RD SHARMA Solutions for Class 12- Maths Chapter 20 - Definite Integrals

Chapter 20 - Definite Integrals Exercise Ex. 20.1

Question 1

Evaluate $\int_4^9 \frac{1}{\sqrt{x}} dx$

Solution 1

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Now,

$$\int_4^9 \frac{1}{\sqrt{x}} dx$$

$$= \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_4^9$$

$$= \left[\frac{\sqrt{x}}{\frac{1}{2}}\right]_4^9$$

$$= 2[\sqrt{9} - \sqrt{4}]$$

$$= 2[3 - 2]$$

$$= 2$$

$$\therefore \int_4^9 \frac{1}{\sqrt{x}} dx = 2$$

Question 2

Evaluate $\int_{-2}^3 \frac{1}{x+7} dx$

Solution 2

We know that $\int \frac{dx}{x} = \log x + C$

Now,

$\int_{-2}^{3} \frac{1}{x+7} dx$

$= \left[\log(x+7)\right]_{-2}^{3}$

$= \left[\log 10 - \log 5\right]_{-2}^{3}$

$= \log \frac{10}{5}$ $\left[\because \log a - \log b = \log \frac{a}{b}\right]$

$= \log 2$

$\therefore \int_{-2}^{3} \frac{1}{x+7} dx = \log 2$

Question 3

Evaluate $\int_{0}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx$

Solution 3

Let $x = \sin\theta$
$\Rightarrow dx = \cos\theta d\theta$

Now,
$x = 0 \Rightarrow \theta = 0$
$x = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$$\therefore \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{\cos\theta d\theta}{\cos\theta}$$
$$= \int_0^{\frac{\pi}{6}} d\theta$$
$$= [\theta]_0^{\frac{\pi}{6}}$$
$$= \left[\frac{\pi}{6} - 0\right]$$
$$= \frac{\pi}{6}$$

$$\therefore \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} = \frac{\pi}{6}$$

Question 4

Evaluate $\int_0^1 \frac{1}{1+x^2} dx$

Solution 4

We have,

$$I = \int_0^1 \frac{1}{1+x^2} dx$$

$$= \left[\tan^{-1} x\right]_0^1$$
$$= \left[\tan^{-1} 1 - \tan^{-1} 0\right]$$
$$= \left[\frac{\pi}{4} - 0\right] \qquad \left[\because \tan^{-1} 1 = \frac{\pi}{4}\right]$$
$$= \frac{\pi}{4}$$

$$\therefore \int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}$$

Question 5

Evaluate $\int_{2}^{3} \frac{x}{x^2+1} dx$

Solution 5

Let $x^2 + 1 = t$

$\Rightarrow 2x\, dx = dt$

$\Rightarrow x\, dx = \frac{dt}{2}$

Now,

$x = 2 \Rightarrow t = 5$

$x = 3 \Rightarrow t = 10$

$$\therefore \int_{2}^{3} \frac{x}{x^2+1} dx = \frac{1}{2}\int_{5}^{10} \frac{dt}{t} = \frac{1}{2}\left[\log t\right]_5^{10}$$

$$= \frac{1}{2}\left[\log 10 - \log 5\right]$$

$$= \frac{1}{2}\left[\log \frac{10}{5}\right]$$

$$= \frac{1}{2}\left[\log 2\right]$$

$$= \log \sqrt{2}$$

$$\therefore \int_{2}^{3} \frac{x}{x^2+1} = \log \sqrt{2}$$

$$\textit{Therefore}, \int_{2}^{3} \frac{x}{x^2+1} = \frac{1}{2}\log 2$$

Question 6

$\int_{0}^{\infty} \frac{1}{a^2 + b^2 x^2} dx$

Solution 6

We have,

$$\int_0^\infty \frac{1}{a^2+b^2x^2}dx = \frac{1}{b^2}\int_0^\infty \frac{1}{\left(\frac{a}{b}\right)^2 + x^2}dx$$

We know that $\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$$\therefore \frac{1}{b^2}\int_0^\infty \frac{1}{\left(\frac{a}{b}\right)^2 + x^2}dx = \frac{1}{b^2}\left[\frac{b}{a}\tan^{-1}\left(\frac{bx}{a}\right)\right]_0^\infty$$

$$= \frac{1}{ab}\left[\tan^{-1}\left(\frac{bx}{a}\right)\right]_0^\infty$$

$$= \frac{1}{ab}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{ab}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{2ab}$$

$$\Rightarrow \int_0^\infty \frac{1}{a^2+b^2x^2}dx = \frac{\pi}{2ab}$$

Question 7

Evaluate $\int_{-1}^{1} \frac{1}{1+x^2}dx$

Solution 7

We have,

$$\int_{-1}^{1} \frac{1}{1+x^2} dx$$

We know that $\int \frac{1}{1+x^2} dx = \tan^{-1} x$

Now,

$$\int_{-1}^{1} \frac{1}{1+x^2} dx$$

$$= \left[\tan^{-1} x\right]_{-1}^{1}$$

$$= \left[\tan^{-1} 1 - \tan^{-1}(-1)\right]$$

$$= \left[\frac{\pi}{4} - \left(\frac{-\pi}{4}\right)\right] \qquad \left[\because \tan^{-1}(-1) = \frac{-\pi}{4}\right]$$

$$= \left[\frac{\pi}{4} + \frac{\pi}{4}\right]$$

$$= \frac{2\pi}{4}$$

$$\therefore \int_{-1}^{1} \frac{1}{1+x^2} dx = \frac{\pi}{2}$$

Question 8

Evaluate $\int_{0}^{\infty} e^{-x} dx$

Solution 8

We have,

$$\int_{0}^{\infty} e^{-x} dx$$

We know that $\int e^{-x} dx = -e^{-x}$

Now,

$$\int_{0}^{\infty} e^{-x} dx$$

$$= \left[-e^{-x}\right]_{0}^{\infty}$$

$$= \left[-e^{-\infty} + e^{-0}\right] \qquad \left[\because e^{\infty} = 0, \quad e^{0} = 1\right]$$

$$= \left[-0 + 1\right]$$

$$\therefore \int_{0}^{\infty} e^{-x} dx = 1$$

Question 9

Evaluate $\int_{0}^{1} \frac{x}{x+1} dx$

Solution 9

We have,

$$\int_0^1 \frac{x}{x+1}dx \qquad \left[\text{Add and subtract 1 in numerator}\right]$$

$$= \int_0^1 \frac{(x+1)-1}{x+1}dx$$

$$= \int_0^1 1dx - \int_0^1 \frac{1}{x+1}dx$$

$$= \left[x\right]_0^1 - \left[\log(x+1)\right]_0^1$$

$$= 1 - \left[\log 2 - \log 1\right]$$

$$= 1 - \log\frac{2}{1}$$

$$= 1 - \log 2$$

$$= \log e - \log 2 \qquad \left[\because \log e = 1\right]$$

$$= \log\frac{e}{2}$$

$$\therefore \int_0^1 \frac{x}{x+1}dx = \log\frac{e}{2}$$

Question 10

Evaluate $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)dx$

Solution 10

We have,

$$\int_0^{\frac{\pi}{2}} (\sin x + \cos x)dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x dx + \int_0^{\frac{\pi}{2}} \cos x dx$$

$$= \left[-cos x\right]_0^{\frac{\pi}{2}} + \left[\sin x\right]_0^{\frac{\pi}{2}}$$

$$= \left[\cos\frac{\pi}{2} + \cos 0\right] + \left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= \left[-0 + 1\right] + 1$$

$$= 1 + 1$$

$$= 2$$

$$\therefore \int_0^{\frac{\pi}{2}} (\sin x + \cos x)dx = 2$$

Question 11

$$\text{Evaluate } \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x dx$$

Solution 11

We have,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x dx$$

We know that $\int \cot x dx = \log(\sin x)$

Now,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x dx$$

$$= \left[\log(\sin x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[\log\left(\sin\frac{\pi}{2}\right) - \log\left(\sin\frac{\pi}{4}\right)\right]$$

$$= \left[\log 1 - \log\frac{1}{\sqrt{2}}\right]$$

$$= \left[0 - (\log 1 - \log\sqrt{2})\right]$$

$$= \log\sqrt{2} \qquad \left[\because \log a^n = n\log a\right]$$

$$= \frac{1}{2}\log 2$$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x dx = \frac{1}{2}\log 2$$

Question 12

$$\text{Evaluate } \int_{0}^{\frac{\pi}{4}} \sec x dx$$

Solution 12

We have,

$$\int_0^{\frac{\pi}{4}} \sec x dx$$

We know that $\int \sec x dx = \log(\sec x + \tan x)$

$$\therefore \int_0^{\frac{\pi}{4}} \sec x dx$$

$$= \left[\log(\sec x + \tan x)\right]_0^{\frac{\pi}{4}}$$

$$= \left[\log(\sqrt{2}+1) - \log(1+0)\right]$$

$$= \log(\sqrt{2}+1) \qquad \left[\because \log 1 = 0\right]$$

$$\therefore \int_0^{\frac{\pi}{4}} \sec x dx = \log(\sqrt{2}+1)$$

Question 13

Evaluate the definite integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \; dx$$

Solution 13

$$\text{Let } I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$$

$$\int \operatorname{cosec} x \; dx = \log\left|\operatorname{cosec} x - \cot x\right| = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}\left(\frac{\pi}{4}\right) - \mathrm{F}\left(\frac{\pi}{6}\right)$$

$$= \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$$

$$= \log\left|\sqrt{2}-1\right| - \log\left|2-\sqrt{3}\right|$$

$$= \log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right)$$

Question 14

Evaluate $\int_0^1 \frac{1-x}{1+x} dx$

Solution 14

We have,

$$\int_0^1 \frac{1-x}{1+x}dx$$

Let $x = \cos 2\theta \Rightarrow dx = -2\sin 2\theta d\theta$

Now,

$x = 0 \Rightarrow \theta = \frac{\pi}{4}$

$x = 1 \Rightarrow \theta = 0$

Now,

$$\int_0^1 \frac{1-x}{1+x}dx$$

$$= \int_{\frac{\pi}{4}}^{0} \frac{1-\cos 2\theta}{1+\cos 2\theta} \times (-2\sin 2\theta)d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{2\sin^2\theta}{2\cos^2\theta} \times 2\sin 2\theta d\theta \qquad \left[\because -\int_a^b f(x)dx = \int_b^a f(x)dx\right]$$

$$= \int_0^{\frac{\pi}{4}} \frac{4\sin^3\theta}{\cos\theta}d\theta$$

Let $\cos\theta = t$

$\Rightarrow -\sin\theta d\theta = dt$

Now,

$\theta = 0 \Rightarrow t = 1$

$\theta = \frac{\pi}{4} \Rightarrow t = \frac{1}{\sqrt{2}}$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{4\sin^3\theta}{\cos\theta}d\theta$$

$$= -4\int_1^{\frac{1}{\sqrt{2}}} \frac{(1-t^2)}{t}dt$$

$$= -4\left[\log t - \frac{t^2}{2}\right]_1^{\frac{1}{\sqrt{2}}}$$

$$= -4\left[\log\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{4} - 0 + \frac{1}{2}\right]$$

$$= -4\left[-\log\sqrt{2} + \frac{1}{4}\right]$$

$$\therefore \int_0^1 \frac{1-x}{1+x}dx = 2\log 2 - 1$$

Question 15

Evaluate $\int_0^{\pi} \frac{1}{1+\sin x} dx$

Solution 15

$I = \int_0^{\pi} \frac{1}{1+\sin x} dx$

Multiplying Numerator and Denominator by $(1-\sin x)$

$$
\begin{aligned}
I &= \int_0^{\pi} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx \\
&= \int_0^{\pi} \frac{(1-\sin x)}{(1^2-\sin^2 x)} dx \\
&= \int_0^{\pi} \frac{1-\sin x}{(\cos^2 x)} dx \\
&= \int_0^{\pi} \frac{1}{\cos^2 x} dx - \int_0^{\pi} \frac{\sin x}{\cos^2 x} dx \\
&= \int_0^{\pi} \sec^2 x dx - \int_0^{\pi} \tan x . \sec x dx \\
&= [\tan x]_0^{\pi} - [\sec x]_0^{\pi} \\
&= [\tan \pi - \tan 0] - [\sec \pi - \sec 0] \\
&= [0-0] - [-1-1] \\
&= 2 \\
I &= 2
\end{aligned}
$$

$\therefore \int_0^{\pi} \frac{1}{1+\sin x} dx = 2$

Question 16

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x} dx$

Solution 16

We have,

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}dx$$

We know,

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\therefore \frac{1}{1+\sin x} = \frac{1}{1+\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} = \frac{1+\tan^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2} = \frac{\sec^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2}$$

$$\Rightarrow \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2}dx$$

If $f(x)$ is an even function $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

So,

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2}dx = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2}dx$$

let $1 + \tan\frac{x}{2} = t$

$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

Now,

$x = -\frac{\pi}{4} \Rightarrow t = 1 - \tan\frac{\pi}{8}$

$x = \frac{\pi}{4} \Rightarrow t = 1 + \tan\frac{\pi}{8}$

$$\therefore 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2}dx = 2\int_{1-\tan\frac{\pi}{8}}^{1+\tan\frac{\pi}{8}} \frac{8dt}{t^2}$$

$$= 2\left[\frac{-1}{t}\right]_{1-\tan\frac{\pi}{8}}^{1+\tan\frac{\pi}{8}}$$

$$= 2\left[\frac{1}{1-\tan\frac{\pi}{8}} - \frac{1}{1+\tan\frac{\pi}{8}}\right]$$

$$= 2\left[\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}\right]$$

$$= 2\tan\frac{\pi}{4} \qquad \left[\because \tan 2A = \frac{2\tan A}{1-\tan^2 A}\right]$$

$$= 2$$

$$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}dx = 2$$

Question 17

Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$$

Solution 17

Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$$\int \cos^2 x \, dx = \int \left(\frac{1+\cos 2x}{2} \right) dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}\left(x + \frac{\sin 2x}{2} \right) = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \left[\mathrm{F}\left(\frac{\pi}{2} \right) - \mathrm{F}(0) \right]$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin \pi}{2} \right) - \left(0 + \frac{\sin 0}{2} \right) \right]$$
$$= \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0 - 0 \right]$$
$$= \frac{\pi}{4}$$

Question 18

Evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x dx$

Solution 18

We have,

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos 3x + 3\cos x}{4} dx \qquad \left[\because \cos 3x = 4\cos^3 x - 3\cos x\right]$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(\cos 3x + 3\cos x\right) dx$$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[\left(\frac{\sin 3\frac{\pi}{2}}{3} + 3\sin\frac{\pi}{2}\right) - \left(\frac{\sin 0}{3} + 3\sin 0\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{-1}{3} + 3\right) - \left(0 + 0\right)\right] = \frac{2}{3}$$

$$= \frac{1}{4}\left[\frac{8}{3}\right]$$

$$= \frac{2}{3}$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^3 x dx = \frac{2}{3}$$

Question 19

Evaluate $\int_0^{\frac{\pi}{6}} \cos x \cos 2x dx$

Solution 19

We have,

$$\int_0^{\frac{\pi}{6}} \cos x \cos 2x dx \qquad \left[\because 2\cos C\cos D = \cos(C+D) - \cos(C-D)\right]$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} 2\cos x \cos 2x dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \left(\cos 3x + \cos x\right) dx$$

$$= \frac{1}{2}\int \left[\frac{\sin 3x}{3} + \sin x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left[\left(\frac{\sin 3\frac{\pi}{6}}{3} + \sin\frac{\pi}{6}\right) - \left(\sin 0 - \sin 0\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin\frac{\pi}{2}}{3} + \sin\frac{\pi}{6}\right]$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{5}{6}\right)$$

$$= \frac{5}{12}$$

$$\therefore \int_0^{\frac{\pi}{6}} \cos x \cos 2x dx = \frac{5}{12}$$

Question 20

Evaluate $\int_0^{\frac{\pi}{2}} \sin x \sin 2x dx$

Solution 20

We have,

$$\int_0^{\frac{\pi}{2}} \sin x \sin 2x dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin x \sin 2x dx \qquad [\because 2\sin C \times \sin D = \cos(D-C) - \cos(D+C)]$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos x - \cos 3x) dx$$

$$= \frac{1}{2}\left[\sin x - \frac{\sin 3x}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\frac{\sin 3\frac{\pi}{2}}{3} - \frac{\sin 0}{3}\right)\right]$$

$$= \frac{1}{2}\left[(1-0) - \left(\frac{-1}{3} - 0\right)\right] \qquad \left[\because \sin 3\frac{\pi}{2} = -1\right]$$

$$= \frac{1}{2} \times \frac{4}{3}$$

$$= \frac{2}{3}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin x \sin 2x dx = \frac{2}{3}$$

Question 21

Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} (\tan x + \cot x)^2 dx$

Solution 21

We have,

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \left(\tan x + \cot x\right)^2 dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \left(\frac{\sin^2 x + \cot^2 x}{\sin x \cos x}\right)^2 dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \left(\frac{1}{\sin x \cos x}\right)^2 dx$$

Multiplying numerator and denominator by 2

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \left(\frac{2}{2 \sin x \cos x}\right)^2 dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \left(\frac{2}{\sin 2x}\right)^2 dx \qquad \left[\because 2 \sin x \cos x = \sin 2x\right]$$

$$= 4\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \operatorname{cosec}^2 x dx$$

$$= 4\left[-\frac{\cot 2x}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{4}}$$

$$= 2\left[-\cot\frac{\pi}{2} + \cot 2\frac{\pi}{3}\right]$$

$$= 2\left[\frac{-1}{\sqrt{3}} - 0\right]$$

$$= \frac{-2}{\sqrt{3}}$$

$$\therefore \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \left(\tan x + \cot x\right)^2 dx = \frac{-2}{\sqrt{3}}$$

Question 22

Evaluate $\int_{0}^{\frac{\pi}{2}} \cos^4 x dx$

Solution 22

We have,

$$\int_0^{\frac{\pi}{2}} \cos^4 x dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(1+\cos 2x\right)^2 dx \qquad \left[\because 2\cos^2 x = 1+\cos 2x\right]$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(1+\cos^2 2x + 2\cos 2x\right)dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(1+\frac{1+\cos 4x}{2} + 2\cos 2x\right)dx$$

$$= \frac{1}{4}\left[x+\frac{1}{2}x+\frac{\sin 4x}{8}+\sin 2x\right]_0^{\frac{\pi}{2}} \qquad \left[\because \int \cos 4x dx = \frac{\sin 4x}{4}\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{2}+\frac{\pi}{4}+0+0-0-0-0-0\right]$$

$$= \frac{1}{4}\times\frac{3\pi}{4}$$

$$= \frac{3\pi}{16}$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^4 x dx = \frac{3\pi}{16}$$

Question 23

Evaluate $\int_0^{\frac{\pi}{2}} (a^2\cos^2 x + b^2\sin^2 x)dx$

Solution 23

We have,

$$\int_0^{\frac{\pi}{2}} \left\{a^2 \cos^2 x + b^2 \left(1 - \cos^2 x\right)\right\} dx$$

$$= \int_0^{\frac{\pi}{2}} \left\{\left(a^2 - b^2\right) \cos^2 x + b^2\right\} dx$$

$$= \frac{a^2 - b^2}{2} \int_0^{\frac{\pi}{2}} \left(1 + \cos 2x\right) dx + b^2 \int_0^{\frac{\pi}{2}} dx$$

$$= \frac{a^2 - b^2}{2} \left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} + b^2 [x]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2 - b^2}{2} \left[\frac{\pi}{2} + 0 - 0 - 0\right] + b^2 \left[\frac{\pi}{2} - 0\right]$$

$$= \frac{a^2 - b^2}{2} \left[\frac{\pi}{2}\right] + b^2 \left[\frac{\pi}{2}\right]$$

$$= a^2 \frac{\pi}{4} + b^2 \left[\frac{\pi}{2} - \frac{\pi}{4}\right]$$

$$= \frac{\pi a^2}{4} + \frac{\pi b^2}{4}$$

$$= \frac{\pi}{4}\left(a^2 + b^2\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} (a^2 \cos^2 x + b^2 \sin^2 x) dx = \frac{\pi}{4}\left(a^2 + b^2\right)$$

Question 24

Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{1 + \sin x}\, dx$

Solution 24

We have,

$$\int_0^{\frac{\pi}{2}} \sqrt{1+\sin x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}\, dx \qquad \text{We use } \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{1+\sin x}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{\frac{\left(1+\tan\frac{x}{2}\right)^2}{1+\tan^2\frac{x}{2}}}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{\left(1+\tan\frac{x}{2}\right)^2}{\sec^2\frac{x}{2}}}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1+\tan\frac{x}{2}}{\sec\frac{x}{2}}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\cos\frac{x}{2}+\sin\frac{x}{2}\right) dx$$

$$= \left[2\sin\frac{x}{2} - 2\cos\frac{x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - 0 + 1\right]$$

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{1+\sin x}\, dx = 2$$

Question 25

Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{1+\cos x}$

Solution 25

We have,

$$\int_0^{\frac{\pi}{2}} \sqrt{1+\cos x}$$

We use $1+\cos x = 2\cos^2\frac{x}{2}$

$$= \int_0^{\frac{\pi}{2}} \sqrt{2\cos^2\frac{x}{2}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{2}\cos\frac{x}{2}dx$$

$$= \sqrt{2}\left[2\sin\frac{x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 2\sqrt{2}\left[\frac{1}{\sqrt{2}}\right]$$

$$= 2$$

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{1+\cos x} = 2$$

Question 26

Evaluate $\int_0^{\frac{\pi}{2}} x\sin x dx$

Solution 26

We have,

$$\int x\sin x\,dx = x\int \sin x dx - \int\left(\int \sin x dx\right)\left(\frac{dx}{dx}\right)dx$$

$$= -x\cos x + \int \cos x dx$$

$$\therefore \int_0^{\frac{\pi}{2}} x\sin x\,dx = \left[-x\cos x + \sin x\right]_0^{\frac{\pi}{2}} = \left(-\frac{\pi}{2}\times 0\right) + 1 + 0 - 0 = 1$$

$$\therefore \int_0^{\frac{\pi}{2}} x\sin x dx = 1$$

Question 27

Evaluate $\int_0^{\frac{\pi}{2}} x\cos x dx$

Solution 27

We have,

$$\int x\cos x\,dx = x\int\cos x\,dx - \int\left(\int\cos x\,dx\right)\frac{dx}{dx}dx = x\sin x - \int\sin x\,dx$$

$$\therefore \int_0^{\frac{\pi}{2}} x\cos x\,dx = \left[x\sin x+\cos x\right]_0^{\frac{\pi}{2}} = \left[\frac{\pi}{2}+0-0-1\right] = \frac{\pi}{2}-1$$

$$\therefore \int_0^{\frac{\pi}{2}} x\cos x\,dx = \frac{\pi}{2}-1$$

Question 28

Evaluate $\int_0^{\frac{\pi}{2}} x^2\cos x\,dx$

Solution 28

We have,

$$\int x^2\cos x\,dx = x^2\int\cos x\,dx - \int(2x)\left(\int\cos x\,dx\right)dx = x^2\sin x - \int\sin x.2x\,dx$$

$$= x^2\sin x - 2\left[x\int\sin x - \int\left(\int\sin x\,dx\right)dx\right]$$

$$= x^2\sin x - 2\left[-x\cos x + \int\cos x\,dx\right]$$

$$\therefore \int_0^{\frac{\pi}{2}} x^2\cos x\,dx = \left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi^2}{4}+0-2-0-0+0\right]$$

$$= \frac{\pi^2}{4}-2$$

$$\therefore \int_0^{\frac{\pi}{2}} x^2\cos x\,dx = \frac{\pi^2}{4}-2$$

Question 29

Evaluate $\int_0^{\frac{\pi}{4}} x^2\sin x\,dx$

Solution 29

We have,

$$\int x^2 \sin x\, dx = x^2 \int \sin x dx - \int 2x \left(\int \sin x dx\right) dx = x^2 \cos x + \int 2x \cos x dx$$

$$= x^2 \cos x + 2\left[x \int \cos x dx - \int \left(\int \cos x dx\right) dx\right]$$

$$= -x^2 \cos x + 2\left[x \sin x - \int \sin x dx\right]$$

$$\therefore \quad \int_0^{\frac{\pi}{4}} x^2 \sin x\, dx = \left[-x^2 \cos x + 2x \sin x + 2\cos x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{-\pi^2}{16} \cdot \frac{1}{\sqrt{2}} + \frac{\pi}{2} \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}} + 0 - 0 - 2$$

$$= \frac{1}{\sqrt{2}}\left[\frac{-\pi^2}{16} + \frac{\pi}{2} + 2\right] - 2$$

$$= \sqrt{2} + \frac{\pi}{2\sqrt{2}} - \frac{\pi^2}{16\sqrt{2}} - 2$$

$$\therefore \int_0^{\frac{\pi}{4}} x^2 \sin x dx = \sqrt{2} + \frac{\pi}{2\sqrt{2}} - \frac{\pi^2}{16\sqrt{2}} - 2$$

Question 30

Evaluate $\int_0^{\frac{\pi}{2}} x^2 \cos 2x dx$

Solution 30

We have,

$$\int x^2 \cos 2x\, dx = x^2 \int \cos 2x dx - \int 2x \left(\int \cos 2x dx\right) dx$$

$$= \frac{x^2 \sin 2x}{2} - \int 2x \times \frac{\sin 2x}{2} dx$$

$$= \frac{x^2 \sin 2x}{2} - \left[x \int \sin 2x dx - \int \left(\int \sin 2x dx\right) dx\right]$$

$$= \frac{x^2 \sin 2x}{2} + \left[\frac{x \cos 2x}{2} - \int \frac{x \cos 2x}{2}\right]$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} x^2 \cos 2x\, dx = \left[\frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi^2}{8} \times 0 + \frac{\pi}{4}(-1) - 0 - 0 - 0 + 0\right]$$

$$= \frac{-\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} x^2 \cos 2x dx = \frac{-\pi}{4}$$

Question 31

Evaluate $\int_0^{\frac{\pi}{2}} x^2 \cos^2 x dx$

Solution 31

We have,

$$\int x^2 \cos^2 x dx = \int x^2 \left(\frac{1+\cos 2x}{2}\right) dx = \frac{1}{2}\int \left(x^2 + x^2 \cos 2x\right) dx = \frac{1}{2}\left[\int x^2 dx + \int x^2 \cos 2x dx\right] \quad ...(A)$$

Now,

$$\int_0^{\frac{\pi}{2}} x^2 dx = \left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}} = \frac{\pi^3}{24} \quad ...(B)$$

$$\int x^2 \cos 2x\, dx = x^2 \int \cos 2x dx - \int 2x \left(\int \cos 2x dx\right) dx = \frac{x^2 \sin 2x}{2} - \int \frac{\sin 2x}{2}.2x dx$$

$$= \frac{x^2 \sin 2x}{2} - \left[x \int \sin 2x - \int \left(\int \sin 2x dx\right) dx\right]$$

$$= \frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \int \frac{\cos 2x}{2} dx$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} x^2 \cos 2x\, dx = \left[\frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{-\pi}{4} \quad -(C)$$

Now, Put (B) & (C) in (A), we get,

$$\int_0^{\frac{\pi}{2}} x^2 \cos^2 x dx = \int_0^{\frac{\pi}{2}} x^2 dx + \int_0^{\frac{\pi}{2}} x^2 \cos 2x\, dx = \frac{1}{2}\left[\frac{\pi^3}{24} - \frac{\pi}{4}\right] = \frac{\pi^3}{48} - \frac{\pi}{8}$$

Question 32

Evaluate $\int_1^2 \log x dx$

Solution 32

We have,

$$\int \log x dx = \int 1.\log x dx = \log x \int dx - \int \left(\int dx\right).\frac{1}{x} dx = x \log x - \int x.\frac{1}{x} dx = x \log x - \int dx$$

$$\therefore \quad \int_1^2 \log x dx = \left[x \log x - x\right]_1^2 = 2\log 2 - 2 - 0 + 1 = 2\log 2 - 1$$

Question 33

Evaluate $\int_1^3 \frac{\log x}{(x+1)^2} dx$

Solution 33

We have,

$$\int \frac{\log x}{(x+1)^2}dx = \int \frac{1}{(x+1)^2}\log x dx = \log x \int \frac{1}{(x+1)^2}dx - \int\left(\int \frac{1}{(x+1)^2}dx\right)\frac{1}{x}dx$$

$$= \frac{-\log x}{(x+1)} + \int \frac{1}{x(x+1)}dx$$

$$= \frac{-\log x}{(x+1)} + \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$\therefore \quad \int_1^3 \frac{\log x}{(x+1)^2}dx = \left[\frac{-\log x}{x+1} + \log x - \log(x+1)\right]_1^3 = \frac{3}{4}\log 3 - \log 2$$

Question 34

Evaluate $\int_1^e \frac{e^x}{x}(1 + x\log x)dx$

Solution 34

Let $I = \int_1^e \frac{e^x}{x}(1 + x\log x)dx$

$I = \int_1^e \frac{e^x}{x}dx + \int_1^e e^x \log x dx$

$I = \left[e^x \log x\right]_1^e - \int_1^e e^x . \log x + \int_1^e e^x \log x$

$I = \left[e^x \log x\right]_1^e$

$I = \left[e^x \log e - e^1 . \log 1\right]$

$I = \left[e^e . 1 - 0\right]$

$I = e^e$

$\therefore \int_1^e \frac{e^x}{x}(1 + x\log x)dx = e^e$

Question 35

Evaluate $\int_1^e \frac{\log x}{x}dx$

Solution 35

We have,

$$\int_1^e \frac{\log x}{x} dx$$

Let $\log x = t$

$$\Rightarrow \frac{1}{x} dx = dt$$

Now,

$x = 1 \Rightarrow t = 0$

$x = e \Rightarrow t = 1$

$$\therefore \int_1^e \frac{\log x}{x} dx = \int_0^1 t\, dt$$

$$= \left[\frac{t^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

$$\therefore \int_1^e \frac{\log x}{x} dx = \frac{1}{2}$$

Question 36

Evaluate $\int_e^{e^2} \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2} dx\right\}$

Solution 36

We have,

$$\int_e^{e^2} \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2} dx\right\}$$

$$I = \int \frac{1}{\log x} .1 dx = \frac{1}{\log x} \int dx - \int \left(\int dx\right). \frac{d}{dx}\left(\frac{1}{\log x}\right) dx = \frac{x}{\log x} + \int \frac{1}{(\log x)^2} .x. \frac{1}{x} dx$$

$$= \frac{x}{\log x} + \int \frac{dx}{(\log x)^2}$$

$$\int_e^{e^2} \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2} dx\right\} = \left[\frac{x}{\log x}\right]_e^{e^2} + \int_e^{e^2} \frac{dx}{(\log x)^2} - \int_e^{e^2} \frac{dx}{(\log x)^2}$$

$$= \left[\frac{x}{\log x}\right]_e^{e^2}$$

$$= \frac{e^2}{2} - e$$

Question 37

Evaluate $\int_1^2 \frac{x+3}{x(x+2)} dx$

Solution 37

We have,

$$\int_1^2 \frac{x+3}{x(x+2)} dx$$

$$= \int_1^2 \frac{x}{x(x+2)} dx + \int_1^2 \frac{3}{x(x+2)} dx$$

$$= \int_1^2 \frac{dx}{(x+2)} + \int_1^2 \frac{3}{x(x+2)} dx$$

$$= \left[\log(x+2)\right]_1^2 + \frac{3}{2}\int_1^2 \frac{1}{x} - \frac{1}{x+2} dx \qquad \left[\text{using partial fraction}\right]$$

$$= \left[\log(x+2)\right]_1^2 + \left[\frac{3}{2}\log x - \frac{3}{2}\log(x+2)\right]_1^2$$

$$= \left[\frac{3}{2}\log x - \frac{1}{2}\log(x+2)\right]_1^2$$

$$= \frac{1}{2}\left[3\log 2 - \log 4 + \log 3\right]$$

$$= \frac{1}{2}\left[3\log 2 - 2\log 2 + \log 3\right] \qquad \left[\because \log 4 = 2\log 2\right]$$

$$= \frac{1}{2}\left[\log 2 + \log 3\right]$$

$$= \frac{1}{2}\left[\log 6\right]$$

$$= \frac{1}{2}\log 6$$

$$\therefore \int_1^2 \frac{x+3}{x(x+2)} dx = \frac{1}{2}\log 6$$

Question 38

Evaluate the definite integral

$$\int_0^1 \frac{2x+3}{5x^2+1} dx$$

Solution 38

Let $I = \int_0^1 \frac{2x+3}{5x^2+1}dx$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x+15}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}dx$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{5}\cdot\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}x\right)$$

$$= \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(1) - \mathrm{F}(0)$$

$$= \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)\right\} - \left\{\frac{1}{5}\log(1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

Question 39

Evaluate the integral in using substitution.

$$\int_0^2 \frac{dx}{x+4-x^2}$$

Solution 39

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2-x-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

Let $x-\frac{1}{2}=t \Rightarrow dx = dt$

When $x=0$, $t=-\frac{1}{2}$ and when $x=2$, $t=\frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2-t^2}$$

$$=\left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)}\log\frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$=\frac{1}{\sqrt{17}}\left[\log\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}}-\frac{\log\frac{\sqrt{17}}{2}-\frac{1}{2}}{\log\frac{\sqrt{17}}{2}+\frac{1}{2}}\right]$$

$$=\frac{1}{\sqrt{17}}\left[\log\frac{\sqrt{17}+3}{\sqrt{17}-3}-\log\frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$=\frac{1}{\sqrt{17}}\log\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{25-17}\right]$$

Question 40

Evaluate $\int_0^1 \frac{1}{2x^2+x+1}dx$

Solution 40

We have,

$$\int_0^1 \frac{1}{2x^2+x+1}dx$$

$$= \frac{1}{2}\int_0^1 \frac{1dx}{\left(x^2+\frac{1}{2}x+\frac{1}{2}\right)}$$

$$= \frac{1}{2}\int_0^1 \frac{dx}{\left(x+\frac{1}{4}\right)^2+\frac{1}{2}-\frac{1}{16}} \qquad \left[\text{Adding } \frac{1}{16} \text{ \& substracting } \frac{1}{16} \text{ in numerator}\right]$$

$$= \frac{1}{2}\int_0^1 \frac{dx}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}}$$

$$= \frac{1}{2}\int_0^1 \frac{dx}{\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{4}{\sqrt{7}}\left[\tan^{-1}\left(\frac{x+\frac{1}{4}}{\frac{\sqrt{7}}{4}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{7}}\left\{\tan^{-1}\frac{5}{\sqrt{7}}-\tan^{-1}\left(\frac{1}{\sqrt{7}}\right)\right\}$$

$$\therefore \int_0^1 \frac{1}{2x^2+x+1}dx = \frac{2}{\sqrt{7}}\left\{\tan^{-1}\frac{5}{\sqrt{7}}-\tan^{-1}\left(\frac{1}{\sqrt{7}}\right)\right\}$$

Question 41

Evaluate $\int_0^1 \sqrt{x(1-x)}\,dx$

Solution 41

Let $I = \int_0^1 \sqrt{x(1-x)}\,dx$

let $x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta.\cos\theta d\theta$

Now,

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore I = \int_0^{\frac{\pi}{2}} \sqrt{\sin^2\theta\left(1-\sin^2\theta\right)}.2\sin\theta.\cos\theta d\theta$

$= \int_0^{\frac{\pi}{2}} 2\sin^2\theta.\cos^2\theta d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 4\sin^2\theta.\cos^2\theta d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\sin^2 2\theta\right) d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 4\theta}{2}\right) d\theta$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(1-\cos 4\theta\right) d\theta$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} d\theta - \frac{1}{4}\int_0^{\frac{\pi}{2}} \cos 4\theta d\theta$

$= \frac{1}{4}\left[\theta\right]_0^{\frac{\pi}{2}} - \frac{1}{4}\left[\frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{4}\left[\frac{\pi}{2} - 0\right] - \frac{1}{16}\left[\sin\pi - \sin 0\right]$

$= \frac{\pi}{8} - \frac{1}{16}\left[0 - 0\right]$

$= \frac{\pi}{8}$

$I = \frac{\pi}{8}$

$\therefore \int_0^1 \sqrt{x(1-x)}\,dx = \frac{\pi}{8}$

Question 42

Evaluate $\int_0^2 \frac{dx}{\sqrt{3+2x-x^2}}$

Solution 42

We have,

$$\int_0^2 \frac{dx}{\sqrt{3+2x-x^2}}$$

$$\int_0^2 \frac{dx}{\sqrt{3+1-\left(x^2-2x+1\right)}} \qquad \left[\text{Add and subtract 1 in denominator}\right]$$

$$= \int_0^2 \frac{dx}{\sqrt{(2)^2\,(x-1)^2}} \qquad \left[\because \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}\right]$$

$$= \left[\sin^{-1}\left(\frac{x-1}{2}\right)\right]_0^2$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(\frac{-1}{2}\right)$$

$$= \sin^{-1}\left(\sin\frac{\pi}{6}\right) - \sin^{-1}\left[\sin\left(\frac{-\pi}{6}\right)\right]$$

$$= \frac{\pi}{6} + \frac{\pi}{6}$$

$$= \frac{\pi}{3}$$

$$\therefore \int_0^2 \frac{dx}{\sqrt{3+2x-x^2}} = \frac{\pi}{3}$$

Question 43

Evaluate $\int_0^4 \frac{dx}{\sqrt{4x-x^2}}$

Solution 43

We have,

$$\int_0^4 \frac{dx}{\sqrt{4x - x^2}}$$

$$= \int_0^4 \frac{dx}{\sqrt{4 - 4 + 4x - x^2}} \qquad \left[\text{Add and subtract 4 in denominator}\right]$$

$$= \int_0^4 \frac{dx}{\sqrt{4 - \left(x^2 - 4x + 4\right)}}$$

$$= \int_0^4 \frac{dx}{\sqrt{\left(2\right)^2 - \left(x - 2\right)^2}}$$

$$= \left[\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^4 \qquad \left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}\right]$$

$$= \sin^{-1}\left(1\right) - \sin^{-1}\left(-1\right)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \frac{2\pi}{2} = \pi$$

$$\therefore \int_0^4 \frac{dx}{\sqrt{4x - x^2}} = \pi$$

Question 44

Evaluate the integral in using substitution.

$$\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5}$$

Solution 44

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}=\int_{-1}^{1}\frac{dx}{\left(x^2+2x+1\right)+4}=\int_{-1}^{1}\frac{dx}{(x+1)^2+(2)^2}$$

Let $x + 1 = t \Rightarrow dx = dt$

When $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

$$\therefore \int_{-1}^{1}\frac{dx}{(x+1)^2+(2)^2}=\int_{0}^{2}\frac{dt}{t^2+2^2}$$

$$=\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$=\frac{1}{2}\tan^{-1}1-\frac{1}{2}\tan^{-1}0$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{8}$$

Question 45

Evaluate $\int_{1}^{4}\frac{x^2+x}{\sqrt{2x+1}}dx$

Solution 45

We have,

$$\int_1^4 \frac{x^2+x}{\sqrt{2x+1}}\,dx$$

Let $2x+1=t^2$

$\Rightarrow 2dx = 2t\,dt$

Now,

$x=1 \Rightarrow t=\sqrt{3}$

$x=4 \Rightarrow t=3$

$$\therefore \int_1^4 \frac{x^2+x}{\sqrt{2x+1}}\,dx = \int_{\sqrt{3}}^{3} \frac{\left(\frac{t^2-1}{2}\right)^2+\left(\frac{t^2-1}{2}\right)}{t}\,t\,dt$$

$$= \frac{1}{4}\int_{\sqrt{3}}^{3}\left(t^4-2t^2+1+2t^2-2\right)dt$$

$$= \frac{1}{4}\int_{\sqrt{3}}^{3} t^4-1$$

$$= \frac{1}{4}\left[\frac{t^5}{5}-t\right]_{\sqrt{3}}^{3}$$

$$= \frac{1}{4}\left[\frac{243-9\sqrt{3}}{5}-3+\sqrt{3}\right]$$

$$= \frac{1}{4}\left[\frac{228}{5}-\sqrt{3}\,(4)\right]$$

$$= \frac{57-\sqrt{3}}{5}$$

$$\therefore \int_1^4 \frac{x^2+x}{\sqrt{2x+1}}\,dx = \frac{57-\sqrt{3}}{5}$$

Question 46

Evaluate $\int_0^1 x\,(1-x)^5\,dx$

Solution 46

We have,

$$\int_0^1 x\left(1-x\right)^5 dx$$

Expanding $\left(1-x\right)^5$ by Binomial theorem

$$\therefore \left(1-x\right)^5 = 1^5 + {}^5C_1\left(-x\right) + {}^5C_2\left(-x\right)^2 + {}^5C_3\left(-x\right)^3 + {}^5C_4\left(-x\right)^4 + {}^5C_5\left(-x\right)^5$$
$$= 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$$

$$= \int_0^1 x\left(1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5\right)dx$$

$$= \left[\frac{x^2}{2} - \frac{5x^3}{3} + \frac{10x^4}{4} - \frac{10x^5}{5} + \frac{5x^6}{6} - \frac{x^7}{7}\right]_0^1$$

$$= \frac{1}{2} - \frac{5}{3} + \frac{10}{4} - \frac{10}{5} + \frac{5}{6} - \frac{1}{7}$$

$$= \frac{1}{42}$$

$$\therefore \int_0^1 x\left(1-x\right)^5 dx = \frac{1}{42}$$

Question 47

Evaluate $\int_1^2 \left(\frac{x-1}{x^2}\right)e^x dx$

Solution 47

We have,

$$\int_1^2 \left(\frac{x-1}{x^2}\right)e^x dx = \int_1^2 \frac{xe^x}{x^2} - \int_1^2 \frac{e^x}{x^2}dx = \int_1^2 \frac{e^x dx}{x} - \int_1^2 \frac{e^x}{x^2}dx$$

