B\sin x).....(i)$$

$$\frac{dy}{dx} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$$
$$\frac{dy}{dx} = e^x\left[(A+B)\cos x - (A-B)\sin x\right]........(ii)$$

$$\frac{d^2y}{dx^2} = e^x\left[(A+B)\cos x - (A-B)\sin x\right] + e^x\left[-(A+B)\sin x - (A-B)\cos x\right]$$
$$\frac{d^2y}{dx^2} = 2e^x(B\cos x - A\sin x)..........(iii)$$

Adding (i) and (iii) we get

$$y + \frac{1}{2}\frac{d^2y}{dx^2} = e^x\left[(A+B)\cos x - (A-B)\sin x\right]$$
$$2y + \frac{d^2y}{dx^2} = 2\frac{dy}{dx}$$
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Hence $y = e^x(A\cos x + B\sin x)$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$

Question 13

Verify that $y = cx + 2c^2$ is a solution of the differential equation

$$2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0$$

Solution 13

$$y = cx + 2c^2 \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$\frac{y}{dx} = c \qquad \text{—(ii)}$$

Now,

$$2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y$$
$$= 2c^2 + xc - cx + 2c^2 \qquad \left[\text{Using equation (i) and(ii)}\right]$$
$$= 0$$

So,

$$2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0$$

Question 14

Verify that $y = -x - 1$ is a solution of the differential equation

$(y - x)\,dy - \left(y^2 - x^2\right)dx = 0$

Solution 14

$$y = -x - 1 \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = -1 \qquad \text{—(ii)}$$

So,

$$(y - x)\,dy - \left(y^2 - x^2\right)dx$$
$$= \left[(y - 2)\frac{dy}{dx} - \left(y^2 - x^2\right)\right]dx$$
$$= \left[(-x - 1 - x)(-1) - \left\{(-x - 1)^2 - x^2\right\}\right]$$

Using equation (i) and (ii),

$$= \left[x + 1 + x - \left(x^2 + 1 + 2x - x^2\right)\right]dx$$
$$= [2x + 1 - 2x - 1]dx$$
$$= 0$$

So,

$$(y - x)\,dy - \left(y^2 - x^2\right)dx = 0$$

Question 15

Verify that $y^2 = 4a(x + a)$ is a solution of the differential equations

$y\left\{1 - \left(\frac{dy}{dx}\right)^2\right\} = 2x\frac{dy}{dx}$

Solution 15

$$y^2 = 4a(x+a) \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$2y\frac{dy}{dx} = 4a(1)$$

$$\frac{dy}{dx} = \frac{2a}{y} \qquad \text{—(ii)}$$

Now,

$$y\left\{1-\left(\frac{dy}{dx}\right)^2\right\}$$

$$= \left[y^2\left\{1-\left(\frac{dy}{dx}\right)^2\right\}\right]\frac{1}{y}$$

$$= \left[4a(x+a) - 4a(x+a)\left(\frac{2a}{y}\right)^2\right]\frac{1}{y}$$

Using equation (i) and (ii)

$$= \left[4ax + 4a^2 - \frac{16a3x}{y^2} - \frac{16a^4}{y^2}\right]\frac{1}{y}$$

$$= \frac{4a}{y^3}\left[xy^2 + ay^2 - 4a^2x - 4a^3\right]$$

$$= \frac{4a}{y^3}\left[y^2(a+x) - 4a^2(x+a)\right]$$

$$= \frac{4a}{y^3}(a+x)\left(y^2 - 4a^2\right)$$

$$= \frac{4a}{y^3}\left(\frac{y^2}{4a}\right)\left(y^2 - 4a^2\right)$$

Using equation (i) and (ii)

$$= \frac{1}{y}\left(y^2 - 4a^2\right)$$

$$= \frac{1}{y}\left[4ax + 4a^2 - 4a\right]$$

$$= \frac{1}{y}(4ax)$$

$$= 2x\left(\frac{2a}{y}\right)$$

$$= 2x\frac{dy}{dx}$$

So,

$$y\left\{1-\left(\frac{dy}{dx}\right)^2\right\} = 2x\frac{dy}{dx}$$

Question 16

Verify that $y = ce^{\tan^{-1}x}$ is a solution of the differential equation

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + (2x-1)\left(\frac{dy}{dx}\right) = 0$$

Solution 16

$$y = ce^{\tan^{-1} x}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = ce^{\tan^{-1} x} \times \left(\frac{1}{1+x^2}\right)$$

$$\left(1+x^2\right)\frac{dy}{dx} = ce^{\tan^{-1} x}$$

$$\left(1+x^2\right)\frac{dy}{dx} = y$$

Again, differentiating it with respeoet to x,

$$2x\frac{dy}{dx} + \left(1+x^2\right)\frac{d^2y}{dx^2} = \frac{dy}{dx}$$

$$2x\frac{dy}{dx} - \frac{dy}{dx} + \left(1+x^2\right)\frac{d^2y}{dx^2} = 0$$

$$(2x-1)\frac{dy}{dx} + \left(1+x^2\right)\frac{d^2y}{dx^2} = 0$$

Question 17

Verify that $y = e^{m\cos^{-1}}$ satisfies the differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

Solution 17

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0$$

$$y=e^{m\cos^{-1}x}$$

$$\frac{dy}{dx}=\frac{me^{m\cos^{-1}x}}{-\sqrt{1-x^2}}$$

$$\frac{dy}{dx}=\frac{-my}{\sqrt{1-x^2}}\,\text{........(i)}$$

$$\frac{d^2y}{dx^2}=\frac{\sqrt{(1-x^2)}.\left(-m\frac{dy}{dx}\right)-(-my)\frac{(-2x)}{2\sqrt{(1-x^2)}}}{(1-x^2)}\ \text{[From (i)]}$$

$$\frac{d^2y}{dx^2}=\frac{(-m)(-my)-x\frac{dy}{dx}}{(1-x^2)}\text{[From (i)]}$$

$$(1-x^2)\frac{d^2y}{dx^2}=m^2y-x\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-m^2y=0$$

Hence Proved

Question 18

Verify that $y=log\left(x+\sqrt{x^2+a^2}\right)^2$ satisfies the differential equation

$$\left(a^2+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$$

Solution 18

$$y = \log\left(x+\sqrt{x^2+a^2}\right)^2$$

Differentiating it with respect to x,

$$\frac{dy}{dx}\frac{1}{\left(x+\sqrt{a^2+x^2}\right)^2}\times 2\left(x+\sqrt{x^2+a^2}\right)\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)$$

$$=\frac{2}{\left(x+\sqrt{a^2+x^2}\right)}\times\left(1+\frac{1}{2\sqrt{x^2+a^2}}(2x)\right)$$

$$=\frac{2}{\left(x+\sqrt{a^2+x^2}\right)}\left(\frac{\sqrt{x^2+a^2}+x}{2\sqrt{x^2+a^2}}\right)$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{a^2+x^2}}$$

$$\sqrt{a^2+x^2}\frac{dy}{dx}=1 \qquad ---(i)$$

Again, differentiating it with respecet to x,

$$\sqrt{1-x^2}\frac{d^2y}{dx^2}+\frac{1}{2\sqrt{1-x^2}}(-2x)\frac{dy}{dx}=-m\frac{dy}{dx}$$

$$\sqrt{1-x^2}\frac{d^2y}{dx^2}-\frac{x}{\sqrt{1-x^2}}\frac{dy}{dx}-m\left(\frac{-e^{m\cos^{-1}x}m}{\sqrt{1-x^2}}\right)=0$$

Using equation (i),

$$\sqrt{a^2+x^2}\frac{d^2y}{dx^2}+\frac{2x}{2\sqrt{a^2+x^2}}\frac{dy}{dx}=0$$

$$\left(a^2+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$$

Question 19

Show that the differential equation of which $y=2\left(x^2-1\right)+ce^{-x^2}$ is a solution is

$$\frac{dy}{dx}+2xy=4x^3$$

Solution 19

$$y = 2\left(x^2 - 1\right) + ce^{-x^2} \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = 2(2x) + ce^{-x^2}(-2x)$$

$$\frac{dy}{dx} = 4x - 2cxe^{-x^2} \qquad \text{—(ii)}$$

Now,

$$\frac{dy}{dx} + 2xy$$

$$= 4x - 2cxe^{-x^2} + 2x\left[2\left(x^2 - 1\right) + ce^{-x^2}\right]$$

Using equation (i) and (ii),

$$= 4x - 2cxe^{-x^2} + 2x\left(2x^2 - 2 + ce^{-x^2}\right)$$

$$= 4x - 2cxe^{-x^2} + 4x^3 - 4x + 2xce^{-x^2}$$

$$= 4x^3$$

So,

$$\frac{dy}{dx} + 2xy = 4x^3$$

Question 20

Show that $y = e^{-x} + ax + b$ is solution of the differential equation $e^x \dfrac{d^2y}{dx^2} = 1$

Solution 20

$$y = e^{-x} + ax + b$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = -e^{-x} + a$$

Differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = e^{-x}$$

$$\frac{1}{e^{-x}}\frac{d^2y}{dx^2} = 1$$

$$e^x\frac{d^2y}{dx^2} = 1$$

Question 21(i)

For the following differential equation verify that the accompanying function is a solution in the mentioned domain (a, b are parameters).

$$x\frac{dy}{dx} = y \qquad y = ax, x \in R - \{0\}$$

Solution 21(i)

$$y = ax \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = a$$

$$= \frac{ax}{x} \qquad [\because x \in R - \{0\}]$$

$$\frac{dy}{dx} = \frac{y}{x} \qquad [\text{Using equation (i)}]$$

$$x\frac{dy}{dx} = y$$

So, $y = ax$ is the solution of the given equation.

Question 21(ii)

For the following differential equation verify that the accompanying function is a solution in the mentioned domain $(a, b$ are parameters$)$.

$$x + y\frac{dy}{dx} = 0; \qquad x \in R, y \neq 0$$

$$y = \pm\sqrt{a^2 - x^2}, \qquad x \in (-a, a)$$

Solution 21(ii)

$$y = \pm\sqrt{a^2 - x^2}$$

Squaring both the sides,

$$y^2 = \left(a^2 - x^2\right)$$

Differentiating it with respect to x,

$$2y\frac{dy}{dx} = -2x$$

$$y\frac{dy}{dx} = -x$$

$$x + y\frac{dy}{dx} = 0$$

So,

$y = \pm\sqrt{a^2 - x^2}$ **is the solution of the given equation.**

Question 21(iii)

Show that $y = \frac{a}{x+a}$ is the solution of the differential equation

$$x\frac{dy}{dx} + y = y^2$$

Solution 21(iii)

$$y = \frac{a}{x+a}$$

$$\frac{dy}{dx} = \frac{a}{(x+a)^2} \times (-1) = -\frac{a}{(x+a)^2}$$

Consider,

$$x\frac{dy}{dx} + y = -\frac{ax}{(x+a)^2} + \frac{a}{x+a} = \frac{-ax + ax + a^2}{(x+a)^2} = \frac{a^2}{(x+a)^2} = y^2$$

$$x\frac{dy}{dx} + y = y^2$$

Hence $y = \frac{a}{x+a}$ is the solution of the differential equation $x\frac{dy}{dx} + y = y^2$.

Question 21(iv)

For the following differential equation verify that the accompanying function is a solution in the mentioned domain $(a, b$ are parameters$)$.

$$x^3\frac{d^2y}{dx^2} = 1, \qquad x \in R - \{0\}$$

$$y = ax + b + \frac{1}{2x}, \qquad x \in R - \{0\}$$

Solution 21(iv)

$$y = ax + b + \frac{1}{2x}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = a - \frac{1}{2x^2}$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = 0 - \frac{(-2)}{2x^3}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^3}$$

$$x^3 \frac{d^2y}{dx^2} = 1$$

So,

$y = ax + b + \frac{1}{2x}$ is the solution of the given equation.

Question 21(v)

For the following differential equation verify that the accompanying function is a solution in the mentioned domain $(a, b$ are parameters$)$.

$y = \left(\frac{dy}{dx}\right)^2, x \in R, y \geq 0 \qquad y = \frac{1}{4}(x \pm a)^2, x \in R$

Solution 21(v)

$$y = \frac{1}{4}(x \pm a)^2$$

Case I:

$$y = \frac{1}{4}(x + a)^2$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = \frac{1}{4}2(x + a)$$

$$\frac{dy}{dx} = \frac{1}{2}(x + a)$$

Squaring both sides,

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(x + a)^2$$

$$\left(\frac{dy}{dx}\right)^2 = y \qquad \left[\text{Using equation (i)}\right]$$

So,

$y = \frac{1}{4}(x + a)$ **is the solution of the given equation.**

Case II:

$$y = \frac{1}{4}(x - a)^2 \qquad \text{---(ii)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = \frac{1}{4}2(x - a)$$

$$\frac{dy}{dx} = \frac{1}{2}(x - a)$$

Squaring both the sides,

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(x - a)^2$$

$$\left(\frac{dy}{dx}\right)^2 = y^2 \qquad \left[\text{Using equation (ii)}\right]$$

So,

$y = \frac{1}{4}(x - a)$ **is the solution of the given equation.**

Chapter 22 - Differential Equations Exercise Ex. 22.4

Question 1

For the following initial value problem verify that the accompanying function is a solution:

$x\frac{dy}{dx} = 1, y(1) = 0 \qquad y = \log x$

Solution 1

Here, $y = \log x$

Differentiating it with respect to x,

$$\frac{dy}{dx} = \frac{1}{x}$$

$$x\frac{dy}{dx} = 1$$

So, $y = \log x$ is a solutionof the equation

If $\quad x = 1, \quad y = \log 1 = 0$

So,

$$y(1) = 0$$

Question 2

For the following initial value problem verify that the accompanying function is a solution:

$\frac{dy}{dx} = y, y(0) = 1 \qquad y = e^x$

Solution 2

Here, $y = e^x$

Differentiating it with respect to x,

$$\frac{dy}{dx} = e^x$$

$$\frac{dy}{dx} = y$$

So, $y = e^x$ is a solution of the equation

If $x = 0, \qquad y = e^0 = 1$

So,

$$y(0) = 1$$

Question 3

For the following initial value problem verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} + y = 0, y(0) = 0, y'(0) = 1 \qquad y = \sin x$$

Solution 3

Here, $y = \sin x$ —(i)

Differentiating it with respect to x,

$$\frac{dy}{dx} = \cos x \quad \text{—(ii)}$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = -\sin x$$

$$\frac{d^2y}{dx^2} = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

So, $y = \sin x$ is a solution of the equation.

Put $x = 0$ in equation (i),

$\Rightarrow \quad y = \sin 0$

$\Rightarrow \quad y = 0$

$\Rightarrow \quad y(0) = 0$

Put $x = 0$ in equation (ii),

$y' = \cos 0$

$y' = 1$

$\Rightarrow \quad y'(0) = 1$

Question 4

For the following initial value problem verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0, y(0) = 2, y'(0) = 1 \qquad y = e^x + 1$$

Solution 4

Here, $y = e^x + 1$ —(i)

Differentiating it with respect to x,

$$\frac{dy}{dx} = e^x$$

$$\frac{dy}{dx} = y - 1 \quad \text{—(ii)}$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$$

It is given differential equation. So,

$y = e^x + 1$ is a solutionof the equation

Put $x = 0$ in equation (i),

$$\Rightarrow \quad y = e^0 + 1 = 2$$

$$y(0) = 2$$

Put $x = 0$ in equation (ii),

$$y' = e^0 = 1$$

$$y'(0) = 1$$

Question 5

For the following initial value problem verify that the accompanying function is a solution:

$$\frac{dy}{dx} + y = 2, y(0) = 3 \quad y = e^{-x} + 2$$

Solution 5

Here, $y = e^{-x} + 2$ —(i)

Differentiating it with respect to x,

$$\frac{dy}{dx} = -e^{-x}$$

$$\frac{dy}{dx} = -(y - 2) \quad [\text{Using equaton (i)}]$$

$$\frac{dy}{dx} + y = 2$$

It is given differential equation. So,

$y = e^{-x} + 2$ is a solutionof the equation

Put $x = 0$ in equation (i),

$$y = e^0 + 2$$

$$= 1 + 2$$

$$y = 3$$

So,

$$y(0) = 3$$

Question 6

For the following initial value problem verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} + y = 0, y(0) = 1, y'(0) = 1 \quad y = \sin x + \cos x$$

Solution 6

$$y = \sin x + \cos x \qquad \text{---(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = \cos x - \sin x \qquad \text{---(ii)}$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = -\sin x - \cos x$$

$$\frac{d^2y}{dx^2} = -(\sin x + \cos x)$$

$$\frac{d^2y}{dx^2} = -y \qquad [\text{Using equation (i)}]$$

$$\frac{d^2y}{dx^2} + y = 0$$

It is the given equation, so

$y = \sin x + \cos x$ **is the solution of the given equation**

Put $x = 0$ in equation (i),

$$y = \sin 0 + \cos 0$$

$$y = 0 + 1$$

$$y = 1$$

So,

$$y(0) = 1$$

Put $x = 0$ in equation (i),

$$\frac{dy}{dx} = \cos 0 - \sin 0$$

$$y' = 1$$

So,

$$y'(0) = 1$$

Question 7

For the following initial value problem verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} - y = 0, y(0) = 2, y'(0) = 0 \qquad y = e^x + e^{-x}$$

Solution 7

$$y = e^x + e^{-x} \qquad \text{---(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = e^x - e^{-x} \qquad \text{---(ii)}$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = e^x + e^{-x}$$

$$\frac{d^2y}{dx^2} = y \qquad \left[\text{Using equation (i)}\right]$$

$$\frac{d^2y}{dx^2} - y = 0$$

It is the given equation, so

$y = e^x + e^{-x}$ is the solution of the given equation.

Put $x = 0$ in equation (i),

$$y = e^0 + e^0$$

$$y = 2$$

So,

$$y(0) = 2$$

Put $x = 0$ in equation (ii),

$$y' = e^0 - e^0$$

$$y' = 0$$

So,

$$y'(0) = 0$$

Question 8

For the following initial value problem verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0, y(0) = 1, y'(0) = 3 \qquad y = e^x + e^{2x}$$

Solution 8

$$y = e^x + e^{2x} \quad \text{—(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = e^x + 2e^{2x} \quad \text{—(ii)}$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = e^x + 4e^{2x}$$
$$= (3-2)e^x + (6-2)e^{2x}$$
$$= 3e^x - 2e^x + 6e^{2x} - 2e^{2x}$$
$$= 3e^x + 6e^{2x} - 2e^x - 2e^{2x}$$
$$= 3\left(e^x + 2e^{2x}\right) - 2\left(e^x + e^{2x}\right)$$
$$\frac{d^2y}{dx^2} = 3\frac{dy}{dx} - 2y \quad \left[\text{Using equation (i) and (ii)}\right]$$
$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

It is the given equation, so

$y = e^x + 2e^{2x}$ is the solution of the given equation.

Put $x = 0$ in equation (i),

$$y = e^0 + e^0$$
$$y = 1 + 1$$
$$y = 2$$

So,

$$y(0) = 2$$

Put $x = 0$ in equation (ii),

$$\frac{dy}{dx} = e^0 + 2e^0$$
$$y' = 1 + 2$$
$$y' = 3$$

So,

$$y'(0) = 3$$

Question 9

For the following initial value problem verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0, y(0) = 1, y'(0) = 2 \qquad y = xe^x + e^x$$

Solution 9

$$y = xe^x + e^x \qquad \text{—(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = \left[x \times \frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}(x)\right] + e^x$$

$$= xe^x + e^x(1) + e^x$$

$$\frac{dy}{dx} = xe^x + 2e^x \qquad \text{—(ii)}$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = x\frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}(x) + 2e^x$$

$$= (2-1)xe^x + (4-1)e^x$$

$$= 2xe^x - xe^x + 4e^x - e^x$$

$$= 2xe^x + 4e^x - xe^x - e^x$$

$$= 2\left(xe^x + 2e^x\right) - \left(xe^x + 1\right)$$

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - y \qquad \left[\text{Using equation (i) and (ii)}\right]$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

It is the given equation, so

$y = xe^x + e^x$ is the solution of the given equation.

Put $y = 0$ in equation (i),

$$y = 0 + e^0$$

$$y = 1$$

So,

$$y(0) = 1$$

Put $y = 0$ in equation (ii),

$$\frac{dy}{dx} = 0 + 2e^0$$

$$y' = 2$$

So,

$$y'(0) = 2$$

Chapter 22 - Differential Equations Exercise Ex. 22.5

Question 1

Solve the following differential equation:

$$\frac{dy}{dx} = x^2 + x - \frac{1}{x}, x \neq 0$$

Solution 1

$$\frac{dy}{dx} = x^2 + x - \frac{1}{x},\ x \neq 0$$
$$\int dy = \int \left(x^2 + x - \frac{1}{x}\right) dx$$
$$y = \frac{x^3}{3} + \frac{x^2}{2} - \log|x| + c,\ x \neq 0$$

Question 2

Solve the following differential equation:

$$\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}, x \neq 0$$

Solution 2

$$\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x},\ x \neq 0$$
$$\int dy = \int \left(x^5 + x^2 - \frac{2}{x}\right) dx$$
$$y = \frac{x^6}{6} + \frac{x^3}{3} - 2\log|x| + c,\ x \neq 0$$

Question 3

Solve the following differential equation:

$$\frac{dy}{dx} + 2x = e^{3x}$$

Solution 3

$$\frac{dy}{dx} + 2x = e^{3x}$$
$$\frac{dy}{dx} = e^{3x} - 2x$$
$$\int dy = \int \left(e^{3x} - 2x\right) dx$$
$$y = \frac{e^{3x}}{3} - \frac{2x^2}{2} + c$$
$$y = \frac{e^{3x}}{3} - x^2 + c$$
$$y + x^2 = \frac{1}{3}e^{3x} + c$$

Question 4

Solve the following differential equation:

$$\left(x^2 + 1\right)\frac{dy}{dx} = 1$$

Solution 4

$$\left(x^2 + 1\right)\frac{dy}{dx} = 1$$
$$\int dy = \int \frac{dx}{x^2 + 1}$$
$$y = \tan^{-1} x + c$$

Question 5

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$$

Solution 5

$$\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$$

$$\frac{dy}{dx} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$\frac{dy}{dx} = \tan^2\frac{x}{2}$$

$$dy = \tan^2\frac{x}{2}dx$$

$$\int dy = \int \tan^2\frac{x}{2}dx$$

$$\int dy = \int\left(\sec^2\frac{x}{2} - 1\right)dx$$

$$y = 2\tan\frac{x}{2} - x + c$$

where $x \neq (2n+1)\pi,\ n \in Z$

Question 6

Solve the following differential equation:

$$(x+2)\frac{dy}{dx} = x^2 + 3x + 7$$

Solution 6

$$(x+2)\frac{dy}{dx} = x^2 + 3x + 7$$

$$dy = \left(\frac{x^2+3x+7}{x+2}\right)dx$$

$$dy = \left(x+1+\frac{5}{x+2}\right)dx$$

$$\int dy = \int\left(x+1+\frac{5}{x+2}\right)dx$$

$$y = \frac{x^2}{2} + x + 5\log|x+2| + c$$

$$x \neq -2$$

Question 7

Solve the following differential equation:

$$\frac{dy}{dx} = \tan^{-1} x$$

Solution 7

$$\frac{dy}{dx} = \tan^{-1} x$$
$$dy = \tan^{-1} x dx$$
$$\int dy = \int \tan^{-1} x dx$$
$$y = \tan^{-1} x \times \int 1 dx - \int \left(\frac{1}{1+x^2} \int dx \right) dx + c$$

Using integration by parts

$$y = x \tan^{-1} x - \int \frac{x}{1+x^2} dx + c$$
$$y = x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx + c$$
$$y = x \tan^{-1} x - \frac{1}{2} \log \left|1+x^2\right| + c$$

where $x \in R$

Question 8

Solve the following differential equation:

$$\frac{dy}{dx} = \log x$$

Solution 8

$$\frac{dy}{dx} = \log x$$
$$\Rightarrow dy = \log x \times dx$$
$$\Rightarrow \int dy = \int \log x dx$$
$$\Rightarrow y = \log x \times \int 1 dx - \int \left(\frac{1}{x} \int 1 dx \right) dx + C \quad [\text{Using integration by parts}]$$
$$\Rightarrow y = x \log x - \int dx + C$$
$$\Rightarrow y = x \log x - x + C$$
$$\Rightarrow y = x(\log x - 1) + C, \text{ where } x \in (0, \infty)$$

Question 9

Solve the following differential equation:

$$\frac{1}{x} \frac{dy}{dx} = \tan^{-1} x, x \neq 0$$

Solution 9

$$\frac{1}{x}\frac{dy}{dx} = \tan^{-1}x$$
$$dy = x\tan^{-1}x\,dx$$
$$\int dy = \int x\tan^{-1}x\,dx$$
$$y = \tan^{-1}x\int x\,dx - \int\left(\frac{1}{1+x^2}\int x\,dx\right)dx + c$$

Using integration by parts

$$y = \frac{x^2}{2}\tan^{-1}x - \int\frac{x^2}{2\left(1+x^2\right)}dx + c$$
$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}dx + c$$
$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1-\frac{1}{x^2+1}\right)dx + c$$
$$y = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + c$$
$$y = \frac{1}{2}\left(x^2+1\right)\tan^{-1}x - \frac{1}{2}x + c$$

Question 10

Solve the following differential equation:

$$\frac{dy}{dx} = \cos^3 x\,\sin^2 x + x\sqrt{2x+1},\; x \in \left[-\frac{1}{2}, \infty\right]$$

Solution 10

$$\frac{dy}{dx} = \cos^3 x \sin^2 x + x\sqrt{2x+1}$$
$$dy = \left(\cos^3 x \sin^2 x + x\sqrt{2x+1}\right) dx$$
$$\int dy = \int \cos^3 x \sin^2 x dx + \int x\sqrt{2x+1} dx$$
$$y = I_1 + I_2 \qquad \text{---(i)}$$
$$I_1 = \int \cos^3 x \sin^2 x dx$$
$$= \int \cos^2 x \times \cos x \times \sin^2 x x dx$$
$$I_1 = \int \left(1 - \sin^2 x\right) \sin^2 x \cos x dx$$
Put $\sin x = t$
$$\cos x dx = dt$$
$$I_1 = \int \left(1 - t^2\right) t^2 dt$$
$$= \int \left(t^2 - t^4\right) dt$$
$$= \frac{t^3}{3} - \frac{t^5}{5} + c_1$$
$$I_1 = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c_1$$
And,
$$I_2 = \int x\sqrt{2x+1} dx$$
Put $2x + 1 = v^2$
$$2dx = 2vdv$$
$$I_2 = \int \left(\frac{v^2 - 1}{2}\right) v \times v dv$$
$$= \frac{1}{2}\int \left(v^4 - v^2\right) dv$$
$$= \frac{1}{2}\left(\frac{v^5}{5} - \frac{v^3}{3}\right) + c_2$$
$$I_2 = \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + c_2$$
Put the I_1 and I_2 in equation (i),
$$y = I_1 + I_2$$
$$y = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + c$$
As $\quad c = c_1 + c_2$

Question 11

Solve the following differential equation:
(sin x + cos x)dy + (cos x - sin x) dx = 0

Solution 11

$$\left(\sin x + \cos x\right)dy + \left(\cos x - \sin x\right)dx = 0$$
$$\left(\sin x + \cos x\right)dy = \left(\sin x - \cos x\right)dx$$
$$dy = \frac{\left(\sin x - \cos x\right)}{\sin x + \cos x}dx$$
$$\int dy = -\int\left(\frac{\cos x - \sin x}{\sin x + \cos x}\right)dx$$
Put $\sin x + \cos x = t$
$$\left(\cos x - \sin x\right)dx = dt$$
$$\int dy = -\int \frac{1}{t}dt$$
$$y = -\log\left|t\right| + c$$
$$y + \log\left|\sin x + \cos x\right| = c$$

Question 12

Solve the following differential equation:

$$\frac{dy}{dx} - x\sin^2 x = \frac{1}{x\log x}$$

Solution 12

$$\frac{dy}{dx} - x\sin^2 x = \frac{1}{x\log x}$$

$$\frac{dy}{dx} = \frac{1}{x \times \log x} + x\sin^2 x$$

$$dy = \left(\frac{1}{x\log x} + x\sin^2 x\right)dx$$

$$\int dy = \int \frac{1}{x\log x}dx + \int x\sin^2 x dx$$

$$y = I_1 + I_2 \qquad \text{---(i)}$$

$$I_1 = \int \frac{1}{x\log x}dx$$

Let $\log x = t$

$$\frac{1}{x}dx = dt$$

$$I_1 = \int \frac{dt}{t}$$

$$= \log|t| + c_1$$

$$I_1 = \log|\log x| + c_1$$

$$I_2 = \int x\sin^2 x dx$$

$$= \int x\frac{(1-\cos 2x)}{2}dx$$

$$= \frac{1}{2}\int (x - x\cos 2x)dx$$

$$= \frac{1}{2}\int xdx - \frac{1}{2}\int x\cos 2xdx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right) - \frac{1}{2}\left[x\int \cos 2xdx - \int\left(1 \times \int \cos 2xdx\right)dx\right] + c_2$$

$$= \frac{x^2}{4} - \frac{1}{2}\left[\frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}dx\right] + c_2$$

$$= \frac{x^2}{4} - \frac{1}{2}\left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right] + c_2$$

$$I_2 = \frac{x^2}{4} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8} + c_2$$

Put the value of I_1 and I_2 in equation (i),

$$y = I_1 + I_2$$

$$y = \log|\log x| + \frac{x^2}{4} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8} + c \text{ as } c_1 + c_2 = c$$

Question 13

Solve the following differential equation:

$$\frac{dy}{dx} = x^5\tan^{-1}(x^3)$$

Solution 13

$$\frac{dy}{dx} = x^5 \tan^{-1}\left(x^3\right)$$
$$dy = x^5 \tan^{-1}\left(x^3\right)dx$$
$$\int dy = \int x^5 \tan^{-1}\left(x^3\right)dx$$

Put $x^3 = t$

$\Rightarrow \quad 3x^2dx = dt$

$\Rightarrow \quad x^2dx = \frac{dt}{3}$

So,

$$\int dy = \frac{1}{3}\left[\tan^{-1} t \int t dt - \int\left(\frac{1}{1+t^2} \times \int t dt\right)\right]dt + c$$

Using integration by parts

$$y = \frac{1}{3}\left[\frac{t^2}{2} + \tan^{-1} t - \int \frac{t^2}{2\left(t^2+1\right)}dt\right] + c$$
$$= \frac{1}{6}t^2 \tan^{-1} t - \frac{1}{6}\int\left(\frac{t^2}{t^2+1}\right)dt + c$$
$$y = \frac{1}{6}t^2 \tan^{-1} t - \frac{1}{6}\int\left(1 - \frac{1}{t^2+1}\right)dt + c$$
$$= \frac{1}{6}t^2 \tan^{-1} t - \frac{1}{6}t + \frac{1}{6}\tan^{-1} t + c$$
$$y = \frac{1}{6}\left(t^2+1\right)\tan^{-1} t - \frac{1}{6}t + c$$
$$y = \frac{1}{6}\left[\left(t^2+1\right)\tan^{-1} t - t\right] + c$$

So,

$$y = \frac{1}{6}\left[\left(x^6+1\right)\tan^{-1}\left(x^3\right) - x^3\right] + c$$

Question 14

Solve the following differential equation:

$$\sin^4 x \frac{dy}{dx} = \cos x$$

Solution 14

$$\sin^4 x \frac{dy}{dx} = \cos x$$

$$dy = \frac{\cos x}{\sin^4 x} dx$$

$$\int dy = \int \frac{\cos x}{\sin^4 x} dx$$

Put $\sin x = t$

$$\cos x dx = dt$$

$$\int dy = \int \frac{dt}{t^4}$$

$$y = \frac{1}{-3t^3} + c$$

$$y = -\frac{1}{3\sin^3 x} + c$$

$$y = -\frac{1}{3} \cos ec^3 x + c$$

Question 15

Solve the following differential equation:

$$\cos x \frac{dy}{dx} - 2\cos 2x = \cos 3x$$

Solution 15

$$\cos x \frac{dy}{dx} - \cos 2x = \cos 3x$$

$$\cos x \frac{dy}{dx} = \cos 3x + \cos 2x$$

$$\frac{dy}{dx} = \frac{4\cos^3 x - 3\cos x + 2\cos^2 x - 1}{\cos x}$$

$$\frac{dy}{dx} = \frac{4\cos^3 x}{\cos x} - \frac{3\cos x}{\cos x} + \frac{2\cos^2 x}{\cos x} - \frac{1}{\cos x}$$

$$\frac{dy}{dx} = 4\cos^2 x - 3 + 2\cos x - \sec x$$

$$\frac{dy}{dx} = 4\left(\frac{\cos 2x + 1}{2}\right) - 3 + 2\cos x - \sec x$$

$$dy = (2\cos 2x + 2 - 3 + 2\cos x - \sec x) dx$$

$$\int dy = \int (2\cos 2x - 1 + 2\cos x - \sec x) dx$$

$$y = \sin 2x - x + 2\sin x - \log\left|\sec x + \tan x\right| + c$$

Question 16

Solve the following differential equation:

$$\sqrt{1 - x^4} dy = x dx$$

Solution 16

$$\sqrt{1-x^4}dy = xdx$$

$$dy = \frac{xdx}{\sqrt{1-x^4}}$$

$$\int dy = \int \frac{xdx}{\sqrt{1-x^4}}$$

Let $x^2 = t$

$$2xdx = dt$$

$$\Rightarrow \quad xdx = \frac{dt}{2}$$

$$\int dy = \int \frac{dt}{2\sqrt{1-t^2}}$$

$$y = \frac{1}{2}\sin^{-1}(t) + c$$

$$y = \frac{1}{2}\sin^{-1}(x^2) + c$$

Question 17

Solve the following differential equation:

$$\sqrt{a+x}dy + xdx = 0$$

Solution 17

$$\sqrt{a+x}dy + xdx = 0$$

$$\sqrt{a+x}dy = -xdx$$

$$dy = \frac{-x}{\sqrt{a+x}}dx$$

$$\int dy = -\int \frac{x}{\sqrt{a+x}}dx$$

Put $a + x = t^2$

$$dx = 2tdt$$

$$\int dy = -\int \left(\frac{t^2-a}{t}\right)2tdt$$

$$\int dy = 2\int (a-t^2)dt$$

$$y = 2\left(at - \frac{t^3}{3}\right) + c$$

$$y + \frac{2}{3}t^3 - 2at = c$$

$$y + \frac{2}{3}(a+x)^{\frac{3}{2}} - 2a\sqrt{a+x} = c$$

Question 18

Solve the following differential equation

$$\left(1+x^2\right)\frac{dy}{dx} - x = 2\tan^{-1}x$$

Solution 18

$$\left(1+x^2\right)\frac{dy}{dx}-x=2\tan^{-1}x$$
$$\left(1+x^2\right)\frac{dy}{dx}=2\tan^{-1}x+x$$
$$dy=\left(\frac{2\tan^{-1}x+x}{1+x^2}\right)dx$$
$$\int dy=\int\left(\frac{2\tan^{-1}x+x}{1+x^2}\right)dx$$
$$y=\int(2t+\tan t)dt \qquad \left[\tan^{-1}x=t\right]$$
$$=\frac{1}{2}\log\left|1+x^2\right|+\left(\tan^{-1}x\right)^2+c$$

Question 19

Solve the following differential equation:

$$\frac{dy}{dx}=x\log x$$

Solution 19

$$\frac{dy}{dx} = x \log x$$
$$dy = x \log x dx$$
$$\int dy = \int x \log x dx$$
$$y = \log|x| \int x dx - \int \left(\frac{1}{x} \int x dx\right) dx + c$$

Using integration by parts

$$= \frac{x^2}{2} \log|x| - \int \frac{x^2}{2x} dx + c$$
$$= \frac{x^2}{2} \log|x| - \frac{1}{2} \int x dx + c$$
$$y = \frac{x^2}{2} \log|x| - \frac{x^2}{4} + c$$

Question 20

Solve the following differential equation:

$$\frac{dy}{dx} = xe^x - \frac{5}{2} + \cos^2 x$$

Solution 20

$$\frac{dy}{dx} = xe^x - \frac{5}{2} + \cos^2 x$$
$$dy = \left(xe^x - \frac{5}{2} + \cos^2 x\right) dx$$
$$\int dy = \int xe^x dx - \frac{5}{2} \int dx + \int \cos^2 x dx$$
$$\int dy = \int xe^x dx - \frac{5}{2} \int dx + \int \left(\frac{1 + \cos 2x}{2}\right) dx$$
$$= \int xe^x - \frac{5}{2} \int dx + \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x dx$$
$$\int dy = \int xe^x - 2 \int dx + \frac{1}{2} \int \cos 2x dx$$
$$y = \left[x \times \int e^x dx - \int \left(1 \times \int e^x dx\right) dx\right] - 2x + \frac{1}{2} \frac{\sin 2x}{2} + c$$

Using integration by parts

$$y = xe^x - e^x - 2x + \frac{1}{4} \sin 2x + c$$
$$y = xe^x - e^x - 2x + \frac{1}{4} \sin 2x + c$$

Question 21

Find the solution of the given differential equation

$$\left(x^3 + x^2 + x + 1\right) \frac{dy}{dx} = 2x^2 + x; \ y = 1 \text{ when } x = 0$$

Solution 21

The given differential equation is:

$$\left(x^3+x^2+x+1\right)\frac{dy}{dx}=2x^2+x$$

$$\Rightarrow \frac{dy}{dx}=\frac{2x^2+x}{\left(x^3+x^2+x+1\right)}$$

$$\Rightarrow dy=\frac{2x^2+x}{(x+1)\left(x^2+1\right)}dx$$

Integrating both sides, we get:

$$\int dy=\int\frac{2x^2+x}{(x+1)\left(x^2+1\right)}dx \qquad ...(1)$$

$$\text{Let } \frac{2x^2+x}{(x+1)\left(x^2+1\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}. \qquad ...(2)$$

$$\Rightarrow \frac{2x^2+x}{(x+1)\left(x^2+1\right)}=\frac{Ax^2+A+(Bx+C)(x+1)}{(x+1)\left(x^2+1\right)}$$

$$\Rightarrow 2x^2+x=Ax^2+A+Bx^2+Bx+Cx+C$$

$$\Rightarrow 2x^2+x=(A+B)x^2+(B+C)x+(A+C)$$

Comparing the coefficients of x^2 and x, we get:

$A+B=2$

$B+C=1$

$A+C=0$

Solving these equations, we get:

$$A=\frac{1}{2},\ B=\frac{3}{2} \text{ and } C=\frac{-1}{2}$$

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{2x^2+x}{(x+1)\left(x^2+1\right)}=\frac{1}{2}\cdot\frac{1}{(x+1)}+\frac{1}{2}\frac{(3x-1)}{\left(x^2+1\right)}$$

Therefore, equation (1) becomes:

$$\int dy = \frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{3x-1}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{2}\int \frac{x}{x^2+1}dx - \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\cdot\int \frac{2x}{x^2+1}dx - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}\left[2\log(x+1) + 3\log(x^2+1)\right] - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}\left[(x+1)^2(x^2+1)^3\right] - \frac{1}{2}\tan^{-1}x + C \qquad ...(3)$$

Now, $y = 1$ when $x = 0$.

$$\Rightarrow 1 = \frac{1}{4}\log(1) - \frac{1}{2}\tan^{-1}0 + C$$

$$\Rightarrow 1 = \frac{1}{4}\times 0 - \frac{1}{2}\times 0 + C$$

$$\Rightarrow C = 1$$

Substituting $C = 1$ in equation (3), we get:

$$y = \frac{1}{4}\left[\log(x+1)^2(x^2+1)^3\right] - \frac{1}{2}\tan^{-1}x + 1$$

Question 22

Solve the following initial value problem:

$$sin\left(\frac{dy}{dx}\right) = k;\ y(0) = 1$$

Solution 22

$$\sin\left(\frac{dy}{dx}\right) = k\,,\quad y(0) = 1$$

$$\frac{dy}{dx} = \sin^{-1} k$$

$$dy = \sin^{-1} k dx$$

$$\int dy = \int \sin^{-1} k dx$$

$$y = x \sin^{-1} k + c \qquad \text{---(i)}$$

Put $x = 0, y = 1$

$$1 = 0 + c$$

$$1 = c$$

Put $c = 1$ in equation (i),

$$y = x \sin^{-1} k + 1$$

$$y - 1 = x \sin^{-1} k$$

Question 23

Solve the following initial value problem:

$$e^{\frac{dy}{dx}} = x + 1; y(0) = 3$$

Solution 23

$$e^{\frac{dy}{dx}} = x + 1, \quad y(0) = 3$$

$$\frac{dy}{dx} = \log(x + 1)$$

$$dy = \log(x + 1)\,dx$$

$$\int dy = \int \log(x + 1)dx$$

$$y = \log(x + 1)\int 1 \times dx - \int\left(\frac{1}{x + 1} \times \int 1 \times dx\right)dx + c$$

Using integration by parts

$$y = x \log(x + 1) - \int\left(\frac{x}{x + 1}\right)dx + c$$

$$= x \log(x + 1) - \int\left(1 - \frac{1}{x + 1}\right)dx + c$$

$$= x \log(x + 1) - x + \log(x + 1) + c$$

$$y = (x + 1)\log(x + 1) - x + c \qquad \text{---(i)}$$

Put $y = 3, x = 0$

$$3 = 0 + c$$

$\Rightarrow$ $c = 3$

Using equation (i),

$$y = (x + 1)\log(x + 1) - x + 3$$

Question 24

Solve the following initial value problem:

$C'(x) = 2 + 0.15x; C(0) = 100$

Solution 24

$$c'(x) = 2 + 0.15x, \qquad c(0) = 100$$

$$c'(x)dx = (2 + 0.15x)dx$$

$$\int c'(x)dx = \int 2dx + 0.15\int xdx$$

$$c(x) = 2x + 0.15\frac{x^2}{2} + c \qquad \text{---(i)}$$

Put $x = 0, c(x) = 100$

$$100 = 2(0) + 0 + c$$

$$100 = c$$

Put $c = 100$ in equation (i),

$$c(x) = 2x + (0.15)\frac{x^2}{2} + 100$$

Question 25

Solve the following initial value problem:

$x\frac{dy}{dx} + 1 = 0;\ y(-1) = 0$

Solution 25

$$x\frac{dy}{dx} + 1 = 0, \quad y(-1) = 0$$

$$x\frac{dy}{dx} = -1$$

$$dy = -\frac{dx}{x}$$

$$\int dy = -\int\frac{dx}{x}$$

$$y = -\log|x| + c \qquad \text{---(i)}$$

Put $x = -1$ and $y = 0$

$$0 = 0 + c$$

$$c = 0$$

Put $c = 0$ in equation (i),

$$y = -\log|x|, x < 0$$

Question 26

solve the following differential equation

$x(x^2 - 1)\frac{dy}{dx} = 1,\ y(2) = 0$

Solution 26

$$x\left(x^2-1\right)\frac{dy}{dx}=1,\ y(2)=0$$

$$\frac{dy}{dx}=\frac{1}{x\left(x^2-1\right)}$$

$$dy=\frac{1}{x\left(x^2-1\right)}dx$$

$$\int dy=\int\left(\frac{1}{x\left(x^2-1\right)}\right)dx$$

$$y=\frac{1}{2}\int\frac{1}{x-1}dx-\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{x+1}dx$$

$$=\frac{1}{2}\log|x-1|-\log|x|+\frac{1}{2}\log|x+1|+c$$

Putting $x=2, y=0$, we have

$$y=\frac{1}{2}\log|x-1|-\log|x|+\frac{1}{2}\log|x+1|+c$$

$$0=\frac{1}{2}\log|2-1|-\log|2|+\frac{1}{2}\log|2+1|+c$$

$$c=\log|2|-\frac{1}{2}\log|3|$$

Putting the value of c, we have

$$y=\frac{1}{2}\log|x-1|-\log|x|+\frac{1}{2}\log|x+1|+c$$

$$=\log\frac{4}{3}\left(\frac{x^2-1}{x^2}\right)$$

Chapter 22 - Differential Equations Exercise Ex. 22.6

Question 1

Solve the following differential equation:

$$\frac{dy}{dx}+\frac{1+y^2}{y}=0$$

Solution 1

$$\frac{dy}{dx} + \frac{1+y^2}{y} = 0, \qquad y \neq 0$$

$$\frac{dy}{dx} = -\frac{1+y^2}{y}$$

$$\int \frac{y}{1+y^2} dy = -\int dx$$

$$\int \frac{2y}{1+y^2} dy = -2\int dx$$

$$\log\left|1+y^2\right| = -2x + c_1$$

$$\frac{1}{2}\log\left|1+y^2\right| + x = c$$

Question 2

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{1+y^2}{y^3}$$

Solution 2

$$\frac{dy}{dx} = \frac{1+y^2}{y^3}, \quad y \neq 0$$

$$\frac{y^3}{1+y^2} dy = dx$$

$$\int \left(y - \frac{y}{y^2+1}\right) dy = \int dx$$

$$\int y dy - \int \frac{y}{y^2+1} dy = \int dx$$

$$\int y dy - \frac{1}{2}\int \frac{2y}{y^2+1} dy = \int dx$$

$$\frac{y^2}{2} - \frac{1}{2}\log\left|y^2+1\right| = x + c$$

Question 3

Solve the following differential equation:

$$\frac{dy}{dx} = \sin^2 y$$

Solution 3

$$\frac{dy}{dx} = \sin^2 y$$

$$\frac{dy}{\sin^2 y} = dx$$

$$\int \cos ec^2 y dy = \int dx$$

$$-\cot x = x + c_1$$

$$x + \cot x = c$$

Question 4

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{1-\cos 2y}{1+\cos 2y}$$

Solution 4

$$\frac{dy}{dx} = \frac{1-\cos 2y}{1+\cos 2y}$$
$$= \frac{2\sin^2 y}{2\cos^2 y}$$
$$\frac{dy}{dx} = \tan^2 y$$
$$\frac{dy}{\tan^2 y} = dx$$
$$\int \cot^2 y dy = \int dx$$
$$\int \left(\cos ec^2 y - 1\right) dy = \int dx$$
$$-\cot y - y + c = x$$
$$c = x + y + \cot y$$

Chapter 22 - Differential Equations Exercise Ex. 22.7

Question 1

Solve the following differential equation:

$$(x-1)\frac{dy}{dx} = 2xy$$

Solution 1

$$(x-1)\frac{dy}{dx} = 2xy$$

Separating the variables,

$$\int \frac{dy}{y} = \int \frac{2x}{x-1} dx$$
$$\int \frac{dy}{y} = \int \left(2 + \frac{2}{x-1}\right) dx$$
$$\log y = 2x + 2\log|x-1| + c$$

Question 2

Solve the following differential equation:

$$\left(1+x^2\right) dy = xy\,dx$$

Solution 2

$$\left(x^2+1\right)dy = xy\,dx$$

$$\int \frac{1}{y}dy = \int \frac{x}{x^2+1}dx$$

$$\int \frac{1}{y}dy = \frac{1}{2}\int \frac{2x}{x^2+1}dx$$

$$\log y = \frac{1}{2}\log\left|x^2+1\right| + \log c$$

$$y = \sqrt{x^2+1} \times c$$

Question 3

Solve the following differential equation:

$$\frac{dy}{dx} = \left(e^x+1\right)y$$

Solution 3

$$\frac{dy}{dx} = \left(e^x+1\right)y$$

$$\int \frac{1}{y}dy = \int \left(e^x+1\right)dx$$

$$\log|y| = e^x + x + c$$

Question 4

Solve the following differential equation:

$$(x-1)\frac{dy}{dx} = 2x^3y$$

Solution 4

$$(x-1)\frac{dy}{dx} = 2x^3y$$

$$\frac{dy}{y} = \frac{2x^3}{x-1}dx$$

$$\int \frac{dy}{y} = 2\int\left(x^2+x+1+\frac{1}{x-1}\right)dx$$

$$\log|y| = 2\left(\frac{x^3}{3}+\frac{x^2}{2}+x+\log|x-1|\right)+c$$

$$\log|y| = \frac{2}{3}x^3+x^2+2x+2\log|x-1|+c$$

Question 5

Solve the following differential equation:

$$xy(y+1)dy = \left(x^2+1\right)dx$$

Solution 5

$$xy(y+1)dy = (x^2+1)dx$$

$$y(y+1)dy = \frac{x^2+1}{x}dx$$

$$\int (y^2+y)dy = \int \left(x+\frac{1}{x}\right)dx$$

$$\frac{y^3}{3}+\frac{y^2}{2} = \frac{x^2}{2}+log|x|+c$$

Question 6

Solve the following differential equation:

$$5\frac{dy}{dx} = e^x y^4$$

Solution 6

$$5\frac{dy}{dx} = e^x y^4$$

$$5\int \frac{dy}{y^4} = \int e^x dx$$

$$5\left(\frac{y^{-4+1}}{-4+1}\right) = e^x + c$$

$$-\frac{5}{3y^3} = e^x + c$$

Question 7

Solve the following differential equation:

$$x \cos y dy = (xe^x \log x + e^x)dx$$

Solution 7

$$x \cos y dy = (xe^x \log x + e^x)dx$$

$$\int \cos y dy = \int e^x \left(\log x + \frac{1}{x}\right)dx$$

$$\sin y = e^x \log x + c$$

Since, $\int (f(x)+f'(x))e^x dx = e^x f(x) + c$

Question 8

Solve the following differential equation:

$$\frac{dy}{dx} = e^{x+y} + x^2 e^y$$

Solution 8

$$\frac{dy}{dx} = e^{x+y} + x^2e^y$$
$$= e^xe^y + x^2e^y$$
$$\frac{dy}{dx} = e^y\left(e^x + x^2\right)$$
$$\int e^{-y}dy = \int\left(e^x + x^2\right)dx$$
$$-e^{-y} = e^x + \frac{x^3}{3} + c$$

Question 9

Solve the following differential equation:

$$x\frac{dy}{dx} + y = y^2$$

Solution 9

$$x\frac{dy}{dx} + y = y^2$$
$$x\frac{dy}{dx} = \left(y^2 - y\right)$$
$$\frac{1}{y^2 - y}dy = \frac{dx}{x}$$
$$\int\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = \int\frac{dx}{x}$$
$$\log|y-1| - \log|y| = \log|x| + \log|c|$$
$$\log\left|\frac{y-1}{y}\right| = |xc|$$
$$y - 1 = xyc$$

Question 10

Solve the following differential equation:

$$\left(e^y + 1\right)\cos x dx + e^y \sin x dy = 0$$

Solution 10

$$\left(e^y + 1\right)\cos x dx + e^y \sin x dy = 0$$
$$\left(e^y + 1\right)\cos x dx = -e^y \sin x dy$$
$$\int\frac{\cos x}{\sin x}dx = -\int\frac{e^y}{e^y + 1}dy$$
$$\int\cot x dx = -\int\frac{e^y}{e^y + 1}dy$$
$$\log|\sin x| = -\log\left|e^y + 1\right| + \log|c|$$
$$\sin x = \frac{c}{e^y + 1}$$
$$\sin x\left(e^y + 1\right) = c$$

Question 11

Solve the following differential equation:

$x\cos^2 y\,dx = y\cos^2 x\,dy$

Solution 11

$$x\cos^2 y\,dx = y\cos^2 x\,dy$$
$$\frac{x}{\cos^2 x}dx = \frac{y}{\cos^2 y}dy$$
$$\int x\sec^2 x\,dx = \int y\sec^2 y\,dy$$
$$x \times \int \sec^2 x - \int\left(1\times\int\sec^2 x\,dx\right)dx = y\int\sec^2 y\,dy - \int\left(1\times\int\sec^2 y\,dy\right)dy$$
$$x\tan x - \int\tan x\,dx = y\tan y - \int\tan y\,dy + c$$
$$x\tan x - \log\left|\sec x\right| = y\tan y - \log\left|\sec y\right| + c$$

Question 12

Solve the following differential equation:

$xy\,dy = (y-1)(x+1)dx$

Solution 12

$$xy\,dy = (y-1)(x+1)dx$$
$$\frac{y}{y-1}dy = \frac{x+1}{x}dx$$
$$\int\left(1+\frac{1}{y-1}\right)dy = \int\left(1+\frac{1}{x}\right)dx$$
$$y + \log\left|y-1\right| = x + \log\left|x\right| + c$$
$$y - x = \log\left|x\right| - \log\left|y-1\right| + c$$

Question 13

Solve the following differential equation:

$x\frac{dy}{dx} + \cot y = 0$

Solution 13

$$x\frac{dy}{dx} + \cot y = 0$$
$$x\frac{dy}{dx} = -\cot y$$
$$\int\tan y\,dy = -\int\frac{dx}{x}$$
$$\log\left|\sec y\right| = -\log\left|x\right| + \log\left|c\right|$$
$$\sec y = \frac{c}{x}$$
$$x\sec y = c$$
$$x = c\cos y$$

Question 14

Solve the following differential equation:

$\frac{dy}{dx} = \frac{xe^x\log x + e^x}{x\cos y}$

Solution 14

$$\frac{dy}{dx} = \frac{xe^x \log x + e^x}{x \cos y}$$

$$\int \cos y dy = \int e^x \left(\log x + \frac{1}{x}\right) dx$$

$$\sin y = e^x \log x + c$$

Since, $\int e^x \left(f(x) + f'(x)\right) dx = e^x f(x) + c$

Question 15

Solve the following differential equation:

$$\frac{dy}{dx} = e^{x+y} + e^y x^3$$

Solution 15

$$\frac{dy}{dx} = e^{x+y} + e^y x^3$$

$$\frac{dy}{dx} = e^y \left(e^x + x^3\right)$$

$$\int e^{-y} dy = \int \left(e^x + x^3\right) dx$$

$$-e^{-y} = e^x + \frac{x^4}{4} + c_1$$

$$e^x + \frac{x^4}{4} + e^{-y} = c$$

Question 16

Solve the following differential equation:

$$y\sqrt{1+x^2} + x\sqrt{1+y^2}\,\frac{dy}{dx} = 0$$

Solution 16

$$y\sqrt{1+x^2} + x\sqrt{1+y^2}\frac{dy}{dx} = 0$$

$$x\sqrt{1+y^2}\frac{dy}{dx} = -y\sqrt{1+x^2}$$

$$\int \frac{\sqrt{1+y^2}}{y}dy = -\int \frac{\sqrt{1+x^2}}{x}dx$$

$$\int \frac{y\sqrt{1+y^2}}{y^2}dy = -\int \frac{x\sqrt{1+x^2}}{x^2}dx$$

Let $1+y^2 = t^2$

$\Rightarrow$ $2ydy = 2tdt$

$1+x^2 = v^2$

$\Rightarrow$ $2xdx = 2vdv$

$$\int \frac{t \times tdt}{t^2-1} = -\int \frac{v \times vdv}{v^2-1}$$

$$\int \frac{t^2dt}{t^2-1} = -\int \frac{v^2dv}{v^2-1}$$

$$\int \left(1+\frac{1}{t^2-1}\right)dt = \int \left(1+\frac{1}{v^2-1}\right)dv$$

$$t + \frac{1}{2}log\left|\frac{t-1}{t+1}\right| = -v - log\left|\frac{v-1}{v+1}\right| + c$$

$$\sqrt{1+y^2} + \frac{1}{2}log\left|\frac{\sqrt{y^2+1}-1}{\sqrt{y^2+1}+1}\right| = -\sqrt{1+x^2} - \frac{1}{2}log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + c$$

$$\sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}log\left|\frac{\sqrt{y^2+1}-1}{\sqrt{y^2+1}+1}\right| + \frac{1}{2}log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| = c$$

Question 17

Solve the following differential equation:

$$\sqrt{1+x^2}dy + \sqrt{1+y^2}dx = 0$$

Solution 17

$$\sqrt{1+x^2}dy + \sqrt{1+y^2}dx = 0$$

$$\sqrt{1+x^2}dy = -\sqrt{1+y^2}dx$$

$$\int \frac{dy}{\sqrt{1+y^2}} = -\int \frac{dx}{\sqrt{1+x^2}}$$

$$log\left|y+\sqrt{1+y^2}\right| = -log\left|x+\sqrt{1+x^2}\right| = log|c|$$

$$\left(y+\sqrt{1+y^2}\right)\left(x+\sqrt{1+x^2}\right) = c$$

Question 18

Solve the following differential equation:

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$

Solution 18

$$\sqrt{1+x^2+y^2+x^2y^2}+xy\frac{dy}{dx}=0$$

$$\sqrt{\left(1+x^2\right)+y^2\left(1+x^2\right)}=-xy\frac{dy}{dx}$$

$$\sqrt{\left(1+x^2\right)\left(1+y^2\right)}=-xy\frac{dy}{dx}$$

$$\frac{ydy}{\sqrt{1+y^2}}=-\frac{\sqrt{1+x^2}}{x}dx$$

$$\int\frac{ydy}{\sqrt{1+y^2}}=-\int\frac{x\sqrt{1+x^2}}{x^2}dx$$

Let $\quad 1+y^2=t^2$

$\Rightarrow \quad 2ydy=2tdt$

$\quad 1+x^2=v^2$

$\Rightarrow \quad 2xdx=2vdv$

$$\int\frac{tdt}{t}=-\int\frac{v\times vdv}{v^2-1}$$

$$\int dt=-\int\frac{v^2}{v^2-1}dv$$

$$-\int dt=\int\left(1+\frac{1}{v^2-1}\right)dv$$

$$-t=v+\frac{1}{2}\log\left|\frac{v-1}{v+1}\right|+c_1$$

$$-\sqrt{1+y^2}=\sqrt{1+x^2}+\frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|+c_1$$

$$\sqrt{1+x^2}+\sqrt{1+y^2}+\frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|=c$$

Question 19

Solve the following differential equation:

$$\frac{dy}{dx}=\frac{e^x\left(\sin^2 x+\sin 2x\right)}{y\left(2\log y+1\right)}$$

Solution 19

$$\frac{dy}{dx} = \frac{e^x\left(\sin^2 x + \sin x 2x\right)}{y\left(2\log y + 1\right)}$$

$$y\left(2\log y + 1\right)dy = e^x\left(\sin^2 x + \sin 2x\right)dx$$

$$\int\left(2y\log y + y\right)dy = \int e^x\left(\sin^2 x + \sin 2x\right)dx$$

$$2\left[\log y \times \int y dy - \int\left(\frac{1}{2}\int y dy\right)dy\right] + \frac{y^2}{2} = e^x \sin^2 x + c$$

Using integration by parts and

$$\int\left(f(x) + f'(x)\right)e^x dx dy + \frac{y^2}{2} = e^x \sin^2 x + c$$

$$y^2 \log y - \frac{y^2}{2} + \frac{y^2}{2} = e^x \sin^2 x + c$$

$$y^2 \log y = e^x \sin^2 x + c$$

Question 20

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{x\left(2\log x + 1\right)}{\sin y + y\cos y}$$

Solution 20

$$\frac{dy}{dx} = \frac{x\left(2\log x + 1\right)}{\sin y + y\cos y}$$

$$\int\left(\sin y + y\cos y\right)dy = \int\left(2x\log x + x\right)dx$$

$$\int \sin y dy + \int y\cos y dy = 2\int x\log x dx + \int x dx$$

$$\int \sin y dy + \left\{y \times \left(\int \cos y dy\right) - \int\left(1 \times \int \cos y dy\right)dy\right\} = 2\left\{\log x\int x dx - \int\left(\frac{1}{x}\int x dx\right)dx\right\} + \int x dx + c$$

$$\int \sin y dy + y\sin y - \int \sin y dy = x^2 \log x - 2\int\frac{x}{2}dx + \int x dx + c$$

$$y\sin y = x^2 \log x + c$$

Question 21

Solve the following differential equation:

$$\left(1 - x^2\right)dy + xy dx = xy^2 dx$$

Solution 21

$$\left(1-x^2\right)dy + xydx = xy^2dx$$

$$\left(1-x^2\right)dy = dx\left(xy^2 - xy\right)$$

$$\left(1-x^2\right)dy = xy\left(y-1\right)dx$$

$$\int \frac{dy}{y\left(y-1\right)} = \int \frac{xdx}{1-x^2}$$

$$\int\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = \frac{1}{2}\int \frac{2x}{1-x^2}dx$$

$$\int\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = -\frac{1}{2}\int \frac{-2x}{1-x^2}dx$$

$$\log\left|y-1\right| - \log\left|y\right| = -\frac{1}{2}\log\left|1-x^2\right| + c$$