Expanding 1^{st} integral by by parts we get

$$= \frac{1}{x}\int_1^2 e^x dx - \int_1^2\left(\int e^x . \frac{d\left(1/x\right)}{dx}dx\right) - \int_1^2 \frac{e^x}{x^2}$$

$$= \left[\frac{e^x}{x}\right]_1^2 + \int_1^2 \frac{e^x}{x^2}dx - \int_1^2 \frac{e^x}{x^2}$$

$$= \left[\frac{e^x}{x}\right]_1^2$$

$$= \frac{e^2}{2} - e$$

$$\therefore \int_1^2 \left(\frac{x-1}{x^2}\right)e^x dx = \frac{e^2}{2} - e$$

Question 48

Evaluate $\int_0^1 \left(xe^{2x} + \sin\frac{\pi x}{2}\right)dx$

Solution 48

We have,

$$\int_0^1 \left(xe^{2x} + \sin\frac{\pi x}{2}\right)dx = \int_0^1 \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{e^{2x}}\,dx + \int_0^1 \sin\frac{\pi x}{2}\,dx$$

Applying by parts in first integral

$$= x\int_0^1 e^{2x}dx - \int_0^1 \left(\int e^{2x}dx.\right)\frac{dx}{dx}dx + \left[\frac{-\cos\frac{\pi x}{2}}{\frac{\pi}{2}}\right]_0^1$$

$$= \frac{xe^{2x}}{2} - \frac{1}{2}\int_0^1 e^{2x}dx + \frac{2}{\pi}[1-0]$$

$$= \frac{xe^{2x}}{2} - \frac{1}{2}\int_0^1 e^{2x}dx + \frac{2}{\pi}[1-0]$$

$$= \left[\frac{xe^{2x}}{2} - \frac{1}{4}e^{2x}\right]_0^1 + \frac{2}{\pi}[1-0]$$

$$= \frac{e^2}{2} - \frac{1}{4}e^2 + \frac{1}{4} + \frac{2}{\pi}[1-0]$$

$$= \frac{e^2}{4} + \frac{2}{\pi} + \frac{1}{4}$$

$$= \frac{e^2}{4} + \frac{1}{4} + \frac{2}{\pi}$$

$$\therefore \int_0^1 \left(xe^{2x} + \sin\frac{\pi x}{2}\right)dx = \frac{e^2}{4} + \frac{1}{4} + \frac{2}{\pi}$$

Question 49

Evaluate $\int_0^1 \left(xe^{x} + \cos\frac{\pi x}{4}\right)dx$

Solution 49

We have,

$$\int_0^1 \left(xe^x + \cos\frac{\pi x}{4} \right) dx$$

$$= \int_0^1 \underset{\mathrm{I}}{x}\, \underset{\mathrm{II}}{e^x}\, dx + \int_0^1 \cos\frac{\pi x}{4} dx$$

Applying by by parts in 1st integral we get,

$$= x\int_0^1 e^x dx - \int_0^1 \left(\int e^x dx \right) \frac{dx}{dx} . dx + \int_0^1 \cos\frac{\pi x}{4} dx$$

$$= \left[xe^x \right]_0^1 - \int_0^1 e^x dx + \left[\frac{\sin\frac{\pi x}{4}}{\frac{\pi}{4}} \right]_0^1$$

$$= \left[xe^x - e^x \right]_0^1 + \frac{4}{\pi}\left[\frac{1}{\sqrt{2}} - 0 \right]$$

$$= \left[e^x (x-1) \right]_0^1 + \frac{4}{\pi}\left[\frac{1}{\sqrt{2}} \right]$$

$$= 0 + 1 + \frac{4}{\pi\sqrt{2}}$$

$$= 1 + \frac{2\sqrt{2}}{\pi}$$

$$\therefore \int_0^1 \left(xe^x + \cos\frac{\pi x}{4} \right) dx = 1 + \frac{2\sqrt{2}}{\pi}$$

Question 50

Evaluate the following definite integrals

$$\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x} \right) dx$$

Solution 50

$$\int_{\frac{\pi}{2}}^{\pi} e^x \frac{1-\sin x}{1-\cos x} dx = \int_{\frac{\pi}{2}}^{\pi} e^x \frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}} dx \qquad \left[1-\cos x = 2\sin^2\frac{x}{2} \right]$$

$$= -\int_{\frac{\pi}{2}}^{\pi} e^x \left(-\frac{1}{2}\csc^2\frac{x}{2} + \cot\frac{x}{2} \right) dx$$

$$= -e^x \cot\frac{x}{2} \Big|_{\frac{\pi}{2}}^{\pi} \qquad \left[\int e^x \left(F(x) + F'(x) \right) dx = e^x F(x) \right]$$

$$= e^{\frac{\pi}{2}}$$

Question 51

Evaluate $\int_0^{2\pi} e^{x/2} \sin\left(\frac{x}{2} + \frac{\pi}{4} \right) dx$

Solution 51

We have,

$$\int_0^{2\pi} e^{x/2}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)dx = \int_0^{2\pi} e^{x/2}\left(\sin\frac{x}{2}\cos\frac{\pi}{4}+\cos\frac{x}{2}\sin\frac{\pi}{4}\right)dx$$

$$= \int_0^{2\pi} e^{x/2}\sin\frac{x}{2}\cdot\frac{1}{\sqrt{2}}dx + \int_0^{2\pi} e^{x/2}\cos\frac{x}{2}\cdot\frac{1}{\sqrt{2}}dx$$

Expanding 1st part by parts, we get,

$$\int_0^{2\pi} e^{x/2}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)dx = \frac{1}{\sqrt{2}}\left\{\sin\frac{x}{2}\int_0^{2\pi} e^{x/2}dx - \int_0^{2\pi}\left(\int_0^{2\pi} e^{x/2}dx\right)\cdot\frac{d\left(\sin\frac{x}{2}\right)}{dx}dx\right\}+\frac{1}{\sqrt{2}}\int_0^{2\pi} e^{x/2}.\cos\frac{x}{2}dx$$

$$= \frac{1}{\sqrt{2}}\left\{\sin\frac{x}{2}.2e^{x/2}\right\}_0^{2\pi} - \frac{1}{\sqrt{2}}\int_0^{2\pi} e^{x/2}2.\frac{1}{2}\cos\frac{x}{2}dx + \frac{1}{\sqrt{2}}\int_0^{2\pi} e^{x/2}\cos\frac{x}{2}dx$$

$$= \frac{1}{\sqrt{2}}\left\{\sin\frac{x}{2}.2e^{x/2}\right\}_0^{2\pi} = \frac{1}{\sqrt{2}}\{0-0\} = 0$$

$$\therefore \int_0^{2\pi} e^{x/2}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)dx = 0$$

Question 52

Evaluate $\int_0^{2\pi} e^x\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$

Solution 52

Let $I = \int_0^{2\pi} e^x\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)dx = \left[\cos\left(\frac{\pi}{4}+\frac{x}{2}\right).e^x\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} e^x\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$

$$\Rightarrow I = \left[\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)e^x\right]_0^{2\pi} + \frac{1}{2}\left[\left\{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right).e^x\right\}_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} e^x\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]dx$$

$$I = \left[\cos\left(\pi+\frac{\pi}{4}\right)e^{2\pi} - \cos\frac{\pi}{4}\right] + \frac{1}{2}\left[\sin\left(\pi+\frac{\pi}{4}\right).e^{2\pi} - \sin\frac{\pi}{4} - \frac{1}{2}I\right]$$

$$I = \left[-\cos\frac{\pi}{4}.e^{2\pi} - \cos\frac{\pi}{4}\right] + \frac{1}{2}\left[-\sin\frac{\pi}{4}.e^{2\pi} - \sin\frac{\pi}{4}\right] - \frac{I}{4}$$

$$\frac{5I}{4} = -\frac{1}{\sqrt{2}}\left(e^{2\pi}+1\right) - \frac{1}{2}.\frac{1}{\sqrt{2}}\left(e^{2\pi}+1\right) = \frac{-3}{2\sqrt{2}}\left(e^{2\pi}+1\right)$$

$$I = \frac{-3\sqrt{2}}{5}\left(e^{2\pi}+1\right)$$

$$\therefore \int_0^{2\pi} e^x\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)dx = \frac{-3\sqrt{2}}{5}\left(e^{2\pi}+1\right)$$

Question 53

Evaluate the definite integral

$$\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Solution 53

$$\text{Let } I = \int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$$

$$I = \int_0^1 \frac{1}{\left(\sqrt{1+x} - \sqrt{x}\right)} \times \frac{\left(\sqrt{1+x} + \sqrt{x}\right)}{\left(\sqrt{1+x} + \sqrt{x}\right)} dx$$

$$= \int_0^1 \frac{\sqrt{1+x} + \sqrt{x}}{1+x-x} dx$$

$$= \int_0^1 \sqrt{1+x}\, dx + \int_0^1 \sqrt{x}\, dx$$

$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}\left[(2)^{\frac{3}{2}} - 1\right] + \frac{2}{3}[1]$$

$$= \frac{2}{3}(2)^{\frac{3}{2}}$$

$$= \frac{2 \cdot 2\sqrt{2}}{3}$$

$$= \frac{4\sqrt{2}}{3}$$

$$\textit{Therefore, } I = \frac{2^{\frac{5}{2}}}{3}$$

Question 54

Evaluate the following definite integrals

$$\int_1^2 \frac{x}{(x+1)(x+2)} dx$$

Solution 54

$$\int_1^2 \frac{x}{(x+1)(x+2)}dx = -\int_1^2 \frac{1}{x+1}dx + \int_1^2 \frac{2}{x+2}dx \qquad \text{[Using Partial Fraction]}$$

$$= -\log(x+1)\Big|_1^2 + 2\log(x+2)\Big|_1^2$$

$$= -(\log 3 - \log 2) + 2(\log 4 - \log 3)$$

$$= -3\log 3 + 5\log 2$$

$$= \log\frac{32}{27}$$

Question 55

$$\int_0^{\pi/2} \sin^3 x \, dx$$

Solution 55

$$\int_0^{\frac{\pi}{2}} (\sin^2 x)\sin x dx$$

$$\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x dx$$

Let cosx =u , Then

$$\frac{du}{dx} = -\sin x$$

$$-du = \sin x dx$$

Hence

$$\Rightarrow \int_1^0 (u^2-1)du$$

$$\Rightarrow (\frac{u^3}{3}-u)_1^0$$

$$\Rightarrow (0)-(\frac{1}{3}-1)$$

$$\Rightarrow \frac{2}{3}$$

Question 56

Evaluate the definite integral

$$\int_0^{\pi}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx$$

Solution 56

$$\text{Let } I = \int_0^{\pi}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx$$

$$= -\int_0^{\pi}\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)dx$$

$$= -\int_0^{\pi}\cos x\ dx$$

$$\int \cos x\ dx = \sin x = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \mathrm{F}(\pi)-\mathrm{F}(0)$$

$$= \sin\pi - \sin 0$$

$$= 0$$

Question 57

Evaluate the integral in using substitution.

$$\int_1^2 \left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Solution 57

$$\int_1^2 \left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Let $2x = t \Rightarrow 2dx = dt$

When $x = 1$, $t = 2$ and when $x = 2$, $t = 4$

$$\therefore \int_1^2 \left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx = \frac{1}{2}\int_2^4 \left(\frac{2}{t}-\frac{2}{t^2}\right)e^{t}dt$$

$$= \int_2^4 \left(\frac{1}{t}-\frac{1}{t^2}\right)e^{t}dt$$

Let $\frac{1}{t} = f(t)$

Then, $f'(t) = -\frac{1}{t^2}$

$$\Rightarrow \int_2^4 \left(\frac{1}{t}-\frac{1}{t^2}\right)e^{t}dt = \int_2^4 e^t\left[f(t)+f'(t)\right]dt$$

$$= \left[e^t f(t)\right]_2^4$$

$$= \left[e^t \cdot \frac{2}{t}\right]_2^4$$

$$= \left[\frac{e^t}{t}\right]_2^4$$

$$= \frac{e^4}{4}-\frac{e^2}{2}$$

$$= \frac{e^4-2e^2}{4}$$

Question 58

$$\int_1^2 \frac{1}{\sqrt{(x-1)(2-x)}}dx$$

Solution 58

$$\int_1^2 \frac{1}{\sqrt{(x-1)(2-x)}}\,dx$$

$$= \int_1^2 \frac{1}{\sqrt{-\left(x-\frac{3}{2}\right)^2+\left(\frac{1}{4}\right)}}\,dx$$

$$= \int_1^2 \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}\,dx$$

$$= \left[\sin^{-1}(2x-3)\right]_1^2$$

$$= \sin^{-1}(1)-\sin^{-1}(-1)$$

$$= \pi$$

Question 59

If $\int_0^k \frac{1}{2+8x^2}\,dx = \frac{\pi}{16}$, find the value of k

Solution 59

$$\int_0^k \frac{1}{2(1+4x^2)}\,dx$$

$$\int_0^k \frac{1}{2(1+(2x)^2)}\,dx$$

$$\left(\frac{\tan^{-1}(2x)}{4}\right)_0^k$$

$$\left(\frac{\tan^{-1}(2k)}{4}-0\right)$$

$$\frac{\tan^{-1}(2k)}{4}$$

Given :

$$\frac{\tan^{-1}(2k)}{4} = \frac{\Pi}{16}$$

$$\tan^{-1}(2k) = \frac{\Pi}{4}$$

$$2k = \tan\frac{\Pi}{4}$$

$$2k = 1$$

$$k = \frac{1}{2}$$

Question 60

If $\int_0^a 3x^2 dx = 8$, find the value of a.

Solution 60

We have,

$$\int_0^a 3x^2 dx = 8$$

$$\Rightarrow \left[x^3\right]_0^a = 8$$

$$\Rightarrow a^3 = 8$$

$$\Rightarrow a = 2$$

Question 61

$$\int_{\pi}^{3\pi/2} \sqrt{1 - \cos 2x}\, dx$$

Solution 61

$$\int_{\pi}^{\frac{3\pi}{2}} \sqrt{1-(1-2\sin^2 x)}dx$$

$$\int_{\pi}^{\frac{3\pi}{2}} \sqrt{2\sin^2 x}dx$$

$$\sqrt{2}\int_{\pi}^{\frac{3\pi}{2}} \sin x dx$$

$$\sqrt{2}(-\cos x)_{\pi}^{\frac{3\pi}{2}}$$

$$= \sqrt{2}$$

Question 62

$$\int_0^{2\pi} \sqrt{1+\sin\frac{x}{2}}dx$$

Solution 62

$$I = \int_0^{2\pi} \sqrt{1+\sin\frac{x}{2}}dx$$

$$\Rightarrow I = \int_0^{2\pi} \sqrt{\sin^2\frac{x}{4}+\cos^2\frac{x}{4}+2\sin\frac{x}{4}\cos\frac{x}{4}}dx$$

$$\Rightarrow I = \int_0^{2\pi} \sqrt{\left(\sin\frac{x}{4}+\cos\frac{x}{4}\right)^2}dx$$

$$\Rightarrow I = \int_0^{2\pi} \left(\sin\frac{x}{4}+\cos\frac{x}{4}\right)dx$$

$$\Rightarrow I = \left[\frac{-\cos\frac{x}{4}}{\frac{1}{4}}+\frac{\sin\frac{x}{4}}{\frac{1}{4}}\right]_0^{2\pi}$$

$$\Rightarrow I = 4(0+1+1-0)$$

$$\Rightarrow I = 8$$

Question 63

$$\int_0^{\pi/4} (\tan x+\cot x)^{-2}dx$$

Solution 63

$$I = \int_0^{\pi/4} (\tan x + \cot x)^{-2}\,dx$$

$$I = \int_0^{\pi/4} \frac{1}{(\tan x + \cot x)^2}\,dx$$

$$I = \int_0^{\pi/4} \frac{1}{\left(\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right)^2}\,dx$$

$$I = \int_0^{\pi/4} (\sin x \cos x)^2\,dx$$

$$I = \int_0^{\pi/4} \sin^2 x(1 - \sin^2 x)\,dx$$

$$I = \int_0^{\pi/4} \sin^2 x\,dx - \int_0^{\pi/4} \sin^4 x\,dx$$

We know that by reduction formula,

$$\int \sin^n x\ dx = \frac{n-1}{n}\int \sin^{n-2} x\ dx - \frac{\cos x \sin^{n-1} x}{n}$$

For $n = 2$

$$\int \sin^2 x\ dx = \frac{2-1}{2}\int 1\ dx - \frac{\cos x \sin x}{2}$$

$$\int \sin^2 x\ dx = \frac{1}{2}x - \frac{\cos x \sin x}{2}$$

For $n = 4$

$$\int \sin^4 x\ dx = \frac{4-1}{4}\int \sin^2 x\ dx - \frac{\cos x \sin^3 x}{4}$$

$$\int \sin^4 x\ dx = \frac{3}{4}\left\{\frac{1}{2}x - \frac{\cos x \sin x}{2}\right\} - \frac{\cos x \sin^3 x}{4}$$

Hence,

$$I = \left\{\frac{1}{2}x - \frac{\cos x \sin x}{2}\right\}_0^{\pi/4} - \left\{\frac{3}{4}\left\{\frac{1}{2}x - \frac{\cos x \sin x}{2}\right\} - \frac{\cos x \sin^3 x}{4}\right\}_0^{\pi/4}$$

$$= \left\{\frac{\pi}{8} - \frac{1}{4}\right\} - \left\{\frac{3}{4}\left(\frac{\pi}{8} - \frac{1}{4}\right) - \frac{1}{16}\right\}$$

$$= \frac{\pi}{32}$$

$$\int_0^{\frac{\pi}{2}} (\sin x \cos x)^2 dx$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x(1-\sin^2 x)dx$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x - \sin^4 x dx$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx - \int_0^{\frac{\pi}{2}} \sin^4 x dx$$

We know , By reduction formula

$$\int \sin^n x dx = \frac{n-1}{n}\int \sin^{n-2} x dx - \frac{\cos x \sin^{n-1} x}{n}$$

For n=2

$$\int \sin^2 x dx = \frac{2-1}{2}\int 1 dx - \frac{\cos x \sin x}{2}$$

$$\int \sin^2 x dx = \frac{1}{2}x - \frac{\cos x \sin x}{2}$$

For n=4

$$\int \sin^4 x dx = \frac{4-1}{4}\int \sin^2 x dx - \frac{\cos x \sin^3 x}{4}$$

$$\int \sin^4 x dx = \frac{3}{4}\left\{\frac{1}{2}x - \frac{\cos x \sin x}{2}\right\} - \frac{\cos x \sin^3 x}{4}$$

Hence

$$\left\{\frac{1}{2}x - \frac{\cos x \sin x}{2}\right\}_0^{\frac{\pi}{2}} - \left\{\frac{3}{4}\left\{\frac{1}{2}x - \frac{\cos x \sin x}{2}\right\} - \frac{\cos x \sin^3 x}{4}\right\}_0^{\frac{\pi}{2}}$$

$$\frac{\Pi}{4} - \frac{3}{4}\left\{\frac{\Pi}{4}\right\}$$

$$\frac{\Pi}{16}$$

Note: Answer given at back is incorrect.

Question 64

$$\int_0^1 x \log(1 + 2x)\, dx$$

Solution 64

Using Integration By parts

$$\int f'g = fg - \int fg'$$

$$f' = x, g = \log(2x+1)$$

$$f = \frac{x^2}{2}, g' = \frac{2}{2x+1}$$

$$\int_0^1 x\log(1+2x)\,dx$$
$$= \left[\frac{x^2\log(1+2x)}{2}\right]_0^1 - \int_0^1 \frac{2x^2}{2(2x+1)}dx$$
$$= \frac{\log(3)}{2} - \int_0^1 \frac{x}{2} - \frac{1}{4} + \frac{1}{4(2x+1)}dx$$
$$= \frac{\log(3)}{2} - \left[\frac{x^2}{4} - \frac{x}{4} + \frac{1}{8}\log|2x+1|\right]_0^1$$
$$= \frac{\log(3)}{2} - \frac{1}{8}\log(3)$$
$$= \frac{3}{8}\log_e(3)$$

Question 65

$$\int_{\pi/6}^{\pi/3} (\tan x + \cot x)^2\,dx$$

Solution 65

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\tan^2 x + 2\tan x\cot x + \cot^2 x)dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left\{\left(\sec^2 x - 1\right) + 2 + \left(\operatorname{cosec}^2 x - 1\right)\right\} dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left\{\sec^2 x + \operatorname{cosec}^2 x\right\} dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec}^2 x dx$$

$$\left\{\tan x\right\}_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \left\{-\cot x\right\}_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$\left\{\sqrt{3} - \frac{1}{\sqrt{3}}\right\} - \left\{\frac{1}{\sqrt{3}} - \sqrt{3}\right\}$$

$$2\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)$$

$$\frac{4}{\sqrt{3}}$$

Question 66

$$\int_0^{\pi/4}\left(a^2\cos^2 x + b^2\sin^2 x\right)dx$$

Solution 66

$$I = \int_0^{\pi/4}\left(a^2\cos^2 x + b^2\sin^2 x\right)dx$$

$$I = \int_0^{\pi/4}\left(a^2\left(1-\sin^2 x\right) + b^2\sin^2 x\right)dx$$

$$I = \int_0^{\pi/4}\left(a^2 - a^2\sin^2 x + b^2\sin^2 x\right)dx$$

$$I = \int_0^{\pi/4} a^2 + \left(b^2 - a^2\right)\sin^2 x\,dx$$

$$I = \int_0^{\pi/4} a^2 + \left(b^2 - a^2\right)\frac{\left(1+\cos 2x\right)}{2}dx$$

$$I = \left[a^2 x + \frac{\left(b^2-a^2\right)}{2}\left(x + \frac{\sin 2x}{2}\right)\right]_0^{\pi/4}$$

$$I = \left[\frac{a^2\pi}{4} + \frac{\left(b^2-a^2\right)}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right)\right]$$

$$I = \frac{\left(b^2+a^2\right)\pi}{8} + \frac{\left(b^2-a^2\right)}{4}$$

Note: Answer given in the book is incorrect.

Question 67

$$\int_0^1 \frac{1}{1+2x+2x^2+2x^3+x^4}dx$$

Solution 67

$$\int_0^1 \frac{1}{x^4+2x^3+2x^2+2x+1}dx$$

$$\int_0^1 \frac{1}{(x+1)^2(x^2+1)}dx$$

$$\int_0^1 \left\{-\frac{x}{2(x^2+1)}+\frac{1}{2(x+1)}+\frac{1}{2(x+1)^2}\right\}dx$$

$$-\int_0^1 \frac{x}{2(x^2+1)}dx+\int_0^1 \frac{1}{2(x+1)}dx+\int_0^1 \frac{1}{2(x+1)^2}dx$$

$$-\left\{\frac{\log(x^2+1)}{4}\right\}_0^1+\left\{\frac{\log(x+1)}{2}\right\}_0^1-\left\{\frac{1}{2(x+1)}\right\}_0^1$$

$$-\frac{\log 2}{4}+\frac{\log 2}{2}-\frac{1}{4}+\frac{1}{2}$$

$$\frac{\log 2}{4}+\frac{1}{4}$$

=(1/4)log(2e)

Chapter 20 - Definite Integrals Exercise Ex. 20.2

Question 1

Evaluate the definite integral

$$\int_2^4 \frac{x}{x^2+1}$$

Solution 1

Let $I=\int_2^4 \frac{x}{x^2+1}dx$

$\int \frac{x}{x^2+1}dx=\frac{1}{2}\int \frac{2x}{x^2+1}dx=\frac{1}{2}\log\left(1+x^2\right)=F(x)$

By the second funamental theorem of calculus, we obtain

$I=F(4)-F(2)$

$=\frac{1}{2}\left[\log\left(1+4^2\right)-\log\left(1+2^2\right)\right]$

$=\frac{1}{2}[\log 17-\log 5]$

$=\frac{1}{2}\log\left(\frac{17}{5}\right)$

Question 2

Evaluate $\int_1^2 \frac{1}{x\left(1+\log x\right)^2}dx$

Solution 2

Let $1+\log x = t$

Differentiating w.r.t. x, we get

$$\frac{1}{x}dx = dt$$

Now, $x = 1 \Rightarrow t = 1$

$x = 2 \Rightarrow t = 1+\log 2$

$$\therefore \int_1^2 \frac{1}{x\left(1+\log x\right)^2}dx = \int_1^{1+\log 2}\frac{dt}{t^2}$$

$$= \left[\frac{-1}{t}\right]_1^{1+\log 2}$$

$$= \left[\frac{-1}{1+\log 2}+1\right]$$

$$= \left[\frac{-1+1+\log 2}{1+\log 2}\right]$$

$$= \left[\frac{\log 2}{1+\log 2}\right] \qquad \left[\because \log e = 1\right]$$

$$= \frac{\log 2}{\log e+\log 2} \qquad \left[\log a+\log b = \log ab\right]$$

$$= \frac{\log 2}{\log 2e}$$

$$\therefore \int_1^2 \frac{1}{x\left(1+\log x\right)^2}dx = \frac{\log 2}{\log 2e}$$

Question 3

Evaluate $\int_1^2 \frac{3x}{9x^2-1}dx$

Solution 3

Let $9x^2 - 1 = t$

Differentiating w.r.t. x, we get

$$18x\,dx = dt$$

$$3x\,dx = \frac{dt}{6}$$

Now, $x = 1 \Rightarrow t = 8$

$x = 2 \Rightarrow t = 35$

$$\therefore \int_1^2 \frac{3x}{9x^2 - 1}dx = \int_8^{35} \frac{dt}{6t}$$

$$= \frac{1}{6}\left[\log t\right]_8^{35}$$

$$= \frac{1}{6}\left(\log 35 - \log 8\right)$$

$$\therefore \int_1^2 \frac{3x}{9x^2 - 1}dx = \frac{1}{6}\left(\log 35 - \log 8\right)$$

Question 4

Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{5\cos x + 3\sin x}$

Solution 4

Put $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{5\cos x + 3\sin x} = \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}dx}{5\left(1-\tan^2\frac{x}{2}\right)+6\tan\frac{x}{2}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}dx}{5-5\tan^2\frac{x}{2}+6\tan\frac{x}{2}}$$

Let $\tan\frac{x}{2} = t$

Differentiating w.r.t. x, we get

$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$

Now, $x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}dx}{5-5\tan^2\frac{x}{2}+6\tan\frac{x}{2}} = \int_0^1 \frac{2dt}{5-5t^2+6t} = \frac{2}{5}\int \frac{dt}{1-t^2+\frac{6}{5}t}$$

Forming perfect square by adding and subtracting $\frac{9}{25}$

$$\frac{2}{5}\int_0^1 \frac{dt}{1-t^2+\frac{6}{5}t}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{\frac{34}{25}-\left(t-\frac{3}{5}\right)^2}$$

$$= \frac{2}{5}\cdot\frac{1}{2}\sqrt{\frac{25}{34}}\log\left(\frac{\sqrt{\frac{34}{25}}+t-\frac{3}{5}}{\sqrt{\frac{34}{25}}-t+\frac{3}{5}}\right)_0^1 \qquad \left[\because \int\frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{x+a}{x-a}\right)\right]$$

$$= \frac{1}{\sqrt{34}}\left\{\log\left(\frac{\sqrt{34}+2}{\sqrt{34}-2}\right) - \log\left(\frac{\sqrt{34}-3}{\sqrt{34}+3}\right)\right\}$$

$$= \frac{1}{\sqrt{34}}\log\left(\frac{\left(\sqrt{34}+2\right)\left(\sqrt{34}-3\right)}{\left(\sqrt{34}-2\right)\left(\sqrt{34}-3\right)}\right)$$

$$= \frac{1}{\sqrt{34}}\log\left(\frac{40+5\sqrt{34}}{40-5\sqrt{34}}\right)$$

$$= \frac{1}{\sqrt{34}}\log\left(\frac{8+\sqrt{34}}{8-\sqrt{34}}\right)$$

Question 5

Evaluate $\int_0^a \frac{x}{\sqrt{a^2+x^2}}dx$

Solution 5

Let $a^2 + x^2 = t^2$

Differentiating w.r.t. x, we get

$$2x\,dx = 2t\,dt$$
$$x\,dx = t\,dt$$

Now, $x = 0 \Rightarrow t = 0$

$$x = a \Rightarrow t = \sqrt{2}a$$

$$\therefore \int_0^a \frac{x\,dx}{\sqrt{a^2+x^2}}dx = \int_a^{\sqrt{2}a} \frac{t\,dt}{t}$$
$$= \int_a^{\sqrt{2}a} dt$$
$$= \left[t\right]_a^{\sqrt{2}a}$$
$$= \left[\sqrt{2}a - a\right]$$
$$= a\left(\sqrt{2}-1\right)$$

$$\therefore \int_0^a \frac{x}{\sqrt{a^2+x^2}}dx = a\left(\sqrt{2}-1\right)$$

Question 6

Evaluate $\int_0^1 \frac{e^x}{1+e^{2x}}dx$

Solution 6

Let $e^x = t$

Differentiating w.r.t. x, we get

$e^x\,dx = dt$

Now, $x = 0 \Rightarrow t = 1$

$x = 1 \Rightarrow t = e$

$$\therefore \int_0^1 \frac{e^x}{1+e^{2x}}dx = \int_1^e \frac{dt}{1+t^2}$$

$$= \left[\tan^{-1} t\right]_1^e \qquad \left[\because \int \frac{dt}{1+t^2} = \tan^{-1} t\right]$$

$$= \left[\tan^{-1} e - \tan^{-1} 1\right] \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$= \tan^{-1} e - \frac{\pi}{4}$$

$$\therefore \int_0^1 \frac{e^x}{1+e^{2x}}dx = \tan^{-1} e - \frac{\pi}{4}$$

Question 7

Evaluate $\int_0^1 x e^{x^2} dx$

Solution 7

Let $x^2 = t$

Differentiating w.r.t. x, we get

$2x\,dx = dt$

Now,

$x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$$\therefore \int_0^1 x e^{x^2} dx = \int_0^1 \frac{e^t dt}{2}$$

$$= \frac{1}{2}\int_0^1 e^t dt$$

$$= \frac{1}{2}\left[e^t\right]_0^1$$

$$= \frac{1}{2}\left[e^1 - e^0\right] \qquad \left[\because e^0 = 1\right]$$

$$= \frac{1}{2}(e-1)$$

$$\therefore \int_0^1 x e^{x^2} dx = \frac{1}{2}(e-1)$$

Question 8

Evaluate $\int_1^3 \frac{\cos(\log x)}{x} dx$

Solution 8

Let $\log x = t$

Differentiating w.r.t. x, we get

$$\frac{1}{x} dx = dt$$

Now,

$x = 0 \Rightarrow t = 0$

$x = 3 \Rightarrow t = \log 3$

$$\int_1^3 \frac{\cos(\log x)}{x} dx$$

$$= \int_0^{\log 3} \cos t \, dt \qquad \left[\because \int \cos t = \sin t\right]$$

$$= [\sin t]_0^{\log 3}$$

$$= \sin(\log 3) - \sin 0$$

$$= \sin(\log 3)$$

$$\int_1^3 \frac{\cos(\log x)}{x} dx = \sin(\log 3)$$

Question 9

Evaluate $\int_0^1 \frac{2x}{1+x^4} dx$

Solution 9

Let $x^2 = t$

Differentiating w.r.t. x, we get

$$2x\,dx = dt$$

Now,

$x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$\therefore \int_0^1 \frac{2x}{1+x^4}\,dx$

$= \int_0^1 \frac{dt}{1+t^2}$

$= \left[\tan^{-1} t\right]_0^1$

$= \left[\tan^{-1} 1 - \tan^{-1} 0\right] \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$

$= \frac{\pi}{4}$

$\therefore \int_0^1 \frac{2x}{1+x^4}\,dx = \frac{\pi}{4}$

Question 10

Evaluate $\int_0^a \sqrt{a^2 - x^2}\,dx$

Solution 10

Let $x = a\sin\theta$

Differentiating w.r.t. x, we get

$$dx = a\cos\theta d\theta$$

Now,

$x = 0 \Rightarrow \theta = 0$

$x = a \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2\left(1 - \sin^2\theta\right)}\, a\cos\theta d\theta$$

$$= a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \qquad \left[\because \left(1 - \sin^2\theta\right) = \cos^2\theta \text{ and } \frac{1 + \cos 2\theta}{2} = \cos 2\theta\right]$$

$$= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} \left(1 + \cos 2\theta\right) d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} + 0 - 0 - 0\right]$$

$$= \frac{\pi a^2}{4}$$

$$\therefore \int_0^a \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{4}$$

Question 11

Evaluate the integral in using substitution.

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\, \cos^5\phi\, d\phi$$

Solution 11

Let $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\cos\phi\, d\phi$

Also, let $\sin\phi = t \Rightarrow \cos\phi\, d\phi = dt$

When $\phi = 0$, $t = 0$ and when $\phi = \frac{\pi}{2}$, $t = 1$

$$\therefore I = \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$

$$= \int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$$

$$= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}}\right]dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154+42-132}{231}$$

$$= \frac{64}{231}$$

Question 12

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\, dx$

Solution 12

Let $\sin x = t$

Differentiating w.r.t. x, we get

$$\cos x \, dx = dt$$

Now,

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x} dx$$

$$= \int_0^1 \frac{dt}{1+t^2}$$

$$= \left[\tan^{-1} t\right]_0^1$$

$$= \left[\tan^{-1} 1 - \tan^{-1} 0\right] \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x} dx = \frac{\pi}{4}$$

Question 13

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin\theta \, d\theta}{\sqrt{1+\cos\theta}}$

Solution 13

Let $1 + \cos\theta = t^2$

Differentiating w.r.t. x, we get

$$-\sin\theta\, d\theta = 2t\, dt$$

$$\sin\theta\, d\theta = -2t\, dt$$

Now,

$x = 0 \Rightarrow t = \sqrt{2}$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta}{\sqrt{1+\cos\theta}}$$

$$= \int_{\sqrt{2}}^{1} \frac{-2t dt}{t}$$

$$= -2 \int_{\sqrt{2}}^{1} dt$$

$$= -2[t]_{\sqrt{2}}^{1}$$

$$= -2\left[1 - \sqrt{2}\right]$$

$$= 2\left[\sqrt{2} - 1\right]$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta}{\sqrt{1+\cos\theta}} = 2\left[\sqrt{2} - 1\right]$$

Question 14

Evaluate $\int_0^{\frac{\pi}{3}} \frac{\cos x}{3 + 4\sin x} dx$

Solution 14

Let $3+4\sin x = t$

Differentiating w.r.t. x, we get

$$4\cos x dx = dt$$

$$\cos x dx = \frac{dt}{4}$$

Now,

$x = 0 \Rightarrow t = 3$

$x = \frac{\pi}{3} \Rightarrow t = 3+2\sqrt{3}$

$$\therefore \int_0^{\frac{\pi}{3}} \frac{\cos x}{3+4\sin x}dx$$

$$= \int_3^{3+2\sqrt{3}} \frac{dt}{4t}$$

$$= \frac{1}{4}\left[\log t\right]_3^{3+2\sqrt{3}}$$

$$= \frac{1}{4}\left[\log\left(3+2\sqrt{3}\right) - \log 3\right]$$

$$= \frac{1}{4}\log\left(\frac{3+2\sqrt{3}}{3}\right)$$

$$\therefore \int_0^{\frac{\pi}{3}} \frac{\cos x}{3+4\sin x}dx = \frac{1}{4}\log\left(\frac{3+2\sqrt{3}}{3}\right)$$

Question 15

Evaluate $\int_0^1 \frac{\sqrt{\tan^{-1} x}}{1+x^2}dx$

Solution 15

Let $\tan^{-1} x = t$

Differentiating w.r.t. x, we get

$$\frac{1}{1+x^2} dx = dt$$

Now,

$x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = \frac{\pi}{4}$

$\therefore \int_0^1 \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx$

$= \int_0^{\frac{\pi}{4}} t^{1/2} dt$

$= \left[\frac{t^{3/2}}{3/2}\right]_0^{\frac{\pi}{4}}$

$= \frac{2}{3}\left[t^{3/2}\right]_0^{\frac{\pi}{4}}$

$= \frac{2}{3}\left[\left(\frac{\pi}{4}\right)^{3/2} - 0\right]$

$= \frac{1}{12}\pi^{3/2}$

$\therefore \int_0^1 \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx = \frac{1}{12}\pi^{3/2}$

Question 16

Evaluate the integral in using substitution.

$$\int_0^2 x\sqrt{x+2} \quad \left(\text{Put } x+2 = t^2\right)$$

Solution 16

$$\int_0^2 x\sqrt{x+2}\,dx$$

Let $x + 2 = t^2 \Rightarrow dx = 2tdt$

When $x = 0$, $t = \sqrt{2}$ and when $x = 2$, $t = 2$

$$\begin{aligned}
\therefore \int_0^2 x\sqrt{x+2}\,dx &= \int_{\sqrt{2}}^2 \left(t^2-2\right)\sqrt{t^2}\,2tdt \\
&= 2\int_{\sqrt{2}}^2 \left(t^2-2\right)t^2 dt \\
&= 2\int_{\sqrt{2}}^2 \left(t^4-2t^2\right)dt \\
&= 2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2 \\
&= 2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right] \\
&= 2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right] \\
&= 2\left[\frac{16+8\sqrt{2}}{15}\right] \\
&= \frac{16\left(2+\sqrt{2}\right)}{15} \\
&= \frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}
\end{aligned}$$

Question 17

Evaluate $\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

Solution 17

Let $x = \tan\theta$

Differentiating w.r.t. x, we get

$$dx = \sec^2\theta d\theta$$

Now,

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\sec^2\theta d\theta \qquad \left[\because \tan^2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right]$$

$$= \int_0^{\frac{\pi}{4}} \tan^{-1}(\tan 2\theta)\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta\sec^2\theta d\theta$$

Applying by parts, we get

$$= 2\left[\theta \int_0^{\frac{\pi}{4}} \sec^2\theta d\theta - \int_0^{\frac{\pi}{4}}\left(\sec^2\theta d\theta\right)\frac{d\theta}{d\theta}d\theta\right]$$

$$=2\left[\theta\tan\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan\theta d\theta]$$

$$= 2\left[\theta\tan\theta + \log(\cos\theta)\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - 0 - 0\right]$$

$$= 2\left[\frac{\pi}{4} + \frac{1}{2}\log 2\right]$$

$$= \frac{\pi}{2} - \log 2$$

$$\therefore \int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx = \frac{\pi}{2} - \log 2$$

Question 18

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{1+\sin^4 x} dx$

Solution 18

Let $\sin^2 x = t$

Differentiating w.r.t. x, we get

$$2\sin x \cos x dx = dt$$

Now,

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{1+\sin^4 x} dx$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{1+t^2}$$

$$= \frac{1}{2}\left[\tan^{-1} t\right]_0^1$$

$$= \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\tan\frac{\pi}{4}\right) - \tan^{-1}(\tan 0)\right]$$

$$= \frac{1}{2} \times \frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{1+\sin^4 x} dx = \frac{\pi}{8}$$

Question 19

Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a\cos x + b\sin x} \qquad a, b > 0$$

Solution 19

Putting $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{1-\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}}$

$$\sin x = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$\therefore \qquad I = \int_0^{\frac{\pi}{2}} \frac{1}{a\cos x + b\sin x}dx = \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{a\left(1-\tan^2\frac{x}{2}\right)+2b\tan^2\frac{x}{2}}dx$$

Put $\tan\frac{x}{2} = t$

$$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

If $x = 0$, $t = 0$ and if $x = \frac{\pi}{2}$, $t = 1$

$$\Rightarrow \qquad I = 2\int_0^1 \frac{dt}{a\left(1-t^2\right)+2bt}$$

$$= 2\int_0^1 \frac{dt}{-at^2+2bt+a}$$

$$= 2\int_0^1 \frac{dt}{-a\left[t^2-\frac{2b}{a}t-1\right]}$$

$$= \frac{2}{a}\int_0^1 \frac{dt}{-\left[\left(t-\frac{b}{a}\right)^2-1-\frac{b^2}{a^2}\right]}$$

$$= \frac{2}{a}\int_0^1 \frac{dt}{\left(\frac{b^2}{a^2}+1\right)-\left(t-\frac{b}{a}\right)^2}$$

$$= \frac{2}{a}\left[\frac{1}{2\sqrt{\frac{b^2+a^2}{a^2}}}\left(\log\left|\frac{\frac{\sqrt{b^2+a^2}}{a^2}+\left(t-\frac{b}{a}\right)}{\frac{\sqrt{b^2+a^2}}{a^2}-\left(t-\frac{b}{a}\right)}\right|\right)\right]_0^1$$

$$= \frac{1}{\sqrt{b^2+a^2}}\log\left(\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}}\right)$$

Question 20

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{5+4\sin x}dx$

Solution 20

We know that $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{5+4\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{5+4\sin\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\dfrac{5\left(1+\tan^2\frac{x}{2}\right)+4\left(2\tan\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1+\tan^2\frac{x}{2}}{\left(5+5\tan^2\frac{x}{2}+8\tan\frac{x}{2}\right)}dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{5+5\tan^2\frac{x}{2}+8\tan\frac{x}{2}}dx$$

Let $\tan\frac{x}{2} = t$

Differentiating w.r.t. x, we get

$$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

Now,

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{5+5\tan^2\frac{x}{2}+8\tan\frac{x}{2}}dx$$

$$= \int_0^1 \frac{2dt}{5 + 5t^2 + 8t}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{1 + t^2 + \frac{8}{5}t}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{1 - \frac{16}{25} + \frac{16}{25} + t^2 + \frac{8}{5}t} \qquad \left[\text{Adding and subtracting } \frac{16}{25}\right]$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{\left(\frac{3}{2}\right)^2 + \left(t + \frac{4}{5}\right)^2}$$

$$= \frac{2}{5}\left[\frac{5}{3}\tan^{-1}\left(t+\frac{4}{5}\right)\times\frac{5}{3}\right]_0^1$$

$$= \frac{2}{3}\left[\tan^{-1}\left(1+\frac{4}{5}\right)\times\frac{5}{3} - \tan^{-1}\frac{4}{5}\times\frac{5}{3}\right]_0^1$$

$$= \frac{2}{3}\left[\tan^{-1} 3 - \tan^{-1}\frac{4}{3}\right]_0^1$$

$$= \frac{2}{3}\left[\tan^{-1}\left(\frac{3-\frac{4}{3}}{1+3\times\frac{4}{3}}\right)\right]_0^1 \qquad \left[\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A - B}{1 + AB}\right)\right]$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{\frac{5}{3}}{5}\right]$$

$$= \frac{2}{3}\tan^{-1}\frac{1}{3}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{5 + 4\sin x}dx = \frac{2}{3}\tan^{-1}\frac{1}{3}$$

Question 21

Evaluate $\int_0^{\pi} \frac{\sin x}{\sin x + \cos x}dx$

Solution 21

We have,

$$\int_0^{\pi} \frac{\sin x}{\sin x + \cos x} dx$$

Let $\sin x = K\left(\sin x + \cos x\right) + L\dfrac{d}{dx}\left(\sin x + \cos x\right)$

$$= K\left(\sin x + \cos x\right) + L\left(\cos x - \sin x\right)$$

$$= \sin x\left(K - L\right) + \cos x\left(K + L\right)$$

Equating similar terms

$$K - L = 1$$

$$K + L = 0$$

$$\Rightarrow K = \frac{1}{2} \text{ and } L = -\frac{1}{2}$$

$$\therefore \int_0^{\pi} \frac{\sin x}{\sin x + \cos x} dx = \frac{1}{2}\int_0^{\pi} dx + \left(\frac{-1}{2}\right)\int_0^{\pi} \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\left[x\right]_0^{\pi} - \frac{1}{2}\left(\log\left|\sin x + \cos x\right|\right)_0^{\pi} = \frac{\pi}{2} - \frac{1}{2}\left(0\right) = \frac{\pi}{2}$$

$$\therefore \int_0^{\pi} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{2}$$

Question 22

Evaluate $\displaystyle\int_0^{\pi} \frac{1}{3 + 2\sin x + \cos x} dx$

Solution 22

We know,

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \quad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\therefore \frac{1}{3+2\sin x+\cos x}$$

$$= \frac{1}{3+2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$= \frac{\left(1+\tan^2\frac{x}{2}\right)}{3\left(1+\tan^2\frac{x}{2}\right)+4\tan\frac{x}{2}+\left(1-\tan^2\frac{x}{2}\right)}$$

$$= \frac{\sec^2\frac{x}{2}dx}{3+3\tan^2\frac{x}{2}+4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}$$

$$\therefore \int_0^{\pi}\frac{1}{3+2\sin x+\cos x}dx = \int_0^{\pi}\frac{\sec^2\frac{x}{2}dx}{2\tan^2\frac{x}{2}+4\tan\frac{x}{2}+4}$$

Let $\tan\frac{x}{2} = t$

Differentiating w.r.t. x, we get

$$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

Now,

$x = 0 \Rightarrow t = 0$

$x = \pi \Rightarrow t = \infty$

$$\therefore \int_0^{\pi} \frac{\sec^2 \frac{x}{2} dx}{2 \tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 4}$$

$$= \int_0^{\infty} \frac{dt}{t^2 + 2t + 2}$$

$$= \int_0^{\infty} \frac{dt}{(t+1)^2 + 1}$$

$$= \left[\tan^{-1}(t+1)\right]_0^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(0+1)$$

$$= \tan^{-1}(\infty) - \tan^{-1}(1)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{2}\right) - \tan^{-1}\left(\tan\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{2\pi - \pi}{4}$$

$$= \frac{\pi}{4}$$

$$\therefore \int_0^{\pi} \frac{1}{3 + 2\sin x + \cos x} dx = \frac{\pi}{4}$$

Question 23

Evaluate $\int_0^1 \tan^{-1} x \, dx$

Solution 23

We have,

$$\int_0^1 1 \cdot \tan^{-1} x \, dx = \tan^{-1} x \int_0^1 dx - \int_0^1 \left(\int dx.\right) \frac{d}{dx}\left(\tan^{-1} x\right) dx$$

$$= \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$$

$$= \left[x \tan^{-1} x - \frac{1}{2}\log\left(1+x^2\right)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}(\log 2 - 0)$$

$$= \frac{\pi}{4} - \frac{1}{2}\log 2$$

$$\therefore \int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{1}{2}\log 2$$

Question 24

$$\int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx$$

Solution 24

Using Integration By parts

$$\int f'g = fg - \int fg'$$

$$f' = \frac{x}{\sqrt{1-x^2}}, g = \sin^{-1} x$$

$$f = -\sqrt{1-x^2}, g' = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} \sin^{-1} x - \int (-1) dx$$

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} \sin^{-1} x + x$$

Hence

$$\int_0^{\frac{1}{2}} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \left\{ x - \sqrt{1-x^2} \sin^{-1} x \right\}_0^{\frac{1}{2}}$$

$$\int_0^{\frac{1}{2}} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \left\{ \frac{1}{2} - \sqrt{1-(\frac{1}{2})^2} \sin^{-1} \frac{1}{2} \right\}$$

$$\int_0^{\frac{1}{2}} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \left\{ \frac{1}{2} - \frac{\sqrt{3}}{2} \frac{\Pi}{6} \right\}$$

Question 25

$$\int_0^{\pi/4} \left(\sqrt{\tan x} + \sqrt{\cot x} \right) dx$$

Solution 25

$$I = \int_0^{\pi/4} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$$

$$I = \int_0^{\pi/4} \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right) dx$$

$$I = \int_0^{\pi/4} \left(\frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\right) dx$$

$$I = \sqrt{2}\int_0^{\pi/4} \left(\frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}}\right) dx$$

$$I = \sqrt{2}\int_0^{\pi/4} \left(\frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}\right) dx$$

Let $\sin x - \cos x = t$

$(\cos x + \sin x)dx = dt$

$x = 0 \Rightarrow t = -1$ and $x = \frac{\pi}{4} \Rightarrow t = 0$

$$I = \sqrt{2}\int_{-1}^{0} \left(\frac{1}{\sqrt{1 - t^2}}\right) dt$$

$$I = \sqrt{2}\left[\sin^{-1} t\right]_{-1}^{0}$$

$$I = \sqrt{2}\left[\sin^{-1}(0) - \sin^{-1}(-1)\right]$$

$$I = \frac{\pi}{\sqrt{2}}$$

Question 26

Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{1 + \cos 2x} dx$

Solution 26

We have,

$$\int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{1+\cos 2x}\,dx = \int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{2\cos^2 x}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x\,dx$$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

Now,

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{4} \Rightarrow t = 1$

$$\therefore \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^2 x \tan^3 x\,dx = \frac{1}{2}\int_0^1 t^3\,dt = \frac{1}{2}\left[\frac{t^4}{4}\right]_0^1 = \frac{1}{8}$$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{1+\cos 2x}\,dx = \frac{1}{8}$$

Question 27

Evaluate $\int_0^{\pi} \frac{1}{5+3\cos x}\,dx$

Solution 27

We know that,

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\frac{1}{5 + 3\cos x} = \frac{1}{5 + 3\left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)} = \frac{1 + \tan^2 \frac{x}{2}}{5\left(1 + \tan^2 \frac{x}{2}\right) + 3\left(1 - \tan^2 \frac{x}{2}\right)} = \frac{\sec^2 \frac{x}{2} dx}{8 + 2\tan^2 \frac{x}{2}}$$

$$\therefore \int_0^{\pi} \frac{dx}{5 + 3\cos x} dx = \frac{1}{2}\int_0^{\pi} \frac{\sec^2 \frac{x}{2}}{2^2 + \tan^2 \frac{x}{2}} dx$$

Let $\tan \frac{x}{2} = t$

Differentiating w.r.t. x, we get

$$\frac{1}{2}\sec^2 \frac{x}{2} dx = dt$$

Now,

$x = 0 \Rightarrow t = 0$

$x = \pi \Rightarrow t = \infty$

$$\therefore \frac{1}{2}\int_0^{\pi} \left(\frac{\sec^2 \frac{x}{2} dx}{2^2 + \tan^2 \frac{x}{2}}\right) dx$$

$$= \int_0^{\infty} \frac{dt}{2^2 + t^2}$$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\tan \frac{\pi}{2}\right) - \tan^{-1}(\tan 0)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{4}$$

$$\therefore \int_0^{\pi} \frac{dx}{5 + 3\cos x} dx = \frac{\pi}{4}$$

Question 28

Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx$

Solution 28

We have,

$$\int_0^{\frac{\pi}{2}} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx$$

Dividing numerator and denominator by $\cos^2 x$

$$\int_0^{\frac{\pi}{2}} \left(\frac{\frac{1}{\cos^2 x}}{a^2 \frac{\sin^2 x}{\cos^2 x} + b^2 \frac{\cos^2 x}{\cos^2 x}} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\sec^2 x}{a^2 \tan^2 x + b^2} \right) dx$$

$$= \frac{1}{a^2} \int_0^{\frac{\pi}{2}} \left(\frac{\sec^2 x}{\tan^2 x + \left(\frac{b}{a}\right)^2} \right) dx$$

Let $\tan x = t$

Differentiating w.r.t. x, we get

$$\sec^2 x \, dx = dt$$

When $x = 0 \Rightarrow t = 0$

$$x = \frac{\pi}{2} \Rightarrow t = \infty$$

$$\therefore \frac{1}{a^2} \int_0^{\frac{\pi}{2}} \left(\frac{\sec^2 x}{\tan^2 x + \left(\frac{b}{a}\right)^2} \right) dx$$

$$= \frac{1}{a^2} \int_0^{\infty} \frac{dt}{\left(\frac{b}{a}\right)^2 + t^2}$$

$$= \frac{1}{a^2} \left[\frac{a}{b} \tan^{-1} \frac{at}{b} \right]_0^{\infty}$$

$$= \frac{1}{a^2} \frac{a}{b} \left[\tan^{-1} \infty - \tan^{-1} 0 \right]$$

$$= \frac{1}{ab} \left[\tan^{-1} \tan \frac{\pi}{2} \right] = \frac{\pi}{2ab}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx = \frac{\pi}{2ab}$$

Question 29

$$\text{Evaluate } I = \int_0^{\frac{\pi}{2}} \frac{x+\sin x}{1+\cos x}dx$$

Solution 29

$$I = \int_0^{\frac{\pi}{2}} \frac{x+\sin x}{1+\cos x}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{x+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{x\sec^2\frac{x}{2}}{2} + \tan\frac{x}{2} \right) dx$$

$$= \left[x\tan\left(\frac{x}{2}\right) - \int_0^{\frac{\pi}{2}} \tan\frac{x}{2}\,dx + \int_0^{\frac{\pi}{2}} \tan\frac{x}{2}\,dx \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{x+\sin x}{1+\cos x}dx = \frac{\pi}{2}$$

Question 30

$$\text{Evaluate } \int_0^1 \frac{\tan^{-1} x}{1+x^2}dx$$

?

Solution 30

$$I = \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$$

Let $t = \tan^{-1} x$

$$dt = \frac{1}{1+x^2} dx$$

$$x = 0, t = 0$$

$$x = 1, t = \frac{\pi}{4}$$

$$I = \int_0^{\frac{\pi}{4}} t dt$$

$$= \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} \frac{\pi^2}{16}$$

$$= \frac{\pi^2}{32}$$

Question 31

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{3 + \sin 2x} dx$$

Solution 31

$$I = \int_0^{\pi/4} \frac{\sin x + \cos x}{3 + \sin 2x} dx$$

$$I = \int_0^{\pi/4} \left(\frac{\sin x + \cos x}{3 + 1 - (\cos x - \sin x)^2}\right) dx$$

$$I = \int_0^{\pi/4} \left(\frac{\sin x + \cos x}{4 - (\cos x - \sin x)^2}\right) dx$$

$$I = \frac{1}{4}\left[\log\left|\frac{2 + \sin x - \cos x}{2 - \sin x + \cos x}\right|\right]_0^{\pi/4}$$

$$I = -\frac{1}{4}\log\left(\frac{1}{3}\right)$$

$$I = \frac{1}{4}\log_e 3$$

Question 32

Evaluate $\int_0^1 x \tan^{-1} x dx$

Solution 32

We have,

$$\int_0^1 x\tan^{-1}x\,dx = \tan^{-1}x\int_0^1 x\,dx - \int_0^1\left(\int x\,dx\right)\frac{d}{dx}\left(\tan^{-1}x\right)dx$$

$$= \left[\frac{x^2}{2}\tan^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1\frac{x^2}{1+x^2}dx$$

$$= \left[\frac{x^2}{2}\tan^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1\frac{1+x^2-1}{1+x^2}dx$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\left[\int_0^1 dx - \int_0^1\frac{dx}{1+x^2}\right]$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1}x\right]_0^1$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[1 - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8}$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

$$\therefore \int_0^1 x\tan^{-1}x\,dx = \frac{\pi}{4} - \frac{1}{2}$$

Question 33

Evaluate $\int_0^1 \frac{1-x^2}{x^4+x^2+1}dx$

Solution 33

Let $I = \int \frac{1-x^2}{x^4+x^2+1}dx = -\int \frac{x^2-1}{x^4+x^2+1}dx.$

Then,

$$I = -\int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx \qquad \left[\begin{array}{l}\text{Dividing the numerator and}\\ \text{denominator by } x^2\end{array}\right]$$

$$\Rightarrow \quad I = -\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-1^2}dx$$

Let, $x+\frac{1}{x} = u$. Then, $d\left(x+\frac{1}{x}\right) = du \Rightarrow \left(1-\frac{1}{x^2}\right)dx = du$

$$\therefore \quad I = -\int \frac{du}{u^2-1^2}$$

$$\Rightarrow \quad I = -\frac{1}{2(1)}\log\left|\frac{u-1}{u+1}\right| + C$$

$$\Rightarrow \quad I = -\frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C = -\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + C$$

$$\therefore \quad \int_0^1 \frac{1-x^2}{x^4+x^2+1}dx = \left[-\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|\right]_0^1 = \left(-\frac{1}{2}\log\left|\frac{1}{3}\right|\right) - \left(-\frac{1}{2}\log|1|\right) = \log\sqrt{3}$$

$$= \log 3^{\frac{1}{2}}$$

$$= \frac{1}{2}\log 3$$

Question 34

Evaluate $\int_0^1 \frac{24x^3}{\left(1+x^2\right)^4}dx$

Solution 34

Let $1 + x^2 = t$

Differentiating w.r.t. x, we get

$2x dx = dt$

Now, $x = 0 \Rightarrow t = 1$

$x = 1 \Rightarrow t = 2$

$$\int_0^1 \frac{24x^3}{\left(1 + x^2\right)^4} dx = \int_1^2 \frac{12\left(t - 1\right)}{t^4} dt$$

$$= 12\int_1^2 \left(\frac{1}{t^3} - \frac{1}{t^4}\right) dt$$

$$= 12\left[-\frac{1}{2t^2} - \frac{1}{3t^3}\right]_1^2$$

$$= 12\left[-\frac{1}{8} + \frac{1}{24} + \frac{1}{2} - \frac{1}{3}\right]$$

$$= 12\left[\frac{-3 + 1 + 12 - 8}{24}\right]$$

$$= \frac{12 \times 2}{24} = 1$$

$$\therefore \int_0^1 \frac{24x^3}{\left(1 + x^2\right)^4} dx = 1$$

Question 35

Evaluate $\int_4^{12} x\left(x - 4\right)^{\frac{1}{3}} dx$

Solution 35

Let $x - 4 = t^3$
Differentiating w.r.t. x, we get

$$dx = 3t^2 dt$$

Now, $x = 4 \Rightarrow t = 0$
$x = 12 \Rightarrow t = 2$

$$\therefore \int_4^{12} x(x-4)^{\frac{1}{3}} dx = \int_0^2 (t^3+1) t.3t^2 dt$$

$$= 3\int_0^2 (t^6 + 4t^3) dt$$

$$= 3\left[\frac{t^7}{7} + t^4\right]_0^2$$

$$= 3\left[\frac{128}{7} + 16\right]$$

$$= \frac{720}{7}$$

$$\therefore \int_4^{12} x(x-4)^{\frac{1}{3}} dx = \frac{720}{7}$$

Question 36

Evaluate $\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx$

Solution 36

We have,

$$\int_0^{\frac{\pi}{2}} \underset{\mathrm{I}}{x^2} \underset{\mathrm{II}}{\sin x}\, dx$$

Using by parts, we get

$$x^2 \int \sin x dx - \int \left(\int \sin x dx\right) \frac{dx^2}{dx} . dx$$
$$= x^2 \cos x + \int \cos x . 2x dx$$

Again applying by parts

$$= x^2 \cos x + 2\left[x \int \cos x dx - \int \left(\int \cos x dx\right) . \frac{dx}{dx} . dx\right]$$
$$= x^2 \cos x + 2\left[x \sin x - \int \sin x dx\right]$$
$$= \left[x^2 \cos x + 2x \sin x + 2 \cos x\right]_0^{\frac{\pi}{2}}$$
$$= \pi + 0 - 0 - 0 - 2$$
$$= \pi - 2$$

$$\therefore \int_0^{\frac{\pi}{2}} x^2 \sin x dx = \pi - 2$$

Question 37

Evaluate $\int_0^1 \sqrt{\frac{1-x}{1+x}} dx$

Solution 37

Let $x = \cos 2\theta$

Differentiating w.r.t. x, we get

$dx = -2\sin 2\theta d\theta$

Now, $x = 0 \Rightarrow \theta = \frac{\pi}{4}$

$x = 1 \Rightarrow \theta = 0$

$$\therefore \int_0^1 \sqrt{\frac{1-x}{1+x}}dx = \int_{\frac{\pi}{4}}^{0} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\left(-2\sin 2\theta\right)d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\left(\ 2\sin 2\theta\right)d\theta \qquad \left[\because \sin 2\theta = 2\sin\theta\cos\theta;\ \text{and}\ \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos\theta}.\sin 2\theta d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sin^2\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \left(1-\cos 2\theta\right)d\theta$$

$$= 2\left[\theta - \frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{\pi}{2} - 1$$

$$\therefore \int_0^1 \sqrt{\frac{1-x}{1+x}}dx = \frac{\pi}{2} - 1$$

Question 38

Evaluate $\int_0^1 \frac{1-x^2}{\left(1+x^2\right)^2}dx$

Solution 38

We have,

$$\int_0^1 \frac{1-x^2}{\left(1+x^2\right)^2}dx = \int_0^1 \frac{-x^2\left(1-\frac{1}{x^2}\right)dx}{x^2\left(x+\frac{1}{x}\right)^2} = -\int_0^1 \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2}$$

Let $x+\frac{1}{x} = t \Rightarrow 1-\frac{1}{x^2}dx = dt$

When $x = 0 \Rightarrow t = \infty$

$x = 1 \Rightarrow t = 2$

$$\therefore \int_0^1 \frac{1-x^2}{\left(1+x^2\right)^2}dx = -\int_\infty^2 \frac{dt}{t^2} = \int_2^\infty \frac{dt}{t^2} = \left[-\frac{1}{t}\right]_2^\infty = \left(\frac{1}{2}-0\right) = \frac{1}{2}$$

Question 39

Evaluate $\int_{-1}^{1} 5x^4 \sqrt{x^2+1}\ dx.$

Solution 39

Put $t = x^5 + 1$, then $dt = 5x^4\ dx$.

Therefore, $\int 5x^4 \sqrt{x^5+1}\ dx = \int \sqrt{t}\ dt = \frac{2}{3}dt = \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}\left(x^5+1\right)^{\frac{3}{2}}$

Hence, $\int_{-1}^{1} 5x^4 \sqrt{x^4+1}\ dx = \frac{2}{3}\left[\left(x^5+1\right)^{\frac{3}{2}}\right]_{-1}^{1}$

$$= \frac{2}{3}\left[\left(1^5+1\right)^{\frac{3}{2}} - \left((-1)^5+1\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}\left(2\sqrt{2}\right) = \frac{4\sqrt{2}}{3}$$

Alternatively, first we transform the integral and then evaluate the transformed integ

Let $t = x^5 + 1$. Then $dt = 5x^4\ dx$.

Note that, when $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

Thus, as x varies from -1 to 1, t varies from 0 to 2

Therefore $\int_{-1}^{1} 5x^4 \sqrt{x^5+1}\ dx = \int_0^2 \sqrt{t}\ dt$

$$= \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}\left(2\sqrt{2}\right) = \frac{4\sqrt{2}}{3}$$

Question 40

$$\int_0^{\pi/2} \frac{\cos^2 x}{1+3\sin^2 x}\ dx$$

Solution 40

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{1+3\sin^2 x}\, dx$$

$$I = \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x\left(\sec^2 x+3\tan^2 x\right)}\, dx$$

Put $\tan x = t$

$\sec^2 x\, dx = dt$

$x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \infty$

$$I = \int_0^{\infty} \frac{1}{\left(1+t^2\right)\left(1+4t^2\right)}\, dt$$

$$I = -\frac{1}{3}\int_0^{\infty}\left[\frac{1}{\left(1+t^2\right)} - \frac{1}{\left(1+4t^2\right)}\right] dt$$

$$I = -\frac{1}{3}\left[\tan^{-1} t - 2\tan^{-1} 2t\right]_0^{\infty}$$

$$I = \frac{\pi}{6}$$

Question 41

Evaluate the follwing integrals:

$$\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\, dt$$

Solution 41

Let $I = \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\, dt$. consider $\int \sin^3 2t \cos 2t\, dt$

Put $\sin 2t = u$ so that $2\cos 2t\, dt = du$ or $\cos 2t\, dt = \frac{1}{2}\, du$

So $\int \sin^3 2t \cos 2t\, dt = \frac{1}{2}\int u^3\, du$

$$= \frac{1}{8}\left[u^4\right] = \frac{1}{8}\sin^4 2t = F(t) \text{ say}$$

Therefore, by the second fundamental theorem of integrals calculus

$$I = F\left(\frac{\pi}{4}\right) - F(o) = \frac{1}{8}\left[\sin^4 \frac{\pi}{2} - \sin^4 0\right] = \frac{1}{8}$$

Question 42

Evaluate $\int_0^{\pi} 5\left(5-4\cos\theta\right)^{\frac{1}{4}} \sin\theta d\theta$

Solution 42

Let $5 - 4\cos\theta = t$

Differentiating w.r.t. x, we get

$4\sin\theta d\theta = dt$

Now, $\theta = 0 \Rightarrow t = 1$

$\theta = \pi \Rightarrow t = 9$

$\therefore \int_0^{\pi} 5(5 - 4\cos\theta)^{\frac{1}{4}} \sin\theta d\theta$

$= \frac{5}{4}\int_1^9 t^{\frac{1}{4}} dt$

$= \frac{5}{4}\left[\frac{4}{5}.t^{\frac{5}{4}}\right]_1^9$

$= 3^{\frac{5}{2}} - 1$

$= 9\sqrt{3} - 1$

$\therefore \int_0^{\pi} 5(5 - 4\cos\theta)^{\frac{1}{4}} \sin\theta d\theta = 9\sqrt{3} - 1$

Question 43

Evaluate $\int_0^{\frac{\pi}{6}} \cos^{-3} 2\theta \sin 2\theta d\theta$

Solution 43

We have,

$$\int_0^{\frac{\pi}{6}} \cos^{-3} 2\theta \sin 2\theta d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{\sin 2\theta}{\cos^3 2\theta} d\theta$$

$$= \int_0^{\frac{\pi}{6}} \tan 2\theta . \sec^2 2\theta d\theta$$

Let $\tan 2\theta = t$

Differentiating w.r.t. x, we get

$$2 \sec^2 2\theta d\theta = dt$$

Now, $\theta = 0 \Rightarrow t = 0$

$$\theta = \frac{\pi}{6} \Rightarrow t = \sqrt{3}$$

$$\therefore \int_0^{\frac{\pi}{6}} \tan 2\theta . \sec^2 2\theta d\theta = \frac{1}{2} \int_0^{\sqrt{3}} t dt = \frac{1}{2} \left[\frac{t^2}{2} \right]_0^{\sqrt{3}}$$

$$= \frac{3}{4}$$

$$\therefore \int_0^{\frac{\pi}{6}} \cos^{-3} 2\theta \sin 2\theta d\theta = \frac{3}{4}$$

Question 44

Evaluate $\int_0^{\pi^{\frac{2}{3}}} \sqrt{x} \cos^2 x^{\frac{3}{2}} dx$

Solution 44

Let $x^{\frac{2}{3}} = t$

Differentiating w.r.t. x, we get

$$\frac{3}{2}\sqrt{x}dx = dt$$

Now, $x = 0 \Rightarrow t = 0$

$x = \pi^{\frac{2}{3}} \Rightarrow t = \pi$

$$\therefore \int_0^{\pi^{\frac{2}{3}}} \sqrt{x}\cos^2 x^{\frac{3}{2}}dx$$

$$= \frac{2}{3}\int_0^{\pi} \cos^2 t\, dt$$

$$= \frac{1}{3}\int_0^{\pi} 1 + \cos 2t\, dt \qquad \left[\because 2\cos^2 t = t + \cos 2t\right]$$

$$= \frac{1}{3}\left[t + \frac{\sin 2t}{t}\right]_0^{\pi}$$

$$= \frac{1}{3}\left[\pi + 0 - 0 - 0\right] = \frac{\pi}{3}$$

$$\therefore \int_0^{\pi^{\frac{2}{3}}} \sqrt{x}\cos^2 x^{\frac{3}{2}}dx = \frac{\pi}{3}$$

Question 45

Evaluate $\int_1^2 \frac{dx}{x(1 + \log x)^2}$

Solution 45

Let $1+\log x = t$

Differentiating w.r.t. x, we get

$$\frac{1}{x}dx = dt$$

When $x = 1 \Rightarrow t = 1$

$x = 2 \Rightarrow t = 1+\log 2$

$$\therefore \int_1^2 \frac{dx}{x(1+\log x)^2}$$

$$= \int_1^{1+\log 2} \frac{dt}{t^2}$$

$$= \left[-\frac{1}{t}\right]_1^{1+\log 2}$$

$$= 1 - \frac{1}{1+\log 2}$$

$$= \frac{\log 2}{1+\log 2}$$

$$\therefore \int_1^2 \frac{dx}{x(1+\log x)^2} = \frac{\log 2}{1+\log 2}$$

Question 46

Evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x \, dx$

Solution 46

We have,

$$\int_0^{\frac{\pi}{2}} \cos^5 x\,dx = \int_0^{\frac{\pi}{2}} \left(1 - \sin^2 x\right)^2 \cos x\,dx$$

Let $\sin x = t$

Differentiating w.r.t. x, we get

$\cos x dx = dt$

When $x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\int_0^{\frac{\pi}{2}} \left(1 - \sin^2 x\right)^2 \cos x\,dx$$

$$= \int_0^1 \left(1 - t^2\right)^2 dt$$

$$= \int_0^1 \left(1 - 2t^2 + t^4\right) dt$$

$$= \left[t - \frac{2}{3}t^3 + \frac{t^5}{5}\right]_0^1$$

$$= 1 - \frac{2}{3} + \frac{1}{5}$$

$$= \frac{8}{15}$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^5 x\,dx = \frac{8}{15}$$

Question 47

Evaluate $\int_4^9 \frac{\sqrt{x}}{\left(30 - x^{\frac{3}{2}}\right)^2}\,dx$

Solution 47

Let $I = \int \frac{\sqrt{x}}{\left(30 - x^{\frac{3}{2}}\right)} dx$. We first find the anti derivative of the integrand.

Pu $30 - x^{\frac{3}{2}} = t$. Then $-\frac{3}{2}\sqrt{x}\, dx = dt$ or $\sqrt{x}\, dx = -\frac{2}{3} dt$

Thus, $\int \frac{\sqrt{x}}{\left(30 - x^{\frac{3}{2}}\right)^2} dx = -\frac{2}{3}\int \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right] = \frac{2}{3}\left[\frac{1}{\left(30 - x^{\frac{3}{2}}\right)}\right] = f(x)$

Therefore, by the second fundamental theorem of calculus, we have

$$I = F(9) - F(4) = \frac{2}{3}\left[\frac{1}{\left(30 - x^{\frac{3}{2}}\right)}\right]_4^9$$

$$= \frac{2}{3}\left[\frac{1}{(30 - 27)} - \frac{1}{30 - 8}\right] = \frac{2}{3}\left[\frac{1}{3} - \frac{1}{22}\right] = \frac{19}{99}$$

Question 48

Evaluate $\int_0^{\pi} \sin^3 x \,(1 + 2\cos x)(1 + \cos x)^2\, dx$

Solution 48

Let $\cos x = t$

Differentiating w.r.t. x, we get

$-\sin x dx = dt$

When $x = 0 \Rightarrow t = 1$

$x = \pi \Rightarrow t = -1$

Now,

$$\int_0^{\pi} \sin^3 x\,(1 + 2\cos x)(1 + \cos x)^2\,dx$$

$$= \int_0^{\pi} \sin^2 x\,(1 + 2\cos x)(1 + \cos x)^2 . \sin x dx$$

$$= -\int_{-1}^{1} \left(1 - t^2\right)(1 + 2t)(1 + t)^2\,dt \qquad \left[\sin^2 x = 1 - \cos^2 x\right]$$

$$= \int_{-1}^{1} \left(1 + 2t - t^2 - 2t^3\right)\left(1 + t^2 + 2t\right)dt$$

$$= \int_{-1}^{1} \left(1 - t^2 + 2t + 2t + 2t^3 + 4t^2 - t^2 - t^4 - 2t^3 - 2t^5 - 4t^4\right)dt$$

$$= \int_{-1}^{1} \left(1 + 4t + 4t^2 - 2t^3 - 5t^4 - 2t^5\right)dt$$

$$= \left[t + 2t^2 + \frac{4}{3}t^3 - \frac{t^4}{2} - t^5 - \frac{t6}{3}\right]_{-1}^{1}$$

$$= \left[2 + 0 + \frac{8}{3} - 0 - 2 - 0\right] = \frac{8}{3}$$

$$\therefore \int_0^{\pi} \sin^3 x\,(1 + 2\cos x)(1 + \cos x)^2\,dx = \frac{8}{3}$$

Question 49

Evaluate $\int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)dx$

Solution 49

$$I = \int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)dx$$

Let $t = \sin x$

$$dt = \cos x dx$$

$$x = 0, t = 0$$

$$x = \frac{\pi}{2}, t = 1$$

$$I = \int_0^1 2t \tan^{-1}(t)dt$$

$$= 2\left[\frac{1}{2}t^2 \tan^{-1} t - \frac{t}{2} + \frac{1}{2}\tan^{-1} t\right]_0^1$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{\pi}{2} - 1$$

$$\therefore I = \int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)dx = \frac{\pi}{2} - 1$$

Question 50

Evaluate $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)dx$

Solution 50

Let $\sin x = t$
Differentiating w.r.t. x, we get
$\cos x\, dx = dt$

Now,
$x = 0 \Rightarrow t = 0$
$x = \frac{\pi}{2} \Rightarrow t = 1$
$\therefore \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)dx = 2\int_0^1 t \tan^{-1} t\, dt \qquad [\because \sin 2x = 2\sin x \cos x]$

Using by parts

$$= 2\left\{\tan^{-1} t \int t dt - \int\left(\int t dt\right)\frac{d \tan^{-1} t}{dt} dt\right\}$$
$$= 2\left\{\frac{t^2}{2}\tan^{-1} - \frac{1}{2}\int \frac{t^2}{1+t^2} dt\right\}$$
$$= 2\left\{\frac{t^2}{2}\tan^{-1} t - \frac{1}{2}\left(\int dt - \int \frac{dt}{1+t^2}\right)\right\}$$
$$= 2\left[\frac{t^2}{2}\tan^{-1} t - \frac{1}{2}\left(t - \tan^{-1} t\right)\right]_0^1$$
$$= 2\left\{\frac{1}{2}\frac{\pi}{4} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right)\right\}$$
$$= 2\left\{\frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8}\right\}$$
$$= 2\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{2} - 1$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)dx = \frac{\pi}{2} - 1$$

Question 51

Evaluate $\int_0^1 \left(\cos^{-1} x\right)^2 dx$

Solution 51

We have,

$$\int_0^1 \left(\cos^{-1}x\right)^2 dx = \left(\cos^{-1}x\right)^2 \int_0^1 dx - \int_0^1 \left(\int dx\right)\frac{d\left(\cos^{-1}x\right)^2}{dx}dx$$

$$= \left[x\left(\cos^{-1}x\right)^2\right]_0^1 + \int_0^1 \frac{x.2\cos^{-1}}{\sqrt{1-x^2}}dx$$

Now,

Let $\cos^{-1}x = t \Rightarrow -\frac{1}{\sqrt{1-x^2}}dx = dt$

When $x = 0 \Rightarrow t = \frac{\pi}{2}$

$x = 1 \Rightarrow t = 0$

$$\therefore \int_0^1 \frac{2x\cos^{-1}x}{\sqrt{1-x^2}}dx = -2\int_{\frac{\pi}{2}}^0 t\cos t\,dt = 2\int_0^{\frac{\pi}{2}} t\cos t\,dt$$

$$= 2\left[t\int \cos t\,dt - \int\left(\cos t\,dt\right)\frac{dt}{dt}.dt\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[t\sin t - \int \sin t\,dt\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[t\sin t + \cos t\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[\frac{\pi}{2} - 1\right]$$

$$\int_0^1 \left(\cos^{-1}x\right)^2 dx = \left[x\left(\cos^{-1}x\right)^2\right]_0^1 + \int_0^1 \frac{x.2\cos^{-1}}{\sqrt{1-x^2}}dx = \left[x\left(\cos^{-1}x\right)^2\right]_0^1 + 2\left(\frac{\pi}{2} - 1\right)$$

$$= 0 - 0 + 2\left(\frac{\pi}{2} - 1\right)$$

$$= \left(\pi - 2\right)$$

$$\therefore \int_0^1 \left(\cos^{-1}x\right)^2 dx = \left(\pi - 2\right)$$

Question 52

Evaluate:

$$\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$$

Solution 52

Let $I = \int \sin^{-1}\sqrt{\dfrac{x}{a+x}}\,dx$

Let $x = a\tan^2\theta$

$$dx = 2a\tan\theta\sec^2\theta\,d\theta$$

$$I = \int\left(\sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}\right)\left(2a\tan\theta\sec^2\theta\right)d\theta$$

$$= \int\left(\sin^{-1}\sqrt{\frac{\tan^2\theta}{\sec^2\theta}}\right)\left(2a\tan\theta\sec^2\theta\right)d\theta$$

$$= \int \sin^{-1}\left(\sin\theta\right)\left(2a\tan\theta\sec^2\theta\right)d\theta$$

$$= \int 2\theta a\tan\theta\sec^2\theta\,d\theta$$

$$= 2a\int\theta\left(\tan\theta\sec^2\theta\right)d\theta$$

$$= 2a\left[\theta\int\tan\theta\sec^2\theta\,d\theta - \int\left(\int\tan\theta\sec^2\theta\,d\theta\right)d\theta\right]$$

$$= 2a\left[\theta\frac{\tan^2\theta}{2} - \int\frac{\tan^2\theta}{2}d\theta\right]$$

$$= a\theta\tan^2\theta - \frac{2a}{2}\int\left(\sec^2\theta - 1\right)d\theta$$

$$= a\theta\tan^2\theta - a\tan\theta + a\theta + c$$

$$= a\left(\tan^{-1}\sqrt{\frac{x}{a}}\right)\frac{x}{a} - a\sqrt{\frac{x}{a}} + a\tan^{-1}\sqrt{\frac{x}{a}} + c$$

$$I = x\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + a\tan^{-1}\sqrt{\frac{x}{a}} + c$$

$$\therefore \int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx = \left[x\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + a\tan^{-1}\sqrt{\frac{x}{a}}\right]_0^a = a\frac{\pi}{4} - a + a\frac{\pi}{4}$$

$$= a\frac{\pi}{2} - a$$

$$= a\left(\frac{\pi}{2} - 1\right)$$

Question 53

Evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{\left(1-\cos x\right)^{\frac{3}{2}}}dx$

Solution 53

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{\frac{3}{2}}}dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{2\cos^2\frac{x}{2}}}{\left(2\sin^2\frac{x}{2}\right)^{\frac{3}{2}}}dx \qquad \left[\begin{array}{l} \because 1+\cos x = 2\cos^2\frac{x}{2} \\ \quad 1-\cos x = 2\sin^2\frac{x}{2} \end{array}\right]$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{2}\cos\frac{x}{2}}{2\sqrt{2}\sin^3\frac{x}{2}}dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\frac{x}{2}\,cosec^2\frac{x}{2}\,dx \qquad \left[\begin{array}{l} \because cosec^2\frac{x}{2} = \frac{1}{\sin^2\frac{x}{2}} \\ \quad \cot\frac{x}{2} = \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} \end{array}\right]$$

Let $\cot\frac{x}{2} = t$

Differentiating w.r.t. x, we get

$$\frac{-1}{2}cosec^2\frac{x}{2} = dt$$

Now, $x = \frac{\pi}{3} \Rightarrow t = \sqrt{3}$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\frac{x}{2}\,cosec^2\frac{x}{2}\,dx = -\int_{\sqrt{3}}^{1} t\,dt = -\left[\frac{t^2}{2}\right]_{\sqrt{3}}^{1} = \frac{-1}{2}[1-3]$$

$$= 1$$

$$\therefore \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{\frac{3}{2}}}dx = 1$$

Question 54

Evaluate $\int_0^a x\sqrt{\frac{a^2-x^2}{a^2+x^2}}dx$

Solution 54

Substitute $x^2 = a^2 \cos 2\theta$

Differentiating w.r.t. x, we get

$$2x\,dx = -2a^2 \sin 2\theta\, d\theta$$

Now, $x = 0 \Rightarrow \theta = \frac{\pi}{4}$

$x = a \Rightarrow \theta = 0$

$$\therefore \int_0^a x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\,dx = \int_{\frac{\pi}{4}}^{0} \sqrt{\frac{a^2\left(1 - \cos 2\theta\right)}{a^2 - \left(1 - \cos 2\theta\right)}}\left(-a^2 \sin 2\theta\right)d\theta$$

$$= -a^2 \int_{\frac{\pi}{4}}^{0} \frac{\sin\theta}{\cos\theta} \sin 2\theta\, d\theta$$

$$= a^2 \int_0^{\frac{\pi}{4}} 2\sin^2\theta\, d\theta$$

$$= a^2 \int_0^{\frac{\pi}{4}} \left(1 - \cos 2\theta\right) d\theta$$

$$= a^2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= a^2\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$\therefore \int_0^a x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\,dx = a^2\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