Question 22

Solve the following differential equation:

$tan\, ydx + sec^2 y\, tan\, xdy = 0$

Solution 22

$$\tan ydx + \sec^2 y \tan xdy = 0$$

$$\tan ydx = -\sec^2 y \tan xdy$$

$$-\frac{dx}{\tan x} = \frac{\sec^2 ydy}{\tan y}$$

$$-\int \cot xdx = \int \frac{\sec^2 ydy}{\tan y}$$

$$-\log\left|\sin x\right| = \log\left|\tan y\right| + \log\left|c\right|$$

$$\frac{1}{\sin x} = c\tan y$$

$$\sin x \tan y = c_1$$

Question 23

Solve the following differential equation:

$(1+x)\left(1+y^2\right)dx + (1+y)\left(1+x^2\right)dy = 0$

Solution 23

$$(1+x)\left(1+y^2\right)dx+(1+y)\left(1+x^2\right)dy=0$$
$$(1+x)\left(1+y^2\right)dx=-(1+y)\left(1+x^2\right)dy$$
$$\frac{(1+y)\,dy}{\left(1+y^2\right)}=-\frac{(1+x)}{\left(1+x^2\right)}dx$$
$$\int\left(\frac{1}{1+y^2}+\frac{y}{1+y^2}\right)dy=-\int\left[\frac{1}{1+x^2}+\frac{x}{1+x^2}\right]dx$$
$$\int\frac{1}{1+y^2}dy+\frac{1}{2}\int\frac{2y}{1+y^2}dy=-\int\frac{1}{1+x^2}dx-\frac{1}{2}\int\frac{2x}{1+x^2}dx$$
$$\tan^{-1}(y)+\frac{1}{2}\log\left|1+y^2\right|=-\tan^{-1}x-\frac{1}{2}\log\left|1+x^2\right|+c$$
$$\tan^{-1}x+\tan^{-1}y+\frac{1}{2}\log\left|\left(1+y^2\right)\left(1+x^2\right)\right|=c$$

Question 24

Solve the following differential equation:

$$tan\,y\,\frac{dy}{dx}=sin\,(x+y)+sin\,(x-y)$$

Solution 24

$$\tan y\frac{dy}{dx}=\sin(x+y)+\sin(x-y)$$
$$\tan y\frac{dy}{dx}=2\sin\left\{\frac{(x+y)+(x-y)}{2}\right\}\cos\left\{\frac{(x+y)-(x-y)}{2}\right\}$$
$$=2\sin\left(\frac{x+y+x-y}{2}\right)\cos\left(\frac{x+y-x+y}{2}\right)$$
$$\tan y\frac{dy}{dx}=2\sin x\cos y$$
$$\frac{\tan y}{\cos y}dy=2\sin x\,dx$$
$$\int\sec y\tan y\,dy=2\int\sin x\,dx$$
$$\sec y=-2\cos x+c$$
$$\sec y+2\cos x=c$$

Question 25

Solve the following differential equation:

$$cos\,x\,cos\,y\,\frac{dy}{dx}=-\,sin\,x\,sin\,y$$

Solution 25

$$\cos x \cos y \frac{dy}{dx} = -\sin x \sin y$$
$$\frac{\cos y}{\sin y} dy = -\frac{\sin x}{\cos x} dx$$
$$\int \cot y dy = -\int \tan x dx$$
$$\log \sin y = \log \cos x + \log c$$
$$\sin y = c \cos x$$

Question 26

Solve the following differential equation:

$$\frac{dy}{dx} + \frac{\cos x \sin y}{\cos y} = 0$$

Solution 26

$$\frac{dy}{dx} + \frac{\cos x \sin y}{\cos y} = 0$$

$$\frac{dy}{dx} = -\cos x \tan y$$

$$\frac{dy}{\tan y} = -\cos x dx$$

$$\int \cot y dy = -\int \cos x dx$$

$$\log|\sin y| = -\sin x + c$$

$$\sin x + \log|\sin y| = c$$

Question 27

Solve the following differential equation:

$$x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$$

Solution 27

$$x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$$
$$x\sqrt{1-y^2}dx = -y\sqrt{1-x^2}dy$$
$$\frac{ydy}{\sqrt{1-y^2}} = -\frac{xdx}{\sqrt{1-x^2}}$$
$$\frac{1}{-2}\int \frac{-2y}{\sqrt{1-y^2}}dy = \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}dx$$
$$-\frac{1}{2}2 \times \sqrt{1-y^2} = \frac{1}{2} \times 2\sqrt{1-x^2} + c_1$$
$$\sqrt{1-y^2} + \sqrt{1-x^2} = c$$

Question 28

Solve the following differential equation:

$$y\left(1+e^x\right)dy = (y+1)e^x dx$$

Solution 28

$$y\left(1+e^x\right)dy = (y+1)e^x dx$$
$$\frac{ydy}{y+1} = \frac{e^x dx}{1+e^x}$$
$$\int\left(1-\frac{1}{y+1}\right)dy = \int\left(\frac{e^x}{1+e^x}\right)dx$$
$$y - \log|y+1| = \log\left|1+e^x\right| + c$$

Question 29

Solve the following differential equation:

$$(y+xy)dx + \left(x - xy^2\right)dy = 0$$

Solution 29

$$(y + xy)dx + (x - xy^2)dy = 0$$

$$y(1 + x)dx = (xy^2 - x)dy$$

$$y(1 + x)dx = x(y^2 - 1)dy$$

$$\frac{(y^2 - 1)dy}{y} = \frac{1 + x}{x}dx$$

$$\int\left(y - \frac{1}{y}\right)dy = \int\left(\frac{1}{x} + 1\right)dx$$

$$\frac{y^2}{2} - \log|y| = \log|x| + x + c_1$$

$$\frac{y^2}{2} - x - \log|y| - \log|x| = c_1$$

$$\log|x| + x + \log|y| - \frac{y^2}{2} = c$$

Question 30

Solve the following differential equation:

$$\frac{dy}{dx} = 1 - x + y - xy$$

Solution 30

$$\frac{dy}{dx} = 1 - x + y - xy$$
$$= (1 - x) + y(1 - x)$$
$$\frac{dy}{dx} = (1 - x)(1 + y)$$
$$\int \frac{dy}{1 + y} = \int (1 - x)dx$$
$$\log|y + 1| = x - \frac{x^2}{2} + c$$

Question 31

Solve the following differential equation:

$$(y^2 + 1)dx - (x^2 + 1)dy = 0$$

Solution 31

$$(y^2 + 1)dx - (x^2 + 1)dy = 0$$
$$(y^2 + 1)dx = (x^2 + 1)dy$$
$$\int \frac{dy}{y^2 + 1} = \int \frac{dx}{x^2 + 1}$$
$$\tan^{-1} y = \tan^{-1} x + c$$

Question 32

Solve the following differential equation:

$$dy + (x + 1)(y + 1)dx = 0$$

Solution 32

$$dy + (x + 1)(y + 1)dx = 0$$
$$dy = -(x + 1)(y + 1)dx$$
$$\int \frac{dy}{y + 1} = -\int (x + 1)dx$$
$$\log|y + 1| = -\frac{x^2}{2} - x + c$$
$$\log|y + 1| + \frac{x^2}{2} + x = c$$

Question 33

Solve the following differential equation:

$$\frac{dy}{dx} = (1 + x^2)(1 + y^2)$$

Solution 33

$$\frac{dy}{dx} = \left(1 + x^2\right)\left(1 + y^2\right)$$

$$\int \frac{dy}{1 + y^2} = \int \left(1 + x^2\right) dx$$

$$\tan^{-1} y = x + \frac{x^3}{3} + c$$

$$\tan^{-1} y - x - \frac{x^3}{3} = c$$

Question 34

Solve the following differential equation:

$$(x - 1)\frac{dy}{dx} = 2x^3 y$$

Solution 34

$$(x - 1)\frac{dy}{dx} = 2x^3 y$$

$$\frac{dy}{y} = \frac{2x^3 dx}{x - 1}$$

$$\int \frac{dy}{y} = 2\int \left(x^2 + x + 1 + \frac{1}{x - 1}\right) dx$$

$$\log|y| = \log e^{\left(\frac{2}{3}x^3 + x^2 + 2x\right)} + \log|x - 1|^2 + \log|c|$$

$$y = c|x - 1|^2 e^{\left(\frac{2}{3}x^3 + x^2 + 2x\right)}$$

Question 35

Solve the following differential equation:

$$\frac{dy}{dx} = e^{x+y} + e^{-x+y}$$

Solution 35

$$\frac{dy}{dx} = e^{x+y} + e^{-x+y}$$

$$= e^{x} \times e^{y} + e^{-x} \times e^{y}$$

$$\frac{dy}{dx} = e^{y}\left(e^{x} + e^{-x}\right)$$

$$\frac{dy}{e^{y}} = \left(e^{x} + e^{-x}\right)dx$$

$$\int e^{-y}dy = \int\left(e^{x} + e^{-x}\right)dx$$

$$-e^{-y} = e^{x} - e^{-x} + c$$

$$e^{-x} - e^{-y} = e^{x} + c$$

Question 36

Solve the following differential equation:

$$\frac{dy}{dx} = \left(cos^2 x - sin^2 x\right)cos^2 y$$

Solution 36

$$\frac{dy}{dx} = \left(\cos^2 x - \sin^2 x\right)\cos^2 y$$

$$\frac{dy}{\cos^2 y} = \left(\cos^2 x - \sin^2 x\right)dx$$

$$\int \sec^2 y dy = \int \cos 2x dx$$

$$\tan y = \frac{\sin 2x}{2} + c$$

Question 37(i)

Solve the following differential equation:

$$\left(xy^2+2x\right)dx+\left(x^2y+2y\right)dy=0$$

Solution 37(i)

$$\left(xy^2+2x\right)dx+\left(x^2y+2y\right)dy=0$$

$$\left(x^2y+2y\right)dy=-\left(xy^2+2x\right)dx$$

$$y\left(x^2+2\right)dy=-x\left(y^2+2\right)dx$$

$$\frac{y}{y^2+2}dy=-\frac{x}{x^2+2}dx$$

$$\int\frac{2y}{y^2+2}dy=-\int\frac{2x}{x^2+2}dx$$

$$\log\left|y^2+2\right|=-\log\left|x^2+2\right|+\log|c|$$

$$\left|y^2+2\right|=\left|\frac{c}{x^2+2}\right|$$

$$y^2+2=\frac{c}{x^2+2}$$

Question 37(ii)

Solve the following differential equation:

$$\operatorname{cosec} x\ \log y\ \frac{dy}{dx}+x^2y^2=0$$

Solution 37(ii)

Consider the given equation

$$\operatorname{cosec} x\ \log y\ \frac{dy}{dx}+x^2y^2=0$$

$$\Rightarrow\frac{\log y\,dy}{y^2}=\frac{-x^2dx}{\operatorname{cosec} x}$$

$$\Rightarrow-\frac{\log y\,dy}{y^2}=x^2\sin x\,dx$$

Integrating on both the sides,

$$\Rightarrow-\int\frac{\log y\,dy}{y^2}=\int x^2\sin x\,dx$$

Using integration by parts on both sides

$$\Rightarrow\frac{\log y+1}{y}=-x^2\cos x+2\left(x\sin x+\cos x\right)+C$$

$$\Rightarrow\frac{\log y+1}{y}+x^2\cos x-2\left(x\sin x+\cos x\right)=C$$

Question 38(i)

Solve the following differential equation:

$$xy\frac{dy}{dx}=1+x+y+xy$$

Solution 38(i)

$$xy\frac{dy}{dx} = 1 + x + y + xy$$
$$= (1+x) + y(1+x)$$
$$xy\frac{dy}{dx} = (1+x)(1+y)$$
$$\int \frac{ydy}{y+1} = \int \frac{1+x}{x}dx$$
$$\int\left(1 - \frac{1}{y+1}\right)dy = \int\left(\frac{1}{x}+1\right)dx$$
$$y - \log|y+1| = \log|x| + x + \log|c|$$
$$y = \log|cx(y+1)| + x$$

Question 38(ii)

Solve the following differential equation:

$$y\left(1-x^2\right)\frac{dy}{dx} = x\left(1+y^2\right)$$

Solution 38(ii)

$$y\left(1-x^2\right)\frac{dy}{dx} = x\left(1+y^2\right)$$
$$\frac{ydy}{\left(1+y^2\right)} = \frac{xdx}{1-x^2}$$
$$-\int \frac{2ydy}{\left(1+y^2\right)} = \int \frac{-2x}{\left(1-x^2\right)}dx$$
$$-\log\left|1+y^2\right| = \log\left|1-x^2\right| + \log|c_1|$$
$$\log|c| = \log\left|1-x^2\right| + \log\left|1+y^2\right|$$
$$c = \left(1-x^2\right)\left(1+y^2\right)$$

Question 38(iii)

$ye^{x/y}\ dx = (xe^{x/y} + y^2)\ dy$, y ¹ 0

Solution 38(iii)

$$ye^{x/y}\ dx = (xe^{x/y} + y^2)dy$$
$$ye^{x/y}\ dx - xe^{x/y}dy = y^2dy$$
$$(ydx - xdy)e^{x/y} = y^2dy$$
$$\left(\frac{ydx - xdy}{y^2}\right)e^{x/y} = dy$$
$$e^{x/y}\ d\left(\frac{x}{y}\right) = dy$$

Integrating on both the sides we get,

$e^{x/y} = y + C$, which is the required solution.

Question 38(iv)

$(1 + y^2) \tan^{-1} x \, dx + 2y (1 + x^2) dy = 0$

Solution 38(iv)

$$(1 + y^2) \tan^{-1} x \, dx + 2y (1 + x^2) dy = 0$$

$$\left(1 + y^2\right) \tan^{-1} x \, dx = -2y\left(1 + x^2\right) dy$$

$$-\frac{\tan^{-1} x}{2\left(1 + x^2\right)} dx = \frac{y}{\left(1 + y^2\right)} dy$$

Integrating on both the sides

$$\int -\frac{\tan^{-1} x}{2\left(1 + x^2\right)} dx = \int \frac{y}{\left(1 + y^2\right)} dy$$

$$-\left(\tan^{-1} x\left(\frac{1}{2}\tan^{-1} x\right) - \int \frac{1}{\left(1 + x^2\right)}\left(\frac{1}{2}\tan^{-1} x\right) dx\right) = \frac{1}{2}\ln\left(y^2 + 1\right) + C$$

$$-\frac{1}{4}\left(\tan^{-1} x\right)^2 = \frac{1}{2}\ln\left(y^2 + 1\right) + C_1$$

$$\frac{1}{2}\left(\tan^{-1} x\right)^2 + \ln\left(y^2 + 1\right) = C$$

Question 39

Solve the following initial value problem:

$$\frac{dy}{dx} = y \tan 2x, y\left(0\right) = 2$$

Solution 39

$$\frac{dy}{dx} = y \tan 2x, \; y\left(0\right) = 2$$

$$\int \frac{dy}{y} = \int \tan 2x dx$$

$$\log|y| = \frac{1}{2}\log|\sec 2x| + \log c$$

$$y = \sqrt{\sec 2x} c \qquad \text{---(i)}$$

Put $x = 0, y = 2$

$$2 = \sqrt{\sec 0} \times c$$

$$2 = c$$

Put $c = 2$ in equation (i),

$$y = 2\sqrt{\sec 2x}$$

$$y = \frac{2}{\sqrt{\cos 2x}}$$

Question 40

Solve the following initial value problem:

$$2x\frac{dy}{dx} = 3y, y\left(1\right) = 2$$

Solution 40

$$2x\frac{dy}{dx} = 3y,\ y(1) = 2$$

$$\int\frac{2dy}{y} = \int\frac{3dx}{x}$$

$$2\log|y| = 3\log|x| + \log c$$

$$y^2 = x^3c \qquad ---(i)$$

Put $x = 1,\ y = 2$

$$4 = c$$

Put $c = 4$ in equation (i),

$$y^2 = 4x^3$$

Question 41

Solve the following initial value problem:

$$xy\frac{dy}{dx} = y + 2, y(2) = 0$$

Solution 41

$$xy\frac{dy}{dx} = y + 2,\ y(2) = 0$$

$$\frac{ydy}{y+2} = \frac{dx}{x}$$

$$\int\left(1 - \frac{2}{y+2}\right)dy = \int\frac{dx}{x}$$

$$y - 2\log|y + 2| = \log|x| + \log|c| \qquad ---(i)$$

Put $y = 0,\ x = 2$

$$0 - 2\log 2 = \log 2 + \log c$$

$$-3\log 2 = \log c$$

$$\log\left(\frac{1}{8}\right) = \log c$$

$$c = \frac{1}{8}$$

Put $c = \frac{1}{8}$ in equation (i),

$$y - 2\log|y + 2| = \log\left|\frac{x}{8}\right|$$

Question 42

Solve the following initial value problem:

$$\frac{dy}{dx} = 2e^x y^3, y(0) = \frac{1}{2}$$

Solution 42

$$\frac{dy}{dx} = 2e^x y^3,\ y(0) = \frac{1}{2}$$

$$\int \frac{dy}{y^3} = \int 2e^x dx$$

$$-\frac{1}{2y^2} = 2e^x + c \qquad \text{---(i)}$$

Put $x = 0,\ y = \frac{1}{2}$

$$-\frac{4}{2} = 2e^0 + c$$

$$-2 = 2 + c$$

$$c = -4$$

Put $c = -4$ in equation (i),

$$-\frac{1}{2y^2} = 2e^x - 4$$

$$-1 = 4e^x y^2 - 8y^2$$

$$-1 = -y^2\left(8 - 4e^x\right)$$

$$y^2\left(8 - 4e^x\right) = 1$$

Question 43

Solve the following initial value problem:

$$\frac{dr}{dt} = -rt, r(0) = r_0$$

Solution 43

$$\frac{dr}{dt} = -rt, \ \ r(0) = r_0$$

$$\int \frac{dr}{r} = -\int t dt$$

$$\log|r| = -\frac{t^2}{2} + c \qquad \text{---(i)}$$

Put $t = 0$, $r = r_0$ inequation (i),

$$\log|r_0| = 0 + c$$

$$\log|r_0| = c$$

Now,

$$\log|r| = -\frac{t^2}{2} + \log|r_0|$$

$$\frac{r}{r_0} = e^{-\frac{t^2}{2}}$$

$$r = r_0 e^{-\frac{t^2}{2}}$$

Question 44

Solve the following initial value problem:

$\frac{dy}{dx} = y \sin 2x, \ y(0) = 1$ 45.

Solution 44

$$\frac{dy}{dx} = y\sin 2x,\ \ y(0) = 1$$

$$\int \frac{dy}{y} = \int \sin 2x dx$$

$$\log|y| = -\frac{\cos 2x}{2} + c \qquad ---(i)$$

Put $y = 1,\ x = 0$

$$\log|1| = -\frac{\cos 0}{2} + c$$

$$0 = -\frac{1}{2} + c$$

$$c = \frac{1}{2}$$

So,

$$\log|y| = -\frac{\cos 2x}{2} + \frac{1}{2}$$

$$= \frac{1 - \cos 2x}{2}$$

$$\log|y| = \sin^2 x$$

$$y = e^{\sin^2 x}$$

Question 45(i)

Solve the following initial value problem:

$$\frac{dy}{dx} = y\tan x, y(0) = 1$$

Solution 45(i)

$$\frac{dy}{dx} = y\tan x,\ \ y(0) = 1$$

$$\int \frac{dy}{y} = \int \tan x dx$$

$$\log|y| = \log|\sec x| + c \qquad ---(i)$$

Put $y = 1,\ x = 0$

$$0 = \log(1) + c$$

$$c = 0$$

Put $c = 0$ in equation (i),

$$\log y = \log|\sec x|$$

$$y = \sec x \qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Question 45(ii)

Solve the following initial value problem:

$$2x\frac{dy}{dx} = 5y, y(1) = 1$$

Solution 45(ii)

$$2x\frac{dy}{dx} = 5y,\ \ y(1) = 1$$

$$\int\frac{2dy}{y} = \int\frac{5dx}{x}$$

$$2\log|y| = 5\log|x| + c \qquad ---(i)$$

Put $x = 1,\ y = 1$

$$2\log(1) = 5\log(1) + c$$

$$0 = c$$

Put $c = 0$ in equation (i),

$$2\log|y| = 5\log|x|$$

$$y^2 = |x|^5$$

$$y = |x|^{\frac{5}{2}}$$

Question 45(iii)

Solve the following initial value problem:

$$\frac{dy}{dx} = 2e^{2x}y^2, y(0) = -1$$

Solution 45(iii)

$$\frac{dy}{dx} = 2e^{2x}y^2,\ y(0) = -1$$
$$\int \frac{dy}{y^2} = \int 2e^{2x}dx$$
$$-\frac{1}{y} = \frac{2e^{2x}}{2} + c$$
$$-\frac{1}{y} = e^{2x} + c \qquad ---(i)$$

Put $y = -1,\ x = 0$

$$1 = e^0 + c$$
$$1 = 1 + c$$
$$c = 0$$

Put $c = 0$ in equation (i),

$$-\frac{1}{y} = e^{2x}$$
$$y = -e^{-2x}$$

Question 45(iv)

Solve the following initial value problem:

$$\cos y\,\frac{dy}{dx} = e^x,\ y(0) = \frac{\pi}{2}$$

Solution 45(iv)

$$\cos y\frac{dy}{dx} = e^x,\ y(0) = \frac{\pi}{2}$$

$$\int \cos y\,dy = \int e^x dx$$

$$\sin y = e^x + c \qquad \text{---(i)}$$

Put $x = 0,\ y = \frac{\pi}{2}$

$$\sin\left(\frac{\pi}{2}\right) = e^0 + c$$

$$1 = 1 + c$$

$$c = 0$$

Put $c = 0$ in equation (i),

$$\sin y = e^x$$

$$y = \sin^{-1}\left(e^x\right)$$

Question 45(v)

Solve the following initial value problem:

$$\frac{dy}{dx} = 2xy,\ y(0) = 1$$

Solution 45(v)

$$\frac{dy}{dx} = 2xy,\ y(0) = 1$$

$$\int \frac{dy}{y} = \int 2x dx$$

$$\log|y| = 2\frac{x^2}{2} + c$$

$$\log|y| = x^2 + c \qquad ---(i)$$

Put $x = 0,\ y = 1$

$$\log(1) = 0 + c$$

$$0 = 0 + c$$

$$c = 0$$

Put $c = 0$ in equation (i),

$$\log y = x^2$$

$$y = e^{x^2}$$

Question 45(vi)

Solve the following initial value problem

$$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2, y(0) = 1$$

Solution 45(vi)

$$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2, y(0) = 1$$
$$= (1 + x^2)(1 + y^2)$$
$$\int \frac{dy}{1+y^2} = \int (1 + x^2)dx$$
$$\tan^{-1} y = x + \frac{x^3}{3} + c \qquad ---(i)$$
Put $x = 0, y = 1$
$$\tan^{-1} y = x + \frac{x^3}{3} + c$$
$$c = \frac{\pi}{4}$$
Put $c = \frac{\pi}{4}$ in equation (i)
$$\tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}$$

Question 45(vii)

Solve the following initial value problem

$$xy\frac{dy}{dx} = (x+2)(y+2), y(1) = -1$$

Solution 45(vii)

$$xy\frac{dy}{dx} = (x+2)(y+2),\ y(1) = -1$$
$$\frac{ydy}{(y+2)} = \frac{(x+2)}{x}dx$$
$$\int\left(1-\frac{2}{y+2}\right)dy = \int\left(1+\frac{2}{x}\right)dx$$
$$y - x - 2\log(y+2) - 2\log x = c$$
$$Put\ x = 1,\ y = -1$$
$$-1-1-2\log(-1+2)-2\log 1 = c$$
$$\Rightarrow -2 = c$$
$$Thus,\ we\ have$$
$$y - x - 2\log(y+2) - 2\log x = -2$$

Question 45(viii)

$$\frac{dy}{dx} = 1 + x + y^2 + xy^2 \text{ when } y = 0,\ x = 0$$

Solution 45(viii)

$$\frac{dy}{dx} = 1 + x + y^2 + xy^2$$
$$\frac{dy}{dx} = (1+x)(1+y^2)$$
$$\frac{1}{(1+y^2)}dy = (1+x)dx$$

Integrating on both the sides we get

$$\int\frac{1}{(1+y^2)}dy = \int(1+x)dx$$
$$\tan^{-1}y = x + \frac{x^2}{2} + C\ ....(i)$$

Put $y = 0,\ x = 0$ then

$$\tan^{-1}0 = 0 + 0 + C$$
$$C = 0$$

From(i) we have

$$\tan^{-1}y = x + \frac{x^2}{2}$$
$$y = \tan\left(x + \frac{x^2}{2}\right)$$

Question 45(ix)

$$2(y+3) - xy\frac{dy}{dx} = 0,\ y(1) = -2$$

Solution 45(ix)

$$2(y+3)-xy\frac{dy}{dx}=0$$

$$2(y+3)=xy\frac{dy}{dx}$$

$$\frac{2}{x}dx=\frac{y}{y+3}dy$$

Integrating on both the sides we get

$$\int\frac{2}{x}dx=\int\frac{y}{y+3}dy$$

$$2\ln|x|=y+3-3\ln|y+3|+C.....(i)$$

Put x = 1 and y = -2 in eq (i)

$$2\ln|1|=-2+3-3\ln|-2+3|+C$$

$$0=1-0+C$$

$$C=-1$$

From eq (i) we have

$$2\ln|x|=y+3-3\ln|y+3|-1$$

$$\ln(|x|)^2=y+2-\ln(|y+3|)^3$$

$$\ln(|x|)^2+-\ln(|y+3|)^3=y+2$$

$$x^2(y+3)^3=e^{y+2}$$

Question 46

Solve the differential equation $x\frac{dy}{dx}+\cot y=0$, given that $y=\frac{\pi}{4}$ when $x=\sqrt{2}$.

Solution 46

$$x\frac{dy}{dx}+\cot y=0,\ y=\frac{\pi}{4}\ \text{at}\ x=\sqrt{2}$$

$$x\frac{dy}{dx}=-\cot y$$

$$\frac{dy}{\cot y}=-\frac{dx}{x}$$

$$\int \tan y\,dy=-\int\frac{dx}{x}+c$$

$$\log\left|\sec y\right|=-\log\left|x\right|+c \qquad \text{---(i)}$$

Put $x=\sqrt{2},\ y=\frac{\pi}{4}$

$$\log\left|\sec\frac{\pi}{4}\right|=-\log\left|\sqrt{2}\right|+c$$

$$\log\left|\sqrt{2}\right|=-\frac{1}{2}\log 2+c$$

$$\frac{1}{2}\log 2=-\frac{1}{2}\log 2+c$$

$$\log 2=c$$

Put c in equation (i),

$$\log\left|\sec y\right|=-\log\left|x\right|+\log 2$$

$$\sec y=\frac{2}{x}$$

$$x=\frac{2}{\sec y}$$

$$x=2\cos y$$

Question 47

Solve the differential equation $\left(1+x^2\right)\frac{dy}{dx}+\left(1+y^2\right)=0$, given that $y=1$ when $x=0$.

Solution 47

$$\left(1+x^2\right)\frac{dy}{dx}+\left(1+y^2\right)=0,\ y=1 \text{ at } x=0$$

$$\left(1+x^2\right)\frac{dy}{dx}=-\left(1+y^2\right)$$

$$\int\frac{dy}{\left(1+y^2\right)}=-\int\frac{dx}{1+x^2}$$

$$\tan^{-1}y=-\tan^{-1}x+c \qquad \text{---(i)}$$

Put $x=0,\ y=1$

$$\tan^{-1}(1)=-\tan^{-1}0+c$$

Put c in equation (1),

$$\tan^{-1}y=-\tan^{-1}x+\frac{\pi}{4}$$

$$\tan^{-1}y=\left(\frac{\pi}{4}-\tan^{-1}x\right)$$

$$y=\tan\left(\frac{\pi}{4}-\tan^{-1}x\right)$$

$$y=\frac{\tan\frac{\pi}{4}-\tan\left(\tan^{-1}x\right)}{1+\tan\frac{\pi}{4}\tan\left(\tan^{-1}x\right)}$$

$$y=\frac{1-x}{1+x}$$

$$y+xy=1-x$$

$$x+y=1-xy$$

Question 48

Solve the differential equation $\frac{dy}{dx}=\frac{2x\left(\log x+1\right)}{\sin y+y\cos y}$, given that $y=0$, when $x=1$.

Solution 48

$$\frac{dy}{dx} = \frac{2x(\log x + 1)}{\sin y + y\cos y},\ y = 0 \text{ at } x = 1$$

$$\int(\sin y + y\cos y)dy = \int 2x(\log x + 1)dx$$

$$\Rightarrow \quad \int \sin y dy + \int y\cos y dy = \int 2x\log x dx + 2\int x dx$$

$$\Rightarrow \quad -\cos y + \left[y \times \int \cos y dy - \int\left(1 \times \int \cos y dy\right)dy\right] = 2\left[\log x \int x dx - \int\left(\frac{1}{x}\int x dx\right)dx\right] + x^2 + c$$

$$\Rightarrow \quad -\cos y + y\sin y - \int \sin y dy = 2\frac{x^2}{2}\log x - 2\int\frac{x}{2}dx + x^2 + c$$

$$\Rightarrow \quad -\cos y + y\sin y + \cos y = x^2\log x - \frac{x^2}{2} + x^2 + c$$

$$y\sin y = x^2\log x + \frac{x^2}{2} + c \qquad \text{---(i)}$$

Put $y = 0,\ x = 1$

$$0 = 0 + \frac{1}{2} + c$$

$$c = -\frac{1}{2}$$

Put $c = -\frac{1}{2}$ in equation (i),

$$y\sin y = x^2\log x + \frac{x^2}{2} - \frac{1}{2}$$

$$2y\sin y = 2x^2\log x + x^2 - 1$$

Question 49

Find the particular solution of $e^{\frac{dy}{dx}} = x + 1$, given that $y = 3$ when $x = 0$.

Solution 49

$$e^{\frac{dy}{dx}} = x + 1$$

$$\frac{dy}{dx} = \log(x+1),\ y = 3 \text{ at } x = 0$$

$$\int dy = \int \log(x+1)\,dx$$

$$y = \log|x+1| \times \int 1 \times dx - \int\left(\frac{1}{x+1} \times \int 1dx\right)dx + c$$

Using integration by parts

$$y = x\log|x+1| - \int \frac{x}{x+1}dx + c$$

$$y = x\log|x+1| - \left(\int\left(1 - \frac{1}{x+1}\right)dx\right) + c$$

$$= x\log|x+1| - (x - \log|x+1|) + c$$

$$y = x\log|x+1| - x + \log|x+1| + c$$

$$y = (x+1)\log|x+1| - x + c \qquad \text{---(i)}$$

Put $y = 3$ and $x = 0$

$$3 = 0 - 0 + c$$

$$c = 3$$

Put $c = 3$ in equation (i),

$$y = (x+1)\log|x+1| - x + 3$$

Question 50

Find the solution of the differential equation $\cos y dy + \cos x \sin y dx = 0$ given that $y = \frac{\pi}{2}$ when $x = \frac{\pi}{2}$.

Solution 50

$$\cos y dy + \cos x \sin y dx = 0$$
$$\cos y dy = -\cos x \sin y dx$$
$$\frac{\cos y}{\sin y} dy = -\cos x dx$$
$$\int \cot y dy = -\int \cos x dx$$
$$\log|\sin y| = -\sin x + c \qquad \text{---(i)}$$

Put $y = \frac{\pi}{2}$ and $x = \frac{\pi}{2}$

$$\log\left|\sin\frac{\pi}{2}\right| = -\sin\frac{\pi}{2} + c$$
$$0 = -1 + c$$
$$c = 1$$

Put $c = 1$ in equation (1),

$$\log|\sin y| = 1 - \sin x$$
$$\log|\sin y| + \sin x = 1$$

Question 51

Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that $y = 1$ when $x = 0$.

Solution 51

$$\frac{dy}{dx} = -4xy^2, \; y = 1 \text{ when } x = 0$$

$$\int \frac{dy}{y^2} = -4\int x dx$$

$$-\frac{1}{y} = -4\frac{x^2}{2} + c \qquad ---(\text{i})$$

Put $y = 1$ and $x = 0$

$$-1 = 0 + c$$

$$c = -1$$

Plut $c = -1$ in equation (i),

$$-\frac{1}{y} = -2x^2 - 1$$

$$\frac{1}{y} = 2x^2 + 1$$

$$y = \frac{1}{2x^2 + 1}$$

Question 52

Find the equation of a curve passing through the point (0,0) and whose differential equation is $\frac{dy}{dx} = e^x \sin x.$

Solution 52

The differential equation of the curve is:

$$y' = e^x \sin x$$
$$\Rightarrow \frac{dy}{dx} = e^x \sin x$$
$$\Rightarrow dy = e^x \sin x$$

Integrating both sides, we get:

$$\int dy = \int e^x \sin x \, dx \qquad ...(1)$$

Let $I = \int e^x \sin x \, dx.$

$$\Rightarrow I = \sin x \int e^x dx - \int\left(\frac{d}{dx}(\sin x) \cdot \int e^x dx\right) dx$$

$$\Rightarrow I = \sin x \cdot e^x - \int \cos x \cdot e^x dx$$
$$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot \int e^x dx - \int\left(\frac{d}{dx}(\cos x) \cdot \int e^x dx\right) dx\right]$$
$$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot e^x - \int(-\sin x) \cdot e^x dx\right]$$
$$\Rightarrow I = e^x \sin x - e^x \cos x - I$$
$$\Rightarrow 2I = e^x (\sin x - \cos x)$$
$$\Rightarrow I = \frac{e^x (\sin x - \cos x)}{2}$$

Substituting this value in equation (1), we get:

$$y = \frac{e^x(\sin x - \cos x)}{2} + C \qquad \ldots(2)$$

Now, the curve passes through point (0, 0).

$$\therefore 0 = \frac{e^0(\sin 0 - \cos 0)}{2} + C$$
$$\Rightarrow 0 = \frac{1(0-1)}{2} + C$$
$$\Rightarrow C = \frac{1}{2}$$

Substituting $C = \frac{1}{2}$ in equation (2), we get:

$$y = \frac{e^x(\sin x - \cos x)}{2} + \frac{1}{2}$$
$$\Rightarrow 2y = e^x(\sin x - \cos x) + 1$$
$$\Rightarrow 2y - 1 = e^x(\sin x - \cos x)$$

Hence, the required equation of the curve is $2y - 1 = e^x(\sin x - \cos x)$.

Question 53

For the differential equation $xy\frac{dy}{dx} = (x+2)(y+2)$, find the solution curve passing through the point (1, −1).

Solution 53

The differential equation of the given curve is:

$$xy\frac{dy}{dx} = (x+2)(y+2)$$
$$\Rightarrow \left(\frac{y}{y+2}\right)dy = \left(\frac{x+2}{x}\right)dx$$
$$\Rightarrow \left(1-\frac{2}{y+2}\right)dy = \left(1+\frac{2}{x}\right)dx$$

Integrating both sides, we get:

$$\int\left(1-\frac{2}{y+2}\right)dy = \int\left(1+\frac{2}{x}\right)dx$$
$$\Rightarrow \int dy - 2\int\frac{1}{y+2}dy = \int dx + 2\int\frac{1}{x}dx$$
$$\Rightarrow y - 2\log(y+2) = x + 2\log x + \mathrm{C}$$
$$\Rightarrow y - x - \mathrm{C} = \log x^2 + \log(y+2)^2$$
$$\Rightarrow y - x - \mathrm{C} = \log\left[x^2(y+2)^2\right] \qquad ...(1)$$

Now, the curve passes through point (1, −1).

$$\Rightarrow -1-1-\mathrm{C} = \log\left[(1)^2(-1+2)^2\right]$$
$$\Rightarrow -2-\mathrm{C} = \log 1 = 0$$
$$\Rightarrow \mathrm{C} = -2$$

Substituting $\mathrm{C} = -2$ in equation (1), we get:

$$y - x + 2 = \log\left[x^2(y+2)^2\right]$$

This is the required solution of the given curve.

Question 54

The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.

Solution 54

Let the rate of change of the volume of the balloon be k (where k is a constant)

$$\Rightarrow \frac{dv}{dt} = k$$
$$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \qquad \left[\text{Volume of sphere} = \frac{4}{3}\pi r^3\right]$$
$$\Rightarrow \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} = k$$
$$\Rightarrow 4\pi r^2\, dr = k\, dt$$

Integrating both sides, we get:

$$4\pi \int r^2 dr = k \int dt$$
$$\Rightarrow 4\pi \cdot \frac{r^3}{3} = kt + C$$
$$\Rightarrow 4\pi r^3 = 3(kt + C) \qquad ...(1)$$

Now, at $t = 0$, $r = 3$:

$4\pi \times 3^3 = 3\ (k \times 0 + C)$

$108\pi = 3C$

$C = 36\pi$

At $t = 3$, $r = 6$:

$4\pi \times 6^3 = 3\ (k \times 3 + C)$

$864\pi = 3\ (3k + 36\pi)$

$3k = -288\pi - 36\pi = 252\pi$

$k = 84\pi$

Substituting the values of k and C in equation (1), we get:

$$4\pi r^3 = 3\left[84\pi t + 36\pi\right]$$
$$\Rightarrow 4\pi r^3 = 4\pi\left(63t + 27\right)$$
$$\Rightarrow r^3 = 63t + 27$$
$$\Rightarrow r = \left(63t + 27\right)^{\frac{1}{3}}$$

Thus, the radius of the balloon after t seconds is $\left(63t + 27\right)^{\frac{1}{3}}$.

Question 55

In a bank, principal increases continuously at the rate of r% per year. Find the value of r if Rs 100 doubles itself in 10 years ($\log_e 2 = 0.6931$).

Solution 55

Let p, t, and r represent the principal, time, and rate of interest respectively.

It is given that the principal increases continuously at the rate of $r\%$ per year.

$$\Rightarrow \frac{dp}{dt} = \left(\frac{r}{100}\right)p$$
$$\Rightarrow \frac{dp}{p} = \left(\frac{r}{100}\right)dt$$

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{r}{100}\int dt$$
$$\Rightarrow \log p = \frac{rt}{100} + k$$
$$\Rightarrow p = e^{\frac{rt}{100}+k} \qquad ...(1)$$

It is given that when $t = 0$, $p = 100$.

$\Rightarrow 100 = e^k$ ... (2)

Now, if $t = 10$, then $p = 2 \times 100 = 200$.

$$200 = e^{\frac{r}{10}+k}$$
$$\Rightarrow 200 = e^{\frac{r}{10}} \cdot e^k$$
$$\Rightarrow 200 = e^{\frac{r}{10}} \cdot 100 \qquad \text{(From (2))}$$
$$\Rightarrow e^{\frac{r}{10}} = 2$$
$$\Rightarrow \frac{r}{10} = \log_e 2$$
$$\Rightarrow \frac{r}{10} = 0.6931$$
$$\Rightarrow r = 6.931$$

Hence, the value of r is 6.93%.

Question 56

In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years $\left(e^{0.5} = 1.648\right)$.

Solution 56

Let p and t be the principal and time respectively.

It is given that the principal increases continuously at the rate of 5% per year.

$$\Rightarrow \frac{dp}{dt} = \left(\frac{5}{100}\right)p$$
$$\Rightarrow \frac{dp}{dt} = \frac{p}{20}$$
$$\Rightarrow \frac{dp}{p} = \frac{dt}{20}$$

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{1}{20}\int dt$$
$$\Rightarrow \log p = \frac{t}{20} + C$$
$$\Rightarrow p = e^{\frac{t}{20}+C} \qquad \ldots(1)$$

Now, when $t = 0$, $p = 1000$.

$1000 = e^C$ … (2)

At $t = 10$, equation (1) becomes:

$$p = e^{\frac{1}{2}+C}$$
$$\Rightarrow p = e^{0.5} \times e^{C}$$
$$\Rightarrow p = 1.648 \times 1000$$
$$\Rightarrow p = 1648$$

Hence, after 10 years the amount will worth Rs 1648.

Question 57

In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

Solution 57

Let y be the number of bacteria at any instant t.

It is given that the rate of growth of the bacteria is proportional to the number present.

$$\therefore \frac{dy}{dt} \propto y$$
$$\Rightarrow \frac{dy}{dt} = ky \text{ (where } k \text{ is a constant)}$$
$$\Rightarrow \frac{dy}{y} = kdt$$

Integrating both sides, we get:

$$\int \frac{dy}{y} = k \int dt$$
$$\Rightarrow \log y = kt + C \qquad ...(1)$$

Let y_0 be the number of bacteria at $t = 0$.

$$\log y_0 = C$$

Substituting the value of C in equation (1), we get:

$$\log y = kt + \log y_0$$
$$\Rightarrow \log y - \log y_0 = kt$$
$$\Rightarrow \log\left(\frac{y}{y_0}\right) = kt$$
$$\Rightarrow kt = \log\left(\frac{y}{y_0}\right) \qquad ...(2)$$

Also, it is given that the number of bacteria increases by 10% in 2 hours.

$$\Rightarrow y = \frac{110}{100} y_0$$
$$\Rightarrow \frac{y}{y_0} = \frac{11}{10} \qquad ...(3)$$

Substituting this value in equation (2), we get:

$$k \cdot 2 = \log\left(\frac{11}{10}\right)$$

$$\Rightarrow k = \frac{1}{2}\log\left(\frac{11}{10}\right)$$

Therefore, equation (2) becomes:

$$\frac{1}{2}\log\left(\frac{11}{10}\right)\cdot t = \log\left(\frac{y}{y_0}\right)$$

$$\Rightarrow t = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} \qquad ...(4)$$

Now, let the time when the number of bacteria increases from 100000 to 200000 be t_1.

$y = 2y_0$ at $t = t_1$

From equation (4), we get:

$$t_1 = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} = \frac{2\log 2}{\log\left(\frac{11}{10}\right)}$$

Hence, in $\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$ hours the number of bacteria increases from 100000 to 200000.

Question 58

If y(x) is a solution of the different equation $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$ and $y(0) = 1$, then find the value of $y\left(\frac{\pi}{2}\right)$.

Solution 58

Consider the given equation

$$\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$$

$$\Rightarrow \frac{dy}{(1+y)} = \frac{-\cos x dx}{(2+\sin x)}$$

Integrating both the sides,

$$\Rightarrow \int\frac{dy}{(1+y)} = \int\frac{-\cos x dx}{(2+\sin x)}$$

$$\Rightarrow \log(1+y) = -\log(2+\sin x) + \log C$$

$$\Rightarrow \log(1+y) + \log(2+\sin x) = \log C$$

$$\Rightarrow \log(1+y)(2+\sin x) = \log C$$

$$\Rightarrow (1+y)(2+\sin x) = C...(1)$$

Given that $y(0) = 1$

$$\Rightarrow (1+1)(2+\sin 0) = C$$

$$\Rightarrow C = 4$$

Substituting the value of C in equation (1), we have,

$$\Rightarrow (1+y)(2+\sin x) = 4$$

$$\Rightarrow (1+y) = \frac{4}{(2+\sin x)}$$

$$\Rightarrow y = \frac{4}{(2+\sin x)} - 1...(2)$$

We need to find the value of $y\left(\frac{\pi}{2}\right)$

Substituting the value of $x=\frac{\pi}{2}$ in equation (2), we get,

$$y = \frac{4}{\left(2+\sin\frac{\pi}{2}\right)} - 1$$

$$\Rightarrow y = \frac{4}{(2+1)} - 1$$

$$\Rightarrow y = \frac{4}{3} - 1$$

$$\Rightarrow y = \frac{1}{3}$$

**Note* : Answer given in the book is incorrect.

Chapter 22 - Differential Equations Exercise Ex. 22.8

Question 1

Solve the following differential equation:

$$\frac{dy}{dx} = (x + y + 1)^2$$

Solution 1

$$\frac{dy}{dx} = (x + y + 1)^2$$

Let $x + y + 1 = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

So,

$$\frac{dv}{dx} - 1 = v^2$$

$$\frac{dv}{dx} = v^2 + 1$$

$$\int \frac{1}{v^2 + 1} = \int dx$$

$$\tan^{-1}(v) = x + c$$

$$\tan^{-1}(x + y + 1) = x + c$$

Question 2

Solve the following differential equation:

$$\frac{dy}{dx} \times \cos(x - y) = 1$$

Solution 2

$$\frac{dy}{dx} \times \cos(x-y) = 1$$

Let $x - y = v$

$$1 - \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{dv}{dx}$$

So,

$$\left(1 - \frac{dv}{dx}\right)\cos v = 1$$

$$1 - \frac{dv}{dx} = \sec v$$

$$1 - \sec v = \frac{dv}{dx}$$

$$dx = \frac{dv}{1 - \sec v}$$

$$dx = \frac{\cos v}{1 - \cos v} dv$$

$$\int dx = \int \frac{\cos^2 \frac{v}{2} - \sin^2 \frac{v}{2}}{2\sin^2 \frac{v}{2}} dv$$

$$\int dx = \int \frac{1}{2}\cot\left(\frac{v}{2}\right) dv - \frac{1}{2} dv$$

$$2\int dx = \int \cot^2\left(\frac{v}{2}\right) dv - \int dv$$

$$2\int dx = \int \left(\cos ec^2 \frac{v}{2} - 1\right) dv - \int dv$$

$$2x = -2\cot\left(\frac{v}{2}\right) dv - v - v + c_1$$

$$2(x + v) = -2\cot\frac{v}{2} + c_1$$

$$x + x - y = -\cot\left(\frac{x-y}{2}\right) + c$$

$$c + y = \cot\left(\frac{x-y}{2}\right)$$

Question 3

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5}$$

Solution 3

$$\frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5}$$

Let $x - y = v$

$$1 - \frac{dy}{dx} = \frac{dv}{dx}$$

So,

$$1 - \frac{dv}{dx} = \frac{v+3}{2v+5}$$

$$\frac{dv}{dx} = 1 - \frac{v+3}{2v+5}$$

$$= \frac{2v+5-v-3}{2v+5}$$

$$\frac{dv}{dx} = \frac{v+2}{2v+5}$$

$$\frac{2v+5}{v+2}dv = dx$$

$$\frac{(2v+4)+1}{v+2}dv = dx$$

$$\int\left(2 + \frac{1}{v+2}\right)dv = \int dx$$

$$2v + \log|v+2| = x + c$$

$$2(x-y) + \log|x-y+2| = x + c$$

Question 4

Solve the following differential equation:

$$\frac{dy}{dx} = (x+y)^2$$

Solution 4

$$\frac{dy}{dx} = (x+y)^2$$

Let $x + y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

So,

$$\frac{dv}{dx} - 1 = v^2$$

$$\frac{dv}{dx} = 1 + v^2$$

$$\int \frac{1}{1+v^2}dv = \int dx$$

$$\tan^{-1} v = x + c$$

$$\tan^{-1}(x+y) = x + c$$

$$x + y = \tan(x+c)$$

Question 5

Solve the following differential equation:

$$(x+y)^2\frac{dy}{dx}=1$$

Solution 5

$$(x+y)^2\frac{dy}{dx}=1$$

Let $x+y=v$

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{dy}{dx}=\frac{dv}{dx}-1$$

So,

$$v^2\left(\frac{dv}{dx}-1\right)=1$$

$$\frac{dv}{dx}=\frac{1}{v^2}+1$$

$$\frac{dv}{dx}=\frac{v^2+1}{v^2}$$

$$\frac{v^2}{v^2+1}dv=dx$$

$$\int\frac{v^2+1-1}{v^2+1}dv=\int dx$$

$$\int\left(1-\frac{1}{v^2+1}\right)dv=\int dx$$

$$v-\tan^{-1}(v)=x+c$$

$$x+y-\tan^{-1}(x+y)=x+c$$

$$y-\tan^{-1}(x+y)=c$$

Question 6

Solve the following differential equation:

$$cos^2(x-2y)=1-2\frac{dy}{dx}$$

Solution 6

$$\cos^2(x-2y)=1-\frac{2dy}{dx}$$

Let $x-2y=v$

$$1-\frac{2dy}{dx}=\frac{dv}{dx}$$

So,

$$\cos^2 v=\frac{dv}{dx}$$

$$\int dx=\int\sec^2 v\,dv$$

$$x=\tan v+c$$

$$x=\tan(x-2y)+c$$

Question 7

Solve the following differential equations:

$$\frac{dy}{dx} = sec\left(x + y\right)$$

Solution 7

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{1}{\cos\left(x + y\right)}$$

Let $x + y = v$. Then,

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Putting $x + y = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ the given differential equation, we get

$$\Rightarrow \quad \frac{dv}{dx} - 1 = \frac{1}{\cos v}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1 + \cos v}{\cos v}$$

$$\Rightarrow \quad \frac{\cos v}{1 + \cos v} dv = dx$$

$$\Rightarrow \quad \frac{\cos v\left(1 - \cos v\right)}{1 - \cos^2 v} dv = dx$$

$$\Rightarrow \quad \left(\cot v \operatorname{cosec} v - \cot^2 v\right) dv = dx$$

$$\Rightarrow \quad \left(\cot v \operatorname{cosec} v - \operatorname{cosec}^2 v + 1\right) dv = dx$$

$$\Rightarrow \quad -\operatorname{cosec} v + \cot v + v = x + C$$

$$\Rightarrow \quad -\operatorname{cosec}\left(x + y\right) + \cot\left(x + y\right) + x + y = x + C$$

$$\Rightarrow \quad -\operatorname{cosec}\left(x + y\right) + \cot\left(x + y\right) + y = C$$

$$\Rightarrow \quad -\frac{1 - \cos\left(x + y\right)}{\sin\left(x + y\right)} + y = C$$

$$\Rightarrow \quad -\tan\left(\frac{x + y}{2}\right) + y = C$$

We have,

$y\left(0\right) = 0$ i.e. $y = 0$ when $x = 0$

Putting $x = 0$ and $y = 0$ in $\left(i\right)$, we get $C = 0$.

Putting $C = 0$ in $\left(i\right)$, we get

$$-\tan\left(\frac{x + y}{2}\right) + y = 0 \Rightarrow y = \tan\left(\frac{x + y}{2}\right),$$ which is the required solution.

Question 8

Solve the following differential equations:

$$\frac{dy}{dx} = tan\left(x + y\right)$$

Solution 8

$$\frac{dy}{dx} = \tan(x+y)$$

Let $x + y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} - 1 = \tan v$$

$$\frac{dv}{dx} = 1 + \tan v$$

$$\frac{1}{1+\tan v} dv = dx$$

$$\frac{\cos v}{\cos v + \sin v} dv = dx$$

$$\left(\frac{2\cos v}{\cos v + \sin v}\right) dv = 2dx$$

$$\left(\frac{\cos v + \sin v + \cos v - \sin v}{\cos v + \sin v}\right) dv = 2dx$$

$$\int dv + \int \left(\frac{\cos v - \sin v}{\cos v + \sin v}\right) dv = 2\int dx$$

$$v + \log\left|\cos v + \sin v\right| = 2x + c$$

$$x + y + \log\left|\cos(x+y) + \sin(x+y)\right| = 2x + c$$

$$y - x + \log\left|\cos(x+y) + \sin(x+y)\right| = c$$

Question 9

Solve the following differential equations:

$(x+y)(dx - dy) = dx + dy$

Solution 9

$$2v - v\frac{dv}{dx} = \frac{dv}{dx}$$

$$\Rightarrow 2v = v\frac{dv}{dx} + \frac{dv}{dx}$$

$$\Rightarrow 2v = (v+1)\frac{dv}{dx}$$

$$\Rightarrow \frac{(v+1)}{v} dv = 2dx$$

$$(x+y)(dx-dy)=dx+dy$$

$$(x+y)\left(1-\frac{dy}{dx}\right)=1+\frac{dy}{dx}$$

Let $x+y=v$

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

So,

$$v\left(1-\left(\frac{dv}{dx}-1\right)\right)=\frac{dv}{dx}$$

$$v\left(2-\frac{dv}{dx}\right)=\frac{dv}{dx}$$

$$2v-v\frac{dv}{dx}=\frac{dv}{dx}$$

$$\Rightarrow 2v=v\frac{dv}{dx}+\frac{dv}{dx}$$

$$\Rightarrow 2v=(v+1)\frac{dv}{dx}$$

$$\Rightarrow \frac{(v+1)}{v}dv=2dx$$

$$\int\left(1+\frac{1}{v}\right)dv=2\int dx$$

$$v+\log|v|=2x+c$$

$$x+y+\log|x+y|=2x+c$$

$$y-x+\log|x+y|=c$$

Question 10

Solve the following differential equations:

$$(x+y+1)\frac{dy}{dx}=1$$

Solution 10

$$(x+y+1)\frac{dy}{dx}=1$$

Let $x+y=v$

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{dy}{dx}=\frac{dv}{dx}-1$$

So,

$$(v+1)\left(\frac{dv}{dx}-1\right)=1$$

$$(v+1)\frac{dv}{dx}-(v+1)=1$$

$$(1+v)\frac{dv}{dx}=1+1+v$$

$$\frac{v+1dv}{2+v}=dx$$

$$\int\left(1-\frac{1}{v+2}\right)dv=\int dx$$

$$v-\log|v+2|=x+\log c$$

$$x+y-\log|x+y+2|=x+\log c$$

$$y=\log c|x+y+2|$$

$$e^y=c(x+y+2)$$

$$ke^y=x+y+2 \qquad [k=1/c]$$

$$x=ke^y-y-2$$

Question 11

Solve the following differential equation.

$$\frac{dy}{dx}+1=e^{x+y}$$

Solution 11

$$\frac{dy}{dx}+1=e^{x+y}$$

Let $x+y=v$

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

$\therefore$ Given differential equation becomes,

$$\frac{dv}{dx}=e^v$$

$$\frac{1}{e^v}dv=dx$$

Integrating on both the sides we get

$$-e^{-v}=x+C$$

$$\therefore -e^{-(x+y)}=x+C$$

Chapter 22 - Differential Equations Exercise Ex. 22.9

Question 1

Solve the following differential equation:

$$x^2dy + y\left(x + y\right)dx = 0$$

Solution 1

Here, $x^2dy + y\left(x + y\right)dx = 0$

$$\frac{dy}{dx} = -\frac{y\left(x + y\right)}{x^2}$$

It is homogeneous equation

Put $y = vx$

and,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So,

$$v + x\frac{dv}{dx} = -\frac{vx\left(x + vx\right)}{x^2}$$

$$v + x\frac{dv}{dx} = -v - v^2$$

$$x\frac{dv}{dx} = -2v - v^2$$

$$\int\frac{1}{v^2 + 2v}dv = -\int\frac{dx}{x}$$

$$\int\frac{1}{v^2 + 2v + 1 - 1}dv = -\int\frac{dx}{x}$$

$$\int\frac{1}{\left(v + 1\right)^2 - \left(1\right)^2}dv = -\int\frac{dx}{x}$$

$$\frac{1}{2}\log\left|\frac{v + 1 - 1}{v + 1 + 1}\right| = -\log\left|x\right| + \log\left|c\right|$$

$$\log\left|\frac{v}{v + 2}\right|^{\frac{1}{2}} = -\log\left|\frac{c}{x}\right|$$

$$\frac{v}{v + 2} = \frac{c^2}{x^2}$$

$$\frac{\frac{y}{x}}{\frac{y}{x} + 2} = \frac{c^2}{x^2}$$

$$\frac{y}{y + 2x} = \frac{c^2}{x^2}$$

$$yx^2 = \left(y + 2x\right)c^2$$

Question 2

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

Solution 2

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

It is homogeneous equation

Put $\quad y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So,

$$v + x\frac{dv}{dx} = \frac{vx - x}{vx + x}$$

$$v + x\frac{dv}{dx} = \frac{v - 1}{v + 1}$$

$$x\frac{dv}{dx} = \frac{v - 1}{v + 1} - v$$

$$= \frac{v - 1 - v^2 - v}{v + 1}$$

$$x\frac{dv}{dx} = -\frac{\left(1 + v^2\right)}{v + 1}$$

$$\int \frac{v + 1}{v^2 + 1}dv = -\int \frac{dx}{x}$$

$$\int \frac{v}{v^2 + 1}dv + \int \frac{1}{v^2 + 1}dv = -\int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{2v}{v^2 + 1}dv + \int \frac{1}{v^2 + 1}dv = -\int \frac{dx}{x}$$

$$\frac{1}{2}\text{ og}\left|v^2 + 1\right| + \tan^{-1} v = -\log|x| + \log|c|$$

$$\log\left|\frac{y^2 + x^2}{x^2}\right| + 2\tan^{-1}\left(\frac{y}{x}\right) = 2\log\left|\frac{c}{x}\right|$$

$$\log\left|x^2 + y^2\right| - 2\log|x| + 2\tan^{-1}\left(\frac{y}{x}\right) = 2\log\left|\frac{c}{x}\right|$$

$$\log\left|x^2 + y^2\right| + 2\tan^{-1}\left(\frac{y}{x}\right) = 2\log(c)$$

$$\log\left|x^2 + y^2\right| + 2\tan^{-1}\left(\frac{y}{x}\right) = k$$

Question 3

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Solution 3

Here, $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$.