Question 55

Evaluate $\int_{-a}^{a} \sqrt{\frac{a - x}{a + x}}\,dx$

Solution 55

Let $x = a\cos 2\theta$

Differentiating w.r.t. x, we get

$dx = -2a\sin 2\theta$

Now, $x = -a \Rightarrow \theta = \frac{\pi}{2}$

$x = a \Rightarrow \theta = 0$

$$\therefore \int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx = \int_{\frac{\pi}{2}}^{0}\sqrt{\frac{a(1-\cos 2\theta)}{a(1+\cos 2\theta)}}(-2\sin 2\theta)d\theta$$

$$= 2a\int_{0}^{\frac{\pi}{2}}\frac{\sin\theta}{\cos\theta}.\sin 2\theta d\theta \qquad \left[\begin{array}{l}\because 1-\cos 2\theta = 2\sin^2\theta \\ \quad 1+\cos 2\theta = 2\cos^2\theta \\ -\int_{a}^{b}f(x)dx = \int_{b}^{a}f(x)dx\end{array}\right]$$

$$= 2a\int_{0}^{\frac{\pi}{2}}\frac{\sin\theta.2\sin\theta\cos\theta}{\cos\theta}$$

$$= 4a\int_{0}^{\frac{\pi}{2}}s\,in^2\theta d\theta$$

$$= 2a\int_{0}^{\frac{\pi}{2}}(1-\cos 2\theta)d\theta$$

$$= 2a\left[\theta - \frac{\sin 2\theta}{2}\right]_{0}^{\frac{\pi}{2}}$$

$$= 2a\left[\theta - \frac{\sin 2\theta}{2}\right]_{0}^{\frac{\pi}{2}}$$

$$= 2a\left[\frac{\pi}{2} - 0 - 0 + 0\right] = \pi a$$

$$\therefore \int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx = \pi a$$

Question 56

Evaluate $\int_{0}^{\frac{\pi}{2}}\frac{\sin x\cos x dx}{\cos^2 x + 3\cos x + 2}$

Solution 56

Let $\cos x = t$

Differentiating w.r.t. x, we get

$$-\sin x dx = dt$$

Now, $x = 0 \Rightarrow t = 1$

$$x = \frac{\pi}{2} \Rightarrow t = 0$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x dx}{\cos^2 x + 3\cos x + 2}$$

$$= -\int_1^0 \frac{tdt}{t^2 + 3t + 2}$$

$$= \int_0^1 \frac{tdt}{(t+2)(t+1)} \qquad \left[\because -\int_a^b f(x) = \int_b^a f(x)\right]$$

$$= \int_0^1 \left(-\frac{1}{t+1} + \frac{2}{t+2}\right) dt \qquad [\text{Applying partial fraction}]$$

$$= \left[-\log|1+t| + 2\log|t+2|\right]_0^1$$

$$= -\log 2 + 2\log 3 + 0 - 2\log 2$$

$$= 2\log 3 - 3\log 2$$

$$= \log\frac{9}{8}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x dx}{\cos^2 x + 3\cos x + 2} = \log\frac{9}{8}$$

Question 57

$$\int_0^{\pi/2} \frac{\tan x}{1 + m^2 \tan^2 x} \, dx$$

Solution 57

$$I = \int_0^{\pi/2} \frac{\tan x}{1 + m^2 \tan^2 x} dx$$

$$I = \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + m^2 \sin^2 x} dx$$

Put $\sin^2 x = t$ then $2\sin x\cos x \, dx = dt$

$x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{2} \Rightarrow t = 1$

$$I = \frac{1}{2}\int_0^1 \frac{1}{(1 - t) + m^2 t} dt$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{(m^2 - 1)t + 1} dt$$

$$I = \frac{1}{2}\left[\frac{1}{m^2 - 1}\log\left|(m^2 - 1)t + 1\right|\right]_0^1$$

$$I = \frac{1}{2}\left[\frac{1}{m^2 - 1}\log\left|m^2\right| - \frac{1}{m^2 - 1}\ln|1|\right]$$

$$I = \frac{1}{2}\left[\frac{\log\left|m^2\right|}{m^2 - 1}\right]$$

$$I = \frac{1}{2}\left[\frac{2\log|m|}{m^2 - 1}\right]$$

$$I = \frac{\log|m|}{m^2 - 1}$$

Question 58

$$\int_0^{1/2} \frac{1}{(1 + x^2)\sqrt{1 - x^2}} dx$$

Solution 58

$$I = \int_0^{1/2} \frac{1}{\left(1+x^2\right)\sqrt{1-x^2}}\,dx$$

Let $x = \sin u$

$dx = \cos u\ du$

$$I = \int_0^{\pi/6} \frac{1}{\left(1+\sin^2 u\right)}\,du$$

$$I = \int_0^{\pi/6} \frac{\sec^2 u}{\left(1+2\tan^2 u\right)}\,du$$

Let $\tan u = v$

$dv = \sec^2 u\,du$

$$I = \int_0^{1/\sqrt{3}} \frac{1}{\left(1+2v^2\right)}\,dv$$

$$I = \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\sqrt{2}v\right)\right]_0^{1/\sqrt{3}}$$

$$I = \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\sqrt{\frac{2}{3}}\right)\right]$$

Question 59

$$\int_{1/3}^{1} \frac{(x-x^3)^{1/3}}{x^4}\,dx$$

Solution 59

$$I = \int_{1/3}^{1} \frac{\left(x - x^3\right)^{1/3}}{x^4} dx$$

$$I = \int_{1/3}^{1} \frac{\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^3} dx$$

Let $\frac{1}{x^2} - 1 = t$

$\frac{-2}{x^3} dx = dt$

$x = \frac{1}{3} \Rightarrow t = 8$ and $x = 1 \Rightarrow t = 0$

$$I = -\frac{1}{2}\int_{8}^{0} (t)^{1/3} dt$$

$$I = -\frac{1}{2}\left[\frac{t^{4/3}}{4/3}\right]_{8}^{0}$$

$$I = -\frac{1}{2}[0 - 12]$$

$I = 6$

Question 60

$$\int_{0}^{\pi/4} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$$

Solution 60

$$\int \sec^2 x \frac{\tan^2 x}{\tan^6 x + 2\tan^3 x + 1} dx$$

$$u = \tan x \to \frac{du}{dx} = \sec^2 x$$

$$\int \frac{u^2}{u^6 + 2u^3 + 1} du$$

$$v = u^3 \to \frac{dv}{du} = 3u^2$$

$$\frac{1}{3}\int \frac{1}{v^2 + 2v + 1} dv$$

$$\frac{1}{3}\int \frac{1}{(v+1)^2} dv$$

$$-\frac{1}{3(v+1)}$$

$$-\frac{1}{3(u^3+1)}$$

$$-\frac{1}{3(\tan^3 x + 1)}$$

$$\left\{-\frac{1}{3(\tan^3 x + 1)}\right\}_0^{\frac{\pi}{4}}$$

$$\left\{-\frac{1}{6} + \frac{1}{3}\right\}$$

$$\frac{1}{6}$$

Question 61

$$\int_0^{\pi/2} \sqrt{\cos x - \cos^3 x}\ (\sec^2 x - 1)\cos^2 x\ dx$$

Solution 61

$$\int_0^{\frac{\pi}{2}} \sqrt{\cos x(1-\cos^2 x)} \tan^2 x \cos^2 x dx$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cos x \sin^2 x} \sin^2 x dx$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cos x} \sin^3 x dx$$

$$\cos x = t \to -\sin x = \frac{dt}{dx}$$

$$-\int_1^0 \sqrt{t}(1-t^2)dt$$

$$\int_0^1 (\sqrt{t} - t^{\frac{5}{2}})dt$$

$$\left\{\frac{2t^{\frac{3}{2}}}{3} - \frac{2t^{\frac{7}{2}}}{7}\right\}_0^1$$

$$\frac{2}{3} - \frac{2}{7}$$

$$\frac{8}{21}$$

Question 62

$$\int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^n} dx$$

Solution 62

$$I = \int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^n} dx$$

$$I = \int_0^{\pi/2} \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^n} dx$$

$$I = \int_0^{\pi/2} \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^{n-1}} dx$$

Let $\cos\frac{x}{2} + \sin\frac{x}{2} = t$

$$\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) dx = 2dt$$

$x = 0 \Rightarrow t = 1$ and $x = \frac{\pi}{2} \Rightarrow t = \sqrt{2}$

$$I = \int_1^{\sqrt{2}} \frac{2}{(t)^{n-1}} dt$$

$$I = \left[\frac{2t^{-n+2}}{-n+2}\right]_1^{\sqrt{2}}$$

$$I = \frac{2}{2-n}\left[\left(\sqrt{2}\right)^{2-n} - 1\right]$$

$$I = \frac{2}{2-n}\left[2^{1-\frac{n}{2}} - 1\right]$$

Chapter 20 - Definite Integrals Exercise Ex. 20.3

Question 26

Evaluate the following integrals:

$$\int_{-\pi/2}^{\pi/2} \frac{-\pi/2}{\sqrt{\cos x \sin^2 x}} dx$$

Solution 26

$$I = \int_{-\pi/2}^{\pi/2} \frac{-\pi/2}{\sqrt{\cos x \sin^2 x}} dx$$

$$I = \frac{-\pi}{2} \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{\cos x}\ |\sin x|} dx$$

Here, $f(-x) = f(x)$

$$\Rightarrow I = \frac{-\pi}{2} \times 2 \int_{0}^{\pi/2} \frac{1}{\sqrt{\cos x}\ |\sin x|} dx$$

$$I = -\pi \int_{0}^{\pi/2} \frac{1}{\sqrt{\cos x}\ \sin x} dx \qquad \left(\because \sin x \text{ is positive between } 0 \text{ and } \frac{\pi}{2} \right)$$

$$I = -\pi \int_{0}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}\ \sin^2 x} dx$$

$$I = -\pi \int_{0}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}\ \left(1 - \cos^2 x\right)} dx$$

Put $\cos x = t^2 \Rightarrow -\sin x dx = 2t dt$

$\Rightarrow \sin x dx = -2t dt$

x	0	$\frac{\pi}{2}$
t	1	0

$$I = -\pi \int_{1}^{0} \frac{-2t dt}{t\left(1 - t^4\right)}$$

$$I = 2\pi \int_{1}^{0} \frac{dt}{\left(1 - t^4\right)}$$

Using partial fraction

$$\Rightarrow I = 2\pi \int_{1}^{0} \frac{dt}{(1-t)(1+t)\left(1 - t^2\right)}$$

$$I = 2\pi \left(\int_{1}^{0} \frac{\frac{1}{4} dt}{(1-t)} + \int_{1}^{0} \frac{\frac{1}{4} dt}{(1+t)} + \int_{1}^{0} \frac{\frac{1}{2} dt}{\left(1 + t^2\right)} \right)$$

$I = \infty$

NOTE: Answer not matching with back answer.

Question 1(i)

Evaluate $\int_{1}^{4} f(x) dx$, Where $f(x) = \begin{cases} 4x + 3, & \text{if } 1 \le x \le 2 \\ 3x + 5, & \text{if } 2 \le x \le 4 \end{cases}$

Solution 1(i)

We have,

$\int_1^4 f(x)dx$

$= \int_1^2 (4x+3)\,dx + \int_2^4 (3x+5)\,dx$

$= \left[\frac{4x^2}{2}+3x\right]_1^2 + \left[\frac{3x^2}{2}+5x\right]_2^4$

$= \left[\left(\frac{16}{2}+6\right)-\left(\frac{4}{2}+3\right)\right]+\left[\left(\frac{48}{2}+20\right)-\left(\frac{12}{2}+10\right)\right]$

$= \left[(14-5)\right]+\left[(44-16)\right]$

$= 9+28$

$= 37$

Question 1(ii)

Evaluate $\int_0^9 f(x)dx$, where $f(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{2} \\ 1, & \frac{\pi}{2} \le x \le 3 \\ e^{x-3}, & 3 \le x \le 9 \end{cases}$

Solution 1(ii)

We have,

$\int_0^9 f(x)dx$

$= \int_0^{\frac{\pi}{2}} \sin x dx + \int_{\frac{\pi}{2}}^3 1\,dx + \int_3^9 e^{x-3}dx$

$= \left[-\cos x\right]_0^{\frac{\pi}{2}} + \left[x\right]_{\frac{\pi}{2}}^3 + \left[e^{x-3}\right]_3^9$

$= \left[-\cos\frac{\pi}{2}+\cos 0\right]+\left[3-\frac{\pi}{2}\right]+\left[e^{9-2}-e^{3-3}\right]$

$= \left[0+1\right]+\left[3-\frac{\pi}{2}\right]+\left[e^6-e^0\right]$

$= 0+1+3-\frac{\pi}{2}+e^6-e^0$

$= 1+3-\frac{\pi}{2}+e^6-1$

$= 3-\frac{\pi}{2}+e^6$

Question 1(iii)

Evaluate the integral

$\int_1^4 f(x)dx$, Where $f(x) = \begin{cases} 7x+3, & \text{if } 1 \le x \le 3 \\ 8x, & \text{if } 3 \le x \le 4 \end{cases}$

Solution 1(iii)

We have,

$$\int_1^4 f(x)dx$$

$$= \int_1^3 (7x+3)dx + \int_3^4 8x\,dx$$

$$= \left[\frac{7x^2}{2}+3x\right]_1^3 + \left[\frac{8x^2}{2}\right]_3^4$$

$$= \left[\left(\frac{7\times 9}{2}+3\times 3\right)-\left(\frac{7\times 1}{2}+3\times 1\right)\right]+\left[\left(\frac{8\times 16}{2}-\frac{8\times 9}{2}\right)\right]$$

$$= \left[\frac{63}{2}+9-\frac{7}{2}-3\right]+\left[64-36\right]$$

$$= 34+28$$

$$= 62$$

Question 2

Evaluate the integral

$$\int_{-4}^{4} |x+2|dx$$

Solution 2

We have,

$$\int_{-4}^{4} |x+2|dx$$

$$= \int_{-4}^{-2} -(x+2)dx + \int_{-2}^{4} (x+2)dx$$

$$= -\left[\frac{x^2}{2}+2x\right]_{-4}^{-2} + \left[\frac{x^2}{2}+2x\right]_{-2}^{4}$$

$$= -\left[\left(\frac{4}{2}-4\right)-\left(\frac{16}{2}-8\right)\right]+\left[\left(\frac{16}{2}+8\right)-\left(\frac{4}{2}-4\right)\right]$$

$$= -\left[(-2)-(0)\right]+\left[(16)-(-2)\right]$$

$$= -\left[-2\right]+\left[16+2\right]$$

$$= 2-18$$

$$= 20$$

$$\therefore \int_{-4}^{4} |x+2|dx = 20$$

Question 3

Evaluate $\int_{-3}^{3} |x+1| dx$

Solution 3

We have,

$\int_{-3}^{3} |x+1| dx$

$= \int_{-3}^{-1} -(x+1)dx + \int_{-1}^{3} (x+1)dx$

$= -\left[\frac{x^2}{2}+x\right]_{-3}^{-1} + \left[\frac{x^2}{2}+x\right]_{-1}^{3}$

$= -\left[\left(\frac{1}{2}-1\right)-\left(\frac{9}{2}-3\right)\right]+\left[\left(\frac{9}{2}+3\right)-\left(\frac{1}{2}-1\right)\right]$

$= -\left[\left(-\frac{1}{2}\right)-\left(1\frac{1}{2}\right)\right]+\left[\left(7\frac{1}{2}\right)-\left(-\frac{1}{2}\right)\right]$

$= -\left[-\frac{1}{2}-1\frac{1}{2}\right]+\left[7\frac{1}{2}+\frac{1}{2}\right]$

$= [-2]+[8]$

$= 2+8$

$= 10$

$\therefore \int_{-3}^{3} |x+1| dx = 10$

Question 4

Evaluate $\int_{-1}^{1} |2x+1| dx$

Solution 4

We have,

$$\int_{-1}^{1}|2x+1|dx$$

$$= \int_{-1}^{-\frac{1}{2}} -(2x+1)dx + \int_{-\frac{1}{2}}^{1}(2x+1)dx$$

$$= -\left[\frac{2x^2}{2}+x\right]_{-1}^{-\frac{1}{2}} + \left[\frac{2x^2}{2}+x\right]_{-\frac{1}{2}}^{1}$$

$$= -\left[\left(\frac{2}{8}-\frac{1}{2}\right)-\left(\frac{2}{2}-1\right)\right]+\left[\left(\frac{2}{2}+1\right)-\left(\frac{2}{8}-\frac{1}{2}\right)\right]$$

$$= -\left[\left(\frac{1}{4}-\frac{1}{2}\right)-(1-1)\right]+\left[(1+1)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]$$

$$= -\left[-\frac{1}{4}\right]+\left[2+\frac{1}{4}\right]$$

$$= \frac{1}{4}+2+\frac{1}{4}$$

$$= 2\frac{1}{2}$$

$$\therefore \int_{-1}^{1}|2x+1|dx = \frac{5}{2}$$

Question 5

$$\int_{-2}^{2}|2x+3|\,dx$$

Solution 5

2x+3 is positive for x>-1.5 . Hence

$$\int_{-2}^{2}|2x+3|dx = -\int_{-2}^{-1.5}(2x+3)dx + \int_{-1.5}^{2}(2x+3)dx$$

$$\int_{-2}^{2}|2x+3|dx = -\left\{x^2+3x\right\}_{-2}^{-1.5} + \left\{x^2+3x\right\}_{-1.5}^{2}$$

$$\int_{-2}^{2}|2x+3|dx = -\left\{(-2.25)-(-2)\right\}+\left\{(10)-(-2.25)\right\}$$

$$\int_{-2}^{2}|2x+3|dx = -\left\{-.25\right\}+\left\{12.25\right\}$$

$$\int_{-2}^{2}|2x+3|dx = 12.5$$

Question 6

$$\int_{0}^{2}|x^2-3x+2|\,dx$$

Solution 6

$$\int_0^2 |x^2-3x+2|\,dx = \int_0^2 |(x-1)(x-2)|\,dx$$

$$\int_0^2 |x^2-3x+2|\,dx = \int_0^1 (x-1)(x-2)dx - \int_1^2 (x-1)(x-2)dx$$

$$\int_0^2 |x^2-3x+2|\,dx = \int_0^1 (x^2-3x+2)dx - \int_1^2 (x^2-3x+2)dx$$

$$\int_0^2 |x^2-3x+2|\,dx = \left\{\frac{x^3}{3}-\frac{3x^2}{2}+2x\right\}_0^1 - \left\{\frac{x^3}{3}-\frac{3x^2}{2}+2x\right\}_1^2$$

$$\int_0^2 |x^2-3x+2|\,dx = \left\{\frac{1}{3}-\frac{3}{2}+2\right\} - \left\{(\frac{8}{3}-6+4)-(\frac{1}{3}-\frac{3}{2}+2)\right\}$$

$$\int_0^2 |x^2-3x+2|\,dx = \left\{\frac{5}{6}\right\} - \left\{(\frac{2}{3})-(\frac{5}{6})\right\}$$

$$\int_0^2 |x^2-3x+2|\,dx = 1$$

Question 7

Evaluate $\int_0^3 |3x-1|dx$

Solution 7

$$\int_0^3 |3x-1|\,dx = \int_0^{\frac{1}{3}} -(3x-1)\,dx + \int_{\frac{1}{3}}^3 (3x-1)\,dx$$

$$= -\left[\frac{3x^2}{2}-x\right]_0^{\frac{1}{3}} + \left[\frac{3x^2}{2}-x\right]_{\frac{1}{3}}^3$$

$$= -\left[\left(\frac{3}{9\times 2}-\frac{1}{3}\right)-(0)\right]+\left[\left(\frac{3\times 9}{2}-3\right)-\left(\frac{3}{9\times 2}-\frac{1}{3}\right)\right]$$

$$= -\left[\left(\frac{1}{6}-\frac{1}{3}\right)\right]+\left[\left(\frac{27}{2}-3\right)-\left(\frac{1}{6}-\frac{1}{3}\right)\right]$$

$$= -\left[\left(-\frac{1}{6}\right)\right]+\left[\left(10\frac{1}{2}\right)-\left(\frac{-1}{6}\right)\right]$$

$$= -\left[\left(-\frac{1}{6}\right)\right]+\left[10\frac{1}{2}+\frac{1}{6}\right]$$

$$= \frac{1}{6}+10\frac{1}{2}+\frac{1}{6}$$

$$= \frac{1}{3}+\frac{21}{2} = \frac{2+63}{6} = \frac{65}{6}$$

$$= \frac{65}{6}$$

$$\therefore \int_0^3 |3x-1|\,dx = \frac{65}{6}$$

Question 8

Evaluate $\int_{-6}^{6} |x+2|\,dx$

Solution 8

$$\int_{-6}^{6} |x+2|\,dx$$

$$= \int_{-6}^{-2} -(x+2)\,dx + \int_{-2}^{6} (x+2)\,dx$$

$$= -\left[\frac{x^2}{2}+2x\right]_{-6}^{-2} + \left[\frac{x^2}{2}+2x\right]_{-2}^{6}$$

$$= -\left[\left(\frac{4}{2}+2(-2)\right)-\left(\frac{36}{2}-12\right)\right]+\left[\left(\frac{36}{2}+12\right)-\left(\frac{4}{2}-4\right)\right]$$

$$= -\left[(2-4)-(18-12)\right]+\left[(18+12)-(2-4)\right]$$

$$= -[-8]+[30+2]$$

$$= 8+32$$

$$= 40$$

$$\therefore \int_{-6}^{6} |x+2|\,dx = 40$$

Question 9

Evaluate $\int_{-2}^{2}|x+1|\,dx$

Solution 9

$$\int_{-2}^{2}|x+1|\,dx = \int_{-2}^{-1}-(x+1)\,dx + \int_{-1}^{2}(x+1)\,dx$$

$$= -\left[\frac{x^2}{2}+x\right]_{-2}^{-1} + \left[\frac{x^2}{2}+x\right]_{-1}^{2}$$

$$= -\left[\left(\frac{1}{2}-1\right)-\left(\frac{4}{2}-2\right)\right]+\left[\left(\frac{4}{2}+2\right)-\left(\frac{1}{2}-1\right)\right]$$

$$= -\left[\left(-\frac{1}{2}\right)-0\right]+\left[4+\frac{1}{2}\right]$$

$$= \frac{1}{2}+4\frac{1}{2}$$

$$= 5$$

$$\therefore \int_{-2}^{2}|x+1|\,dx = 5$$

Question 10

Evaluate the integral.

$$\int_{1}^{2}|x-3|\,dx$$

Solution 10

$$\int_{1}^{2}|x-3|\,dx = \int_{1}^{2}-(x-3)\,dx \qquad [x-3<0 \text{ for } 1>x>2]$$

$$= -\left[\frac{x^2}{2}-3x\right]_{1}^{2}$$

$$= -\left[\left(\frac{4}{2}-6\right)-\left(\frac{1}{2}-3\right)\right]$$

$$= -\left[(-4)-\left(-2\frac{1}{2}\right)\right]$$

$$= -\left[-4+2\frac{1}{2}\right]$$

$$= -\left[-\frac{3}{2}\right]$$

$$= \frac{3}{2}$$

$$\therefore \int_{1}^{2}|x-3|\,dx = \frac{3}{2}$$

Question 11

Evaluate $\int_{0}^{\frac{\pi}{2}}|\cos 2x|\,dx$

Solution 11

$$\int_0^{\frac{\pi}{2}} |\cos 2x| dx$$

$$= \int_0^{\frac{\pi}{4}} -\cos 2x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} +\cos 2x \, dx$$

$$= \left[\frac{+\sin 2x}{2}\right]_0^{\frac{\pi}{4}} + \left[\frac{-\sin 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin 0\right] + \frac{1}{2}\left[\sin \pi + \sin\frac{\pi}{2}\right]$$

$$= \frac{1}{2}[1] + \frac{1}{2}[1]$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

$$\therefore \int_0^{\frac{\pi}{2}} |\cos 2x| dx = 1$$

Question 12

Evaluate $\int_0^{2\pi} |\sin x| dx$

Solution 12

$$\int_0^{2\pi} |\sin x| dx = \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} -\sin x dx$$
$$= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi}$$
$$= [1+1] + [1+1]$$
$$\int_0^{2\pi} |\sin x| dx = 4$$

Question 13

Evaluate $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} |\sin x| dx$

Solution 13

$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} |\sin x| dx$$
$$= \int_{\frac{-\pi}{4}}^{0} -\sin x \, dx + \int_0^{\frac{\pi}{4}} \sin x \, dx$$
$$= [\cos x]_{\frac{-\pi}{4}}^{0} + [-\cos x]_0^{\frac{\pi}{4}}$$
$$= \left(1 - \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}} - 1\right)$$
$$= \left(2 - \sqrt{2}\right)$$

$$\therefore \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} |\sin x| dx = 2 - \sqrt{2}$$

Question 14

Evaluate $\int_2^8 |x - 5| dx$

Solution 14

We have,

$$I = \int_2^8 |x-5|\,dx$$

We have,

$$|x-5| = \begin{cases} x-5 & \text{if } x \in (5, 8) \\ -(x-5) & \text{if } x \in (2, 5) \end{cases}$$

Hence,

$$I = \int_2^5 -(x-5)\,dx + \int_5^8 (x-5)\,dx$$

$$= -\left[\frac{x^2}{2} - 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$$

$$= -\left[\left(\frac{25}{2} - 25\right) - \left(\frac{4}{2} - 10\right)\right] + \left[\left(\frac{64}{2} - 40\right) - \left(\frac{25}{2} - 25\right)\right]$$

$$= -\left[-\frac{25}{2} + 8\right] + \left[(-8) + \left(\frac{25}{2}\right)\right]$$

$$= \frac{25}{2} - 8 - 8 + \frac{25}{2}$$

$$= 25 - 16 = 9$$

$$\therefore \int_2^8 |x-5|\,dx = 9$$

Question 15

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{\sin|x| + \cos|x|\}\,dx$$

Solution 15

We have,

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{\sin|x| + \cos|x|\}\,dx$$

Let $f(x) = \sin|x| + \cos|x|$

$Then, f(x) = f(-x)$

$\therefore$ f(x) is an even function.

So, $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{\sin|x| + \cos|x|\}\,dx = 2\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\,dx = 2\left[-\cos x + \sin x\right]_0^{\frac{\pi}{2}} = 4$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{\sin|x| + \cos|x|\}\,dx = 4$$

Question 16

By using the properties of definite integrals, evaluate the integral

$$\int_0^4 |x-1|\,dx$$

Solution 16

$$I = \int_0^4 |x-1|\,dx$$

It can be seen that, $(x-1) \le 0$ when $0 \le x \le 1$ and $(x-1) \ge 0$ when $1 \le x \le 4$

$$I = \int_0^1 |x-1|\,dx + \int_1^4 |x-1|\,dx \qquad \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)$$

$$= \int_0^1 -(x-1)\,dx + \int_1^4 (x-1)\,dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$$

$$= 1 - \frac{1}{2} + \frac{(4)^2}{2} - 4 - \frac{1}{2} + 1$$

$$= 1 - \frac{1}{2} + 8 - 4 - \frac{1}{2} + 1$$

$$= 5$$

Question 17

$$\int_1^4 \{|x-1| + |x-2| + |x-4|\}dx$$

Evaluate the integral

Solution 17

$$Let\ I=\int_1^4\{|x-1|+|x-2|+|x-4|\}dx$$

$$=\int_1^2\{(x-1)-(x-2)-(x-4)\}dx+\int_2^4\{(x-1)+(x-2)-(x-4)\}dx$$

$$=\int_1^2\{(x-1-x+2-x+4)\}dx+\int_2^4\{(x-1+x-2-x+4)\}dx$$

$$=\int_1^2(5-x)dx+\int_2^4(x+1)dx$$

$$=\left[5x-\frac{x^2}{2}\right]_1^2+\left[\frac{x^2}{2}+x\right]_2^4$$

$$=\left[10-2-5+\frac{1}{2}\right]+[8+4-2-2]$$

$$=\frac{7}{2}+8$$

$$I=\frac{23}{2}$$

Question 18

$$\int_{-5}^{0} f(x)\,dx,\ \text{where } f(x)=|x|+|x+2|+|x+5|$$

Solution 18

We have,

$$I=\int_{-5}^{0}\left(|x|+|x+2|+|x+5|\right)dx=\int_{-5}^{0}|x|dx+\int_{-5}^{0}|x+2|dx+\int_{-5}^{0}|x+5|dx$$

$$\Rightarrow I=\int_{-5}^{0}-x\,dx+\int_{-5}^{-2}-(x+2)\,dx+\int_{-2}^{0}(x+2)\,dx+\int_{-5}^{0}(x+5)\,dx$$

$$=\left[\frac{-x^2}{2}\right]_{-5}^{0}+\left[\frac{-x^2}{2}-2x\right]_{-5}^{-2}+\left[\frac{x^2}{2}+2x\right]_{-2}^{0}+\left[\frac{x^2}{2}+5x\right]_{-5}^{0}$$

$$=\left[+\frac{25}{2}\right]-\left[\frac{4}{2}-4-\frac{25}{2}+10\right]+\left[0+0-\frac{4}{2}+4\right]+\left[0+0-\frac{25}{2}+25\right]$$

$$=\frac{25}{2}-\left[8-\frac{25}{2}\right]+[2]+\left[25-\frac{25}{2}\right]$$

$$=\frac{25}{2}-8+\frac{25}{2}+2+25-\frac{25}{2}$$

$$=19+\frac{25}{2}=31\frac{1}{2}$$

$$I=\frac{63}{2}$$

Question 19

$$\int_0^4\left(|x|+|x-2|+|x-4|\right)dx$$

Solution 19

$$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x<0 \end{cases}$$

$$|x-2| = \begin{cases} x-2, & x \ge 2 \\ 2-x, & x<2 \end{cases}$$

$$|x-4| = \begin{cases} x-4, & x \ge 4 \\ 4-x, & x < 4 \end{cases}$$

Splitting the limits of the integral, we get

$$\int_0^4 \left(|x| + |x-2| + |x-4|\right) dx$$

$$= \int_0^2 \left(|x| + |x-2| + |x-4|\right) dx + \int_2^4 \left(|x| + |x-2| + |x-4|\right) dx$$

$$= \int_0^2 (x + 2 - x + 4 - x) dx + \int_2^4 (x + x - 2 + 4 - x) dx$$

$$= \int_0^2 (6-x) dx + \int_2^4 (2+x) dx$$

$$= \left[6x - \frac{x^2}{2}\right]_0^2 + \left[2x + \frac{x^2}{2}\right]_2^4$$

$$= [12-2] + [16-6]$$

$$= 10 + 10$$

$$= 20$$

Question 20

$$\int_{-1}^{2} (|x+1| + |x| + |x-1|) dx$$

Solution 20

$$\int_{-1}^{2} |x+1| dx + \int_{-1}^{2} |x| dx + \int_{-1}^{2} |x-1| dx$$

$$\int_{-1}^{2} (x+1) dx - \int_{-1}^{0} x dx + \int_{0}^{2} x dx - \int_{-1}^{1} (x-1) dx + \int_{1}^{2} (x-1) dx$$

$$\left\{\frac{x^2}{2} + x\right\}_{-1}^{2} - \left\{\frac{x^2}{2}\right\}_{-1}^{0} + \left\{\frac{x^2}{2}\right\}_{0}^{2} - \left\{\frac{x^2}{2} - x\right\}_{-1}^{1} + \left\{\frac{x^2}{2} - x\right\}_{1}^{2}$$

$$\left\{(4) - (-\frac{1}{2})\right\} - \left\{-\frac{1}{2}\right\} + \{2\} - \left\{(-\frac{1}{2}) - (\frac{3}{2})\right\} + \left\{(0) - (-\frac{1}{2})\right\}$$

$$\left\{4 + \frac{1}{2}\right\} + \left\{\frac{1}{2}\right\} + \{2\} + \{2\} + \left\{\frac{1}{2}\right\}$$

$$\frac{19}{2}$$

Question 21

$$\int_{-2}^{2} xe^{|x|}dx$$

Solution 21

$$\int_{-2}^{0} xe^{-x}dx + \int_{0}^{2} xe^{x}dx$$

For

$$\int_{-2}^{0} xe^{-x}dx$$

Using Integration By parts

$$\int f'g = fg - \int fg'$$

$$f' = e^{-x}, g = x$$

$$f = -e^{-x}, g' = 1$$

$$\int_{-2}^{0} xe^{-x}dx = \left\{-xe^{-x}\right\}_{-2}^{0} + \int_{-2}^{0} e^{-x}dx$$

$$\int_{-2}^{0} xe^{-x}dx = \left\{-xe^{-x} - e^{-x}\right\}_{-2}^{0}$$

$$\int_{-2}^{0} xe^{-x}dx = \left\{(-1) - (2e^{2} - e^{2})\right\}$$

$$\int_{-2}^{0} xe^{-x}dx = \left\{-1 - e^{2}\right\}$$

For

$$\int_{0}^{2} xe^{x}dx$$

Using Integration By parts

$$\int f'g = fg - \int fg'$$

$$f' = e^{x}, g = x$$

$$f = e^{x}, g' = 1$$

$$\int_{0}^{2} xe^{x}dx = \left\{xe^{x}\right\}_{0}^{2} - \int_{0}^{2} e^{x}dx$$

$$\int_{0}^{2} xe^{x}dx = \left\{xe^{x} - e^{x}\right\}_{0}^{2}$$

$$\int_{0}^{2} xe^{x}dx = 2e^{2} - e^{2} + 1$$

$$\int_{0}^{2} xe^{x}dx = e^{2} + 1$$

Hence answer is,

$$\int_{-2}^{2} xe^{|x|}dx = -1 - e^{2} + e^{2} + 1 = 0$$

Question 22

$$\int_{-\pi/4}^{\pi/2} \sin x \mid \sin x \mid dx$$

Solution 22

$$-\int_{-\frac{\pi}{4}}^{0} \sin^2 x dx + \int_{0}^{\frac{\pi}{2}} \sin^2 x dx$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$-\int_{-\frac{\pi}{4}}^{0} \frac{1-\cos 2x}{2} dx + \int_{0}^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx$$

$$-\frac{1}{2}\left\{x - \frac{\sin 2x}{2}\right\}_{-\frac{\pi}{4}}^{0} + \frac{1}{2}\left\{x - \frac{\sin 2x}{2}\right\}_{0}^{\frac{\pi}{2}}$$

$$-\frac{1}{2}\left\{-(-\frac{\Pi}{4} + \frac{1}{2})\right\} + \frac{1}{2}\left\{\frac{\Pi}{2}\right\}$$

$$\left\{-\frac{\Pi}{8} + \frac{1}{4}\right\} + \left\{\frac{\Pi}{4}\right\}$$

$$\frac{\Pi}{8} + \frac{1}{4}$$

$$\frac{\Pi + 2}{8}$$

Question 23

$$\int_{0}^{\pi} \cos x \mid \cos x \mid dx$$

Solution 23

$$\int_0^{\frac{\Pi}{2}} \cos^2 x dx - \int_{\frac{\Pi}{2}}^{\Pi} \cos^2 x dx$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\int_0^{\frac{\Pi}{2}} \frac{1+\cos 2x}{2} dx - \int_{\frac{\Pi}{2}}^{\Pi} \frac{1+\cos 2x}{2} dx$$

$$\frac{1}{2}\left\{x+\frac{\sin 2x}{2}\right\}_0^{\frac{\Pi}{2}} - \frac{1}{2}\left\{x+\frac{\sin 2x}{2}\right\}_{\frac{\Pi}{2}}^{\Pi}$$

$$\frac{\Pi}{4} - \frac{\Pi}{4}$$

$$0$$

Question 24

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} (2\sin|x| + \cos|x|) dx$$

Solution 24

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} (2\sin|x| + \cos|x|) dx$$

$$= \int_{-\frac{\pi}{4}}^{0} (-2\sin x + \cos x) dx + \int_0^{\frac{\pi}{2}} (2\sin x + \cos x) dx$$

$$= [2\cos x + \sin x]_{-\frac{\pi}{4}}^{0} + [-2\cos x + \sin x]_0^{\frac{\pi}{2}}$$

$$= 2 + 0 - 0 + 1 + 0 + 1 + 2 - 0$$

$$= 6$$

Question 25

$$\int_{-\pi/2}^{\pi} \sin^{-1}(\sin x)\, dx$$

Solution 25

$$\int_{-\frac{\Pi}{2}}^{\Pi} \sin^{-1}(\sin x) dx = \int_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}} x dx + \int_{\frac{\Pi}{2}}^{\Pi} (\Pi - x) dx$$

$$\Rightarrow \left\{\frac{x^2}{2}\right\}_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}} + \left\{\Pi x - \frac{x^2}{2}\right\}_{\frac{\Pi}{2}}^{\Pi}$$

$$\Rightarrow\left\{(\Pi^2-\frac{\Pi^2}{2})-\left(\frac{\Pi^2}{2}-\frac{\Pi^2}{8}\right)\right\}$$

$$\Rightarrow\left\{\frac{\Pi^2}{2}-\frac{3\Pi^2}{8}\right\}$$

$$\Rightarrow\frac{\Pi^2}{8}$$

Question 27

$$\int_0^2 2x[x]dx$$

Solution 27

[x]=0 for 0 < x

and [x]=1 for 1< x < 2

Hence

$$\int_0^1 0+\int_1^2 2xdx$$

$$\left\{x^2\right\}_1^2$$

$$3$$

Question 28

$$\int_0^{2\pi}\cos^{-1}(\cos x)\,dx$$

Solution 28

$$\int_0^{2\pi}\cos^{-1}(\cos x)\,dx$$

$$=-\int_0^{\pi}\cos^{-1}(\cos x)\,dx+\int_{\pi}^{2\pi}\cos^{-1}(\cos x)\,dx$$

$$=-\int_0^{\pi}x\,dx+\int_{\pi}^{2\pi}x\,dx$$

$$=-\left[\frac{x^2}{2}\right]_0^{\pi}+\left[\frac{x^2}{2}\right]_{\pi}^{2\pi}$$

$$=-\frac{\pi^2}{2}+\frac{4\pi^2}{2}-\frac{\pi^2}{2}$$

$$=\pi^2$$

Chapter 20 - Definite Integrals Exercise Ex. 20.4A

Question 1

$$\int_0^{2\pi}\frac{e^{\sin x}}{e^{\sin x}+e^{-\sin x}}dx$$

Solution 1

We know

$$\int_0^{2\Pi}f(x)dx=\int_0^{2\Pi}f(2\Pi-x)dx$$

Hence

$$\int_0^{2\Pi} \frac{e^{\sin x}}{e^{\sin x}+e^{-\sin x}}dx = \int_0^{2\Pi} \frac{e^{\sin(2\Pi-x)}}{e^{\sin(2\Pi-x)}+e^{-\sin(2\Pi-x)}}dx$$

We know

$$\sin(2\Pi - x) = -\sin x$$

$$\int_0^{2\Pi} \frac{e^{\sin x}}{e^{\sin x}+e^{-\sin x}}dx = \int_0^{2\Pi} \frac{e^{-\sin x}}{e^{-\sin x}+e^{\sin x}}dx$$

If

$$I = \int_0^{2\Pi} \frac{e^{\sin x}}{e^{\sin x}+e^{-\sin x}}dx$$

Then also

$$I = \int_0^{2\Pi} \frac{e^{-\sin x}}{e^{-\sin x}+e^{\sin x}}dx$$

Hence

$$2I = \int_0^{2\Pi} \frac{e^{-\sin x}}{e^{-\sin x}+e^{\sin x}}dx + \int_0^{2\Pi} \frac{e^{\sin x}}{e^{-\sin x}+e^{\sin x}}dx$$

$$2I = \int_0^{2\Pi} \frac{e^{-\sin x}}{e^{-\sin x}+e^{\sin x}} + \frac{e^{\sin x}}{e^{-\sin x}+e^{\sin x}}dx$$

$$2I = \int_0^{2\Pi} dx$$

$$2I = 2\Pi$$

$$I = \Pi$$

Question 2

$$\int_0^{2\pi} \log(\sec x + \tan x)dx$$

Solution 2

We know

$$\int_0^{2\Pi} f(x)dx = \int_0^{2\Pi} f(2\Pi - x)dx$$

Hence

$$\int_0^{2\Pi} \log(\sec x + \tan x)dx = \int_0^{2\Pi} \log(\sec(2\Pi - x) + \tan(2\Pi - x))dx$$

$$\int_0^{2\Pi} \log(\sec x + \tan x)dx = \int_0^{2\Pi} \log(\sec x - \tan x)dx$$

If

$$I = \int_0^{2\Pi} \log(\sec x + \tan x)dx$$

Then

$$I = \int_0^{2\Pi} \log(\sec x - \tan x)dx$$

$$2I = \int_0^{2\pi} \log(\sec x + \tan x)dx + \int_0^{2\pi} \log(\sec x - \tan x)dx$$

$$2I = \int_0^{2\pi} \log(\sec x + \tan x) + \log(\sec x - \tan x)dx$$

$$2I = \int_0^{2\pi} \log(\sec^2 x - \tan^2 x)dx$$

$$2I = \int_0^{2\pi} \log(1)dx$$

$$2I = 0$$

$$I = 0$$

Question 3

$$\int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$$

Solution 3

We know

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Hence

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan(\frac{\Pi}{2} - x)}}{\sqrt{\tan(\frac{\Pi}{2} - x)} + \sqrt{\cot(\frac{\Pi}{2} - x)}} dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$$

If

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$$

Then

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$$

So

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}}\,dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} + \frac{\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}}\,dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\,dx$$

$$2I = \frac{\Pi}{6}$$

$$I = \frac{\Pi}{12}$$

Question 4

$$\int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

Solution 4

We know

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Hence

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin(\frac{\Pi}{2}-x)}}{\sqrt{\sin(\frac{\Pi}{2}-x)}+\sqrt{\cos(\frac{\Pi}{2}-x)}}\,dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

If

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

Then

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

Hence

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 dx$$

$$2I = \frac{\Pi}{6}$$

$$I = \frac{\Pi}{12}$$

Question 5

$$\int_{-\pi/4}^{\pi/4} \frac{\tan^2 x}{1+e^x} dx$$

Solution 5

We know

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Hence

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x}{1+e^x} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2(-x)}{1+e^{-x}} dx$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x}{1+e^x} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x}{1+e^{-x}} dx$$

If

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x}{1+e^x} dx$$

Then

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x}{1+e^{-x}} dx$$

So

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x}{1+e^x} + \frac{\tan^2 x}{1+e^{-x}} dx$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x}{1+e^x} + \frac{\tan^2 x}{1+e^{-x}} dx$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x}{1+e^x} + \frac{e^x \tan^2 x}{1+e^x} dx$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x + e^x \tan^2 x}{1+e^x} dx$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+e^x)\tan^2 x}{1+e^x} dx$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x dx$$

$$I = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x dx$$

We know

If f(x) is even

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

If f(x) is odd

$$\int_{-a}^{a} f(x)dx = 0$$

Here

$$f(x) = \tan^2 x$$

f(x) is even, hence

$$I = \int_{0}^{\frac{\pi}{4}} \tan^2 x dx$$

$$I = \int_0^{\frac{\pi}{4}} \sec^2 x - 1 dx$$

$$I = \{\tan x - x\}_0^{\frac{\pi}{4}}$$

$$I = 1 - \frac{\Pi}{4}$$

Note: Answer given in the book is incorrect.