It is a homogeneous equation

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So,

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2xvx}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - \frac{v}{1}$$

$$= \frac{v^2 - 1 - 2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{-1 - v^2}{2v}$$

$$\int \frac{2v}{1 + v^2} dv = -\int \frac{dx}{x}$$

$$\log\left|1 + v^2\right| = -\log|x| + \log|c|$$

$$1 + v^2 = \frac{c}{x}$$

$$1 + \frac{y^2}{x^2} = \frac{c}{x}$$

$$x^2 + y^2 = cx$$

Question 4

Solve the following differential equation:

$$x\frac{dy}{dx} = x + y$$

Solution 4

Here, $\frac{x dy}{dx} = x + y,\ x \neq 0$

$$\frac{dy}{dx} = \frac{x + y}{x}$$

It is a homogeneous equation

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So,

$$v + x\frac{dv}{dx} = \frac{x + vx}{x}$$

$$v + x\frac{dv}{dx} = 1 + v$$

$$\int dv = \int\frac{dx}{x}$$

$$v = \log|x| + c$$

$$\frac{y}{x} = \log|x| + c$$

$$y = x\log|x| + cx$$

Question 5

Solve the following differential equation:

$$\left(x^2 - y^2\right)dx - 2xy\,dy = 0$$

Solution 5

Here, $\left(x^2 - y^2\right)dx - 2xydy = 0$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

It is a homogeneous equation

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So,

$$v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{2xvx}$$

$$x\frac{dv}{dx} = \frac{1 - v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1 - v^2 - 2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1 - 3v^2}{2v}$$

$$\int \frac{2v}{1 - 3v^2}dv = \int \frac{dx}{x}$$

$$\frac{1}{-3}\int \frac{-6v}{1 - 3v^2}dv = \int \frac{dx}{x}$$

$$\int \frac{-6v}{1 - 3v^2} = -3\int \frac{dx}{x}$$

$$\log\left|1 - 3v^2\right| = -3\log|x| + \log|c|$$

$$1 - 3v^2 = \frac{c}{x^3}$$

$$x^3\left(1 - \frac{3y^2}{x^2}\right) = c$$

$$\frac{x^3\left(x^2 - 3y^2\right)}{x^2} = c$$

$$x\left(x^2 - 3y^2\right) = c$$

Question 6

Solve the following initial value problem

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

Solution 6

$$\frac{dy}{dx}=\frac{x+y}{x-y}$$

Here it is a homogeneous equation

Put $\quad y=vx$

And

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

So,

$$v+x\frac{dv}{dx}=\frac{1+v}{1-v}$$

$$x\frac{dv}{dx}=\frac{1+v}{1-v}-v$$

$$x\frac{dv}{dx}=\frac{1+v^2}{1-v}$$

$$\frac{1-v}{1+v^2}dv=\frac{dx}{x}$$

$$\int\frac{1-v}{1+v^2}dv=\int\frac{dx}{x}$$

$$\int\frac{1}{1+v^2}dv-\frac{1}{2}\int\frac{2v}{1+v^2}dv=\int\frac{dx}{x}$$

$$\tan^{-1}v-\frac{1}{2}\log\left(1+v^2\right)=\log x+c$$

$$\tan^{-1}\frac{y}{x}=\frac{1}{2}\log\left(x^2+y^2\right)+c$$

Question 7

Solve the following differential equation:

$$2xy\frac{dy}{dx}=x^2+y^2$$

Solution 7

Here, $2xy\frac{dy}{dx} = x^2 + y^2$

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

It is a homogeneous equation

Put $y = vx$

and, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2xvx}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv = \frac{dx}{x}$$

$$\int\frac{-2v}{1-v2}dv = -\int\frac{dx}{x}$$

$$\log\left|1-v^2\right| = -\log|x| + \log c$$

$$\left(1-v^2\right) = \frac{c}{x}$$

$$x\left(1-\frac{y^2}{x^2}\right) = c$$

$$\frac{x\left(x^2-y^2\right)}{x^2} = c$$

$$x^2 - y^2 = cx$$

Question 8

Solve the differential equation $x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$

Solution 8

Consider the given differential equation

$$x^2\frac{dy}{dx}=x^2-2y^2+xy$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^2-2y^2+xy}{x^2}$$

This is a homogeneous differential equation.

Substituting $y=vx$ *and* $\frac{dy}{dx}=v+x\frac{dv}{dx}$, *we have*

$$v+x\frac{dv}{dx}=\frac{x^2-2v^2\times x^2+x\times v\times x}{x^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=1-2v^2+v$$

$$\Rightarrow x\frac{dv}{dx}=1-2v^2$$

$$\Rightarrow \frac{dv}{1-2v^2}=\frac{dx}{x}$$

$$\Rightarrow \frac{dv}{v^2-\frac{1}{2}}=-2\frac{dx}{x}$$

$$\Rightarrow \int\frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2-v^2}=2\int\frac{dx}{x}$$

$$\Rightarrow \int\frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2-v^2}=2\int\frac{dx}{x}$$

$$\Rightarrow \frac{\sqrt{2}}{2}\log\left(\frac{\frac{1}{\sqrt{2}}+v}{\frac{1}{\sqrt{2}}-v}\right)=2\log x+\log C$$

$$\Rightarrow \frac{1}{\sqrt{2}}\log\left(\frac{\frac{1}{\sqrt{2}}+\frac{y}{x}}{\frac{1}{\sqrt{2}}-\frac{y}{x}}\right)=2\log x+\log C$$

$$\Rightarrow \frac{1}{\sqrt{2}}\log\left(\frac{x+y\sqrt{2}}{x-y\sqrt{2}}\right)=2\log x+\log C$$

$$\Rightarrow \frac{1}{\sqrt{2}}\log\left(\frac{x+y\sqrt{2}}{x-y\sqrt{2}}\right)=\log x^2+\log C$$

$$\Rightarrow \log\left(\frac{x+y\sqrt{2}}{x-y\sqrt{2}}\right)^{\frac{1}{\sqrt{2}}}=\log Cx^2$$

$$\Rightarrow \left(\frac{x+y\sqrt{2}}{x-y\sqrt{2}}\right)^{\frac{1}{\sqrt{2}}}=Cx^2$$

$$\Rightarrow \left(\frac{x+y\sqrt{2}}{x-y\sqrt{2}}\right)=\left(Cx^2\right)^{\sqrt{2}}$$

Question 9

Solve the following differential equation:

$$xy\frac{dy}{dx} = x^2 - y^2$$

Solution 9

Here, $xy\dfrac{dy}{dx} = x^2 - y^2$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$$

It is a homogeneous equation

Put $y = vx$

and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{xvx}$$

$$x\frac{dv}{dx} = \frac{1 - v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{1 - v^2 - v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1 - 2v^2}{v}$$

$$\frac{v}{1 - 2v^2}dv = \frac{dx}{x}$$

$$\int\frac{-4v}{1 - 2v^2}dv = -4\int\frac{dx}{x}$$

$$\log\left|1 - 2v^2\right| = -4\log|x| + \log c$$

$$\left(1 - 2\frac{y^2}{x^2}\right) = \frac{c}{x^4}$$

$$\left(\frac{x^2 - 2y^2}{x^2}\right) = \frac{c}{x^4}$$

$$x^2\left(x^2 - 2y^2\right) = c$$

Question 10

Solve the following differential equation:

$$ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y\right)dy$$

Solution 10

Here, $ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y\right)dy$

$$\frac{dx}{dy} = \frac{xe^{\frac{x}{y}} + y}{ye^{\frac{x}{y}}}$$

It is a homogeneous equation

Put $x = vy$

and $\frac{dx}{dy} = v + y\frac{dv}{dy}$

So,

$$v + y\frac{dv}{dy} = \frac{vye^{\frac{vy}{y}} + y}{ye^{\frac{vy}{y}}}$$

$$v + y\frac{dv}{dy} = \frac{ve^{v} + 1}{e^{v}}$$

$$y\frac{dv}{dy} = \frac{ve^{v} + 1}{e^{v}} - v$$

$$y\frac{dv}{dy} = \frac{ve^{v} + 1 - ve^{v}}{e^{v}}$$

$$y\frac{dv}{dy} = \frac{1}{e^{v}}$$

$$\int evdv = \int\frac{dy}{y}$$

$$e^{v} = \log|y| + c$$

$$e^{\frac{x}{y}} = \log y + c$$

Question 11

Solve the following differential equation:

$$x^2\frac{dy}{dx} = x^2 + xy + y^2$$

Solution 11

Here, $x^2\frac{dy}{dx} = x^2 + xy + y^2$

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$$

It is a homogeneous equaton

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{x^2 + xvx + v^2x^2}{x^2}$$

$$x\frac{dv}{dx} = 1 + v + v^2 - v^2$$

$$x\frac{dv}{dx} = 1 + v^2$$

$$\int\frac{dv}{1+v^2} = \int\frac{dx}{x}$$

$$\tan^{-1} v = \log|x| + c$$

$$\tan^{-1}\frac{y}{x} = \log|x| + c$$

Question 12

Solve the following differential equation:

$$\left(y^2 - 2xy\right)dx = \left(x^2 - 2xy\right)dy$$

Solution 12

Here, $\left(y^2 - 2xy\right)dx = \left(x^2 - 2xy\right)dy$

$$\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy}$$

It is a homogeneous equation

Put $\quad y = vx$

and $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - 2xvx}{x^2 - 2xvx}$$

$$v + x\frac{dv}{dx} = \frac{v^2 - 2v}{1 - 2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 2v}{1 - 2v} - v$$

$$= \frac{v^2 - 2v - v + 2v^2}{1 - 2v}$$

$$x\frac{dv}{dx} = \frac{3v^2 - 3v}{1 - 2v}$$

$$\frac{1 - 2v}{3\left(v^2 - v\right)}dv = \frac{dx}{x}$$

$$\frac{-}{3}\frac{(2v - 1)}{v^2 - v}\,dv = \frac{dx}{x}$$

$$\int\frac{2v - 1}{v^2 - v}dv = -3\int\frac{dx}{x}$$

$$\log\left|v^2 - v\right| = -3\log|x| + \log c$$

$$v^2 - v = \frac{c}{x^3}$$

$$\frac{y^2}{x^2} - \frac{y}{x} = \frac{c}{x^3}$$

$$y^2 - xy = \frac{c}{x}$$

$$x\left(y^2 - xy\right) = c$$

Question 13

Solve the following differential equation:

$$2xydx + \left(x^2 + 2y^2\right)dy = 0$$

Solution 13

Here, $2xydx + \left(x^2 + 2y^2\right)dy = 0$

$$\frac{dy}{dx} = \frac{2xy}{x^2 + 2y^2}$$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{2xvx}{x^2 + 2v^2x^2}$$

$$v + x\frac{dv}{dx} = \frac{2v}{1 + 2v^2}$$

$$x\frac{dv}{dx} = \frac{2v}{1 + 2v^2} - v$$

$$= \frac{2v - v - 2v^3}{1 + 2v^2}$$

$$x\frac{dv}{dx} = \frac{v - 2v^3}{1 + 2v^2}$$

$$\int \frac{1 + 2v^2}{v - 2v^3}dv = \int \frac{dx}{x} \qquad \text{---(i)}$$

$$\frac{1 + 2v^2}{v - 2v^3} = \frac{1 + 2v^2}{v\left(1 - 2v^2\right)}$$

$$\frac{1 + 2v^2}{v\left(1 - 2v^2\right)} = \frac{A}{v} + \frac{Bv + c}{1 - 2v^2}$$

$$\frac{1 + 2v^2}{v\left(1 - 2v^2\right)} = \frac{A\left(1 - 2v^2\right) + \left(Bv + c\right)v}{v\left(1 - 2v^2\right)}$$

$$1 + 2v^2 = A - 2Av^2 + Bv^2 + cv$$

$$1 + 2v^2 = v^2\left(-2A + B\right) + cv + A$$

Comparing the coefficients of like powers of v,

$A = 1$

$c = 0$

$-2A + B = 2$

$-2 + B = 0$

$B = 4$

$$\frac{1 + 2v^2}{v - 2v^3} = \frac{1}{v} + \frac{4v}{1 - 2v^2}$$

$$\frac{1 + 2v^2}{v - 2v^3} = \frac{1}{v} - \frac{\left(-4v\right)}{\left(1 - 2v^2\right)}$$

Now using (i),

$$\int \frac{1}{v} dv - \int \frac{-4v}{1-2v^2} dv = \int \frac{dx}{x}$$

$$\log v - \log\left|1-2v^2\right| = \log|x| + \log c$$

$$\frac{v}{1-2v^2} = xc$$

$$\frac{\frac{y}{x}}{1-2\frac{y^2}{x^2}} = xc$$

$$xy = \left(x^2 - 2y^2\right)xc$$

$$y = \left(x^2 - 2y^2\right)c$$

Question 14

Solve the following differential equation:

$$3x^2 dy = \left(3xy + y^2\right)dx$$

Solution 14

Here, $3x^2 dy = \left(3xy + y^2\right)dx$

$$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2}$$

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{3xvx + v^2x^2}{3x^2}$$

$$v + x\frac{dv}{dx} = \frac{3v + v^2}{3}$$

$$x\frac{dv}{dx} = \frac{3v + v^2}{3} - v$$

$$x\frac{dv}{dx} = \frac{3v + v^2 - 3v}{3}$$

$$x\frac{dv}{dx} = \frac{v^2}{3}$$

$$3\int \frac{1}{v^2} dv = \int \frac{dx}{x}$$

$$3\left(-\frac{1}{v}\right) = \log|x| + c$$

$$-\frac{3x}{y} = \log|x| + c$$

Question 15

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{x}{2y + x}$$

Solution 15

Here, $\dfrac{dy}{dx} = \dfrac{x}{2y + x}$

It is a homogeneous equation

Put $y = vx$

and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{x}{2vx + x}$$

$$v + x\frac{dv}{dx} = \frac{1}{2v + 1}$$

$$x\frac{dv}{dx} = \frac{1}{2v + 1} - v$$

$$x\frac{dv}{dx} = \frac{1 - 2v^2 - v}{2v + 1}$$

$$\int \frac{2v + 1}{1 - v - 2v^2}dv = \int \frac{dx}{x}$$

$$-\int \frac{2v + 1}{2v^2 + v - 1}dv = \int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{4v + 2}{2v^2 + v - 1}dv = -\int \frac{dx}{x}$$

$$\int \frac{4v + 1 + 1}{2v^2 + v - 1}dv = -2\int \frac{dx}{x}$$

$$\int \frac{4v + 1}{2v^2 + v - 1}dv + \int \frac{1}{2v^2 + v - 1}dv = -2\int \frac{dx}{x}$$

$$\int \frac{4v + 1}{2v^2 + v - 1}dv + \frac{1}{2}\int \frac{1}{v^2 + \frac{v}{2} - \frac{1}{2}}dv = -2\int \frac{dx}{x}$$

$$\int \frac{4v + 1}{2v^2 + v - 1}dv + \frac{1}{2}\int \frac{dv}{v^2 + 2v\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - \frac{1}{2}} = -2\int \frac{dx}{x}$$

$$\int \frac{4v + 1}{2v^2 + v - 1}dv + \frac{1}{2}\int \frac{dv}{\left(v + \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2} = -2\int \frac{dx}{x}$$

$$\log\left|2v^2 + v - 1\right| + \frac{1}{2} \times \frac{1}{2\left(\frac{3}{4}\right)}\log\left|\frac{v + \frac{1}{4} - \frac{3}{4}}{v + \frac{1}{4} + \frac{3}{4}}\right| = -2\log|x| + \log c$$

$$\log\left|2v^2+v-1\right|+\frac{1}{3}\log\left|\frac{v-\frac{1}{2}}{v+1}\right|=\log\frac{c}{x^2}$$

$$\log\left|2v^2+v-1\right|+\frac{1}{3}\log\left|\frac{2v-1}{2v+2}\right|=\log\frac{c}{x^2}$$

$$\log\left(2v^2+v-1\right)\left(\frac{2v-1}{2v+2}\right)^{\frac{1}{3}}=\log\frac{c}{x^2}$$

$$\frac{(2v-1)(v+1)(2v-1)^{\frac{1}{3}}}{2^{\frac{1}{3}}(v+1)^{\frac{1}{3}}}=\frac{c}{x^2}$$

$$\frac{(2v-1)^3(v+1)^3(2v-1)}{2(v+1)}=\frac{c^3}{x^6}$$

$$(2v-1)^4(v+1)^2=\frac{c_1}{x^6}$$

$$(2v-1)^2(v+1)=\frac{D}{x^3}$$

$$x^3\left(\frac{2y-x}{x}\right)^2\left(\frac{y+x}{x}\right)=D$$

$$(2y-x)^2(y+x)=D$$

Question 16

Solve the following differential equation:

$$(x+2y)dx-(2x-y)dy=0$$

Solution 16

Here, $(x+2y)dx-(2x-y)dy=0$

$$\frac{dy}{dx}=\frac{(x+2y)}{(2x-y)}$$

It is a homogeneous equation

Put $y=vx$

$$\therefore \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

So,

$$v+x\frac{dv}{dx}=\frac{x+2vx}{2x-vx}$$

$$v+x\frac{dv}{dx}=\frac{1+2v}{2-v}$$

$$x\frac{dv}{dx}=\frac{1+2v}{2-v}-\frac{v}{1}$$

$$x\frac{dv}{dx}=\frac{1+2v-2v+v^2}{2-v}$$

$$x\frac{dv}{dx}=\frac{1+v^2}{2-v}$$

$$\frac{2-v}{1+v^2}=\frac{dx}{x}$$

$$\int\frac{2-v}{1+v^2}dv=\int\frac{dx}{x}$$

$$\int\frac{2}{1+v^2}dv-\int\frac{v}{1+v^2}dv=\int\frac{dx}{x}$$

$$2\tan^{-1}v-\frac{1}{2}\log\left|1+v^2\right|=\log|x|+\log c$$

$$2\tan^{-1}v=\log xc+\log\left|1+v^2\right|^{\frac{1}{2}}$$

$$e^{2\tan^{-1}v}=\left(1+v^2\right)^{\frac{1}{2}}xc$$

$$e^{2\tan^{-1}\frac{y}{x}}=\left\{\frac{\left(y^2+x^2\right)^{\frac{1}{2}}}{x}\right\}xc$$

$$e^{2\tan^{-1}\frac{y}{x}}=\left(y^2+x^2\right)^{\frac{1}{2}}c$$

Question 17

Solve the following differential equation:

$$\frac{dy}{dx}=\frac{y}{x}-\sqrt{\frac{y^2}{x^2}-1}$$

Solution 17

Here, $\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{vx}{\mathbf{x}} - \sqrt{\frac{v^2x^2}{x^2} - 1}$$

$$v + x\frac{dv}{dx} = v - \sqrt{v^2 - 1}$$

$$x\frac{dv}{dx} = v - \sqrt{v^2 - 1} - v$$

$$x\frac{dv}{dx} = -\sqrt{v^2 - 1}$$

$$\int \frac{dv}{\sqrt{v^2 - 1}} - \int \frac{dx}{x}$$

$$\log\left|v + \sqrt{v^2 - 1}\right| = -\log|x| + \log c$$

$$\left(\frac{y}{x} + \sqrt{v^2 - 1}\right) = \frac{c}{x}$$

$$y + \sqrt{y^2 - x^2} = c$$

Question 18

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{y}{x}\left\{log\left(\frac{y}{x}\right) + 1\right\}$$

Solution 18

$$\frac{dy}{dx} = \frac{y}{x}\left\{\log\left(\frac{y}{x}\right) + 1\right\}$$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{vx}{x}\left\{\log\left(\frac{vx}{x}\right) + 1\right\}$$

$$v + x\frac{dv}{dx} = v\log v + v$$

$$x\frac{dv}{dx} = v\log v$$

$$\int\frac{1}{v\log v}dv = \int\frac{dx}{x}$$

$$\log\log v = \log|x| + \log c$$

$$\log v = xc$$

$$\log\frac{y}{x} = xc$$

$$\frac{y}{x} = e^{xc}$$

$$y = xe^{xc}$$

Question 19

Solve the following initial value problem

$$\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$$

Solution 19

$$\frac{dy}{dx}=\frac{y}{x}+\sin\left(\frac{y}{x}\right)$$

Here it is a homogeneous equation

Put $y = vx$

And

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

So,

$$v+x\frac{dv}{dx}=v+\sin v$$

$$x\frac{dv}{dx}=\sin v$$

$$\operatorname{cosec} v\,dv=\frac{dx}{x}$$

$$\int \operatorname{cosec} v\,dv=\int\frac{dx}{x}$$

$$\log\tan\frac{v}{2}=\log x+\log c$$

$$\tan\frac{v}{2}=Cx$$

$$\tan\frac{y}{2x}=Cx$$

Question 20

Solve the following differential equation:

$$y^2dx+\left(x^2-xy+y^2\right)dy=0$$

Solution 20

Here, $y^2dx + \left(x^2 - xy + y^2\right)dy = 0$

$$\frac{dy}{dx} = \frac{-y^2}{x^2 - xy + y^2}$$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2 - xvx + v^2x^2}$$

$$v + x\frac{dv}{dx} = \frac{-v^2}{1 - v + v^2}$$

$$x\frac{dv}{dx} = \frac{-v^2}{1 - v + v^2} - \frac{v}{1}$$

$$= \frac{-v^2 - v + v^2 - v^3}{1 - v + v^2}$$

$$x\frac{dv}{dx} = \frac{-v - v^3}{v^2 - v + 1}$$

$$\frac{v^2 - v + 1}{-v\left(1 + v^2\right)}dv = \frac{dx}{x}$$

$$\left(\frac{1}{1+v^2} - \frac{1}{v}\right)dv = \frac{dx}{x}$$

$$-\int \frac{1}{v}dv + \int \frac{1}{1+v^2}dv = \int \frac{dx}{x}$$

$$-\log|v| + \tan^{-1}v = \log|x| + \log c$$

$$\log\left|\frac{x}{y}\right| + \tan^{-1}\left(\frac{y}{x}\right) = \log c$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \log xc - \log\frac{x}{y}$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \log\left(\frac{xcy}{x}\right)$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \log(cy)$$

$$e^{\tan^{-1}\left(\frac{y}{x}\right)} = cy$$

Question 21

Solve the following differential equation:

$$\left[x\sqrt{x^2 + y^2} - y^2\right]dx + xydy = 0$$

Solution 21

Here, $\left[x\sqrt{x^2+y^2}-y^2\right]dx + xydy = 0$

$$\frac{dy}{dx} = \frac{\left[y^2 - x\sqrt{x^2+y^2}\right]}{xy}$$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{\left[v^2x^2 - x\sqrt{x^2+v^2x^2}\right]}{xvx}$$

$$v + x\frac{dv}{dx} = \frac{\left[v^2 - \sqrt{1+v^2}\right]}{v}$$

$$x\frac{dv}{dx} = \frac{v^2-\sqrt{1+v^2}}{v} - v$$

$$= \frac{v^2-\sqrt{1+v^2}-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{-\sqrt{1+v^2}}{v}$$

$$\int\frac{v}{\sqrt{1+v^2}}dv = -\int\frac{dx}{x}$$

$$\frac{1}{2}\int\frac{2v}{\sqrt{1+v^2}}dv = -\int\frac{dx}{x}$$

Let $1+v^2 = t$

$$2vdv = dt$$

$$\frac{1}{2}\int\frac{1}{\sqrt{t}}dt = -\int\frac{dx}{x}$$

$$\frac{1}{2}\times 2\sqrt{t} = -\log|x| + \log c$$

$$\sqrt{1+v^2} = \log\left|\frac{c}{x}\right|$$

$$\frac{\sqrt{x^2+y^2}}{x} = \log\left|\frac{c}{x}\right|$$

$$\sqrt{x^2+y^2} = x\log\left|\frac{c}{x}\right|$$

Question 22

Solve the following differential equation:

$$x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$$

Solution 22

Here, $x\dfrac{dy}{dx} = y - x\cos^2\left(\dfrac{y}{x}\right)$

$$\frac{dy}{dx} = \frac{y - x\cos^2\left(\frac{y}{x}\right)}{x}$$

It is a homogeneous equation

Put $y = vx$

and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{vx - x\cos^2\left(\frac{vx}{x}\right)}{x}$$

$$v + x\frac{dv}{dx} = v - \cos^2 v$$

$$x\frac{dv}{dx} = v - \cos^2 v - v$$

$$x\frac{dv}{dx} = -\cos^2 v$$

$$\frac{dv}{\cos^2 v} = -\frac{dx}{x}$$

$$\int \sec^2 v\,dv = -\int \frac{dx}{x}$$

$$\tan v = -\log|x| + \log c$$

$$\tan\frac{y}{x} = \log\left|\frac{c}{x}\right|$$

Question 23

Solve the following differential equation:

$$\frac{y}{x}cos\left(\frac{y}{x}\right)dx - \left\{\frac{x}{y}sin\left(\frac{y}{x}\right) + cos\left(\frac{y}{x}\right)\right\}dy = 0$$

Solution 23

Here, $\frac{y}{x}\cos\left(\frac{y}{x}\right)dx - \left\{\frac{x}{y}\sin\left(\frac{y}{x}\right)+\cos\left(\frac{y}{x}\right)\right\}dy = 0$

$$\frac{dy}{dx} = \frac{\frac{y}{x}\cos\left(\frac{y}{x}\right)}{\frac{x}{y}\sin\left(\frac{y}{x}\right)+\cos\left(\frac{y}{x}\right)}$$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{\frac{vx}{x}\cos\left(\frac{vx}{x}\right)}{\frac{x}{vx}\sin\left(\frac{vx}{x}\right)+\cos\left(\frac{vx}{x}\right)}$$

$$= \frac{v\cos v}{\frac{1}{v}\sin v + \cos v}$$

$$v + x\frac{dv}{dx} = \frac{v^2\cos v}{\sin v + v\cos v}$$

$$x\frac{dv}{dx} = \frac{v^2\cos v}{\sin v + v\cos v} - v$$

$$x\frac{dv}{dx} = \frac{v^2\cos v - v\sin v - v^2\cos v}{\sin v + v\cos v}$$

$$x\frac{dv}{dx} = \frac{-v\sin v}{\sin v + v\cos v}$$

$$\frac{\sin v + v\cos v}{v\sin v}dv = -\frac{dx}{x}$$

$$\int\left(\frac{1}{v} + \cot v\right)dv = -\log|x| + \log c$$

$$\log|v| + \log|\sin v| = \log\left|\frac{c}{x}\right|$$

$$\log|v\sin v| = \log\left|\frac{c}{x}\right|$$

$$|v\sin v| = \left|\frac{c}{x}\right|$$

$$\left|x\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)\right| = |c|$$

$$\left|y\sin\frac{y}{x}\right| = c$$

Question 24

Solve the following differential equation:

$$xy\,log\left(\frac{x}{y}\right)dx + \left\{y^2 - x^2\,log\left(\frac{x}{y}\right)\right\}dy = 0$$

Solution 24

Here, $xy\log\left(\frac{x}{y}\right)dx + \left\{y^2 - x^2\log\left(\frac{x}{y}\right)\right\}dy = 0$

$$\frac{dy}{dx} = \frac{x^2\log\left(\frac{x}{y}\right) - y^2}{xy\log\left(\frac{x}{y}\right)}$$

It is a homogeneous equation

Put $x = vy$

and $\frac{dx}{dy} = v + y\frac{dv}{dy}$

So,

$$v + y\frac{dv}{dy} = \frac{v^2y^2\log\left(\frac{vy}{y}\right) - y^2}{vy\log\left(\frac{vy}{y}\right)}$$

$$v + y\frac{dv}{dy} = \frac{v^2\log v - 1}{v\log v}$$

$$y\frac{dv}{dy} = \frac{v^2\log v - 1}{v\log v} - v$$

$$y\frac{dv}{dy} = \frac{v^2\log v - 1 - v^2\log v}{v\log v}$$

$$y\frac{dv}{dy} = \frac{-1}{v\log v}$$

$$\int v\log v\,dv = -\int\frac{dy}{y}$$

$$\log v \times \int v\,dv - \int\frac{1}{v}\times\int v\,dv\,dv = -\log|y| + \log c$$

Itegrating it by parts

$$\frac{v^2}{2}\log v\int\frac{1}{v}\times\frac{v^2}{2}dv = \log\left|\frac{c}{y}\right|$$

$$\frac{v^2}{2}\log v - \frac{1}{2}\int v\,dv = \log\left|\frac{c}{y}\right|$$

$$\frac{v^2}{2}\log v - \frac{v^2}{4} = \log\left|\frac{c}{y}\right|$$

$$\frac{v^2}{2}\left[\log v - \frac{1}{2}\right] = \log\left|\frac{c}{y}\right|$$

$$v^2\left[\log v - \frac{1}{2}\right] = 2\log\left|\frac{c}{y}\right|$$

$$\frac{x^2}{y^2}\left\{\log\frac{x}{y} - \frac{1}{2}\right\} = -\log y^2 + \log c^2$$

$$\frac{x^2}{y^2}\left[\log\frac{x}{y} - \frac{1}{2}\right] + \log y^2 = \log k$$

Question 25

Solve the following initial value problem

$$\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$$

Solution 25

$$\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$$

Here it is a homogeneous equation

Put $\quad x=vy$

And

$$\frac{dx}{dy}=v+y\frac{dv}{dy}$$

So,

$$\frac{dx}{dy}=-\frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{\left(1+e^{\frac{x}{y}}\right)}$$

$$v+y\frac{dv}{dy}=-\frac{e^{\frac{vy}{y}}\left(1-\frac{vy}{y}\right)}{\left(1+e^{\frac{vy}{y}}\right)}$$

$$=-\frac{e^{v}(1-v)}{\left(1+e^{v}\right)}$$

$$y\frac{dv}{dy}=-\frac{e^{v}(1-v)}{\left(1+e^{v}\right)}-v$$

$$=\frac{-e^{v}(1-v)-v\left(1+e^{v}\right)}{\left(1+e^{v}\right)}$$

$$\frac{\left(1+e^{v}\right)}{-e^{v}(1-v)-v\left(1+e^{v}\right)}dv=\frac{dy}{y}$$

$$x+ye^{x/y}=c$$

Question 26

Solve the following differential equation:

$$\left(x^{2}+y^{2}\right)\frac{dy}{dx}=8x^{2}-3xy+2y^{2}$$

Solution 26

Here, $\left(x^2+y^2\right)\frac{dy}{dx}=8x^2-3xy+2y^2$

$$\frac{dy}{dx}=\frac{8x^2-3xy+2y^2}{x^2+y^2}$$

It is a homogeneous equation

Put $y=vx$

and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

So,

$$v+x\frac{dv}{dx}=\frac{8x^2-3xvx+2v^2x^2}{x^2+v^2x^2}$$

$$v+x\frac{dv}{dx}=\frac{8-3v+2v^2}{1+v^2}$$

$$x\frac{dv}{dx}=\frac{8-3v+2v^2}{1+v^2}-v$$

$$=\frac{8-3v+2v^2-v-v^3}{1+v^2}$$

$$x\frac{dv}{dx}=\frac{8-4v+2v^2-v^3}{1+v^2}$$

$$\frac{1+v^2}{8-4v+2v^2-v^3}dv=\frac{dx}{x}$$

$$\frac{1+v^2}{4(2-v)+v^2(2-v)}dv=\frac{dx}{x}$$

$$\frac{1+v^2}{4(2-v)+v^2(2-v)}dv=\frac{dx}{x}$$

$$\int\frac{1+v^2}{\left(4+v^2\right)(2-v)}dv=\int\frac{dx}{x} \qquad \text{---(A)}$$

$$\frac{1+v^2}{\left(4+v^2\right)(2-v)}=\frac{Av+B}{4+v^2}+\frac{c}{2-v}$$

$$\frac{1+v^2}{\left(4+v^2\right)(2-v)}=\frac{(Av+B)(2-v)+c\left(4+v^2\right)}{\left(4+v^2\right)(2-v)}$$

$$1+v^2=2Av-Av^2+2B-Bv+4c+cv^2$$

$$1+v^2=v^2(-A+c)+v(2A-B)+2B+4c$$

Comparing the coefficients of like powers of v

$$-A+c=1 \qquad \text{---(i)}$$

$$2A-B=0$$

$$\Rightarrow \quad B=2A \qquad \text{---(ii)}$$

$$2B+4c=1 \qquad \text{---(iii)}$$

Solving equation (i), (ii) and (iii)

$$A=-\frac{3}{8}, B=-\frac{3}{4}, C=\frac{5}{8}$$

Using equation (A)

$$\int \frac{\left(-\frac{3}{8}v-\frac{3}{4}\right)}{4+v^2}dv+\frac{5}{8}\int\frac{1}{2-v}dv=\int\frac{dx}{x}$$

$$-\frac{3}{8}\int\frac{v+2}{4+v^2}dv+\frac{5}{8}\int\frac{1}{2-v}dv=\int\frac{dx}{x}$$

$$-\frac{3}{8}\int\frac{v}{4+v^2}dv-\frac{3}{8}\int\frac{1}{4+v^2}dv+\frac{5}{8}\int\frac{1}{2-v}dv=\int\frac{dx}{x}$$

$$-\frac{3}{16}\log\left|4+v^2\right|-\frac{3}{8}\tan^{-1}\frac{v}{2}-\frac{5}{8}\log|2-v|=\log|x|+loc$$

$$-\left[\log\left|4+v^2\right|^{\frac{3}{16}}+\log e^{\frac{3}{8}\tan^{-1}\left(\frac{v}{2}\right)}+\log(2-v)^{\frac{5}{8}}\right]=\log|xc|$$

$$\left(4+v^2\right)^{\frac{3}{16}}\times e^{\frac{3}{8}\tan^{-1}\left(\frac{v}{2}\right)}\times(2-v)^{\frac{5}{8}}=\frac{c}{x}$$

$$\frac{\left(4x^2+y^2\right)^{\frac{3}{16}}}{x^{\frac{3}{8}}}\times e^{\frac{3}{8}\tan^{-1}\left(\frac{y}{2x}\right)}\frac{(2x-y)^{\frac{5}{8}}}{x^{\frac{5}{8}}}=\frac{c}{x}$$

$$\left(4x^2+y^2\right)^{\frac{3}{16}}\times(2x-y)^{\frac{5}{8}}=c\,e^{\frac{-3}{8}\tan^{-1}\left(\frac{y}{2x}\right)}$$

Question 27

Solve the following differential equation:

$$\left(x^2-2xy\right)dy+\left(x^2-3xy+2y^2\right)dx=0$$

Solution 27

Here, $\left(x^2 - 2xy\right)dy + \left(x^2 - 3xy + 2y^2\right)dx = 0$

$$\frac{dy}{dx} = \frac{x^2 - 3xy + 2y^2}{2xy - x^2}$$

It is a homogeneous equation

Put $y = vx$

and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{x^2 - 3xvx + 2v^2x^2}{2xvx - x^2}$$

$$x\frac{dv}{dx} = \frac{1 - 3v + 2v^2}{2v - 1} - v$$

$$x\frac{dv}{dx} = \frac{1 - 3v + 2v^2 - 2v^2 + v}{2v - 1}$$

$$x\frac{dv}{dx} = \frac{1 - 2v}{2v - 1}$$

$$\frac{2v - 1}{1 - 2v}dv = \frac{dx}{x}$$

$$\frac{1 - 2v}{1 - 2v}dv = -\int\frac{dx}{x}$$

$$\int dv = -\int\frac{dx}{x}$$

$$v = -\log|x| + C$$

$$y/x + \log x = C$$

Question 28

Solve the following differential equation:

$$x\frac{dy}{dx} = y - \cos^2\left(\frac{y}{x}\right)$$

Solution 28

Here, $x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$

$$\frac{dy}{dx} = \frac{y - x\cos^2\left(\frac{y}{x}\right)}{x}$$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{vx - x\cos^2\left(\frac{vx}{x}\right)}{x}$$

$$v + x\frac{dv}{dx} = v - \cos^2 v$$

$$x\frac{dv}{dx} = v - \cos^2 v - v$$

$$x\frac{dv}{dx} = -\cos^2 v$$

$$\frac{dv}{\cos^2 v} = -\frac{dx}{x}$$

$$\int \sec^2 v\,dv = -\int\frac{dx}{x}$$

$$\tan v = -\log|x| + \log c$$

$$\tan\frac{y}{x} = \log\left|\frac{c}{x}\right|$$

Question 29

Solve the following differential equation:

$$x\frac{dy}{dx} - y = 2\sqrt{y^2 - x^2}$$

Solution 29

Here, $x\frac{dy}{dx} - y = 2\sqrt{y^2 - x^2}$

$$\frac{dy}{dx} = \frac{2\sqrt{y^2 - x^2} + y}{x}$$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{2\sqrt{v^2x^2 - x^2} + vx}{x}$$

$$v + x\frac{dv}{dx} = 2\sqrt{v^2 - 1} + v$$

$$x\frac{dv}{dx} = 2\sqrt{v^2 - 1}$$

$$\int \frac{dv}{\sqrt{v^2 - 1}} = 2\int \frac{dx}{x}$$

$$\log\left|v + \sqrt{v^2 - 1}\right| = 2\log|x| + \log|c|$$

$$\log\left|v + \sqrt{v^2 - 1}\right| = \log\left|cx^2\right|$$

$$v + \sqrt{v^2 - 1} = \left|cx^2\right|$$

$$\frac{y}{x} + \sqrt{\frac{y^2}{x^2} - 1} = \left|cx^2\right|$$

$$\left(y + \sqrt{y^2 - x^2}\right) = cx^3$$

Question 30

Solve the following differential equation:

$$x\cos\left(\frac{y}{x}\right) \times (ydx + xdy) = y\sin\left(\frac{y}{x}\right) \times (xdy - ydx)$$

Solution 30

Here, $x\cos\left(\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)(xdy - ydx)$

$$yx\cos\left(\frac{y}{x}\right) + x^2\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = xy\sin\left(\frac{y}{x}\right) - y^2\sin\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \left(\frac{-y^2\sin\left(\frac{y}{x}\right) - xy\cos\left(\frac{y}{x}\right)}{x^2\cos\left(\frac{y}{x}\right) - xy\sin\left(\frac{y}{x}\right)}\right)$$

$$\frac{dy}{dx} = \frac{-xy\cos\left(\frac{y}{x}\right) - y^2\sin\left(\frac{y}{x}\right)}{x^2\cos\left(\frac{y}{x}\right) - xy\sin\left(\frac{y}{x}\right)}$$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{-xvx\cos\left(\frac{vx}{x}\right) - v^2x^2\sin\left(\frac{vx}{x}\right)}{x^2\cos\left(\frac{vx}{x}\right) - xvx\sin\left(\frac{vx}{x}\right)}$$

$$x\frac{dv}{dx} = \frac{-v\cos v - v^2\sin v}{\cos v - v\sin v} - v$$

$$x\frac{dv}{dx} = \frac{-v\cos v - v^2\sin v - v\cos v + v^2\sin v}{\cos v - v\sin v}$$

$$x\frac{dv}{dx} = \frac{-2v\cos v}{\cos v - v\sin v}$$

$$\int\frac{\cos v - v\sin v}{v\cos v}dv = -2\int\frac{dx}{x}$$

$$\int\left(\frac{1}{v} - \tan v\right)dv = -2\int\frac{dx}{x}$$

$$\log|v| - \log|\sec v| = -2\log|x| + \log|c|$$

$$\frac{v}{\sec v} = \frac{c}{x^2}$$

$$v\cos v = \frac{c}{x^2}$$

$$\frac{y}{x}\cos\left(\frac{y}{x}\right) = \frac{c}{x^2}$$

$$xy = c\sec\left(\frac{y}{x}\right)$$

Question 31

Solve the following differential equation:

$$\left(x^2 + 3xy + y^2\right)dx - x^2dy = 0$$

Solution 31

Here, $\left(x^2+3xy+y^2\right)dx - x^2dy = 0$

$$\frac{dy}{dx} = \frac{x^2+3xy+y^2}{x^2}$$

It is a homogeneous equation

Put $y = vx$

and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{x^2+3xvx+v^2x^2}{x^2}$$

$$v + x\frac{dv}{dx} = 1+3v+v^2$$

$$x\frac{dv}{dx} = 1+2v+v^2$$

$$x\frac{dv}{dx} = (v+1)^2$$

$$\int\frac{1}{(v+1)^2}dv = \int\frac{dx}{x}$$

$$-\frac{1}{v+1} = \log|x| - c$$

$$\frac{x}{x+y} + \log|x| = c$$

Question 32

Solve the following differential equation:

$$(x-y)\frac{dy}{dx} = x+2y$$

Solution 32

Here, $(x-y)\frac{dy}{dx} = x+2y$

$$\frac{dy}{dx} = \frac{x+2y}{x-y}$$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{x+2vx}{x-vx}$$

$$x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

$$x\frac{dv}{dx} = \frac{1+2v-v+v^2}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v+v^2}{1-v}$$

$$\frac{1-v}{v^2+v+1}dv = \frac{dx}{x}$$

$$-\frac{v-1}{v^2+v+1}dv = \frac{dx}{x}$$

$$\frac{1}{2}\times\frac{2v-2}{v^2+v+1}dv = \frac{-dx}{x}$$

$$\int\frac{(2v+1)-3}{v^2+v+1}dv = -\int\frac{2dx}{x}$$

$$\int\frac{2v+1}{v^2+v+1}dv - \int\frac{3}{v^2+2v\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1} = -2\int\frac{dx}{x}$$

$$\int\frac{2v+1}{v^2+v+1}dv - \int\frac{3}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dv = -2\int\frac{dx}{x}$$

$$\log\left|v^2+v+1\right| - 3\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = -2\log|x| + c$$

$$\log\left|y^2+xy+x^2\right| = 2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{x\sqrt{3}}\right) + c$$

Question 33

Solve the following differential equation:

$$\left(2x^2y+y^3\right)+\left(xy^2-3x^3\right)dy = 0$$

Solution 33

$$\left(2x^2y+y^3\right)dx+\left(xy^2+3x^3\right)dy=0$$

$$\frac{dy}{dx}=\frac{2x^2y+y^3}{3x^3-xy^2}$$

It is a homogeneous equation

Put $\quad y=vx$

and $\quad \frac{dy}{dx}=v+x\frac{dv}{dx}$

So,

$$v+x\frac{dv}{dx}=\frac{2x^2vx+v^3x^3}{3x^3-xv^2x^2}$$

$$x\frac{dv}{dx}=\frac{2v+v^3}{3-v^2}-v$$

$$=\frac{2v+v^3-3v+v^3}{3-v^2}$$

$$x\frac{dv}{dx}=\frac{2v^3-v}{3-v^2}$$

$$\int\frac{3-v^2}{2v^3-v}dv=\int\frac{dx}{x} \qquad ---(\text{i})$$

$$\frac{3-v^2}{v\left(2v^2-1\right)}=\frac{A}{(v)}+\frac{Bv+c}{\left(2v^2-1\right)}$$

$$3-v^2=A\left(2v^2-1\right)+(Bv+c)(v)$$

$$=2Av^2-A+Bv^2+cv$$

$$3-v^2=(2A+B)v^2cv-A$$

Comparing the coefficient of like powers of v

$A=-3$

$C=0$

and $\quad 2A+B=-1$

$\Rightarrow \quad 2(-3)+B=-1$

$\Rightarrow \quad B=5$

So,

$$\int\frac{-3}{v}dv+\int\frac{5v}{2v^2-1}dv=\int\frac{dx}{x}$$

$$-3\int\frac{1}{v}dv+\frac{5}{4}\int\frac{4v}{2v^2-1}dv=\int\frac{dx}{x}$$

$$-3\log|v|+\frac{5}{4}\log\left|2v^2-1\right|=\log|x|+\log|c|$$

$$-12\log|v|+5\log\left|2v^2-1\right|=4\log|x|+4\log|c|$$

$$\frac{\left|2v^2 - 1\right|^5}{v^{12}} = x^4c^4$$

$$\frac{\left|2y^2 - x^2\right|^5}{x^{10}} = x^4c^4\left(\frac{y}{x}\right)^{12}$$

$$\left|2y^2 - x^2\right|^5 = x^{14}c^4\,\frac{y^{12}}{x^{12}}$$

$$x^2c^4y^{12} = \left|2y^2 - x^2\right|^5$$

Question 34

Solve the following differential equation:

$$x\frac{dy}{dx} - y + x\,sin\left(\frac{y}{x}\right) = 0$$

Solution 34

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$

$$\frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{vx - x\sin\left(\frac{vx}{x}\right)}{x}$$

$$x\frac{dv}{dx} = v - \sin v - v$$

$$\int \operatorname{cosec} v\,dv = -\int\frac{dx}{x}$$

$$\log\left|\operatorname{cosec} v + \cot v\right| = -\log\frac{c}{x}$$

$$\log\left|\operatorname{cosec} v + \cot v\right| = \log\frac{x}{c}$$

$$\operatorname{cosec}\left(\frac{y}{x}\right) + \frac{\cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)} = \frac{x}{c}$$

$$\frac{\left(1 + \cos\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)} = \frac{x}{c}$$

$$x\sin\left(\frac{y}{x}\right) = c\left(1 + \cos\frac{y}{x}\right)$$

Question 35

Solve the following differential equation:

$$ydx + \left\{x\log\left(\frac{y}{x}\right)dy - 2xdy = 0\right\}$$

Solution 35

$$ydx + \left\{x \log\left(\frac{y}{x}\right)\right\}dy - 2xdy = 0$$

$$y + x\log\left(\frac{y}{x}\right)\frac{dy}{dx} - 2x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

It is a homogeneous equation

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So,

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log\left(\frac{vx}{x}\right)}$$

$$x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

$$x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$$

$$\int \frac{\log v - 2}{v(\log v - 1)}dv = -\int \frac{dx}{x}$$

Let $\log v - 1 = t$

$$\frac{1}{v}dv = dt$$

$$\int\left(\frac{t-1}{t}\right)dt = -\int\frac{dx}{x}$$

$$t - \log|t| = \log\left|\frac{c}{x}\right|$$

$$\log v - 1\log(\log v - 1) = \log\left|\frac{c}{x}\right|$$

$$\log e^{\log v - 1} - \log|\log v - 1| = \log\left|\frac{c}{x}\right|$$

$$e^{\log\left(\frac{v}{e}\right)} = \frac{c}{x}|\log v - 1|$$

$$\frac{v}{e} = \frac{c}{x}|\log v - 1|$$

$$y = c_1\left\{\log\left|\frac{y}{x}\right| - 1\right\}$$

Question 36(i)

Solve the following initial value problem:

$$\left(x^2 + y^2\right)dx = 2xydy,\ y(1) = 0$$

Solution 36(i)

$$\left(x^2 + y^2\right)dx = 2xydy,\ y(1) = 0$$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

It is a homogenues equation

Put $\quad y = vx$

and $\quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2xvx}$$

$$x\frac{dv}{dx} = \frac{1 + v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1 + v^2 - 2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1 - v^2}{2v}$$

$$\int \frac{2v}{1 - v^2} = \int \frac{dx}{x}$$

$$\log\left|1 - v^2\right| = -\log|x| + \log|c|$$

$$\log\left|1 - v^2\right| = \log\left|\frac{c}{x}\right|$$

$$\left|\frac{x^2 - y^2}{x^2}\right| = \left|\frac{c}{x}\right|$$

$$\left|x^2 - y^2\right| = |cx| \qquad \text{---(i)}$$

Put $y = 0,\ x = 1$

$$1 - 0 = c$$

$$c = 1$$

Put the value of c in equation (i),

$$\left|x^2 - y^2\right| = |x|$$

$$\left(x^2 - y^2\right)^2 = x^2$$

Question 36(ii)

Solve the following initial value problem:

$$xe^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0,\ y(e) = 0$$

Solution 36(ii)

Here, $xe^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0,\ y(e) = 0$

$$\frac{dy}{dx} = \frac{y - xe^{\frac{y}{x}}}{x}$$

It is a homogeneous equation

Put $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{vx - xe^{\frac{vx}{x}}}{x}$$

$$x\frac{dv}{dx} = v - e^{v} - v$$

$$x\frac{dv}{dx} = -e^{v}$$

$$\int -e^{-v}dv = \int\frac{dx}{x}$$

$$e^{v} = \log|xc|$$

$$v = \log\left(\log|xc|\right)$$

$$\frac{y}{x} = \log\log|x| + k$$

$$y = x\log\left(\log|x|\right) + k \qquad \text{---(i)}$$

Put $y = 0,\ x = e$

$$0 = e\log(\log e) + k$$

$$0 = e \times 0 + k$$

$$0 = k$$

Using equation (i),

$$y = x\log\left(\log|x|\right)$$

Question 36(iii)

Solve the following initial value problem

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\frac{y}{x} = 0, y(1) = 0$$

Solution 36(iii)

$$\frac{dy}{dx}-\frac{y}{x}+\operatorname{cosec}\frac{y}{x}=0, y(1)=0$$

Here it is a homogeneous equation

Put $y=vx$

And

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

So,

$$v+x\frac{dv}{dx}=\frac{vx}{x}-\operatorname{cosec}\frac{vx}{x}$$

$$x\frac{dv}{dx}=v-\operatorname{cosec} v-v$$

$$=-\operatorname{cosec} v$$

$$\frac{dv}{\operatorname{cosec} v}=-\frac{dx}{x}$$

$$\sin v\,dv=-\frac{dx}{x}$$

$$-\cos v=-\log|x|+c$$

$$-\cos\frac{y}{x}=-\log|x|+c$$

Now putting $y=0, x=1$, we have

$$c=-1$$

Now

$$-\cos\frac{y}{x}+1=-\log|x|$$

$$\log|x|=\cos\frac{y}{x}-1$$

Question 36(iv)

Solve the following initial value problem:

$$\left(xy-y^2\right)dx-x^2dy=0, y(1)=1$$

Solution 36(iv)

$$\left(xy - y^2\right)dx - x^2dy = 0,\ \ y\left(1\right) = 1$$

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$$

It is a homogeneous equation

Put $\quad y = vx$

and $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{xvx - v^2x^2}{x^2}$$

$$x\frac{dv}{dx} = v - v^2 - v$$

$$x\frac{dv}{dx} = -v^2$$

$$-\int\frac{1}{v^2}dv = \int\frac{dx}{x}$$

$$-\left(-\frac{1}{v}\right) = \log\left|x\right| + c$$

$$\frac{x}{y} = \log\left|x\right| + c \qquad \text{---(i)}$$

Put $y = 1,\ x = 1$

$$1 = c$$

Using equation $\left(1\right)$,

$$x = y\left[\log\left|x\right| + 1\right]$$

$$y = \frac{x}{\left[\log\left|x\right| + 1\right]}$$

Question 36(v)

Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{y\left(x + 2y\right)}{x\left(2x + y\right)},\ y\left(1\right) = 2$$

Solution 36(v)

$$\frac{dy}{dx} = \frac{y(x+2y)}{x(2x+y)},\ y(1) = 2$$

It is a homogeneous equation

Put $\quad y = vx$

and $\quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{vx(x+2vx)}{x(2x+vx)}$$

$$x\frac{dv}{dx} = \frac{v(1+2v)}{(2+v)} - v$$

$$x\frac{dv}{dx} = \frac{v+2v^2-2v-v^2}{2+v}$$

$$x\frac{dv}{dx} = \frac{v^2-v}{2+v}$$

$$\frac{2+v}{v^2-v}dv = \frac{dx}{x}$$

$$\int\frac{2+v}{v^2-v}dv = \int\frac{dx}{x} \qquad \text{---(i)}$$

$$\frac{2+v}{v(v-1)} = \frac{A}{v} + \frac{B}{v-1}$$

$$\frac{2+v}{v(v-1)} = \frac{A(v-1)+Bv}{v(v-1)}$$

$$2+v = (A+B)v - A$$

Comparing the coefficients of like powers of v,

$$A = -2$$

$$A + B = 1$$

$\Rightarrow \quad -2 + B = 1$

$\Rightarrow \quad B = 3$

Using equation(i),

$$\int\frac{-2}{v}dv + 3\int\frac{1}{v-1}dv = \int\frac{dx}{x}$$

$$-2\log|v| + 3\log|v-1| = \log|cx|$$

$$|v-1|^3 = v^2cx$$

$$\frac{|y-x|^3}{x^3} = \frac{y^2}{x^2}cx$$

$$|y-x|^3 = y^2cx^2 \qquad \text{---(ii)}$$

Put $y = 2,\ x = 1$

$1 = 4c$

$\Rightarrow \quad c = \dfrac{1}{4}$

Using equation (ii),

$$4|y-x|^3 = y^2x^2$$

$$2|y-x|^{\frac{3}{2}} = xy$$

Question 36(vi)

Solve the following initial value problem:

$$\left(y^4 - 2x^3y\right)dx + \left(x^4 - 2xy^3\right)dy = 0,\ y(1) = 1$$

Solution 36(vi)

$$\left(y^4 - 2x^3y\right)dx + \left(x^4 - 2xy^3\right)dy = 0$$

$$\frac{dy}{dx} = \frac{2x^3y - y^4}{x^4 - 2xy^3}$$

It is a homogeneous equation

Put $\quad y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So,

$$v + x\frac{dv}{dx} = \frac{2x^3vx - x^4v^4}{x^4 - 2xv^3x^3}$$

$$x\frac{dv}{dx} = \frac{2v - v^4}{1 - 2v^3} - v$$

$$x\frac{dv}{dx} = \frac{2v - v^4 - v + 2v^4}{1 - 2v^3}$$

$$x\frac{dv}{dx} = \frac{v^4 + v}{1 - 2v^3}$$

$$\int \frac{1 - 2v^3}{v\left(v^3 + 1\right)}dv = \int \frac{dx}{x} \qquad \text{---(i)}$$

$$\frac{1 - 2v^3}{v\left(v + 1\right)\left(v^2 - v + 1\right)} = \frac{A}{v} + \frac{B}{V + 1} + \frac{cv + D}{v^2 - v + 1}$$

$$1 - 2v^3 = A\left(v^3 + 1\right) + Bv\left(v^2 - v + 1\right) + \left(cv + D\right)\left(v^2 + v\right)$$

$$= Av^3 + A + cv^3 - Bv^2 + cv + cv^3 + cv^2 + Dv^2 + Dv$$

$$1 - 2v^3 = v^3\left(A + B + C\right) + v^2\left(-B + C + D\right) + v\left(B + D\right) + A$$

Comparing the coefficients of like powers of v

$$A = 1 \qquad \text{---(ii)}$$

$$B + D = 0 \qquad \text{---(iii)}$$

$$-B + C + D = 0 \qquad \text{---(iv)}$$

$$A + B + C = -2 \qquad \text{---(v)}$$

Solution of equation $\left(\text{ii}\right), \left(\text{iii}\right), \left(\text{iv}\right), \left(\text{v}\right)$ gives

$$A = 1,\ b = -1,\ c = -2,\ x = 1$$

Using equation $\left(\text{i}\right)$,

$$\int \frac{1}{v}dv - \int \frac{1}{v + 1}dv - \int \frac{2v - 1}{v^2 - v + 1}dv = \int \frac{dx}{x}$$

$$\log\left|v\right| - \log\left|v + 1\right| - \log\left|v^2 - v + 1\right| = \log\left|xc\right|$$

$$\log\left|\frac{v}{v^3 + 1}\right| = \log\left|xc\right|$$

$$|v| = \left|\left(v^3 + 1\right)(xc)\right|$$

$$\left|\frac{y}{x}\right| = \left|\frac{\left(y^3 + x^3\right)}{x^3}(xc)\right|$$

$$|dy| = \left|\left(x^3 + y^3\right)c\right|$$

Put $\quad y = 1,\ x = 1$

$$1 = 2c$$

$\Rightarrow \quad c = \frac{1}{2}$

So,

$$|xy| = \left|\left(x^3 + y^3\right)\frac{1}{2}\right|$$

$$4x^2y^2 = \left(x^3 + y^3\right)^2$$

Question 36(vii)

Solve the following initial value problem:

$$x\left(x^2 + 3y^2\right)dx + y\left(y^2 + 3x^2\right)dy = 0,\ y(1) = 1$$

Solution 36(vii)

Here, $x\left(x^2+3y^2\right)dx+y\left(y^2+3x^2\right)dy=0,\ y(1)=1$

$$\frac{dy}{dx}=-\frac{x\left(x^2+3y^2\right)}{y\left(y^2+3x^2\right)}$$

It is a homogeneous equation

Put $y=vx$

and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

So,

$$v+x\frac{dv}{dx}=-\frac{x\left(x^2+3v^2x^2\right)}{vx\left(v^2x^2+3x^2\right)}$$

$$x\frac{dv}{dx}=-\frac{\left(1+3v^2\right)}{v\left(v^2+3\right)}-v$$

$$x\frac{dv}{dx}=\frac{-1-3v^2-v^4-3v^2}{v\left(v^2+3\right)}$$

$$=\frac{-v^4-6v^2-1}{v\left(v^2+3\right)}$$

$$\frac{v\left(v^2+3\right)}{v^4+6v^2+1}dc=-\frac{dx}{x}$$

$$\int\frac{4v^3+12v}{v^4+6v^2+1}dv=-4\int\frac{dx}{x}$$

$$\log\left|v^4+6v^2+1\right|=\log\left|\frac{c}{x^4}\right|$$

$$\left|v^4+6v^2+1\right|=\left|\frac{c}{x^4}\right|$$

$$\left|y^4+6y^2x^2+x^4\right|=|c| \qquad \text{---(i)}$$

Put $y=1,\ x=1$

$(1+6+1)=c$

$\Rightarrow$ $c=8$

Put $c=8$ in equation (i),

$$\left(y^4+x^4+6x^2y^2\right)=8$$

Question 36(viii)

$$\left\{x\sin^2\left(\frac{y}{x}\right)-y\right\}dx+xdy=0,\ y(1)=\frac{\pi}{4}$$

Solution 36(viii)

$$\left\{x\sin^2\left(\frac{y}{x}\right)-y\right\}dx+xdy=0$$

$$\left\{x\sin^2\left(\frac{y}{x}\right)-y\right\}dx=-xdy$$

$$\sin^2\left(\frac{y}{x}\right)+\frac{y}{x}=\frac{dy}{dx}\ldots\ldots\ldots(i)$$

Let $v = \dfrac{y}{x}$

$$v+x\frac{dv}{dx}=\frac{dy}{dx}$$

From eq (i)

$$\sin^2 v+v=v+x\frac{dv}{dx}$$

$$\frac{1}{\sin^2 v}dv=\frac{1}{x}dx$$

Integrating on both the sides we have,

$$\int\frac{1}{\sin^2 v}dv=\int\frac{1}{x}dx$$

$$-\cot v=\log(x)+C$$

$$-\cot\left(\frac{y}{x}\right)=\log(x)+C\ldots\ldots\ldots(ii)$$

Put x= 1 y = $\dfrac{\pi}{4}$ in eq (ii)

$$-\cot\left(\frac{\pi}{4}\right)=\log(1)+C$$

$$C=-1$$

From eq (ii) we have

$$-\cot\left(\frac{y}{x}\right)=\log(x)-1$$

Question 36(ix)

Solve the following initial value problem

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0, y(2) = \pi$$

Solution 36(ix)

$$x\frac{dy}{dx}-y+x\sin\left(\frac{y}{x}\right)=0, y(2)=\pi$$

Here it is a homogeneous equation

Put $y=vx$

And

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

So,

$$v+x\frac{dv}{dx}=\frac{vx}{x}-\sin\left(\frac{vx}{x}\right)$$

$$x\frac{dv}{dx}=-\sin v$$

$$\frac{dv}{\sin v}=-\frac{dx}{x}$$

$$\operatorname{cosec} v\,dv=-\frac{dx}{x}$$

$$-\log(\operatorname{cosec} v+\cot v)=-\log x+c$$

Now putting $y=\pi, x=2$, we have

$$c=0.301$$

Now

$$-\log\left(\operatorname{cosec}\left(\frac{y}{x}\right)+\cot\left(\frac{y}{x}\right)\right)=-\log x+0.301$$

Question 37

Find the particular solution of the differential equation

$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x$, given that when $x=1, y=\frac{\pi}{4}$

Solution 37

Consider the given equation

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x$$

This is a homogeneous differential equation.

Thus, substiuting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

in the above equation, we get,

$$x\cos\left(\frac{vx}{x}\right)\left(v+x\frac{dv}{dx}\right) = vx\cos\left(\frac{vx}{x}\right)+x$$

$$\Rightarrow \cos v\left(v+x\frac{dv}{dx}\right)=v\cos\left(\frac{vx}{x}\right)+1$$

$$\Rightarrow v\cos v+x\cos v\frac{dv}{dx}=v\cos v+1$$

$$\Rightarrow x\cos v\frac{dv}{dx}=1$$

$$\Rightarrow \cos v dv=\frac{dx}{x}$$

Integrating both the sides,

$$\Rightarrow \int\cos v dv=\int\frac{dx}{x}$$

$$\Rightarrow \sin v = \log x + C$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = \log x + C\ldots(1)$$

Given that when $x = 1$, $y = \frac{\pi}{4}$

Substituting the values, $x = 1$ and $y = \frac{\pi}{4}$

in equation (1), we get,

$$\Rightarrow \sin\left(\frac{\frac{\pi}{4}}{1}\right) = \log 1 + C$$

$$\Rightarrow \sin\left(\frac{\pi}{4}\right) = 0 + C$$

$$\Rightarrow \frac{1}{\sqrt{2}} = C$$

Subsituting the value of C, in equation (1) we get,

$$\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{\sqrt{2}}$$

Question 38

Find the particular solution of the differential equation

$(x-y)\frac{dy}{dx} = x+2y$, given that when $x = 1, y = 0$.

Solution 38

consider the given equation

$$(x-y)\frac{dy}{dx} = x+2y$$

This is a homogeneous equation.

Substiuting y=vx and $\frac{dy}{dx} = \left(v + x\frac{dv}{dx}\right)$ in

the above equation, we have,

$$(x-vx)\left(v + x\frac{dv}{dx}\right) = x + 2vx$$

$$\Rightarrow (1-v)\left(v + x\frac{dv}{dx}\right) = 1 + 2v$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+2v-v(1-v)}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+2v-v+v^2}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v+v^2}{1-v}$$

$$\Rightarrow \frac{(1-v)dv}{(1+v+v^2)} = \frac{dx}{x}$$

Integrating on both the sides, we have,

$$\Rightarrow \int \frac{(1-v)dv}{(1+v+v^2)} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{3}{2}\int \frac{dv}{(1+v+v^2)} - \int \frac{1}{2}\frac{(2v+1)dv}{(1+v+v^2)} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{3}{2}\int \frac{dv}{v^2+\frac{1}{4}+v+\frac{3}{4}} - \frac{1}{2}\int \frac{(2v+1)dv}{(1+v+v^2)} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{3}{2}\int \frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{2}\int \frac{(2v+1)dv}{(1+v+v^2)} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{3}{2}\times\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}} - \frac{1}{2}\log(1+v+v^2) = \log x + C$$

$$\Rightarrow \sqrt{3}\tan^{-1}\frac{2v+1}{\sqrt{3}} - \frac{1}{2}\log(1+v+v^2) = \log x + C$$

$$\Rightarrow \sqrt{3}\tan^{-1}\frac{2\left(\frac{y}{x}\right)+1}{\sqrt{3}} - \frac{1}{2}\log\left(1+\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\right) = \log x + C\ldots(1)$$

Given that when x = 1, y = 0

Substituting the values, in the above equation, we get,

$$\Rightarrow \sqrt{3}\tan^{-1}\frac{2\times 0+1}{\sqrt{3}} - \frac{1}{2}\log(1+0+0^2) = \log 1 + C$$

$$\Rightarrow \sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}} - \frac{1}{2}\times 0 = 0 + C$$

$$\Rightarrow C = \sqrt{3}\times\frac{\pi}{6}$$

$$\Rightarrow C = \frac{\pi}{2\sqrt{3}}$$

Thus, equation (1) becomes,

$$\sqrt{3}\tan^{-1}\frac{2\left(\frac{y}{x}\right)+1}{\sqrt{3}} - \frac{1}{2}\log\left(1+\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\right) = \log x + \frac{\pi}{2\sqrt{3}}$$

$$\Rightarrow \sqrt{3}\tan^{-1}\frac{2y+x}{x\sqrt{3}} - \frac{\pi}{2\sqrt{3}} = \log x + \frac{1}{2}\log\left(1+\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\right)$$

$$\Rightarrow 2\sqrt{3}\tan^{-1}\frac{2y+x}{x\sqrt{3}} - \frac{\pi}{\sqrt{3}} = \log x^2 + \log\left(\frac{x^2+xy+y^2}{x^2}\right)$$

$$\Rightarrow 2\sqrt{3}\tan^{-1}\frac{2y+x}{x\sqrt{3}} - \frac{\pi}{\sqrt{3}} = \log(x^2+xy+y^2)$$

Question 39

Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$

given that y = 1 when x = 0.

Solution 39

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

$$\frac{dy}{dx} = \left(\frac{1}{\frac{x}{y} + \frac{y}{x}}\right) \ldots\ldots(i)$$

Let $v = \frac{y}{x}$

$$x\frac{dv}{dx} + v = \frac{dy}{dx}$$

From (i) we have,

$$x\frac{dv}{dx} + v = \left(\frac{1}{\frac{1}{v} + v}\right)$$

$$\left(-\frac{1}{v^3} - \frac{1}{v}\right)dv = \frac{1}{x}dx$$

Integrating on both the sides we have

$$\frac{1}{2v^2} - \log v = \log x + C$$

$$\Rightarrow \frac{x^2}{2y^2} = \log\left(\frac{y}{x} \times x\right) + C \ldots\ldots\ldots(ii)$$

Put x =0 , y = 1

$$0 = \log(1) + C$$

$$C = 0$$

From eq (ii) we have

$$\frac{x^2}{2y^2} = \log(y)$$

Chapter 22 - Differential Equations Exercise Ex. 22.10

Question 1

Solve the following differential equation:

$$\frac{dy}{dx} + 2y = e^{3x}$$

Solution 1

Here, $\frac{dy}{dx}+2y=e^{3x}$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx}+Py=Q$$

$$P=2, Q=e^{3x}$$

I.F. $=e^{\int P dx}$

$=e^{\int 2dx}$

$=e^{2x}$

Multiplying both the sides by I.F.

$$e^{2x}\frac{dy}{dx}+e^{2x}2y=e^{2x}\times e^{3x}$$

$$e^{2x}\frac{dy}{dx}+e^{2x}2y=e^{5x}$$

Integrating it with respect to x,

$$ye^{2x}=\int e^{5x}dx+c$$

$$ye^{2x}=\frac{e^{5x}}{5}+c$$

$$y=\frac{e^{3x}}{5}+ce^{-2x}$$

Question 2

Solve the following differential equation:

$$4\frac{dy}{dx}+8y=5e^{-3x}$$

Solution 2

Here, $4\frac{dy}{dx}+8y = 5e^{-3x}$

$$\frac{dy}{dx}+2y = \frac{5}{4}e^{-3x}$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx}+Py = Q$$

$$P = 2, Q = \frac{5}{4}e^{-3x}$$

I.F. $= e^{\int P dx}$

$$= e^{\int 2dx}$$

$$= e^{2x}$$

Solution of the equation is given by

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$ye^{2x} = \int \frac{5}{4}e^{-3x} \times e^{2x} dx + c$$

$$ye^{2x} = \int \frac{5}{4}e^{-x} dx + c$$

$$ye^{2x} = \frac{-5}{4}e^{-x} + c$$

$$y = \frac{-5}{4}e^{-3x} + ce^{-2x}$$

Question 3

Solve the following differential equation:

$$\frac{dy}{dx}+2y = 6e^{x}$$

Solution 3

Here, $\frac{dy}{dx}+2y = 6e^{x}$

It is a linear differential equation, comparing it with

$$\frac{dy}{dx}+Py = Q$$

$$P = 2, Q = 6e^{x}$$

I.F. $= e^{\int P dx}$

$$= e^{\int 2dx}$$

$$= e^{2x}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y \times \left(e^{2x}\right) = \int 6e^{x} \times e^{2x} dx + c$$

$$= \int 6e^{3x} dx + c$$

$$ye^{2x} = \frac{6}{3}e^{3x} + c$$

$$ye^{2x} = 2e^{3x} + c$$

$$y = 2e^{x} + ce^{-2x}$$

Question 4

Solve the following differential equation:

$$\frac{dy}{dx} + y = e^{-2x}$$

Solution 4

Here, $\frac{dy}{dx} + y = e^{-2x}$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = 1, Q = e^{-2x}$$

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int 2 dx}$$

$$= e^{x}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y \times e^{x} = \int e^{-2x} \times e^{x} dx + c$$

$$= \int e^{-x} + c$$

$$ye^{x} = \frac{e^{-x}}{-1} + c$$

$$y = -e^{-2x} + ce^{-x}$$

Question 5

Solve the following differential equation:

$$x\frac{dy}{dx} = x + y$$

Solution 5

Here, $x\frac{dy}{dx} = x + y$

It is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = -\frac{1}{x}, Q = 1$$

I.F. $= e^{\int P dx}$

$$= e^{\int -\frac{1}{x} dx}$$

$$= e^{-\log x}$$

$$= e^{\log\left(\frac{1}{x}\right)}$$

$$= \frac{1}{x}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y\left(\frac{1}{x}\right) = \int 1 \times \left(\frac{1}{x}\right) dx + c$$

$$y\left(\frac{1}{x}\right) = \log|x| + c$$

$$y = x\log|x| + cx, \ x \neq 0$$

Question 6

Solve the following differential equation:

$$\frac{dy}{dx} + 2y = 4x$$

Solution 6

Here, $\frac{dy}{dx} + 2y = 4x$

It is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = 2, Q = 4x$$

I.F. $= e^{\int P dx}$

$= e^{\int 2dx}$

$= e^{2x}$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y \times e^{2x} = \int 4x \times e^{2x} dx + c$$

$$= 4\left[x \times \int e^{2x} dx - \int \left(1 \times \int e^{2x} dx\right) dx\right] + c$$

Using integration by parts

$$y \times e^{2x} = 4\left[x \times \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} dx\right] + c$$

$$= 2xe^{2x} - 2\frac{e^{2x}}{2} + c$$

$$ye^{2x} = 2xe^{2x} - e^{2x} + c$$

$$ye^{2x} = (2x - 1)e^{2x} + c$$

$$y = (2x - 1) + ce^{-2x}$$

Question 7

Solve the following differential equation:

$$x\frac{dy}{dx} + y = xe^x$$

Solution 7

Here, $x\frac{dy}{dx}+y = xe^x$

$$\frac{dy}{dx}+\frac{y}{x} = e^x$$

It is a linear differential equation, comparing it with

$$\frac{dy}{dx}+Py = Q$$

$$P = \frac{1}{x}, Q = e^x$$

I.F. $= e^{\int P dx}$

$$= e^{\int \frac{1}{x} dx}$$

$$= e^{\log x}$$

$$= x$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y \times (x) = \int e^x \times x dx + c$$

$$xy = x\int e^x dx - \int\left(1 \times \int e^x dx\right) dx + c$$

Using integration by parts

$$= xe^x - \int e^x dx + c$$

$$= xe^x - e^x + c$$

$$xy = (x-1)e^x + c$$

$$y = \left(\frac{x-1}{x}\right)e^x + \frac{c}{x},\ x > 0$$

Question 8

Solve the following differential equation:

$$\frac{dy}{dx}+\frac{4x}{x^2+1}y = -\frac{1}{\left(x^2+1\right)^2}$$

Solution 8

Here, $\frac{dy}{dx} + \frac{4x}{x^2+1}y = -\frac{1}{\left(x^2+1\right)^2}$

It is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{4x}{x^2+1}, Q = -\frac{1}{\left(x^2+1\right)^2}$$

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int \frac{4x}{x^2+1}dx}$$

$$= e^{2\int \frac{2x}{x^2+1}dx}$$

$$= e^{2\log\left|x^2+1\right|}$$

$$= \left(x^2+1\right)^2$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y\left(x^2+1\right)^2 = \int -\frac{1}{\left(x^2+1\right)^2}\left(x^2+1\right)^2 x dx + c$$

$$y\left(x^2+1\right)^2 = \int -dx + c$$

$$y\left(x^2+1\right)^2 = -x + c$$

$$y = -\frac{x}{\left(x^2+1\right)^2} + \frac{c}{\left(x^2+1\right)^2}$$

Question 9

Solve the following differential equation:

$$x\frac{dy}{dx} + y = x \log x$$

Solution 9

Here, $x\frac{dy}{dx}+y = x\log x$

$$\frac{dy}{dx}+\frac{y}{x} = \log x$$

It is a linear differential equation, comparing it with

$$\frac{dy}{dx}+Py = Q$$

$$P = \frac{1}{x}, Q = \log x$$

I.F. $= e^{\int P dx}$

$$= e^{\int \frac{1}{x} dx}$$

$$= e^{\log|x|}$$

$$= x,\ x > 0$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y \times x = \int (\log x)(x)\, dx + c$$

$$yx = \log x \times \int x dx - \int \left(\frac{1}{x} \times \int x dx\right) dx + c$$

$$= \frac{x^2}{2}\log x - \int \frac{x^2}{2x} dx + c$$

$$= \frac{x^2}{2}\log x - \int \frac{x}{2} dx + c$$

$$yx = \frac{x^2}{2}\log x - \frac{x^2}{4} + c$$

$$y = \frac{x}{2}\log x - \frac{x}{4} + \frac{c}{x},\ x > 0$$

Question 10

Solve the following differential equation:

$$x\frac{dy}{dx} - y = (x-1)e^x$$

Solution 10

Here, $x\dfrac{dy}{dx} - y = (x-1)e^x$

$$\frac{dy}{dx} - \frac{y}{x} = \left(\frac{x-1}{x}\right)e^x$$

It is a linear differential equation. Comparing the equation by,

$$\frac{dy}{dx} + Py = Q$$

$$P = -\frac{1}{x}, Q = \left(\frac{x-1}{x}\right)e^x$$

I.F. $= e^{\int P dx}$

$$= e^{-\int \frac{1}{x}dx}$$

$$= e^{-log|x|}$$

$$= \frac{1}{x},\ x > 0$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y\left(\frac{1}{x}\right) = \int \left(\frac{x-1}{x}\right)e^x\left(\frac{1}{x}\right)dx + c$$

$$\frac{y}{x} = \int\left(\frac{1}{x} - \frac{1}{x^2}\right)e^x dx + c$$

$$\frac{y}{x} = \frac{1}{x}e^x + c$$

Since $\int\left[f(x) + f'(x)\right]e^x dx = f(x)e^x + c$

$$y = e^x + Cx\ ,\ x > 0$$

Question 11

Solve the following differential equation:

$$\frac{dy}{dx} + \frac{y}{x} = x^3$$

Solution 11

Here, $\frac{dy}{dx} + \frac{y}{x} = x^3$

It is a linear differential equation. Comparing the equation by,

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{1}{x}, Q = x^3$$

I.F. $= e^{\int P dx}$

$$= e^{\int \frac{1}{x} dx}$$

$$= e^{\log|x|}$$

$$= x, \ x > 0$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y \times x = \int x^3 \times (x) dx + c$$

$$xy = \frac{x^5}{5} + c$$

$$y = \frac{x^4}{5} + \frac{c}{x}, \ x > 0$$

Question 12

Solve the differential equation

$$\frac{dy}{dx} + y = \sin x$$

Solution 12

$$\frac{dy}{dx}+y=\sin x$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py=Q$$

$$p=1, Q=\sin x$$

I.F.

$$=e^{\int p\,dx}$$

$$=e^{\int dx}$$

$$=e^{x}$$

Solution of the equation is given by,

$$y\times(\text{I.F})=\int Q\times(\text{I.F})dx+c$$

$$y\left(e^{x}\right)=\int \sin x\times\left(e^{x}\right)dx+c$$

$$ye^{x}=\frac{e^{x}}{2}(\sin x-\cos x)+c$$

Question 13

Solve the following differential equation:

$$\frac{dy}{dx}+y=\cos x$$

Solution 13

Here, $\frac{dy}{dx} + y = \cos x$

It is a linear differential equation. Comparing the equation by,

$$\frac{dy}{dx} + Py = Q$$

$$P = 1, Q = \cos x$$

I.F. $= e^{\int P dx}$

$$= e^{\int dx}$$

$$= e^{x}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y\left(e^{x}\right) = \int (\cos x)\left(e^{x}\right) + c_1 \qquad \text{---(i)}$$

Let $I = \int e^{x} \cos x dx$

$$= \cos x \times \int e^{x} dx \int \left(\sin x \int e^{x} dx\right) dx + c_2$$

Using integration by parts

$$I = e^{x} \cos x + \int \sin x e^{x} dx + c$$

$$= e^{x} \cos x + \left[\sin x \int e^{x} dx - \int \left(\cos x \int e^{x} dx\right) dx\right] + c_2$$

$$I = e^{x} \cos x + \sin e^{x} - I + c_2$$

$$2I = e^{x} (\cos x + \sin x) + c_2$$

$$I = \frac{e^{x}}{2} (\cos x + \sin x) + \frac{c_2}{2}$$

$$I = \frac{e^{x}}{2} (\cos x + \sin x) + c_3$$

Putting I in equation (i),

$$ye^{x} = \frac{e^{x}}{2} (\cos x + \sin x) + c_1 + c_3$$

$$ye^{x} = \frac{e^{x}}{2} (\cos x + snx) + c$$

$$y = \frac{1}{2} (\cos x + \sin x) + ce^{-x}$$

Question 14

Solve the differential equation

$$\frac{dy}{dx}+2y=\sin x$$

Solution 14

$$\frac{dy}{dx}+2y=\sin x$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py=Q$$

$$p=2, Q=\sin x$$

I.F.

$$=e^{\int pdx}$$

$$=e^{\int dx}$$

$$=e^{2x}$$

Solution of the equation is given by,

$$y\times(\text{I.F})=\int Q\times(\text{I.F})dx+c$$

$$y\left(e^{2x}\right)=\int \sin x\times\left(e^{2x}\right)dx+c$$

$$ye^{2x}=\frac{e^{2x}}{5}(2\sin x-\cos x)+c$$

Question 15

Solve the following differential equation:

$$\frac{dy}{dx}-y\tan x=-2\sin x$$

Solution 15

Here, $\frac{dy}{dx} - y\tan x = -2\sin x$

It is a linear differential equation. Comparing the equation by,

$$\frac{dy}{dx} + Py = Q$$

$$P = -\tan x,\ Q = -2\sin x$$

I.F. $= e^{\int P dx}$

$$= e^{-\int \tan x dx}$$

$$= e^{-\log \sec x}$$

$$= \frac{1}{\sec x}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$\frac{y}{\sec x} = \int -\frac{2\sin x}{\sec x} dx + c$$

$$y\cos x = -\int 2\sin x \cos x dx + c$$

$$y\cos x = -\int \sin 2x dx + c$$

$$y\cos x = \frac{\cos 2x}{2} + c$$

$$y = \frac{\cos 2x}{2\cos x} + \frac{c}{\cos x}$$

Question 16

Solve the following differential equation:

$$\left(1+x^2\right)\frac{dy}{dx}+y=tan^{-1}x$$

Solution 16

Here, $\left(1+x^2\right)\frac{dy}{dx}+y = tan^{-1}x$

$$\frac{dy}{dx}+\frac{y}{1+x^2}=\frac{tan^{-1}x}{1+x^2}$$

It is a linear differential equation. Comparing the equation by,

$$\frac{dy}{dx}+Py = Q$$

$$P = \frac{1}{1+x^2}, Q = \frac{tan^{-1}x}{1+x^2}$$

I.F. $= e^{\int P dx}$

$$= e^{\int \frac{1}{1+x^2}dx}$$

$$= e^{tan^{-1}x}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F})dx + c$$

$$y\left(e^{tan^{-1}x}\right) = \int \frac{tan^{-1}x}{1+x^2}e^{tan^{-1}x}dx + c$$

Let $tan^{-1}x = t$

$$\frac{1}{1+t^2}dx = dt$$

So,

$$ye^t = \int t \times e^t dt + c$$

$$= t \times \int e^t dt - \int\left(1 \times e^t dt\right)dt + c$$

Using integration by parts

$$ye^t = te^t - e^t + c$$

$$y = (t-1)ce^{-t}$$

$$y = \left(tan^{-1}x - 1\right) + ce^{-tan^{-1}x}$$

Question 17

Solve the following differential equation:

$$\frac{dy}{dx}+y\,tan\,x = cos\,x$$

Solution 17

Here, $\frac{dy}{dx} + y\tan x = \cos x$

It is a linear differential equation. Comparing the equation by,

$$\frac{dy}{dx} + Py = Q$$

$$P = \tan x, Q = \cos x$$

I.F. $= e^{\int P dx}$

$= e^{\int \tan x dx}$

$= e^{\log|\sec x|}$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y \sec x = \int \cos x (\sec x) dx + c$$

$$\frac{y}{\cos x} = \int dx + c$$

$$\frac{y}{\cos x} = x + c$$

$$y = x\cos x + c\cos x$$

Question 18

Solve the differential equation

$$\frac{dy}{dx} + y\cot x = x^2\cot x + 2x$$

Solution 18

$$\frac{dy}{dx} + y\cot x = x^2 \cot x + 2x$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$p = \cot x, Q = x^2 \cot x + 2x$$

I.F.

$$= e^{\int p dx}$$

$$= e^{\int \cot dx}$$

$$= e^{\log \sin x}$$

$$= \sin x$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y(\sin x) = \int (x^2 \cos x + 2x \sin x) dx + c$$

$$y \sin x = \int x^2 \cos x dx + \int 2x \sin x dx + C$$

$$= x^2 \sin x + C$$

Question 19

Solve the following differential equation:

$$\frac{dy}{dx} + y \tan x = x^2 \cos^2 x$$

Solution 19

Here, $\frac{dy}{dx} + y \tan x = x^2 \cos^2 x$

It is a linear differential equation. Comparing the equation by,

$$\frac{dy}{dx} + Py = Q$$

$$P = \tan x, Q = x^2 \cos^2 x$$

I.F. $= e^{\int P dx}$

$= e^{\int \tan x dx}$

$= e^{\log|\sec x|}$

$= \sec x$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y \sec x = \int x^2 \cos^2 x (\sec x) dx + c$$

$$= \int x^2 \cos x dx + c$$

$$= x^2 \int \cos x dx - \int \left(2x \int \cos x dx\right) dx + c$$

Using integration by parts

$$y (\sec x) = x^2 \sin x - 2 \int x \sin x dx + c$$

$$= x^2 \sin x - 2\left[x \times \int \sin x dx - \int \left(1 \times \int \sin x dx\right) dx\right] + c$$

$$y \sec x = x^2 \sin x + 2x \cos x - 2 \sin x + c$$

$$y = x^2 \sin x \cos x + 2x \cos^2 x - 2 \sin x \cos x + c \cos x$$

Question 20

Solve the following differential equation:

$$\left(1 + x^2\right) \frac{dy}{dx} + y = e^{\tan^{-1} x}$$

Solution 20

Here, $\left(1+x^2\right)\frac{dy}{dx}+y=e^{\tan^{-1}x}$

$$\frac{dy}{dx}+\frac{y}{1+x^2}=\frac{e^{\tan^{-1}x}}{1+x^2}$$

It is a linear differential equation. Comparing the equation by,

$$\frac{dy}{dx}+Py=Q$$

$$P=\frac{1}{1+x^2},\ Q=\frac{e^{\tan^{-1}x}}{1+x^2}$$

I.F. $=e^{\int Pdx}$

$$=e^{\int\frac{1}{1+x^2}dx}$$

$$=e^{\tan^{-1}x}$$

Solution of the equation is given by,

$$y\times(\text{I.F})=\int Q\times(\text{I.F})dx+c$$

$$y\left(e^{\tan^{-1}x}\right)=\int\frac{e^{\tan^{-1}x}}{1+x^2}\times e^{\tan^{-1}x}dx+c$$

Let $e^{\tan^{-1}x}=t$

$$e^{\tan^{-1}x}*\frac{1}{1+x^2}dx=dt$$

$$y(t)=\int tdt+c$$

$$yt=\frac{t^2}{2}+c$$

$$y=\frac{t}{2}+\frac{c}{t}$$

$$y=\left(\frac{1}{2}e^{\tan^{-1}x}+ce^{-\tan^{-1}x}\right)$$

Question 21

Solve the following differential equation:

$xdy=\left(2y+2x^4+x^2\right)dx$

Solution 21

Here, $xdy = \left(2y + 2x^4 + x^2\right)dx$

$$x\frac{dy}{dx} = 2y + 2x^4 + x^2$$

$$\frac{dy}{dx} - \frac{2}{x}y = 2x^3 + x$$

It is a linear differential equation. Comparing it with equation,

$$\frac{dy}{dx} + Py = Q$$

$$P = -\frac{2}{x}, Q = 2x^3 + x$$

I.F. $= e^{\int P dx}$

$$= e^{-2\int \frac{1}{x}dx}$$

$$= e^{-2\log|x|}$$

$$= e^{\log\left(\frac{1}{x^2}\right)}$$

$$= \frac{1}{x^2}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F})\, dx + c$$

$$y\left(\frac{1}{x^2}\right) = \int \left(2x^3 + x\right)\left(\frac{1}{x^2}\right)dx + c$$

$$\frac{y}{x^2} = \int \left(2x + \frac{1}{x}\right)dx + c$$

$$\frac{y}{x^2} = 2\frac{x^2}{2} + \log|x| + c$$

$$y = x^4 + x^2\log|x| + cx^2$$

Question 22

Solve the following differential equation:

$$\left(1 + y^2\right) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$$

Solution 22

Here, $\left(1+y^2\right)+\left(x-e^{tan^{-1}y}\right)\frac{dy}{dx}=0$

$$\left(x-e^{tan^{-1}y}\right)\frac{dy}{dx}=-\left(1+y^2\right)$$

$$e^{tan^{-1}y}-x=\left(1+y^2\right)\frac{dy}{dx}$$

$$\left(1+y^2\right)\frac{dx}{dy}+x=e^{tan^{-1}y}$$

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{tan^{-1}y}}{1+y^2}$$

It is a linear differential equation. Comparing the equation with,

$$\frac{dx}{dy}+Px=Q$$

$$P=\frac{1}{1+y^2},\ Q=\frac{e^{tan^{-1}y}}{1+y^2}$$

I.F. $=e^{\int Pdy}$

$$=e^{\int\frac{1}{1+y^2}dy}$$

$$=e^{tan^{-1}y}$$

Solution of the equation is given by,

$$x\times\left(\text{I.F}\right)=\int Q\times\left(\text{I.F}\right)dy+c$$

$$x\left(e^{tan^{-1}y}\right)=\int\frac{e^{tan^{-1}y}}{1+y^2}\left(e^{tan^{-1}y}\right)dy+c$$

Let $e^{tan^{-1}y}=t$

$$e^{tan^{-1}y}\left(\frac{1}{1+y^2}\right)dy=dt$$

$$xt=\int tdt+c$$

$$xt=\frac{t^2}{t}+c$$

$$x=\frac{1}{2}t+\frac{c}{t}$$

$$x=\frac{1}{2}e^{tan^{-1}y}+ce^{-tan^{-1}y}$$

Question 23

Solve the following differential equation:

$$y^2\frac{dx}{dy}+x-\frac{1}{y}=0$$

Solution 23

Here, $y^2\dfrac{dx}{dy}+x-\dfrac{1}{y}=0$

$$\frac{dx}{dy}+\frac{x}{y^2}=\frac{1}{y^3}$$

It is a linear differential equation. Comparing the equation with,

$$\frac{dx}{dy}+Px=Q$$

$$P=\frac{1}{y^2},\ Q=\frac{1}{y^3}$$

I.F. $=e^{\int P\,dy}$

$$=e^{\int\frac{1}{y^2}dy}$$

$$=e^{-\frac{1}{y}}$$

Solution of the equation is given by,

$$x\times(\text{I.F})=\int Q\times(\text{I.F})\,dy+c$$

$$x\left(e^{-\frac{1}{y}}\right)=\int\frac{1}{y^3}\left(e^{-\frac{1}{y}}\right)dy+c$$

Let $e^{-\frac{1}{y}}=t$

$\Rightarrow$ $\dfrac{1}{y}=-\log t$

$$e^{-\frac{1}{y}}\times\frac{1}{y^2}dy=dt$$

$$x(t)=\int\frac{1}{y}dt+c$$

$$=-\int\log +dt+c$$

$$=-\left[\log t\times\int 1\times dt-\int\left(\frac{1}{t}\int 1\times dt\right)dt\right]+c$$

$$=-\left[t\log t-\int\frac{t}{t}dt\right]+c$$

$$x(t)=-t\log t+t+c$$

$$x(t)=-t\left[\log t-1\right]+c$$

$$x=-\left[-\frac{1}{y}-1\right]+ce^{\frac{1}{y}}$$

$$x=\frac{1}{y}+1+ce^{\frac{1}{y}}$$

$$x=\left(\frac{1+y}{y}\right)+ce^{\frac{1}{y}}$$

Question 24

Solve the following differential equation:

$$\left(2x-10y^3\right)\frac{dy}{dx}+y=0$$

Solution 24

Here, $\left(2x - 10y^3\right)\frac{dy}{dx} + y = 0$

$$y\frac{dx}{dy} + 2x - 10y^3 = 0$$

$$\frac{dx}{dy} = \frac{2}{y}x = 10y^2$$

It is a linear differential equation. Comparing the equation with,

$$\frac{dx}{dy} + Px = Q$$

$$P = \frac{2}{y},\ Q = 10y^2$$

I.F. $= e^{\int P dy}$

$$= e^{\int \frac{2}{y} dy}$$

$$= e^{2\log|y|}$$

$$= y^2$$

Solution of the equation is given by,

$$x \times (\text{I.F}) = \int Q \times (\text{I.F})\, dy + c$$

$$x\left(y^2\right) = \int 10y^2\left(y^2\right)dy + c$$

$$xy^2 = 10\frac{y^5}{5} + c$$

$$xy^2 = 2y^5 + c$$

$$x = 2y^3 + \frac{c}{y^2}$$

$$x = 2y^3 + cy^{-2}$$

Question 25

Solve the following differential equation:

$(x + \tan y)dy = \sin 2y dx$

Solution 25

Here, $(x+\tan y)dy = \sin 2y dx$

$$x+\tan y = \sin 2y \frac{dx}{dy}$$

$$\sin 2y \frac{dx}{dy} - x = \tan y$$

$$\frac{dx}{dy} - \operatorname{cosec} 2y x = \frac{\tan y}{\sin 2y}$$

It is a linear differential equation. Comparing it with,

$$\frac{dx}{dy} + Px = Q$$

$$P = -\operatorname{cosec} 2y, \; Q = \frac{\tan y}{\sin 2y}$$

I.F. $= e^{-\int \operatorname{cosec} 2y dy}$

$$= e^{-\int \operatorname{cosec} 2y dy}$$

$$= e^{-\frac{1}{2}\log \tan y}$$

$$= e^{\log \sqrt{\cot y}}$$

$$= \sqrt{\cot y}$$

Solution of the equation is given by,

$$x \times (\text{I.F}) = \int Q \times (\text{I.F})\, dy + c$$

$$x\sqrt{\cot y} = \int \frac{\tan y}{\sin 2y}\sqrt{\cot y}\, dy + c$$

$$= \int \frac{\sqrt{\tan y}}{\left(\frac{2\tan y}{1+\tan^2 y}\right)} dy + c$$

$$x\sqrt{\cot y} = \int \frac{1+\tan^2 y}{2\sqrt{\tan y}} dy + c$$

$$\frac{x}{\sqrt{\tan y}} = \frac{1}{2}\int \frac{\sec^2 y}{\sqrt{\tan y}} dy + c$$

Put $\tan y = t$

$$\sec^2 y \times dy = dt$$

$$\frac{x}{\sqrt{\tan y}} = \frac{1}{2}\int \frac{dt}{\sqrt{t}} + c$$

$$= \frac{1}{2} \times 2\sqrt{t} + c$$

$$\frac{x}{\sqrt{\tan y}} = \sqrt{\tan y} + c$$

$$x = \tan y + c\sqrt{\tan y}$$

Question 26

Solve the following differential equation:

$$dx + x dy = e^{-y} \sec^2 y dy$$

Solution 26

Here, $dx + xdy = e^{-y} \sec^2 ydy$

$$\frac{dx}{dy} + x = e^{-y} \sec^2 y$$

It is a linear differential equation. Comparing it with,

$$\frac{dx}{dy} + Px = Q$$

$$P = 1, Q = e^{-y} \sec^2 y$$

I.F. $= e^{\int P dy}$

$= e^{\int dy}$

$= e^{y}$

Solution of the equation is given by,

$$x \times (\text{I.F}) = \int Q \times (\text{I.F})\, dy + c$$

$$xe^{y} = \int e^{-y} \sec^2 y e^{y} dy + c$$

$$= \int \sec^2 ydy + c$$

$$xe^{y} = \int \tan y + c$$

$$x = e^{-y} (\tan y + c)$$

Question 27

Solve the following differential equation:

$$\frac{dy}{dx} = y \tan x - 2 \sin x$$

Solution 27

Here, $\frac{dy}{dx} = y \tan x - 2 \sin x$

$$\frac{dy}{dx} - y \tan x = -2 \sin x$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = -\tan x, Q = -2 \sin x$$

I.F. $= e^{\int P dx}$

$$= e^{-\int \tan x dx}$$

$$= e^{-\log \sec x}$$

$$= \frac{1}{\sec x}$$

$$= \cos x$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y \cos x = -\int 2 \sin x \cos x dx + c$$

Let $\sin x = t$

$$\cos x dx = dt$$

$$y (\cos x) = -\int 2t dt + c$$

$$= -t^2 + c$$

$$y \cos x = -\sin^2 x + c$$

$$y = \sec x \left(-\sin^2 x + c\right)$$

Question 28

Solve the following differential equation:

$$\frac{dy}{dx} + y \cos x = \sin x \cos x$$

Solution 28

Here, $\frac{dy}{dx} + y\cos x = \sin x \cos x$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = \cos x,\ Q = \sin x \cos x$$

I.F. $= e^{\int P dx}$

$$= e^{\int \cos x dx}$$

$$= e^{\sin x}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y\left(e^{\sin x}\right) = \int \sin x \cos x e^{\sin x} dx + c$$

Let $\sin x = t$

$$\cos x dx = dt$$

$$ye^{t} = \int t \times e^{t} dt + c$$

$$= t \times \int e^{t} dt - \int\left(1 \int e^{t} dt\right) dt + c$$

$$ye^{t} = te^{t} - e^{t} + c$$

$$ye^{t} = e^{t}(t - 1) + c$$

$$y = t - 1 + ce^{-t}$$

$$y = \sin x - 1 + ce^{-\sin x}$$

Question 29

Solve the following differential equation:

$$\left(1 + x^2\right)\frac{dy}{dx} - 2xy = \left(x^2 + 2\right)\left(x^2 + 1\right)$$

Solution 29

Here, $\left(1+x^2\right)\frac{dy}{dx} - 2xy = \left(x^2+2\right)\left(x^2+1\right)$

$$\frac{dy}{dx} - \frac{2x}{x^2+1}y = \left(x^2+2\right)$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = -\frac{2x}{x^2+1}, Q = x^2+2$$

I.F. $= e^{\int P dx}$

$$= e^{-\int \frac{2x}{x^2+1}dx}$$

$$= e^{-\log\left|x^2+1\right|}$$

$$= \frac{1}{\left(x^2+1\right)}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y\left(\frac{1}{x^2+1}\right) = \int\left(\frac{x^2+2}{x^2+1}\right)dx + c$$

$$= \int\left(1+\frac{1}{x^2+1}\right)dx + c$$

$$\frac{y}{\left(x^2+1\right)} = x + \tan^{-1}x + c$$

$$y = \left(x^2+1\right)\left(x+\tan^{-1}x+c\right)$$

Question 30

Solve the following differential equation:

$$(\sin x)\frac{dy}{dx} + y\cos x = 2\sin^2 x\cos x$$

Solution 30

Here, $(\sin x)\frac{dy}{dx} + y\cos x = 2\sin^2 x\cos x$

$$\frac{dy}{dx} + y\cot x = 2\sin x\cos x$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = \cot x, Q = 2\sin x\cos x$$

I.F. $= e^{\int P dx}$

$$= e^{\int \cot x dx}$$

$$= e^{\log \sin x}$$

$$= \sin x$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F}) dx + c$$

$$y(\sin x) = \int 2\sin x\cos x(\sin x)dx + c$$

ysin x = (2/3)sin^3x+C

Question 31

Solve the following differential equation:

$$\left(x^2 - 1\right)\frac{dy}{dx} + 2\left(x + 2\right)y = 2\left(x + 1\right)$$

Solution 31

Here, $\left(x^2-1\right)\frac{dy}{dx}+2\left(x+2\right)y = 2\left(x+1\right)$

$$\frac{dy}{dx}+\frac{2\left(x+2\right)}{\left(x^2-1\right)}y = \frac{2\left(x+1\right)}{\left(x^2-1\right)}$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py = Q$$

$$P = \frac{2\left(x+2\right)}{x^2-1}, Q = \frac{2\left(x+1\right)}{\left(x^2-1\right)}$$

I.F. $= e^{\int P dx}$

$$= e^{2\int\frac{(x+2)}{(x^2-1)}dx}$$

$$= e^{\int\frac{2x}{x^2-1}dx+4\int\frac{1}{x^2-1}dx}$$

$$= e^{log\left|x^2-1\right|+4\times\frac{1}{2}log\left|\frac{x-1}{x+1}\right|}$$

$$= e^{log\left|x^2-1\right|+log\left|\frac{x-1}{x+1}\right|^2}$$

$$= e^{log\frac{\left(x^2-1\right)\left(x-1\right)^2}{\left(x+1\right)^2}}$$

I.F. $= \frac{\left(x+1\right)\left(x-1\right)\left(x-1\right)^2}{\left(x+1\right)^2}$

$$= \frac{\left(x-1\right)^3}{\left(x+1\right)}$$

Solution of the equation is given by,

$$4\times\left(\text{I.F}\right) = \int Q\times\left(\text{I.F.}\right)dx+c$$

$$y\left(x-1\right)^3 = \int 2\left(x+1\right)\left(\ldots\right.$$

$$\frac{y\left(x-1\right)^3}{\left(x+1\right)} = 2\frac{x^2}{2}-6x+8\,log\left|x+1\right|+c$$

$$\frac{y\left(x-1\right)^3}{x+1} = x^2-6x+8\,log\left|x+1\right|+c$$

$$y = \frac{x+1}{\left(x-1\right)^3}\left[x^2-6x+8\,log\left|x+1\right|+c\right]$$

Question 32

Solve the following differential equation:

$$\frac{dy}{dx}+\frac{2y}{x}=\cos x$$

Solution 32

Here, $\frac{dy}{dx}+\frac{2y}{x}=\cos x$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py=Q$$

$$P=\frac{2}{x},\ Q=\cos x$$

I.F. $= e^{\int P dx}$

$$= e^{2\int \frac{1}{x}dx}$$

$$= e^{2\log|x|}$$

$$= x^2$$

Solution of the equation is given by,

$$y\times(\text{I.F})=\int Q\times(\text{I.F.})dx+c$$

$$y\left(x^2\right)=\int \cos x\left(x^2\right)dx+c$$

$$yx^2=\int x^2\cos x dx+c$$

$$=x^2\int\cos x-\int\left(2x\times\int\cos x dx\right)dx+c$$

Using integration by parts

$$yx^2=x^2\sin x-\int 2x\sin x dx+c$$

$$=x^2\sin x-2\left[x\times\int\sin x dx-\int\left(1\times\int\sin x dx\right)dx\right]+c$$

$$=x^2\sin x+2x\cos x-2\int\cos x dx+c$$

$$yx^2=x^2\sin x+2x\cos x-2\sin x+c$$

$$y=\sin x+\frac{2}{x}\cos x-\frac{2}{x^2}\sin x+\frac{c}{x^2}$$

Question 33

Solve the following differential equation:

$$\frac{dy}{dx}-y=xe^x$$

Solution 33

Here, $\frac{dy}{dx} - y = xe^{x}$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = -1, Q = xe^{x}$$

I.F. $= e^{\int P dx}$

$= e^{-\int dx}$

$= e^{-x}$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.}) dx + c$$

$$ye^{-x} = \int xe^{x} \times e^{-x} dx + c$$

$$= \int x dx + c$$

$$ye^{-x} = \frac{x^2}{2} + c$$

$$y = e^{x}\left(\frac{x^2}{2} + c\right)$$

Question 34

Solve the following differential equation:

$$\frac{dy}{dx} + 2y = xe^{4x}$$

Solution 34

Here, $\frac{dy}{dx} + 2y = xe^{4x}$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = 2, Q = xe^{4x}$$

I.F. $= e^{\int P dx}$

$= e^{\int 2dx}$

$= e^{2x}$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.}) dx + c$$

$$y\left(e^{2x}\right) = \int xe^{4x}\left(e^{2x}\right) dx + c$$

$$= \int xe^{6x} dx + c$$

$$= x \times \int e^{6x} dx - \int \left(1 \int e^{6x} \times dx\right) + c$$

Using integration by parts

$$ye^{2x} = x \times \frac{e^{6x}}{6} - \int \frac{e^{6x}}{6} dx + c$$

$$ye^{2x} = \frac{x}{6} e^{6x} - \frac{e^{6x}}{36} + c$$

$$y = \frac{x}{6} e^{4x} - \frac{e^{4x}}{36} + ce^{-2x}$$

Question 35

Solve the differential equation $\left(x + 2y^2\right) = \frac{dy}{dx} = y$, given that when $x = 2, y = 1$

Solution 35

Here, $\left(x+2y^2\right)\dfrac{dy}{dx}=y$

$$y\frac{dx}{dy}-x=2y^2$$

$$\frac{dx}{dy}-\frac{x}{y}=2y$$

It is a linear differential equation. Comparing it with,

$$\frac{dx}{dy}+Px=Q$$

$$P=-\frac{1}{y},\ Q=2y$$

I.F. $=e^{\int P dy}$

$$=e^{-\int \frac{1}{y}dy}$$

$$=e^{-\log|y|}$$

$$=\frac{1}{y},\ y>0$$

Solution of the equation is given by,

$$x\times(\text{I.F})=\int Q\times(\text{I.F.})\,dx+c$$

$$x\left(\frac{1}{y}\right)=\int 2y\left(\frac{1}{y}\right)dy+c$$

$$=\int 2dy+c$$

$$x\left(\frac{1}{y}\right)=2y+c \qquad \text{---(i)}$$

Given, when $x=2,\ y=1$

So,

$$2=2+c$$

$$c=0$$

Put the value of c in equation (i),

$$x=2y^2$$

Question 36(i)

Find one-parameter families of solution curves of the following differential equation:

$\dfrac{dy}{dx}+3y=e^{mx}$, m is a given real number

Solution 36(i)

Here, $\frac{dy}{dx} + 3y = e^{mx}$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = 3, Q = e^{mx}$$

I.F. $= e^{\int P dx}$

$$= e^{\int 3 dx}$$

$$= e^{3x}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.})\, dx + c$$

$$y\left(e^{3x}\right) = \int e^{mx} e^{3x} dx + c$$

$$= \int e^{(m+3)x} dx + c$$

$$y\left(e^{3x}\right) = \frac{e^{(m+3)x} dx}{(m+3)} + c$$

Question 36(ii)

Find one-parameter families of solution curves of the following differential equation:

$\frac{dy}{dx} - y = \cos 2x.$

Solution 36(ii)

Here, $\frac{dy}{dx} - y = \cos 2x$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = -1, Q = \cos 2x$$

$$\text{I.F.} \quad = e^{\int P dx}$$

$$= e^{-\int dx}$$

$$= e^{-x}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.})\, dx + c$$

$$y \times e^{-x} = \int \cos 2x \times e^{-x} dx + c \qquad \text{---(i)}$$

$$I = \int \cos 2x e^{-x} dx = \cos 2x \times \left(-e^{-x}\right) - \int \left(\frac{\sin 2x}{2}\right) e^{-x} dx \qquad \text{[Using integration by parts]}$$

$$I = -e^{-x} \cos 2x - \frac{1}{2}\left[\left(-\sin 2x e^{-x}\right) + \int \frac{\cos 2x}{2} e^{-x} dx\right]$$

$$I = -e^{-x} \cos 2x + \frac{1}{2} \sin 2x e^{-x} - \frac{1}{4} I$$

$$\frac{5}{4} I = \frac{e^{-x}}{2}\left(\sin 2x - 2\cos 2x\right)$$

$$I = \frac{2}{5} e^{-x} \left(\sin 2x - 2\cos 2x\right)$$

So, solution of the equation is given by

$$y = \frac{2}{5}\left(\sin 2x - 2\cos 2x\right) + ce^{x}$$

Question 36(iii)

Find one-parameter families of solution curves of the following differential equation:

$x \frac{dy}{dx} - y = (x + 1)e^{-x}$.

Solution 36(iii)

Here, $x\frac{dy}{dx}-y=(x+1)e^{-x}$

$$\frac{dy}{dx}-\frac{y}{x}=\left(\frac{x+1}{x}\right)e^{-x}$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py=Q$$

$$P=-\frac{1}{x},\ Q=\left(\frac{x+1}{x}\right)e^{-x}$$

I.F. $=e^{\int P\,dx}$

$$=e^{-\int\frac{1}{x}dx}$$

$$=e^{-\log|x|}$$

$$=e^{\log\left(\frac{1}{x}\right)}$$

$$=\frac{1}{x},\quad x>0$$

Solution of the equation is given by,

$$y\times(\text{I.F})=\int Q\times(\text{I.F.})\,dx+c$$

$$y\times\left(\frac{1}{x}\right)=\int\left(\frac{x+1}{x}\right)e^{-x}\times\left(\frac{1}{x}\right)dx+c$$

$$\frac{y}{x}=\int\left(\frac{1}{x}+\frac{1}{x^2}\right)e^{-x}dx+c$$

Let $-x=t$

$$-dx=dt$$

$$y\left(-\frac{1}{x}\right)=\int\left(-\frac{1}{t}+\frac{1}{t^2}\right)e^{t}dt+c$$

$$y\left(-\frac{1}{x}\right)=-\frac{1}{t}e^{t}+c$$

$\left[\text{Since }\int\{f(x)+f'(x)\}e^{x}dx=f(x)e^{x}+c\right]$

$$-\frac{y}{x}=\frac{1}{x}e^{-x}+c$$

$$y=-\left(e^{-x}+cx\right)$$

$$y=-e^{-x}+c_1x$$

Question 36(iv)

Find one-parameter families of solution curves of the following differential equation:

$x\frac{dy}{dx}+y=x^4.$

Solution 36(iv)

Here, $x\dfrac{dy}{dx}+y=x^4$

$$\frac{dy}{dx}+\frac{y}{x}=x^3$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py=Q$$

$$P=\frac{1}{x},\ Q=x^3$$

I.F. $=e^{\int P\,dx}$

$=e^{\int \frac{1}{x}dx}$

$=e^{\log|x|}$

$=x$

Solution of the equation is given by,

$$y\times(\text{I.F})=\int Q\times(\text{I.F.})\,dx+c$$

$$yx=\int x^3(x)\,dx+c$$

$$xy=\frac{x^5}{5}+c$$

$$y=\frac{x^4}{5}+\frac{c}{x},\ x>0$$

Question 36(v)

Find one-parameter families of solution curves of the following differential equation:

$(x\log x)\dfrac{dy}{dx}+y=\log x.$

Solution 36(v)

Here, $\left(x\log x\right)\dfrac{dy}{dx}+y=\log x$

$$\frac{dy}{dx}+\frac{y}{x\log x}=\frac{1}{x}$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py=Q$$

$$P=\frac{1}{x\log x},\ Q=\frac{1}{x}$$

I.F. $=e^{\int P\,dx}$

$$=e^{\int\frac{1}{x\log x}dx}$$

$$=e^{\log|\log x|}$$

$$=\log x$$

Solution of the equation is given by,

$$y\times\left(\text{I.F}\right)=\int Q\times\left(\text{I.F.}\right)dx+c$$

$$y\left(\log x\right)=\int\frac{1}{x}\left(\log x\right)dx+c$$

$$y\left(\log x\right)=\frac{\left(\log x\right)^2}{2}+c$$

$$y=\frac{1}{2}\log x+\frac{c}{\log x},\ \ x>0,\ \ x\neq 1$$

Question 36(vi)

Find one-parameter families of solution curves of the following differential equation:

$$\frac{dy}{dx}-\frac{2x}{1+x^2}y=x^2+2.$$

Solution 36(vi)

Here, $\frac{dy}{dx} - \frac{2x}{1+x^2}y = x^2 + 2$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = -\frac{2x}{1+x^2}, Q = x^2 + 2$$

I.F. $= e^{-\int \frac{2x}{1+x^2}dx}$

$$= e^{-\log\left|1+x^2\right|}$$

$$= \frac{1}{1+x^2}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.})\, dx + c$$

$$y\left(\frac{1}{1+x^2}\right) = \int \left(x^2+2\right)\left(\frac{1}{1+x^2}\right)dx + c$$

$$= \int \left(\frac{x^2+2}{1+x^2}\right)dx + c$$

$$= \int \left(1 + \frac{1}{1+x^2}\right)dx + c$$

$$y\left(\frac{1}{1+x^2}\right) = x + \tan^{-1}x + c$$

$$y = \left(1+x^2\right)\left(x + \tan^{-1}x + c\right)$$

Question 36(vii)

Find one-parameter families of solution curves of the following differential equation:

$\frac{dy}{dx} + y\cos x = e^{\sin x}\cos x.$

Solution 36(vii)

Here, $\frac{dy}{dx} + y\cos x = e^{\sin x} \times \cos x$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = \cos x, Q = e^{\sin x}\cos x$$

I.F. $= e^{\int P dx}$

$$= e^{\int \cos x dx}$$

$$= e^{\sin x}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.}) dx + c$$

$$ye^{\sin x} = \int e^{\sin x} \times \cos x e^{\sin x} dx + c$$

Let $e^{\sin x} = t$

$$e^{\sin x}\cos x dx = dt$$

$$yt = \int t \times dt + c$$

$$yt = \frac{t^2}{t} + c$$

$$y = \frac{1}{2}t + \frac{c}{t}$$

$$y = \frac{1}{2}e^{\sin x} + ce^{-\sin x}$$

Question 36(viii)

Find one-parameter families of solution curves of the following differential equation:

$(x+y)\frac{dy}{dx} = 1.$

Solution 36(viii)

Here, $(x+y)\frac{dy}{dx} = 1$

$$\frac{dx}{dy} - x = y$$

It is a linear differential equation. Comparing it with,

$$\frac{dx}{dy} + Px = Q$$

$$P = -1, Q = y$$

I.F. $= e^{\int P dy}$

$$= e^{-\int dy}$$

$$= e^{-y}$$

Solution of the equation is given by,

$$x \times (\text{I.F}) = \int Q \times (\text{I.F.}) dy + c$$

$$x\left(e^{-y}\right) = \int y\left(e^{-y}\right) dy + c$$

$$= y \int e^{-y} dy - \int \left(1 \int e^{-y} dy\right) dy + c$$

$$= -ye^{-y} + \int e^{-y} dy + c$$

$$xe^{-y} = -ye^{-y} + e^{-y} + c$$

$$(x + y - 1)e^{-y} = c$$

$$x + y - 1 = ce^{y}$$

Question 36(ix)

Find one-parameter families of solution curves of the following differential equation:

$\frac{dy}{dx}\cos^2 x = \tan x - y.$

Solution 36(ix)

Here, $\frac{dy}{dx}\cos^2 x = \tan x - y$

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$

$$\frac{dy}{dx} + \sec^2 xy = \tan x \sec^2 x$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = \sec^2 x, Q = \tan x \sec^2 x$$

I.F. $= e^{\int P dx}$

$$= e^{\int \sec^2 x dx}$$

$$= e^{\tan x}$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.})\, dx + c$$

$$y\left(e^{\tan x}\right) = \int \tan x \sec^2 x e^{\tan x} dx + c$$

Let $e^{\tan x} = t$

$$e^{\tan x} \times \sec^2 x dx = dt$$

$$y \times t = \int (\log t) \times dt + c$$

$$= \log t \times \int 1 \times dt - \int \left(\frac{1}{t} \times \int 1 \times dt\right) dt + c$$

$$= t \log t - \int \frac{1}{t} \times t dt + c$$

$$yt = t \log t - t + c$$

$$y = \log t - 1 + \frac{c}{t}$$

$$y = \tan x - 1 + ce^{-\tan x}$$

Question 36(x)

Find one-parameter families of solution curves of the following differential equation:

$e^{-y} \sec^2 y dy = dx + x dy.$

Solution 36(x)

Here, $e^{-y} \sec^2 y dy = dx + x dy$

$$e^{-y} \sec^2 y = \frac{dx}{dy} + x$$

$$\frac{dx}{dy} + x = e^{-y} \sec^2 y$$

It is a linear differential equation. Comparing it with,

$$\frac{dx}{dy} + Px = Q$$

$$P = 1, Q = e^{-y} \sec^2 y$$

I.F. $= e^{\int P dy}$

$= e^{\int dy}$

$= e^{y}$

Solution of the equation is given by,

$$x \times (\text{I.F}) = \int Q \times (\text{I.F.}) dy + c$$

$$x(e^{y}) = \int e^{-y} \sec^2 y (e^{y}) dy + c$$

$$xe^{y} = \tan y + c$$

$$x = (\tan y + c) e^{-y}$$

Question 36(xi)

Find one-parameter families of solution curves of the following differential equation:

$x \log x \dfrac{dy}{dx} + y = 2 \log x.$

Solution 36(xi)

Here, $x\log x\dfrac{dy}{dx}+y=2\log x$

$$\frac{dx}{dy}+\frac{y}{x\log x}=\frac{2}{x}$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py=Q$$

$$P=\frac{1}{x\log x},\ Q=\frac{2}{x}$$

I.F. $=e^{\int P\,dx}$

$$=e^{\int\frac{1}{x\log x}dx}$$

$$=e^{\log\log x}$$

$$=\log x$$

Solution of the equation is given by,

$$y\times(\text{I.F})=\int Q\times(\text{I.F.})\,dx+c$$

$$y(\log x)=\int\frac{2}{x}\times\log x\,dx+c$$

$$y(\log x)=(\log x)^2+c$$

$$y=\log x+\frac{c}{\log x}$$

Question 36(xii)

Find one-parameter families of solution curves of the following differential equation:

$x\dfrac{dy}{dx}+2y=x^2\log x.$

Solution 36(xii)

Here, $x\frac{dy}{dx}+2y=x^2\log x$

$$\frac{dx}{dy}+\frac{2y}{x}=x\log x$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py=Q$$

$$P=\frac{2}{x},\ Q=x\log x$$

I.F. $=e^{\int P dx}$

$$=e^{\int \frac{2}{x}dx}$$

$$=e^{2\log x}$$

$$=e^{\log x^2}$$

$$=x^2$$

Solution of the equation is given by,

$$y\times(\text{I.F})=\int Q\times(\text{I.F.})\,dx+c$$

$$y\left(x^2\right)=\int(x\log x)\left(x^2\right)dx+c$$

$$yx^2=\int x^3\log x\,dx+c$$

$$=\log x\int x^3dx-\int\left(\frac{1}{x}\times\int x^3dx\right)dx+c$$

$$=\frac{x^4}{4}\log x-\int\frac{1}{x}\times\frac{x^4}{4}dx+c$$

$$=\frac{x^4}{4}\log x-\frac{1}{16}x^4+c$$

$$y=\frac{x^2}{4}\log x-\frac{1}{16}x^2+\frac{c}{x^2}$$

$$y=\frac{x^2}{16}(4\log x-1)+\frac{c}{x^2}$$

Question 37(i)

Solve the following initial value problem:

$$y'+y=e^x,\ y(0)=\frac{1}{2}$$

Solution 37(i)

Here, $y' + y = e^x$

It is a linear differential equation. Comparing it with,

$$y' + Py = Q$$

$$P = 1, Q = e^x$$

I.F. $= e^{\int P dx}$

$$= e^{\int dx}$$

$$= e^x$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.})\, dx + c$$

$$y\left(e^x\right) = \int e^x \left(e^x\right) dx + c$$

$$y\left(e^x\right) = \frac{e^{2x}}{2} + c \qquad \text{---(i)}$$

Here, $y = \frac{1}{2}$ when $x = 0$

$$\frac{1}{2}\left(e^0\right) = \frac{e^0}{2} + c$$

$$\frac{1}{2} = \frac{1}{2} + c$$

$\Rightarrow$ $c = 0$

Put $c = 0$ in equation (i),

$$y\left(e^x\right) = \frac{e^{2x}}{2} + c$$

$$y = \frac{e^x}{2}$$

Question 37(ii)

Solve the following initial value problem:

$$x\frac{dy}{dx} - y = \log x, y(1) = 0$$

Solution 37(ii)

Here, $x\frac{dy}{dx} - y = log\,x$

$$\frac{dy}{dx} - \frac{y}{x} = \frac{log\,x}{x}$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = -\frac{1}{x}, Q = \frac{log\,x}{\mathbf{x}}$$

I.F. $= e^{\int P dx}$

$$= e^{-\int \frac{1}{x} dx}$$

$$= e^{-\int log|x|}, \qquad x > 0$$

$$= e^{log\left(\frac{1}{x}\right)}, \qquad x > 0$$

$$= \frac{1}{x}, \qquad x > 0$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.}) dx + c$$

$$y \times (\text{I.F}) = \int \frac{log\,x}{x}\left(\frac{1}{x}\right) dx + c$$

$$y\left(\frac{1}{x}\right) = \int \frac{log\,x}{x^2} dx + c$$

$$= log\,x \times \left(-\frac{1}{x}\right) - \int\left(\frac{1}{x} \times \left(-\frac{1}{x}\right)\right) dx + c$$

$$y\left(\frac{1}{x}\right) = -\frac{1}{x} log\,x + \left(-\frac{1}{x}\right) + c$$

$$y = -log\,x - 1 + cx \qquad \text{---(i)}$$

Put $y = 0$, $x = 1$

$$0 = -log\,1 - 1 + c$$

$$0 = 0 - 1 + c$$

$$c = 0$$

Put $c = 1$ in equation (i),

$$y = -log\,x - 1 + x$$

$$y = x - 1 - log\,x$$

Question 37(iii)

Solve the following initial value problem:

$$\frac{dy}{dx} + 2y = e^{-2x} sin\,x, y(0) = 0$$

Solution 37(iii)

Here, $\frac{dy}{dx} + 2y = e^{-2x} \sin x, y(0) = 0$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = 2, Q = e^{-2x} \sin x$$

I.F. $= e^{\int P dx}$

$= e^{\int 2 dx}$

$= e^{2x}$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.}) dx + c$$

$$ye^{2x} = \int e^{-2x} \sin x \left(e^{2x}\right) dx + c$$

$$= \int \sin x dx + c$$

$$ye^{2x} = -\cos x + c \qquad \text{---(i)}$$

Put $x = 0$, $y = 0$

$$0 = -1 + c$$

$$c = 1$$

Put $c = 1$ in equation (i),

$$ye^{2x} = -\cos x + 1$$

$$y = (1 - \cos x) e^{-2x}$$

Question 37(iv)

Solve the following initial value problem:

$x \frac{dy}{dx} - y = (x + 1) e^{-x}, \; y(1) = 0$

Solution 37(iv)

Here, $x\frac{dy}{dx} - y = (x+1)e^{-x}$, $y(1) = 0$

$$\frac{dy}{dx} - \frac{y}{x} = \frac{(x+1)}{x}e^{-x}$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = -\frac{1}{x}, Q = \frac{(x+1)}{x}e^{-x}$$

I.F. $= e^{\int P dx}$

$$= e^{-\int \frac{1}{x} dx}$$

$$= e^{-\log|x|}$$

$$= \frac{1}{x}, \quad x > 0$$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.}) dx + c$$

$$\frac{y}{x} = \int \left(\frac{x+1}{x}\right) e^{-x} \left(\frac{1}{x}\right) dx + c$$

$$\frac{y}{x} = \int \left(\frac{1}{x} + \frac{1}{x^2}\right) e^{-x} + dx + c$$

Let $-x = t$

$$-dx = dt$$

$$y\left(-\frac{1}{t}\right) = -\int \left(-\frac{1}{t} + \frac{1}{t^2}\right) e^t dt + c$$

$$-\frac{y}{t} = -\left[-\frac{1}{t}e^t + c\right]$$

$\left[\text{Since } \int \{f(x) + f'(x)\} e^x dx = f(x)e^x + c\right]$

$$-y = e^t - c^t$$

$$-y = e^{-x} + cx \qquad \text{---(i)}$$

Put $y = 0$, $x = 1$

$$0 = e^{-1} + c(1)$$

$$0 = c + \frac{1}{e}$$

$$c = -\frac{1}{e}$$

Put the value of c in equation (i),

$$-y = e^{-x} - \frac{x}{e}$$

$$-y = e^{-x} - xe^{-1}$$

$$y = xe^{-1} - e^{-x}$$

Question 37(v)

Solve the following initial value problem:

$$\left(1+y^2\right)dx+\left(x-e-\tan^{-1}t\right)dy=0,\ y(0)=0$$

Solution 37(v)

Here, $\left(1+y^2\right)dx+\left(x-e-\tan^{-1}t\right)dy=0$

$$\left(1+y^2\right)\frac{dx}{dy}+x=e^{-\tan^{-1}y}$$

It is a linear differential equation. Comparing it with,

$$\frac{dx}{dy}+Px=Q$$

$$P=\frac{1}{1+y^2},\ Q=\frac{e^{-\tan^{-1}y}}{1+y^2}$$

I.F. $=e^{\int P\,dy}$

$$=e^{\int\frac{1}{1y^2}dy}$$

$$=e^{\tan^{-1}y}$$

Solution of the equation is given by,

$$x\times(\text{I.F})=\int Q\times(\text{I.F.})dy+c$$

$$x\left(e^{\tan^{-1}y}\right)=\int\frac{\left(e^{-\tan^{-1}y}\right)}{1+y^2}e^{\tan^{-1}y}dy+c$$

Let $e^{-\tan^{-1}y}$

$$e^{-\tan^{-1}y}\left(-\frac{1}{1+y^2}\right)dy=dt$$

$$x\left(\frac{1}{t}\right)=-\int\frac{1}{t}dt+c$$

$$=-\log|t|+c$$

$$xe^{\tan^{-1}y}=-\log\left|e^{-\tan^{-1}y}\right|+c \qquad \text{---(i)}$$

Put $x=0,\ y=0$

$$0=0+c$$

$$c=0$$

Put $c=0$ in equation (i)

$$xe^{\tan^{-1}y}=\tan^{-1}y$$

Question 37(vi)

Solve the following initial value problem:

$$\frac{dy}{dx}+y\tan x=2x+x^2\tan x,\ y(0)=1$$

Solution 37(vi)

Here, $\frac{dy}{dx} + y\tan x = 2x + x^2 \tan x,\ y(0) = 1$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx} + Py = Q$$

$$P = \tan x, Q = 2x + x^2 \tan x$$

I.F. $= e^{\int P dx}$

$= e^{\int \tan x dx}$

$= e^{\log|\sec x|}$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F.}) dx + c$$

$$y \sec x = \int (2x + x^2 \tan x) \sec dx + c$$

$$= \int 2x \sec x dx + \int x^2 \sec x \tan x dx + c$$

$$= \int 2x \times \sec x dx + \left[x^2 \times \int \sec x \tan x dx - \int \left(2x \int \sec \tan x dx\right) dx\right] + c$$

$$y \sec x = \int 2x \sec dx + x^2 \sec x - \int 2x \sec x dx + c$$

$$y \sec x = x^2 \sec x + c \qquad \text{---(i)}$$

Put $x = 0,\ y = 1$

$$1 = 0 + c$$

$$c = 1$$

Put $c = 1$ in equation (i),

$$y \sec x = x^2 \sec x + 1$$

$$y = x^2 + \frac{1}{\sec x}$$

$$y = x^2 + \cos x$$

Question 37(vii)

Solve the following initial value problem:

$$x\frac{dy}{dx} + y = x\cos x + \sin x, \left(\frac{\pi}{2}\right) = 1$$

Solution 37(vii)

Here, $x\dfrac{dy}{dx}+y = x\cos x+\sin x$

$$\frac{dy}{dx}+\frac{y}{x} = \cos x+\frac{\sin x}{x}$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py = Q$$

$$P = \frac{1}{x}, Q = \cos x+\frac{\sin x}{x}$$

I.F. $= e^{\int P dx}$

$$= e^{\int \frac{1}{x}dx}$$

$$= e^{\log|x|}, \qquad x > 0$$

$$= x, \qquad x > 0$$

Solution of the equation is given by,

$$y\times(\text{I.F}) = \int Q\times(\text{I.F.})dx+c$$

$$y(x) = \int\left(\cos x+\frac{\sin x}{x}\right)x dx+c$$

$$= \int(x\cos x+\sin x)dx+c$$

$$xy = \int x\cos x dx+\int \sin x dx+c$$

$$= x\int\cos x dx-\int\left(1\times\int\cos x dx\right)dx-\cos x+c$$

$$= x\sin x-\int\sin x dx-\cos x+c$$

$$= x\sin x+\cos x-\cos x+c$$

$$xy = x\sin x+c \qquad \text{---(i)}$$

Put $x = \dfrac{\pi}{2}$, $y = 1$

$$\frac{\pi}{2} = \frac{\pi}{2}+c$$

$$c = 0$$

Put $c = 0$ in equation (i),

$$xy = x\sin x$$

$$y = \sin x$$

Question 37(viii)

Solve the following initial value problem

$$\frac{dy}{dx} + y\cot x = 4x\,\text{cosec}\,x,\; y\left(\frac{\pi}{2}\right) = 0$$

Solution 37(viii)

$$\frac{dy}{dx}+y\cot x=4x\operatorname{cosec}x, y\left(\frac{\pi}{2}\right)=0$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py=Q$$

$$p=\cot x, Q=4x\operatorname{cosec}x$$

I.F.

$$=e^{\int p\,dx}$$

$$=e^{\int \cot x\,dx}$$

$$=e^{\log \sin x}$$

$$=\sin x$$

Solution of the equation is given by,

$$y\times(\text{I.F})=\int Q\times(\text{I.F})dx+c$$

$$y(\sin x)=\int 4x\operatorname{cosec}x\times(\sin x)\,dx+c$$

$$=\int 4x\,dx+c$$

$$y\sin x=4\frac{x^2}{2}+c$$

$$=2x^2+c$$

Put $y=0, x=\frac{\pi}{2}$

$$0=\frac{\pi}{2}^2+c$$

$$c=-\frac{\pi}{2}^2$$

Now

$$y\sin x=2x^2-\frac{\pi^2}{2}$$

Question 37(ix)

$$\frac{dy}{dx}+2y\tan x=\sin x;\ y=0 \text{ when } x=\frac{\pi}{3}$$

Solution 37(ix)

$$\frac{dy}{dx}+2y\tan x=\sin x$$

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where $P=2\tan x$ and $Q=\sin x$.