Question 6

$$\int_{-a}^{a} \frac{1}{1+a^x} dx, a > 0$$

Solution 6

We know

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Hence

$$\int_{-a}^{a} \frac{1}{1+a^x} dx = \int_{-a}^{a} \frac{1}{1+a^{-x}} dx$$

If

$$I = \int_{-a}^{a} \frac{1}{1+a^x} dx$$

Then

$$I = \int_{-a}^{a} \frac{1}{1+a^{-x}} dx$$

So

$$2I = \int_{-a}^{a} \frac{1}{1+a^x} + \frac{1}{1+a^{-x}} dx$$

$$2I = \int_{-a}^{a} \frac{1}{1+a^x} + \frac{a^x}{1+a^x} dx$$

$$2I = \int_{-a}^{a} 1 dx$$

$$2I = 2a$$

$$I = a$$

Question 7

$$\int_{-\pi/3}^{\pi/3} \frac{1}{1+e^{\tan x}} dx$$

Solution 7

We know

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Hence

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{1+e^{\tan x}} dx = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{1+e^{-\tan x}} dx$$

If

$$I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{1+e^{\tan x}} dx$$

Then

$$I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{1+e^{-\tan x}} dx$$

So

$$2I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{1+e^{\tan x}} + \frac{1}{1+e^{-\tan x}} dx$$

$$2I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{1+e^{\tan x}} + \frac{e^{\tan x}}{1+e^{\tan x}} dx$$

$$2I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 1 dx$$

$$2I = \frac{2\Pi}{3}$$

$$I = \frac{\Pi}{3}$$

Question 8

$$\int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+e^x} dx$$

Solution 8

We know

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

Hence

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2 x}{1+e^x} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2(-x)}{1+e^{-x}} dx$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2 x}{1+e^x} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2 x}{1+e^{-x}} dx$$

If

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^2 x}{1+e^x}dx$$

Then

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^2 x}{1+e^{-x}}dx$$

So

$$2I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^2 x}{1+e^x}+\frac{\cos^2 x}{1+e^{-x}}dx$$

$$2I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^2 x}{1+e^x}+\frac{e^x\cos^2 x}{1+e^x}dx$$

$$2I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(1+e^x)\cos^2 x}{1+e^x}dx$$

$$2I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 x dx$$

$$2I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2x}{2}dx$$

$$I=\frac{1}{4}\left\{x+\frac{\sin 2x}{2}\right\}_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$I=\frac{1}{4}\left\{\left(\frac{\Pi}{2}\right)-\left(-\frac{\Pi}{2}\right)\right\}$$

$$I=\frac{\Pi}{4}$$

Note: Answer given in the book is incorrect.

Question 9

$$\int_{-\pi/4}^{\pi/4}\frac{x^{11}-3x^9+5x^7-x^5+1}{\cos^2 x}dx$$

Solution 9

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^{11}-3x^9+5x^7-x^5+1}{\cos^2 x}dx$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^{11}-3x^9+5x^7-x^5}{\cos^2 x}+\frac{1}{\cos^2 x}dx$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^{11}-3x^9+5x^7-x^5}{\cos^2 x}dx+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x dx$$

If f(x) is even

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

If f(x) is odd

$$\int_{-a}^{a} f(x)dx = 0$$

Here

$\frac{x^{11}-3x^9+5x^7-x^5}{\cos^2 x}$ is odd and

$\sec^2 x$ is even. Hence

$$0+2\int_{0}^{\frac{\pi}{4}} \sec^2 x dx$$

$$2\left\{\tan x\right\}_0^{\frac{\pi}{4}}$$

$$2$$

Question 10

$$\int_a^b \frac{x^{1/n}}{x^{1/n}+(a+b-x)^{1/n}}dx,\ n\in N, n\geq 2$$

Solution 10

$$I = \int_a^b \frac{x^{1/n}}{x^{1/n} + (a+b-x)^{1/n}}\,dx$$

$$I = \int_a^b \frac{(a+b-x)^{1/n}}{(a+b-x)^{1/n} + x^{1/n}}\,dx$$

$$2I = \int_a^b \frac{x^{1/n}}{x^{1/n} + (a+b-x)^{1/n}}\,dx + \int_a^b \frac{(a+b-x)^{1/n}}{(a+b-x)^{1/n} + x^{1/n}}\,dx$$

$$2I = \int_a^b \frac{x^{1/n} + (a+b-x)^{1/n}}{x^{1/n} + (a+b-x)^{1/n}}\,dx$$

$$I = \frac{1}{2}\int_a^b dx$$

$$I = \frac{b-a}{2}$$

Question 11

Evaluate $\int_0^{\frac{\pi}{2}} \left(2\log\cos x - \log\sin 2x\right)dx$

Solution 11

We have,

$$I = \int_0^{\frac{\pi}{2}} \left(2\log\cos x - \log\sin 2x\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\log\cos^2 x - \log\sin 2x\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log\frac{\cos^2 x}{\sin x}dx$$

$$= \int_0^{\frac{\pi}{2}} \log\frac{\cos^2 x}{2\sin x.\cos x}dx$$

$$= \int_0^{\frac{\pi}{2}} \log\frac{\cos x}{2\sin x}dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\log\cos x - \log\sin x - \log 2\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log\cos x\,dx - \int_0^{\frac{\pi}{2}} \log\sin x\,dx - \int_0^{\frac{\pi}{2}} \log 2$$

We know that $\int_0^{\frac{\pi}{2}} \log\cos x\,dx = \int_0^{\frac{\pi}{2}} \log\sin x\,dx \qquad -(i)$

Hence from equation (i)

$$I = -\int_0^{\frac{\pi}{2}} \log 2 = -\frac{\pi}{2}\log 2$$

Question 12

By using the properties of definite integrals, evaluate the integral

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$$

Solution 12

$$\text{Let } I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx \qquad ...(1)$$

It is known that, $\left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$$

$$\Rightarrow 2I = \int_0^a 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^a$$

$$\Rightarrow 2I = a$$

$$\Rightarrow I = \frac{a}{2}$$

Question 13

Evaluate $\int_0^5 \frac{\sqrt[4]{x+4}}{\sqrt[4]{x+4}+\sqrt[4]{9-x}}dx$

Solution 13

Let $I = \int_0^5 \frac{\sqrt[4]{x+4}}{\sqrt[4]{x+4}+\sqrt[4]{9-x}}\,dx$ --(i)

We know that $\int_0^a f(x) = \int_0^a f(a-x)$

So,

$$I = \int_0^5 \frac{\sqrt[4]{(5-x)+4}}{\sqrt[4]{(5-x)+4}+\sqrt[4]{9-(5-x)}}\,dx$$

$$I = \int_0^5 \frac{\sqrt[4]{9-x}}{\sqrt[4]{9-x}+\sqrt[4]{4+x}}\,dx \qquad \text{--(ii)}$$

Adding (i) & (ii)

$$2I = \int_0^5 \frac{\sqrt[4]{x+4}}{\sqrt[4]{x+4}+\sqrt[4]{9-x}}\,dx + \int_0^5 \frac{\sqrt[4]{9-x}}{\sqrt[4]{9-x}+\sqrt[4]{4+x}}\,dx$$

$$2I = \int_0^5 \frac{\sqrt[4]{x+4}+\sqrt[4]{9-x}}{\sqrt[4]{x+4}+\sqrt[4]{9-x}}\,dx$$

$$2I = \int_0^5 dx$$

$$2I = [x]_0^5$$

$$I = \frac{1}{2}[5-0] = \frac{5}{2}$$

$$\therefore \int_0^5 \frac{\sqrt[4]{x+4}}{\sqrt[4]{x+4}+\sqrt[4]{9-x}}\,dx = \frac{5}{2}$$

Question 14

Evaluate $\int_0^7 \frac{\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[3]{7-x}}\,dx$

Solution 14

Let $I = \int_0^7 \frac{\sqrt[3]{x}}{\sqrt[3]{x} + \sqrt[3]{7-x}} dx \quad --(i)$

We know that $\int_0^a f(x) = \int_0^a f(a-x)$

Hence,

$I = \int_0^7 \frac{\sqrt[3]{7-x}}{\sqrt[3]{7-x} + \sqrt[3]{x}} dx \quad --(ii)$

Adding (i) & (ii)

$$2I = \int_0^7 \frac{\sqrt[3]{x}}{\sqrt[3]{x} + \sqrt[3]{7-x}} dx + \int_0^7 \frac{\sqrt[3]{7-x}}{\sqrt[3]{7-x} + \sqrt[3]{x}} dx$$

$$2I = \int_0^7 \frac{\sqrt[3]{x} + \sqrt[3]{7-x}}{\sqrt[3]{x} + \sqrt[3]{7-x}} dx$$

$$2I = \int_0^7 dx$$

$$2I = [x]_0^7$$

$$I = \frac{7}{2}$$

Question 15

Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}} dx$

Solution 15

$$\text{Let } I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\,dx$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \qquad --(i)$$

$$\text{We know that } \int_a^b f(x) = \int_a^b f(a+b-x)\,dx$$

Hence,

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}\,dx$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \qquad --(ii)$$

Adding (i) & (ii)

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx$$

$$2I = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$I = \frac{\pi}{12}$$

Question 16

$$\text{If } f(a+b-x) = f(x), \text{ then } \int_a^b x f(x)\,dx = \frac{a+b}{2}\int_a^b x f(x)\,dx$$

Solution 16

$$I = \int_a^b x f(x)\,dx$$

$$I = \int_a^b (a+b-x) f(a+b-x)\,dx$$

$$I = \int_a^b (a+b-x) f(x)\,dx \ldots\ldots\ldots \left[\because f(a+b-x) = f(x)\right]$$

$$I = \int_a^b (a+b) f(x)\,dx - \int_a^b f(x)\,dx$$

$$I = (a+b)\int_a^b f(x)\,dx - I$$

$$2I = (a+b)\int_a^b f(x)\,dx$$

$$I = \frac{(a+b)}{2}\int_a^b f(x)\,dx$$

$$\therefore \int_a^b x f(x)\,dx = \frac{(a+b)}{2}\int_a^b f(x)\,dx$$

Chapter 20 - Definite Integrals Exercise Ex. 20.4B

Question 1

Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x} dx$

Solution 1

We have,

$$\frac{1}{1+\tan x}=\frac{1}{1+\frac{\sin x}{\cos x}}=\frac{\cos x}{\cos x+\sin x}$$

$$\therefore \int_0^{\frac{\pi}{2}}\frac{dx}{1+\tan x}=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}dx$$

Let

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}dx \qquad --(\mathrm{I})$$

So,

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)}dx \qquad \left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\cos x+\sin x}dx \qquad --(\mathrm{II})$$

Hence, adding (I) & (II)

$$2I=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}dx+\int_0^{\frac{\pi}{2}}\frac{\sin x}{\cos x+\sin x}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{\cos x+\sin x}{\cos x+\sin x}dx$$

$$=\int_0^{\frac{\pi}{2}}dx$$

$$2I=\left[x\right]_0^{\frac{\pi}{2}}$$

$$2I=\left[\frac{\pi}{2}-0\right] \qquad \Rightarrow \quad I=\frac{\pi}{4}$$

Question 2

Evaluate $\int_0^{\frac{\pi}{2}}\frac{1}{1+\cot x}dx$

Solution 2

We have,

$$\frac{1}{1+\cot x} = \frac{1}{1+\frac{\cos x}{\sin x}} = \frac{\sin x}{\sin x + \cos x}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$

Let

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx \qquad --(\mathrm{I})$$

So,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \left[\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)\,dx\right]\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx \qquad --(\mathrm{II})$$

Adding (I) & (II)

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} dx$$

$$= [x]_0^{\frac{\pi}{2}}$$

$$2I = \left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{4}$$

Question 3

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}\,dx$

Solution 3

We have,

$$\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}=\frac{\sqrt{\frac{\cos x}{\sin x}}}{\sqrt{\frac{\cos x}{\sin x}}+\sqrt{\frac{\sin x}{\cos x}}}=\frac{\sqrt{\frac{\cos x}{\sin x}}}{\frac{\cos x+\sin x}{\sqrt{\sin x}.\sqrt{\cos x}}}=\sqrt{\frac{\cos x}{\sin x}}\times\frac{\sqrt{\sin x}.\sqrt{\cos x}}{\cos x+\sin x}$$

$$=\frac{\cos x}{\cos x+\sin x}$$

$$\therefore \int_0^{\frac{\pi}{2}}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}dx$$

Let

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}dx \qquad --(\mathrm{I})$$

So,

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)}dx \qquad \left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\cos x+\sin x}dx \qquad --(\mathrm{II})$$

Adding (I) & (II)

$$2I=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}dx+\int_0^{\frac{\pi}{2}}\frac{\sin x}{\cos x+\sin x}dx$$

$$2I=\int_0^{\frac{\pi}{2}}\frac{\cos x+\sin x}{\cos x+\sin x}dx$$

$$2I=\int_0^{\frac{\pi}{2}}dx$$

$$2I=\left[x\right]_0^{\frac{\pi}{2}}$$

$$2I=\left[\frac{\pi}{2}-0\right]$$

$$I=\frac{\pi}{4}$$

B

Question 4

By using the properties of definite integrals, evaluate the integral

$$\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}x\,dx}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}$$

Solution 4

$$\text{Let } I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx \qquad \ldots(1)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1\cdot dx$$

$$\Rightarrow 2I=\left[x\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

Question 5

Evaluate $\int_0^{\frac{\pi}{2}}\frac{\sin^n x}{\sin^n x+\cos^n x}dx$

Solution 5

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx \qquad --(\text{i})$$

So,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^n \left(\frac{\pi}{2} - x\right)}{\sin^n \left(\frac{\pi}{2} - x\right) + \cos^n \left(\frac{\pi}{2} - x\right)} dx \qquad \left[\because \int_0^a f(x) dx = \int_0^a f(a - x) dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x} \qquad --(\text{II})$$

Adding (I) & (II)

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^n + \cos^n x}{\sin^n + \cos^n x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} dx$$

$$2I = \left[x\right]_0^{\frac{\pi}{2}}$$

$$2I = \left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{4}$$

Question 6

$$\text{Evaluate } \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sqrt{\tan x}} dx$$

Solution 6

We have,

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sqrt{\tan x}}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

Let

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \qquad --(i)$$

So

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}\,dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} \qquad --(ii)$$

Adding (i) & (ii)

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} dx$$

$$2I = [x]_0^{\frac{\pi}{2}}$$

$$I = \frac{\pi}{4}$$

Question 7

Evaluate $\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}\,dx$

Solution 7

$$Let\ I = \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$$

Let $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

Now, $x = 0 \Rightarrow \theta = 0$

$x = a \Rightarrow \theta = \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \frac{a\cos\theta\, d\theta}{a\sin\theta + a\cos\theta}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta + \cos\theta} \qquad --(\text{i})$$

So,

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right) + \cos\left(\frac{\pi}{2} - \theta\right)} d\theta \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta + \sin\theta} \qquad --(\text{ii})$$

Adding (i) & (ii) we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta} d\theta$$

$$2I = \int_0^{\frac{\pi}{2}} d\theta$$

$$2I = \frac{1}{2}\left[\theta\right]_0^{\frac{\pi}{2}}$$

$$I = \frac{\pi}{4}$$

Question 8

Evaluate $\int_0^{\infty} \frac{\log x}{1 + x^2} dx$

Solution 8

Put $\quad x = \tan\theta$

$\Rightarrow \quad dx = \sec^2\theta d\theta$

If $\quad x = 0,\ \theta = 0$

If $\quad x = \infty,\ \theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^\infty \frac{\log x}{1+x^2}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\log(\tan\theta)\sec^2\theta d\theta}{1+\tan^2\theta}$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}} \log(\tan\theta)d\theta \qquad ---(i)$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}} \log\tan\left(\frac{\pi}{2}-\theta\right)d\theta$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}} \log\cot(\theta)d\theta \qquad ---(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} (\log\tan\theta + \log\cot\theta)d\theta$$

$$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}} \log 1 \times dx = \int_0^{\frac{\pi}{2}} 0 \times dx = 0$$

$$\Rightarrow \quad I = 0$$

Question 9

Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

Solution 9

Let $x = \tan\theta$

$\Rightarrow \quad dx = \sec^2\theta d\theta$

If $\quad x = 0, \ \theta = 0$

If $\quad x = 1, \ \theta = \frac{\pi}{4}$

$\therefore \quad \int_0^1 \frac{\log(1+x)}{1+x^2}dx$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{4}} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\}d\theta$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{4}} \log\left\{1+\frac{1-\tan\theta}{1+\tan\theta}\right\}d\theta$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{4}} \left(\log 2 - \log(1+\tan\theta)\right)d\theta$$

$$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{4}} \log 2 \times d\theta = \frac{\pi}{4}\log 2$$

$$\Rightarrow \quad I = \frac{\pi}{8}\log 2$$

Question 10

Evaluate $\int_0^{\infty} \frac{x}{(1+x)(1+x^2)}dx$

Solution 10

$$I = \int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx$$

Let,

$$\frac{x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

$$\Rightarrow x = A(1+x^2) + (Bx+C)(1+x)$$

Equating coeffcients, we get

$$A + B = 0 \Rightarrow A = -B$$
$$B + C = 1 \Rightarrow -2A = 1$$
$$A + C = 0 \Rightarrow A = -C$$

$$\therefore\ A = -\frac{1}{2},\ B = \frac{1}{2},\ C = \frac{1}{2}$$

So,

$$I = \int_0^{\infty} \left(\frac{-\frac{1}{2}}{1+x} + \frac{1}{2} \frac{x+1}{x^2+1} \right) dx$$

$$= \int_0^{\infty} -\frac{1}{2} \frac{dx}{1+x} + \frac{1}{2} \int_0^{\infty} \frac{x}{x^2+1} dx + \frac{1}{2} \int_0^{\infty} \frac{dx}{1+x^2}$$

$$= \left[-\frac{1}{2} \log|1+x| + \frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1} x \right]_0^{\infty}$$

$$= 0 + 0 + \frac{\pi}{4} + 0 - 0 - 0$$

$$= \frac{\pi}{4}$$

$$\because \int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx = \frac{\pi}{4}$$

Question 11

Evaluate $\displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x} dx$

Solution 11

We have,

$$I = \int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x} dx$$

$$I = \int_0^{\pi} \frac{x\left(\frac{\sin x}{\cos x}\right)}{\left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right)} dx$$

$$I = \int_0^{\pi} x \sin^2 x \, dx \qquad \ldots(i)$$

$$I = \int_0^{\pi} (\pi - x) \sin^2 (\pi - x) dx \qquad \left[\because \int_0^{a} f(x) dx = \int_0^{a} f(a - x) dx\right]$$

$$I = \int_0^{\pi} (\pi - x) \sin^2 x \, dx \qquad --(ii)$$

Add (i) and (ii), we get

$$2I = \int_0^{\pi} (\pi) \sin^2 x \, dx = \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} dx = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{\pi}{2}[\pi - 0 - 0 + 0] = \frac{\pi^2}{2}$$

$$\therefore \int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x} dx = \frac{\pi^2}{4}$$

Question 12

Evaluate $\int_0^{\pi} x \sin x \cos^4 x \, dx$

Solution 12

Let $I = \int_0^{\pi} x \sin x . \cos^4 x \, dx$ --(i)

So,

$I = \int_0^{\pi} (\pi - x) \sin(\pi - x) . \cos^4 (\pi - x) \, dx$ $\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx\right]$

$= \int_0^{\pi} (\pi - x) \sin x . \cos^4 x \, dx$

$= \int_0^{\pi} \pi \sin x . \cos^4 x \, dx - \int_0^{\pi} x \sin x . \cos^4 x \, dx$

So from equation (i)

$I = \int_0^{\pi} \pi \sin x . \cos^4 x \, dx - I$

$2I = \pi \int_0^{\pi} \sin x . \cos^4 x \, dx$

Let $t = \cos x \, dx$

$dt = - \sin \; x \, dx$

As,

$x = 0 \quad t = 1$

$x = \pi \quad t = -1$

Hence

$2I = \pi \int_{-1}^{+1} t^4 \, dt = \pi \left[\frac{t^5}{5}\right]_{-1}^{1} = \pi \left[\frac{1}{5} + \frac{1}{5}\right]$

$I = \frac{\pi}{5}$

Question 13

Evaluate $\int_0^{\pi} x \sin^3 x \, dx$

Solution 13

$$\text{Let } I = \int_0^\pi x \sin^3 x\, dx$$

$$= \int_0^\pi (\pi - x) \sin^3 (\pi - x)\, dx \qquad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^\pi \pi \sin^3 x\, dx - \int_0^\pi x \sin^3 x\, dx$$

$$\therefore I = \int_0^\pi \pi \sin^3 x\, dx - I$$

$$\Rightarrow 2I = \pi \int_0^\pi \sin^3 x\, dx$$

$$\Rightarrow 2I = \pi \int_0^\pi \frac{3 \sin x - \sin 3x}{4}\, dx$$

$$= \frac{\pi}{4} \int_0^\pi (3 \sin x - \sin 3x)\, dx$$

$$= \frac{\pi}{4} \left[-3 \cos x + \frac{\cos 3x}{3}\right]_0^\pi$$

$$= \frac{\pi}{4} \left[\left(-3 \cos \pi + \frac{\cos 3\pi}{3}\right) - \left(-3 \cos 0 + \frac{\cos 0}{3}\right)\right]$$

$$= \frac{\pi}{4} \left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right]$$

$$= \frac{\pi}{4} \left[3 - \frac{1}{3} + 3 - \frac{1}{3}\right]$$

$$\frac{\pi}{4} \left[6 - \frac{2}{3}\right]$$

$$= \frac{\pi}{4} \times \frac{16}{3} \quad = \quad \frac{4\pi}{3}$$

$$\therefore I = \frac{2\pi}{3}$$

Question 14

$$\text{Evaluate } \int_0^\pi x \log \sin x\, dx$$

Solution 14

We have,

$$I = \int_0^{\pi} x \log \sin x \, dx = \int_0^{\pi} (\pi - x) \log \sin(\pi - x) \, dx$$

$$I = \pi \int_0^{\pi} \log \sin(x) \, dx - \int_0^{\pi} x \log \sin x \, dx$$

$$2I = \pi \int_0^{\pi} \log \sin x \, dx$$

Since $f(x) = f(-x)$, f(x) is an even function.

$$\therefore 2I = 2\pi \int_0^{\frac{\pi}{2}} \log \sin x \, dx$$

$$I = \pi \int_0^{\frac{\pi}{2}} \log \sin x \, dx \qquad \ldots(\text{i})$$

$$\Rightarrow I = \pi \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = \pi \int_0^{\frac{\pi}{2}} \log \cos x \, dx \quad \ldots(\text{ii})$$

Now adding (i) & (ii) we get

$$2I = \pi \int_0^{\frac{\pi}{2}} \log \sin x \, dx + \pi \int_0^{\frac{\pi}{2}} \log \cos x \, dx = \pi \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) \, dx = \pi \int_0^{\frac{\pi}{2}} \log \sin x . \cos x \, dx$$

$$\Rightarrow 2I = \pi \int_0^{\frac{\pi}{2}} \log\left(\frac{2 \sin x . \cos x}{2}\right) dx = \pi \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx = \pi \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \pi \int_0^{\frac{\pi}{2}} \log 2 \, dx \qquad \ldots(\text{iii})$$

Now let $I = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx$

Putting $2x = t$ we get

$$I_1 = \int_0^{\pi} \log \sin t \frac{dt}{2} = \frac{1}{2} \int_0^{\pi} \log \sin t . dt = \frac{1}{2} \times 2\pi \int_0^{\frac{\pi}{2}} \log \sin t \, dt = \pi \int_0^{\frac{\pi}{2}} \log \sin x \, dx = I$$

So from (iii) we get

$$2I = I - \pi \frac{\pi}{2} \log 2$$

$$I = -\frac{\pi^2}{2} \log 2$$

Question 15

Evaluate $\int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx$

Solution 15

$$\text{Let } I = \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\sin x} dx \qquad \left[\because \int_0^{a} f(x) dx = \int_0^{a} f(a-x) dx\right]$$

$$I = \int_0^{\pi} \frac{\pi \sin x}{1+\sin x} dx - \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} \times \frac{(1-\sin x)}{(1-\sin x)} dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{1+\sin^2 x} dx$$

$$2I = \pi \int_0^{\pi} \frac{(\sin x - \sin^2 x)}{\cos^2 x} dx$$

$$2I = \pi \int_0^{\pi} (\tan x . \sec x - \tan^2 x) dx$$

$$2I = \pi \int_0^{\pi} \left[\tan x . \sec x - (\sec^2 x - 1)\right] dx$$

$$2I = \pi \int_0^{\pi} (\sec x . \tan x - \sec^2 x + 1) dx$$

$$2I = \pi \int_0^{\pi} (\sec x . \tan x - \sec^2 x + 1) dx$$

$$2I = \pi \left[\sec x - \tan x + x\right]_0^{\pi}$$

$$2I = \pi \left[(\sec \pi - \tan \pi + \pi) - (\sec 0 - \tan 0 + 0)\right]$$

$$2I = \pi \left[(-1 - 0 + \pi) - (1 - 0 + 0)\right]$$

$$2I = \pi (\pi - 1 - 1)$$

$$I = \frac{\pi}{2}(\pi - 2)$$

$$\therefore \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx = \pi\left(\frac{\pi}{2} - 1\right)$$

Question 16

$$\text{Evaluate } \int_0^{\pi} \frac{x}{1+\cos\alpha \sin x} dx, \quad 0 < \alpha < \pi$$

Solution 16

We have

$$I=\int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\sin x}\qquad --(i)$$

$$\therefore \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$$

$$I=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos\alpha\sin(\pi-x)}=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos\alpha\sin x}\qquad --(ii)$$

Adding (i) & (ii) we get

$$2I=\pi\int_0^{\pi}\frac{\pi}{1+\cos\alpha.\sin x}dx$$

Substituting $\sin x=\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$2I=\pi\int_0^{\pi}\frac{\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2}\,2\cos\alpha.\tan\frac{x}{2}}dx=\pi\int_0^{\pi}\frac{\sec^2\frac{x}{2}dx}{1-\cos^2\alpha+\left(\cos\alpha.\tan\frac{x}{2}\right)^2}dx$$

Let $\tan\frac{x}{2}=t\Rightarrow\frac{1}{2}\sec^2\frac{x}{2}dx=dt$

When $x=0\quad t=0$

$\pi\ \Rightarrow t=\alpha$

$$2I=\int_0^{\alpha}\frac{dt}{\left(1+\cos^2\alpha\right)+\left(\cos\alpha+t\right)^2}dx=2\pi.\frac{1}{\sqrt{1+\cos^2\alpha}}.\left[\tan^{-1}\left(\frac{\cos\alpha+1}{\sqrt{1+\cos^2\alpha}}\right)\right]_0^{\alpha}$$

$$=\frac{2\pi}{\sin\alpha}\left[\frac{\pi}{2}-\tan^{-1}\cot\alpha\right]$$

$$=\frac{2\pi}{\sin\alpha}\left[\cot^{-1}(\cot\alpha)\right]$$

$$=\frac{2\pi}{\sin\alpha}.\alpha$$

$$\Rightarrow I=\frac{\pi\alpha}{\sin\alpha}$$

Question 17

Evaluate $\int_0^{\pi}x\cos^2x\,dx$

Solution 17

Let $I = \int_0^{\pi} x \cos^2 x \, dx$

$I = \int_0^{\pi} (\pi - x)\cos^2(\pi - x)\, dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)\, dx\right]$

$I = \pi\int_0^{\pi} \cos^2 x \, dx - \int_0^{\pi} x \cos^2 x \, dx$

$2I = \pi\int_0^{\pi} \cos^2 x \, dx$

$= \pi\int_0^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx \qquad$ Since $\cos^2 x = \frac{1+\cos 2x}{2}$

$= \frac{\pi}{2}\int_0^{\pi}(1+\cos 2x)\, dx$

$= \frac{\pi}{2}\left[x + \left(-\frac{\sin 2x}{2}\right)\right]_0^{\pi}$

$\therefore\ 2I = \frac{\pi}{2}\left[\pi - \frac{\sin 2.\pi}{2} - 0 + \frac{\sin 0}{2}\right]$

$\Rightarrow 2I = \frac{\pi}{2}\left[\pi - 0 - 0 + 0\right]$

$I = \frac{\pi^2}{4}$

Question 18

$$\int_{\pi/6}^{\pi/3} \frac{1}{1+\cot^{3/2} x}\, dx$$

Solution 18

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\cot^{3/2} x} dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sin^{3/2} x}{\sin^{3/2} x+\cos^{3/2} x} dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sin^{3/2}\left(\frac{\pi}{2}-x\right)}{\sin^{3/2}\left(\frac{\pi}{2}-x\right)+\cos^{3/2}\left(\frac{\pi}{2}-x\right)} dx = \int_{\pi/6}^{\pi/3} \frac{\cos^{3/2}(x)}{\cos^{3/2}(x)+\sin^{3/2}(x)} dx$$

$$\therefore 2I = \int_{\pi/6}^{\pi/3} \frac{\sin^{3/2} x}{\sin^{3/2} x+\cos^{3/2} x} dx + \int_{\pi/6}^{\pi/3} \frac{\cos^{3/2}(x)}{\cos^{3/2}(x)+\sin^{3/2}(x)} dx$$

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sin^{3/2} x+\cos^{3/2} x}{\sin^{3/2} x+\cos^{3/2} x} dx$$

$$I = \frac{1}{2}\int_{\pi/6}^{\pi/3} dx$$

$$I = \frac{\pi}{12}$$

Question 19

$$\int_0^{\pi/2} \frac{\tan^7 x}{\tan^7 x + \cot^7 x} dx$$

Solution 19

$$I = \int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\tan^7 x+\cot^7 x} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\tan^7\left(\frac{\Pi}{2}-x\right)}{\tan^7\left(\frac{\Pi}{2}-x\right)+\cot^7\left(\frac{\Pi}{2}-x\right)} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cot^7 x}{\tan^7 x+\cot^7 x} dx$$

Hence

$$2I = \int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\tan^7 x + \cot^7 x} + \frac{\cot^7 x}{\tan^7 x + \cot^7 x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1 dx$$

$$2I = \frac{\Pi}{2}$$

$$I = \frac{\Pi}{4}$$

Question 20

$$\int_2^8 \frac{\sqrt{10 - x}}{\sqrt{x} + \sqrt{10 - x}} dx$$

Solution 20

$$I = \int_2^8 \frac{\sqrt{10 - x}}{\sqrt{x} + \sqrt{10 - x}} dx$$

$$I = \int_2^8 \frac{\sqrt{10 - (8 + 2 - x)}}{\sqrt{(8 + 2 - x)} + \sqrt{10 - (8 + 2 - x)}} dx$$

$$I = \int_2^8 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{10 - x}} dx$$

$$2I = \int_2^8 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{10 - x}} + \frac{\sqrt{10 - x}}{\sqrt{x} + \sqrt{10 - x}} dx$$

$$2I = \int_2^8 1 dx$$

$$2I = 6$$

$$I = 3$$

Question 21

$$\int_0^{\pi} x \sin x \cos^2 x \, dx$$

Solution 21

$$\int_0^{\Pi} x\sin x\cos^2 x dx = \int_0^{\Pi} (\Pi - x)\sin(\Pi - x)\cos^2(\Pi - x)dx$$

$$\int_0^{\Pi} x\sin x\cos^2 x dx = \int_0^{\Pi} (\Pi - x)\sin x\cos^2 x dx$$

$$\int_0^{\Pi} x\sin x\cos^2 x dx = \int_0^{\Pi} \Pi\sin x\cos^2 x dx - \int_0^{\Pi} x\sin x\cos^2 x dx$$

$$2\int_0^{\Pi} x\sin x\cos^2 x dx = \int_0^{\Pi} \Pi\sin x\cos^2 x dx$$

$$\int_0^{\Pi} x\sin x\cos^2 x dx = \frac{\Pi}{2}\int_0^{\Pi} \sin x\cos^2 x dx$$

Now

$$\int_0^{\Pi} \sin x\cos^2 x dx$$

Let cosx=t

$$\sin x dx = -dt$$

$$-\int_1^{-1} t^2 dt$$

$$\int_{-1}^{1} t^2 dt$$

$$\left\{\frac{t^3}{3}\right\}_{-1}^{1}$$

$$\frac{2}{3}$$

$$\therefore \int_0^{\pi} x\sin x\cos^2 x dx = \frac{\pi}{2} \times \frac{2}{3} = \frac{\pi}{3}$$

Question 22

Evaluate $\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x} dx$

Solution 22

We have,

$$I = \int_0^{\frac{\pi}{2}} \frac{x \sin x . \cos x}{\sin^4 x + \cos^4 x} dx \qquad --(i)$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \cos x . \sin x}{\cos^4 x + \sin^4 x} dx \qquad --(ii)$$

Adding (i) & (ii)

$$2I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x . \cos x}{\cos^4 x + \sin^4 x} dx$$

$$2I = \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{2 \sin x . \cos x}{\cos^4 x + \sin^4 x} dx$$

Let $t = \sin^2 x$

$$\Rightarrow 2I = \frac{\pi}{4} \int_0^1 \frac{1}{(1-t)^2 + t^2} dt$$

$$\Rightarrow 2I = \frac{\pi}{8} \int_0^1 \frac{1}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} dt$$

$$\Rightarrow 2I = \frac{\pi}{8} \times 2 \left[\tan^{-1}(2t-1) \right]_0^1$$

$$\Rightarrow I = \frac{\pi}{8} \left[\frac{\pi}{4} + \frac{\pi}{4} \right] \quad = \quad \frac{\pi^2}{16}$$

Question 23

Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \, dx$

Solution 23

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \, dx$

$f(-x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3(-x) \, dx$

$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \, dx$

Here $f(x) = -f(+x)$

Hence $f(x)$ is odd function.

So,
$I = 0$

Question 24

Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x \, dx$

Solution 24

We have,

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^4 x\,dx \qquad \left[\because \sin^4 x \text{ is an even function}\right]$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\sin^2 x\right)^2 dx$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1-\cos 2x\right)^2 dx$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \left(1+\cos^2 2x - 2\cos 2x\right) dx\right]$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \left(1-2\cos 2x + \frac{1+\cos 4x}{2}\right) dx\right]$$

$$= \frac{1}{4}\left[\int_0^{\frac{\pi}{2}} \left(3-4\cos 2x + \cos 4x\right) dx\right]$$

$$= \frac{1}{4}\left[3x - \frac{4\sin 2x}{2} + \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[\left\{\frac{3\pi}{2} - 2\sin\pi + \frac{1}{4}\sin 2\pi\right\} - \left\{0-0+0\right\}\right]$$

$$= \frac{1}{4}\left[\frac{3\pi}{2} - 0 + 0\right] = \frac{1}{4}\times\frac{3\pi}{2}$$

$$= \frac{3\pi}{8}$$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x\,dx = \frac{3\pi}{8}$$

Question 25

Evaluate $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx$

Solution 25

We have,

$$I = \int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx$$

Since, $\log\left\{\frac{2-(-x)}{2+(-x)}\right\} = -\log\left(\frac{2-x}{2+x}\right)$ $\therefore$ This is an odd function.

Hence,

$I = 0$

Question 26

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\, dx$

Solution 26

We have,

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\, dx$$

$\sin^2 x$ is even function.