$\therefore$ I.F. $=e^{\int 2\tan x\,dx}=e^{2\log\sec x}=\sec^2 x$

Multiplying both the sides of $\frac{dy}{dx}+2y\tan x=\sin x$ by $\sec^2 x$, we get

$$\sec^2 x\frac{dy}{dx}+2y\tan x\sec^2 x=\sin x\sec^2 x$$

Integrating on both the sides with respect to x,

$$y\sec^2 x=\int \tan x\sec x\,dx+C$$

$$y\sec^2 x=\sec x+C..........(i)$$

Putting $x=\frac{\pi}{3}$, and $y=0$ in (i)

$$0=2+C$$

$$C=-2$$

From (i) we get

$$y\sec^2 x=\sec x-2$$

$$y=\cos x-2\cos^2 x$$

Question 37(x)

$$\frac{dy}{dx}-3y\cot x=\sin 2x;\ y=0 \text{ when } x=\frac{\pi}{2}$$

Solution 37(x)

$$\frac{dy}{dx} - 3y\cot x = \sin 2x;$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where P = -3cotx and Q = sin2x.

$\therefore$ I.F. $= e^{\int -3\cot x\, dx} = e^{-3\log \sin x} = \text{cosec}^3 x$

Multiplying both the sides of $\frac{dy}{dx} - 3y\cot x = \sin 2x$ by $\text{cosec}^3 x$, we get

$\text{cosec}^3 x \frac{dy}{dx} - 3y\cot x\, \text{cosec}^3 x = \sin 2x\, \text{cosec}^3 x$

Integrating on both the sides with respect to x,

$y\text{cosec}^3 x = 2\int \text{cosec}x \cot x\; dx + C$

$y\text{cosec}^3 x = -2\text{cosec}x + C$

$y = -2\sin^2 x + C\sin^3 x$..........(i)

Putting $x = \frac{\pi}{2}$, and y = 2 in (i)

$2 = -2 + C$

$C = 4$

From (i) we get

$y = -2\sin^2 x + 4\sin^3 x$

Question 37(xi)

$$\frac{dy}{dx} + y\cot x = 2\cos x, y\left(\frac{\pi}{3}\right) = 0$$

Solution 37(xi)

$$\frac{dy}{dx} + y \cot x = 2 \cos x$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \cot x$ and $Q = \cos x$.

$\therefore$ I.F. $= e^{\int \cot x \, dx} = e^{\log \sin x} = \operatorname{cosec} x$

Multiplying both the sides of $\frac{dy}{dx} + y \cot x = 2 \cos x$ by $\operatorname{cosec} x$, we get

$$\operatorname{cosec} x \frac{dy}{dx} + y \operatorname{cosec} x \cot x = 2 \operatorname{cosec} x \cos x$$

Integrating on both the sides with respect to x,

$$y \operatorname{cosec} x = 2\int \operatorname{cosec} x \cos x \, dx + C$$

$$y = -\cot x \cos x + C \sin x \text{..........(i)}$$

Putting $x = \frac{\pi}{2}$, and $y = 0$ in (i)

$0 = 0 + C$

$C = 0$

From (i) we get

$y = -\cot x \cos x$

Question 37(xii)

dy = cos x (2 - y cosec x) dx

Solution 37(xii)

dy = cos x (2 - y cosec x) dx

$$\frac{dy}{dx} + y\operatorname{cosec}x = 2\cos x$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py$

= Q, where P = cosecx and Q = cosx.

$$\therefore \text{I.F.} = e^{\int \operatorname{csec} dx} = e^{\log\left|\tan\left(\frac{x}{2}\right)\right|} = \left|\tan\left(\frac{x}{2}\right)\right|$$

Multiplying both the sides of $\frac{dy}{dx} + y\operatorname{cosec}x = 2\cos x$ by $\left|\tan\left(\frac{x}{2}\right)\right|$, we get

$$\left|\tan\left(\frac{x}{2}\right)\right|\frac{dy}{dx} + y\operatorname{cosec}x\left|\tan\left(\frac{x}{2}\right)\right| = 2\cos x\left|\tan\left(\frac{x}{2}\right)\right|$$

Integrating on both the sides with respect to x,

$$y\frac{1}{\operatorname{cosec}x} = 2\int \cos x\left|\tan\left(\frac{x}{2}\right)\right|dx + C$$

$$y\sin x = -\frac{1}{2}\cos 2x + \frac{3}{2}$$

Question 38

Find the general solution of the differential equation $x\frac{dy}{dx} + 2y = x^2\ (x \neq 0)$

Solution 38

The given differential equation is

$$x\frac{dy}{dx} + 2y = x^2\ (x \neq 0) \qquad ----(i)$$

Dividing both sides of (i) by x, we get

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

which is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$

where $P = \frac{2}{x}$ and $Q = x$

So, $\quad \text{I.F.} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

$\therefore$ Solution of the given equation is given by

$$y.\,x^2 = \int x.x^2\,dx + C = \int x^3dx + C$$

$$\therefore \qquad y = \frac{x^2}{4} + Cx^{-2}$$

which is the general solution of the given differential equation.

Question 39

Find the general solution of the differential equation $\frac{dy}{dx} - y = \cos x$

Solution 39

Given differential equation is $\frac{dy}{dx} - y = \cos x$

It is of the form $\frac{dy}{dx} + Py = Q$, where $P = -1$ and $Q = \cos x$

$\therefore \qquad \text{I.F.} = e^{\int -1\,dx} = e^{-x}$

Multiplying both sides of equation by I.F. , we get

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\cos x$$

$$\Rightarrow \qquad \frac{d}{dx}\left(ye^{-x}\right) = e^{-x}\cos x$$

On integrating both sides w.r.t. x , we get

$$ye^{-x} = \int e^{-x}\cos x\,dx + C \qquad ------(i)$$

Let $\quad I = \int e^{-x}\cos x\,dx$

$$= \cos x\left(\frac{e^{-x}}{-1}\right) - \int(-\sin x)\left(-e^{-x}\right)dx$$

$$= -\cos x\,e^{-x} - \int \sin x\,e^{-x}\,dx$$

$$= -\cos x\,e^{-x} - \left[\sin x\left(-e^{-x}\right) - \int\left(-e^{-x}\right)\cos x\,dx\right]$$

$$= -\cos x\,e^{-x} + \sin x\,e^{-x} - \int e^{-x}\cos x\,dx$$

$$= (\sin x - \cos x)e^{-x} - I$$

$$\Rightarrow \quad 2I = (\sin x - \cos x)e^{-x}$$

$$\Rightarrow \quad I = \frac{(\sin x - \cos x)e^{-x}}{2}$$

Substituting the value of I in (i),

$$ye^{-x} = \frac{(\sin x - \cos x)e^{-x}}{2} + C$$

Or $\qquad y = \left(\frac{\sin x - \cos x}{2}\right) + Ce^{x}$

Question 40

Solve the differential equation

$$\left(y+3x^2\right)\frac{dx}{dy}=x$$

Solution 40

$$\left(y+3x^2\right)\frac{dx}{dy}=x$$

$$\frac{dx}{dy}=\frac{x}{y+3x^2}$$

$$\frac{dy}{dx}=\frac{y+3x^2}{x}$$

$$\frac{dy}{dx}-\frac{y}{x}=3x$$

It is a linear differential equation. Comparing it with,

$$\frac{dy}{dx}+Py=Q$$

$$p=-\frac{1}{x},Q=3x$$

I.F.

$$=e^{\int pdx}$$

$$=e^{-\int\frac{1}{x}dx}$$

$$=e^{-\log x}$$

$$=\frac{1}{x}$$

Solution of the equation is given by,

$$y\times(\text{I.F})=\int Q\times(\text{I.F})dx+c$$

$$y\left(\frac{1}{x}\right)=\int 3x\times\left(\frac{1}{x}\right)dx+c$$

$$\frac{y}{x}=3x+c$$

Question 41

Find the particular solution of the differential equation

$$\frac{dx}{dy}+x\cot y=2y+y^2\cot y,\ y\neq 0$$

given that x=0 when $y=\frac{\pi}{2}$

Solution 41

Consider the given differential equation,

$$\frac{dx}{dy} + x\cot y = 2y + y^2 \cot y,\ y \neq 0$$

This is a differential equation of the form,

$$\frac{dx}{dy} + Rx = S,\ \text{where R and S are functions of y or constants.}$$

Here, $R = \cot y$ and $S = 2y + y^2 \cot y$

Thus, $\text{I.F.} = e^{\int R dy} = e^{\int \cot y\, dy}$

$\Rightarrow \text{I.F} = e^{\log(\sin y)} = \sin y$

Thus, the solution is given by

$$x(\text{I.F.}) = \int S(\text{I.F.})dy + C$$

$$\Rightarrow x(\sin y) = \int \left(2y + y^2 \cot y\right)(\sin y)dy + C \ldots(1)$$

Evaluating the integral on the right hand side, we have,

$$\int \left(2y + y^2 \cot y\right)(\sin y)dy$$

$$= \int 2y \sin y dy + \int y^2 \cot y \sin y dy$$

$$= \int 2y \sin y dy + \int y^2 \cos y dy$$

Applying integration by parts and solving, we get,

$$\int \left(2y + y^2 \cot y\right)(\sin y)dy$$

$$= -2y\cos y + 2\sin y + y^2 \sin y - 2\left[y(-\cos y) + \int \cos y dy\right]$$

$$= -2y\cos y + 2\sin y + y^2 \sin y + 2y\cos y - 2\sin y$$

$$= y^2 \sin y$$

Therefore equation (1) becomes,

$$x(\sin y) = y^2 \sin y + C \ldots(2)$$

Given that $x = 0$ when $y = \frac{\pi}{2}$

Substituting the above values in equation (2),

$$0 = \left(\frac{\pi}{2}\right)^2 \sin\frac{\pi}{2} + C$$

$$\Rightarrow C = -\frac{\pi^2}{4}$$

Thus equation (2) becomes,

$$x(\sin y) = y^2 \sin y - \frac{\pi^2}{4}$$

Chapter 22 - Differential Equations Exercise Ex. 22.11

Question 1

The surface area of a balloon being inflated, changes at a rate proportional to time t. If initially its radius is 1 unit and after 3 seconds it is 2 units, find the radius after time t.

Solution 1

Let A be the surface area of balloon, so

$$\frac{dA}{dt} \propto t$$

$$\Rightarrow \quad \frac{dA}{dt} = \lambda t$$

$$\Rightarrow \quad \frac{d}{dt}\left(4\pi r^2\right) = \lambda t$$

$$\Rightarrow \quad 8\pi r \frac{dr}{dt} = \lambda t$$

$$\Rightarrow \quad 8\pi r\, dr = \lambda t$$

$$\Rightarrow \quad 8\pi \int r\, dr = \lambda \int t\, dt$$

$$\Rightarrow \quad 8\pi \frac{r^2}{2} = \frac{\lambda t^2}{2} + c$$

$$\Rightarrow \quad 4\pi r^2 = \frac{\lambda t^2}{2} + c \text{ - - - - } (1)$$

Given $r = 1$ unit when $t = 0$, so

$$4\pi (1)^2 = 0 + c$$

$$\Rightarrow \quad 4\pi = c$$

Using it is equation (i),

$$4\pi r^2 = \frac{\lambda t^2}{2} + 4\pi \text{ - - - - } (2)$$

Also, given $r = 2$ units when $t = 3$ sec.

$$4\pi (2)^2 = \frac{\lambda (3)^2}{2} + 4\pi$$

$$\Rightarrow \quad 16\pi = \frac{9}{2}\lambda + 4\pi$$

$$\Rightarrow \quad \frac{9}{2}\lambda = 12\pi$$

$$\Rightarrow \quad \lambda = \frac{24}{9}\pi$$

$$\Rightarrow \quad \lambda = \frac{8}{3}\pi$$

Now,equation (2) becomes

$$4\pi r^2 = \frac{8\pi}{6}t^2 + 4\pi$$

$$\Rightarrow \quad 4\pi\left(r^2 - 1\right) = \frac{4}{3}\pi t^2$$

$$\Rightarrow \quad r^2 - 1 = \frac{1}{3}t^2$$

$$\Rightarrow \quad r^2 = 1 + \frac{1}{3}t^2$$

$$\therefore \quad r = \sqrt{\left(1 + \frac{1}{3}t^2\right)}$$

Question 2

A population grows at the rate of 5% per year. How long does it take for the population to double?

Solution 2

Let the population after time t be P and initial population be P_0.

So,

$$\frac{dP}{dt} = 5\% \times P$$

$$\Rightarrow \quad \frac{dP}{dt} = \frac{P}{20}$$

$$\Rightarrow \quad 20\frac{dP}{P} = dt$$

$$\Rightarrow \quad 20\int \frac{dP}{P} = \int dt$$

$$\Rightarrow \quad 20\log|P| = t + c \text{ ----- } (1)$$

Given $P = P_0$ when $t = 0$

$$20\log(P_0) = 0 + c$$

$$\Rightarrow \quad 20\log(P_0) = c$$

Now, equation (1) becomes

$$20\log(P) = t + 20\log(P_0)$$

$$\Rightarrow \quad 20\log\left(\frac{P}{P_0}\right) = t$$

Let time is t, when $P = 2P_0$, so,

$$20\log\left(\frac{2P}{P_0}\right) = t_1$$

$$\Rightarrow \quad 20\log 2 = t_1$$

Required time period $= 20\log 2$ years

Question 3

The rate of growth of population is proportional to the number present. If the population of a city doubled in the past 25 years, and the present polulation is 100000, when will the city have a population of 500000?

$\left[\text{Given } \log_e 5 = 1.609, \log_e 2 = 0.6931.\right]$

Solution 3

Let P be the population at any time t and P_0 be the initial population.
So

$$\frac{dP}{dt} \propto P$$

$$\Rightarrow \quad \frac{dP}{dt} = \lambda P$$

$$\Rightarrow \quad \frac{dP}{dt} = \lambda\, dt$$

$$\Rightarrow \quad \int \frac{dP}{dt} = \lambda \int dt +$$

$$\Rightarrow \quad \log P = \lambda t + c \; --- (1)$$

Here, $P = P_0$ t when $t = 0$,

$$\log(P_0) = 0 + c$$

$$\Rightarrow \quad c = \log(P_0)$$

Now, equation (1) becomes

$$\log(P) = \lambda t + \log(P_0)$$

$$\Rightarrow \quad \log\left(\frac{P}{P_0}\right) = \lambda t \; --- (2)$$

Given $P = 2P_0$ when $t = 25$

$$\log\left(\frac{2P_0}{P_0}\right) = 25\lambda$$

$$\Rightarrow \quad \log 2 = 25\lambda$$

$$\Rightarrow \quad \lambda = \frac{\log 2}{25}$$

Now equation (2) becomes

$$\log\left(\frac{P}{P_0}\right) = \left(\frac{\log 2}{25}\right)t$$

let t_1 be the time to become population 500000 from 100000, so,

$$\log\left(\frac{500000}{100000}\right) = \frac{\log 2}{25} t_1$$

$$\Rightarrow \quad t_1 = \frac{25 \log 5}{\log 2}$$

$$\Rightarrow \quad = \frac{25(1.609)}{(0.6931)} = 58$$

Required time = 58 years

Question 4

In a culture, the bacteria count is 100000. *The number is increased by* 10% *in* 2 *hours*. *In how many hours will the count reach* 200000, *if the rate of growth of bacteria is proportional to the number present?*

Solution 4

Let C be the count of bacteria at any time t.
It is given that

$$\frac{dC}{dt} \propto C$$

$$\Rightarrow \frac{dC}{dt} = \lambda C, \text{ where } \lambda \text{ is a constant of proportionality}$$

$$\Rightarrow \frac{dC}{C} = \lambda dt$$

$$\Rightarrow \int \frac{dC}{C} = \lambda \int dt$$

$$\Rightarrow \log C = \lambda t + \log K \ldots.(1)$$

Initially, at $t = 0$, $C = 100000$
Thus, *we have*,

$$\log 100000 = \lambda \times 0 + \log K \ldots.(2)$$

$$\Rightarrow \log 100000 = \log K \ldots.(3)$$

At $t = 2$, $C = 100000 + 100000 \times \frac{10}{100} = 110000$
Thus, *from* (1), *we have*,

$$\log 110000 = \lambda \times 2 + \log K \ldots.(4)$$

Subtracting equation (2) *from* (4), *we have*,

$$\log 110000 - \log 100000 = 2\lambda$$

$$\Rightarrow \log 11 \times 10000 - \log 10 \times 10000 = 2\lambda$$

$$\Rightarrow \log \frac{11 \times 10000}{10 \times 10000} = 2\lambda$$

$$\Rightarrow \log \frac{11}{10} = 2\lambda$$

$$\Rightarrow \lambda = \frac{1}{2} \log \frac{11}{10} \ldots.(5)$$

We need to find the time 't' in which the count reaches 200000.
Substituting the values of λ *and* K *from equations* (3) *and* (5) *in equation* (1), *we have*

$$\log 200000 = \frac{1}{2} \log \frac{11}{10} t + \log 100000$$

$$\Rightarrow \frac{1}{2} \log \frac{11}{10} t = \log 200000 - \log 100000$$

$$\Rightarrow \frac{1}{2} \log \frac{11}{10} t = \log \frac{200000}{100000}$$

$$\Rightarrow \frac{1}{2} \log \frac{11}{10} t = \log 2$$

$$\Rightarrow t = \frac{2 \log 2}{\log \frac{11}{10}} \text{ hours}$$

Question 5

If the interst is compounded continuously at 6% per annum, how much worth Rs. 1000 will be after 10 years? How long will it take to double Rs. 1000?

$$\left[\text{Given } e^{0.6} = 1.822\right]$$

Solution 5

Given that, interest is compounded 6% per annum. Let P be principal

$$\frac{dP}{dt} = \frac{\Pr}{100}$$

$$\frac{dP}{dt} = \frac{r}{100} dt$$

$$\int \frac{dP}{P} = \int \frac{r}{100} dt$$

$$\log P = \frac{rt}{100} + c - - - (1)$$

Let P_0 be the initial principal at $t = 0$,

$$\log(P_0) = 0 + c$$

$$c = \log(P_0)$$

Put value of C is equation (1)

$$\log(P) = \frac{rt}{100} + \log(P_0)$$

$$\log\left(\frac{P}{P_0}\right) = \frac{rt}{100}$$

Case I:

Here, $P_0 = 1000, t = 10$ years and $r = 6$

$$\log\left(\frac{P}{1000}\right) = \frac{6 \times 10}{100}$$

$$\log P - \log 1000 = 0.6$$

$$\log\ P = \log e^{0.6} + \log 1000$$

$$= \log\left(e^{0.6} + 1000\right)$$

$$= \log(1.822 + 1000)$$

$$\log P = \log\ 1822$$

so,

$$P = \text{Rs}1822$$

Rs 1000 will be Rs 1822 after 10 years

Case II: let t_1 be the time to double Rs 1000, so

$P = 2000,\ P_0 = 1000,\ r = 6\%$

$$\log\left(\frac{P}{P_0}\right) = \frac{rt}{100}$$

$$\log\left(\frac{2000}{1000}\right) = \frac{6t_1}{100}$$

$$\frac{100 \log 2}{6} = t_1$$

$$\frac{100 \times 0.6931}{6} = t_1$$

$$11.55 \text{ years} = t_1$$

It will take approximatily 12 years to double

Question 6

The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given the number triples in 5 hrs. find how many bacteria will be present after 10 hours. Also find the time necessary for the number of vacteria to be 10 times the number of initial present.

$$\left[\text{Given } \log_e 3 = 1.0986, e^{2.1972} = 9\right]$$

Solution 6

Let A be the amount of bacteria present at time t and A_0 be the initial amount of bacteria. Here,

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = \lambda A$$

$$\int \frac{dA}{A} = \int \lambda dt$$

$$\log A = \lambda t + c \text{ --- } (1)$$

When $t = 0$, $A = A_0$

$$\log(A_0) = 0 + c$$

$$c = \log A_0$$

Using equation (1),

$$\log A = \lambda t + \log A_0$$

$$\log\left(\frac{A}{A_0}\right) = \lambda t \text{ ----- } (2)$$

Given, bacteria triples is 5 hours, so $A = 3A_0$, when $t = 5$

so, $$\log\left(\frac{3A_0}{A_0}\right) = 5\lambda$$

$$\log 3 = 5\lambda$$

$$\lambda = \frac{\log 3}{5}$$

Putting the value of λ in equation (2)

$$\log\left(\frac{A}{A_0}\right) = \frac{\log 3}{5} t$$

Case I: let A_1 be the number of bacteria present 10 hours, os

$$\log\left(\frac{A_1}{A_0}\right) = \frac{\log 3}{5} \times 10$$

$$\log\left(\frac{A_1}{A_0}\right) = 2\log 3$$

$$\log\left(\frac{A_1}{A_0}\right) = 2(1.0986)$$

$$\log\left(\frac{A_1}{A_0}\right) = 2.1972$$

$$A_1 = A_0 e^{2.1972}$$

$$A_1 = A_0 9$$

thus

There will be 9 times the bateria present is 10 hours.

Case II: let t_1 be the time necessary for the bacteria to be 10 times, os

$$\log\left(\frac{A}{A_0}\right) = \frac{\log 3}{5} \times t$$

$$\log\left(\frac{10A_0}{A_0}\right) = \frac{\log 3}{5} \times t_1$$

$$5 \log 10 = \log 3 t_1$$

$$5\frac{\log 10}{\log 3} = t_1$$

Required time is $\frac{5\log 10}{\log 3}$ hours

Question 7

The population of a city increses at a rate proportional to the number of inhabitants present at any time t. *If the population of the city was* 200000 *in* 1990 *and* 250000 *in* 2000, *what will be the population in* 2010?

Solution 7

Let P be the population of the city at any time t.

It is given that

$$\frac{dP}{dt} \propto P$$

$$\Rightarrow \frac{dP}{dt} = \lambda P, \textit{ where } \lambda \textit{ is a constant of proportionality}$$

$$\Rightarrow \frac{dP}{P} = \lambda dt$$

$$\Rightarrow \int \frac{dP}{P} = \lambda \int dt$$

$$\Rightarrow \log P = \lambda t + \log K \text{....(1)}$$

Initially, at $t = 1990$, $P = 200000$

Thus, we have,

$$\log 200000 = \lambda \times 1990 + \log K \text{....(2)}$$

At $t = 2000$, $P = 250000$

Thus, from (1), *we have,*

$$\log 250000 = \lambda \times 2000 + \log K \text{....(3)}$$

Subtracting equation (2) *from* (3), *we have,*

$$\log 250000 - \log 200000 = 10\lambda$$

$$\Rightarrow \log \frac{4}{5} = 10\lambda$$

$$\Rightarrow \lambda = \frac{1}{10} \log \frac{4}{5} \text{....(4)}$$

Substituting the value of λ *from equation* (4) *in equation* (1), *we have*

$$\log 200000 = 1990 \times \frac{1}{10} \log \frac{4}{5} + \log K$$

$$\Rightarrow \log K = \log 200000 - 199 \times \log \frac{4}{5} \text{....(5)}$$

Substituting the value of λ, $\log K$ *and* $t = 2010$ *in equation* (1), *we have*

$$\log P = \left\{ \frac{1}{10} \log \frac{4}{5} \right\} 2010 + \log 200000 - 199 \times \log \frac{4}{5}$$

$$\Rightarrow \log P = \log \left\{ \frac{4}{5} \right\}^{201} + \log \left(200000 \times \left(\frac{5}{4} \right)^{199} \right)$$

$$\Rightarrow P = \left\{ \frac{4}{5} \right\}^{201} \times 200000 \times \left(\frac{5}{4} \right)^{199}$$

$$\Rightarrow P = \left(\frac{5}{4} \right)^{2} \times 200000 = \frac{25}{16} \times 200000 = 312500$$

Question 8

If the marginal cost of manufaturing a certain item is given by $C'(x) = \frac{dC}{dx} = 2 + 0.15x$.

Find the total cost function $C(x)$, given that $C(0) = 100$.

Solution 8

Given,

$$C'(x) = \frac{dC}{dx} = 2 + 0.15x$$

$$dC = (2 + 0.15x)\,dx$$

$$\int dC = \int (2 + 0.15x)dx$$

$$C = 2x + \frac{0.15x^2}{2} + \lambda \text{ ----} (1)$$

Given $C = 100$ when $x = 0$, so

$$100 = 0 + 0 + \lambda$$

$$\lambda = 100$$

Put the value of λ in equation (1) total cost function is

$$C(x) = 2x + \frac{0.15x^2}{2} + 100$$

$$C(x) = 2x + 0.075x^2 + 100$$

Question 9

A bank pays interest by continuous compounding, that is, by treating the interest rate as the instantaneous rate of change of principal. Suppose in an account interest accruces at 8% per year, compounded continuously. Calculate the percentage increase in such an account over one year. $\left[\text{Take } e^{0.08} = 1.0833\right]$

Solution 9

Let P be principal at any time t at the rate of $r\%$ per annum, so

$$\frac{dP}{dt} = \frac{\Pr}{100}$$

$$\frac{dP}{P} = \frac{r}{100}dt$$

$$\int \frac{dP}{P} = \frac{r}{100}\int dt$$

$$\log P = \frac{rt}{100} + c \text{ --- } (1)$$

Let P_0 be the initial amount, so

$$\log(P_0) = 0 + c$$

$$c = \log(P_0)$$

Put the value of C in equation (1),

$$\log P = \frac{rt}{100} + \log P_0$$

$$\log P - \log P_0 = \frac{rt}{100}$$

$$\log\left(\frac{P}{P_0}\right) = \frac{rt}{100}$$

For $t = 1, r = 8\%$

$$\log\left(\frac{P}{P_0}\right) = \frac{8 \times 1}{100}$$

$$\log\frac{P}{P_0} = 0.08$$

$$\frac{P}{P_0} = e^{0.08}$$

$$\frac{P}{P_0} = 1.0833$$

$$\frac{P}{P_0} - 1 = 1.0833 - 1$$

$$\frac{P - P_0}{P_0} = 0.0833$$

percentage increase in amount in one year

$$= 0.0833 \times 100$$

$$= 8.33\%$$

Required percentage $= 8.33\%$

Question 10

In a simple circuit of resistance R, self inductance L and voltage E, the current i at any time t is given by $L\frac{\mathrm{di}}{\mathrm{dt}} + Ri = E$. If E is constant and initially no current passes through the circuit, prove that $i = \frac{E}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$

Solution 10

Here,

$$L\frac{di}{dt}+Ri=E$$

$$\frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}$$

It is a linear differential equation. Compound it with $\frac{dy}{dx}+Py=Q$

$$P=\frac{R}{L},\ Q=\frac{E}{L}$$

$$I.F.=e^{\int P dt}$$

$$=e^{\int \frac{P}{L}dt}$$

$$I.F.==e^{\left(\frac{R}{L}\right)t}$$

Solution of the equation is given by

$$i\left(I.F.\right)=\int Q\left(I.F.\right)dt+c$$

$$i\left(e^{\left(\frac{R}{L}\right)t}\right)=\int\frac{E}{L}\left(e^{\left(\frac{R}{L}\right)t}\right)dt+c$$

$$i\left(e^{\left(\frac{R}{L}\right)t}\right)=\frac{E}{L}\times\frac{L}{R}\left(e^{\left(\frac{R}{L}\right)t}\right)+c$$

$$i\left(e^{\left(\frac{R}{L}\right)t}\right)=\frac{E}{L}\left(e^{\left(\frac{R}{L}\right)t}\right)+c$$

$$i=\left(\frac{E}{L}\right)+c\left(e^{\left(\frac{R}{L}\right)t}\right)------(1)$$

Initiatially there was no current, so put $i=0, t=0$

$$0=\frac{F}{R}+ce^{0}$$

$$0=\frac{F}{R}+c$$

$$c=-\frac{F}{R}$$

Using Equation (1)

$$i=\frac{F}{R}-\frac{F}{R}e^{\left(-\frac{R}{L}\right)t}$$

$$i=\frac{F}{R}\left(1-e^{\left(-\frac{R}{L}\right)t}\right)$$

Question 11

The decay rate of radius at any time t is proportional to its mass at that time. Find the time when the mass will be halved of its initial mass.

Solution 11

Let A be the quantity of mass at any time t, so

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dP} = -\lambda A$$

$$\frac{dA}{A} = -\lambda dt$$

$$\int \frac{dA}{A} = -\lambda \int dt$$

$$\log A = -\lambda t + c - - - (1)$$

Let initial quantity of mass be A_0, so

$$\log A_0 = -\lambda(0) + c$$

$$\log(A_0) = c$$

Now, equation (1) becames,

$$\log\ A = -\lambda t + \log A_0$$

$$\log\left(\frac{A}{A_0}\right) = -\lambda t$$

Let t_1 be the required time to half the mass , so $A = \frac{1}{2}A_0$,

Now, $$\log\left(\frac{A}{A_0}\right) = -\lambda t$$

$$\log\left(\frac{A}{2A}\right) = -\lambda t$$

$$-\log 2 = -\lambda t$$

$$\frac{1}{\lambda}\log 2 = t$$

Required time is $\frac{1}{\lambda}$ log2 units where λ is constant of proportionality.

Question 12

Exepriments show that radius disintegrates at a rate proportional to the amount of radius present at the moment. Its half-life is 1590 years. What percentage will disappear in one year?

$$\left[\text{Use: } e^{-\frac{\log 2}{1590}} = 0.9996\right]$$

Solution 12

Let A be the quantity of radius at any time t, so

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dP} = -\lambda A$$

$$\frac{dA}{A} = -\lambda t$$

$$\int \frac{dA}{A} = -\lambda \int dt$$

$$\log A = -\lambda t + c - - - (1)$$

Let A_o be the initial amount of radius percentage , so

$$\log A_o = -\lambda (0) + c$$

$$c = \log (A_o)$$

Using, equation (1),

$$\log\ A = -\lambda t + \log A_o$$

$$\log \left(\frac{A}{A_o}\right) = -\lambda t - - - - - (2)$$

Given,its half-life is 1590 years, so

$$\log \left(\frac{\frac{1}{2} A_o}{A_o}\right) = -\lambda (1590)$$

$$\log \left(\frac{1}{2}\right) = -\lambda (1590)$$

$$-\log 2 = -\lambda (1590)$$

$$\log 2 = \lambda (1590)$$

$$\frac{\log 2}{1590} = \lambda$$

Now, equation (1) becomes

$$\log \left(\frac{A}{A_o}\right) = -\frac{\log 2}{1590} t$$

Now, put $t = 1$

$$\log\left(\frac{A}{A_o}\right) = -\frac{\log 2}{1590}$$

$$\frac{A}{A_o} = e^{-\frac{\log 2}{1590}}$$

$$\frac{A}{A_o} = 0.9996$$

$$1 - \frac{A}{A_o} = 1 - 0.9996$$

$$\frac{A_o - A}{A_o} = 0.0004$$

percentage to be disappear is one year

$$= \frac{A_o - A}{A_o} \times 100$$

$$= 0.0004 \times 100$$

$$= 0.04\%$$

Required percentage = 0.04%

Question 13

The slopeof the tangent at a point $P(x, y)$ on a curve is $-\frac{x}{y}$. If the curve passes through the point $(3, -4)$, find the equation of the curve.

Solution 13

Slope of tangent at point $(x, y) = -\frac{x}{y}$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y\,dy = -x dx$$

$$\int y\,dt = -\int x\,dx$$

$$\frac{y^2}{2} + \frac{x^2}{2} = c_1$$

$$x^2 + y^2 = c \text{ - - - - - } (1)$$

Given, curve is passing through $(3, -4)$, so

$$(3)^2 + (-4)^2 = c$$

$$9 + 16 = c$$

$$c = 25$$

So, using equation (1),

$$x^2 + y^2 = 25$$

$$x^2 + y^2 = 25$$

Question 14

Find the equation of curve which passes through the point $(2,2)$ and satisfies the differential equation $y - x\dfrac{dy}{dx} = y^2 + \dfrac{dy}{dx}$.

Solution 14

$$y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$$

$$\frac{dy}{dx} + x\frac{dy}{dx} = y - y^2$$

$$(1+x)\frac{dy}{dx} = y - y^2$$

$$\frac{dy}{y - y^2} = \frac{dx}{1+x}$$

$$\frac{dy}{y(1-y)} = \frac{dx}{1+x}$$

$$\int\left(\frac{1}{y} + \frac{1}{1-y}\right)dx = \int\frac{dx}{1+x}$$

$$\log|y| - \log|1 - y = \log|1+x\| + \log|c|$$

$$\frac{y}{1-y} = c(1+x)$$

$$y = (1-y)c(1+x) ----(1)$$

It is passing through $(2,2)$ so,

$$2 = (1-2)c(1+2)$$

$$2 = -3c$$

$$c = -\frac{2}{3}$$

Now,equation (1) becomes,

$$y = -\frac{2}{3}(1-y)(1+x)$$

$$3y = -2(1 + x - y - xy)$$

$$3y + 2 + 2x - 2y - 2xy = 0$$

$$y + 2x - 2xy + 2 = 0$$

$$2xy - 2x - 2 - y = 0$$

Question 15

Find the equation of curve which passes through the point $\left(1, \frac{\pi}{4}\right)$ and tangent at any point of which makes an angle $\tan^{-1}\left(\frac{y}{x} - \cos^2\frac{y}{x}\right)$ with x – axis.

Solution 15

Given that, tangent makes an angle (say, θ) with x-axis

$$\theta = \tan^{-1}\left(\frac{y}{x} - \cos^2\frac{y}{x}\right)$$

slope of tangent $= \tan\theta$

$$\frac{dy}{dx} = \tan\left\{\tan^{-1}\left(\frac{y}{x} - \cos^2\frac{y}{x}\right)\right\}$$

$$\frac{dy}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$$

It is a homogeneous equation, so

let $\quad y = vx$

and $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now,

$$x\frac{dv}{dx} + v = \frac{vx}{x} - \cos^2\left(\frac{vx}{x}\right)$$

$$x\frac{dv}{dx} + v = v - \cos^2 v$$

$$x\frac{dv}{dx} = -\cos^2 v$$

$$\frac{dv}{\cos^2 v} = -\frac{dx}{x}$$

$$\int \sec^2 v\, dv = -\int \frac{dx}{x}$$

$$\tan v = -\log|x| + c$$

$$\tan\left(\frac{y}{x}\right) = -\log|x| + c \; ---(1)$$

It is passing through $\left(1, \frac{\pi}{4}\right)$, so,

$$\tan\left(\frac{\pi}{4}\right) = -\log|1| + c$$

$$1 = 0 + c$$

$$c = 1$$

Now, equation (1) becomes

$$\tan\left(\frac{y}{x}\right) = -\log|x| + 1$$

Therefore,

$$\tan\left(\frac{y}{x}\right) = \log\left|\frac{e}{x}\right|$$

Question 16

Find the curve for which the intercept cut off by a tangent on x-axis is equal to four times the ordinate of the point of contact.

Solution 16

Let $P(x, y)$ be the point of contact of tangent and curve $y = f(x)$. and It cuts axes at A and B so, equatin of tangent at $P(x, y)$

$$Y - y = \frac{dy}{dx}(X - x)$$

Put $X = 0$

$$Y - y = \frac{dy}{dx}(-x)$$

$$Y = y - x\frac{dy}{dx}$$

So, coordinate of $A = \left(0, y - x\frac{dy}{dx}\right)$

Put $Y = 0$,

$$0 - y = \frac{dy}{dx}(X - x)$$

$$-y\frac{dx}{dy} = X - x$$

$$X = x - y\frac{dx}{dy}$$

Coordinate of $B = \left(x - y\frac{dx}{dy}, 0\right)$

Given, (intercept on x - axis) = 4(ordinate)

$$x - y\frac{dx}{dy} = 4y$$

$$y\frac{dx}{dy} + 4y = x$$

$$\frac{dx}{dy} + 4 = \frac{x}{y}$$

$$\frac{dx}{dy} - \frac{x}{y} = -4$$

It is a linear different equation. Comparing it with $\frac{dx}{dy} + Px = Q$

$P = -\frac{1}{y}, \; Q = -4$

$$I.F. = e^{\int p\,dy}$$

$$= e^{-\int \frac{1}{y} dy}$$

$$= e^{-\log y}$$

$$= \frac{1}{y}$$

Solution of the equation is given by,

$$x\left(I.F.\right) = \int Q\left(I.F.\right) dy + \log c$$

$$x\left(\frac{1}{y}\right) = \int (-4)\left(\frac{1}{y}\right) dy + \log c$$

$$\frac{x}{y} = -4\log y + \log c$$

$$e^{\frac{x}{y}} = \frac{c}{y^4}$$

Question 17

Show that the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin is $y + 2\left(x + 1\right) = 2e^{2x}$.

Solution 17

Slope at any point = $y + 2x$

$$\frac{dy}{dx} = y + 2x$$

$$\frac{dy}{dx} - y = 2x$$

It is a linear differential equation. comparing it with $\frac{dy}{dx} + Py = Q$

$P = -1, Q = 2x$

$$I.F. = e^{\int P dx}$$

$$= e^{\int(-1)dx}$$

$$= e^{-x}$$

Solution of the equation is given by

$$(I.F.) = \int Q\,(I.F.)\,dx + c$$

$$y\left(e^{-x}\right) = \int (2x)\left(e^{-x}\right) dx + c$$

$$y\left(e^{-x}\right) = 2\int xe^{-x} dx + c$$

$$y\left(e^{-x}\right) = 2\left[x\left(-e^{-x}\right) + \int 1e^{-x} dx\right] + c$$

$$y\left(e^{-x}\right) = -2xe^{-x} - 2e^{-x} + c$$

$$y = -2x - 2 + ce^{x}$$

$$y + 2(x + 1) = ce^{x} \quad ---(1)$$

It is passing through origin,

$$0 + 2(0 + 1) = ce^{0}$$

$$2 = c$$

Now, equation (1) becomes,

$$y + 2(x + 1) = 2e^{x}$$

Question 18

The tangent at any point (x, y) of a curve makes s an angle $\tan^{-1}(2x + 3y)$ with x – axis. Find the equation of the curve if it passes through $(1,2)$.

Solution 18

Given, tangent makes on angle $\tan^{-1}(2x+3y)$ with x-axis,

Slope of tangent $= \tan\theta$

$$\frac{dy}{dx} = \tan\left(\tan^{-1}(2x+3y)\right)$$

$$\frac{dy}{dx} = 2x + 3y$$

$$\frac{dy}{dx} - 3y = 2x$$

It is a linear differetial equation comparing it with $\frac{dy}{dx} + Py = Q$

$P = -3, Q = 2x$

$$I.F. = e^{\int P dx}$$

$$= e^{-\int 3 dx}$$

$$= e^{-3x}$$

Solution of the equation on given by

$$y\,(\text{I.F.}) = \int Q\,(I.F.)\,dx + c$$

$$y\left(e^{-3x}\right) = \int 2xe^{-3x}dx + c$$

$$= 2\left[x\left(\frac{-e^{-3x}}{3}\right) - \int 1.\left(\frac{-e^{-3x}}{3}\right)dx\right] + c$$

$$= -\frac{2}{3}xe^{-3x} + \frac{2}{3}\int e^{-3x}dx + c$$

$$y\left(e^{-3x}\right) = -\frac{2}{3}xe^{-3x} + \frac{2}{9}e^{-3x} + c$$

$$y = -\frac{2}{3}x - \frac{2}{9} + ce^{3x} \quad -----(1)$$

It is passing through $(1,2)$,

$$2 = -\frac{2}{3} - \frac{2}{9} + ce^{3}$$

$$2 = -\frac{8}{9} + ce^{3}$$

$$\frac{26}{9} = ce^{3}$$

$$c = \frac{26}{9}e^{-3}$$

Now equation (1) becomes,

$$ye^{-3x} = \left(-\frac{2}{3}x - \frac{2}{9}\right)e^{-3x} + \frac{26}{9}e^{-3}$$

Question 19

Find the equation of the curve such that the portion of the x-axis cut off between the origin and the tangent at a point is twice the abscissa and wahich passes through the point $(1,2)$.

Solution 19

Let $P(x, y)$ be the point of contact of tangent whit curve $y = f(x)$ equatin of tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

Put Y = 0

$$-y = \frac{dy}{dx}(X - x)$$

$$X = X - \frac{ydx}{dx}$$

Coordinate of $B = \left(x - y\frac{dx}{dy}, 0\right)$

Given, (intercept on x - axis) = $4x$

$$x - y\frac{dx}{dy} = 2x$$

$$-y\frac{dx}{dy} = 2x - x$$

$$-y\frac{dx}{dy} = x$$

$$-\frac{dx}{x} = \frac{dy}{y}$$

$$-\int\frac{dx}{x} = \int\frac{dy}{y}$$

$$-\log x = \log y + c - - - (1)$$

It is passing through $(1,2)$

$$-\log 1 = \log 2 + c$$

$$c = -\log 2$$

Put c in equation (1)

$$-\log x = \log y - \log 2$$

$$\frac{1}{x} = \frac{y}{2}$$

$$xy = 2$$

Question 20

Find the equation of the curve satisfying $x(x+1)\frac{dy}{dx} - y = x(x+1)$ passing through $(1,0)$.

Solution 20

$$x(x+1)\frac{dy}{dx} - y = x(x+1)$$
$$\frac{dy}{dx} - \frac{y}{x(x+1)} = \frac{x(x+1)}{x(x+1)}$$
$$\frac{dy}{dx} - \frac{y}{x(x+1)} = 1$$

It is linear differential equation coparing it with $\frac{dy}{dx} + Py = Q$

$$P = -\frac{1}{x(x+1)}, \qquad Q = 1$$
$$I.F. = e^{\int \frac{1}{x(x+1)} dx}$$
$$= e^{\int \left(\frac{1}{x} - \frac{1}{(x+1)}\right) dx}$$
$$= e^{-\log|x| + \log|x+1|}$$
$$= e^{\log\left(\frac{x+1}{x}\right)}$$
$$= \frac{x+1}{x}$$

Solution of the equation is given by

$$y(I.F.) = \int Q(I.F.)\,dx + c$$
$$y\left(\frac{x+1}{x}\right) = \int \left(\frac{x+1}{x}\right) dx + c$$
$$y\left(\frac{x+1}{x}\right) = \int \left(1 + \frac{1}{x}\right) dx + c$$
$$y\left(\frac{x+1}{x}\right) = x + \log|x| + c \text{ ---} (1)$$

It is passing through $(1,0)$, so

$$0 = 1 + \log(1) + c$$
$$-1 = c$$

Now,equation (1) becomes,

$$y\left(\frac{x+1}{x}\right) = x + \log|x| - 1$$

$$y(x+1) = x(x + \log x - 1)$$

Question 21

Find the equation of the curve which passes through the poing $(3,-4)$ and has the slope $\frac{2y}{x}$ at any point (x, y) on it.

Solution 21

Slope of the curve $= \frac{2y}{x}$

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\frac{dy}{y} = \frac{2}{x}dx$$

$$\int \frac{dy}{y} = 2\int \frac{1}{x}dx$$

$$\log|y| = 2\log|x| + \log|c|$$

$$y = x^2 c \quad ---(1)$$

It is passing through $(3, -4)$ so,

$$-4 = (3)^2 c$$

$$-4 = 9c$$

$$c = -\frac{4}{9}$$

Now, equation (1) becomes,

$$y = -\frac{4}{9}x^2$$

$$9y = -4x^2$$

$$9y + 4x^2 = 0$$

Question 22

Find the equation of the curve which passes through the origin and has the scope $x + 3y - 1$ at any point (x, y) on it.

Solution 22

Given,

Slope of the equation $= x + 3y - 1$

$$\frac{dy}{dx} = x + 3y - 1$$

$$\frac{dy}{dx} - 3y = x - 1$$

It is a linear differential equation. Camparing it with $\frac{dy}{dx} + Py = Q$

$P = -3, Q = x - 1$

$$I.F. = e^{\int P dx}$$

$$= e^{\int -3 dx}$$

$$= e^{-3x}$$

Solution of the equation is given by,

$$y\left(I.F.\right) = \int Q\left(I.F.\right) dx + c$$

$$y\left(e^{-3x}\right) = \int\left(x - 1\right)\left(e^{-3x}\right) dx + c$$

$$y\left(e^{-3x}\right) = \left(x - 1\right)\left(-\frac{1}{3}e^{-3x}\right) - \int\left(1\right)\left(\frac{-e^{-3x}}{3}\right) dx + c$$

$$y\left(e^{-3x}\right) = -\frac{\left(x - 1\right)}{3}e^{-3x} + \left(-\frac{e^{-3x}}{9}\right) + c$$

$$y = -\frac{x}{3} + \frac{1}{3} - \frac{1}{9} + ce^{3x}$$

$$y = -\frac{x}{3} + \frac{2}{9} + ce^{3x}$$

It is passing through origin, so

$$0 = 0 + \frac{2}{9} + ce^{3(0)}$$

$$0 = \frac{2}{9} + c$$

$$c = -\frac{2}{9}$$

Now, equation (1) becomes,

$$y = -\frac{x}{3} + \frac{2}{9} - \frac{2}{9}e^{3x}$$

$$9y = -3x + 2 - 2e^{3x}$$

$$3\left(3y + x\right) = 2\left(1 - e^{3x}\right)$$

Question 23

At every point on a curve the slope is the sum of the abscissa and the product of the ordinate and the abscissa, and the curve passes through $(0,1)$. Find the equation of the curve.

Solution 23

Given,

Slope at point $(x, y) = x + xy$

$$\frac{dy}{dx} = x(y+1)$$

$$\frac{dy}{y+1} = x\,dx$$

$$\int \frac{dy}{y+1} = \int x\,dx$$

$$\log|y+1| = \frac{x^2}{2} + c ------(1)$$

It is passing through $(0, 1)$, so,

$$\log z = 0 + c$$

$$c = \log 2$$

Now,equation (2) becomes,

$$\log|y+1| = \frac{x^2}{2} + \log 2$$

$$y + 1 = 2e^{\frac{x^2}{2}}$$

Question 24

A curve is such that the lenght of the perpendicular from the origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is $y^2 - 2xy\frac{dy}{dx} - x^2 = 0$, and hence find the curve.

Solution 24

$$y^2 - 2xy\frac{dy}{dx} - x^2 = 0$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

It is a homeganeous equation.

put, $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now,

$$x\frac{dv}{dx} + v = \frac{v^2x^2 - x^2}{2xvx}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$

$$x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{-v^2 - 1}{2v}$$

$$\int\frac{2v}{v^2 + 1}dv = -\int\frac{dx}{x}$$

$$\log\left|v^2 + 1\right| = -\log|x| + \log|c|$$

$$v^2 + 1 = \frac{c}{x}$$

$$\frac{y^2 + x^2}{x^2} = \frac{c}{x}$$

$$y^2 + x^2 = cx$$

$$y^2 + x^2 - cx = 0$$

Differentiating it with respect to x,

$$2x + 2y\frac{dy}{dx} - c = 0$$

$$\frac{dy}{dx} = \frac{c - 2x}{2y}$$

Let (h,k) be the point where tangent passes through origin and length is equal to h. so, equation of tangent at (h,k) is

$$(y-k)=\left(\frac{dy}{dx}\right)_{(h,k)}(x-h)$$

$$(y-k)=\left(\frac{c-2h}{2k}\right)(x-h)$$

$$2ky-2k^2=xc-2hx-hc+2h^2$$

$$x(c-2h)-2ky+2k^2-hc+2h^2=0$$

$$x(c-2h)-2ky+2\left(k^2+h^2\right)-hc=0$$

$$x(c-2h)-2ky+2(ch)-hc=0$$

$$\left[\text{Since } h^2+k^2=ch \text{ as } (h,k) \text{ is on the curve}\right]$$

$$x(c-2h)-2ky+hc=0$$

length of perpendicular as tangent from origin is

$$L=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

$$=\left|\frac{(0)(c-2h)+(0)(-2k)+hc}{\sqrt{(c-2h)^2+(-2k)^2}}\right|$$

$$=\frac{hc}{\sqrt{c^2+4h^2+4k^2-4ch}}$$

$$L=\frac{hc}{\sqrt{c^2+4\left(h^2+k^2-ch\right)}}$$

$$=\frac{hc}{\sqrt{c^2+4(0)}}$$

$$=\frac{hc}{c}$$

$$=c$$

Hence,

$$x^2+y^2=cx \text{ is the required curve}$$

Question 25

Find the equation of the curve which passes through the point $(1,2)$ and the distance between the foot of the ordinate of the point of contact and the point of intersection of the tangent with x-axis is twice the abscissa of the point of contact.

Solution 25

Let $P(x, y)$ be the point of contact of tangent and curve $y = f(x)$. Equation tangent at $P(x, y)$ is

$$Y - y = \frac{dx}{dy}(X - x)$$

put $Y = 0$

$$-y = \frac{dx}{dy}(X - x)$$

$$-y = \frac{dx}{dy}(X - x)$$

$$X = x - y\frac{dx}{dy}$$

coordinate of $B = \left(x - y\frac{dx}{dy}, 0\right)$

Given,

Distance between foot of ordinate of the point of contact and the point of intersection of tangent and x – axis = $2x$

$$BC = 2x$$

$$\sqrt{\left(x - y\frac{dx}{dy} - x\right)^2 + (0)^2} = 2x$$

$$y\frac{dx}{dy} = 2x$$

$$y\frac{dx}{x} = 2\frac{dy}{y}$$

$$\int\frac{dx}{x} = 2\int\frac{dy}{y}$$

$$\log x = 2\log y + \log c \text{ ---} (1)$$

It is passing through $(1,2)$,

$$\log 1 = 2\log 2 + \log c$$

$$-2\log 2 = \log c$$

$$\log\left(\frac{1}{4}\right) = \log c$$

$$c = \frac{1}{4}$$

Put value of c in equation (1),

$$\log x = 2\log y + \log\left(\frac{1}{4}\right)$$

$$x = \frac{y^2}{4}$$

$$y^2 = 4x$$

Question 26

The normal to a given curve at each point (x, y) on the curve passes through the point $(3,0)$. If the curve contains the point $(3,4)$ find its equation.

Solution 26

Equation of normal on point (x, y) on the curve

$$Y - y = \frac{-dx}{dy}(X - x)$$

Itis passing through $(3,0)$

$$0 - y = \frac{-dx}{dy}(3 - x)$$

$$y = \frac{dx}{dy}(3 - x)$$

$$ydy = (3 - x)dx$$

$$\int y\,dy = \int (3 - x)\,dx$$

$$\frac{y^2}{2} = 3x - \frac{x^2}{2} + c \text{ --- } (1)$$

It passing through $(3,4)$, so,

$$\frac{16}{2} = 9 - \frac{9}{2} + c$$

$$\frac{16}{2} = \frac{9}{2} + c$$

$$c = 7$$

Put $c = 7$ is equation (1)

$$\frac{y^2}{2} = 3x - \frac{x^2}{2} + \frac{7}{2}$$

$$y^2 = 6x - x^2 + 7$$

Question 27

The rate of increase of bacteria in a culture is proportional to the number of bacteria present and it is found that the number doubles in 16 hours. Prove that the bacteria becomes 8 times at the end of 18 hours.

Solution 27

Let A be the quantity of bacteria present in culture at any time t and initial quantity of bacteria is A_0.

$$\frac{dA}{dt} \propto A$$
$$\frac{dA}{dt} = \lambda A$$
$$\frac{dA}{A} = \lambda dt$$
$$\int \frac{dA}{A} = \lambda \int dt$$
$$\log A = \lambda t + c \text{ ----} (1)$$

Initially, $A = A_0, t = 0$

$$\log A_0 = 0 + c$$
$$\log A_0 = c$$

Now eqution (1) becomes,

$$\log A = \lambda t + \log A_0$$
$$\log\left(\frac{A}{A_0}\right) = \lambda t \text{ -----} (2)$$

Given $A = 2A_0$ when $t = 6$ hours

$$\log\left(\frac{A}{A_0}\right) = 6\lambda$$
$$\frac{\log 2}{6} = \lambda$$

Now equation (2) becomes,

$$\log\left(\frac{A}{A_0}\right) = \frac{\log 2}{6}t$$

Now, $A = 8A_0$

so, $$\log\left(\frac{8A_0}{A_0}\right) = \frac{\log 2}{6}t$$
$$\log 2^3 = \frac{\log 2}{6}t$$
$$3\log 2 = \frac{\log 2}{6}t$$
$$18 = t$$

Therefore,

Bacteria becomes 8 times in 18 hours

Question 28

Radium decomposes at a rate proporational to the quantity of radium present. It is found that in 25 year, approximately 1.1% of a certain quantity of radium has decomposed. Determine approximately how long it will take for one-half of the original amount of radium to decompose?

$\left[\text{Given } \log_e \ 0.989 = 0.1106 \text{ and } \log_e 2 = 06931\right]$

Solution 28

Let A be the quantity of radium present at time t and A_0 be the initial quantity of radium.

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = -\lambda A$$

$$\frac{dA}{A} = -\lambda dt$$

$$\int \frac{dA}{A} = -\lambda \int dt$$

$$\log A = -\lambda t + c - - - - (2)$$

Now, $A = A_0$ when $t = 0$

$$\log A_0 = 0 + c$$

$$c = \log A_0$$

Put value of c in equation

$$\log A = -\lambda t + \log A_0$$

$$\log\left(\frac{A}{A_0}\right) = -\lambda t - - - (2)$$

Given that,

In 25 years bacteria decomposes 1.1% ,so

$A = (100 - 1.1)\% = 98.9\% = 0.989A_0, t = 5$

$$\log\left(\frac{0.989A_0}{A_0}\right) = -\lambda 25$$

$$\log(0.989) = -25\lambda$$

$$\lambda = -\frac{1}{25}\log(0.989)$$

Now, equation (2) becomes,

$$\log\left(\frac{A}{A_0}\right) = \left\{\frac{1}{25}\log(0.989)\right\}t$$

Now $A = \frac{1}{2}A_0$

$$\log\left(\frac{A}{2A}\right) = \frac{1}{25}\log(0.989)t$$

$$\frac{-\log 2 \times 25}{\log(0.989)} = t$$

$$-\frac{0.6931 \times 25}{0.01106} = t$$

$t = 1567$ years.

Required time $= 1567$ years

Question 29

Show that all curves for which the slope at any point (x, y) on it is $\frac{x^2+y^2}{2xy}$ are rectangular hyperbola.

Solution 29

Given,

$$\text{Slope of tangent} = \frac{x^2+y^2}{2xy}$$

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

It is a homeganeous equation.

put, $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now,

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2xvx}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1+v^2-2v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{v}$$

$$\frac{v}{1-v^2}dv = \frac{dx}{x}$$

$$\int \frac{v}{1-v^2}dv = \int \frac{dx}{x}$$

$$\int \frac{-2v}{1-v^2}dv = \int \frac{-2dx}{x}$$

$$\log\left|1-v^2\right| = -2\log x + \log c$$

$$1 - \frac{y^2}{x^2} = \frac{c}{x^2}$$

$$\frac{x^2-y^2}{x^2} = \frac{c}{x^2}$$

$$x^2 - y^2 = c$$

It is equation of rectangular hyperbola.

Question 30

The slope of the tangent at each point of curve is equal to sum of the coordinates of the point. Find the curve that passes through the origin.

Solution 30

Given,

Slope of tangent at $(x, y) = x + y$

$$\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} - y = x$$

It is a linear differential equation.Comparing it with $\frac{dy}{dx} + Py = Q$

$P = -1, Q = x$

$$I.F. = e^{\int P dx}$$

$$= e^{\int (-1) dx}$$

$$= e^{-x}$$

Solution of equation is given by,

$$y\left(I.F.\right) = \int Q\left(I.F.\right) dx + c$$

$$y\left(e^{-x}\right) = \int xe^{-x} dx + c$$

$$ye^{-x} = x\left(e^{-x}\right) + \int\left(1 \times e^{-x}\right) dx + c$$

[Using integration by parts]

$$ye^{-x} = -xe^{-x} - e^{-x} + c$$

$$y = -x - 1 + ce^{x} \text{ - - - - - } (1)$$

It is passing through origin

$$0 = 0 - 1 + ce^{0}$$

$$1 = c$$

Put $c = 1$ is equation

$$y = -x - 1 + e^{x}$$

$$y + x + 1 = e^{x}$$

Question 31

Find the equation of a curve passing through the point $(0,1)$.If the slope of the tangent to the curve at any point (x,y) is equal to the sum of the x coordinate and the product of the x coordinate and y coordinate of that point.

Solution 31

We know that the slope of the tangent to the curve is $\frac{dy}{dx}$.

$\therefore \qquad \frac{dy}{dx} = x + xy$

$\Rightarrow \qquad \frac{dy}{dx} - xy = y \qquad ------(i)$

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$

where $P = -x$ and $Q = x$.

So, $\qquad I.F. = e^{\int -x dx} = e^{\frac{-x^2}{2}}$

$\therefore$ Solution of the given equation is given by

$$y.\, e^{\frac{-x^2}{2}} = \int x.e^{\frac{-x^2}{2}}\, dx + C \qquad ------(ii)$$

Let $\qquad I = \int x.e^{\frac{-x^2}{2}}\, dx$

Let $\frac{-x^2}{2} = t$, then $-x\,dx = dt$ or $x\,dx = -dt$

$\therefore \qquad I = \int x.e^{\frac{-x^2}{2}}\, dx = \int -e^t\, dt = -e^t = -e^{\frac{-x^2}{2}}$

Substituting the value of I in (ii), we get

$$y.\, e^{\frac{-x^2}{2}} = -e^{\frac{-x^2}{2}} + C$$

or $\qquad y = -1 + Ce^{\frac{x^2}{2}} \qquad ------(iii)$

This equation (iii) passes through (0,1)

$\therefore \qquad 1 = -1 + Ce^0 \quad \Rightarrow \quad C = 2$

Substituting the value of C in (iii), we get

$$y = -1 + 2e^{\frac{x^2}{2}}$$

which is the equation of the required curve.

Question 32

The slope of a curve at each of its points is equal to the square of the abscissae of the point. Find the particular curve through the point $(-1, 1)$.

Solution 32

Given,

Slope of tangent at $(x, y) = x^2$

$$\frac{dy}{dx} = x^2$$

$$dy = x^2 dx$$

$$\int dy = \int x^2 dx$$

$$y = \frac{x^3}{3} + c - - - - (1)$$

It is passing through $(-1, 1)$

$$1 = \frac{(-1)}{3} + c$$

$$1 = -\frac{1}{3} + c$$

$$c = 1 + \frac{1}{3}$$

$$c = \frac{4}{3}$$

Put is equation

$$y = \frac{x^3}{3} + \frac{4}{3}$$

$$3y = x^3 + 4$$

Question 33

Find the equation of the curve that passes through the point $(0, a)$ and is such that at any point (x, y) on it, the product of its slope and the ordinate is equal to the abscissa.

Solution 33

Given,

$$y\left(\text{Slope of tangent}\right) = x$$

$$y\frac{dy}{dx} = x$$

$$ydy = xdx$$

$$\int ydy = \int xdx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c - - - - (1)$$

It is passing through $(0, a)$

$$\frac{a^2}{2} = 0 + c$$

$$c = \frac{a^2}{2}$$

Put $c = \frac{a^2}{2}$ is equation (1)

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{a^2}{2}$$

$$y^2 = x^2 + a^2$$

Question 34

The x-intercept of the tangent line to a curve is equal to the ordinate of the point of contact. Find the particular curve through the point $(1, 1)$.

Solution 34

Let $P(x, y)$ be the point on the curve $y = f(x)$ such that tangent at P cuts the coordinate axes at A and B.

The equation of tangent is,

$$Y - y = \frac{dy}{dx}(X - x)$$

Put $Y = 0$

$$-y = \frac{dy}{dx}(X - x)$$

$$-y\frac{dy}{dx} + x = X$$

Coordinate of $B = \left(-y\frac{dy}{dx} + x, 0\right)$

Here, x intercept of tangent $= y$

$$-y\frac{dx}{dy} + x = y$$

$$\frac{dx}{dy} - \frac{x}{y} = -1$$

It is a linear differential equation on comparing it with $\frac{dx}{dy} + py = Q$

$P = \frac{1}{y}, Q = -1$

$$I.F. = e^{\int\left(\frac{1}{y}\right)dy}$$

$$= e^{\log y}$$

$$= \frac{1}{y}$$

Solution of the equation is given by,

$$x(I.F.) = \int Q(I.F)dy + c$$

$$x\left(\frac{1}{y}\right) = \int(-1)\left(\frac{1}{y}\right)dy + c$$

$$x\left(\frac{1}{y}\right) = -\log y + c ----(1)$$

It is passing through $(1,1)$

$$\frac{1}{1} = -\log 1 + c$$

$$c = 1$$

put $c = 1$ is equation (1),

$$\frac{x}{y} = -\log y + 1$$

$$x = y - y\log y$$

$$x + y\log y = y$$

Chapter 22 - Differential Equations Exercise Ex. 22RE

Question 1(i)

Determine order and degree(if defined) of differential equation $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$

Solution 1(i)

$$\left(\frac{ds}{dt}\right)^4 + 3\frac{d^2s}{dt^2} = 0$$

The highest order derivative present in the given differential equation is $\frac{d^2s}{dt^2}$. Therefore, its order is two.

It is a polynomial equation in $\frac{d^2s}{dt^2}$ and $\frac{ds}{dt}$. The power raised to $\frac{d^2s}{dt^2}$ is 1.

Hence, its degree is one.

Question 1(ii)

Determine order and degree(if defined) of differential equation $y''' + 2y'' + y' = 0$

Solution 1(ii)

$$y''' + 2y'' + y' = 0$$

The highest order derivative present in the differential equation is y'''. Therefore, its order is three.

It is a polynomial equation in y''', y'' and y'. The highest power raised to y''' is 1. Hence, its degree is 1.

Question 1(iii)

Determine order and degree(if defined) of differential equation $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

Solution 1(iii)

$$(y''')^2 + (y'')^3 + (y') + y^5 = 0$$

The highest order derivative present in the differential equation is y'''. Therefore, its order is three.

The given differential equation is a polynomial equation in y''', y'', and y'.

The highest power raised to y''' is 2. Hence, its degree is 2.

Question 1(iv)

Determine order and degree(if defined) of differential equation $y''' + 2y'' + y' = 0$

Solution 1(iv)

$y''' + 2y'' + y' = 0$

The highest order derivative present in the differential equation is y'''. Therefore, its order is three.

It is a polynomial equation in y''', y'' and y'. The highest power raised to y''' is 1. Hence, its degree is 1.

Question 1(v)

Determine order and degree(if defined) of differential equation $y'' + (y')^2 + 2y = 0$

Solution 1(v)

$y'' + (y')^2 + 2y = 0$

The highest order derivative present in the differential equation is y''. Therefore, its order is two.

The given differential equation is a polynomial equation in y'' and y' and the highest power raised to y'' is one.

Hence, its degree is one.

Question 1(vi)

Determine order and degree(if defined) of differential equation $y'' + 2y' + \sin y = 0$

Solution 1(vi)

$y'' + 2y' + \sin y = 0$

The highest order derivative present in the differential equation is y''. Therefore, its order is two.

This is a polynomial equation in y'' and y' and the highest power raised to y'' is one. Hence, its degree is one.

Question 1(vii)

Determine the order and degree (if defined) of the differential equation:

$y''' + y^2 + e^{y'} = 0$

Solution 1(vii)

The highest order derivative present in the differential equation is y''', so its order is 3.

The given differential equation is not a polynomial equation in its derivatives, so its degree is not defined.

Question 2

Verify that the function $y = e^{-3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

Solution 2

$$y = e^{-3x}$$

Differentiating with respect to x,

$$\frac{dy}{dx} = -3e^{-3x}$$

Again, differentiating with respect to x,

$$\frac{d^2y}{dx^2} = 9e^{-3x}$$

$$= -\left(-3e^{-3x} - 6e^{-3x}\right)$$

$$\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx} - 6y\right)$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

So, $y = e^{-3x}$ is the solution of the given equation.

Question 3(i)

Verify that the given function (explicit or implicit) is a solution of the differential equation:

$y = e^x + 1 \qquad y'' - y' = 0$

Solution 3(i)

$$y = e^x + 1$$

Differentiating with respect to x,

$$y' = e^x$$

Differentiating with respect to x,

$$y'' = e^{3x}$$

$$y'' = y'$$

$$y'' - y' = 0$$

So, $y = e^x + 1$ is the solution of the given equation.

Question 3(ii)

Verify that the given function (explicit or implicit) is a solution of the differential equation:

$y = x^2 + 2x + c \qquad y' - 2x - 2 = 0$

Solution 3(ii)

$$y = x^2 + 2x + c$$

Differentiating with respect to x,

$$y' = 2x + 1$$

$$y' - 2x - 2 = 0$$

So, $y = x^2 + 2x + c$ is the solution of the given equation.

Question 3(iii)

Verify that the given function (explicit or implicit) is a solution of the differential equation:

$y = \cos x + c \qquad y' + \sin x = 0$

Solution 3(iii)

$$y = \cos x + c$$

Differentiating with respect to x,

$$y' = -\sin x$$

$$y' + \sin x = 0$$

So, $y = \cos x + c$ is the solution of the given equation.

Question 3(iv)

Verify that the given function (explicit or implicit) is a solution of the differential equation:

$y = \sqrt{1 + x^2} \qquad y' \frac{xy}{1 + x^2}$

Solution 3(iv)

$$y = \sqrt{1 + x^2}$$

Squaring both the sides,

$$y^2 = 1 + x^2$$

Differentiating with respect to x,

$$2yy' = 2x$$

$$y' = \frac{2x}{2y}$$

$$y' = \frac{x}{y}$$

$$= \frac{xy}{y^2}$$

$$y' = \frac{xy}{\left(1 + x^2\right)}$$

So, $y = \sqrt{1 + x^2}$ is the solution of the given equation.

Question 3(v)

Verify that the given function (explicit or implicit) is a solution of the differential equation:

$y = x \sin x \qquad xy' = y + x\sqrt{x^2 - y^2}\ (x \neq 0 \text{ and } x > y \text{ or } x < -y)$

Solution 3(v)

$$y = x \sin x$$

Differentiating, we get,

$$y' = x \cos x + \sin x$$

Now,

$$y + x\sqrt{x^2 - y^2}$$
$$= x \sin x + x\sqrt{x^2 - x^2 \sin^2 x}$$
$$= x \sin x + x\sqrt{x^2\left(1 - \sin^2 x\right)}$$
$$= x \sin x + x\sqrt{x^2 \cos^2 x}$$
$$= x \sin x + x^2 \cos x$$
$$= x\left(\sin x + x \cos x\right)$$
$$= xy'$$

So,

$$y + x\sqrt{x^2 - y^2} = xy'$$

So, $y = x \sin x$ is the solution of the given equation.

Question 3(vi)

Verify that the given function (explicit or implicit) is a solution of the differential equation:

$y = \sqrt{a^2 - x^2}, x \in (-a, a) \qquad x + y\frac{dy}{dx} = 0, y \neq 0$

Solution 3(vi)

$$y = \sqrt{a^2 - x^2}$$

Squaring both the sides,

$$y^2 = a^2 - x^2$$

Differentiating with respect to x,

$$2y\frac{dy}{dx} = -2x$$
$$y\frac{dy}{dx} = -x$$
$$x + y\frac{dy}{dx} = 0$$

So, $y = \sqrt{a^2 - x^2}$ is the solution of the given equation.

Question 4

Form the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant.

Solution 4

$$y = mx \qquad \text{---(i)}$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = m$$

Using equation (i),

$$\frac{dy}{dx} = \frac{y}{x}$$

$$x\frac{dy}{dx} = y$$

$$x\frac{dy}{dx} - y = 0$$

Question 5

Form the differential equation representing the family of curves $y = a\,sin(x + b)$, where a, b are arbitrary constant.

Solution 5

$$y = a\sin(x + b)$$

Differentiating it with respect to x,

$$\frac{dy}{dx} = a\cos(x + b)$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = -a\sin(x + b)$$

$$\frac{d^2y}{dx^2} = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

Question 6

Form the differential equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.

Solution 6

We know that, equation of parabolas having vertex at origin and axis along positive direction of x-axis.

$$y^2 = 4ax \qquad \text{---(i)}$$

Differentiating it with respect to x,

$$2y\frac{dy}{dx} = 4a$$

$$2y\frac{dy}{dx} = \frac{y^2}{x}$$

Using equation(i),

$$2xy\frac{dy}{dx} = y^2$$

$$y^2 - 2xy\frac{dy}{dx} = 0$$

Question 7

Form the differential equation of the family of circles having centre on y-axis and radius 3 unit.

Solution 7

We know that, equation of family of circles with centre (h,k) and radius r is given by,

$$(x-h)^2+(y-k)^2=r^2$$

Centre is an y-axis, so $h=0$ and $r=3$

$$x^2+(y-k)^2=9 \qquad \text{---(i)}$$

Differentiating it with respect to x,

$$2x+2(y-k)\frac{dy}{dx}=0$$

$$x+(y-)\frac{dy}{dx}=0$$

$$(y-k)=\frac{-x}{\left(\frac{dy}{dx}\right)}$$

Using equation (i),

$$\left(x^2-9\right)+\left(\frac{-x}{\frac{dy}{dx}}\right)^2=0$$

$$\left(x^2-9\right)\left(\frac{dy}{dx}\right)^2+x^2=0$$

$$\left(x^2-9\right)(y')^2+x^2=0$$

Question 8

Form the differential equation of the family of the familyof parabolas having vertex at origin and axis along positive y-axis.

Solution 8

We know that, equation of family of parabolas having vertex at origin and axis along positive y-axis,

$$x^2=4ay$$

Differentiating it with respect to x,

$$2x=4a\frac{dy}{dx}$$

$$2x=\frac{x^2}{y}\frac{dy}{dx}$$

Using equation (i),

$$2y=xy'$$

$$xy'-2y=0$$

Question 9

Form the differential equation of the family of ellipses having foci on y-axis and centre at the origin.

Solution 9

Equation of family of ellipses with foci an y-axis and centre at origin.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad b>a$$

Differentiating it with respect to x,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{x}{a^2}+\frac{y}{b^2}\frac{dy}{dx}=0 \qquad \text{---(i)}$$

Differetntiating it with respect to x,

$$\frac{1}{a^3}+\frac{1}{b^2}\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)+\frac{y}{b^2}\frac{d^2y}{dx^2}=0$$

Multiplying with x,

$$\frac{x}{a^2}+\frac{x}{b^2}\left(\frac{dy}{dx}\right)^2+\frac{xy}{b^2}\frac{d^2y}{dx^2}=0$$

Using equation (i),

$$\frac{-y}{b^2}\frac{dy}{dx}+\frac{x}{b^2}\left(\frac{dy}{dx}\right)^2+\frac{xy}{b^2}\left(\frac{d^2y}{dx^2}\right)=0$$

$$-yy'+x(y')^2+xy(y'')=0$$

$$xy(y'')+x(y')^2-yy'=0$$

Question 10

Form the differential equation of the family of hyperbolas having foci on x-axis and centre at the origin.

Solution 10

We know that family of hyperbola is given by,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Differentiating it with respect to x,

$$\frac{2x}{a^2}-\frac{2y}{b^2}(y')=0 \qquad \text{---(i)}$$

Again, differetntiating it with respect to x,

$$\frac{2}{a^2}-\frac{2}{b^2}[y'\times y'+y\times y'']=0$$

Multiplying with x,

$$\frac{2x}{a^2}-\frac{2x}{b^2}\left[(y')^2+y\times y'\right]=0$$

Using equation (i),

$$\frac{2y}{b^2}(y')-\frac{2x}{b^2}\left[(y')^2+y\times y''\right]=0$$

$$yy'-x(y')^2-xyy''=0$$

Question 11

Verify that $xy = ae^x + be^{-x} + x^2$ is a solution of the differential equation

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0.$$

Solution 11

$$xy = ae^x + be^{-x} + x^2 \quad \text{---(i)}$$

Differentiating with respect to x,

$$xy' + y = ae^x - be^{-x} + 2x$$

Again, differentiating with respect to x,

$$xy'' + y' + y' = \left(ae^x + be^{-x}\right) + 2$$

$$xy'' + 2y' = xy - x^2 + 2$$

Using equatoin (i),

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$$

So, $xy = ae^x + be^{-x} + x^2$ is the solution of the given equation.

Question 12

Show that $y = cx + 2c^2$ is a solution of the differential equation

$$2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0.$$

Solution 12

$$y = cx + 2c^2 \quad \text{---(i)}$$

Differentiating with respect to x,

$$\frac{dy}{dx} = c$$

From equation (i),

$$y = x\left(\frac{dy}{dx}\right) + 2\left(\frac{dy}{dx}\right)^2$$

$$2\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) - y = 0$$

So, $y = cx + 2c^2$ is the solutoin of the given equation.

Question 13

Show that $y^2 - x^2 - xy = a$ is a solution of the differential equation

$$(x - 2y)\frac{dy}{dx} + 2x + y = 0.$$

Solution 13

$$y^2 - x^2 - xy = a$$

Differentiating with respect to x,

$$2y\frac{dy}{dx} - 2x - \left[x\frac{dy}{dx} + y\right] = 0$$

$$x\frac{dy}{dx} - 2y\frac{dy}{dx} + 2x + y = 0$$

$$(x - 2y)\frac{dy}{dx} + 2x + y = 0$$

So, $y^2 - x^2 - xy = a$ is the solution of the given equation.

Question 14

Verify that $y = A\cos x + \sin x$ satisfies the differential equation $\cos x\frac{dy}{dx} + (\sin x)y = 1$.

Solution 14

$$y = A\cos x + \sin x$$

Differentiating with respect to x,

$$\frac{dy}{dx} = -A\sin x + \cos x$$

Multiplying both the sides by $\cos x$,

$$\cos x\frac{dy}{dx} = -A\sin x\cos x + \cos^2 x$$
$$= -A\sin x\cos x + 1 - \sin^2 x$$
$$= -A\sin x\cos x - \sin^2 x + 1$$
$$= -\sin x\left(A\cos x + six\right) + 1$$

$$\cos x\frac{dy}{dx} = -(\sin x)y + 1$$

$$\cos x\frac{dy}{dx} + (\sin x)y = 1$$

So, $y = A\cos x + \sin x$ is the solution of the given equation.

Question 15

Find the differential equation corresponding to $y = ae^{2x} + be^{-3x} + ce^{x}$, *where* a, b, c *are arbitrary constants*.

Solution 15

Consider the given equation $y = ae^{2x} + be^{-3x} + ce^{x}$
Differentiating the above equation with respect to x, *we have*,

$$\frac{dy}{dx} = 2ae^{2x} - 3be^{-3x} + ce^{x} \ldots.(1)$$

$$\Rightarrow 7\frac{dy}{dx} = 14ae^{2x} - 21be^{-3x} + 7ce^{x} \ldots.(2)$$

Differentiating equation (1) *with respect to* x, *we have*,

$$\frac{d^2y}{dx^2} = 4ae^{2x} + 9be^{-3x} + ce^{x} \ldots.(3)$$

Again differentiating the above equation with respect to x, *we have*,

$$\frac{d^3y}{dx^3} = 8ae^{2x} - 27be^{-3x} + ce^{x} \ldots.(4)$$

Now consider the following expression

$$\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y$$

$$= 8ae^{2x} - 27be^{-3x} + ce^{x} - 14ae^{2x} + 21be^{-3x} - 7ce^{x} + 6\left(ae^{2x} + be^{-3x} + ce^{x}\right)$$

$$= 8ae^{2x} - 27be^{-3x} + ce^{x} - 14ae^{2x} + 21be^{-3x} - 7ce^{x} + 6ae^{2x} + 6be^{-3x} + 6ce^{x}$$

$$= 0$$

Thus, the required differential equation corresponding to
$y = ae^{2x} + be^{-3x} + ce^{x}$ *is*

$$\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$$

Question 16

Show that the differential equation of all parabolas which have their axes parallel to y-axis is $\frac{d^3y}{dx^3} = 0$.

Solution 16

Equation of parabolas parallel to y-axis is given by,

$$(X-h)^2 = 4a(Y-k)$$

Differentiating it with respect to x,

$$2(x-h) = 4a\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{2a}(x-h)$$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = \frac{1}{2a}$$

Again, differentiating it with respect to x,

$$\frac{d^3y}{dx^3} = 0$$

Question 17

From $x^2 + y^2 + 2ax + 2by + c = 0$, derive a differential equation not containing a, b and c.

Solution 17

$$x^2 + y^2 + 2ax + 2by + c = 0 \qquad \text{---(i)}$$

Differentiating it again and again three times,

$$2x + 2y\frac{dy}{dx} + 2a + 2b\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} + a + b\frac{dy}{dx} = 0$$

$$(y-b)\frac{dy}{dx} + x + a = 0 \qquad \text{---(ii)}$$

Now,

$$(y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0 \qquad \text{---(iii)}$$

Again,

$$(y-b)\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = 0$$

$$-\left\{\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right\}\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\frac{d^2y}{dx^2} = 0$$

Since, using equation (iii),

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{d^3y}{dx^3} - 3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)^2 = 0$$

Question 18

Solve the following differential equation:

$$\frac{dy}{dx} = sin^3 x\, cos^4 x + x\sqrt{x+1}$$

Solution 18

$$\frac{dy}{dx} = \sin^3 x \cos^4 x + x\sqrt{x+1}$$

$$\int dy = \int \sin^3 x \cos^4 x + \int x\sqrt{x+1}dx$$

$$y = I_1 + I_2 + c_1 \qquad \text{---(i)}$$

$$I_1 = \int \sin^2 x \cos^4 x \sin x dx$$

$$= \int \left(1 - \cos^2 x\right) \cos^4 x \sin x dx$$

Put $\cos x = t$

$$-\sin x dx = dt$$

$$I_1 = \int -\left(1 - t^2\right)t^4 dt$$

$$= \int \left(t^6 - t^4\right) dt$$

$$= \frac{t^7}{7} - \frac{t^5}{5} + c_2$$

$$I_1 = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + c_2$$

$$I_2 = \int x\sqrt{x+1}dx$$

Let $x + 1 = t^2$

$$dx = 2tdt$$

$$= \int \left(t^2 - 1\right) \times t \times 2tdt$$

$$= 2\int \left(t^4 - t^2\right)dt$$

$$= 2\left(\frac{t^5}{5} - \frac{t^3}{3}\right) + c_3$$

$$I_2 = \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c_3$$

Put the value of I_1 and I_2 in equation (i),

$$y = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$$

Question 19

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{1}{x^2 + 4x + 5}$$

Solution 19

$$\frac{dy}{dx} = \frac{1}{x^2 + 4x + 5}$$

$$dy = \frac{1}{x^2 + 4x + 5}dx$$

$$\int dy = \int \frac{1}{x^2 + 2x(2) + (2)^2 - (2)^2 + 5}dx$$

$$\int dy = \int \frac{1}{(x+2)^2 + (1)^2}dx$$

$$y = \tan^{-1}(x+2) + c$$

Question 20

Solve the following differential equation:

$$\frac{dy}{dx} = y^2 + 2y + 2$$

Solution 20

$$\frac{dy}{dx} = y^2 + 2y + 2$$

$$\frac{dy}{y^2 + 2y + 2} = dx$$

$$\int \frac{dy}{y^2 + 2y + 2} = \int dx$$

$$\int \frac{dy}{y^2 + 2y + (1)^2 - (1)^2 + 2} = \int dx$$

$$\int \frac{dy}{(y+1)^2 + (1)^2} = \int dx$$

$$\tan^{-1}(y+1) = x + c$$

Question 21

Solve the following differential equation:

$$\frac{dy}{dx} + 4x = e^x$$

Solution 21

$$\frac{dy}{dx} + 4x = e^x$$

$$\frac{dy}{dx} = e^x - 4x$$

$$\int dy = \int \left(e^x - 4x\right) dx$$

$$y = e^x - \frac{4x^2}{2} + c$$

$$y = e^x - 2x^2 + c$$

$$y + 2x^2 = e^x + c$$

Question 22

Solve the following differential equation:

$$\frac{dy}{dx} = x^2 e^x$$

Solution 22

$$\frac{dy}{dx} = x^2e^x$$

$$\frac{dy}{dx} = x^2e^x dx$$

$$\int dy = \int x^2e^x dx$$

Using integration by parts

$$y = x^2 \times \int x^x dx - \int \left(2x\int e^x dx\right)dx + c$$

$$= x^2e^x - 2\int xe^x dx + c$$

$$= x^2e^x - 2\left[x\int e^x dx - \int \left(1 \times \int e^x dx\right)dx\right] + c$$

$$y = x^2e^x - 2xe^x + 2e^x + c$$

$$y = \left(x^2 - 2x + 2\right)e^x + c$$

Question 23

Solve the following differential equation:

$$\frac{dy}{dx} - x\sin^2 x = \frac{1}{x\log x}$$

Solution 23

$$\frac{dy}{dx} - x\sin^2 x = \frac{1}{x\log x}$$

$$\frac{dy}{dx} = \frac{g1}{x\log x} + x\sin^2 x$$

$$dy = \left(\frac{1}{x\log x} + x\sin^2 x\right)dx$$

$$dy\int = \int\frac{1}{x\log x}dx + \int x\sin^2 x dx$$

$$y = I_1 + I_2 + c_1 \qquad \text{---(i)}$$

$$I_1 = \int\frac{1}{x\log x}dx$$

Put $\log x = t$

$$\frac{1}{x}dx = dt$$

$$I_1 = \int\frac{1}{t}dt$$

$$= \log|t| + c_2$$

$$I_1 = \log|\log x| + c_2$$

Now,

$$I_2 = \int x\sin^2 x dx$$

$$\int x\left(\frac{1-\cos^2 x}{2}\right)dx$$

$$= \frac{1}{2}\int x dx - \int\frac{1}{2}x\cos 2x dx$$

$$= \frac{1}{2}\frac{x^2}{2} - \frac{1}{2}\left[x\int\cos 2x dx - \int\left(1\int\cos 2x dx\right)dx\right] + c_2$$

$$= \frac{x^2}{4} - \frac{1}{2}\left[x\left(\frac{\sin 2x}{2}\right) - \int\frac{\sin 2x}{2}dx\right] + c_2$$

$$I_2 = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{\cos 2y}{8} + c_2$$

Put I_1 and I_2 in equation (i),

$$y = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{\cos 2x}{8} + \log|\log x| + c$$

Question 24

Solve the following differential equation:

$$\left(tan^2 x + 2\,tan\,x + 5\right)\frac{dy}{dx} = 2\left(1 + tan\,x\right)sec^2 x$$

Solution 24

$$\left(\tan^2 x + 2\tan x + 5\right)\frac{dy}{dx} = 2\left(1+\tan x\right)\sec^2 x$$

$$\int dy = \int \frac{2\left(1+\tan x\right)}{\tan^2 x + 2\tan x + 5}\sec^2 x dx$$

Put $\tan x = t$

$$\sec^2 x dx = dt$$

$$\int dy = \int \frac{2\left(1+t\right)}{t^2+2t+5}dt$$

Put $t^2 + 2t + 5 = du$

$$\left(2t+2\right)dt = du$$

$$\int dy = \int \frac{du}{u}$$

$$y = \log\left|u\right| + c$$

$$= \log\left|t^2 + 2t + 5\right| + c$$

$$y = \log\left|\tan^2 x + 2\tan x + 5\right| + c$$

Question 25

Solve the following differential equation:

$$\frac{dy}{dx} = sin^3 x\, cos^2 x + xe^x$$

Solution 25

$$\frac{dy}{dx} = \sin^3 x \cos^2 x + xe^x$$

$$\frac{dy}{dx} = \left(\sin^3 x \cos^2 x + xe^x\right) dx$$

$$\int dy = \int \sin^3 x \cos^2 x + dx + \int xe^x dx$$

$$y = I_1 + I_2 + c_1 \qquad \text{---(i)}$$

$$I_1 = \int \sin^3 x \cos^2 x dx$$

$$= \int \sin^2 x \cos^2 x \sin x dx$$

$$= \int \left(1 - \cos^2 x\right) \cos^2 x \sin x dx$$

Put $\cos x = t$

$$-\sin x dx = dt$$

$$I_1 = \int -\left(1 - t^2\right) t^2 dt$$

$$= \int \left(t^4 - t^2\right) dt$$

$$= \frac{t^5}{5} - \frac{t^3}{3} + c_2$$

$$I_1 = \frac{1}{5} cso^5 x - \frac{1}{3} \cos^3 x + c_2$$

Now,

$$I_2 = \int xe^x dx$$

$$= x \int e^x dx - \int \left(1 - \int e^x dx\right) dx + c_2$$

$$I_2 = xe^x - e^x + c_2$$

Using equation (i) and I_1, I_2

$$y = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + xe^x - e^x + c$$

Question 26

Solve the following differential equation:

$\tan y dx + \tan x dy = 0$

Solution 26

$$\tan y dx + \tan x dy = 0$$

$$\tan y dx = -\tan x dy$$

$$\frac{1}{\tan x} dx = -\frac{dy}{\tan y}$$

$$\int \cot x dx = -\int \cot y dy$$

$$\log|\sin x| = -\log|\sin y| + \log|c|$$

$$\sin x = \frac{c}{\sin y}$$

$$\sin x \sin y = c$$

Question 27

Solve the following differential equation:

$(1 + x) y dx + (1 + y) x dy = 0$

Solution 27

$$(1+x)\,ydx + (1+y)\,xdy = 0$$
$$(1+x)\,ydx = -(1+y)\,xdy$$
$$\left(\frac{1+x}{x}\right)dx = -\frac{1+y}{y}dy$$
$$\int\left(\frac{1}{x}+1\right)dx = -\int\left(\frac{1}{y}+1\right)dy$$
$$\log|x| + x + \log|y| = c$$
$$\log|xy| + x + y = c$$

Question 28

Solve the following differential equation:

$$x\cos^2 ydx = y\cos^2 xdy$$

Solution 28

$$x\cos^2 ydx = y\cos^2 xdy$$
$$\frac{xdx}{\cos^2 x} = \frac{y}{\cos^2 y}dy$$
$$x\sec^2 xdx = y\sec^2 ydy$$
$$\int x\sec^2 xdx = \int y\sec^2 ydy$$
$$x\int\sec^2 xdx - \int\left(1\int\sec^2 xdx\right)dx = y\int\sec^2 ydy - \int 1\int\sec^2 ydy$$
$$x\tan x - \int\tan dx = y\tan y - \int\tan ydy + c$$
$$x\tan x - \log|\sec x| = y\tan y - \log|\sec y| + c$$
$$x\tan x - y\tan y = \log|\sec x| - \log|\sec y| + c$$

Question 29

Solve the following differential equation:

$$\cos y\log(\sec x + \tan x)dx = \cos x\log(\sec y + \tan y)dy$$

Solution 29

$$\cos y \log(\sec x + \tan x)\,dx = \cos x \log(\sec y + \tan y)\,dy$$

$$\int \frac{\log(\sec x + \tan x)\,dx}{\cos x} = \int \frac{\log(\sec y + \tan y)}{\cos y}dy$$

$$\int \sec x \log(\sec x + \tan x)dx = \int \sec y \log(\sec y + \tan y)dy$$

$$I_1 = I_2 \qquad \text{---(i)}$$

$$I_1 = \log(\sec x) + \tan x \times \int \sec x dx - \int\left(\frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \times \int \sec x dx\right)dx + c_1$$

$$I_1 = \log(\sec x + \tan x)\log(\sec x + \tan x) - \int \sec x \log(\sec x + \tan x)dx + c_1$$

$$I_1 = \{\log(\sec x + \tan x)\}^2 - I_1 + c_1$$

$$2I_1 = \{\log(\sec x + \tan x)\}^2 + c_1$$

$$I_1 = \frac{1}{2}\{\log(\sec x + \tan x)\}^2 + \frac{c_1}{2} \qquad \text{---(ii)}$$

Similarly,

$$I_2 = \frac{1}{2}\{\log(\sec y + \tan y)\}^2 + \frac{c_2}{2} \qquad \text{---(iii)}$$

Using equation (i), (ii) and (iii),

$$\frac{1}{2}\{\log(\sec x + \tan x)\}^2 + \frac{c_1}{2} = \frac{1}{2}\{\log(\sec y + \tan y)\}^2 + \frac{c_2}{2}$$

$$\{\log(\sec x + \tan x)\}^2 = \{\log(\sec y + \tan y)\}^2 + c$$

Question 30

Solve the following differential equation:

$cosecx\,(log\,y)\,dy + x^2 y dx = 0$

Solution 30

$$\operatorname{cosec} x\,(\log y)\,dy + x^2 y dx = 0$$

$$(\log y)\,dy = -\frac{x^2 y}{\operatorname{cosec} x}dx$$

$$\frac{\log y}{y}dy = -x^2 \sin x dx$$

$$\int \frac{\log y}{y}dy = -\int x^2 \sin x dx$$

$$\frac{(\log y)^2}{2} = -\left[x^2 \int \sin x dx - \int\left(2x \int \sin x dx\right)dx + c\right]$$

$$= -\left[-x^2 \cos x + 2\int x \cos x dx\right] + c$$

$$= x^2 \cos x - 2\left[x \int \cos x dx - \int\left(1 \int \cos x dx\right)dx\right] + c$$

$$= x^2 \cos x - 2x \sin x + 2\int \sin x dx + c$$

$$\frac{1}{2}(\log y)^2 = x^2 \cos x - 2x \sin x - 2\cos x + c$$

$$\frac{1}{2}(\log)^2 = \left(x^2 - 2\right)\cos x - 2x \sin x + c$$

Question 31

Solve the following differential equation:

$\left(1-x^2\right)dy + xydx = xy^2dx$

Solution 31

$$\left(1-x^2\right)dy + xydx = xy^2dx$$

$$\left(1-x^2\right)dy = xy^2dx - xydx$$

$$\left(1-x^2\right)dy = xdx\left(y^2-y\right)$$

$$\frac{1}{y^2-y}dy = \frac{x}{1-x^2}dx$$

$$\frac{-2}{y\left(y-1\right)}dy = \frac{-2x}{1-x^2}dx$$

$$-2\int\left[\frac{1}{y-1}-\frac{1}{y}\right]dy = \int\frac{-2x}{\left(1-x^2\right)}dx$$

$$-2\log\left|y-1\right| + 2\log\left|y\right| = \log\left|1-x^2\right| + \log\left|c\right|$$

$$\log\left(\frac{y}{y-1}\right)^2 = \log\left(1-x^2\right)c$$

$$\frac{y^2}{\left(y-1\right)^2} = \left(1-x^2\right)c$$

$$y^2 = \left(y-1\right)^2\left(1-x^2\right)c$$

Question 32

Solve the following differential equations:

$\frac{dy}{dx} = \frac{\sin x + x\cos x}{y\left(2\log y + 1\right)}$

Solution 32

$$\frac{dy}{dx} = \frac{\sin x + x\cos x}{y\left(2\log y + 1\right)}$$

$$\int y\left(2\log y + 1\right)dy = \int\left(\sin x + x\cos x\right)dx$$

$$\Rightarrow \quad 2\int y\log y\,dy + \int y\,dy = \int\sin x + \int x\cos x\,dx$$

$$\Rightarrow \quad 2\left[\log y \times \frac{y^2}{2} - \int\frac{1}{y}\times\frac{y^2}{2}dy\right] + \frac{y^2}{2} = -\cos x + \left[x\sin x - \int\sin x\,dx\right] + c$$

$$\Rightarrow \quad 2\left[\frac{y^2}{2}\log y - \frac{y^2}{4}\right] + \frac{y^2}{2} = -\cos x + x\sin x + \cos x + c$$

$$\Rightarrow \quad y^2\log y - \frac{y^2}{2} + \frac{y^2}{2} = x\sin x + c$$

$$y^2\log y = x\sin x + c$$

Question 33

Solve the following differential equation:

$x\left(e^{2y}-1\right)dy + \left(x^2-1\right)e^ydx = 0$

Solution 33

$$x\left(e^{2y}-1\right)dy+\left(x^2-1\right)e^y dx=0$$

$$x\left(e^{2y}-1\right)dy=\left(1-x^2\right)e^y dx$$

$$\left(\frac{e^{2y}-1}{e^y}\right)dy=\left(\frac{1-x^2}{x}\right)dx$$

$$\int\left(e^y-e^{-y}\right)dy=\int\left(\frac{1}{x}-x\right)dx$$

$$\left(e^y+e^{-y}\right)=\log|x|-\frac{x^2}{2}+c$$

Question 34

Solve the following differential equation:

$$\frac{dy}{dx}+1=e^{x+y}$$

Solution 34

$$\frac{dy}{dx}+1=e^{x+y}$$

$$\frac{dy}{dx}=e^{x+y}-1$$

Let $x+y=v$

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{dy}{dx}=\frac{dv}{dx}-1$$

Now,

$$\frac{dv}{dx}-1=e^v-1$$

$$\frac{dv}{dx}=e^v$$

$$\int e^{-v}dv=\int dx$$

$$-e^{-v}=x+c$$

$$-1=(x+c)e^v$$

$$-1=(x+c)e^{x+y}$$

Question 35

Solve the following differential equation:

$$\frac{dy}{dx}=(x+y)^2$$

Solution 35

$$\frac{dy}{dx} = (x + y)^2$$

Let $x + y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

So,

$$\frac{dv}{dx} - 1 = v^2$$

$$\frac{dv}{dx} = v^2 + 1$$

$$\frac{1}{1 + v^2} dv = dx$$

$$\int \frac{1}{1 + v^2} dv = \int dx$$

$$\tan^{-1} v = x + c$$

$$\tan^{-1}(x + y) = x + c$$

Question 36

Solve the following differential equation:

$cos(x + y)dy = dx$

Solution 36

$$\cos(x+y)\,dy = dx$$

Let $x + y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

So,

$$\cos v\left(\frac{dv}{dx} - 1\right) = 1$$

$$\frac{dv}{dx} - 1 = \frac{1}{\cos v}$$

$$\frac{dv}{dx} = \sec v + 1$$

$$\frac{1}{1+\sec v}dv = dx$$

$$\frac{\cos v}{1+\cos v}dv = dx$$

$$\frac{\cos^2\frac{v}{2} - \sin^2\frac{v}{2}}{2\cos^2\frac{v}{2}}dv = dx$$

$$\left(\frac{1}{2} - \frac{1}{2}\tan^2\frac{v}{2}\right)dv = dx$$

$$\frac{1}{2}\int\left(1 - \tan^2\frac{v}{2}\right)dv = \int dx$$

$$\frac{1}{2}\int\left\{1 - \left(\sec^2\frac{v}{2} - 1\right)\right\}dv = \int dx$$

$$\frac{1}{2}\int\left\{1 - \sec^2\frac{v}{2} + 1\right\}dv = dx$$

$$\frac{1}{2}\left[2v - 2\tan\frac{v}{2}\right] = x + c$$

$$v - \tan\frac{v}{2} = x + c$$

$$(x+y) - \tan\left(\frac{x+y}{2}\right) = x + c$$

$$y = c + \tan\left(\frac{x+y}{2}\right)$$

Question 37

Solve the following differential equation:

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$$

Solution 37

$$\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$$

Let $\frac{y}{x}=v$

$$y=xv$$

$$\frac{dy}{dx}=x\frac{dv}{dx}+v$$

So,

$$\left(x\frac{dv}{dx}+v\right)+v=v^2$$

$$x\frac{dv}{dx}=v^2-2v$$

$$\int\frac{dv}{v^2-2v}=\int\frac{dx}{x}$$

$$\int\frac{dv}{v^2-2v+(1)^2-(1)^2}=\int\frac{dx}{x}$$

$$\int\frac{dv}{(v-1)^2-1^2}=\int\frac{dx}{x}$$

$$\frac{1}{2}\log\left|\frac{v-1-1}{v-1+1}\right|=\log|x|+\log|c|$$

$$\log\left|\frac{v-2}{v}\right|=\log\left|x^2c^2\right|$$

$$v-2=vx^2c^2$$

$$\frac{y}{x}-2=\left(\frac{y}{x}\right)x^2c^2$$

$$\frac{(y-2x)}{x}=xyc^2$$

$$y-2x=x^2yc^2$$

Question 38

Solve the following differential equation:

$$\frac{dy}{dx}=\frac{y(x-y)}{x(x+y)}$$

Solution 38

$$\frac{dy}{dx} = \frac{y(x-y)}{x(x+y)}$$

It is a homogeneous equation

Put $y = vx$

$$\frac{dy}{dx}$$

Let $\frac{y}{x} = v$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

So,

$$x\frac{dv}{dx} + v = \frac{vx(x-vx)}{x(x+vx)}$$

$$x\frac{dv}{dx} = \frac{v(1-v)}{(1+v)} - v$$

$$= \frac{v-v^2-v-v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{-2v^2}{1+v}$$

$$\frac{1+v}{v^2}dv = \frac{-2}{x}dx$$

$$\int\left(\frac{1+v}{v^2}\right)dv = -\int\frac{dx}{x}$$

$$\int\left(\frac{1}{v^2}+\frac{1}{v}\right)dv = -2\int\frac{dx}{x}$$

$$-\frac{1}{v} + \log|v| = -2\log|x| + \log|c|$$

$$\log\left|\frac{v}{e^{\frac{1}{v}}}\right| = \log\left|\frac{c}{x^2}\right|$$

$$v = e^{\frac{1}{v}}\frac{c}{x^2}$$

$$\frac{y}{x} = e^{\frac{x}{y}}\frac{c}{x^2}$$

$$xy = e^{\frac{x}{y}}c$$

Question 39

Solve the following differential equation:

$(x+y-1)dy = (x+y)dx$

Solution 39

$$(x+y-1)\,dy = (x+y)\,dx$$

$$(x+y-1)\frac{dy}{dx} = (x+y)$$

Let $x+y=v$

$$1+\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

So,

$$(v-1)\left(\frac{dv}{dx}-1\right) = v$$

$$\left(\frac{dv}{dx}-1\right) = \frac{v}{v-1}$$

$$\frac{dv}{dx} = \frac{v}{v-1} + 1$$

$$= \frac{v+v-1}{v-1}$$

$$\frac{dv}{dx} = \frac{2v-1}{v-1}$$

$$\int \frac{v-1}{2v-1}dv = \int dx$$

$$\int \left(\frac{1}{2} - \frac{\frac{1}{2}}{2v-1}\right)dv = \int dx$$

$$\frac{1}{2}\int dv - \frac{1}{2}\int \frac{1}{2v-1}dv = \int dx$$

$$\frac{1}{2}v - \frac{1}{2}\times\frac{\log|2v-1|}{2} = x + c_1$$

$$\frac{1}{2}v - \frac{1}{4}\log|2v-1| = x + c_1$$

$$\frac{1}{2}(x+y) - \frac{1}{4}\log[2x+2y-1] = x + c_1$$

$$2(x+y) - \log|2x+2y-1| = 4x + 4c_1$$

$$2x+2y-4x-\log|2x+2y-1| = 4c_1$$

$$2(y-x) - \log|2x+2y-1| = c$$

Question 40

Solve the following differential equation:

$$\frac{dy}{dx} - y\cot x = \operatorname{cosec} x$$

Solution 40

$$\frac{dy}{dx} - y\cot x = \operatorname{cosec} x$$

It is a linear differential equation.

Comparing it with $\frac{dy}{dx} + Py = Q$

$P = -\cot x, Q = \operatorname{cosec} x$

I.F. $= e^{\int P dx}$

$= e^{-\int \cot x dx}$

$= e^{-\log|\sin x|}$

$= e^{\log|\operatorname{cosec} x|}$

$= \operatorname{cosec} x$

Solution of the given equation is given by,

$y\,(\text{I.F.}) = \int Q \times (\text{I.F.})dx + c$

$y \operatorname{cosec} x = \int \operatorname{cosec} x \times \operatorname{cosec} x dx + c$

$y \operatorname{cosec} x = \int \operatorname{cosec}^2 x dx + c$

$y \operatorname{cosec} x = -\cot x + c$

Question 41

Solve the following differential equation:

$$\frac{dy}{dx} - y\tan x = -2\sin x$$

Solution 41

$$\frac{dy}{dx} - y\tan x = -2\sin x$$

It is a linear differential equation.

Comparing it with $\frac{dy}{dx} + Py = Q$

$P = -\tan x, Q = -2\sin x$

I.F. $= e^{\int P dx}$

$= e^{-\int \tan x dx}$

$= e^{\log \cos x}$

$= \cos x$

Solution of the given equation is given by,

$y\,(\text{I.F.}) = \int Q \times (\text{I.F.})dx + c$

$= \int -2\sin x \cos x dx + c$

$= -\int \sin 2x dx + c$

$y \cos x = \frac{1}{2}\cos 2x + c$

Question 42

Solve the following differential equation:

$$\frac{dy}{dx} - y\tan x = e^x \sec x$$

Solution 42

$$\frac{dy}{dx} - y\tan x = e^x \sec x$$

It is a linear differential equation.

Comparing it with $\frac{dy}{dx} + Py = Q$

$$P = -\tan x, Q = e^x \sec x$$

I.F. $= e^{\int P dx}$

$= e^{-\int \tan x dx}$

$= e^{-\log \sec x}$

$= \cos x$

Solution of the given equation is given by,

$$y\left(\text{I.F.}\right) = \int Q \times \left(\text{I.F.}\right) dx + c$$

$$= \int e^x \sec x \cos x dx + c$$

$$= \int e^x dx + c$$

$$y \cos x = e^x + c$$

Question 43

Solve the following differential equation:

$$\frac{dy}{dx} - y \tan x = e^x$$

Solution 43

$$\frac{dy}{dx} - y\tan x = e^x$$

It is a linear differential equation.

Comparing it with $\frac{dy}{dx} + Py = Q$

$$P = -\tan x, Q = e^x$$

I.F. $= e^{\int P dx}$

$$= e^{-\int \tan x dx}$$

$$= e^{-\log \sec x}$$

$$= \cos x$$

Solution of the given equation is given by,

$$y\,(\text{I.F.}) = \int Q \times (\text{I.F.})\,dx + c_1$$

$$y\cos x = \int e^x \sec x \cos x dx + c_1$$

$$y\cos x = I + c_1 \qquad \text{---(i)}$$

$$I = \int e^x \cos x dx$$

Using equation by parts

$$I = e^x \int \cos x dx - \int \left(e^x \int \cos x dx\right) dx + c_2$$

$$= e^x \sin x - \int e^x \sin x dx + c_2$$

$$= e^x \sin x - \left[e^x \int \sin x dx - \int \left(e^x \int \sin x dx\right) dx\right] + c_2$$

$$I = e^x \sin x + e^x \cos x - \int e^x \cos x dx + c_2$$

$$I = e^x (\sin x + \cos x) - I + c_2$$

$$2I = e^x (\sin x + \cos x) + c_2$$

$$I = \frac{1}{2} e^x (\sin x + \cos x) + c_3$$

Using equation (i),

$$y\cos x = \frac{1}{2} e^x (\sin x + \cos x) + c$$

Question 44

Solve the following differential equation:

$$\left(1 + y + x^2 y\right)dx + \left(x + x^3\right)dy = 0$$

Solution 44

$$\left(1 + y + x^2 y\right) dx + \left(x + x^3\right) dy = 0$$
$$\left\{1 + y\left(1 + x^2\right)\right\} dx = -x\left(1 + x^2\right) dy$$
$$\left\{\frac{1}{\left(1 + x^2\right)} + y\right\} dx = -x dy$$
$$\frac{1}{1 + x^2} + y = -x\frac{dy}{dx}$$
$$\frac{dy}{dx} + \frac{y}{x} = -\frac{1}{x\left(1 + x^2\right)}$$

It is a linear differential equation.

Comparing it with $\frac{dy}{dx} + Py = Q$

$$P = \frac{1}{x}, Q = -\frac{1}{x\left(1 + x^2\right)}$$

I.F. $= e^{\int P dx}$

$= e^{\int \frac{1}{x} dx}$

$= e^{\log|x|}$

$= x$

Solution of the equation is given by,

$$y \times (\text{I.F}) = \int Q\,(\text{I.F.})dx + c$$
$$yx = \int \frac{1}{x\left(1 + x^2\right)}(x)\,dx + c$$
$$yx = \int \frac{1}{1 + x^2} dx + c$$
$$xy = \tan^{-1} x + c$$

Question 45

Solve the following differential equation:

$\left(x^2 + 1\right) dy + (2y - 1) dx = 0$

Solution 45

$$\left(x^2+1\right)dy+(2y-1)\,dx=0$$

$$\left(x^2+1\right)\frac{dy}{dx}+(2y-1)=0$$

$$\frac{dy}{dx}+\frac{2}{x^2+1}y=\frac{1}{x^2+1}$$

It is a linear differential equation.

Comparing it with $\frac{dy}{dx}+Py=Q$

$$P=\frac{2}{x^2+1},\ Q=\frac{1}{x^2+1}$$

I.F. $= e^{\int P\,dx}$

$$= e^{\int \frac{2}{1+x^2}dx}$$

$$= e^{2\tan^{-1}x}$$

Solution of the equation is given by,

$$y\times(\text{I.F}) = \int Q\,(\text{I.F.})dx + c$$

$$ye^{2\tan^{-1}x} = \int \frac{1}{x^2+1}e^{2\tan^{-1}x}dx + c$$

Let $2\tan^{-1}x = t$

$$\frac{2}{1+x^2}dx = dt$$

So,

$$ye^t = \frac{1}{2}\int e^t dt + c$$

$$ye^t = \frac{1}{2}e^t + c$$

$$y = \frac{1}{2}+ce^{-t}$$

$$y = \frac{1}{2}+ce^{-2\tan^{-1}x}$$

Question 46

Solve the following differential equation:

$$y\sec^2 x+(y+7)\tan x\frac{dy}{dx}=0\left(x^2+1\right)dy+(2y-1)dx=0$$

Solution 46

$$y\sec^2 x + (y+7)\tan x\frac{dy}{dx} = 0$$

$$y\frac{\sec^2 x}{\tan x} + (y+7)\frac{dy}{dx} = 0$$

$$(y+7)\frac{dy}{dx} = -y\frac{\sec^2 x}{\tan x}$$

$$\frac{y+7}{y}dy = -\frac{\sec^2 x}{\tan x}dx$$

$$\int\left(1+\frac{7}{y}\right)dy = -\int\frac{\sec^2 x}{\tan x}dx$$

Let $\tan x = t$

$$\sec^2 x dx = dt$$

$$\int\left(1+\frac{7}{y}\right)dy = -\int\frac{dt}{t}$$

$$= -\log|t| + \log|c|$$

$$y + 7\log|y| = -\log|\tan x| + loc|c|$$

$$e^y \times e^7 = \frac{c}{\tan x}$$

$$\tan x \times y^7 = ce^{-y}$$

Question 47

Solve the following differential equation:

$$(2ax + x^2)\frac{dy}{dx} = a^2 + 2ax$$

Solution 47

$$\left(2ax + x^2\right)\frac{dy}{dx} = a^2 + 2ax$$

$$\left(2ax + x^2\right)dy = \left(a^2 + 2ax\right)dx$$

$$dy = \frac{\left(a^2 + 2ax\right)}{\left(2ax + x^2\right)}dx$$

$$\int dy = \int \frac{a(a+2x)}{\left(2ax + x^2\right)}dx$$

$$= a\int \frac{a+2x}{\left(2ax + x^2\right)}dx$$

$$= a\int \frac{2a+2x-a}{\left(2ax + x^2\right)}dx$$

$$\int dy = a\int \frac{(2a+2x)}{\left(2ax + x^2\right)}dx - a^2\int \frac{1}{x^2 + 2ax}dx$$

$$\int dy = a\int \frac{2a+2x}{2ax + x^2}dx - a^2\int \frac{1}{(x+a)^2 - a^2}dx$$

$$y = a\log\left|2ax + x^2\right| - a^2\,\frac{1}{2a}\log\left|\frac{x+a-a}{x+a+a}\right| + c$$

$$y = a\log\left|x(2a+x)\right| - \frac{a}{2}\log|x| + \frac{a}{2}\log|x+2a| + c$$

$$y = a\log|x| + a\log|2a+x| - \frac{a}{2}\log|x| + \frac{a}{2}\log|x+2a| + c$$

$$= \frac{a}{2}\log|x| + \frac{3a}{2}\log|2a+x| + c$$

$$y = \frac{a}{2}\left\{\log|x| + 3\log|x+2a|\right\} + c$$

Question 49

Solve the following differential equation:

$$x^2dy + \left(x^2 - xy + y^2\right)dx = 0$$

Solution 49

$$x^2dy + \left(x^2 - xy + y^2\right)dx = 0$$

$$\frac{dy}{dx} = -\frac{\left(x^2 - xy + y^2\right)}{x^2}$$

It is a homogeneous equation.

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So,

$$v + x\frac{dv}{dx} = -\frac{\left(x^2 + x^2v + v^2x^2\right)}{x^2}$$

$$x\frac{dv}{dx} = -1 + v - v^2 - v^2$$

$$x\frac{dv}{dx} = -\left(1 + v^2\right)$$

$$\int\frac{dv}{1 + v^2} = -\int\frac{dx}{x}$$

$$\tan^{-1}v = -\log|x| + \log|c|$$

$$e^{\tan^{-1}v} = \frac{c}{x}$$

$$xe^{\tan^{-1}\frac{y}{x}} = c$$

Question 50

Solve the following differential equation:

$$y - x\frac{dy}{dx} = b\left(1 + x^2\frac{dy}{dx}\right)$$

Solution 50

$$y - x\frac{dy}{dx} = b\left(1 + x^2\frac{dy}{dx}\right)$$

$$y - x\frac{dy}{dx} - bx^2\frac{dy}{dx} = b$$

$$y - \left(x + bx^2\right)\frac{dy}{dx} = b$$

$$-\left(x + bx^2\right)\frac{dy}{dx} = \left(b - y\right)$$

$$\frac{dy}{y - b} = \frac{dy}{bx^2 + x}$$

$$\int\frac{dy}{y - b} = \int\frac{1}{bx^2 + x}dx$$

$$= \frac{1}{b}\int\frac{1}{x^2 + \frac{x}{b}}dx$$

$$\int\frac{dy}{y - b} = \frac{1}{b}\int\frac{dx}{x^2 + 2x\left(\frac{1}{2b}\right) + \left(\frac{1}{2b}\right)^2 - \left(\frac{1}{2b}\right)^2}$$

$$b\int\frac{dy}{y - b} = \int\frac{dx}{\left(x + \frac{1}{2b}\right)^2 - \left(\frac{1}{2b}\right)^2}$$

$$b\log\left|y - b\right| = \frac{1}{2\left(\frac{1}{2b}\right)}\log\left|\frac{x + \frac{1}{2b} - \frac{1}{2b}}{x + \frac{1}{2b} + \frac{1}{2b}}\right| + \log\left|c\right|$$

$$\log\left|y - b\right| = \log\left|\frac{x}{x + \frac{1}{b}}\right| + \frac{\log\left|c\right|}{b}$$

$$\log\left|y - b\right| = \log\left(\left|\frac{bx}{bx + 1}\right| \times k\right)$$

$$\left(bx + 1\right)\left(y - b\right) = bxk$$

Question 51

Solve the following differential equation:

$$\frac{dy}{dx} + 2y = sin\,3x$$

Solution 51

$$\frac{dy}{dx} + 2y = \sin 3x$$

It is a linear differential equation comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = 2, Q = \sin 3x$$

I.F. $= e^{\int P dx}$

$$= e^{\int 2dx}$$

$$= e^{2x}$$

Solution of the equation is given by,

$$y \times (\text{I.F.}) = \int Q\,(\text{I.F.})dx + c$$

$$ye^{2x} = \int \sin 3x e^{2x} dx + c$$

$$ye^{2x} = I + c_1 \qquad \text{---(i)}$$

$$I = \int \sin 3x e^{2x} dx$$

Using integration by parts,

$$= \sin 3x \frac{e^{2x}}{2} - \int 3 \cos x \frac{e^{2x}}{2} dx + c_2$$

$$I = \frac{1}{2} \sin 3x e^{2x} - \frac{3}{2} \int \cos 3x e^{2x} dx + c_2$$

$$= \frac{1}{2} \sin 3x e^{2x} - \frac{3}{2}\left[\cos 3x \frac{e^{2x}}{2} + \int 3 \sin 3x \frac{e^{2x}}{2} dx\right] + c_2$$

$$I = \frac{1}{2} \sin 3x e^{2x} - \frac{3}{4} \cos 3x e^{2x} - \frac{9}{4} I + c_2$$

$$I + \frac{9}{4} I = \frac{e^{2x}}{4}(2 \sin 3x - 3 \cos 3x) + c_2$$

$$I = \frac{1}{13} e^{2x}(2 \sin 3x - 3 \cos 3x) + c_3$$

Using equation (i),

$$ye^{2x} = \frac{1}{13} e^{2x}(2 \sin 3x - 3 \cos 3x) + c$$

$$y = \frac{3}{13} e^{2x}\left(\frac{2}{3} \sin 3x - \cos 3x\right) + ce^{-2x}$$

Question 52

Solve the following differential equation:

$$\frac{dy}{dx} + y = 4x$$

Solution 52

$$\frac{dy}{dx} + y = 4x$$

It is a linear differential equation comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = 1, Q = 4x$$

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int 1 dx}$$

$$= e^{x}$$

Solution of the equation is given by,

$$y \times (\text{I.F.}) = \int Q\,(\text{I.F.})dx + c$$

$$ye^{x} = \int 4xe^{x}dx + c$$

$$= 4\left[xe^{x} - \int e^{x}dx\right] + c$$

$$ye^{x} = 4xe^{x} - 4e^{x} + c$$

$$y = 4(x-1) + ce^{-x}$$

Question 53

Solve the following differential equation:

$$\frac{dy}{dx} + 5y = \cos 4x$$

Solution 53

$$\frac{dy}{dx} + 5y = \cos 4x$$

It is a linear differential equation

Comparing it with $\frac{dy}{dx} + Py = Q$

$$P = 5, Q = \cos 4x$$

$$\text{I.F.} \quad = e^{\int P dx}$$

$$= e^{\int 5 dx}$$

$$= e^{5x}$$

Solution of the equation is given by,

$$y \times (\text{I.F.}) = \int Q\,(\text{I.F.})dx + c$$

$$ye^{5x} = \int \cos 4x e^{5x} dx + c$$

$$ye^{5x} = I + c_1 \qquad \text{---(i)}$$

$$I = \int \cos 4x e^{5x} dx$$

$$= \cos 4x \frac{e^{5x}}{5} + \int 4 \sin 4x \frac{e^{5x}}{5} dx + c_2$$

$$= \frac{1}{5} \cos 4x e^{5x} + \frac{4}{5}\left[\sin 4x \frac{e^{5x}}{5} - \int 4 \cos 4x \frac{e^{5x}}{5} dx\right] + c_2$$

$$I = \frac{1}{5} \cos 4x e^{5x} \frac{4}{25} \sin 4x e^{5x} - \frac{16}{25} I + c_2$$

$$\frac{41}{25} I = \frac{e^{5x}}{25}(5 \cos 4x + 4 \sin 4x) + c_2$$

$$I = \frac{4e^{5x}}{41}\left(\sin 4x + \frac{5}{4} \cos 4x\right) + c_3$$

Put I in equation (i),

$$y = \frac{4}{41}\left(\sin 4x + \frac{5}{4} \cos 4x\right) + c$$

Question 54

Solve the following differential equation:

$$x \frac{dy}{dx} + x\, cos^2\left(\frac{y}{x}\right) = y$$

Solution 54

$$x\frac{dy}{dx}+x\cos^2\left(\frac{y}{x}\right)=y$$

$$\frac{dy}{dx}=\frac{y-x\cos^2\left(\frac{y}{x}\right)}{x}$$

It is a homogeneous equation.