Hence,

$$I = 2\int_{0}^{\frac{\pi}{4}} \sin^2 x\, dx = 2\int_{0}^{\frac{\pi}{4}} \left(\frac{1-\cos 2x}{2}\right) dx = \frac{2}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{2\pi}{4} - \sin\frac{\pi}{2} - 0 + \sin 0\right]$$

$$= \frac{1}{2}\left[\frac{2\pi}{4} - 1\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

$$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\, dx = \frac{\pi}{4} - \frac{1}{2}$$

Question 27

Evaluate $\int_{0}^{\pi} \log(1 - \cos x)\, dx$

Solution 27

$$I = \int_0^{\pi} \log(1 - \cos x)\, dx$$

$$= \int_0^{\pi} \log\left(2\sin^2 \frac{x}{2}\right) dx$$

$$= \int_0^{\pi} \log 2\, dx + \int_0^{\pi} \log \sin^2 \frac{x}{2}\, dx$$

$$= \int_0^{\pi} \log 2\, dx + 2\int_0^{\pi} \log \sin \frac{x}{2}\, dx$$

$$I = \log 2[x]_0^{\pi} + 4\int_0^{\frac{\pi}{2}} \log \sin t\, dt \qquad \left[\text{Put} \quad t = \frac{x}{2} \Rightarrow dt = \frac{1}{2}dx\right]$$

$$I = \pi \log 2 + 4I_1 \qquad \ldots(i)$$

$$I_1 = \int_0^{\frac{\pi}{2}} \log \sin t\, dt \qquad \ldots(ii)$$

$$= \int_0^{\frac{\pi}{2}} \log \cos t\, dt \qquad \ldots(iii)$$

Adding (ii) & (iii) we get

$$2I_1 = \int_0^{\frac{\pi}{2}} \log \sin t . \cos t\, dt = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2t}{2}\right) dt = \int_0^{\frac{\pi}{2}} \log \sin 2t\, dt - \frac{\pi}{2}\log 2$$

We know the property $\int_a^b f(x) = \int_a^b f(t)$

$$2I_1 = I_1 - \frac{\pi}{2}\log 2$$

$$\Rightarrow I_1 = -\frac{\pi}{2}\log 2 \qquad \ldots(iv)$$

Putting the value from (iv) to (i)

$$I = \pi \log 2 + 4\left(-\frac{\pi}{2}\log 2\right) = \pi \log 2 - 2\pi \log 2 = -\pi \log 2$$

$$I = -\pi \log 2$$

Question 28

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\left(\frac{2 - \sin x}{2 + \sin x}\right) dx$

Solution 28

We have,

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\left(\frac{2-\sin x}{2+\sin x}\right) dx$$

Let $f(x) = \log\left(\frac{2-\sin x}{2+\sin x}\right)$

Then,

$$f(-x) = \log\left(\frac{2-\sin(-x)}{2+\sin(-x)}\right) = -\log\left(\frac{2-\sin x}{2+\sin x}\right) = -f(x)$$

Thus, f(x) is an odd function.

$$\therefore I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\left(\frac{2-\sin x}{2+\sin x}\right) dx = 0$$

Question 29

$$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\, dx$$

Solution 29

$$I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\, dx$$

$$I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\, dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\, dx$$

$$I = 0 + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\, dx \ldots\ldots\ldots \left[\because \frac{2x}{1+\cos^2 x} \text{ is an odd function}\right]$$

$$I = 2\int_{0}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\, dx \ldots\ldots\ldots \left[\because \frac{2x\sin x}{1+\cos^2 x} \text{ is an even function}\right]$$

$$I = 4\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx$$

$$I = 2\pi\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx \ldots\ldots\ldots \left[\because \int_{0}^{a} xf(x)\,dx = \frac{a}{2}\int_{0}^{a} f(x)\,dx\right]$$

Put $\cos x = t$ then $-\sin x\, dx = dt$

$$I = -2\pi\int_{1}^{-1} \frac{1}{1+t^2}\, dt$$

$$I = -2\pi\left[\tan^{-1} t\right]_{1}^{-1}$$

$$I = \pi^2$$

Question 30

$$\int_{-a}^{a} \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right) d\theta,\ a>0$$

Solution 30

$$I = \int_{-a}^{a} \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right) d\theta$$

$$\text{Let } f(\theta) = \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right)$$

$$f(-\theta) = \log\left(\frac{a-\sin(-\theta)}{a+\sin(-\theta)}\right) = -\log\left(\frac{a-\sin\theta}{a+\sin\theta}\right) = -f(\theta)$$

$$\therefore\ f(\theta) = \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right) \text{ is an odd function.}$$

$$\therefore\ I = \int_{-a}^{a} \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right) d\theta = 0$$

Question 31

$$\int_{-2}^{2} \frac{3x^3+2|x|+1}{x^2+|x|+1}\, dx$$

Solution 31

$$I = \int_{-2}^{2} \frac{3x^3+2|x|+1}{x^2+|x|+1}\, dx$$

$$I = \int_{-2}^{2} \frac{3x^3}{x^2+|x|+1}\, dx + \int_{-2}^{2} \frac{2|x|+1}{x^2+|x|+1}\, dx$$

$$I = 0 + \int_{-2}^{2} \frac{2|x|+1}{x^2+|x|+1}\, dx \ldots\ldots\ldots \left[\because \frac{3x^3}{x^2+|x|+1} \text{ is an odd function}\right]$$

$$I = 2\int_{0}^{2} \frac{2|x|+1}{x^2+|x|+1}\, dx \ldots\ldots\ldots \left[\because \frac{2|x|+1}{x^2+|x|+1} \text{ is an even function}\right]$$

$$I = 2\left[\log\left(x^2+|x|+1\right)\right]_0^2$$

$$I = 2\left[\log(4+2+1)-\log(1)\right]$$

$$I = 2\log_e(7)$$

Question 32

$$\int_{-3\pi/2}^{-\pi/2} \left\{\sin^2(3\pi+x)+(\pi+x)^3\right\}\, dx$$

Solution 32

$$I = \int_{-3\pi/2}^{-\pi/2} \left\{\sin^2(3\pi + x) + (\pi + x)^3\right\} \; dx$$

Substitute $\pi + x = u$ then $dx = du$

$$I = \int_{-\pi/2}^{\pi/2} \left\{\sin^2(2\pi + u) + (u)^3\right\} \; du$$

$$I = \int_{-\pi/2}^{\pi/2} \left\{\sin^2(u) + (u)^3\right\} \; du$$

$$I = \left[\frac{1}{2}\left(u - \frac{1}{2}\sin(2u)\right) + \frac{u^4}{4}\right]_{-\pi/2}^{\pi/2}$$

$$I = \frac{\pi}{2}$$

Question 33

By using the properties of definite integrals, evaluate the integral

$$\int_0^2 x\sqrt{2-x}\,dx$$

Solution 33

$$\text{Let } I = \int_0^2 x\sqrt{2-x}\,dx$$

$$I = \int_0^2 (2-x)\sqrt{x}\,dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$= \int_0^2 \left\{2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right\}dx$$

$$= \left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^2$$

$$= \frac{4}{3}(2)^{\frac{3}{2}} - \frac{2}{5}(2)^{\frac{5}{2}}$$

$$= \frac{4\times 2\sqrt{2}}{3} - \frac{2}{5}\times 4\sqrt{2}$$

$$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

$$= \frac{40\sqrt{2} - 24\sqrt{2}}{15}$$

$$= \frac{16\sqrt{2}}{15}$$

Question 34

$$\int_0^1 \log\left(\frac{1}{x} - 1\right)dx$$

Evaluate the integral

Solution 34

$$Let\ I=\int_0^1 \log\left(\frac{1}{x}-1\right)dx$$

$$=\int_0^1 \log\left(\frac{1-x}{x}\right)dx$$

$$=\int_0^1 \log(1-x)dx-\int_0^1 \log(x)dx$$

$$Applying\ the\ property,\ \int_0^a f(x)dx=\int_0^a f(a-x)dx$$

$$Thus,\ I=\int_0^1 \log(1-(1-x))dx-\int_0^1 \log(x)dx$$

$$=\int_0^1 \log(1-1+x)dx-\int_0^1 \log(x)dx$$

$$=\int_0^1 \log(x)dx-\int_0^1 \log(x)dx$$

$$=0$$

Question 35

$$\int_{-1}^{1} |\, x \cos \pi\ x \,|\ dx$$

Solution 35

$$I = \int_{-1}^{1} |x\cos \pi x|\,dx$$

Let $f(x) = |x\cos \pi x|$

$$f(-x) = |-x\cos(-\pi x)| = |-x\cos(\pi x)| = |x\cos \pi x| = f(x)$$

$$\therefore\ I = \int_{-1}^{1} |x\cos \pi x|\,dx = 2\int_{0}^{1} |x\cos \pi x|\,dx$$

Now,

$$f(x) = |x\cos \pi x| = \begin{cases} x\cos \pi x\ ,\ \text{if } 0 \le x \le \frac{1}{2} \\ -x\cos \pi x\ ,\ \text{if } \frac{1}{2} < x < 1 \end{cases}$$

$$\therefore\ I = 2\int_{0}^{1} |x\cos \pi x|\,dx$$

$$\Rightarrow I = 2\left[\int_{0}^{\frac{1}{2}} x\cos \pi x\ dx + \int_{\frac{1}{2}}^{1} -x\cos \pi x\,dx\right]$$

$$\Rightarrow I = 2\left\{\left[\frac{x}{\pi}\sin \pi x + \frac{1}{\pi^2}\cos \pi x\right]_0^{\frac{1}{2}} - \left[\frac{x}{\pi}\sin \pi x + \frac{1}{\pi^2}\cos \pi x\right]_{\frac{1}{2}}^{1}\right\}$$

$$\Rightarrow I = 2\left\{\left[\frac{1}{2\pi} - \frac{1}{\pi^2}\right] - \left[-\frac{1}{\pi^2} - \frac{1}{2\pi}\right]\right\}$$

$$\Rightarrow I = \frac{2}{\pi}$$

Question 36

$$\int_0^{\pi}\left(\frac{x}{1+\sin^2 x} + \cos^7 x\right)dx$$

Solution 36

$$I = \int_0^{\Pi}\left(\frac{x}{1+\sin^2 x} + \cos^7 x\right)dx$$

$$I = \int_0^{\Pi}\left(\frac{\Pi - x}{1+\sin^2(\Pi - x)} + \cos^7(\Pi - x)\right)dx$$

$$I = \int_0^{\Pi}\left(\frac{\Pi - x}{1+\sin^2 x} - \cos^7 x\right)dx$$

$$2I = \int_0^{\Pi}\left(\frac{\Pi}{1+\sin^2 x}\right)dx$$

$$2I = \Pi\int_0^{\Pi} \frac{1}{1+\sin^2 x}\,dx$$

$$2I = \Pi\int_0^{\Pi} \frac{1}{1+2\tan^2 x}\sec^2 x\,dx$$

$$I = \pi\int_0^{\frac{\pi}{2}} \frac{1}{1+2\tan^2 x}\sec^2 x\,dx \ldots\ldots\ldots\ldots \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)\right]$$

Let $\tan x = v$

$dv = \sec^2 x\ dx$

$$\Rightarrow I = \pi\int_0^{\infty} \frac{1}{1+2v^2}\,dv$$

$$\Rightarrow I = \pi\left[\frac{\tan^{-1}\left(\sqrt{2}v\right)}{\sqrt{2}}\right]_0^{\infty}$$

$$\Rightarrow I = \pi\left[\frac{\pi}{2\sqrt{2}}\right]$$

$$\Rightarrow I = \frac{\pi^2}{2\sqrt{2}}$$

Question 37

$$\int_0^{\pi} \frac{x}{1+\cos\alpha\sin x}\,dx$$

Solution 37

$$I=\int_0^{\pi}\frac{x}{1+\cos\alpha\sin x}dx$$

Then,

$$I=\int_0^{\pi}\frac{(\pi-x)}{1+\cos\alpha\sin(\pi-x)}dx$$

$$I=\int_0^{\pi}\frac{(\pi-x)}{1+\cos\alpha\sin x}dx$$

$$2I=\pi\int_0^{\pi}\frac{1}{1+\cos\alpha\sin x}dx$$

$$2I=\pi\int_0^{\pi}\frac{1+\tan^2\left(\frac{x}{2}\right)}{\left(1+\tan^2\left(\frac{x}{2}\right)\right)+2\cos\alpha\tan\left(\frac{x}{2}\right)}dx$$

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{\sec^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)+2\cos\alpha\tan\left(\frac{x}{2}\right)+1}dx$$

Put $\tan\left(\frac{x}{2}\right)=t$ then $\sec^2\left(\frac{x}{2}\right)dx=2dt$

$x=0\Rightarrow t=0$ and $x=\pi\Rightarrow t=\infty$

$$I=\frac{\pi}{2}\int_0^{\infty}\frac{2}{t^2+2t\cos\alpha+1}dt$$

$$I=\pi\int_0^{\infty}\frac{1}{(t+\cos\alpha)^2+(1-\cos^2\alpha)}dt$$

$$I=\pi\int_0^{\infty}\frac{1}{(t+\cos\alpha)^2+\sin^2\alpha}dt$$

$$I=\frac{\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^{\infty}$$

$$I=\frac{\pi\alpha}{\sin\alpha}$$

Question 38

$$\int_0^{2\pi}\sin^{100}x\cos^{101}x\,dx$$

Solution 38

We know

$$\int_0^{2a}f(x)dx=\int_0^{a}f(x)+\int_0^{a}f(2a-x)dx$$

Also here

$$f(x) = f(2\Pi - x)$$

So

$$I = \int_0^{2\pi} \sin^{100} x \cos^{101} x dx = 2\int_0^{\pi} \sin^{100} x \cos^{101} x dx$$

$$I = 2\int_0^{\pi} \sin^{100}(\Pi - x)\cos^{101}(\Pi - x)dx$$

$$I = -2\int_0^{\pi} \sin^{100} x \cos^{101} x dx$$

Hence

$$2I = 0$$

$$I = 0$$

Question 39

$$\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx$$

Solution 39

$$I = \int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx$$

Then,

$$I = \int_0^{\pi/2} \frac{a\sin\left(\frac{\pi}{2} - x\right) + b\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$I = \int_0^{\pi/2} \frac{a\cos x + b\sin x}{\cos x + \sin x} dx$$

$$2I = \int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{a\cos x + b\sin x}{\cos x + \sin x} dx$$

$$2I = (a+b)\int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$I = \frac{(a+b)}{2}\int_0^{\pi/2} 1 dx$$

$$I = \frac{(a+b)\pi}{4}$$

Question 40

If f is an integrable function such that $f(2a - x) = f(x)$, then prove that

$$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$$

Solution 40

We have,

$$I = \int_0^{2a} f(x)\,dx$$

Then

$$I = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$$

$$I = \int_0^{a} f(x)\,dx + I_1$$

where, $I_1 = \int_a^{2a} f(x)\,dx$

Let $2a - t = x$ then $dx = -dt$

If $t = a \Rightarrow x = a$

If $t = 2a \Rightarrow x = 0$

$$I_1 = \int_0^{2a} f(x)\,dx = \int_a^{0} f(2a-t)(-dt) = -\int_a^{0} f(2a-t)\,dt$$

$$I_1 = \int_0^{a} f(2a-t)\,dt = \int_0^{a} f(2a-x)\,dx$$

$$\therefore\ I = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$$

$$I = \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad [f(2a-x) = f(x)]$$

Hence Proved.

Question 41

If $f(2a-x) = -f(x)$, prove that $\int_0^{2a} f(x)\,dx = 0$

Solution 41

We have,

$$I = \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$

$$I = \int_0^a f(x)\,dx + I_1$$

Let $2a - t = x$ then $dx = -dt$

$t = a,\ x = a$

$t = 2a \quad x = 0$

$$I_1 = \int_0^{2a} f(x) = \int_a^0 f(2a - t)(-dt)$$

$$= -\int_a^0 f(2a - t)\,dt$$

$$I_1 = \int_0^a f(2a - t)\,dt = \int_0^a f(2a - x)\,dx$$

$$I = \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx$$

$$I = \int_0^a f(x)\,dx - \int_0^a f(x) \qquad \left[\because f(2a - x) = -f(x)\right]$$

$$I = 0$$

Hence,

$$\int_0^{2a} f(x)\,dx = 0$$

Question 42

If $f(x)$ is an integrable function, show that

(i) $\int_{-a}^{a} f(x^2)\,dx = 2\int_0^a f(x^2)\,dx$

(ii) $\int_{-a}^{a} x f(x^2)\,dx = 0$

Solution 42

(i) We have,

$$I = \int_{-a}^{a} f\left(x^2\right) dx$$

Clearly $f\left(x^2\right)$ is an even function.

So,

$$\int_{-a}^{a} f\left(t\right) = 2\int_{0}^{a} f\left(t\right)$$

$$I = 2\int_{0}^{a} f\left(x^2\right) dx$$

(ii) We have,

$$I = \int_{-a}^{a} xf\left(x^2\right) dx$$

Clearly, $xf\left(x^2\right)$ is odd function.

So, $I = 0$

$$\therefore \int_{-a}^{a} xf\left(x^2\right) dx = 0$$

Question 43

If $f\left(x\right)$ is a continuous function defined on $\left[0, 2a\right]$. Then, prove that

$$\int_{0}^{2a} f\left(x\right) dx = \int_{0}^{a} \left\{f\left(x\right) + f\left(2a - x\right)\right\} dx$$

Solution 43

We have from LHS,

$$I = \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx \qquad \ldots(i)$$

Let $x = 2a - t$, then $dx = -dt$

$x = a \Rightarrow t = a$, and $x = 2a \Rightarrow t = 0$

$$\therefore \int_0^{2a} f(x)\,dx = -\int_a^{0} f(2a-t)\,dt$$

$$\Rightarrow \int_0^{2a} f(x)\,dx = \int_0^{a} f(2a-t)\,dt$$

$$\Rightarrow \int_0^{2a} f(x)\,dx = \int_0^{a} f(2a-x)\,dx$$

Substituting $\int_0^{2a} f(x)\,dx = \int_0^{a} f(2a-x)\,dx$ in (i)

we get,

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$$

$$\Rightarrow \int_0^{2a} f(x)\,dx = \int_0^{a} \{f(x) + f(2a-x)\}\,dx$$

Question 44

If $f(a+b-x) = f(x)$, then prove that

$$\int_a^b x f(x)\,dx = \left(\frac{a+b}{2}\right)\int_a^b f(x)\,dx$$

Solution 44

$$I = \int_a^b x f(x)\,dx$$

$$\Rightarrow I = \int_a^b (a+b-x)\,f(a+b-x)\,dx$$

$$\Rightarrow I = \int_a^b (a+b-x)\,f(x)\,dx \ldots\ldots\ldots\ldots [\text{Given that } f(a+b-x) = f(x)]$$

$$\Rightarrow I = \int_a^b (a+b)\,f(x)\,dx - \int_a^b x f(x)\,dx$$

$$\Rightarrow I = \int_a^b (a+b)\,f(x)\,dx - I$$

$$\Rightarrow 2I = \int_a^b (a+b)\,f(x)\,dx$$

$$\Rightarrow I = \frac{a+b}{2}\int_a^b f(x)\,dx$$

Question 45

If $f(x)$ is a continuous function defined on $[-a, a]$, then, prove that

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a}\{f(x) + f(-x)\}dx$$

Solution 45

We have,

$$I = \int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

Let $x = -t$ then $dx = -dt$

$x = -a \Rightarrow t = a$

$x = 0 \Rightarrow t = 0$

$$\therefore \int_{-a}^{a} f(x)dx = \int_{a}^{0} f(-t)(-dt) = -\int_{a}^{0} f(-t)dt$$

$$\Rightarrow \int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-t)dt$$

$$\Rightarrow \int_{-a}^{0} f(x)dx = \int_{0}^{a} f(-x)dx$$

$$\therefore \int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx$$

Hence,

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a}\{f(-x) + f(x)\}dx$$

Proved

Question 46

Prove that: $\int_{0}^{\pi} x\, f(\sin x)dx = \frac{\pi}{2}\int_{a}^{b} f(\sin x)\, dx.$

Solution 46

$$I = \int_{0}^{\Pi} xf(\sin x)dx$$

$$I = \int_{0}^{\Pi} (\Pi - x)f(\sin(\Pi - x))dx$$

$$I = \int_{0}^{\Pi} (\Pi - x)f(\sin x)dx$$

$$2I = \int_{0}^{\Pi} \Pi f(\sin x)dx$$

$$I = \frac{\Pi}{2}\int_{0}^{\Pi} f(\sin x)dx$$

Chapter 20 - Definite Integrals Exercise Ex. 20.5

Question 1

Evaluate $\int_0^3 (x+4)\,dx$

Solution 1

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----+f(a+(n-1)h)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=0,\ b=3$ and $f(x)=(x+4)$

$$h=\frac{3}{n}\Rightarrow nh=3$$

Thus, we have,

$$\Rightarrow I=\int_0^3 (x+4)\,dx$$

$$\Rightarrow I=\lim_{h\to 0} h\left[f(0)+f(h)+f(2h)+----f((n-1)h)\right]$$

$$\Rightarrow I=\lim_{h\to 0} h\left[4+(h+4)+(2h+4)+----+((n-1)h+4)\right]$$

$$\Rightarrow I=\lim_{h\to 0} h\left[4n+h(1+2+3+----+(n-1))\right]$$

$$\Rightarrow I=\lim_{h\to 0} h\left[4n+h\left(\frac{n(n-1)}{2}\right)\right] \qquad \left[\because h\to 0\ \&\ h=\frac{3}{n}\Rightarrow n\to\infty\right]$$

$$\Rightarrow I=\lim_{n\to\infty}\frac{3}{n}\left[4n+\frac{3}{n}\frac{(n^2-1)}{2}\right]$$

$$\Rightarrow I=\lim_{n\to\infty} 12+\frac{9}{2}\left(1-\frac{1}{n}\right)$$

$$=12+\frac{9}{2}=\frac{33}{2}$$

$$\therefore \int_0^3 (x+4)\,dx=\frac{33}{2}$$

Question 2

Evaluate $\int_0^2 (x+3)\,dx$

Solution 2

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----+f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here $a = 0,\ b = 2$

$$\Rightarrow h = \frac{2}{n}\ \&\ f(x) = x+3$$

Thus, we have,

$$I = \int_0^2 (x+3)\,dx$$

$$\Rightarrow I = \lim_{h\to 0} h\left[f(0)+f(h)+f(2h)+----f((n-1)h)\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[3+(h+3)+(2h+3)+(3h+3)----+(n-1)h+3\right]$$

$$= \lim_{h\to 0} h\left[3n+h(1+2+3+----(n-1))\right]$$

$$= \lim_{h\to 0} h\left[3n+h\frac{n(n-1)}{2}\right]$$

$$\because h = \frac{2}{n}\ \&\text{if } h\to 0 \Rightarrow n\to\infty$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[3n+\frac{2}{n}\frac{n(n-1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[6+\frac{2}{n}n^2\left(1-\frac{1}{n}\right)\right]$$

$$= 6+2 = 8$$

$$\therefore \int_0^2 (x+3)\,dx = 8$$

Question 3

Evaluate $\int_1^3 (3x-2)\,dx$

Solution 3

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----+f(a+(n-1)h)\right]$$

where $h = \frac{b-a}{n}$

Here $a = 1,\ b = 3$ and $f(x) = 3x-2$

$$h = \frac{2}{n} \Rightarrow nh = 2$$

Thus, we have,

$$I = \int_1^3 (3x-2)dx$$

$$\Rightarrow I = \lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h)+----f(1+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[1+\{3(1+h)-2\}+\{3(1+2h)-2\}+-----+\ \{3(1+(n-1)h)-2\}\right]$$

$$= \lim_{h\to 0} h\left[n+3h(1+2+3+----(n-1))\right]$$

$$= \lim_{h\to 0} h\left[n+3h\frac{n(n-1)}{2}\right]$$

$\because h = \frac{2}{n}$ $\quad\therefore$ if $h\to 0 \Rightarrow n\to\infty$

$$\therefore \lim_{n\to\infty} \frac{2}{n}\left[n+\frac{6}{n}\frac{n(n-1)}{2}\right]$$

$$= \lim_{n\to\infty} 2+\frac{6}{n^2}n^2\left(1-\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} 2+6 = 8$$

$$\therefore \int_1^3 (3x-2)dx = 8$$

Question 4

Evaluate $\int_{-1}^{1} (x+3)dx$

Solution 4

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + ----+f(a+(n-1)h)\right]$$

where $h = \dfrac{b-a}{n}$

Here $a = -1$, $b = 1$ and $f(x) = x + 3$

$$\therefore h = \frac{2}{n} \Rightarrow nh = 2$$

Thus, we have,

$$I = \int_{-1}^{1}(x+3)dx$$

$$I = \lim_{h\to 0} h\left[f(-1) + f(-1+h) + f(-1+2h) + ----+f(-1+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[2 + (2+h) + (2+2h) + ----+\{(n-1)h+2\}\right]$$

$$= \lim_{h\to 0} h\left[2n + h(1+2+3+----)\right]$$

$$= \lim_{h\to 0} h\left[2n + h\frac{n(n-1)}{2}\right] \qquad \left[\because h = \frac{2}{n} \text{ \& if } h \to 0 \Rightarrow n \to \infty\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[2n + \frac{2}{n}\frac{n(n-1)}{2}\right]$$

$$= \lim_{n\to\infty} 4 + \frac{2n^2}{n^2}\left(1 - \frac{1}{n}\right)$$

$$= 4 + 2 = 6$$

$$\therefore \int_{-1}^{1}(x+3)dx = 6$$

Question 5

Evaluate $\int_0^5 (x+1)dx$

Solution 5

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

where $h = \frac{b-a}{n}$

Here $a = 0,\ b = 5$

and $f(x) = (x+1)$

$\therefore h = \frac{5}{n} \Rightarrow nh = 5$

Thus, we have,

$$I = \int_0^5 (x+1)dx$$

$$I = \lim_{h\to 0} h\left[f(0)+f(h)+f(2h)+----f\{(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\left[1+(h+1)+(2h+1)+-----+((n-1)h+1)\right]$$

$$= \lim_{h\to 0} h\left[n+h(1+2+3+----(n-1))\right]$$

$\because h = \frac{5}{n}$ and if $h \to 0,\ n \to \infty$

$$= \lim_{n\to\infty} \frac{5}{n}\left[n+\frac{5}{n}\frac{n(n-1)}{2}\right]$$

$$= \lim_{n\to\infty} 5+\frac{25}{2n^2}n^2\left(1-\frac{1}{n}\right)$$

$$= 5+\frac{25}{2}$$

$$\therefore \int_0^5 (x+1)dx = \frac{35}{2}$$

Question 6

Evaluate $\int_1^3 (2x+3)dx$

Solution 6

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

$$\text{where } h=\frac{b-a}{n}$$

Here, $a=1$, $b=3$

and $f(x)=(2x+3)$

$\therefore\ h=\dfrac{2}{n}\Rightarrow nh=2$

Thus, we have,

$$I=\int_1^3(2x+3)\,dx$$

$$=\lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h)+----f(1+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[2+3+\{2(1+h)+3\}+\{2(1+2h)+3\}----+2\{1+(n-1)+3\}\right]$$

$$=\lim_{h\to 0} h\left[5+(5+2h)+(5+4h)+----5+2(n-1)h\right]$$

$$=\lim_{h\to 0} h\left[5n+2h(1+2+3+----(n-1))\right]$$

$\because h=\dfrac{2}{n}$ and if $h\to 0\Rightarrow n\to\infty$

$$\therefore \lim_{n\to\infty}\frac{2}{n}\left[5n+\frac{4}{n}\frac{n(n-1)}{2}\right]$$

$$=\lim_{n\to\infty}\left[10+\frac{4}{n^2}\frac{n(n-1)}{2}\right]\ =14$$

$$\therefore \int_1^3(2x+3)\,dx=14$$

Question 7

Evaluate $\displaystyle\int_3^5(2-x)\,dx$

Solution 7

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here, $a = 3$, $b = 5$

and $f(x) = (2-x)$

$\therefore\ h = \frac{2}{n} \Rightarrow nh = 2$

Thus, we have,

$$I = \int_3^5 (2-x)\,dx$$

$$= \lim_{h\to 0} h\left[f(3)+f(3+h)+f(3+2h)+----f(3+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[(2-3)+\{2-(3+h)\}+\{2-(3+2h)\}+----\{2-(3+(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\left[-1+(-1-h)+(-1-2h)+----\{-1-(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\left[-n-h(1+2+----(n-1)h\right]$$

$\because h = \frac{2}{n}$ & if $h \to 0 \Rightarrow n \to \infty$

$$\therefore \lim_{n\to\infty} \frac{2}{n}\left[-n-\frac{2}{n}\frac{n(n-1)}{2}\right]$$

$$= \lim_{n\to\infty} -2-\frac{2}{n^2}n^2\left(1-\frac{1}{n}\right) = -2-2 = -4$$

$$\therefore \int_3^5 (2-x)\,dx = -4$$

Question 8

Evaluate $\int_0^2 (x^2+1)\,dx$

Solution 8

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here $a = 0,\ b = 2$ and $f(x) = (x^2+1)$

$\therefore\ h = \frac{2}{n} \Rightarrow nh = 2$

Thus, we have,

$$I = \int_0^2 (x^2+1)dx$$

$$= \lim_{h\to 0} h\left[f(0)+f(h)+f(2h)+----f\{(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\left[1+(h^2+1)+\{(2h)^2+1\}+----\{((n-1)h)^2+1\}\right]$$

$$= \lim_{h\to 0} h\left[n+h^2\left(1+2^2+3^2+-----+(n-1)^2\right)\right]$$

$$\because h = \frac{2}{n} \ \&\ \text{if } h \Rightarrow 0 \Rightarrow n \to \infty$$

$$\therefore \lim_{n\to\infty} \frac{2}{n}\left[n+\frac{4}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} 2+\frac{4}{3n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= 2+\frac{4}{3}\times 2 = \frac{14}{3}$$

$$\therefore \int_0^2 (x^2+1)dx = \frac{14}{3}$$

Question 9

Evaluate $\int_1^2 x^2 dx$

Solution 9

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + ---- f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here $a = 1,\ b = 2$ and $f(x) = x^2$

$\therefore\ h = \frac{1}{n} \Rightarrow nh = 1$

Thus, we have,

$$I = \int_1^2 x^2\,dx$$

$$= \lim_{h\to 0} h\left[f(1) + f(1+h) + f(1+2h) + ---- f(1+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[1 + (1+h)^2 + (1+2h)^2 + ---- + \left(1+(n-1)h\ \right)^2\right]$$

$$= \lim_{h\to 0} h\left[1 + \left(1+2h+h^2\right) + \left(1+2\times 2h + 2\times 2h^2\right) + ---- \left(1 + 2\times(n-1)h + (1-n)^2 h^2\right)\right]$$

$$= \lim_{h\to 0} h\left[n + 2h\left\{1+2+3 --- (n-1)\right\} + h^2\left\{1^2 + 2^2 + 3^2 + --- (n-1)^2\right\}\right]$$

$\because h = \frac{1}{n}$ & if $h \to 0 \Rightarrow n \to \infty$

$$= \lim_{n\to\infty} \frac{1}{n}\left[n + \frac{2}{n}\frac{n(n-1)}{2} + \frac{1}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} 1 + \frac{n^2}{n^2}\left(1-\frac{1}{n}\right) + \frac{1}{6n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= 1 + 1 + \frac{2}{6} = \frac{7}{3}$$

$$\therefore \int_1^2 x^2\,dx = \frac{7}{3}$$

Question 10

Evaluate $\int_2^3 \left(2x^2 + 1\right) dx$

Solution 10

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here $a = 2,\ b = 3$ and $f(x) = 2x^2+1$

$\therefore\ h = \frac{1}{n} \Rightarrow nh = 1$

Thus, we have,

$$I = \int_2^3 \left(2x^2+1\right)dx$$

$$= \lim_{h\to 0} h\left[f(2)+f(2+h)+f(2+2h)+----f(2+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[\left(2\times 2^2+1\right)\left\{2(2+h)^2+1\right\}+\left\{2(2+2h)^2+1\right\}+----+\left\{2(2+(n-1)h)^2+1\right\}\right]$$

$$= \lim_{h\to 0} h\left[9n+8h(1+2+3---)+2h^2\left(1^2+2^2+3^2+--\right)\right]$$

$\because h = \frac{1}{n}$ & if $h\to 0 \Rightarrow n\to\infty$

$$= \lim_{n\to\infty} \frac{1}{n}\left[9n+\frac{8}{n}\frac{n(n-1)}{2}+\frac{2}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty}\ 9+\frac{4}{n^2}n^2\left(1-\frac{1}{n}\right)+\frac{1}{3n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= 9+4+\frac{2}{3} = \frac{41}{3}$$

$$\therefore \int_2^3\left(2x^2+1\right)dx = \frac{41}{3}$$

Question 11

Evaluate $\int_1^2 \left(x^2-1\right)dx$

Solution 11

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f\left(a+(n-1)h\right)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here $a = 1,\ b = 2$ and $f(x) = x^2 - 1$

$\therefore\ h = \dfrac{1}{n} \ \Rightarrow\ nh = 1$

Thus, we have,

$$I = \int_1^2 \left(x^2-1\right)dx$$

$$= \lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h)+----f\left(1+(n-1)h\right)\right]$$

$$= \lim_{h\to 0} h\left[\left(1^2-1\right)\left\{(1+h)^2-1\right\}+\left\{(1+2h)^2-1\right\}+-----+\left\{\left(1+(n-1)h\right)^2-1\right\}\right]$$

$$= \lim_{h\to 0} h\left[0+2h(1+2+3+---)+h^2\left(1+2^2+3^2+--\right)\right]$$

$$\because h = \frac{1}{n} \ \&\ \text{if } h\to 0 \Rightarrow n\to\infty$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{2}{n}\frac{n(n-1)}{2}+\frac{1}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^2}n^2\left(1-\frac{1}{n}\right)+\frac{1}{6n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= 1+\frac{2}{6} = \frac{4}{3}$$

$$\therefore\ \int_1^2\left(x^2-1\right)dx = \frac{4}{3}$$

Question 12

Evaluate $\int_0^2 \left(x^2+4\right)dx$

Solution 12

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + ---- f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here $a = 0$, $b = 2$ and $f(x) = x^2 + 4$

$\therefore\ h = \dfrac{2}{n} \Rightarrow nh = 2$

Thus, we have,

$$I = \int_0^2 \left(x^2 + 4\right) dx$$

$$= \lim_{h\to 0} h\left[f(0) + f(h) + f(2h) + ---- f(0 + (n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[4\left(h^2 + 4\right) + \left\{(2h)^2 + 4\right\} + ---- \left\{(n-1)h^2 + 4\right\}\right]$$

$\because h = \dfrac{2}{n}$ & if $h \to 0 \Rightarrow n \to \infty$

$$= \lim_{n\to\infty} \frac{2}{n}\left[4n + \frac{4}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} 8 + \frac{4}{3n^2}n^3\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)$$

$$= 8 + \frac{4\times 2}{3} = \frac{32}{3}$$

$$\therefore\ \int_0^2 \left(x^2 + 4\right) dx = \frac{32}{3}$$

Question 13

Evaluate $\int_1^4 \left(x^2 - x\right) dx$

Solution 13

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here $a = 1$, $b = 4$ and $f(x) = x^2 - x$

$$h = \frac{3}{n} \Rightarrow nh = 3$$

Thus, we have,

$$I = \int_1^4 (x^2 - x)\,dx$$

$$= \lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h)+----f(1+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[(1^2-1)+\left\{(1+h)^2-(1+h)\right\}+\left\{(1+2h)^2-(1+h)\right\}+----\right]$$

$$= \lim_{h\to 0} h\left[0+(h+h^2)+\left\{2h+(2h)^2\right\}+----\right]$$

$$= \lim_{h\to 0} h\left[h+(1+2+3+---(n-1))+h^2\left\{(1+2^2+3^2+---(n-1)^2\right\}\right]$$

$$\because h = \frac{3}{n} \text{ \& if } h\to 0 \Rightarrow n\to\infty$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{3}{n}\frac{n(n-1)}{2}+\frac{9}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} \frac{9}{n^2}n^2\left(1-\frac{1}{n}\right)+\frac{3}{2n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= \frac{9}{2}+3 = \frac{27}{2}$$

$$\therefore \int_1^4 (x^2 - x)\,dx = \frac{27}{2}$$

Question 14

Evaluate $\int_0^1 (3x^2 + 5x)\,dx$

Solution 14

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

where $h = \dfrac{b-a}{n}$

Here, $a = 0,\ b = 1$ and $f(x) = 3x^2+5x$

$h = \dfrac{1}{n} \Rightarrow nh = 1$

Thus, we have,

$$I = \int_0^1 \left(3x^2+5x\right)dx$$

$$= \lim_{h\to 0} h\left[f(0)+f(0+h)+f(0+2h)+----f(0+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[\left\{0+\left(3h^2+5h\right)+\left\{3(2h)^2+5(2h)\right\}+----\right]\right.$$

$$= \lim_{h\to 0} h\left[\left\{3h^2\left(1+2^2+3^2+---(n-1)^2\right)\right\}+5h\left\{1+2+3+---(n-1)\right\}\right]$$

$\because h = \dfrac{1}{n}$ if $h\to 0 \Rightarrow n\to\infty$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{3}{n^2}\frac{n(n-1)(2n-1)}{6}+\frac{5}{n}\frac{n(n-1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n^3}\frac{n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{5}{2n^2}n^2\left(1-\frac{1}{n}\right)$$

$$= \frac{3\times 2}{6}+\frac{5}{2} = \frac{7}{2}$$

$$\therefore \int_0^1 \left(3x^2+5x\right)dx = \frac{7}{2}$$

Question 15

Evaluate the following in tegrals as limit of sums

$$\int_0^2 e^x dx$$

Solution 15

We have

$$\int_a^b f(x) = dx \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+...+f(a+(n-1)h)\right]$$

Where $h=\dfrac{b-a}{n}$

Here

$a=0, b=2$ and $f(x)=e^x$

Now

$$h=\frac{2}{n}$$

$$nh=2$$

Thus, we have

$$I=\int_0^2 e^x dx$$

$$=\lim_{h\to 0} h\left[f(0)+f(h)+f(2h)+...+f((n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[1+e^h+e^{2h}+...+e^{(n-1)h}\right]$$

$$=\lim_{h\to 0} h\left\{\frac{\left(e^h\right)^n-1}{e^h-1}\right\}$$

$$=\lim_{h\to 0} h\left\{\frac{e^{nh}-1}{e^h-1}\right\}$$

$$=\lim_{h\to 0} h\left\{\frac{e^2-1}{e^h-1}\right\} \quad [nh=2]$$

$$=\lim_{h\to 0}\left\{\frac{e^2-1}{\frac{e^h-1}{h}}\right\}$$

$$=e^2-1$$

Question 16

Evaluate $\int_a^b e^x dx$

Solution 16

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here, $a = a,\ b = b$ and $f(x) = e^x$

$$\therefore \quad h = \frac{b-a}{n} \quad \Rightarrow \ nh = b-a$$

Thus, we have,

$$I = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[e^a + e^{a+h} + e^{a+2h} + ----e^{a+(n-1)h}\right]$$

$$= \lim_{h\to 0} he^a\left[1+e^h+e^{2h}+e^{3h}+----e^{(n-1)h}\right]$$

$$= \lim_{h\to 0} he^a\left[1+e^h+\left(e^h\right)^2+\left(e^h\right)^3+----\left(e^h\right)^{n-1}\right]$$

$$= \lim_{h\to 0} he^a\left\{\frac{\left(e^h\right)^n-1}{e^h-1}\right\} \qquad \left[\because a+ar+ar^2+----ar^{n-1} = a\left\{\frac{r^n-1}{r-1}\right\} \text{ if } r>1\right]$$

$$= \lim_{h\to 0} he^a n\left\{\frac{e^{nh}-1}{nh}\right\}\times\left(\frac{h}{e^{h-1}}\right) \qquad \left[\because \lim_{\theta\to 0}\frac{e^\theta-1}{\theta} = 1 \quad \& \quad nh = b-a\right]$$

$$\therefore \lim_{h\to 0}\left(e^{b-a}-1\right) = e^b - e^a$$

$$\therefore \int_a^b e^x dx = e^b - e^a$$

Question 17

Evaluate the integral as limit of sum $\int_a^b \cos x dx$.

Solution 17

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f\left(a+(n-1)h\right)\right],\ \text{where } h = \frac{b-a}{n}.$$

Since we have to find $\int_a^b \cos x dx$

We have, $\quad f(x) = \cos x$

$$\therefore \quad I = \int_a^b \cos x dx$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[\cos a + \cos(a+h) + \cos(a+2h) + \ldots + \cos\left(a+(n-1)h\right)\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[\frac{\cos\left(a+(n-1)\frac{h}{2}\right)\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right] = \lim_{h\to 0} h\left[\frac{\cos\left(a+\frac{nh}{2}-\frac{h}{2}\right)\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[\frac{\cos\left(a+\frac{b-a}{2}-\frac{h}{2}\right)\sin\left(\frac{b-a}{2}\right)}{\sin\frac{h}{2}}\right] \qquad [\because nh = b-a]$$

$$\Rightarrow \quad I = \lim_{h\to 0}\left[\frac{\frac{h}{2}}{\sin\frac{h}{2}} \times 2\cos\left(\frac{a+b}{2}-\frac{h}{2}\right)\sin\left(\frac{b-a}{2}\right)\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0}\left(\frac{\frac{h}{2}}{\sin\frac{h}{2}}\right) \times \lim_{h\to 0} 2\cos\left(\frac{a+b}{2}-\frac{h}{2}\right)\sin\left(\frac{b-a}{2}\right) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$\Rightarrow \quad I = \sin b - \sin a \qquad [\because 2\cos A\sin B = \sin(A-B) - \sin(A+B)]$$

Question 18

Evaluate $\int_0^{\frac{\pi}{2}} \sin x\, dx$

Solution 18

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here, $a = 0,\ b = \frac{\pi}{2}$ and $f(x) = \sin x$

$$\therefore \quad h = \frac{\frac{\pi}{2}-0}{n} = \frac{\pi}{2n} \qquad nh = \frac{2}{\pi}$$

Thus, we have,

$$I = \int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$= \lim_{h\to 0} h\left[f(0)+f(0+h)+f(0+2h)+----f\{0+(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\left[\sin 0+\sin h+\sin 2h+----\sin(n-1)h\right]$$

$$= \lim_{h\to 0} h\left[\frac{\sin\left(\frac{nh}{2}-\frac{h}{2}\right)\times\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$= \lim_{h\to 0} h\left[\frac{\sin\left(\frac{\pi}{4}-\frac{h}{2}\right)\times\sin\frac{\pi}{4}}{\sin\frac{h}{2}}\right]$$

$$\left[\therefore \lim_{h\to 0}\frac{\sin\theta}{\theta} = 1\right] \qquad \therefore \lim_{h\to 0}\frac{h}{\sin\frac{h}{2}}\left[\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\right]$$

$$= 2\times\frac{1}{2} = 1$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin x\, dx = 1$$

Question 19

Evaluate the integral as limit of sum $\int_0^{\frac{\pi}{2}} \cos x dx$.