Put $y=vx$

$$\frac{dy}{dx}=x\frac{dv}{dx}+v$$

So,

$$x\frac{dv}{dx}+v=\frac{vx-x\cos^2\left(\frac{vx}{v}\right)}{x}$$

$$x\frac{dv}{dx}+v=v-\cos^2 v$$

$$x\frac{dv}{dx}=v-\cos^2 v-v$$

$$x\frac{dv}{dx}=-\cos^2 v$$

$$\frac{1}{\cos^2 v}dv=-\frac{dx}{x}$$

$$\int\sec^2 v dv=-\int\frac{dx}{x}$$

$$\tan v=-\log|x|+\log|c|$$

$$c^{\tan v}=\frac{c}{x}$$

$$xe^{\tan\frac{y}{x}}=c$$

Question 55

Solve the following differential equation:

$$\cos^2 x\frac{dy}{dx}+y=\tan x$$

Solution 55

$$\cos^2 x\frac{dy}{dx}+y = \tan x$$

$$\frac{dy}{dx}+\sec^2 xy = \tan x\sec^2 x$$

It is a linear differential equation.

Comparing it with $\frac{dy}{dx}+Py = Q$

$$P = \sec^2 x, Q = \tan x\sec^2 x$$

I.F. $= e^{\int P dx}$

$= e^{\int \sec^2 x dx}$

$= e^{\tan x}$

Solution of the equation is given by,

$$y\left(\text{I.F.}\right) = \int Q\left(\text{I.F.}\right)dx + c$$

$$ye^{\tan x} = \int \tan x\sec^2 xe^{\tan x}dx + c$$

Let $\tan x = t$

$$\sec^2 xdx = dt$$

$$ye^t = \int te^t dt + c$$

$$= te^t - \int 1e^t dt + c$$

$$= te^t - e^t + c$$

$$ye^t = e^t\left(t-1\right)+c$$

$$y = \left(t-1\right)+ce^{-t}$$

$$y = \left(\tan x - 1\right)+ce^{-\tan x}$$

Question 56

Solve the following differential equation:

$$x\cos x\frac{dy}{dx}+y\left(x\sin x + \cos x\right) = 1$$

Solution 56

$$x\cos x\frac{dy}{dx}+y\left(x\sin x+\cos x\right)=1$$

$$\frac{dy}{dx}+y\frac{\left(x\sin x+\cos x\right)}{x\cos x}=\frac{1}{x\cos x}$$

$$\frac{dy}{dx}+y\left(\tan x+\frac{1}{x}\right)=\frac{1}{x\cos x}$$

It is a linear differential equation.

Comparing it with $\frac{dy}{dx}+Py=Q$

$$P=\tan x+\frac{1}{x}, Q=\frac{1}{x\cos x}$$

I.F. $=e^{\int Pdx}$

$$=e^{\int\left(1+\tan x+\frac{1}{x}\right)dx}$$

$$=e^{\log|\sec x|+\log|x|}$$

$$=e^{\log x\sec x}$$

I.F. $=x\sec x$

Solution of the equation is given by,

$$y\left(\text{I.F.}\right)=\int Q\left(\text{I.F.}\right)dx+c$$

$$y\left(x\sec x\right)=\int\frac{1}{x\cos x}x\sec x\,dx+c$$

$$=\int\sec^2 x\,dx+c$$

$$xy\times\sec x=\tan x+c$$

Question 57

Solve the following differential equation:

$$\left(1+y^2\right)+\left(x-e^{-\tan^{-1}y}\right)\frac{dy}{dx}=0$$

Solution 57

$$\left(1+y^2\right)+\left(x-e^{-\tan^{-1}y}\right)\frac{dy}{dx}=0$$

$$\left(1+y^2\right)\frac{dx}{dy}+x-e^{-\tan^{-1}y}=0$$

$$\left(1+y^2\right)\frac{dx}{dy}+x=e^{-\tan^{-1}y}$$

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{-\tan^{-1}y}}{1+y^2}$$

It is a linear differential equation.

Comparing it with $\frac{dy}{dx}+Px=0$

$$P=\frac{1}{1+y^2},\ Q=\frac{e^{-\tan^{-1}y}}{1+y^2}$$

I.F. $= e^{\int P dy}$

$$= e^{\int \frac{1}{1+y^2} dy}$$

$$e = ^{\tan^{-1}y}$$

Solution of the equation is given by,

$$x\left(\text{I.F.}\right)=\int Q\left(\text{I.F.}\right)dy+c$$

$$x\left(e^{\tan^{-1}y}\right)=\int\frac{e^{-\tan^{-1}y}}{1+y^2}e^{\tan^{-1}y}dy+c$$

$$xe^{\tan^{-1}y}=\int\frac{1}{1+y^2}dy+c$$

$$xe^{\tan^{-1}y}=\tan^{-1}y+c$$

Question 58

Solve the following differential equation:

$$y^2+\left(x+\frac{1}{y}\right)\frac{dy}{dx}=0$$

Solution 58

$$y^2 + \left(x + \frac{1}{y}\right)\frac{dy}{dx} = 0$$

$$y^2\frac{dx}{dy} + x + \frac{1}{y} = 0$$

$$\frac{dy}{dx} + \frac{x}{y^2} = -\frac{1}{y^2}$$

It is linear differential equation.

Comparing it with $\frac{dx}{dy} + Px = Q$

$$P = \frac{1}{y^2}, Q = -\frac{1}{y^3}$$

I.F. $= e^{\int P dy}$

$$= e^{\int \frac{1}{y^2} dy}$$

$$= e^{-\frac{1}{y}}$$

Solution of the equation is given by,

$$x\,(\text{I.F.}) = \int Q\,(\text{I.F.})dy + c$$

$$xe^{-\frac{1}{y}} = \int\left(-\frac{1}{y^3}\right)\left(e^{-\frac{1}{y}}\right)dy + c$$

Let $-\frac{1}{y} = t$

$$\frac{1}{y^2}dy = dt$$

$$xe^t = \int te^t dt + c$$

$$= t\int e^t dt - \int\left(1e^t\right)dt + c$$

$$= te^t - e^t + c$$

$$xe^t = e^t\left(t-1\right) + c$$

$$x = t - 1 + ce^{-t}$$

$$x = -\frac{1}{y} - 1 + ce^{\frac{1}{y}}$$

$$x + \frac{1}{y} + 1 = ce^{\frac{1}{y}}$$

Question 59

Solve the following differential equation:

$2\cos x\frac{dy}{dx} + 4y\,six = \sin 2x$, given that $y = 0$ when $x = \frac{\pi}{3}$

Solution 59

$$2\cos x\frac{dy}{dx}+4ysix=\sin 2x \text{ and } y=0 \text{ when } x=\frac{\pi}{3}$$

$$\frac{dy}{dx}+2\tan xy=\frac{2\sin x\cos x}{2\cos x}$$

$$\frac{dy}{dx}+2\tan xy=\sin x$$

It is linear differential equation.

Comparing it with $\frac{dx}{dy}+Px=Q$

$$P=2\tan x, Q=\sin x$$

I.F. $=e^{\int Pdy}$

$=e^{2\int \tan x dx}$

$=e^{2\log|\sec x|}$

$=e^{\log \sec^2 x}$

$=\sec^2 x$

Solution of the equation is given by,

$$x(\text{I.F.})=\int Q(\text{I.F.})dx+c$$

$$y\left(\sec^2 x\right)=\int \sin x\left(\sec^2 x\right)dx+c$$

$$=\int\frac{\sin x}{\cos x}\sec x dx+c$$

$$=\int \sec x\tan x dx+c$$

$$y\left(\sec^2 x\right)=\sec x+c$$

Put $y=0$ and $x=\frac{\pi}{3}$

$$0\left(\sec^2\frac{\pi}{3}\right)=\sec\frac{\pi}{3}+c$$

$$0(4)=2+c$$

$$c=-2$$

So,

$$y\left(\sec^2 x\right)=\sec x-2$$

$$y=\cos x-2\cos^2 x$$

Question 60

Solve the following differential equation:

$$\left(1+y^2\right)dx=\left(\tan^{-1}y-x\right)dy$$

Solution 60

$$\left(1+y^2\right)dx = \left(\tan^{-1} y - x\right)dy$$

$$\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$$

$$\left(1+y^2\right)\frac{dx}{dy} + x = \tan^{-1} y$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

It is linear differential equation.

Comparing it with $\frac{dx}{dy} + Px = Q$

$$P = \frac{1}{1+y^2},\ Q = \frac{\tan^{-1} t}{1+y^2}$$

I.F. $= e^{\int \frac{1}{1+y^2} dx}$

$= e^{\tan^{-1} y}$

Solution of the equation is given by,

$$x\left(\text{I.F.}\right) = \int Q\left(\text{I.F.}\right)dx + c$$

$$x\left(e^{\tan^{-1} y}\right) = \int \frac{\left(\tan^{-1} y\right)}{1+y^2} dy + c$$

Let $\tan^{-1} y = t$

$$\frac{1}{1+y^2} dy = dt$$

$$xe^t = \int te^t dt + c$$

$$= te^t - \int e^t dt + c$$

$$= te^t - e^t + c$$

$$xe^t = e^t\left(t-1\right) + c$$

$$x = t - 1 + ce^{-t}$$

$$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$$

Question 61

Solve the following differential equation:

$\frac{dy}{dx} + y + \tan x = x^n \cos x,\ n \neq -1$

Solution 61

$$\frac{dy}{dx} + y + \tan x = x^n \cos x,\ \ n \neq -1$$

It is linear differential equation.

Comparing it with $\frac{dx}{dy} + Px = Q$

$$P = \tan x, Q = x^n \cos x$$

I.F. $= e^{\int P dx}$

$= e^{\int \tan x dx\ t}$

$= e^{\log|\sec x|}$

$= \sec x$

Solution of the equation is given by,

$$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})dx + c$$

$$y\,(\sec x) = \int x^n \cos x\,(\sec x)dx + c$$

$$= \int x^n dx + c$$

$$y \sec x = \frac{x^{n+1}}{n+1} + c$$

Question 62

Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x+1}{2-y}, y \neq 2.$

Solution 62

$$\frac{dy}{dx} = \frac{x+1}{2-y},\ \ y \neq 2$$

$$(2-y)dy = (x+1)dx$$

$$\int (2-y)dy = \int (x+1)dx$$

$$2y - \frac{y^2}{2} = \frac{x^2}{2} + x + c$$

$$4y - y^2 - x^2 - 2x = c$$

Question 63

Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that $y = 1$, when $x = 0$.

Solution 63

$$\frac{dy}{dx} = -4xy^2,\ \ y = 1 \text{ when } x = 0$$

$$\frac{dy}{y^2} = -4xdx$$

$$\int \frac{dy}{y^2} = -4\int xdx$$

$$-\frac{1}{y} = -4\frac{x^2}{2} + c$$

Plut $y = 1$ at $x = 0$

$$-1 = 0 + c$$

$$c = -1$$

So,

$$-\frac{1}{y} = -2x^2 - 1$$

$$\frac{1}{y} = \left(2x^2 + 1\right)$$

$$y = \frac{1}{2x^2 + 1}$$

Question 64(i)

Find the general solution of the given differential equation

$$\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$$

Solution 64(i)

The given differential equation is:

$$\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} = \tan^2 \frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \left(\sec^2 \frac{x}{2} - 1\right)$$

Separating the variables,we get:

$$dy = \left(\sec^2 \frac{x}{2} - 1\right) dx$$

Now, integrating both sides of this equation, we get:

$$\int dy = \int\left(\sec^2 \frac{x}{2} - 1\right) dx = \int \sec^2 \frac{x}{2} dx - \int dx$$

$$\Rightarrow y = 2\tan \frac{x}{2} - x + \mathrm{C}$$

This is the required general solution of the given differential equation.

Question 64(ii)

Find the general solution of the given differential equation

$$\frac{dy}{dx} = \sqrt{4-y^2} \quad (-2 < y < 2)$$

Solution 64(ii)

The given differential equation is:

$$\frac{dy}{dx} = \sqrt{4-y^2}$$

Separating the variables, we get:

$$\Rightarrow \frac{dy}{\sqrt{4-y^2}} = dx$$

Now, integrating both sides of this equation, we get:

$$\int \frac{dy}{\sqrt{4-y^2}} = \int dx$$

$$\Rightarrow \sin^{-1}\frac{y}{2} = x + C$$

$$\Rightarrow \frac{y}{2} = \sin(x + C)$$

$$\Rightarrow y = 2\sin(x + C)$$

This is the required general solution of the given differential equation.

Question 64(iii)

Find the general solution of the given differential equation

$$\frac{dy}{dx} = \left(1+x^2\right)\left(1+y^2\right)$$

Solution 64(iii)

The given differential equation is:

$$\frac{dy}{dx} = \left(1+x^2\right)\left(1+y^2\right)$$
$$\Rightarrow \frac{dy}{1+y^2} = \left(1+x^2\right)dx$$

Integrating both sides of this equation, we get:

$$\int \frac{dy}{1+y^2} = \int \left(1+x^2\right)dx$$
$$\Rightarrow \tan^{-1} y = \int dx + \int x^2 dx$$
$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + C$$

This is the required general solution of the given differential equation.

Question 64(iv)

Find the general solution of the given differential equation

$$y \log y \, dx - x \, dy = 0$$

Solution 64(iv)

The given differential equation is:

$$y \log y \, dx - x \, dy = 0$$
$$\Rightarrow y \log y \, dx = x \, dy$$
$$\Rightarrow \frac{dy}{y \log y} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{dy}{y \log y} = \int \frac{dx}{x} \qquad \text{...(1)}$$

Let $\log y = t$.

$$\therefore \frac{d}{dy}(\log y) = \frac{dt}{dy}$$
$$\Rightarrow \frac{1}{y} = \frac{dt}{dy}$$
$$\Rightarrow \frac{1}{y} dy = dt$$

Substituting this value in equation (1), we get:

$$\int \frac{dt}{t} = \int \frac{dx}{x}$$
$$\Rightarrow \log t = \log x + \log C$$
$$\Rightarrow \log(\log y) = \log Cx$$
$$\Rightarrow \log y = Cx$$
$$\Rightarrow y = e^{Cx}$$

This is the required general solution of the given differential equation.

Question 64(v)

Find the general solution of the given differential equation

$$\frac{dy}{dx} = \sin^{-1} x$$

Solution 64(v)

The given differential equation is:

$$\frac{dy}{dx} = \sin^{-1} x$$
$$\Rightarrow dy = \sin^{-1} x \ dx$$

Integrating both sides, we get:

$$\int dy = \int \sin^{-1} x \, dx$$
$$\Rightarrow y = \int \left(\sin^{-1} x \cdot 1\right) dx$$
$$\Rightarrow y = \sin^{-1} x \cdot \int (1)\, dx - \int \left[\left(\frac{d}{dx}\left(\sin^{-1} x\right) \cdot \int (1)\, dx \right) \right] dx$$

$$\Rightarrow y = \sin^{-1} x \cdot x - \int \left(\frac{1}{\sqrt{1-x^2}} \cdot x \right) dx$$
$$\Rightarrow y = x \sin^{-1} x + \int \frac{-x}{\sqrt{1-x^2}} dx \qquad ...(1)$$

Let $1 - x^2 = t$.

$$\Rightarrow \frac{d}{dx}\left(1 - x^2\right) = \frac{dt}{dx}$$
$$\Rightarrow -2x = \frac{dt}{dx}$$
$$\Rightarrow x\, dx = -\frac{1}{2} dt$$

Substituting this value in equation (1), we get:

$$y = x \sin^{-1} x + \int \frac{1}{2\sqrt{t}} dt$$
$$\Rightarrow y = x \sin^{-1} x + \frac{1}{2} \cdot \int (t)^{-\frac{1}{2}} dt$$
$$\Rightarrow y = x \sin^{-1} x + \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + \mathrm{C}$$
$$\Rightarrow y = x \sin^{-1} x + \sqrt{t} + \mathrm{C}$$
$$\Rightarrow y = x \sin^{-1} x + \sqrt{1-x^2} + \mathrm{C}$$

This is the required general solution of the given differential equation.

Question 64(vi)

Find the general solution of the given differential equation

$$\frac{dy}{dx}+y=1\ (y\neq 1)$$

Solution 64(vi)

The given differential equation is:

$$\frac{dy}{dx}+y=1$$
$$\Rightarrow dy+y\ dx=dx$$
$$\Rightarrow dy=(1-y)dx$$

Separating the variables, we get:

$$\Rightarrow \frac{dy}{1-y}=dx$$

Now, integrating both sides, we get:

$$\int\frac{dy}{1-y}=\int dx$$
$$\Rightarrow \log(1-y)=x+\log C$$
$$\Rightarrow -\log C-\log(1-y)=x$$
$$\Rightarrow \log C(1-y)=-x$$
$$\Rightarrow C(1-y)=e^{-x}$$
$$\Rightarrow 1-y=\frac{1}{C}e^{-x}$$
$$\Rightarrow y=1-\frac{1}{C}e^{-x}$$
$$\Rightarrow y=1+Ae^{-x}\ \left(\text{where } A=-\frac{1}{C}\right)$$

This is the required general solution of the given differential equation.

Question 65(i)

Find the solution of the given differential equation

$$x\left(x^2-1\right)\frac{dy}{dx}=1;\ y=0 \text{ when } x=2$$

Solution 65(i)

$$x\left(x^2-1\right)\frac{dy}{dx}=1$$
$$\Rightarrow dy=\frac{dx}{x\left(x^2-1\right)}$$
$$\Rightarrow dy=\frac{1}{x(x-1)(x+1)}dx$$

Integrating both sides, we get:

$$\int dy=\int\frac{1}{x(x-1)(x+1)}dx \qquad ...(1)$$

$$\text{Let } \frac{1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}. \qquad ...(2)$$
$$\Rightarrow \frac{1}{x(x-1)(x+1)}=\frac{A(x-1)(x+1)+Bx(x+1)+Cx(x-1)}{x(x-1)(x+1)}$$
$$=\frac{(A+B+C)x^2+(B-C)x-A}{x(x-1)(x+1)}$$

Comparing the coefficients of x^2, x, and constant, we get:

$$A=-1$$
$$B-C=0$$
$$A+B+C=0$$

Solving these equations, we get $B=\frac{1}{2}$ and $C=\frac{1}{2}$.

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{1}{x(x-1)(x+1)}=\frac{-1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$$

Therefore, equation (1) becomes:

$$\int dy=-\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{x-1}dx+\frac{1}{2}\int\frac{1}{x+1}dx$$
$$\Rightarrow y=-\log x+\frac{1}{2}\log(x-1)+\frac{1}{2}\log(x+1)+\log k$$
$$\Rightarrow y=\frac{1}{2}\log\left[\frac{k^2(x-1)(x+1)}{x^2}\right] \qquad ...(3)$$

Now, $y = 0$ when $x = 2$.

$$\Rightarrow 0 = \frac{1}{2}\log\left[\frac{k^2(2-1)(2+1)}{4}\right]$$

$$\Rightarrow \log\left(\frac{3k^2}{4}\right) = 0$$

$$\Rightarrow \frac{3k^2}{4} = 1$$

$$\Rightarrow 3k^2 = 4$$

$$\Rightarrow k^2 = \frac{4}{3}$$

Substituting the value of k^2 in equation (3), we get:

$$y = \frac{1}{2}\log\left[\frac{4(x-1)(x+1)}{3x^2}\right]$$

$$y = \frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$$

Question 65(ii)

Find the solution of the given differential equation

$$\cos\left(\frac{dy}{dx}\right) = a(a \in R); y = 1 \text{ when } x = 0$$

Solution 65(ii)

$$\cos\left(\frac{dy}{dx}\right) = a$$
$$\Rightarrow \frac{dy}{dx} = \cos^{-1} a$$
$$\Rightarrow dy = \cos^{-1} a \, dx$$

Integrating both sides, we get:

$$\int dy = \cos^{-1} a \int dx$$
$$\Rightarrow y = \cos^{-1} a \cdot x + C$$
$$\Rightarrow y = x \cos^{-1} a + C \qquad \ldots(1)$$

Now, $y = 1$ when $x = 0$.

$$\Rightarrow 1 = 0 \cdot \cos^{-1} a + C$$
$$\Rightarrow C = 1$$

Substituting $C = 1$ in equation (1), we get:

$$y = x \cos^{-1} a + 1$$
$$\Rightarrow \frac{y-1}{x} = \cos^{-1} a$$
$$\Rightarrow \cos\left(\frac{y-1}{x}\right) = a$$

Question 65(iii)

Find the solution of the given differential equation

$$\frac{dy}{dx} = y \tan x;\ y = 1 \text{ when } x = 0$$

Solution 65(iii)

$$\frac{dy}{dx} = y\tan x$$
$$\Rightarrow \frac{dy}{y} = \tan x\,dx$$

Integrating both sides, we get:

$$\int\frac{dy}{y} = -\int\tan x\,dx$$
$$\Rightarrow \log y = \log(\sec x) + \log C$$
$$\Rightarrow \log y = \log(C\sec x)$$
$$\Rightarrow y = C\sec x \qquad ...(1)$$

Now, $y = 1$ when $x = 0$.

$$\Rightarrow 1 = C \times \sec 0$$
$$\Rightarrow 1 = C \times 1$$
$$\Rightarrow C = 1$$

Substituting $C = 1$ in equation (1), we get:

$$y = \sec x$$

Question 66(i)

Solve the following differential equation

$$(x-y)\frac{dy}{dx} = x + 2y$$

Solution 66(i)

The given differential equation is

$$(x-y)\frac{dy}{dx} = x + 2y$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{x+2y}{x-y} = \frac{1+2\frac{y}{x}}{1-\frac{y}{x}}$$

Clearly this is a homogenous differential equation.

Let $y = vx$

Differentiating $y = vx$ w.r.t x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \qquad v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{1+2v}{1-v} - v = \frac{v^2+v+1}{1-v}$$

$$\Rightarrow \qquad \frac{v-1}{v^2+v+1}dv = -\frac{dx}{x}$$

$$\therefore \qquad \int\frac{v-1}{v^2+v+1}dv = \int -\frac{dx}{x}$$

$$\Rightarrow \qquad \frac{1}{2}\int\frac{2v+1-3}{v^2+v+1}dv = -\log|x| + C_1$$

$$\Rightarrow \frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv - \frac{3}{2}\int\frac{1}{v^2+v+1}dv = -\log|x| + C_1$$

$$\Rightarrow \frac{1}{2}\log\left|v^2+v+1\right| - \frac{3}{2}\int\frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dv = -\log|x| + C_1$$

$$\Rightarrow \frac{1}{2}\log\left|v^2+v+1\right| - \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = -\log|x| + C_1$$

$$\Rightarrow \frac{1}{2}\log\left|v^2+v+1\right| + \frac{1}{2}\log x^2 = \sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) + C_1$$

Replacing v by $\frac{y}{x}$, we get

$$\frac{1}{2}\log\left|\left(\frac{y}{x}\right)^2 + \frac{y}{x} + 1\right| + \frac{1}{2}\log x^2 = \sqrt{3}\tan^{-1}\left(\frac{2\frac{y}{x}+1}{\sqrt{3}}\right) + C_1$$

$$\Rightarrow \frac{1}{2}\log\left|\left\{\left(\frac{y}{x}\right)^2 + \frac{y}{x} + 1\right\}x^2\right| = \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + C_1$$

$$\Rightarrow \log\left|y^2+xy+x^2\right| = 2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + 2C_1$$

$$\Rightarrow \log\left|x^2+xy+y^2\right| = 2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right) + C$$

Question 66(ii)

Solve the following differential equation:

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$

Solution 66(ii)

The given differential equation is :

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

Clearly this is a hmogenous differential equation.

Let $y = vx$

Differentiating $y = vx$ w.r.t x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v = \frac{1}{\cos v}$$

$$\Rightarrow \quad \cos v\, dv = \frac{dx}{x}$$

$$\therefore \quad \int \cos v\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \sin v = \log|x| + \log|C|$$

$$\Rightarrow \quad \sin v = \log|Cx|$$

Replacing v by $\frac{y}{x}$, we get

$$\sin\left(\frac{y}{x}\right) = \log|Cx|$$

Question 66(iii)

Show that the given differential equation is homogeneous and solve it

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

Solution 66(iii)

$$y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$$

$$\Rightarrow y\,dx = \left[2x - x\log\left(\frac{y}{x}\right)\right]dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \qquad \ldots(1)$$

Let $F(x,y) = \dfrac{y}{2x - x\log\left(\frac{y}{x}\right)}$.

$$\therefore\ F(\lambda x, \lambda y) = \frac{\lambda y}{2(\lambda x) - (\lambda x)\log\left(\frac{\lambda y}{\lambda x}\right)} = \frac{y}{2x - \log\left(\frac{y}{x}\right)} = \lambda^0 \cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx)$$
$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log v}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{2 - \log v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$
$$\Rightarrow x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{v\log v - v}{2 - \log v}$$
$$\Rightarrow \frac{2 - \log v}{v(\log v - 1)}dv = \frac{dx}{x}$$
$$\Rightarrow \left[\frac{1 + (1 - \log v)}{v(\log v - 1)}\right]dv = \frac{dx}{x}$$

$$\Rightarrow \left[\frac{1}{v(\log v-1)}-\frac{1}{v}\right]dv=\frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{1}{v(\log v-1)}dv-\int \frac{1}{v}dv=\int \frac{1}{x}dx$$

$$\Rightarrow \int \frac{dv}{v(\log v-1)}-\log v=\log x+\log C \qquad ...(2)$$

$$\Rightarrow \text{Let } \log v-1=t$$

$$\Rightarrow \frac{d}{dv}(\log v-1)=\frac{dt}{dv}$$

$$\Rightarrow \frac{1}{v}=\frac{dt}{dv}$$

$$\Rightarrow \frac{dv}{v}=dt$$

Therefore, equation (1) becomes:

$$\Rightarrow \int \frac{dt}{t}-\log v=\log x+\log C$$

$$\Rightarrow \log t-\log\left(\frac{y}{x}\right)=\log(Cx)$$

$$\Rightarrow \log\left[\log\left(\frac{y}{x}\right)-1\right]-\log\left(\frac{y}{x}\right)=\log(Cx)$$

$$\Rightarrow \log\left[\frac{\log\left(\frac{y}{x}\right)-1}{\frac{y}{x}}\right]=\log(Cx)$$

$$\Rightarrow \frac{x}{y}\left[\log\left(\frac{y}{x}\right)-1\right]=Cx$$

$$\Rightarrow \log\left(\frac{y}{x}\right)-1=Cy$$

This is the required solution of the given differential equation.

Question 66(iv)

Find the general solution of the differential equation $\frac{dy}{dx}-y=\cos x$

Solution 66(iv)

Given differential equation is $\frac{dy}{dx} - y = \cos x$

It is of the form $\frac{dy}{dx} + Py = Q$, where $P = -1$ and $Q = \cos x$

$\therefore \quad \text{I.F.} = e^{\int -1\,dx} = e^{-x}$

Multiplying both sides of equation by I.F. , we get

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\cos x$$

$$\Rightarrow \quad \frac{d}{dx}\left(ye^{-x}\right) = e^{-x}\cos x$$

On integrating both sides w.r.t. x , we get

$$ye^{-x} = \int e^{-x}\cos x\,dx + C \qquad ------(i)$$

Let $\quad I = \int e^{-x}\cos x\,dx$

$$= \cos x\left(\frac{e^{-x}}{-1}\right) - \int\left(-\sin x\right)\left(-e^{-x}\right)dx$$

$$= -\cos x\,e^{-x} - \int \sin x\,e^{-x}\,dx$$

$$= -\cos x\,e^{-x} - \left[\sin x\left(-e^{-x}\right) - \int\left(-e^{-x}\right)\cos x\,dx\right]$$

$$= -\cos x\,e^{-x} + \sin x\,e^{-x} - \int e^{-x}\cos x\,dx$$

$$= (\sin x - \cos x)e^{-x} - I$$

$$\Rightarrow \quad 2I = (\sin x - \cos x)e^{-x}$$

$$\Rightarrow \quad I = \frac{(\sin x - \cos x)e^{-x}}{2}$$

Substituting the value of I in (i),

$$ye^{-x} = \frac{(\sin x - \cos x)e^{-x}}{2} + C$$

Or $\quad y = \left(\frac{\sin x - \cos x}{2}\right) + Ce^{x}$

Question 66(v)

Find the general solution of the differential equation $x\frac{dy}{dx} + 2y = x^2 \; (x \neq 0)$

Solution 66(v)

The given differential equation is

$$x\frac{dy}{dx}+2y=x^2\ (x\neq 0) \qquad ----(i)$$

Dividing both sides of (i) by x, we get

$$\frac{dy}{dx}+\frac{2}{x}y=x$$

which is a linear differential equation of the type $\frac{dy}{dx}+Py=Q$

where $P=\frac{2}{x}$ and $Q=x$

So, $\quad I.F.=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$

$\therefore$ Solution of the given equation is given by

$$y.\,x^2=\int x.x^2\,dx+C=\int x^3dx\ +C$$

$$\therefore \quad y\ =\frac{x^2}{4}+Cx^{-2}$$

which is the general solution of the given differential equation.

Question 66(vi)

Find the general solution of the differential equation

$$\frac{dy}{dx}+2y=\sin x$$

Solution 66(vi)

The given differential equation is $\frac{dy}{dx}+2y=\sin x$.

This is in the form of $\frac{dy}{dx}+py=Q$ (where $p=2$ and $Q=\sin x$).

Now, I.F $=e^{\int p\,dx}=e^{\int 2\,dx}=e^{2x}$.

The solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})dx+\text{C}$$
$$\Rightarrow ye^{2x}=\int\sin x\cdot e^{2x}dx+\text{C} \qquad ...(1)$$
$$\text{Let } I=\int\sin x\cdot e^{2x}.$$
$$\Rightarrow I=\sin x\cdot\int e^{2x}dx-\int\left(\frac{d}{dx}(\sin x)\cdot\int e^{2x}dx\right)dx$$
$$\Rightarrow I=\sin x\cdot\frac{e^{2x}}{2}-\int\left(\cos x\cdot\frac{e^{2x}}{2}\right)dx$$
$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x\cdot\int e^{2x}-\int\left(\frac{d}{dx}(\cos x)\cdot\int e^{2x}dx\right)dx\right]$$
$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x\cdot\frac{e^{2x}}{2}-\int\left[(-\sin x)\cdot\frac{e^{2x}}{2}\right]dx\right]$$
$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}-\frac{1}{4}\int(\sin x.e^{2x})dx$$
$$\Rightarrow I=\frac{e^{2x}}{4}(2\sin x-\cos x)-\frac{1}{4}I$$
$$\Rightarrow\frac{5}{4}I=\frac{e^{2x}}{4}(2\sin x-\cos x)$$
$$\Rightarrow I=\frac{e^{2x}}{5}(2\sin x-\cos x)$$

Therefore, equation (1) becomes:

$$ye^{2x}=\frac{e^{2x}}{5}(2\sin x-\cos x)+\text{C}$$
$$\Rightarrow y=\frac{1}{5}(2\sin x-\cos x)+\text{C}e^{-2x}$$

This is the required general solution of the given differential equation.

Question 66(vii)

Find the general solution of the differential equation

$$\frac{dy}{dx}+3y=e^{-2x}$$

Solution 66(vii)

The given differential equation is $\frac{dy}{dx}+py=Q$ (where $p=3$ and $Q=e^{-2x}$).

Now, I.F $=e^{\int p\,dx}=e^{\int 3\,dx}=e^{3x}$.

The solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})\,dx+\text{C}$$
$$\Rightarrow ye^{3x}=\int\left(e^{-2x}\times e^{3x}\right)+\text{C}$$
$$\Rightarrow ye^{3x}=\int e^{x}\,dx+\text{C}$$
$$\Rightarrow ye^{3x}=e^{x}+\text{C}$$
$$\Rightarrow y=e^{-2x}+\text{C}e^{-3x}$$

This is the required general solution of the given differential equation.

Question 66(viii)

Find the general solution of the differential equation

$$\frac{dy}{dx}+\frac{y}{x}=x^{2}$$

Solution 66(viii)

The given differential equation is:

$$\frac{dy}{dx} + py = Q \text{ (where } p = \frac{1}{x} \text{ and } Q = x^2\text{)}$$

Now, I.F $= e^{\int p\,dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x.$

The solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\,dx + \text{C}$$

$$\Rightarrow y(x) = \int (x^2 \cdot x)\,dx + \text{C}$$

$$\Rightarrow xy = \int x^3 dx + \text{C}$$

$$\Rightarrow xy = \frac{x^4}{4} + \text{C}$$

This is the required general solution of the given differential equation.

Question 66(ix)

Find the general solution of the differential equation

$$\frac{dy}{dx} + \sec x y = \tan x \left(0 \le x < \frac{\pi}{2}\right)$$

Solution 66(ix)

The given differential equation is:

$$\frac{dy}{dx} + py = Q \text{ (where } p = \sec x \text{ and } Q = \tan x\text{)}$$

Now, I.F $= e^{\int p\,dx} = e^{\int \sec x\,dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x.$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\,dx + \text{C}$$

$$\Rightarrow y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\,dx + \text{C}$$

$$\Rightarrow y(\sec x + \tan x) = \int \sec x \tan x\,dx + \int \tan^2 x\,dx + \text{C}$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \int (\sec^2 x - 1)\,dx + \text{C}$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \tan x - x + \text{C}$$

Question 66(x)

Find the general solution of the differential equation

$$x\frac{dy}{dx}+2y=x^2\log x$$

Solution 66(x)

The given differential equation is:

$$x\frac{dy}{dx}+2y=x^2\log x$$

$$\Rightarrow\frac{dy}{dx}+\frac{2}{x}y=x\log x$$

This equation is in the form of a linear differential equation as:

$$\frac{dy}{dx}+py=Q\ (\text{where } p=\frac{2}{x} \text{ and } Q=x\log x)$$

$$\text{Now, I.F } =e^{\int p\,dx}=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2.$$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})\,dx+\text{C}$$

$$\Rightarrow y\cdot x^2=\int\left(x\log x\cdot x^2\right)dx+\text{C}$$

$$\Rightarrow x^2y=\int\left(x^3\log x\right)dx+\text{C}$$

$$\Rightarrow x^2y=\log x\cdot\int x^3dx-\int\left[\frac{d}{dx}(\log x)\cdot\int x^3dx\right]dx+\text{C}$$

$$\Rightarrow x^2y=\log x\cdot\frac{x^4}{4}-\int\left(\frac{1}{x}\cdot\frac{x^4}{4}\right)dx+\text{C}$$

$$\Rightarrow x^2y=\frac{x^4\log x}{4}-\frac{1}{4}\int x^3dx+\text{C}$$

$$\Rightarrow x^2y=\frac{x^4\log x}{4}-\frac{1}{4}\cdot\frac{x^4}{4}+\text{C}$$

$$\Rightarrow x^2y=\frac{1}{16}x^4\left(4\log x-1\right)+\text{C}$$

$$\Rightarrow y=\frac{1}{16}x^2\left(4\log x-1\right)+\text{C}x^{-2}$$

Question 66(xi)

Find the general solution of the differential equation

$$x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$$

Solution 66(xi)

The given differential equation is:

$$x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$$
$$\Rightarrow\frac{dy}{dx}+\frac{y}{x\log x}=\frac{2}{x^2}$$

This equation is the form of a linear differential equation as:

$$\frac{dy}{dx}+py=Q\ (\text{where } p=\frac{1}{x\log x}\ \text{ and } Q=\frac{2}{x^2})$$

Now, I.F $=e^{\int p\,dx}=e^{\int\frac{1}{x\log}dx}=e^{\log(\log x)}=\log x.$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})\,dx+\text{C}$$
$$\Rightarrow y\log x=\int\left(\frac{2}{x^2}\log x\right)dx+\text{C}\qquad ...(1)$$

Now, $\int\left(\frac{2}{x^2}\log x\right)dx = 2\int\left(\log x\cdot\frac{1}{x^2}\right)dx.$

$$= 2\left[\log x\cdot\int\frac{1}{x^2}dx - \int\left\{\frac{d}{dx}(\log x)\cdot\int\frac{1}{x^2}dx\right\}dx\right]$$

$$= 2\left[\log x\left(-\frac{1}{x}\right) - \int\left(\frac{1}{x}\cdot\left(-\frac{1}{x}\right)\right)dx\right]$$

$$= 2\left[-\frac{\log x}{x} + \int\frac{1}{x^2}dx\right]$$

$$= 2\left[-\frac{\log x}{x} - \frac{1}{x}\right]$$

$$= -\frac{2}{x}(1+\log x)$$

Substituting the value of $\int\left(\frac{2}{x^2}\log x\right)dx$ in equation (1), we get:

$$y\log x = -\frac{2}{x}(1+\log x) + C$$

This is the required general solution of the given differential equation.

Question 66(xii)

Find the general solution of the differential equation

$$\left(1+x^2\right)dy + 2xy\ dx = \cot x dx\left(x \neq 0\right)$$

Solution 66(xii)

$$\left(1+x^2\right)dy+2xy\ dx=\cot x dx$$
$$\Rightarrow \frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{\cot x}{1+x^2}$$

This equation is a linear differential equation of the form:

$$\frac{dy}{dx}+py=Q \text{ (where } p=\frac{2x}{1+x^2} \text{ and } Q=\frac{\cot x}{1+x^2})$$
$$\text{Now, I.F } =e^{\int p\,dx}=e^{\int \frac{2x}{1+x^2}dx}=e^{\log\left(1+x^2\right)}=1+x^2.$$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})dx+\text{C}$$
$$\Rightarrow y\left(1+x^2\right)=\int\left[\frac{\cot x}{1+x^2}\times\left(1+x^2\right)\right]dx+\text{C}$$
$$\Rightarrow y\left(1+x^2\right)=\int\cot x dx+\text{C}$$
$$\Rightarrow y\left(1+x^2\right)=\log\left|\sin x\right|+\text{C}$$

Question 66(xiii)

Find the general solution of the differential equation

$$(x+y)\frac{dy}{dx}=1$$

Solution 66(xiii)

$$(x+y)\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x+y}$$

$$\Rightarrow \frac{dx}{dy}=x+y$$

$$\Rightarrow \frac{dx}{dy}-x=y$$

This is a linear differential equation of the form:

$\frac{dy}{dx}+px=Q$ (where $p=-1$ and $Q=y$)

Now, I.F $=e^{\int p\,dy}=e^{\int -dy}=e^{-y}$.

The general solution of the given differential equation is given by the relation,

$$x(\text{I.F.})=\int(\text{Q}\times\text{I.F.})\,dy+\text{C}$$

$$\Rightarrow xe^{-y}=\int\left(y\cdot e^{-y}\right)dy+\text{C}$$

$$\Rightarrow xe^{-y}=y\cdot\int e^{-y}dy-\int\left[\frac{d}{dy}(y)\int e^{-y}dy\right]dy+\text{C}$$

$$\Rightarrow xe^{-y}=y\left(-e^{-y}\right)-\int\left(-e^{-y}\right)dy+\text{C}$$

$$\Rightarrow xe^{-y}=-ye^{-y}+\int e^{-y}dy+\text{C}$$

$$\Rightarrow xe^{-y}=-ye^{-y}-e^{-y}+\text{C}$$

$$\Rightarrow x=-y-1+\text{C}e^{y}$$

$$\Rightarrow x+y+1=\text{C}e^{y}$$

Question 66(xiv)

Find the general solution of the differential equation

$$y\,dx+\left(x-y^2\right)dy=0$$

Solution 66(xiv)

$$y\,dx + \left(x - y^2\right)dy = 0$$
$$\Rightarrow ydx = \left(y^2 - x\right)dy$$
$$\Rightarrow \frac{dx}{dy} = \frac{y^2 - x}{y} = y - \frac{x}{y}$$
$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = y$$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + px = Q \text{ (where } p = \frac{1}{y} \text{ and } Q = y\text{)}$$

Now, I.F $= e^{\int p\,dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y.$

The general solution of the given differential equation is given by the relation,

$$x\left(\text{I.F.}\right) = \int \left(\text{Q} \times \text{I.F.}\right)dy + \text{C}$$
$$\Rightarrow xy = \int \left(y \cdot y\right)dy + \text{C}$$
$$\Rightarrow xy = \int y^2 dy + \text{C}$$
$$\Rightarrow xy = \frac{y^3}{3} + \text{C}$$
$$\Rightarrow x = \frac{y^2}{3} + \frac{\text{C}}{y}$$

Question 66(xv)

Find the general solution of the differential equation

$$\left(x + 3y^2\right)\frac{dy}{dx} = y\left(y > 0\right)$$

Solution 66(xv)

$$\left(x+3y^2\right)\frac{dy}{dx}=y$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x+3y^2}$$

$$\Rightarrow \frac{dx}{dy}=\frac{x+3y^2}{y}=\frac{x}{y}+3y$$

$$\Rightarrow \frac{dx}{dy}-\frac{x}{y}=3y$$

This is a linear differential equation of the form:

$$\frac{dx}{dy}+px=Q \text{ (where } p=-\frac{1}{y} \text{ and } Q=3y)$$

$$\text{Now, I.F } = e^{\int p\,dy}=e^{-\int\frac{dy}{y}}=e^{-\log y}=e^{\log\left(\frac{1}{y}\right)}=\frac{1}{y}.$$

The general solution of the given differential equation is given by the relation,

$$x(\text{I.F.})=\int(\text{Q}\times\text{I.F.})\,dy+\text{C}$$

$$\Rightarrow x\times\frac{1}{y}=\int\left(3y\times\frac{1}{y}\right)dy+\text{C}$$

$$\Rightarrow \frac{x}{y}=3y+\text{C}$$

$$\Rightarrow x=3y^2+\text{C}y$$

Question 67(i)

Find the particular solution of the differential equation

$$\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2};\ y=0 \text{ when } x=1$$

Solution 67(i)

$$\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$$
$$\Rightarrow \frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{1}{\left(1+x^2\right)^2}$$

This is a linear differential equation of the form:

$$\frac{dy}{dx}+py=Q \text{ (where } p=\frac{2x}{1+x^2} \text{ and } Q=\frac{1}{\left(1+x^2\right)^2}\text{)}$$

$$\text{Now, I.F } = e^{\int p\,dx} = e^{\int\frac{2x\,dx}{1+x^2}} = e^{\log\left(1+x^2\right)} = 1+x^2.$$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int(\text{Q}\times\text{I.F.})\,dx+\text{C}$$
$$\Rightarrow y\left(1+x^2\right)=\int\left[\frac{1}{\left(1+x^2\right)^2}\cdot\left(1+x^2\right)\right]dx+\text{C}$$
$$\Rightarrow y\left(1+x^2\right)=\int\frac{1}{1+x^2}\,dx+\text{C}$$
$$\Rightarrow y\left(1+x^2\right)=\tan^{-1}x+\text{C} \qquad \ldots(1)$$

Now, $y=0$ at $x=1$.

Therefore,

$$0=\tan^{-1}1+\text{C}$$
$$\Rightarrow \text{C}=-\frac{\pi}{4}$$

Substituting $\text{C}=-\frac{\pi}{4}$ in equation (1), we get:

$$y\left(1+x^2\right)=\tan^{-1}x-\frac{\pi}{4}$$

This is the required general solution of the given differential equation.

Question 67(ii)

Find the particular solution of differential equation satisfying the given condition

$$(x+y)\,dy+(x-y)\,dy=0;\ y=1 \text{ when } x=1$$

Solution 67(ii)

$$(x+y)\,dy+(x-y)\,dx=0$$
$$\Rightarrow (x+y)\,dy=-(x-y)\,dx$$
$$\Rightarrow \frac{dy}{dx}=\frac{-(x-y)}{x+y} \qquad ...(1)$$

Let $F(x,y)=\dfrac{-(x-y)}{x+y}$.

$$\therefore\ F(\lambda x,\lambda y)=\frac{-(\lambda x-\lambda y)}{\lambda x-\lambda y}=\frac{-(x-y)}{x+y}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y=vx$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$
$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\dfrac{dy}{dx}$ in equation (1), we get:

$$v+x\frac{dv}{dx}=\frac{-(x-vx)}{x+vx}$$
$$\Rightarrow v+x\frac{dv}{dx}=\frac{v-1}{v+1}$$
$$\Rightarrow x\frac{dv}{dx}=\frac{v-1}{v+1}-v=\frac{v-1-v(v+1)}{v+1}$$
$$\Rightarrow x\frac{dv}{dx}=\frac{v-1-v^2-v}{v+1}=\frac{-(1+v^2)}{v+1}$$
$$\Rightarrow \frac{(v+1)}{1+v^2}\,dv=-\frac{dx}{x}$$
$$\Rightarrow \left[\frac{v}{1+v^2}+\frac{1}{1+v^2}\right]dv=-\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\log\left(1+v^2\right)+\tan^{-1}v=-\log x+k$$
$$\Rightarrow \log\left(1+v^2\right)+2\tan^{-1}v=-2\log x+2k$$
$$\Rightarrow \log\left[\left(1+v^2\right)\cdot x^2\right]+2\tan^{-1}v=2k$$
$$\Rightarrow \log\left[\left(1+\frac{y^2}{x^2}\right)\cdot x^2\right]+2\tan^{-1}\frac{y}{x}=2k$$
$$\Rightarrow \log\left(x^2+y^2\right)+2\tan^{-1}\frac{y}{x}=2k \qquad ...(2)$$

Now, $y = 1$ at $x = 1$.

$$\Rightarrow \log 2+2\tan^{-1}1=2k$$
$$\Rightarrow \log 2+2\times\frac{\pi}{4}=2k$$
$$\Rightarrow \frac{\pi}{2}+\log 2=2k$$

Substituting the value of $2k$ in equation (2), we get:

$$\log\left(x^2+y^2\right)+2\tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{2}+\log 2$$

This is the required solution of the given differential equation.

Question 67(iii)

Find the particular solution of differential equation satisfying the given condition

$$x^2dy+\left(xy+y^2\right)dx=0;\ y=1 \text{ when } x=1$$

Solution 67(iii)

$$x^2 dy + \left(xy + y^2\right) dx = 0$$

$$\Rightarrow x^2 dy = -\left(xy + y^2\right) dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left(xy + y^2\right)}{x^2} \qquad \ldots(1)$$

Let $F\left(x, y\right) = \frac{-\left(xy + y^2\right)}{x^2}$.

$$\therefore\ F\left(\lambda x, \lambda y\right) = \frac{\left[\lambda x \cdot \lambda y + \left(\lambda y\right)^2\right]}{\left(\lambda x\right)^2} = \frac{-\left(xy + y^2\right)}{x^2} = \lambda^0 \cdot F\left(x, y\right)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{d}{dx}\left(y\right) = \frac{d}{dx}\left(vx\right)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{-\left[x \cdot vx + (vx)^2\right]}{x^2} = -v - v^2$$

$$\Rightarrow x\frac{dv}{dx} = -v^2 - 2v = -v(v+2)$$

$$\Rightarrow \frac{dv}{v(v+2)} = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{(v+2)-v}{v(v+2)}\right]dv = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{v} - \frac{1}{v+2}\right]dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\left[\log v - \log(v+2)\right] = -\log x + \log \mathrm{C}$$

$$\Rightarrow \frac{1}{2}\log\left(\frac{v}{v+2}\right) = \log\frac{\mathrm{C}}{x}$$

$$\Rightarrow \frac{v}{v+2} = \left(\frac{\mathrm{C}}{x}\right)^2$$

$$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+2} = \left(\frac{C}{x}\right)^2$$

$$\Rightarrow \frac{y}{y+2x} = \frac{C^2}{x^2}$$

$$\Rightarrow \frac{x^2 y}{y+2x} = C^2 \qquad ...(2)$$

Now, $y = 1$ at $x = 1$.

$$\Rightarrow \frac{1}{1+2} = C^2$$

$$\Rightarrow C^2 = \frac{1}{3}$$

Substituting $C^2 = \frac{1}{3}$ in equation (2), we get:

$$\frac{x^2 y}{y+2x} = \frac{1}{3}$$

$$\Rightarrow y + 2x = 3x^2 y$$

This is the required solution of the given differential equation.

Question 68

Find the equation of the curve passing through the point $(1,1)$ whose differential equation is $x\,dy = (2x^2 + 1)\,dx$, $x \neq 0$.

Solution 68

Given differential equation is $x\,dy = \left(2x^2+1\right)dx$, $x \neq 0$.

$$\therefore \quad dy = \left(\frac{2x^2+1}{x}\right)dx$$

$$\Rightarrow \quad dy = \left(2x+\frac{1}{x}\right)dx$$

Integrating both sides, we get

$$\int dy = \int\left(2x+\frac{1}{x}\right)dx$$

$$\Rightarrow \quad y = x^2 + \log|x| + C$$

Given that the above equation passes through $(1,1)$

$$\therefore \quad 1 = 1^2 + \log 1 + C$$

$$\Rightarrow \quad C = 0$$

$\therefore$ The equation of the required curve is

$$y = x^2 + \log|x|$$

Question 69

Find the equation of a curve passing through the point $(-2,3)$, given that the slope of the tangent to the curve at any point (x,y) is $\frac{2x}{y^2}$

Solution 69

We know the slope of the tangent at any on a curve is $\frac{dy}{dx}$

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{y^2}$$

Separating the variables, the above equation can be written as

$$y^2\,dy = 2x\,dx$$

Integrating both sides, we get

$$\int y^2\,dy = \int 2x\,dx$$

$$\Rightarrow \quad \frac{y^3}{3} = x^2 + C$$

Substituting $x = -2$, $y = 3$ in above equation

$$\frac{3^3}{3} = (-2)^2 + C$$

$$\Rightarrow \quad C = 5$$

Substituting the value of C in (i), we get the equation of the required curve

$$\frac{y^3}{3} = x^2 + 5$$

or $$y = \left(3x^2+15\right)^{\frac{1}{3}}$$

Question 70

Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$.

Solution 70

The differential equation of the curve is:

$$y' = e^x \sin x$$
$$\Rightarrow \frac{dy}{dx} = e^x \sin x$$
$$\Rightarrow dy = e^x \sin x$$

Integrating both sides, we get:

$$\int dy = \int e^x \sin x \, dx \qquad \ldots(1)$$

Let $I = \int e^x \sin x \, dx$.

$$\Rightarrow I = \sin x \int e^x dx - \int\left(\frac{d}{dx}(\sin x) \cdot \int e^x dx\right) dx$$

$$\Rightarrow I = \sin x \cdot e^x - \int \cos x \cdot e^x dx$$
$$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot \int e^x dx - \int\left(\frac{d}{dx}(\cos x) \cdot \int e^x dx\right) dx\right]$$
$$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot e^x - \int(-\sin x) \cdot e^x dx\right]$$
$$\Rightarrow I = e^x \sin x - e^x \cos x - I$$
$$\Rightarrow 2I = e^x (\sin x - \cos x)$$
$$\Rightarrow I = \frac{e^x (\sin x - \cos x)}{2}$$

Substituting this value in equation (1), we get:

$$y = \frac{e^x(\sin x - \cos x)}{2} + C \qquad ...(2)$$

Now, the curve passes through point (0, 0).

$$\therefore 0 = \frac{e^0(\sin 0 - \cos 0)}{2} + C$$
$$\Rightarrow 0 = \frac{1(0-1)}{2} + C$$
$$\Rightarrow C = \frac{1}{2}$$

Substituting $C = \frac{1}{2}$ in equation (2), we get:

$$y = \frac{e^x(\sin x - \cos x)}{2} + \frac{1}{2}$$
$$\Rightarrow 2y = e^x(\sin x - \cos x) + 1$$
$$\Rightarrow 2y - 1 = e^x(\sin x - \cos x)$$

Hence, the required equation of the curve is $2y - 1 = e^x(\sin x - \cos x)$.

Question 71

At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (–4, –3). Find the equation of the curve given that it passes through (–2, 1).

Solution 71

It is given that (x, y) is the point of contact of the curve and its tangent.

The slope (m_1) of the line segment joining (x, y) and $(-4, -3)$ is $\frac{y+3}{x+4}$.

We know that the slope of the tangent to the curve is given by the relation,

$$\frac{dy}{dx}$$

$$\therefore \text{Slope } (m_2) \text{ of the tangent } = \frac{dy}{dx}$$

According to the given information:

$$m_2 = 2m_1$$
$$\Rightarrow \frac{dy}{dx} = \frac{2(y+3)}{x+4}$$
$$\Rightarrow \frac{dy}{y+3} = \frac{2dx}{x+4}$$

Integrating both sides, we get:

$$\int \frac{dy}{y+3} = 2\int \frac{dx}{x+4}$$
$$\Rightarrow \log(y+3) = 2\log(x+4) + \log C$$
$$\Rightarrow \log(y+3) \log C(x+4)^2$$
$$\Rightarrow y+3 = C(x+4)^2 \qquad ...(1)$$

This is the general equation of the curve.

It is given that it passes through point $(-2, 1)$.

$$\Rightarrow 1+3 = C(-2+4)^2$$
$$\Rightarrow 4 = 4C$$
$$\Rightarrow C = 1$$

Substituting $C = 1$ in equation (1), we get:

$$y + 3 = (x + 4)^2$$

This is the required equation of the curve.

Question 72

Show that the family of curves for which the slope of the tangent at any point (x,y) on it is $\frac{x^2+y^2}{2xy}$, is given by $x^2 - y^2 = cx$.

Solution 72

We know the slope of the tangent at any on a curve is $\frac{dy}{dx}$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

$$= \frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}}$$

Clearly this is a hmogenous differential equation.

Let $y = vx$

Differentiating $y = vx$ w.r.t x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\Rightarrow \quad \frac{2v}{1-v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \quad \frac{2v}{v^2-1}dv = -\frac{dx}{x}$$

$$\therefore \quad \int\frac{2v}{v^2-1}dv = \int-\frac{dx}{x}$$

$$\Rightarrow \quad \log\left|v^2-1\right| = -\log|x| + \log|C_1|$$

$$\Rightarrow \quad \log\left|\left(v^2-1\right)(x)\right| = \log|C_1|$$

$$\Rightarrow \quad \left(v^2-1\right)x = \pm C_1$$

Replacing v by $\frac{y}{x}$, we get

$$\left(\frac{y^2}{x^2}-1\right)x = \pm C_1$$

$$\Rightarrow \quad \left(y^2-x^2\right) = \pm C_1 x \quad \text{or} \quad x^2 - y^2 = Cx$$

Question 73

Find the equation of a curve passing through the point $(0,1)$. If the slope of the tangent to the curve at any point (x,y) is equal to the sum of the x coordinate and the product of the x coordinate and y coordinate of that point.

Solution 73

We know that the slope of the tangent to the curve is $\frac{dy}{dx}$.

$$\therefore \qquad \frac{dy}{dx} = x + xy$$

$$\Rightarrow \qquad \frac{dy}{dx} - xy = y \qquad \text{------(i)}$$

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$

where $P = -x$ and $Q = x$.

So, $\qquad \text{I.F.} = e^{\int -x dx} = e^{\frac{-x^2}{2}}$

$\therefore$ Solution of the given equation is given by

$$y.\, e^{\frac{-x^2}{2}} = \int x.e^{\frac{-x^2}{2}}\, dx + C \qquad \text{------(ii)}$$

Let $\qquad I = \int x.e^{\frac{-x^2}{2}}\, dx$

Let $\frac{-x^2}{2} = t$, then $-x\, dx = dt$ or $x\, dx = -dt$

$$\therefore \qquad I = \int x.e^{\frac{-x^2}{2}}\, dx = \int -e^t\, dt = -e^t = -e^{\frac{-x^2}{2}}$$

Substituting the value of I in (ii), we get

$$y.\, e^{\frac{-x^2}{2}} = -e^{\frac{-x^2}{2}} + C$$

$$\text{or} \qquad y = -1 + Ce^{\frac{x^2}{2}} \qquad \text{------(iii)}$$

This equation (iii) passes through $(0,1)$

$$\therefore \qquad 1 = -1 + Ce^0 \quad \Rightarrow \quad C = 2$$

Substituting the value of C in (iii), we get

$$y = -1 + 2e^{\frac{x^2}{2}}$$

which is the equation of the required curve.

Question 74

Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

Solution 74

Let $F(x, y)$ be the curve passing through the origin.

At point (x, y), the slope of the curve will be $\frac{dy}{dx}$.

According to the given information:

$$\frac{dy}{dx} = x + y$$
$$\Rightarrow \frac{dy}{dx} - y = x$$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + py = Q \text{ (where } p = -1 \text{ and } Q = x)$$

Now, I.F $= e^{\int p\,dx} = e^{\int(-1)dx} = e^{-x}$.

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F.}) = \int(\text{Q} \times \text{I.F.})\,dx + \text{C}$$
$$\Rightarrow ye^{-x} = \int xe^{-x}dx + \text{C} \qquad \ldots(1)$$

Now, $\int xe^{-x}dx = x\int e^{-x}dx - \int\left[\frac{d}{dx}(x)\cdot\int e^{-x}dx\right]dx.$

$$= -xe^{-x} - \int -e^{-x}dx$$
$$= -xe^{-x} + \left(-e^{-x}\right)$$
$$= -e^{-x}(x+1)$$

Substituting in equation (1), we get:

$$ye^{-x} = -e^{-x}(x+1) + \mathrm{C}$$
$$\Rightarrow y = -(x+1) + \mathrm{C}e^{x}$$
$$\Rightarrow x + y + 1 = \mathrm{C}e^{x} \qquad ...(2)$$

The curve passes through the origin.

Therefore, equation (2) becomes:

1 = C

C = 1

Substituting C = 1 in equation (2), we get:

$$x + y + 1 = e^{x}$$

Hence, the required equation of curve passing through the origin is $x + y + 1 = e^{x}$.

Question 75

Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

Solution 75

Let $F(x, y)$ be the curve and let (x, y) be a point on the curve. The slope of the tangent to the curve at (x, y) is $\frac{dy}{dx}$.

According to the given information:

$$\frac{dy}{dx}+5=x+y$$
$$\Rightarrow \frac{dy}{dx}-y=x-5$$

This is a linear differential equation of the form:

$$\frac{dy}{dx}+py=Q \text{ (where } p=-1 \text{ and } Q=x-5)$$

Now, I.F $=e^{\int p dx}=e^{\int(-1)dx}=e^{-x}$.

The general equation of the curve is given by the relation,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})dx+\text{C}$$
$$\Rightarrow y\cdot e^{-x}=\int(x-5)e^{-x}dx+\text{C} \qquad ...(1)$$

$$\text{Now, } \int (x-5)e^{-x}dx = (x-5)\int e^{-x}dx - \int\left[\frac{d}{dx}(x-5).\int e^{-x}dx\right]dx.$$
$$= (x-5)(-e^{-x}) - \int(-e^{-x})dx$$
$$= (5-x)e^{-x} + (-e^{-x})$$
$$= (4-x)e^{-x}$$

Therefore, equation (1) becomes:

$$ye^{-x} = (4-x)e^{-x} + C$$
$$\Rightarrow y = 4 - x + Ce^{x}$$
$$\Rightarrow x + y - 4 = Ce^{x} \qquad ...(2)$$

The curve passes through point (0, 2).

Therefore, equation (2) becomes:

$$0 + 2 - 4 = Ce^{0}$$
$$\Rightarrow -2 = C$$
$$\Rightarrow C = -2$$

Substituting $C = -2$ in equation (2), we get:

$$x + y - 4 = -2e^{x}$$
$$\Rightarrow y = 4 - x - 2e^{x}$$

This is the required equation of the curve.

Question 76

The slope of the tangent to the curve at any point is the reciprocal of twice the ordinate at that point. The curve passes through the point $(4,3)$. Determine the equation.

Solution 76

Given, the slope of the tangent to the curve at any point is the reciprocal of twice the ordinate at that point.

$\therefore \quad \frac{dy}{dx} = \frac{1}{2y}$

$\Rightarrow \quad 2y\,dy = dx$

Integrating both sides, we get

$$\int 2y\,dy = \int dx$$

$\Rightarrow \quad 2 \times \frac{y^2}{2} = x + C$

$\therefore \quad y^2 = x + C$

The curve passes through the point $(4,3)$.

$\therefore \quad 3^2 = 4 + C$

$\Rightarrow \quad C = 5$

$\therefore$ The required equation of the curve is $y^2 = x + 5$

Question 77

The decay rate of radium at any time t is proportional to its mass at that time. Find the time when the mass will be halved of its initial mass.

Solution 77

Let A_0 be the original amount of radium and A be the amount at any time t,

So,

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = -\lambda A, \ \lambda \text{ is a positive constant}$$

$$\frac{dA}{A} = -\lambda dt$$

$$\int \frac{dA}{A} = -\lambda \int dt$$

$$\log A = -\lambda t + c$$

At $t = 0$, $A = A_0$

$$\log A_0 = 0 + c$$

$$c = \log A_0$$

So,

$$\log A = -\lambda t + \log A_0$$

$$\log\left(\frac{A}{A_0}\right) = -\lambda t$$

To find time t when m 255 is half of original, so

$$A = \frac{1}{2} A_0$$

$$2A = A_0$$

So,

$$\log\left(\frac{A}{2A}\right) = -\lambda t$$

$$\log\left(\frac{1}{2}\right) = -\lambda t$$

$$\log 2 = \lambda t$$

$$y = \frac{1}{\lambda} \log 2$$

Question 78

Experiments show that radium disintegratest at a rate proportional to the amount of radium present at the moment. Its half-life is 1590 years. What percentage will disappear in one year?

Solution 78

Let A_0 be the initial amount of radium and A be the amount of that time,

So,

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = -\lambda A$$

$$\frac{dA}{A} = -\lambda dt$$

$$\int \frac{dA}{A} = -\lambda \int dt$$

$$\log A = -\lambda t + c$$

Initial amount of radim is A_0 when $t = 0$

$$\log A_0 = c$$

So,

$$\log A = -\lambda t + \log A_0$$

$$\log\left(\frac{A}{A_0}\right) = -\lambda t \qquad \text{---(i)}$$

Half life is 1590 years, so

So,

$$\log\left(\frac{A}{2A}\right) = -\lambda 1590$$

$$\log\left(\frac{1}{2}\right) = -\lambda 1590$$

$$\log 2 = \lambda 1590$$

$$\frac{\log 2}{1590} = \lambda$$

Put λ in equation (i),

$$\log\left(\frac{A}{A_0}\right) = -\left(\frac{\log 2}{1590}\right)t$$

$$\frac{A}{A_0} = e^{-\left(\frac{\log 2}{1590}\right)t}$$

$$A = A_0 e^{-\left(\frac{\log 2}{1590}\right)t}$$

Put $t = 1$

$$A = A_0(0.9996)$$

So, amount that disappears in 1 year

$$A_0 - A = A_0 - 0.9996A_0$$

$$A_0 - A = 0.0004A_0$$

$$\frac{A_0 - A}{A_0} \times 100 = \frac{0.0004A_0 \times 100}{A_0}$$

$$= 0.04\%$$

Amount disappear in one year = 0.04%.

Question 79

A wet porous substance in the open air loeses its moisture at a rate proportional to the moisture content. If a sheet hung in the wind loses half of its moisture during the first hour, when will it have lost 95% moisture, wheather conditions remaining the same.

Solution 79

Let moisture content at a given time be A and the initial amount of moisture by A_0

So,

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = -\lambda A$$

$$\frac{dA}{A} = -\lambda dt$$

$$\int \frac{dA}{A} = -\lambda \int dt$$

$$\log A = -\lambda t + c$$

When $t = 0$, $A = A_0$

$$\log A_0 = 0 + c$$

$$\log A_0 = c$$

So,

$$\log A = -\lambda t + \log A_0$$

$$\log\left(\frac{A}{A_0}\right) = -\lambda t$$

If $t = 1$, $A = \frac{1}{2} A_0$

$$\log\left(\frac{A}{2A}\right) = -\lambda$$

$$\log 2 = \lambda$$

So,

$$\log\left(\frac{A}{A_0}\right) = -\log 2t$$

Now, let is t, time it moisture lost 95% So,

$$A = \frac{5}{100} A_0$$

$$A = \frac{1}{2_0} A_0$$

So,

$$\log\left(\frac{A_0}{20A_0}\right) = -\log 2t_1$$

$$\log 20 = \log 2t_1$$

$$t_1 = \frac{\log 20}{\log 2}$$

So,

$$\text{Required time} = \frac{\log 20}{\log 2} \text{ hours.}$$

Chapter 22 - Differential Equations Exercise MCQ

Question 1

Mark the correct alternative in each of the following

The integrating factor of the differential equation

$(x \log x) \dfrac{dy}{dx} + y = 2\log x$, is given by

(a) $\log(\log x)$

(b) e^{x}

(c) $\log x$

(d) x

Solution 1

Correct option: (c)

Given differential equation

$$(x\log x)\frac{dy}{dx} + y = \frac{2\log x}{x\log x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x\log x}y = 2\log x$$

Comparing with $\dfrac{dy}{dx} + Py = Q$

$$\Rightarrow P = \frac{1}{x\log x},\ Q = \frac{2\log x}{x\log x}$$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$$

Question 2

The general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x}$ is

a. $\log y = kx$
b. $y = kx$
c. $xy = k$
d. $y = k \log x$

Solution 2

Correct option: (b)

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

Integrating on both sides,

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\log|y| = \log|x| + \log k$$

$$\Rightarrow \log\left(\frac{y}{x}\right) = \log k$$

$$\Rightarrow y = kx$$

Question 3

Integrating factor of the differential equation $\cos x \frac{dy}{dx} + y \sin x = 1$, is

a. Sin x
b. Sec x
c. Tan x
d. Cos x

Solution 3

Correct option: (b)

$$\cos x \frac{dy}{dx} + y \sin x = 1$$

$$\Rightarrow \frac{dy}{dx} + \frac{y \sin x}{\cos x} = \frac{1}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} + y \tan x = \sec x$$

Comparing with $\frac{dy}{dx} + Py = Q$

$P = \tan x,\ Q = \sec x$

Integrating factor $= e^{\int P dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

Question 4

The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right) = y^3$, is

a. ½
b. 2
c. 3
d. 4

Solution 4

Correct option:(b)

Degree is the power of highest order derivative.

Highest order is 2 and its power is 2.
Hence, degree of differential equation is 2.

Question 5

The degree of the differential equation

$$\left\{5+\left(\frac{dy}{dx}\right)^2\right\}^{5/3} = x^5\left(\frac{d^2y}{dx^2}\right), \text{is}$$

a. 4
b. 2
c. 5
d. 10

Solution 5

$$\left\{5+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{5}{3}} = x^5\left(\frac{d^2y}{dx^2}\right)$$

$$\Rightarrow \left\{5+\left(\frac{dy}{dx}\right)^2\right\}^{5} = \left\{x^5\left(\frac{d^2y}{dx^2}\right)\right\}^3$$

Degree is 3.

NOTE: Answer not matching with back answer.

Question 6

The general solution of the different equation $\frac{dy}{dx} + y \cot x$ $= \operatorname{cosec} x$, is

a. x + y sin x =C
b. x + y cos x = C
c. y+ x (sin x + cos x) = C
d. y sin x = x + C

Solution 6

Correct option:(d)

$$\frac{dy}{dx} + y \cot x = \operatorname{cosec} x$$

Comparing with $\frac{dy}{dx} + Py = Q$

$\Rightarrow P = \cot x,\ Q = \operatorname{cosec} x$

$\text{I.F.} = e^{\int P dx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$

Multiplying on both sides by $\sin x$

$$\sin x \frac{dy}{dx} + y \cos x = 1$$

$$\Rightarrow \frac{d}{dx}(y \sin x) = 1$$

$$\Rightarrow y \sin x = \int 1 dx$$

$$\Rightarrow y \sin x = x + C$$

Question 7

The different equation obtained on eliminating A and B from y = A cos ωt + B sin ωt ,is

a. $Y'' + y' = 0$
b. $Y'' - \omega^2 y = 0$
c. $Y'' = -\omega^2 y$
d. $Y'' + y = 0$

Solution 7

Correct option: (c)

$$y = A\cos\omega t + B\sin\omega t$$

$$\frac{dy}{dx} = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$\frac{d^2y}{dx^2} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$y'' = -\omega^2 y$$

Question 8

The equation of the curve whose slope is given by $\frac{dy}{dx} = \frac{2y}{x}$;

$x > 0, y > 0$ and which passes through the point $(1,1)$ is

a. $x^2 = y$
b. $y^2 = x$
c. $x^2 = 2y$
d. $y^2 = 2x$

Solution 8

Correct option: (a)

$$\frac{dy}{dx} = \frac{2y}{x}$$
$$\Rightarrow \frac{dy}{2y} = \frac{dx}{x}$$
$$\Rightarrow \int\frac{dy}{2y} = \int\frac{dx}{x}$$
$$\Rightarrow \log|y| = 2\log|x| + \log c$$
$$\Rightarrow \log|y| = \log x^2 + \log c$$
$$\Rightarrow \log y = \log(x^2 c)$$
$$\Rightarrow y = cx^2$$
Tangent passing through $(1, 1)$
$$\Rightarrow c = 1$$
$$\Rightarrow y = x^2$$

Question 9

The order of the different equation whose general solution is given by $y = c_1 \cos(2x + c_2) - (c_3 + c_4)\, a^{x+c_5} + c_6 \sin(x - c_7)$ is

a. 3
b. 4
c. 5
d. 2

Solution 9

Correct option: (c)
Here, constants are c_1, c_2, c_3, c_4, c_5, c_6.
But c_3+c_4 is also constant. Hence, total 5 constants.

Question 10

The solution of the different equation $\frac{dy}{dx} = \frac{ax + g}{by + f}$

representsa circle when

a. a = b
b. a = -b
c. a =-2b
d. a =2b

Solution 10

Correct option: (b)

$$\frac{dy}{dx} = \frac{ax+g}{by+f}$$

$$\Rightarrow (by+f)dy = (ax+g)dx$$

$$\Rightarrow \frac{by^2}{2} + fy = \frac{ax^2}{2} + gx + C$$

$$\Rightarrow by^2 + 2fy = ax^2 + 2gx + C$$

$$\Rightarrow by^2 + 2fy - ax^2 - 2gx - C = 0$$

$$\Rightarrow -ax^2 + by^2 - 2gx + 2fy - C = 0$$

This is general equation for the circle

if coefficients of x^2 and y^2 are equal.

$-a = b$

$a = -b$

Question 11

The solution of the diffetential equation $\frac{dy}{dx} + \frac{2y}{x} = 0$

with y (1) =1 is given by

(a) $y = \frac{1}{x^2}$

(b) $x = \frac{1}{y^2}$

(c) $x = \frac{1}{y}$

(d) $y = \frac{1}{x}$

Solution 11

Correct option: (a)

$$\frac{dy}{dx}+\frac{2y}{x}=0$$
$$\Rightarrow \frac{dy}{dx}=-\frac{2y}{x}$$
$$\Rightarrow \frac{dy}{2y}=-\frac{dx}{x}$$
$$\Rightarrow \int\frac{dy}{2y}=-\int\frac{dx}{x}$$
$$\Rightarrow \frac{1}{2}\log|y|=-\log|x|+\log c$$
$$\Rightarrow \sqrt{y}x=c$$
$$\Rightarrow yx^2=c$$
Given that $y(1)=1 \Rightarrow x=y=1$
$$\Rightarrow c=1$$
$$\Rightarrow yx^2=1$$
$$\Rightarrow y=\frac{1}{x^2}$$

Question 12

The solution of the different equation $\frac{dy}{dx}-\frac{y(x+1)}{x}=0$ is given by

(a) $y = xe^{x+C}$
(b) $x = ye^{x}$
(c) $y = x+C$
(d) $xy = e^{x}+C$

Solution 12

Correct option: (a)

$$\frac{dy}{dx} - \frac{y(x+1)}{x} = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{y(x+1)}{x}$$
$$\Rightarrow \frac{dy}{y} = \frac{x+1}{x} dx$$
$$\Rightarrow \int \frac{dy}{y} = \int \frac{x+1}{x} dx$$
$$\Rightarrow \int \frac{dy}{y} = \int \left(1 + \frac{1}{x}\right) dx$$
$$\Rightarrow \log y = x + \log x + c$$
$$\Rightarrow \log\left(\frac{y}{x}\right) = x + c$$
$$\Rightarrow \frac{y}{x} = e^{x+c}$$
$$\Rightarrow y = xe^{x+c}$$

Question 13

The order of the differential equation satisfying

$\sqrt{1-x^4} + \sqrt{1-y^4} = a(x^2 - y^2)$ is

a. 1
b. 2
c. 3
d. 4

Solution 13

Correct option: (a)

Differential equation contains only one constant hence,

Order of differential equation is 1.

Question 14

The solution of the differential equation $y_1\ y_3 = y_2^2$ is

a. $x = C_1 e^{C_2 y} + C_3$
b. $y = C_1\ e^{C_2 x} + C_3$
c. $2x = C_1\ e^{C_2 y} + C_3$
d. None of these

Solution 14

Correct option: (b)

$$y_1 y_3 = y_2^2$$

$$\Rightarrow \frac{y_1}{y_2} = \frac{y_2}{y_3}$$

$$\Rightarrow \frac{\frac{dy}{dx}}{\frac{d^2y}{dx^2}} = \frac{\frac{d^2y}{dx^2}}{\frac{d^3y}{dx^3}}$$

$$\Rightarrow \frac{\frac{d^2y}{dx^2}}{\frac{dy}{dx}} = \frac{\frac{d^3y}{dx^3}}{\frac{d^2y}{dx^2}}$$

$$\Rightarrow \int \frac{\frac{d^2y}{dx^2}}{\frac{dy}{dx}} = \int \frac{\frac{d^3y}{dx^3}}{\frac{d^2y}{dx^2}}$$

$$\Rightarrow \log\frac{dy}{dx} = \log\frac{d^2y}{dx^2} + \log C$$

$$\Rightarrow C\frac{dy}{dx} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \int C dx = \int \frac{\frac{d}{dx}\left(\frac{dy}{dx}\right)}{\frac{dy}{dx}}$$

$$\Rightarrow Cx + C_1 = \log\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{Cx+C_1}$$

$$\Rightarrow \int dy = \int e^{Cx+C_1} dx$$

$$\Rightarrow y = C_4 e^{C_5 x} + C_3$$

$$\Rightarrow y = C_1 e^{C_2 x} + C_3$$

Question 15

The general solution of the differential equation $\frac{dy}{dx} + y\ g'\ (x)$ $= g\ '(x)$, where $g\ (x)$ is a given function of x, is

(a) $g\ (x) + \log\ \{1 + y + g(x)\} = C$

(b) $g\ (x) + \log\ \{1 + y - g(x)\} = C$

(c) $g\ (x) - \log\ \{1 + y - g(x)\} = C$

(d) none of these

Solution 15

Correct option: (b)

$$\frac{dy}{dx} + y\ g'(x) = g'(x)$$

Comparing with $\frac{dy}{dx} + y\ P = Q$

$P = Q = g'(x)$

$$\text{I. F.} = e^{\int g'(x)dx} = e^{g(x)}$$

Multiplying by $e^{g(x)}$ on both sides,

$$e^{g(x)}\left(\frac{dy}{dx} + y\ g'(x)\right) = e^{g(x)}g(x)g'(x)$$

$$\frac{d}{dx}ye^{g(x)} = e^{g(x)}g(x)g'(x)$$

$$\int dye^{g(x)} = \int e^{g(x)}g(x)g'(x)dx + c$$

Put $g(x) = t \Rightarrow g'(x)dx = dt$

$$ye^{g(x)} = \int e^{t}tdt + c$$

$$ye^{g(x)} = te^{t} - e^{t} + c$$

$$ye^{g(x)} = g(x)e^{g(x)} - e^{g(x)} + c$$

$$\left[y - 1 - g(x)\right]e^{g(x)} = c$$

$$y - 1 - g(x) = ce^{-g(x)}$$

Taking log on both sides,

$$\log\left[y - 1 - g(x)\right] = K - g(x) \quad (\because \log c = k)$$

$$g(x) + \log\left[y - 1 - g(x)\right] = K$$

Question 16

The solution of the differential equation $\frac{dy}{dx} = 1 + x + y^2 + xy^2, y(0) = 0$ is

(a) $y^2 = \exp\left(x + \frac{x^2}{2}\right) - 1$

(b) $y^2 = 1 + C\ \exp\left(x + \frac{x^2}{2}\right)$

(c) $y = \tan\ (C+x+x^2)$

(d) $y = \tan\left(x + \frac{x^2}{2}\right)$

Solution 16

Correct option: (d)

$$\frac{dy}{dx} = 1 + x + y^2 + xy^2$$

$$\Rightarrow \frac{dy}{dx} = (1 + x)(1 + y^2)$$

$$\Rightarrow \frac{dy}{1 + y^2} = (1 + x)dx$$

$$\Rightarrow \int \frac{dy}{1 + y^2} = \int (1 + x)dx$$

$$\Rightarrow \tan^{-1} y = x + \frac{x^2}{2} + c$$

Given $y(0) = 0 \Rightarrow x = 0,\ y = 0$

$$\Rightarrow c = 0$$

$$\Rightarrow \tan^{-1} y = x + \frac{x^2}{2}$$

$$\Rightarrow y = \tan\left(x + \frac{x^2}{2}\right)$$

Question 17

The differential equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = C$ is

(a) $\frac{y''}{y'} + \frac{y'}{y} - \frac{1}{x} = 0$

(b) $\frac{y''}{y'} + \frac{y'}{y} + \frac{1}{x} = 0$

(c) $\frac{y''}{y'} - \frac{y'}{y} - \frac{1}{x} = 0$

(d) none of these

Solution 17

Correct option: (a)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=C$$

$$\Rightarrow b^2x^2+a^2y^2=a^2b^2C$$

$$\Rightarrow b^2 2x+a^2 2y\frac{dy}{dx}=0$$

$$\Rightarrow 2b^2+2a^2y\frac{d^2y}{dx^2}+2a^2\frac{dy}{dx}\frac{dy}{dx}=0$$

$$\Rightarrow 2a^2\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)=-2b^2$$

$$\Rightarrow \left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)=\frac{-b^2}{a^2}$$

$$\Rightarrow \left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)=\frac{y}{x}\frac{dy}{dx}$$

$$\Rightarrow yy''+(y')^2=\frac{y}{x}y'$$

$$\Rightarrow xyy''+x(y')^2-yy'=0$$

$$\Rightarrow xyy'\left(\frac{y''}{y'}+\frac{y'}{y}-\frac{1}{x}\right)=0$$

$$\Rightarrow \frac{y''}{y'}+\frac{y'}{y}-\frac{1}{x}=0$$

Question 18

Solution of the differential equation $\frac{dy}{dx}+\frac{y}{x}=\sin x$ is

a. x(y + cos x) = sin x +C
b. x(y - cos x) = sin x +C
c. x(y + cos x) = cos x +C
d. None of these

Solution 18

Correct option: (a)

$$\frac{dy}{dx} + \frac{y}{x} = \sin x$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x} y = \sin x$$

Comparing with $\frac{dy}{dx} + Py = Q$

$$\Rightarrow P = \frac{1}{x},\ Q = \sin x$$

$$\text{I. F.} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Multiplying by x on both sides,

$$x\left(\frac{dy}{dx} + \frac{y}{x}\right) = x \sin x$$

$$x\frac{dy}{dx} + y = x \sin x$$

$$\frac{d(xy)}{dx} = x \sin x$$

$$\int d(xy) = \int x \sin x\, dx$$

$$xy = x\int \sin x\, dx - \int\left(\frac{dx}{dx}\int \sin x\, dx\right)dx$$

$$xy = -x\cos x - \int -\cos x\, dx$$

$$xy = -x\cos x + \sin x + C$$

$$x(y + \cos x) = \sin x + C$$

Question 19

The equation of the curve satisfying the differential equation $y(x+y^3)\,dx = x(y^3-x)\,dy$ and passing through the point (1,1) is

a. $y^3-2x+3x^2y=0$
b. $y^3+2x+3x^2y=0$
c. $y^3+2x-3x^2y=0$
d. None of these

Solution 19

Correct option: (c)

$$y\left(x+y^3\right)dx = x\left(y^3-x\right)dy$$
$$xydx + y^4dx = xy^3dy - x^2dy$$
$$xydx - xy^3dy + y^4dx + x^2dy = 0$$
$$xydx + x^2dy + y^4dx - xy^3dy = 0$$
$$x\left(ydx + x^2\right) + y^3\left(ydx - xdy\right) = 0$$
$$x\left(ydx + x^2\right) + x^2y^3\frac{\left(ydx - xdy\right)}{x^2} = 0$$
$$x\left(ydx + x^2\right) - x^2y^3\frac{\left(xdy - ydx\right)}{x^2} = 0$$
$$x\left(ydx + x^2\right) - x^2y^3d\left(\frac{y}{x}\right) = 0$$
$$x\left(ydx + x^2\right) = x^2y^3d\left(\frac{y}{x}\right)$$
$$\frac{x\left(ydx + x^2\right)}{x^3y^2} = \frac{y}{x}d\left(\frac{y}{x}\right)$$
$$\int\frac{x\left(ydx + x^2\right)}{x^3y^2} = \int\frac{y}{x}d\left(\frac{y}{x}\right)$$
$$\int\frac{d(xy)}{x^2y^2} = \int\frac{y}{x}d\left(\frac{y}{x}\right)$$
$$\frac{-1}{xy} = \frac{\left(\frac{y}{x}\right)^2}{2} + c$$
$$\frac{1}{xy} + \frac{\left(\frac{y}{x}\right)^2}{2} + c = 0$$
$$\Rightarrow y^3 + 2x + 2cx^2y = 0$$
Curve passes through $(1, 1)$
$$1 + 2 + 2c = 0$$
$$1 + 2 + 2c = 0$$
$$c = \frac{-3}{2}$$
$$\Rightarrow y^3 + 2x - 3x^2y = 0$$

Question 20

The solution of the differential equation $2x\frac{dy}{dx} - y = 3$ represents

a. Circles
b. Straight lines
c. Ellipses

d. Parabolas

Solution 20

Correct option: (d)

$$2x\frac{dy}{dx}-y=3$$

$$2x\frac{dy}{dx}=y+3$$

$$\frac{dy}{y+3}=\frac{dx}{2x}$$

$$\int\frac{dy}{y+3}=\int\frac{dx}{2x}$$

$$2\log|y+3|=\log|x|+\log c$$

$$(y+3)^2=xc$$

This is general equation of parabolas.

Question 21

The solution of the differential equation $x\frac{dy}{dx}=y+x\tan\frac{y}{x}$, is

(a) $\sin\frac{x}{y}=x+C$

(b) $\sin\frac{y}{x}=Cx$

(c) $\sin\frac{x}{y}=Cy$

(d) $\sin\frac{y}{x}=Cy$

Solution 21

Correct option: (b)

$$x\frac{dy}{dx} = y + x\tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \ldots\ldots\ldots(i)$$

Put $\frac{y}{x} = v$

$y = vx$

Differentiating on both sides,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + \tan v = v + x\frac{dv}{dx} \quad \text{(from (i))}$$

$$\tan v = x\frac{dv}{dx}$$

$$\frac{dx}{x} = \cot v\,dv$$

$$\int\frac{dx}{x} = \int\cot v\,dv$$

$$\log|x| + \log C = \log|\sin v|$$

$$Cx = \sin v$$

$$\sin\left(\frac{y}{x}\right) = Cx$$

Question 22

The different equation satisfied by $ax^2+by^2=1$ is

a. $xyy_2+y_1^2+yy_1=0$

b. $xyy_2+xy_1^2-yy_1=0$

c. $xyy_2-xy_1^2+yy_1=0$

d. none of these

Solution 22

Correct option: (b)

$$ax^2 + by^2 = 1$$
$$\Rightarrow 2ax + 2by\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{-ax}{by}$$
Consider,
$$ax + by\frac{dy}{dx} = 0$$
$$a + by\frac{d^2y}{dx^2} + b\left(\frac{dy}{dx}\right)^2 = 0$$
$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\frac{a}{b}$$
$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{y}{x}\frac{dy}{dx}$$
$$\Rightarrow yy_2 + \left(y_1\right)^2 = \frac{y}{x}y_1$$
$$\Rightarrow xyy_2 + x\left(y_1\right)^2 = yy_1$$
$$\Rightarrow xyy_2 + x\left(y_1\right)^2 - yy_1 = 0$$

Question 23

The different equation which represents the family of curves $y = e^{Cx}$ is

a. $y_1 = C^2y$
b. $xy_1 - \ln y = 0$
c. $x \ln y = yy_1$
d. $y \ln y = xy_1$

Solution 23

Correct option: (d)

$$y = e^{Cx}$$
$$\Rightarrow \log y = Cx \Rightarrow C = \frac{\log y}{x}\ldots\ldots(i)$$
$$\Rightarrow \frac{1}{y}y_1 = \frac{\log y}{x} \quad (\because \text{From (i)})$$
$$\Rightarrow y_1x = y\log y$$

Note: log is considered same as ln.

Question 24

Which of the following transformation reduce the differential equation $\frac{dz}{dx} + \frac{z}{x}\log\ z = \frac{z}{x^2}(\log z)^2$ into the form

$\frac{du}{dx} + P(x)\,u = Q(x)$

a. u = log x
b. $u = e^z$
c. $u = (\log z)^{-1}$
d. $u = (\log z)^{2}$

Solution 24

Correct option: (c)

$$\frac{dz}{dx} + \frac{z}{x}\log\ z = \frac{z}{x^2}(\log z)^2$$

$$\frac{dz}{dx} = \frac{z}{x^2}(\log z)^2 - \frac{z}{x}\log\ z \ldots\ldots(i)$$

Put $v = (\log z)^{-1}$

$$\frac{dv}{dx} = \frac{-1}{(\log z)^2}\frac{1}{z}\frac{dz}{dx}$$

$$\frac{dz}{dx} = -z(\log z)^2\frac{dv}{dx}\ldots\ldots\ldots(ii)$$

From (i) and (ii)

$$-z(\log z)^2\frac{dv}{dx} = \frac{z}{x^2}(\log z)^2 - \frac{z}{x}\log\ z$$

$$(\log z)\frac{dv}{dx} = -\frac{1}{x^2}(\log z) + \frac{1}{x}$$

$$\frac{dv}{dx} = -\frac{1}{x^2} + \frac{u}{x}$$

$$\frac{dv}{dx} - \frac{u}{x} = -\frac{1}{x^2}$$

$$P(x) = \frac{-1}{x},\ q(x) = -\frac{1}{x^2}$$

Given differential equation can be reduced using $v = (\log z)^{-1}$

Question 25

The solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$ is

(a) $\phi\left(\frac{y}{x}\right) = kx$

(b) $x\phi\left(\frac{y}{x}\right) = k$

(c) $\phi\left(\frac{y}{x}\right) = ky$

(d) $Y\,\phi\left(\frac{y}{x}\right) = k$

Solution 25

Correct option:(a)

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)} \ldots\ldots\ldots\ldots(i)$$

Put $v = \frac{y}{x}$

$\Rightarrow x\frac{dv}{dx} + v = \frac{dy}{dx}$

$\Rightarrow x\frac{dv}{dx} + v = v + \frac{\phi(v)}{\phi'(v)}$ From (i)

$\Rightarrow x\frac{dv}{dx} = \frac{\phi(v)}{\phi'(v)}$

$\Rightarrow \frac{\phi'(v)}{\phi(v)}dv = \frac{dx}{x}$

$\Rightarrow \int\frac{\phi'(v)}{\phi(v)}dv = \int\frac{dx}{x}$

$\Rightarrow \log\phi(v) = \log|x| + \log k$

$\Rightarrow \log\phi\left(\frac{y}{x}\right) - \log|x| = \log k$

$\Rightarrow \log\left[\frac{\phi\left(\frac{y}{x}\right)}{x}\right] = \log k$

$\Rightarrow \phi\left(\frac{y}{x}\right) = kx$

Question 26

If m and n are the order and degree of the differential equation

$$(y_2)^5 + \frac{4(y_2)^3}{y_3} + y_3 = x^2 - 1, \text{then}$$

a. m = 3, n = 3
b. m = 3, n =2
c. m = 3, n =5
d. m =3, n =1

Solution 26

Correct option: (b)

$$(y_2)^5 + \frac{4(y_2)^3}{y_3} + y_3 = x^2 - 1$$

$$\Rightarrow y_3(y_2)^5 + 4(y_2)^3 + y_3^2 = \left(x^2 - 1\right)y_3$$

Here, highest order derivative is 3 = m,

power of 3[rd] order derivative is 2 = n.

Question 27

The solution of the differential equation $\frac{dy}{dx} + 1 = e^{x+y}$, is

(a) $(x+y)\, e^{x+y} = 0$

(b) $(x+C)\, e^{x+y} = 0$

(c) $(x-C)\, e^{x+y} = 1$

(d) $(x-C)\, e^{x+y} + 1 = 0$

Solution 27

Correct option: (d)

$$\frac{dy}{dx} + 1 = e^{x+y}$$

$$\Rightarrow \frac{dy}{dx} = e^{x+y} - 1 \text{..........(i)}$$

Let, $x + y = v$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow e^{v} = \frac{dv}{dx} \qquad \text{..............from (i)}$$

$$\Rightarrow dx = e^{-v}dv$$

$$\Rightarrow \int dx = \int e^{-v}dv$$

$$\Rightarrow x = -e^{-v} + c$$

$$\Rightarrow x - c = -e^{-(x+y)}$$

$$\Rightarrow (x - c)e^{x+y} + 1 = 0$$

Question 28

The solution of $x^2 + y^2 \frac{dy}{dx} = 4$, is

(a) $x^2 + y^2 = 12x + C$

(b) $x^2 + y^2 = 3x + C$

(c) $x^3 + y^3 = 3x + C$

(d) $x^3 + y^3 = 12x + C$

Solution 28

Correct option: (d)

$$x^2 + y^2 \frac{dy}{dx} = 4$$

$$y^2 \frac{dy}{dx} = 4 - x^2$$

$$y^2 dy = \left(4 - x^2\right) dx$$

$$\Rightarrow \int y^2 dy = \int \left(4 - x^2\right) dx$$

$$\Rightarrow \frac{y^3}{3} = 4x - \frac{x^3}{3} + c$$

$$\Rightarrow y^3 = 12x - x^3 + c$$

$$\Rightarrow x^3 + y^3 = 12x + c$$

Question 29

The family of curves in which the subtangent at any point of a curve is double the abscisae, is given by

a. $x = Cy^2$
b. $y = Cx^2$
c. $x^2 = Cy^2$
d. $y = Cx$

Solution 29

Correct option: (a)

Sub tangent is $\frac{y}{\frac{dy}{dx}}$.

Given that sub tangent = 2x

$\Rightarrow \frac{y}{\frac{dy}{dx}} = 2x$

$\Rightarrow y = 2x\frac{dy}{dx}$

$\Rightarrow \frac{dx}{x} = \frac{2dy}{y}$

$\Rightarrow \int\frac{dx}{x} = \int\frac{2dy}{y}$

$\Rightarrow \log|x| = 2\log|y| + \log C$

$\Rightarrow x = Cy^2$

Question 30

The solution of the differential equation x dx +y dy =x^2y dy -y^2 x dx , is

a. $x^2-1 = C\ (1+y^2)$
b. $x^2+1=C\ (1-y^2)$
c. $x^3-1=C\ (1+y^3)$
d. $x^3+1=C\ (1-y^3)$

Solution 30

Correct option:(a)

$xdx + ydy = x^2ydy - y^2xdx$

$xdx + y^2xdx = x^2ydy - ydy$

$x\left(1+y^2\right)dx = y\left(x^2-1\right)dy$

$\frac{xdx}{x^2-1} = \frac{ydy}{1+y^2}$

$\int\frac{xdx}{x^2-1} = \int\frac{ydy}{1+y^2}$

$\frac{1}{2}\int\frac{2xdx}{x^2-1} = \frac{1}{2}\int\frac{2ydy}{1+y^2}$

$\frac{1}{2}\log\left(x^2-1\right) = \frac{1}{2}\log\left(1+y^2\right) + \log c$

$\log\left(x^2-1\right) = \log\left(1+y^2\right) + \log c$

$x^2 - 1 = \left(1+y^2\right)c$

Question 31

The solution of the differential equation $\left(x^2+1\right)\frac{dy}{dx}+(y^2+1)=0$, is

(a) $y=2+x^2$

(b) $y\frac{1+x}{1-x}$

(c) $y=x(x-1)$

(d) $y=\frac{1-x}{1+x}$

Solution 31

Correct option:

$$\left(x^2+1\right)\frac{dy}{dx}+y^2+1=0$$

$$\left(x^2+1\right)\frac{dy}{dx}=-\left(y^2+1\right)$$

$$\frac{dy}{y^2+1}=-\frac{dx}{x^2+1}$$

$$\int\frac{dy}{y^2+1}=-\int\frac{dx}{x^2+1}$$

$$\tan^{-1}y=-\tan^{-1}x+\tan^{-1}c$$

$$\tan^{-1}y+\tan^{-1}x=\tan^{-1}c$$

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1}c$$

$$\frac{x+y}{1-xy}=c$$

Answer not matching with back answer.

Question 32

The different equation $x\frac{dy}{dx}-y=x^2$, has the general solution

(a) $y-x^3=2cx$

(b) $2y-x^3=cx$

(c) $2y+x^2=2cx$

(d) $y+x^2=2cx$

Solution 32

Correct option: (b)

$$x\frac{dy}{dx} - y = x^2$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = x$$

Comparing with $\frac{dy}{dx} - Py = Q$

$$\Rightarrow P = \frac{-1}{x},\ Q = x$$

$$\text{I. F.} = e^{\int P dx} = e^{\int \frac{-1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

Multiplying $\frac{1}{x}$ on both sides,

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 1$$

$$\frac{d}{dx}\frac{y}{x} = 1$$

$$\int d\frac{y}{x} = \int x dx$$

$$\frac{y}{x} = \frac{x^2}{2} + c$$

$$2y = x^3 + cx$$

$$2y - x^3 = cx$$

Question 33

The solution of the differential equation $\frac{dy}{dx} - ky = 0, y(0) = 1$ approches to zero when $x \to \infty$, if

a. k =0
b. k > 0
c. k < 0
d. none of these

Solution 33

Correct option:(c)

$$\frac{dy}{dx} - ky = 0$$
$$\frac{dy}{dx} = ky$$
$$\frac{dy}{y} = kdx$$
$$\int \frac{dy}{y} = k\int dx$$
$$\log|y| = kx + c$$
$$y(0) = 1 \Rightarrow x = 0,\ y = 1$$
$$\Rightarrow c = 0$$
$$\Rightarrow \log|y| = kx$$
$$\Rightarrow e^{kx} = y$$
Given that $e^{k\infty} = 0$

as $e^{-\infty} = 0$

$\Rightarrow k < 0$

Question 34

The solution of the differential equation $(1+x^2)\frac{dy}{dx} + 1 + y^2 = 0$, is

a. $\tan^{-1} x - \tan^{-1} y = \tan^{-1} C$
b. $\tan^{-1} y - \tan^{-1} x = \tan^{-1} C$
c. $\tan^{-1} y \pm \tan^{-1} x = \tan C$
d. $\tan^{-1} y + \tan^{-1} x = \tan^{-1} C$

Solution 34

Correct option: (d)

$$\left(1+x^2\right)\frac{dy}{dx} + 1 + y^2 = 0$$
$$\left(1+x^2\right)\frac{dy}{dx} = -\left(1 + y^2\right)$$
$$\frac{dy}{1+y^2} = -\frac{dx}{1+x^2}$$
$$\int\frac{dy}{1+y^2} = -\int\frac{dx}{1+x^2}$$
$$\tan^{-1} y = -\tan^{-1} x + \tan^{-1} c$$
$$\tan^{-1} y + \tan^{-1} x = \tan^{-1} c$$

Question 35

The solution of the differential equation $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$, is

(a) $\tan^{-1}\left(\frac{x}{y}\right) = \log y + C$

(b) $\tan^{-1}\left(\frac{y}{x}\right) = \log x + C$

(c) $\tan^{-1}\left(\frac{x}{y}\right) = \log x + C$

(d) $\tan^{-1}\left(\frac{y}{x}\right) = \log y + C$

Solution 35

Correct option: (b)

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$$

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x^2} \ldots\ldots\ldots\ldots(i)$$

Let, $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$1 + v + v^2 = v + x\frac{dv}{dx} \ldots\ldots\ldots\text{from (i)}$$

$$1 + v^2 = x\frac{dv}{dx}$$

$$\frac{dx}{x} = \frac{dv}{1 + v^2}$$

$$\int\frac{dx}{x} = \int\frac{dv}{1 + v^2}$$

$$\log|x| = \tan^{-1} v + c$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \log x + c$$

Question 36

The differential equation $\frac{dy}{dx} + Py = Qy^n, n > 2$ can be reduced to linear form by substituting

(a) $z = y^{n-1}$

(b) $z = y^n$

(c) $z = y^{n+1}$

(d) $z = y^{1-n}$

Solution 36

Correct option:(d)

$$\frac{dy}{dx} + Py = Qy^n$$

$$\Rightarrow y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

$$\Rightarrow y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Let, $y^{1-n} = v$

$$\Rightarrow \frac{dv}{dx} = (1-n)v\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{1-n}\frac{dv}{dx} + Pv = Q$$

$$\Rightarrow \frac{dv}{dx} + Pv(1-n) = Q(1-n)$$

which is linear form.

$z = y^{1-n}$ can be reduced to linear form.

Question 37

If p and q are the order and degree of the differential equation

$$y\frac{dy}{dx} + x^3\frac{d^2y}{dx^2} + xy = \cos x, \text{ then}$$

a. $p < q$
b. $p = q$
c. $p > q$
d. none of these

Solution 37

Correct option: (c)

$$y\frac{dy}{dx} + x^3\frac{d^2y}{dx^2} + xy = \cos x$$

Highest derivative is $2 = p$ and degree is $1 = q$.

$\Rightarrow p > q$

Note: Answer not matching with back answer.

Question 38

which of the following is the integrating factor of

$$(x\log x)\frac{dy}{dx} + y = 2\log x?$$

a. x
b. e^x
c. log x

d. log (log x)

Solution 38

Correct option: (c)

$$(x \log x)\frac{dy}{dx} + y = 2\log x$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x}$$

Comparing with $\frac{dy}{dx} + Py = Q$

$$P = \frac{1}{x\log x},\ Q = \frac{2}{x}$$

$$\text{I. F.} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$$

Question 39

What is integrating factor of $\frac{dy}{dx} + y \sec x = \tan x$?

a. sec x + tan x
b. log (sec x+ tan x)
c. $e^{\sec x}$
d. sec x

Solution 39

Correct option: (a)

$$\frac{dy}{dx} + y\sec x = \tan x$$

Comparing with $\frac{dy}{dx} + Py = Q$

P=secx, Q=tanx

$$\text{I. F.} = e^{\int \sec x dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

Question 40

Itegrating factor of the differential equation $\cos x \frac{dy}{dx} + y\sin x = 1$, is

(a) cos x
(b) tan x
(c) sec x
(d) sin x

Solution 40

Correct option: (c)

$$\cos x\frac{dy}{dx}+y\sin x=1$$
$$\Rightarrow \frac{dy}{dx}+y\tan x=\sec x$$
Comparing with $\frac{dy}{dx}+yP=Q$

$\Rightarrow P=\tan x,\ Q=\sec x$

$I.F.=e^{\int \tan x dx}=e^{\log \sec x}=\sec x$

Question 41

The degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0, \text{is}$$

(a) 3
(b) 2
(c) 1
(d) Not defined

Solution 41

Correct option: (d)
Highest order derivative is 2 but equation cannot be expressed as a polynomial in differential equation.
Hence, it is not defined.

Question 42

The order of the differential equation $2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$ is

(a) 2
(b) 1
(c) 0
(d) Not defined

Solution 42

Correct option:(a)
Highest order of the derivative is 2.

Question 43

The number of arbitrary constants in the general solution of differential equation of fourth order is
(a) 0
(b) 2
(c) 3
(d) 4

Solution 43

Correct option: (d)

In the general solution of differential equation of order n has n number of arbitrary constants.

Question 44

The number of arbitrary constants in the particular solution of a differential equation of third order is

(a) 3

(b) 2

(c) 1

(d) 0

Solution 44

Correct option: (d)

The number of arbitrary constants in the particular solution of a differential equation of third order is always zer0.

Question 45

Which of the following differential equations has

$y = C_1e^x + C_2e^{-x}$ as the general solution?

(a) $\frac{d^2y}{dx^2} + y = 0$

(b) $\frac{d^2y}{dx^2} - y = 0$

(c) $\frac{d^2y}{dx^2} + 1 = 0$

(d) $\frac{d^2y}{dx^2} - 1 = 0$

Solution 45

Correct option: (b)

$$y = C_1e^x + C_2e^{-x}$$

$$\frac{dy}{dx} = C_1e^x - C_2e^{-x}$$

$$\frac{d^2y}{dx^2} = C_1e^x + C_2e^{-x}$$

$$\frac{d^2y}{dx^2} = y$$

$$\frac{d^2y}{dx^2} - y = 0$$

Question 46

Which of the following differential equation has y = x as one of its particular solution?

(a) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$

(b) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = x$

(c) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$

(d) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$

Solution 46

Correct option: (c)

$y = x$

$\frac{dy}{dx} = 1$

$\frac{d^2y}{dx^2} = 0$

$x^2 + \frac{d^2y}{dx^2} = x^2$

$xy + \frac{d^2y}{dx^2} = 1 \times x^2$

$xy + \frac{d^2y}{dx^2} = \frac{dy}{dx} \times x^2$

$xy + \frac{d^2y}{dx^2} - \frac{dy}{dx} \times x^2 = 0$

Question 47

The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$, is

$[0, 1]$

(a) $e^x + e^{-y} = C$

(b) $e^x + e^y = C$

(c) $e^{-x} + e^y = C$

(d) e-^{x} +e^{-y} =C

Solution 47

Correct option: (a)

$$\frac{dy}{dx} = e^{x+y}$$
$$e^{-y}dy = e^{x}dx$$
$$\int e^{-y}dy = \int e^{x}dx$$
$$-e^{-y} = e^{x} + c$$
$$e^{x} + e^{-y} = c$$

Question 48

A homogeneous differential equation of the form $\frac{dy}{dx} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution

(a) y = vx
(b) v = yx
(c) x = vy
(d) x = v

Solution 48

Correct option: (c)

A homogeneous differential equation of the form $\frac{dy}{dx} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution

$$\frac{x}{y} = v$$
$$\Rightarrow x = vy$$

Question 49

Which of the following is a homogeneous differential equation?
(a) $(4x+6y+5)\,dy-(3y+2x+4)\,dx=0$
(b) $xy\,dx-(x^3+y^3)dy=0$
(c) $(x^3+2y^2)dx+2xy\,dy=0$
(d) $y^2\,dx+(x^2-xy-y^2)\,dy=0$

Solution 49

Correct option: (d)

A differential equation is homogenous if all the terms in the equation have equal degree and it can be written in the form $\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}$

Question 50

The integrating factor of the differential equation $x\frac{dy}{dx} - y = 2x^2$

a. e^{-x}
b. e^{-y}
c. 1/x
d. x

Solution 50

Correct option: (c)

$$x\frac{dy}{dx} - y = 2x^2$$

$$\frac{dy}{dx} - \frac{1}{x}y = 2x$$

Comparing with $\frac{dy}{dx} + Py = Q$

$$P = -\frac{1}{x},\ Q = 2x$$

$$\text{I. F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

Question 51

The integrating factor of the differential equation

$(1-y^2)\frac{dx}{dy} + yx = ay(-1 < y < 1)$ is

(a) $\frac{1}{y^2 - 1}$

(b) $\frac{1}{\sqrt{y^2 - 1}}$

(c) $\frac{1}{1 - y^2}$

(d) $\frac{1}{\sqrt{1 - y^2}}$

Solution 51

Correct option:(d)

$$(1-y^2)\frac{dx}{dy} + yx = \frac{ay}{1-y^2}$$

$$\frac{dx}{dy} + \frac{y}{1-y^2}x = Q$$

Comparing with $\frac{dx}{dy} + Px = Q$

$$P = \frac{y}{1-y^2}, \; Q = ay$$

$$\text{I. F.} = e^{\int \frac{y}{1-y^2}dy}$$

$$= e^{\frac{-1}{2}\int \frac{-2y}{1-y^2}dy}$$

$$= e^{\frac{-1}{2}\log\left|1-y^2\right|}$$

$$= e^{\log\left|\frac{1}{\sqrt{1-y^2}}\right|}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

Question 52

The general solution of the differential equation $\frac{y\,dx - x\,dy}{y} = 0$, is

(a) $xy = C$
(b) $x = Cy^2$
(c) $y = Cx$
(d) $y = Cx^2$

Solution 52

Correct option:(c)

$$\frac{y\,dx - x\,dy}{y} = 0$$

$$\frac{y - x\dfrac{dy}{dx}}{\dfrac{y}{dx}} = 0$$

$$y - x\frac{dy}{dx} = 0$$

$$y = x\frac{dy}{dx}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\int\frac{dx}{x} = \int\frac{dy}{y}$$

$$\log x + \log C = \log y$$

$$C = \frac{y}{x}$$

$$y = Cx$$

Question 53

The general solution of a differential equation of the type

$\frac{dy}{dx} + P_1\ x = Q_1$ is

(a) $y\ e^{\int P_1 dy} = \int\left\{Q_1 e^{\int P_1\ dy}\right\} dy + C$

(b) $y\ e^{\int P_1\ dx} = \int\left\{Q_1 e^{\int P_1\ dx}\right\} dx + C$

(c) $x\ e^{\int P_1\ dy} = \int\left\{Q_1 e^{\int P_1\ dy}\right\} dy + C$

(d) $x\ e^{\int P_1\ dx} = \int\left\{Q_1 e^{\int P_1\ dx}\right\} dx + C$

Solution 53

Correct option: (c)

$$\frac{dy}{dx} + P_1\ x = Q_1$$

$$\text{I. F.} = e^{\int P dx}$$

$$e^{\int P dx}\left(\frac{dy}{dx} + P_1\ x\right) = e^{\int P dx} Q_1$$

$$\Rightarrow x\ e^{\int P_1\ dy} = \int\left\{Q_1 e^{\int P_1\ dy}\right\} dy + C$$

Question 54

The general solution of the differential equation $e^x dy + (ye^x + 2x)\,dx = 0$ is

(a) x $e^y+x^2=C$
(b) x $e^y+y^2=C$
(c) y $e^x+x^2=C$
(d) y $e^y+x^2=C$

Solution 54

Correct option: (c)

$$e^x dy + \left(ye^x + 2x\right)dx = 0$$

$$e^x dy = -\left(ye^x + 2x\right)dx$$

$$\frac{dy}{dx} = \frac{-\left(ye^x + 2x\right)}{e^x}$$

$$\frac{dy}{dx} = -y - 2xe^{-x}$$

$$\frac{dy}{dx} + y = -2xe^{-x}$$

$$\text{I. F.} = e^{\int dx} = e^x$$

$$e^x\left(\frac{dy}{dx} + y\right) = -2xe^{-x}e^x$$

$$\frac{d}{dx}e^x y = -2x$$

$$\int de^x y = -2\int x dx$$

$$e^x y = -2\frac{x^2}{2} + C$$

$$ye^x + x^2 = C$$

Chapter 22 - Differential Equations Exercise Ex. 22VSAQ

Question 1

Define a differential equation.

Solution 1

Differential equation :

An equation containing an independent variable, dependent variable and differential coefficients of dependent variable with respect to independent variable is called a differential equation.

Question 2

Define order of a differential equation.

Solution 2

Order of a differential equation :

The order of a differential equation is the order of the highest order derivation appearing is the equation the order of a differential equation is a positive integer.

Question 3

Define degree of a differential equation.

Solution 3

Degree of a differential equation :

The degree of a differential equation is two degree of the highest order derivative, when differential coefficients are made free from radicols and fractions.

or

The degree of a differential equation is the power of the highest order derivative occuring is a differential equation when it is written as a polynomial in differential coeffcients.

Question 4

Write the differential equation representing the family of straight lines $y = Cx + 5$, where C is an arbitrary contant.

Solution 4

Here,

$$y = Cx + 5 ----(1)$$

Differentiating it with respect to x

$$\frac{dy}{dx} = C$$

$$\frac{dy}{dx} = \frac{C-5}{x} \qquad \left[\text{Using equation } (1)\right]$$

$$x\frac{dy}{dx} = y - 5$$

$$x\frac{dy}{dx} - y + 5 = 0$$

Question 5

Write the differential equation obtained by eliminating the arbitrary contant C in the equation $x^2 - y^2 = C^2$.

Solution 5

$$x^2 - y^2 = C^2$$

Differentiating it with respect to x

$$2x - 2y\frac{dy}{dx} = 0$$

$$2x\,dx - 2y\,dy = 0$$

$$x\,dx - y\,dy = 0$$

Question 6

Write the differential equation obtained by eliminating the arbitrary contant C in the equation $xy = C^2$.

Solution 6

$$xy = C^2$$

Differentiating it with respect to x

$$x\frac{dy}{dx} + y(1) = 0$$

$$x\,dy + y\,dx = 0$$

Question 7

Write the degree of the differential equation $a^2\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{4}}$.

Solution 7

$$a^2\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{4}}$$

$$\left(a^2\frac{d^2y}{dx^2}\right)^4 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\left(a^8\left(\frac{d^2y}{dx^2}\right)^4\right) - \left(\frac{dy}{dx}\right)^2 - 1 = 0$$

The highest order differential coefficient is $\frac{d^2y}{dx^2}$ and its power is 4, so

Degree of equation $= 4$

Question 8

Write the order of the differential equation $1 + \left(\frac{dy}{dx}\right)^2 = 7\left(\frac{d^2y}{dx^2}\right)^3$

Solution 8

$$1+\left(\frac{dy}{dx}\right)^2 = 7\left(\frac{d^2y}{dx^2}\right)^3$$

$$7\left(\frac{d^2y}{dx^2}\right)^3 - \left(\frac{dy}{dx}\right)^2 - 1 = 0$$

The highest order differential coefficient is $\frac{d^2y}{dx^2}$

order of the differential of equation is 2

Question 9

Write the order and degree of the differential equation

$$y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Solution 9

$$y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(y - x\frac{dy}{dx}\right) = a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Square both the sides,

$$y^2 - x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) = a^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

$$x^2\left(\frac{dy}{dx}\right)^2 - a^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + y^2 - a^2 = 0$$

$$\left(\frac{dy}{dx}\right)^2\left(x^2 - a^2\right) - 2xy\left(\frac{dy}{dx}\right) + y^2 - a^2 = 0$$

The highest order differential coefficient is $\left(\frac{dy}{dx}\right)$ and differential equation is 2 so,

Order of differential equation is 1
Degree of differential equation is 2

Question 10

Write the degree of the differential equation

$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right).$$

Solution 10

$$\left(\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right)$$

Given differential equation is not a polynomail in differential coefficient, so

Degre of the differential of equation is not defined.

Question 11

Write the order of the differential equation of the family of circles touching X-axis at the origin.

Solution 11

We have, equation of family of circles with radius r and centre $(4, k)$

$$(x - h)^2 + (y - k)^2 = r^2 - - - - - (1)$$

Here, circle is touching x-axis at origin.so, radius of circle is $r = k$ and $h = 0$

so, from equation as(1)

$$(x - 0)^2 + (y - k)^2 = k^2$$

$$x^2 + (y - k)^2 = k^2$$

$$x^2 + y^2 - 2ky + k^2 = k^2$$

$$k = \frac{x^2 + y^2}{2y}$$

Differentiating it with respect to x,

$$0 = \frac{2y\left[2x + 2y\frac{dy}{dx}\right] - \left(x^2 + y^2\right)2\frac{dy}{dx}}{(2\mathrm{y})^2}$$

$$4y + 4y^2\frac{dy}{dx} - \left(x^2 + y^2\right)2\frac{dy}{\mathrm{dx}} = 0$$

$$2y^2\frac{dy}{dx} - \left(x^2 + y^2\right)\frac{dy}{\mathrm{dx}} + 2y = 0$$

$$\frac{dy}{dx}\left(2y^2 - x^2 - y^2\right) + 2y = 0$$

$$\left(y^2 - x^2\right)\frac{dy}{dx} + 2y = 0$$

The highest order differential coefficient is $\frac{dy}{dx}$ and its power is 1, so,

Order of the differential of equation is 1.

Question 12

Write the order of the differential equation of all non-horizontal lines in plane.

Solution 12

We know that, equation of all non-horizontal lines is a plane is given by

$$ax + by + c = 0$$

Differentiating it with respect to x,

$$a + b\frac{dy}{dx} = 0$$

Again differentiating it with respect to x,

$$b\frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = 0$$

It is a second order differential equation

Question 13

If $\sin x$ is an integrating factor of the differential equation $\frac{dy}{dx} + Py = Q$, then wirte the value of P.

Solution 13

We know that,

$I.F.$ of the equation $\frac{dy}{dx} + Py = Q$ is given by,

$$I.F. = e^{\int P dx}$$

$$\sin x = e^{\int P dx}$$

$$\log \sin x = \int P dx$$

Differentiating it with respect to x,

$$\frac{1}{\sin x}(\cos x) = P$$

$$P = \cot x$$

Question 14

Write the order of the differential equation of the family of circle of radius r.

Solution 14

We know that, family of circle with given radius r and centre (h,k) is given by,

$$(x-h)^2+(y-k)^2=r^2 ------(1)$$

Differentiating it with respect to x,

$$2(x-h)+2(y-k)\frac{dy}{dx}=0$$

$$(x-h)+(y-k)\frac{dy}{dx}=0------(2)$$

Again, differentiating it with respect to x,

$$1+(y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

$$(y-k)=\frac{-\left(\frac{dy}{dx}\right)^2-1}{\frac{d^2y}{dx^2}}------(3)$$

put the value of $(y-k)$ from equation (3) in equation (2),

$$(x-h)+\left[\frac{-\left(\frac{dy}{dx}\right)^2-1}{\frac{d^2y}{dx^2}}\right]\frac{dy}{dx}=0$$

$$(x-h)-\left[\frac{\left(\frac{dy}{dx}\right)^2+1}{\frac{d^2y}{dx^2}}\right]\frac{dy}{dx}=0$$

$$(x-h)=\left[\frac{\left(\frac{dy}{dx}\right)^2+1}{\frac{d^2y}{dx^2}}\right]\frac{dy}{dx}$$

Put the value of $(x-h)$ and $(y-k)$ is equation (1),

$$\left\{\left[\frac{\left(\frac{dy}{dx}\right)^2+1}{\frac{d^2y}{dx^2}}\right]\left(\frac{dy}{dx}\right)\right\}^2+\left\{-\left[\frac{\left(\frac{dy}{dx}\right)^2+1}{\frac{d^2y}{dx^2}}\right]\right\}^2=r^2$$

$$\left[\left(\frac{dy}{dx}\right)^2+1\right]^2+\left[\left(\frac{dy}{dx}\right)^2+1\right]=r^2\frac{d^2y}{dx^2}$$

$$r^2\frac{d^2y}{dx^2}-\left[\left(\frac{dy}{dx}\right)^2+1\right]^3=0$$

Order of the differential equation is 2

Question 15

Write the order of the differential equation whose solution is

$$y=a\cos x+b\sin x+ce^{-x}.$$

Solution 15

$$y = a\cos x + b\sin x + ce^{-x}$$

Since their are three arbitrary constants is the solutaion. it means,

The order of the differential equation formed by it is 3

Question 16

Write the order of the differential equation associated wwith the primitive $y = C_1 + C_2 e^{x} + C_3 e^{-2x} + C_4$ Where C_1, C_2, C_3, and C_4 are arbitrary constants.

Solution 16

Given,

$$y = C_1 + C_2 e^{x} + C_3 e^{-2x} + C_4$$

Where C_1, C_2, C_3, and C_4 are arbitrary constants.

It has four arbitrary constants but one of the arbitrary constants (C_4) is part of exponential.

So, the order of the equation will be decided by the remaining number of arbitrary constants. so,

The order of the differential equation equation formed by the given solution is 3

Question 17

What is the degree of the following differential equation?

$$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$$

Solution 17

$$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$$

Here, the highest order differential coefficient is $\frac{d^2y}{dx^2}$ and its power is 1. so,

The degree of the differential equation is 1

Question 18

Write the degree of the differential equation $\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0.$

Solution 18

Consider the given differential equation,

$$\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0$$

The degree of a differential equation is the degree of the highest order derivative.

In the given differential equation, the power of the highest order differential coefficient is 1.

Hence its degree is 1.

Question 19

Write the degree of the differential equation $x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0.$

Solution 19

Consider the given differential equation,

$$x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0.$$

The degree of a differential equation is the degree of the highest order derivative.

In the given differential equation, the power of the highest order differential coefficient is 3.

Hence its degree is 3.

Question 20

Write the differential equation representing family of curves y = mx, where m is arbitrary constant.

Solution 20

The equation of family of curves is

$y = mx$...(1)

Clearly, there is only one arbitrary constant, 'm', in the above equation.

Therefore, we shall differentiate the above equation with respect to x, only once.

$\frac{dy}{dx} = m$...(2)

Substituting the value of 'm' from equation (2) in equation (1), we have,

$$y = x\frac{dy}{dx}$$

Thus, the differential equation representing family of curves $y = mx$ is $y = x\frac{dy}{dx}$.

Question 21

Write the degree of the differential equation $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0.$

Solution 21

Consider the given differential equation,

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$

The degree of a differential equation is the degree of the highest order derivative.

In the given differential equation, the power of the highest order differential coefficient is 2.

Hence its degree is 2.