Solution 19

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

where $h = \dfrac{b-a}{n}$

Here, $a = 0$, $b = \dfrac{\pi}{2}$ and $f(x) = \cos x$

$$\therefore \quad h = \frac{\frac{\pi}{2}-0}{n} = \frac{\pi}{2n} \qquad nh = \frac{2}{\pi}$$

Thus, we have,

$$I = \int_0^{\frac{\pi}{2}} \cos x\, dx$$

$$= \lim_{h\to 0} h\left[f(0)+f(0+h)+f(0+2h)+----f\{0+(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\left[\cos 0+\cos h+\cos 2h+----\cos(n-1)h\right]$$

$$= \lim_{h\to 0} h\left[\frac{\cos\left(\frac{nh}{2}-\frac{h}{2}\right)\times\cos\frac{nh}{2}}{\cos\frac{h}{2}}\right]$$

$$= \lim_{h\to 0} h\left[\frac{\cos\left(\frac{\pi}{4}-\frac{h}{2}\right)\times\cos\frac{\pi}{4}}{\cos\frac{h}{2}}\right]$$

$$\left[\therefore \lim_{h\to 0}\frac{\cos\theta}{\theta} = 1\right] \qquad \therefore \lim_{h\to 0}\frac{h}{\cos\frac{h}{2}}\left[\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\right]$$

$$= 2\times\frac{1}{2} = 1$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos x\, dx = 1$$

Question 20

Evaluate $\int_1^4 \left(3x^2+2x\right)dx$

Solution 20

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

where $h = \dfrac{b-a}{n}$

Here, $a = 1,\ b = 4$ and $f(x) = 3x^2 + 2x$

$$I = \lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h)+----f(a+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[(3+2)+\left\{3(1+h)^2+2(1-+h)+\left\{3(1+2h)^2+2(1+2h)\right\}+----\right.\right]$$

$$= \lim_{h\to 0} h\left[5+8h(1+2+3+--)+3h^2\left(1+2^2+3^2+---\right)\right]$$

$$\because h = \frac{3}{n} \text{ \& if } h \to 0 \Rightarrow n \to \infty$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[5n+\frac{24}{n}\frac{n(n-1)}{2}+\frac{27}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty}\ 15+\frac{36}{n^2}n^2\left(1-\frac{1}{n}\right)+\frac{27}{2n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= 15+36+27 \quad = 78$$

$$\therefore \int_1^4 \left(3x^2+2x\right)dx = 78$$

Question 21

Evaluate $\int_0^2 \left(3x^2-2\right)dx$

Solution 21

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here, $a = 0$, $b = 2$ and $f(x) = 3x^2 - 2$

$$\therefore \quad h = \frac{2}{n} \quad \Rightarrow \quad nh = 2$$

Thus, we have,

$$I = \int_0^2 \left(3x^2 - 2\right)dx$$

$$= \lim_{h\to 0} h\left[f(0)+f(0+h)+f(0+2h)+----f\left\{0+(n-1)h\right\}\right]$$

$$= \lim_{h\to 0} h\left[-2+\left(3h^2-2\right)+\left\{3(2h)^2-2\right\}+----\right]$$

$$= \lim_{h\to 0} h\left[-2h+3h^2\left(1+2^2+3^2+----\right)\right]$$

$$\because h = \frac{2}{n} \quad \text{if } h\to 0 \Rightarrow n\to\infty$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[-2n+\frac{12}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} -4+\frac{4}{n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) = -4+8 = 4$$

$$\therefore \quad \int_0^2 \left(3x^2-2\right)dx = 4$$

Question 22

Evaluate $\int_0^2 \left(x^2+2\right)dx$

Solution 22

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

where $h = \dfrac{b-a}{n}$

Here, $a = 0,\ b = 2$ and $f(x) = x^2 + 2$

$\therefore\quad h = \dfrac{2}{n} \quad\Rightarrow\quad nh = 2$

Thus, we have,

$$I = \int_0^2 (x^2+2)dx$$

$$= \lim_{h\to 0} h\left[f(0)+f(0+h)+f(2h)+----f(0+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[2+(h^2+2)+\{(2h)^2+2\}+----\right]$$

$$= \lim_{h\to 0} h\left[2h+h^2(1+2^2+3^2+----(n-1)^2\right]$$

$\because h = \dfrac{2}{n}$ & if $h\to 0 \Rightarrow n\to\infty$

$$= \lim_{n\to\infty} \frac{2}{n}\left[2n+\frac{4}{n^2}\,\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} 4+\frac{4}{3n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= 4+\frac{8}{3} = \frac{20}{3}$$

$$\therefore\quad \int_0^2 (x^2+2)dx = \frac{20}{3}$$

Question 23

Evaluate the following definite integrals as limit of sums

$$\int_0^4 \left(x+e^{2x}\right)dx$$

Solution 23

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right],\ \text{where } h=\frac{b-a}{n}$$

Here, $a = 0,\ b = 4,$ and $f(x) = x + e^{2x}$

$$\therefore h = \frac{4-0}{n} = \frac{4}{n}$$

$$\Rightarrow \int_0^4 \left(x+e^{2x}\right)dx = (4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(h)+f(2h)+\ldots+f((n-1)h)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left(0+e^{0}\right)+\left(h+e^{2h}\right)+\left(2h+e^{2\cdot 2h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[1+\left(h+e^{2h}\right)+\left(2h+e^{4h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h+2h+3h+\ldots+(n-1)h\right\}+\left(1+e^{2h}+e^{4h}+\ldots+e^{2(n-1)h}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1+2+\ldots(n-1)\right\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(h(n-1)n\right)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}\cdot\frac{(n-1)n}{2}+\left(\frac{e^{8}-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$= 4(2)+4\lim_{n\to\infty}\frac{\left(e^{8}-1\right)}{\left(\dfrac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$= 8+\frac{4\cdot\left(e^{8}-1\right)}{8} \qquad \left(\lim_{x\to 0}\frac{e^{x}-1}{x}=1\right)$$

$$= 8+\frac{e^{8}-1}{2}$$

$$= \frac{15+e^{8}}{2}$$

Question 24

Evaluate $\int_0^2 \left(x^2+x\right)dx$

Solution 24

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + ---- f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Thus, we have,

$$I = \int_0^2 (x^2 + x)dx$$

$$= \lim_{h\to 0} h\left[f(0) + f(0+h) + f(0+2h) + ---- f\{(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\left[0 + (h^2 + h) + \{(2h)^2 + 2h\} + ----\right]$$

$$= \lim_{h\to 0} h\left[\{h^2(1 + 2^2 + 3^2 + ---- (n-1)^2\} + h\{ \quad 1 + 2 + 3 --- (n-1)\}\right]$$

$$\because h = \frac{2}{n} \quad \& \text{ if } h \to 0 \Rightarrow n \to \infty$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\frac{n(n-1)(2n-1)}{6} + \frac{2}{n}\frac{n(n-1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{4}{3n^3} n^3\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + \frac{2}{n^2} n^2\left(1 - \frac{1}{n}\right)$$

$$= \frac{8}{3} + 2 \quad = \frac{14}{3}$$

$$\therefore \int_0^2 (x^2 + x)dx = \frac{14}{3}$$

Question 25

Evaluate $\int_0^2 (x^2 + 2x + 1)dx$

Solution 25

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

$$\text{where } h=\frac{b-a}{n}$$

Here, $a=0,\ b=2$ and $f(x)=x^2+2x+1$

$$\therefore\quad h=\frac{2}{n}\quad\Rightarrow\quad nh=2$$

Thus, we have,

$$I=\int_0^2\left(x^2+2x+1\right)dx$$

$$=\lim_{h\to 0} h\left[f(0)+f(h)+f(2h)+----f(0+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[1+\left(h^2+2h+1\right)+\left\{(2h)^2+2\times 2h+1\right\}+----\right]$$

$$=\lim_{h\to 0} h\left[n+h^2\left(1+2^2+3^2+----(n-1)^2+2h\left(1+2+3----(n-1)\right]\right.\right.$$

$$\because h=\frac{2}{n}\quad \text{if } h\to 0\Rightarrow n\to\infty$$

$$=\lim_{n\to\infty}\frac{2}{n}\left[n+\frac{4}{n^2}\frac{n(n-1)(2n-1)}{6}+\frac{4}{n}\frac{n(n-1)}{2}\right]$$

$$=\lim_{n\to\infty} 2+\frac{4}{3n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{4}{n^2}n^2\left(1-\frac{1}{n}\right)$$

$$=\ 2+\frac{8}{3}+4\ =\frac{26}{3}$$

$$\therefore\quad \int_0^2\left(x^2+2x+1\right)dx=\frac{26}{3}$$

Question 26

Evaluate $\int_0^3\left(2x^2+3x+5\right)dx$

Solution 26

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

where $h = \dfrac{b-a}{n}$

Here, $a = 0$, $b = 3$ and $f(x) = 2x^2 + 3x + 5$

$$\therefore \quad h = \frac{3}{n} \Rightarrow nh = 3$$

Thus, we have,

$$I = \int_0^3 (2x^2 + 3x + 5)dx$$

$$= \lim_{h\to 0} h\left[f(0)+f(h)+f(2h)+----f((n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[5+(2h^2+3h+5)+\{2(2h)^2+3\times 2h+5\}+----\right]$$

$$= \lim_{h\to 0} h\left[5n+2h^2(1+2^2+3^2+----(n-1)^2+3h(1+2+3----(n-1))\right]$$

$$\because h = \frac{3}{n} \ \& \text{ if } h \to 0 \Rightarrow n \to \infty$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[5n + \frac{18}{n^2}\frac{n(n-1)(2n-1)}{6} + \frac{9}{n}\frac{n(n-1)}{2}\right]$$

$$= \lim_{n\to\infty} 15 + \frac{9}{n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + \frac{27}{2n^2}n^2\left(1-\frac{1}{n}\right)$$

$$= 15+18+\frac{27}{2} = \frac{93}{2}$$

$$\therefore \int_0^3 (2x^2+3x+5)dx = \frac{93}{2}$$

Question 27

Evaluate the following definite integrals as limit of sums

$$\int_a^b x\,dx$$

Solution 27

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right],\text{ where } h=\frac{b-a}{n}$$

Here, $a=a$, $b=b$, and $f(x)=x$

$$\therefore \int_a^b x\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[a+(a+h)\ldots(a+2h)\ldots a+(n-1)h\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[\Big(\underbrace{a+a+a+\ldots+a}_{n\text{ times}}\Big)+\big(h+2h+3h+\ldots+(n-1)h\big)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\big(1+2+3+\ldots+(n-1)\big)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{n(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{n}{n}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)(b-a)}{2n}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right]$$

$$=(b-a)\left[a+\frac{(b-a)}{2}\right]$$

$$=(b-a)\left[\frac{2a+b-a}{2}\right]$$

$$=\frac{(b-a)(b+a)}{2}$$

$$=\frac{1}{2}\left(b^2-a^2\right)$$

Question 28

Evaluate the following definite integrals as limit of sums

$$\int_0^5 (x+1)\,dx$$

Solution 28

$$\text{Let } I = \int_0^5 (x+1)\,dx$$

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)\ldots f(a+(n-1)h)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=0$, $b=5$, and $f(x)=(x+1)$

$$\Rightarrow h=\frac{5-0}{n}=\frac{5}{n}$$

$$\therefore \int_0^5 (x+1)\,dx = (5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+\ldots+f\left((n-1)\frac{5}{n}\right)\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+\ldots\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+1\ldots1}_{n\text{ times}}\right)+\left[\frac{5}{n}+2\cdot\frac{5}{n}+3\cdot\frac{5}{n}+\ldots(n-1)\frac{5}{n}\right]\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3\ldots(n-1)\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\cdot\frac{(n-1)n}{2}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5(n-1)}{2}\right]$$

$$=5\lim_{n\to\infty}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right]$$

$$=5\left[1+\frac{5}{2}\right]$$

$$=5\left[\frac{7}{2}\right]$$

$$=\frac{35}{2}$$

Question 29

Evaluate the following definite integrals as limit of sums

$$\int_2^3 x^2\,dx$$

Solution 29

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+2h)\ldots f\{a+(n-1)h\}\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=2,\ b=3,$ and $f(x)=x^2$

$$\Rightarrow h=\frac{3-2}{n}=\frac{1}{n}$$

$$\therefore \int_2^3 x^2\,dx = (3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)\ldots f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\ldots\left(2+\frac{(n-1)}{n}\right)^2\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2\cdot2\cdot\frac{1}{n}\right\}+\ldots+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2\cdot2\cdot\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{2^2+\ldots+2^2}_{n\text{ times}}\right)+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+\left(\frac{n-1}{n}\right)^2\right\}+2\cdot2\cdot\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\ldots+\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{1^2+2^2+3^2\ldots+(n-1)^2\right\}+\frac{4}{n}\left\{1+2+\ldots+(n-1)\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{4n-4}{2}\right]$$

$$=\lim_{n\to\infty}\left[4+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+2-\frac{2}{n}\right]$$

$$=4+\frac{2}{6}+2$$

$$=\frac{19}{3}$$

Question 30

Evaluate the following in tegrals as limit of sums

$$\int_{1}^{3}\left(x^{2}+x\right)dx$$

Solution 30

We have

$$\int_a^b f(x) = dx \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+...+f(a+(n-1)h)\right]$$

Where $h=\frac{b-a}{n}$

Here

$$a=1, b=3 \text{ and } f(x)=x^2+x$$

Now

$$h=\frac{2}{n}$$
$$nh=2$$

Thus, we have

$$I=\int_1^3 \left(x^2+x\right)dx$$
$$=\lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h)+...+f(1+(n-1)h)\right]$$
$$=\lim_{h\to 0} h\left[\left(1^2+1\right)+\left\{(1+h)^2+(1+h)\right\}+\left\{(1+2h)^2+(1+2h)\right\}+...\right]$$
$$=\lim_{h\to 0} h\left[\left(1^2+(1+h)^2+(1+2h)^2+...\right)+\left\{1+(1+h)+(1+2h)+...\right\}\right]$$
$$=\lim_{h\to 0} h\left[\left(n+2h(1+2+3+...)+h^2\left(1+2^2+3^3+...\right)\right)+\left(n+h(1+2+3+...)\right)\right]$$
$$=\lim_{h\to 0} h\left[\left(2n+3h(1+2+3+...+(n-1))+h^2\left(1+2^2+3^3+...+(n-1)^2\right)\right)\right]$$
$$\because h=\frac{2}{n} \text{ \& if } h\to 0 \Rightarrow n\to\infty$$
$$=\lim_{h\to\infty}\frac{2}{n}\left[2n+\frac{9}{n}\frac{n(n-1)}{2}+\frac{9}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$
$$=\frac{38}{3}$$

Question 31

Evaluate the following in tegrals as limit of sums

$$\int_0^2 (x^2 - x)\,dx$$

Solution 31

We have

$$\int_a^b f(x) = dx \lim_{k \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f\left(a+(n-1)h\right)\right]$$

Where $h = \dfrac{b-a}{n}$

Here

$$a = 0, b = 2 \text{ and } f(x) = x^2 - x$$

Now

$$h = \frac{2}{n}$$

$$nh = 2$$

Thus, we have

$$I = \int_0^2 \left(x^2 - x\right) dx$$

$$= \lim_{k \to 0} h\left[f(0) + f(h) + f(2h) + \ldots + f\left((n-1)h\right)\right]$$

$$= \lim_{k \to 0} h\left[\left\{(0)^2 - (0)\right\} + \left\{(h)^2 - (h)\right\} + \left\{(2h)^2 - (2h)\right\} + \ldots\right]$$

$$= \lim_{k \to 0} h\left[\left((h)^2 + (2h)^2 + \ldots\right) - \left\{(h) + (2h) + \ldots\right\}\right]$$

$$= \lim_{k \to 0} h\left[h^2\left(1 + 2^2 + 3^3 + \ldots + (n-1)^2\right) - h\left\{1 + 2 + 3 + \ldots + (n-1)\right\}\right]$$

$$\because h = \frac{2}{n} \text{ \& if } h \to 0 \Rightarrow n \to \infty$$

$$= \lim_{k \to \infty} \frac{2}{n}\left[\frac{9}{n^2}\frac{n(n-1)(2n-1)}{6} - \frac{9}{n}\frac{n(n-1)}{2}\right]$$

$$= \frac{2}{3}$$

Question 32

Evaluate the following in tegrals as limit of sums

$$\int_1^3 \left(2x^2 + 5x\right) dx$$

Solution 32

We have

$$\int_a^b f(x) = dx \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+...+f(a+(n-1)h)\right]$$

Where $h=\dfrac{b-a}{n}$

Here

$$a=1, b=3 \text{ and } f(x)=2x^2+5x$$

Now

$$h=\frac{2}{n}$$

$$nh=2$$

Thus, we have

$$I=\int_1^3 (2x^2+5x)dx$$

$$=\lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h)+...+f(1+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[(2+5)+\left\{2(1+h)^2+5(1+h)\right\}+\left\{2(1+2h)^2+5(1+2h)\right\}+...\right]$$

$$=\lim_{h\to 0} h\left[\left(7n+9h(1+2+3+...)+2h^2\left(1+2^2+3^3+...\right)\right)\right]$$

$$\because h=\frac{2}{n} \text{ \& if } h\to 0 \Rightarrow n\to\infty$$

$$=\lim_{h\to\infty}\frac{2}{n}\left[7n+\frac{18}{n}\frac{n(n-1)}{2}+\frac{8}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$=\frac{112}{3}$$

Question 33

Evaluate the following integral as limit of sums:

$$\int_1^3 (3x^2+1)dx$$

Solution 33

Given,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)\right],$$

where $h = \dfrac{b-a}{n}$

Here, $f(x) = 3x^2 + 1$, $a = 1$, $b = 3$. Therefore, $h = \dfrac{3-1}{n} = \dfrac{2}{n}$

$$\therefore I = \int_1^3 (3x^2 + 1)dx$$

$$\Rightarrow I = \lim_{h\to 0} h\left[f(1) + f(1+h) + f(1+2h) + \ldots + f(1+(n-1)h)\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[3(1)^2 + 1 + 3(1+h)^2 + 1 + 3(1+2h)^2 + 1 + \ldots + 3(1+(n-1)h)^2 + 1\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[3n + n + 6h(1+2+3+\ldots+(n-1)) + 3h^2\left(1^2 + 2^2 + \ldots + (n-1)^2\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[4n + \frac{12}{n}(1+2+3+\ldots+(n-1)) + 3\times\frac{4}{n^2}\left(1^2 + 2^2 + \ldots + (n-1)^2\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[8 + \frac{24}{n^2}\times\frac{n(n-1)}{2} + \frac{24}{n^3}\times\frac{(n-1)(n)(2n-1)}{6}\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[8 + 12\left(1 - \frac{1}{n}\right) + 4\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = 8 + 12 + 4\times 2 = 28$$

Chapter 20 - Definite Integrals Exercise Ex. 20RE

Question 1

Evaluate $\int_0^4 x\sqrt{4-x}\,dx$

Solution 1

$$\text{Let } x = 4\sin^2\theta$$
$$dx = 8\sin\theta\cos\theta\, d\theta$$

$$\text{Now, } x = 0 \Rightarrow \theta = 0$$
$$x = 4 \Rightarrow \theta = \frac{\pi}{2}$$

$$\int_0^4 x\sqrt{4-x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} 4\sin^2\theta \times 2\cos\theta \times 8\sin\theta\cos\theta\, d\theta \qquad \left[\because \sqrt{4-4\sin^2\theta} = 2\sqrt{\left(1-\sin^2\theta\right)} = 2\sqrt{\cos^2\theta} = 2\cos\theta\right]$$

$$= 64\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\, d\theta$$

$$= 64\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2 x\sin\theta\, d\theta$$

$$= 64\int_0^{\frac{\pi}{2}} \left(1-\cos^2\theta\right)\cos^2 x\sin\theta\, d\theta$$

$$\text{Let } \cos\theta = t$$
$$-\sin\theta\, d\theta = dt$$

$$\text{Now, } \theta = 0 \Rightarrow t = 1$$
$$\theta = \frac{\pi}{2} \Rightarrow t = 0$$

$$\therefore 64\int_0^{\frac{\pi}{2}} \left(1-\cos^2\theta\right)\cos^2 x\sin\theta\, d\theta$$

$$= -64\int_1^0 \left(1-t^2\right)t^2 dt$$

$$= 64\int_0^1 t^2 - t^4 dt$$

$$= 64\int_0^1 \left(t^2 - t^4\right)dt$$

$$= 64\left[\frac{t^3}{3} - \frac{t^5}{5}\right]_0^1$$

$$= 64\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= 64 \times \frac{5-3}{15}$$

$$= 64 \times \frac{2}{15}$$

$$= \frac{128}{15}$$

$$\therefore \int_0^4 x\sqrt{4-x}\,dx = \frac{128}{15}$$

Question 2

Evaluate $\int_1^2 x\sqrt{3x-2}\,dx$

Solution 2

Let $3x - 2 = t^2$

$\Rightarrow 3dx = 2t\,dt$

Now, $x = 1 \Rightarrow t = 1$

$x = 2 \Rightarrow t = 2$

$$\therefore \int_1^2 x\sqrt{3x - 2}\,dx = \frac{2}{3}\int_1^2 \left(\frac{t^2 + 2}{3}\right) t^2 dt$$

$$= \frac{2}{9}\int_1^2 \left(t^4 + 2t^2\right) dt$$

$$= \frac{2}{9}\left[\frac{t^5}{5} + \frac{2}{3}t^3\right]_1^2$$

$$= \frac{2}{9}\left[\frac{31}{5} + \frac{14}{3}\right] = \frac{326}{135}$$

$$\therefore \int_1^2 x\sqrt{3x - 2}\,dx = \frac{326}{135}$$

Question 3

Evaluate $\int_1^5 \frac{x}{\sqrt{2x - 1}}dx$

Solution 3

Let $2x - 1 = t^2$

$\Rightarrow 2dx = 2t\,dt$

When $x = 1 \Rightarrow t = 1$

$x = 5 \Rightarrow t = 3$

$$\therefore \int_1^5 \frac{x\,dx}{\sqrt{2x - 1}} = \frac{1}{2}\int_1^3 \frac{\left(t^2 + 1\right)t\,dt}{t}$$

$$= \frac{1}{2}\int_1^3 \left(t^2 + 1\right)dt$$

$$= \frac{1}{2}\left[\frac{t^3}{3} + t\right]_1^3$$

$$= \frac{1}{2}\left[\frac{26}{3} + 2\right]$$

$$= \frac{16}{3}$$

$$\therefore \int_1^5 \frac{x}{\sqrt{2x - 1}}dx = \frac{16}{3}$$

Question 4

Evaluate the following integrals

$$\int_0^1 \cos^{-1} x dx$$

Solution 4

$$\int_0^1 \cos^{-1} x dx = \int_0^1 \cos^{-1} x \cdot 1 dx$$

$$= \cos^{-1} x \int_0^1 dx - \int_0^1 \left\{ \frac{d}{dx} \cos^{-1} x \int dx \right\} dx \quad \text{[Using Partial Fraction]}$$

$$= x \cos^{-1} x \Big|_0^1 - \int_0^1 \left\{ -\frac{x}{\sqrt{1-x^2}} \right\} dx$$

$$= \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{t dt}{t} \qquad \left[1 - x^2 = t^2\right]$$

$$= 1$$

Question 5

Evaluate $\int_0^1 \tan^{-1} x \, dx$

Solution 5

We have,

$$\int \tan^{-1} x\, dx = \int 1 \times \tan^{-1} x\, dx$$

Using by parts, we get

$$\tan^{-1} x \int 1 \times dx - \int \left(\int 1 \times dx\right) \frac{d \tan^{-1} x}{dx} dx$$

$$= x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

$$\therefore \int_0^1 \tan^{-1} x\, dx$$

$$= \left[x \tan^{-1} x - \frac{1}{2} \log \left|1+x^2\right|\right]_0^1$$

$$= \left[\left(1 + \tan^{-1}(1) - \frac{1}{2} \log |1+1|\right) - \left(0 \times \tan^{-1}(0) - \frac{1}{2} \log |1+0|\right)\right]$$

$$= \left[\left(1 \times \tan^{-1}\left(\tan \frac{\pi}{2}\right) - \frac{1}{2} \log 2\right) - \left(0 \times \tan^{-1}(\tan 0) - \frac{1}{2} \log 1\right)\right]$$

$$= \left[\frac{\pi}{4} - \frac{1}{2} \log 2 - 0 + 0\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \log 2\right)$$

$$\therefore \int_0^1 \tan^{-1} x\, dx = \frac{1}{2}\left(\frac{\pi}{2} - \log 2\right)$$

Question 6

Evaluate $\int_0^1 \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$

Solution 6

Let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

Now, $x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\int_0^1 \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^{-1}\left(\cos 2\theta\right)\sec^2\theta\, d\theta \qquad \left[\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$

$$= \int_0^{\frac{\pi}{4}} 2\theta \sec^2\theta\, d\theta$$

Using by parts, we get

$$\int 2\theta \sec^2\theta\, d\theta$$

$$= 2\left[\theta \int \sec^2\theta\, d\theta - \int\left(\int \sec^2\theta\, d\theta\right)\frac{d\theta}{d\theta} \times d\theta\right]$$

$$= 2\left[\theta \tan\theta - \int \tan\theta\, d\theta\right]$$

$$\therefore \int_0^{\frac{\pi}{4}} 2\theta \sec^2\theta\, d\theta$$

$$= 2\left[\theta \tan\theta + \log\cos\theta\right]_0^{\frac{\pi}{4}} \qquad \left[\because \int \tan\theta\, d\theta = \log\cos\theta\right]$$

$$= 2\left[\left(\frac{\pi}{4}\tan\frac{\pi}{4} + \log\cos\frac{\pi}{4}\right) - \left(0 \times \tan 0 + \log\cos 0\right)\right]$$

$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - 0 - 0\right]$$

$$= 2\left(\frac{\pi}{4} + \log\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{2} - \log 2$$

$$\therefore \int_0^1 \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx = \frac{\pi}{2} - \log 2$$

Question 7

Evaluate $\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

Solution 7

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

Now, $x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore \int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\sec^2\theta\, d\theta \qquad \left[\because \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right]$$

$$= \int_0^{\frac{\pi}{4}} \tan^{-1}(\tan 2\theta)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta\sec^2\theta\, d\theta$$

Using by parts, we get

$$\int 2\theta\sec^2\theta\, d\theta$$

$$= 2\left[\theta\int\sec^2\theta\, d\theta - \int\left(\int\sec^2\theta\, d\theta\right)\frac{d\theta}{d\theta}\times d\theta\right]$$

$$= 2\left[\theta\tan\theta - \int\tan\theta\, d\theta\right]$$

$$= 2\left[\theta\tan\theta + \log\cos\theta\right] \qquad \left[\because \int\tan\theta\, d\theta = -\log\cos\theta\right]$$

$$\therefore \int_0^{\frac{\pi}{4}} 2\theta\sec^2\theta\, d\theta = 2\left[\theta\tan\theta + \log\cos\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\left\{\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left(\cos\frac{\pi}{4}\right)\right\} - \left\{0\times\tan 0 + \log\cos 0\right\}\right]$$

$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - 0 - 0\right]$$

$$= 2\left(\frac{\pi}{4} + \log\frac{1}{\sqrt{2}}\right)$$

$$= \left(\frac{\pi}{2} - \log 2\right)$$

$$\therefore \int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx = \frac{\pi}{2} - \log 2$$

Question 8

Evaluate $\int_0^{\frac{1}{\sqrt{3}}} \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)dx$

Solution 8

Let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

Now, $x = 0 \Rightarrow \theta = 0$

$x = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

$$\therefore \int_0^{\frac{1}{\sqrt{3}}} \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)dx$$

$$= \int_0^{\frac{\pi}{6}} \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2}\right)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \tan^{-1}(\tan 3\theta)\sec^2\theta\, d\theta \qquad \left[\because \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2}\right]$$

$$= \int_0^{\frac{\pi}{6}} 3\theta \sec^2\theta\, d\theta$$

Using by parts, we get

$$\int 3\theta \sec^2\theta\, d\theta$$

$$= 3\left[\theta\int \sec^2\theta\, d\theta - \int\left(\int \sec^2\theta\, d\theta\right)\frac{d\theta}{d\theta} \times d\theta\right]$$

$$= 3\left[\theta\tan\theta - \int \tan\theta\, d\theta\right]$$

$$\therefore \int_0^{\frac{\pi}{6}} 3\theta\sec^2\theta\, d\theta = 3\left[\theta\tan\theta + \log\cos\theta\right]_0^{\frac{\pi}{6}}$$

$$= 3\left[\left(\frac{\pi}{6}\tan\frac{\pi}{6} + \log\cos\frac{\pi}{6}\right) - \left(0 \times \tan 0 + \log\cos 0\right)\right]$$

$$= 3\left[\frac{\pi}{6} \times \frac{1}{\sqrt{3}} + \log\left(\frac{\sqrt{3}}{2}\right) - 0 + \log 1\right]$$

$$= \frac{3\pi}{6\sqrt{3}} + \frac{3}{2}\log\frac{3}{4}$$

$$= \frac{\pi}{2\sqrt{3}} - \frac{3}{2}\log\frac{4}{3}$$

$$\therefore \int_0^{\frac{1}{\sqrt{3}}} \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)dx = \frac{\pi}{2\sqrt{3}} - \frac{3}{2}\log\frac{4}{3}$$

Question 9

Evaluate $\int_0^1 \frac{1-x}{1+x}dx$

Solution 9

Let $x = \tan 2\theta \Rightarrow dx = -2\sin 2\theta\, d\theta$

Now, $x = 0 \Rightarrow \theta = \frac{\pi}{4}$

$x = 1 \Rightarrow \theta = 0$

$$\therefore \int_0^1 \frac{1-x}{1+x}dx$$

$$= \int_{\frac{\pi}{4}}^{0} \frac{1-\cos 2\theta}{1+\cos 2\theta}\left(-2\sin 2\theta\, d\theta\right)$$

$$= \int_0^{\frac{\pi}{4}} \frac{2\sin^2\theta}{2\cos^2\theta} \times 4\sin\theta\cos\theta\, d\theta \qquad \left[\because \begin{array}{l} 1-\cos 2\theta = 2\sin^2\theta \\ 1+\cos 2\theta = 2\cos^2\theta \end{array}\right]$$

$$= 4\int_0^{\frac{\pi}{4}} \frac{\sin^3\theta}{\cos\theta}d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \frac{\left(1-\cos^2\theta\right)\sin\theta}{\cos\theta}d\theta \qquad \left[\because \sin^2\theta = 1-\cos^2\theta\right]$$

Let $\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$

Now, $\theta = 0 \Rightarrow t = 1$

$\theta = \frac{\pi}{4} \Rightarrow t = \frac{1}{\sqrt{2}}$

$$\therefore 4\int_0^{\frac{\pi}{4}} \frac{\left(1-\cos^2\theta\right)\sin\theta}{\cos\theta}d\theta$$

$$= -4\int_1^{\frac{1}{\sqrt{2}}} \frac{\left(1-t^2\right)dt}{t} = 2\int_{\frac{1}{\sqrt{2}}}^{1} \left(\frac{1}{t}-t\right)dt \qquad \left[\because -\int_a^b f(x)dx = \int_b^a f(x)dx\right]$$

$$= 4\left[\log t - \frac{t^2}{2}\right]_{\frac{1}{\sqrt{2}}}^{1}$$

$$= 4\left[0 - \frac{1}{2} - \log\frac{1}{\sqrt{2}} + \frac{1}{4}\right]$$

$$= 4\left[\frac{1}{2}\log 2 - \frac{1}{4}\right]$$

$$= 2\log 2 - 1$$

$$\therefore \int_0^1 \frac{1-x}{1+x}dx = 2\log 2 - 1$$

Question 10

Evaluate $\int_0^{\frac{\pi}{3}} \frac{\cos x}{3+4\sin x}dx$

Solution 10

Let $3+4\sin x = t$

$4\cos x\, dx = dt$

Now, $x = 0 \Rightarrow t = 3$

$x = \frac{\pi}{3} \Rightarrow t = 3 + \frac{4\sqrt{3}}{2} = 3 + 2\sqrt{3}$

$$\therefore \int_0^{\frac{\pi}{3}} \frac{\cos x}{3+4\sin x}dx$$

$$= \frac{1}{4}\int_3^{3+2\sqrt{3}} \frac{dt}{t} \quad = \frac{1}{4}\left[\log t\right]_3^{3+2\sqrt{3}}$$

$$= \frac{1}{4}\left[\log\left(3+2\sqrt{3}\right) - \log 3\right]$$

$$= \frac{1}{4}\left[\log\frac{3+2\sqrt{3}}{3}\right]$$

$$\therefore \int_0^{\frac{\pi}{3}} \frac{\cos x}{3+4\sin x}dx = \frac{1}{4}\left(\log\frac{3+2\sqrt{3}}{3}\right)$$

Question 11

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\left(1+\cos x\right)^2}dx$

Solution 11

We have,

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{(1+\cos x)^2} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{\left(2\cos^2\frac{x}{2}\right)^2} dx \qquad \left[\begin{array}{l} \because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \\ \& \ 1+\cos x = 2\cos^2\frac{x}{2} \end{array}\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{\cos^4\frac{x}{2}} dx$$

$$= \int_0^{\frac{\pi}{2}} \tan^2\frac{x}{2} dx$$

Add and subtract 1, we get

$$\int_0^{\frac{\pi}{2}} \left(1+\tan^2\frac{x}{2}\right) - 1\, dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\sec^2\frac{x}{2} - 1\right) dx \qquad \left[\because \int \sec^2\frac{x}{2} dx = 2\tan\frac{x}{2}\right]$$

$$= \left[2\tan\frac{x}{2} - x\right]_0^{\frac{\pi}{2}} = \left(2\tan\frac{\pi}{4} - \frac{\pi}{2} - 0 + 0\right)$$

$$= 2 - \frac{\pi}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{(1+\cos x)^2} dx = 2 - \frac{\pi}{2}$$

Question 12

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1+\cos x}} dx$

Solution 12

We have,

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1+\cos x}}\,dx$$

We know that

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

and $1+\cos x = 2\cos^2\frac{x}{2}$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1+\cos x}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sqrt{2\cos^2\frac{x}{2}}}$$

$$= \frac{2}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{\sin\frac{x}{2}\cos\frac{x}{2}}{\cos\frac{x}{2}}\,dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \sin\frac{x}{2}\,dx$$

$$= \sqrt{2}\left[-2\cos\frac{x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 2\sqrt{2}\left[1-\frac{1}{\sqrt{2}}\right]$$

$$= 2\left(\sqrt{2}-1\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1+\cos x}}\,dx = 2\left(\sqrt{2}-1\right)$$

Question 13

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx$

Solution 13

Let $\sin x = t$

$\cos x \, dx = dt$

Now, $x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x} dx$$

$$= \int_0^1 \frac{dt}{1 + t^2}$$

$$= \left[\tan^{-1} t\right]_0^1 \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$= \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x} dx = \frac{\pi}{4}$$

Question 14

Evaluate $\int_0^{\pi} \sin^3 x \left(1 + 2\cos x\right)\left(1 + \cos x\right)^2 dx$

Solution 14

Let $\cos x = t$

$\Rightarrow -\sin x\, dx = dt$

Now, $x = 0 \Rightarrow t = 1$

$x = \pi \Rightarrow t = -1$

$$\therefore \int_0^{\pi} \sin^3 x \left(1 + 2\cos x\right)\left(1 + \cos x\right)^2 dx$$

$$= -\int_1^{-1} \left(1 - t^2\right)\left(1 + 2t\right)\left(1 + t\right)^2 dt \qquad \left[\because -\int_a^b f\left(x\right)dx = \int_b^a f\left(x\right)dx\right]$$

$$= \int_{-1}^{1} \left(1 - 2t - t^2 - 2t^5\right)\left(1 + t^2 + 2t\right)dt$$

$$= \int_{-1}^{1} \left(1 + 4t + 4t^2 - 2t^3 - 5t^4 - 2t^5\right)dt$$

$$= \left[t + 2t^2 + \frac{4}{3}t^3 - \frac{t^4}{2} - t^5 - \frac{t^6}{3}\right]_{-1}^{1}$$

$$= \left(1 + 2 + \frac{4}{3} - \frac{1}{2} - 1 - \frac{1}{3}\right) - \left(-1 + 2 - \frac{4}{3} - \frac{1}{2} + 1 - \frac{1}{3}\right)$$

$$= \frac{8}{3}$$

$$\therefore \int_0^{\pi} \sin^3 x \left(1 + 2\cos x\right)\left(1 + \cos x\right)^2 dx = \frac{8}{3}$$

Question 15

Evaluate $\int_0^{\infty} \frac{x}{\left(1 + x\right)\left(1 + x^2\right)} dx$

Solution 15

We have,

$$\int_0^\infty \frac{x}{(1+x)(1+x^2)}dx$$

Add and subtract 1 in numerator.

$$\int_0^\infty \frac{(1+x)-1}{(1+x)(1+x^2)}dx$$

$$= \underbrace{\int_0^\infty \frac{dx}{(1+x^2)}}_{A} - \underbrace{\int_0^\infty \frac{dx}{(1+x)(1+x^2)}}_{B}$$

Now,

$$\int_0^\infty \frac{dx}{1+x^2} \quad = \left[\tan^{-1} x\right]_0^\infty$$

$$= \frac{\pi}{2} \quad ......(A) \qquad \left[\because \tan\frac{\pi}{2} = \infty\right]$$

Applying partial fraction, we get

$$\frac{1}{(1+x)(1+x^2)} = \frac{1}{2}\left[\frac{1}{1+x} + \frac{1-x}{1+x^2}\right]$$

$$\therefore \int_0^\infty \frac{dx}{(1+x)(1+x^2)}$$

$$= \frac{1}{2}\left[\int_0^\infty \frac{dx}{1+x} + \int_0^\infty \frac{1-x}{1+x^2}dx\right]$$

$$= \frac{1}{2}\left[\log(1+x)+\tan^{-1} x - \frac{1}{2}\log(1+x^2)\right]_0^\infty$$

$$\frac{1}{2}\left[\frac{\pi}{2}0 - 0 - 0\right]$$

$$= \frac{\pi}{4} \quad ......(B)$$

$\therefore$ Adding A and B.

$$\int_0^\infty \frac{x\ dx}{(1+x)(1+x^2)} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Question 16

Evaluate $\int_0^{\frac{\pi}{4}} \sin 2x \sin 3x\, dx$

Solution 16

We have,

$$\int_0^{\frac{\pi}{4}} \sin 2x \sin 3x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sin 2x \sin 3x \, dx$$

We know that $\cos D - \cos C = 2\sin\left(\frac{D-C}{2}\right)\sin\left(\frac{D+C}{2}\right)$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\cos x - \sin 5x\right) dx$$

$$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} - \frac{1}{5}\left(-\frac{1}{\sqrt{2}}\right) - 0 + 0\right]$$

$$= \frac{1}{2}\left[\frac{5+1}{5\sqrt{2}}\right] = \frac{3}{5\sqrt{2}}$$

$$\therefore \int_0^{\frac{\pi}{4}} \sin 2x \sin 3x \, dx = \frac{3}{5\sqrt{2}}$$

Question 17

Evaluate $\int_0^1 \sqrt{\frac{1-x}{1+x}} \, dx$

Solution 17

We have,

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$$

Let $x = \cos 2\theta \Rightarrow dx = -2\sin 2\theta\, d\theta$

When $x = 0 \Rightarrow \theta = \frac{\pi}{4}$

$x = 1 \Rightarrow \theta = 0$

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx = -2\int_{\frac{\pi}{4}}^{0} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \times \sin 2\theta\, d\theta$$

$$\left[\begin{array}{c} \because -\int_a^b f(x)dx = \int_b^a f(x)dx \\ 1-\cos 2\theta = 2\sin^2\theta \\ 1+\cos 2\theta = 2\cos^2\theta \\ \& \ \sin 2\theta = 2\sin\theta\cos\theta \end{array}\right]$$

$$= 2\int_0^{\frac{\pi}{4}} \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \times \sin 2\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{cis\theta} \times 2\sin\theta\cos\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \left(2\sin^2\theta\right) d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \left(1-\cos 2\theta\right) d\theta$$

$$= 2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{\pi}{2} - 1$$

$$\therefore \int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx = \frac{\pi}{2} - 1$$

Question 18

Evaluate $\int_1^2 \frac{1}{x^2} e^{\frac{-1}{x}} dx$

Solution 18

We have,

Let $e^{\frac{-1}{x}} = t$

$$\frac{e^{\frac{-1}{x}}}{x^2}dx = dt$$

When $x = 1 \Rightarrow t = e^{-1}$

$$x = 2 \Rightarrow t = e^{\frac{-1}{2}}$$

$$\int_0^2 \frac{1}{x^2}e^{\frac{-1}{x}}dx$$

$$= \int_{e^{-1}}^{e^{\frac{-1}{2}}} dt \quad = [t]_{e^{-1}}^{e^{\frac{-1}{2}}}$$

$$= e^{\frac{-1}{2}} - e^{-1} = \frac{1}{\sqrt{e}} - \frac{1}{e}$$

$$= \frac{\sqrt{e}-1}{e}$$

$$\therefore \int_0^2 \frac{1}{x^2}e^{\frac{-1}{x}}dx \quad = \frac{\sqrt{e}-1}{e}$$

Question 19

Evaluate $\int_0^{\frac{\pi}{4}} \cos^4 x \sin^3 x \, dx$

Solution 19

We have,

$$\int_0^{\frac{\pi}{4}} \cos^4 x \sin^3 x \, dx$$

$$\int_0^{\frac{\pi}{4}} \cos^4 x \left(1 - \sin^2 x\right) \sin x \, dx$$

Let $\cos x = t$

$\Rightarrow -\sin x \, dx = dt$

When $x = 0 \Rightarrow t = 1$

$$x = \frac{\pi}{4} \Rightarrow t = \frac{1}{\sqrt{2}}$$

$$\therefore \int_0^{\frac{\pi}{4}} \cos^4 x \left(1 - \cos^2 x\right) \sin x \, dx$$

$$= -\int_1^{\frac{1}{\sqrt{2}}} t^4 \left(1 - t^2\right) dt$$

$$= \int_{\frac{1}{\sqrt{2}}}^{1} t^4 - t^6 dt$$

$$= \left[\frac{t^5}{5} - \frac{t^7}{7}\right]_{\frac{1}{\sqrt{2}}}^{1}$$

$$= \left(\frac{1}{5} - \frac{1}{7} - \frac{1}{20\sqrt{2}} + \frac{1}{56\sqrt{2}}\right)$$

$$= \frac{2}{35}$$

$$\therefore \int_0^{\frac{\pi}{4}} \cos^4 x \sin^3 x \, dx = \frac{2}{35}$$

Question 20

Evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1 + \cos x}}{\left(1 - \cos x\right)^{\frac{5}{2}}} dx$

Solution 20

We have,

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{\left(1-\cos x\right)^{\frac{5}{2}}}\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{2\cos^2\frac{x}{2}}}{\left(2\sin^2\frac{x}{2}\right)^{\frac{5}{2}}}\,dx \qquad \left[\because 1+\cos x = 2\cos^2\frac{x}{2};\ 1-\cos x = 2\sin^2\frac{x}{2}\right]$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos\frac{x}{2}}{4\sin^5\frac{x}{2}}\,dx$$

Let $\sin^2\frac{x}{2} = t$

$$\frac{1}{2}\cos^2\frac{x}{2}\,dx = dt$$

Now, $x = \frac{\pi}{3} \Rightarrow t = \frac{1}{\sqrt{2}}$

$x = \frac{\pi}{2} \Rightarrow t = \frac{1}{\sqrt{2}}$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos\frac{x}{2}}{4\sin^5\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{dt}{t^5}$$

$$= \frac{1}{2}\left[-\frac{1}{4t^5}\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{8}\left[-4+16\right]$$

$$= \frac{12}{8}$$

$$= \frac{3}{2}$$

$$\therefore \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{\left(1-\cos x\right)^{\frac{5}{2}}}\,dx = \frac{3}{2}$$

Question 21

$$\int_0^{\frac{\pi}{2}} x^2\cos 2x\,dx$$

Evaluate the integral

Solution 21

$$\int_0^{\frac{\pi}{2}} x^2 \cos 2x\,dx = x^2\int_0^{\frac{\pi}{2}} \cos 2x\,dx - \int_0^{\frac{\pi}{2}}\left\{\frac{d}{dx}x^2\int\cos 2x\,dx\right\}dx \text{ [Using by parts]}$$

$$= x^2\frac{\sin 2x}{2}\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\left\{2x\frac{\sin 2x}{2}\right\}dx$$

$$= -\int_0^{\frac{\pi}{2}}\{x\sin 2x\}dx$$

$$= x\frac{\cos 2x}{2}\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\frac{\sin 2x}{2}\Big|_0^{\frac{\pi}{2}} \qquad \text{[Using by parts again]}$$

$$= -\frac{\pi}{4}$$

Question 22

Evaluate $\int_0^1 \log(1+x)\,dx$

Solution 22

We have,

Using by parts

$$\int 1 \times \log(1+x)dx$$
$$= \log(1+x)\int dx - \int\left(\int dx\right)\frac{d}{dx}\left(\log(1+x)\right)dx$$
$$= x\log(1+x) - \int \frac{x}{1+x}dx$$
$$= x\log(1+x) - \int \frac{1+x-x}{1+x}dx$$
$$= x\log(1+x) - \int dx + \int \frac{dx}{1+x}$$
$$= x\log(1+x) - x + \log(1+x)$$

$$\therefore \int_0^1 1 \times \log(1+x)dx$$
$$= \left[\log(1+x)(1+x) - x\right]_0^1$$
$$= 2\log 2 - 1$$

$$\therefore \int_0^1 \log(1+x)dx = 2\log 2 - 1$$

Question 23

Evaluate $\int_2^4 \frac{x^2+x}{\sqrt{2x+1}}dx$

Solution 23

Let $2x + 1 = t^2$

$2dx = 2t\,dt$

Now, $x = 2 \Rightarrow t = \sqrt{5}$

$x = 4 \Rightarrow t = 3$

$$\therefore \int_2^4 \frac{x^2 + x}{\sqrt{2x+1}}\,dx$$

$$= \int_{\sqrt{5}}^{3} \frac{\frac{\left(t^2-1\right)^2}{4} + \frac{t^2-1}{2}}{t} \times t\,dt$$

$$= \frac{1}{4}\int_{\sqrt{5}}^{3} \left(t^4 - 2t^2 1 + 2t^2 - 2\right) dt$$

$$= \frac{1}{4}\int_{\sqrt{5}}^{3} \left(t^4 - 1\right) dt$$

$$= \frac{1}{4}\left[\frac{t^5}{5} - t\right]_{\sqrt{5}}^{3}$$

$$= \frac{1}{4}\left[\frac{243}{5} - 3 - \frac{25\sqrt{5}}{2} + \sqrt{5}\right]$$

$$= \frac{1}{4}\left[\frac{243 - 15 - 25\sqrt{5} + 5\sqrt{5}}{5}\right]$$

$$= \frac{1}{4}\left[\frac{228}{5} - 4\sqrt{5}\right] = \frac{57}{5} - \sqrt{5}$$

$$\therefore \int_2^4 \frac{x^2 + x}{\sqrt{2x+1}}\,dx = \frac{57}{5} - \sqrt{5}$$

Question 24

Evaluate $\int_0^1 x\left(\tan^{-1} x\right)^2 dx$

Solution 24

Let $\tan^{-1} x = t$

$\Rightarrow \dfrac{1}{1+x^2}dx = dt$

$\Rightarrow dx = \left(1+\tan^2 t\right)dt$

$\Rightarrow dx = \sec^2 t\,dt$

$\therefore \int_0^1 x\left(\tan^{-1} x\right)^2 dx$

$= \int_0^{\frac{\pi}{4}} \tan t \times t^2 \sec^2 t\,dt$

$= \int_0^{\frac{\pi}{4}} \tan t \times t^2 \sec^2 t\,dt$

$= \int_0^{\frac{\pi}{4}} \tan t \times \sec^2 t \times t^2\,dt$

Using by parts, we get

$\int \tan t \times \sec^2 t \times t^2\,dt$

$= t^2 \int \tan t \times \sec^2 t - \int\left(\int \tan t \times \sec^2 t\,dt\right)\dfrac{dt^2}{dt} \times dt$

$= \dfrac{t^2}{2}\tan^2 t - \dfrac{1}{2}\int 2\tan^2 t \times t\,dt$

Again applying by parts

$= \dfrac{t^2}{2}\tan^2 t - \left[t\int \tan^2 t\,dt - \int\left(\tan^2 t\,dt\right)\dfrac{dt}{dt} \times dt\right]$

$= \dfrac{t^2}{2}\tan^2 t - \left[t\left(\tan t - t\right) + \int\left(\tan t - t\right)dt\right]$

$\therefore \int_0^{\frac{\pi}{4}} \tan t \times \sec^2 t \times t^2\,dt$

$= \dfrac{t^2}{2}\tan^2 t - \left[t\left(\tan t - t\right)\right] - \log\cos t - \dfrac{t^2}{2}\Big]_0^{\frac{\pi}{4}}$

$= \left(\dfrac{\pi^2}{32} - \left(\dfrac{\pi}{4}\left(1 - \dfrac{\pi}{4}\right)\right) - \log\dfrac{1}{\sqrt{2}} - \dfrac{\pi^2}{32}\right) - \left(0 - 0 - 0 + 0\right)$

$= \dfrac{\pi^2}{16} - \dfrac{\pi}{4} + \dfrac{1}{2}\log 2$

$= \dfrac{\pi^2}{16} - \dfrac{\pi}{4} + \dfrac{1}{2}\log 2$

$\therefore \int_0^1 x\left(\tan^{-1} x\right)^2 dx = \dfrac{\pi^2}{16} - \dfrac{\pi}{4} + \dfrac{1}{2}\log 2$

Question 25

Evaluate $\int_0^1 \left(\cos^{-1} x\right)^2 dx$

Solution 25

Let $\cos^{-1} x = t$

$$\Rightarrow \frac{-1}{\sqrt{1-x^2}} dx = dt$$

$$\Rightarrow -dx = \sqrt{1-x^2} dx$$

$$= \sqrt{1-\cos^2 t}\, dt = \sin t\, dt$$

Now, $x = 0 \Rightarrow t = \frac{\pi}{2}$

$x = 1 \Rightarrow t = 0$

$$\therefore \int_0^1 \left(\cos^{-1} x\right)^2 dx = -\int_{\frac{\pi}{2}}^{0} t^2 \sin t\, dt$$

$$= \int_0^{\frac{\pi}{2}} t^2 \sin t\, dt \qquad \left[\because -\int_0^a f(x)\, dx = \int_a^0 f(x) dx\right]$$

Integrating by parts, we get

$$\int t^2 \sin t\, dt = t^2 \int \sin t\, dt - \int \left(\int \sin t\, dt\right) \frac{dt^2}{dt} \times dt$$

$$= -t^2 \cos t + \int 2 + \cos t\, dt$$

Again integrating by parts

$$= -t^2 \cos t + 2\left[t \int \cos t\, dt - \int \left(\int \cos t\, dt\right) \frac{dt}{dt} dt\right]$$

$$= -t^2 \cos t + 2\left[t \sin t - \int \sin t\, dt\right]$$

$$\therefore \int_0^{\frac{\pi}{2}} t^2 \sin t\, dt = \left[-t^2 \cos t + 2\left[t \sin t + \cos t\right]\right\}_0^{\frac{\pi}{2}}$$

$$= 0 + 2\left[\frac{\pi}{2} + 0\right] - 0 - 0 - 2$$

$$= \pi - 2$$

$$\therefore \int_0^1 \left(\cos^{-1} x\right)^2 dx = \pi - 2$$

Question 26

Evaluate $\int_1^2 \frac{x+3}{x(x+2)} dx$

Solution 26

Let,

$$\frac{x+3}{x(x+2)} = \frac{A}{x}+\frac{B}{x+2}$$

$$\Rightarrow A(x+2)+Bx = x+3$$

If, $x = -2$, we get

$$-2B = 1 \Rightarrow B = -\frac{1}{2}$$

If, $x = 0$ we get

$$2A = 3 \Rightarrow A = \frac{3}{2}$$

$$\therefore \int_1^2 \frac{x+3}{x(x+2)}dx = \int_1^2 \frac{\frac{3}{2}}{x} - \frac{\frac{1}{2}}{x+2}dx$$

$$= \left[\frac{3}{2}\log x - \frac{1}{2}\log|x+2|\right]_1^2$$

$$= \left(\frac{3}{2}\log 2 - \frac{1}{2}\log 4 - 0 + \frac{1}{2}\log 3\right)$$

$$= \frac{3}{2}\log 2 - \log 2 + \frac{1}{2}\log 3$$

$$= \frac{1}{2}\log 6 \qquad [\because \log a + \log b = \log ab]$$

$$\therefore \int_1^2 \frac{x+3}{x(x+2)}dx = \frac{1}{2}\log 6$$

Question 27

Evaluate $\int_0^{\frac{\pi}{4}} e^x \sin x \, dx$

Solution 27

Integrating by parts we get

$$I = \int e^x \sin x\, dx = \sin x \int e^x dx - \int \left(\int e^x dx \right) \frac{d \sin x}{dx} dx$$

$$= e^x \sin x - \int e^x \cos x\, dx$$

Integrating by parts again

$$= e^x \sin x - \left[\cos x \int e^x dx - \int \left(\int e^x dx \right) \frac{d \cos x}{dx} dx \right]$$

$$= e^x \sin x - \left(e^x \cos x + \int e^2 \sin x dx \right)$$

$$= e^x \sin x - e^x \cos x - I$$

$$\Rightarrow 2I = \left[e^x \left(\sin x - \cos x \right) \right]_0^{\frac{\pi}{4}}$$

$$= \left[e^{\frac{\pi}{4}} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) - (0 - 1) \right]$$

$$= 1$$

$$\therefore I = \int_0^{\frac{\pi}{4}} e^x \sin x\, dx = \frac{1}{2}$$

Question 28

Evaluate the following integrals

$$\int_0^{\frac{\pi}{4}} \tan^4 x dx$$

Solution 28

$$\int_0^{\frac{\pi}{4}} \tan^4 x dx = \int_0^{\frac{\pi}{4}} \tan^2 x \tan^2 x dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^2 x \left(\sec^2 x - 1\right) dx \qquad \left[\tan^2 x = \sec^2 x - 1\right]$$

$$= \int_0^{\frac{\pi}{4}} \left(\tan^2 x \sec^2 x - \tan^2 x\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\tan^2 x \sec^2 x - \sec^2 x + 1\right) dx \ \left[\tan^2 x = \sec^2 x - 1\right]$$

$$= \int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x dx - \int_0^{\frac{\pi}{4}} \sec^2 x dx + \int_0^{\frac{\pi}{4}} dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} - \frac{2}{3}$$

Question 29

Evaluate $\int_0^1 |2x - 1| dx$

Solution 29

We have

$$\int_0^1 \left|2x-1\right|dx$$

$$= \int_0^{\frac{1}{2}} -\left(2x-1\right)dx + \int_{\frac{1}{2}}^{1}\left(2x-1\right)dx \qquad \left[\because 2x-1 \geq 0 \quad \text{if } x \geq \frac{1}{2}\right]$$

$$= -\left[x^2 - x\right]_0^{\frac{1}{2}} + \left[x^2 - x\right]_{\frac{1}{2}}^{1}$$

$$= -\left[\frac{1}{4} - \frac{1}{2}\right] + \left[0 - \frac{1}{4} + \frac{1}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{4} \quad = \frac{1}{2}$$

$$\therefore \int_0^1 \left|2x-1\right|dx = \frac{1}{2}$$

Question 30

Evaluate $\int_1^3 \left|x^2 - 2x\right|dx$

Solution 30

We have

$$\int_1^3 \left|x^2 - 2x\right|dx$$

$$= \int_1^2 -\left(x^2 - 2x\right)dx + \int_2^3 \left(x^2 - 2x\right)dx \qquad \left[\because x^2 - 2x \geq 0 \quad \text{if } x\left(x-2\right) \geq 0 \quad \text{or } x \geq 2\right]$$

$$= -\left[\frac{x^3}{3} - x^2\right]_1^2 + \left[\frac{x^3}{3} - x^2\right]_2^3$$

$$= -\left[\frac{8}{3} - 4 - \frac{1}{3} + 1\right] + \left[9 - 9 - \frac{8}{3} + 4\right]$$

$$= -\left[\frac{7}{3} - 3\right] + \frac{4}{3}$$

$$= \quad 2$$

$$\therefore \int_1^3 \left|x^2 - 2x\right|dx = 2$$

Question 31

Evaluate $\int_0^{\frac{\pi}{2}} \left|\sin x - \cos x\right|dx$

Solution 31

We have,

$$\int_0^{\frac{\pi}{2}} \left|\sin x - \cos x\right| dx$$

$$= \int_0^{\frac{\pi}{4}} -\left(\sin x - \cos x\right) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\sin x - \cos x\right) dx \quad \left[\because \sin x - \cos x \geq 0 \;\Rightarrow\; \tan x \geq 1 \Rightarrow\; x \geq \frac{\pi}{4}\right]$$

$$= -\left[-\cos x - \sin x\right]_0^{\frac{\pi}{4}} + \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -\left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 1 + 0\right] + \left[-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right]$$

$$= -\left[-\sqrt{2} + 1\right] + \left[\sqrt{2} - 1\right]$$

$$= \;\; 2\left[\sqrt{2} - 1\right]$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \left|\sin x - \cos x\right| dx = 2\left(\sqrt{2} - 1\right)$$

Question 32

Evaluate $\int_0^1 \left|\sin 2\pi x\right| dx$

Solution 32

We have,

$$\int_0^1 \left|\sin 2\pi x\right| dx$$

$$= \int_0^{\frac{1}{2}} \sin 2\pi x \, dx + \int_{\frac{1}{2}}^1 -\sin 2\pi x dx \qquad \left[\because \sin 2\pi x \geq 0 \;\text{ if }\; 0 \leq x \leq \frac{1}{2}\right]$$

$$= \left[\frac{-\cos 2\pi x}{2\pi}\right]_0^{\frac{1}{2}} - \left[\frac{-\cos 2\pi x}{2\pi}\right]_{\frac{1}{2}}^1$$

$$= \frac{1}{2\pi}\left[1 - \cos \pi\right] - \frac{1}{2\pi}\left[\cos \pi - \cos 2\pi\right]$$

$$= \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$$

$$\therefore \quad \int_0^1 \left|\sin 2\pi x\right| dx = \frac{2}{\pi}$$

Question 33

Evaluate $\int_1^3 \left|x^2 - 4\right| dx$

Solution 33

We have,

$$\int_1^3 \left|x^2 - 4\right| dx$$

$$= \int_1^2 -\left(x^2 - 4\right) dx + \int_2^3 \left(x^2 - 4\right) dx \qquad \left[\because x^2 - 4 \ge 0 \text{ if } x \ge 2\right]$$

$$= -\left[\frac{x^3}{3} - 4x\right]_1^2 + \left[\frac{x^3}{3} - 4x\right]_2^3$$

$$= -\left[\frac{8}{3} - 8 - \frac{1}{3} + 4\right] + \left[9 - 12 - \frac{8}{3} + 8\right]$$

$$= -\left[-\frac{5}{3}\right] + \frac{7}{3}$$

$$= 4$$

$$\therefore \int_1^3 \left|x^2 - 4\right| dx = 4$$

Question 34

Evaluate $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^9(x)\, dx$

Solution 34

We have,

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^9(x)\, dx$$

We know that $\int_{-a}^{a} f(x)\, dx = 0$, if $f(x)$ is odd function.

Here,

$f(x) = \sin^9(x)$ is an odd function

$$\therefore \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^9(x)\, dx = 0$$

Question 35

Evaluate $\int_{\frac{-1}{2}}^{\frac{1}{2}} \cos x \log\left(\frac{1+x}{1-x}\right)$

Solution 35

Let $I=\int_{\frac{-1}{2}}^{\frac{1}{2}} \cos x \log\left(\frac{1+x}{1-x}\right)$

We know that $\int_{-a}^{a} f(x)dx = 0$ if $f(x)$ is an odd function.

Hence

$f(x) = \int_{\frac{-1}{2}}^{\frac{1}{2}} \cos x \log\left(\frac{1+x}{1-x}\right)$

which is the combination of an even and an odd function, so the combined function becomes odd.

$\therefore \int_{\frac{-1}{2}}^{\frac{1}{2}} \cos x \log\left(\frac{1+x}{1-x}\right) = 0$

Question 36

Evaluate $\int_{-a}^{a} \frac{xe^{x^2}}{1+x^2}dx$

Solution 36

We have,

$\int_{-a}^{a} \frac{xe^{x^2}}{1+x^2}dx$

We know that $\int_{-a}^{a} f(x)dx = 0$ if $f(x)$ is an odd function

Hence, $f(x) = x$ is an odd function

and $\frac{xe^{x^2}}{1+x^2}$ is an even function.

$\therefore$ the combined function $\frac{x \times e^{x^2}}{1+x^2}$ is odd.

$\therefore \int_{-a}^{a} \frac{xe^{x^2}}{1+x^2}dx = 0$

Question 37

Evaluate $\int_{0}^{\frac{\pi}{2}} \frac{dx}{1+\cot^7 x}dx$

Solution 37

We have,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cot^7 x}$$

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^7 x\, dx}{\sin^7 x + \cos^7 x}$ --(i) $\left[\text{Using } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^7\left(\frac{\pi}{2}-x\right)dx}{\sin^7\left(\frac{\pi}{2}-x\right)+\cos^7\left(\frac{\pi}{2}-x\right)}$$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\cos^7 x}{\cos^7 x + \sin^7 x}dx = I \qquad \text{--(ii)}$$

Add (i) & (ii) we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^7 x + \sin^7 x}{\cos^7 x + \sin^7 x}dx$$

$$= \int_0^{\frac{\pi}{2}} dx$$

$$= [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\cot^7 x} = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{1+\cot^7 x} = \frac{\pi}{4}$$

Question 38

Evaluate the following integrals

$$\int_{0}^{2\pi} \cos^7 x\,dx$$

Solution 38

Let $f(x) = \cos^7 x$. Then $f(2\pi - x) = \{\cos(2\pi - x)\}^7 = \cos^7 x$

$$\int_0^{2\pi} \cos^7 x dx = 2\int_0^{\pi} \cos^7 x dx$$

Now

$$f(\pi - x) = \{\cos(\pi - x)\}^7 = -\cos^7 x = -f(x)$$

Therefore

$$\int_0^{\pi} \cos^7 x dx = 0$$

Hence

$$\int_0^{2\pi} \cos^7 x dx = 2\int_0^{\pi} \cos^7 x dx = 0$$

Question 39

Evaluate $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx$

Solution 39

Let,

$$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx \qquad --(i)$$

$$\therefore I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx \qquad --(ii) \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

Add (i) and (ii)

$$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx$$

$$\therefore 2I = \int_0^a dx \quad = [x]_0^a = a$$

$$\Rightarrow I = \frac{a}{2}$$

$$\therefore \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx = \frac{a}{2}$$

Question 40

Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^3 x} dx$

Solution 40

Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^3 x}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{1}{\frac{\cos^3 x + \sin^3 x}{\cos^3 x}} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} dx \qquad --(1)$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\cos^3\left(\frac{\pi}{2} - x\right)}{\cos^3\left(\frac{\pi}{2} - x\right) + \sin^3\left(\frac{\pi}{2} - x\right)} dx \qquad \left[\text{Using } \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]\right]$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx = I \qquad --(2)$$

Adding (1) and (2) we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x} dx = \int_0^{\frac{\pi}{2}} dx \qquad = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^3 x} = \frac{\pi}{4}$$

Question 41

Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

Solution 41

Let $I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$ (i)

$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$ $\left[\because \int f(x)dx = \int f(a - x)dx\right]$

$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx$

Let $\cos x = t$

$- \sin x \, dx = dt$

When $x = 0 \Rightarrow t = 1$

$x = \pi \Rightarrow t = -1$

$= -\pi \int_1^{-1} \frac{dt}{1 + t^2} = -\pi\left[\tan^{-1} t\right]_1^{-1}$

$= -\pi\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$

$= -\pi\left[\tan^{-1}\left(\tan\frac{-\pi}{4}\right) - \tan^{-1}\left(\tan\frac{\pi}{4}\right)\right]$

$= -\pi\left[\frac{-\pi}{4} - \frac{\pi}{4}\right]$

$= \frac{\pi^2}{2}$

$\therefore \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{2}$

Question 42

Evaluate $\int_0^{\pi} x \sin x \cos^4 x \, dx$

Solution 42

We have

$$I = \int_0^{\pi} x \sin x \cos^4 x \, dx \qquad --(1)$$

$$I = \int_0^{\pi} (\pi - x)\sin(\pi - x)\cos^4(\pi - x)dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\therefore I = \int_0^{\pi} (\pi - x)\sin x \cos^4 x \, dx \qquad --(2)$$

Adding (1) and (2) we get

$$2I = \int_0^{\pi} \pi \sin x \cos^4 x \, dx$$

$$= \pi \int_0^{\pi} \sin x \cos^4 x \, dx$$

Let $\cos x = t \Rightarrow -\sin x \, dx = dt$

Now, $x = 0 \Rightarrow t = 1$

$x = \pi \Rightarrow t = -1$

$$\therefore \pi \int_0^{\pi} \sin x \cos^4 x \, dx = -\pi \int_1^{-1} t^4 dt$$

$$= -\pi \left[\frac{t^5}{5}\right]_1^{-1}$$

$$= -\pi \left[\frac{-1}{5} - \frac{1}{5}\right]$$

$$= \frac{2\pi}{5}$$

$$\therefore \int_0^{\pi} x \sin x \cos^4 x \, dx = \frac{\pi}{5}$$

Question 43

Evaluate $\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$

Solution 43

Let,

$$I=\int_0^{\pi}\frac{x}{a^2\cos^2 x+b^2\sin^2 x}dx \qquad --(\text{i})$$

$$\therefore\ I=\int_0^{\pi}\frac{(\pi-x)}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}dx \qquad \left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\Rightarrow I=\int_0^{\pi}\frac{(\pi-x)}{a^2\cos^2 x+b^2\sin^2 x}dx \qquad --(\text{ii})$$

Adding (i) and (ii) we get,

$$2I=\int_0^{\pi}\frac{\pi}{a^2\cos^2 x+b^2\sin^2 x}dx$$
$$=\pi\int_0^{\pi}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$$

Divide numerator and denominator by $\cos^2 x$

$$=2\pi\int_0^{\pi}\frac{\sec^2 x\,dx}{a^2+b^2\tan^2 x} \qquad \left[\because \int_0^{2a} f(x)dx=2\int_0^a f(x)dx\right]$$

Let $\tan x=t$

$\sec^2 x\,dx=dt$

Now, $x=0\Rightarrow t=0$

$x=\frac{\pi}{2}\Rightarrow t=\infty$

$$\therefore\ 2\pi\int_0^{\pi}\frac{\sec^2 x\,dx}{a^2+b^2\tan^2 x}$$
$$=2\pi\int_0^{\infty}\frac{dt}{a^2+b^2t^2x}$$
$$=\frac{2\pi}{b^2}\left[\frac{b}{a}\tan^{-1}\frac{bt}{a}\right]_0^{\infty}$$
$$\therefore=\frac{2\pi}{ab}\left[\frac{\pi}{2}-0\right]$$
$$=\frac{\pi^2}{ab}$$
$$I=\frac{\pi^2}{2ab}$$

$$\therefore\ \int_0^{\pi}\frac{x}{a^2\cos^2 x+b^2\sin^2 x}dx=\frac{\pi^2}{2ab}$$

Question 44

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left|\tan x\right| dx$

Solution 44

$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} |\tan x|\,dx = -\int_{\frac{-\pi}{4}}^{0} \tan x\,dx + \int_{0}^{\frac{\pi}{4}} \tan x\,dx \qquad \begin{bmatrix} \because \tan x \ge 0 & \text{When } 0 < x < \frac{\pi}{4} \\ \tan x \le 0 & \text{When } \frac{-\pi}{4} < x < 0 \end{bmatrix}$$

$$= [\log \sec x]_{\frac{-\pi}{4}}^{0} - [\log \sec x]_{0}^{\frac{\pi}{4}}$$

$$= \left[0 - \log \frac{1}{\sqrt{2}}\right] - \left[\log \frac{1}{\sqrt{2}} - 0\right]$$

$$= -2\log \frac{1}{\sqrt{2}}$$

$$= 2 \times \frac{1}{2} \log 2$$

$$= \log 2$$

$$\therefore \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} |\tan x|\,dx = \log 2$$

Question 45

Evaluate $\int_{0}^{1.5} [x^2]\,dx$

Solution 45

We have,

$$\int_0^{1.5}\left[x^2\right]dx$$

$$= \int_0^1\left[x^2\right]dx + \int_1^{\sqrt{2}}\left[x^2\right]dx + \int_{\sqrt{2}}^{1.5}\left[x^2\right]dx$$

$$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\times dx + \int_{\sqrt{2}}^{1.5} 2\times dx$$

$$= 0 + \left[x\right]_1^{\sqrt{2}} + 2\left[x\right]_{\sqrt{2}}^{1.5}$$

$$= 0 + \left(\sqrt{2} - 1\right) + 2\left[1.5 - \sqrt{2}\right]$$

$$= \sqrt{2} - 1 + 3 - 2\sqrt{2}$$

$$= 2 - \sqrt{2}$$

$$\therefore \int_0^{1.5}\left[x^2\right]dx = 2 - \sqrt{2}$$

Question 46

Evaluate $\displaystyle\int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\sin x}$

Solution 46

We have, $\int_0^{\pi} \frac{x\,dx}{1+\cos\alpha\sin x}$

Let

$$I = \int_0^{\pi} \frac{x\,dx}{1+\cos\alpha\sin x} \qquad --(\text{i})$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)dx}{1+\cos\alpha\sin(\pi - x)} \qquad \left[\because \int_0^{a} f(x)dx = \int_0^{a} f(a-x)dx\right]$$

$$\therefore I = \int_0^{\pi} \frac{(\pi - x)dx}{1+\cos\alpha\sin x} \qquad --(\text{ii})$$

Adding (i) and (ii) we get,

$$2I = \int_0^{\pi} \frac{\pi\,dx}{1+\cos\alpha\sin x}$$

Substituting $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$= \pi\int_0^{\pi} \frac{\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2}+2\cos\alpha\tan\frac{x}{2}}$$

$$= \pi\int_0^{\pi} \frac{\sec^2\frac{x}{2}dx}{\left(1-\cos^2\alpha\right)+\left(\cos\alpha+\tan\frac{x}{2}\right)^2}$$

Add and subtract $\cos\alpha$ in numerator

Let $\tan\frac{x}{2} = t$

$$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

When $x = 0 \Rightarrow t = 0$

$x = \pi \Rightarrow t = \infty$

$$2\pi\int_0^\infty \frac{dt}{\left(1-\cos^2\alpha\right)+\left(\cos\alpha+t\right)^2}$$

$$= \left[2\pi\times\frac{1}{\sqrt{1-\cos^2\alpha}}\tan^{-1}\left(\frac{\cos\alpha+t}{\sqrt{1-\cos^2\alpha}}\right)\right]_0^\infty$$

$$= \frac{2\pi}{\sin\alpha}\left[\frac{\pi}{2}-\tan^{-1}\cot\alpha\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\cot^{-1}\left(\cot\alpha\right)\right]$$

$$= \frac{2\pi}{\sin\alpha}\times\alpha \qquad \left[\because \frac{\pi}{2}-\tan^{-1}\theta = \cot^{-1}\theta\right]$$

$$\therefore I = \frac{\pi\alpha}{\sin\alpha}$$

Question 47

Evaluate $\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$

Solution 47

We have,

$$I = \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx \qquad \text{(i)}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \sin x \cos x}{\sin^4 x + \cos^4 x} dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get,

$$2I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

Divide Numerator and Denominator by $\cos^4 x$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\tan x \sec^2 x\, dx}{1 + \tan^4 x\, dx} dx$$

Let $\tan^2 x = t$

$2 \tan x \sec^2 x\, dx = dt$

When $x = 0 \quad \Rightarrow t = 0$

$x = \frac{\pi}{2} \quad \Rightarrow t = \infty$

$$= \frac{\pi}{2} \times \frac{1}{2} \int_0^{\infty} \frac{dt}{1 + t^2}$$

$$= \frac{\pi}{4} \left[\tan^{-1} t\right]_0^{\infty}$$

$$= \frac{\pi}{4} \times \frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

$$I = \frac{\pi^2}{16}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} = \frac{\pi^2}{16}$$

Question 48

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x} dx$

Solution 48

Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x} dx \qquad --(i)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx \qquad --(ii)$$

Adding (i) and (ii) we get,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x + \sin^2 x}{\sin x + \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x} \qquad \left[\because \sin^2 x + \cos^2 x = 1\right]$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sec^2 \frac{x}{2} dx}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}} \qquad \left[\because \sin x = \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} \text{ and } \cos x = \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right]$$

Let $\tan\frac{x}{2} = t$

$$-\frac{1}{2}\sec^2\frac{x}{2} dx = dt$$

When $x = 0 \Rightarrow t = 1$

$$x = \frac{\pi}{2} \Rightarrow t = 0$$

$$= -\int_1^0 \frac{2\,dt}{2t+1-t^2}$$

Forming the perfect square by adding & subtracting 1 in Denominator.

$$+\int_0^1 \frac{2\,dt}{2-\left(1+t\right)^2}$$

$$= 2\left[\frac{1}{2\sqrt{2}}\log\frac{\sqrt{2}+1+t}{\sqrt{2}-1-t}\right]_0^1$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)-\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log\frac{1}{1-\sqrt{2}}-\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(-\frac{1}{\sqrt{2}+1}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

$$\therefore\ I = \frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x+\cos x}dx = \frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

Question 49

Evaluate $\int_0^{\pi} \cos 2x \log \sin x \, dx$

Solution 49

We have,

Integrating by parts we get,

$$\int \cos 2x \log \sin x \, dx = \log \sin x \int \cos 2x - \int \left(\int \cos 2x \, dx \right) \frac{d \left(\log \sin x \right)}{dx} dx$$

$$= \frac{\sin 2x}{2} \log \sin x - \frac{1}{2} \int \sin 2x \times \frac{\cos x}{\sin x} dx$$

$$= \frac{1}{2} \sin 2x \log \sin x - \frac{1}{2} \times 2 \int \frac{\sin x \cos x \times \cos x}{\sin x} dx$$

$$= \frac{1}{2} \sin 2x \log \sin x - \int \cos^2 x \, dx \qquad \left[\because \cos^2 x = \frac{1 + \cos 2x}{2} \right]$$

$$= \frac{1}{2} \sin 2x \log \sin x - \int \frac{1 + \cos 2x}{2} dx$$

$$\int_0^{\pi} \cos 2x \log \sin x \, dx = \left[\frac{1}{2} \sin 2x \log \sin x \right]_0^{\pi} - \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= 0 - \frac{1}{2} \left[\pi + 0 \right] - 0 + \frac{1}{2} \left[0 + 0 \right]$$

$$= \frac{-\pi}{2}$$

$$\therefore \int_0^{\pi} \cos 2x \log \sin x \, dx = \frac{-\pi}{2}$$

Question 50

Evaluate $\int_0^{\pi} \frac{x}{a^2 - \cos^2 x} dx$

Solution 50

Let,

$$I = \int_0^{\pi} \frac{x}{a^2 - \cos^2 x} dx \qquad --\text{(i)}$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{a^2 - \cos^2 (\pi - x)} dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\therefore\ I = \int \frac{(\pi - x)dx}{a^2 - \cos^2 x} \qquad --\text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{dx}{a^2 - \cos^2 x}$$

$$= \pi \int_0^{\pi} \frac{dx}{a^2 - \cos^2 x}$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{a^2 - \cos^2 x} \qquad \left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx\right]$$

Dividing Numerator and Denominator by $\cos^2 x$ and

Let $\tan x = t$

$\sec^2 x\, dx = dt$

When $x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = \infty$

$$\therefore\ 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{a^2 - \sec^2 x - 1}$$

$$= 2\pi \int_0^{\infty} \frac{dt}{(1 + t^2) - 1}$$

$$= 2\pi \int_0^{\infty} \frac{dt}{(a^2 - 1) + dt^2}$$

$$= \frac{2\pi}{a^2} \int_0^{\infty} \frac{dt}{\left(\frac{\sqrt{a^2 - 1}}{a}\right)^2 + t^2}$$

$$= \frac{2\pi}{a^2} \left[\frac{a}{\sqrt{a^2 - 1}} \tan^{-1} \frac{at}{\sqrt{a^2 - 1}}\right]_0^{\infty}$$

$$= \frac{2\pi}{a\sqrt{a^2 - 1}} \left\{\frac{\pi}{2} - 0\right\}$$

$$2I = \frac{\pi^2}{a\sqrt{a^2-1}}$$

$$\therefore\ I = \frac{\pi^2}{2a\sqrt{a^2-1}}$$

$$\therefore \int_0^{\pi} \frac{x}{a^2-\cos^2(\pi-x)}dx = \frac{\pi^2}{2a\sqrt{a^2-1}}$$

Question 51

Evaluate $\int_0^{\pi} \frac{x\ \tan x}{\sec x + \tan x}dx$

Solution 51

We have,

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx \qquad --(i)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \sin(\pi - x)} dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\therefore I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \sin x} dx \qquad --(ii)$$

Adding (i) and (ii) we get

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1 + \sin x}$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x}$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{1 + \sin x - 1}{1 + \sin x}$$

$$= 2\pi \left[\int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x}\right] \text{ ...... (A)}$$

Now, $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x} = \int_0^{\frac{\pi}{2}} \frac{\sec^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} + 2\tan\frac{x}{2}}$

$$= 2\int_0^1 \frac{dt}{(1 + t)^2}$$

$$= 2\left[-\frac{1}{1+t}\right]_0^1$$

$$= 2\left(1 - \frac{1}{2}\right) = 1$$

$\therefore$ From (A)

$$2I = 2\pi \left\{ [x]_0^{\frac{\pi}{2}} - 1 \right\}$$

$$= 2\pi \left\{\frac{\pi}{2} - 1\right\}$$

$$= \pi(\pi - 2)$$

$$\therefore I = \frac{\pi}{2}(\pi - 2)$$

$$\therefore \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} = \frac{\pi}{2}(\pi - 2)$$

Question 52

Evaluate $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$

Solution 52

We have,

$$I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx \quad = I \qquad --\text{(i)}$$

$$\therefore\ I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx \qquad --\text{(ii)} \qquad \left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

Adding (i) and (ii) we get,

$$2I = \int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx$$
$$= \int_2^3 dx$$
$$= [x]_2^3$$
$$= 1$$
$$I = \frac{1}{2}$$

$$\therefore \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx = \frac{1}{2}$$

Question 53

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}dx$

Solution 53

We have,

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx \qquad --(i)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx \qquad \left[\because \int_0^a f(x) dx = \int_0^a f(a - x) dx\right]$

$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x \, dx}{\cos x + \sin x} dx \qquad --(ii)$

Adding (i) and (ii) we get,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x + \sin^2 x}{\cos x + \sin x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{\cos x + \sin x}$$

Substituting $\quad \sin x = \dfrac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 \frac{x}{2} dx}{2 \tan \frac{x}{2} + 1 - \tan \frac{x}{2}}$$

Let $\tan \frac{x}{2} = t$

$$-\frac{1}{2} \sec^2 \frac{x}{2} dx = dt$$

When $x = 0 \Rightarrow t = 1$

$$x = \frac{\pi}{2} \Rightarrow t = 0$$

$$\therefore \int \frac{\sec^2 \frac{x}{2} \quad dx}{2 \tan \frac{x}{2} + 1 - \tan \frac{x}{2}}$$

$$= -\int_1^0 \frac{2\,dt}{2t+1-t^2}$$

$$= +\int_0^1 \frac{2\,dt}{2-(1+t)^2} \qquad \text{[Forming the perfect square by adding and subtracting 1 in Denominator.]}$$

$$= 2\left[\frac{1}{2\sqrt{2}}\left(\log\frac{\sqrt{2}+1+t}{\sqrt{2}-1-t}\right)\right]_0^1$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right) - \log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log\frac{1}{1-\sqrt{2}} - \log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(-\frac{1}{\sqrt{2}+1}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

Question 54

Evaluate $\int_0^{\frac{\pi}{2}} \frac{x}{\cos^2 x + \sin^2 x}\,dx$

Solution 54

We have,

$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{\cos^2 x + \sin^2 x}$$

$$= \int_0^{\frac{\pi}{2}} x\,dx \qquad \left[\because \cos^2 x + \sin^2 x = 1\right]$$

$$= \left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi^2}{8} - 0\right]$$

$$= \frac{\pi^2}{8}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{x}{\cos^2 x + \sin^2 x}\,dx = \frac{\pi^2}{8}$$

Question 55

Evaluate $\int_{-\pi}^{\pi} x^{10}\sin^7 x\,dx$

Solution 55

We know,

$$\int_{-a}^{a} f(x)\,dx = 0 \qquad \text{if } f(x) \text{ is an odd function}$$

Here, $f(x) = x^{10}\sin^7 x$ is odd fuction

as $f(-x) = -x^{10}\sin^7 x$

$= f(x)$

$$\therefore \int_{-\pi}^{\pi} x^{10}\sin^7 x\,dx = 0$$

Question 56

Evaluate $\int_0^1 \cot^{-1}\left(1 - x + x^2\right)dx$

Solution 56

We have,

$$\int_0^1 \cot^{-1}\left(1-x+x^2\right)dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx \qquad \left[\because \cot^{-1}\theta = \tan^{-1}\frac{1}{\theta}\right]$$

$$= \int_0^1 \tan^{-1}\left(\frac{x+1-x}{1-x\left(1-x\right)}\right)dx \qquad \left[\text{Add subt. } x \text{ in Numerator}\right]$$

$$= \int_0^1 \tan^{-1}x + \int_0^1 \tan^{-1}\left(1-x\right)dx \qquad \left[\because \tan^{-1}A + \tan^{-1}B = \left(\frac{A+B}{1-AB}\right)\right]$$

$$= \int_0^1 \tan^{-1}x + \int_0^1 \tan^{-1}\left(1-1+x\right)dx \qquad \left[\because \int_0^a f\left(a\right)dx = \int_0^a f\left(a-x\right)dx\right]$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}x\,dx$$

$$= 2\int_0^1 \tan^{-1}x\,dx$$

Integrating by parts, we get

$$2\int \tan^{-1}x\,dx = 2\left[\tan^{-1}x\int dx - \int\left(\int dx\right)\frac{d\tan^{-1}x}{dx}\times dx\right]$$

$$= 2\left[x\tan^{-1}x - \int\frac{x}{1+x^2}dx\right]$$

$$\therefore\ 2\int_0^1 \tan^{-1}x\,dx = 2\left[x\tan^{-1}x - \frac{1}{2}\log\left(1+x^2\right)\right]_0^1$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right] = \frac{\pi}{2} - \log 2$$

$$\therefore\ \int_0^1 \cot^{-1}\left(1-x+x^2\right)dx = \frac{\pi}{2} - \log 2$$

Question 57

Evaluate $\int_0^{\pi}\frac{dx}{6-\cos x}$

Solution 57

$$\int_0^{\pi} \frac{dx}{6-\cos x} = \int_0^{\pi} \frac{dx}{6-\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} \qquad \left[\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]$$

$$= \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{6\left(1+\tan^2\frac{x}{2}\right)-\left(1-\tan^2\frac{x}{2}\right)} \qquad \left[\because 1+\tan^2\frac{x}{2} = \sec^2\frac{x}{2}\right]$$

$$= \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{5+7\tan^2\frac{x}{2}}dx$$

$$= \frac{1}{7}\int_0^{\pi} \frac{\sec^2\frac{x}{2}}{\frac{5}{7}+\tan^2\frac{x}{2}}dx$$

Let $\tan\frac{x}{2} = t$

$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$

When $x = 0 \Rightarrow t = 0$

$x = \pi \Rightarrow t = \infty$

$$= \frac{2}{7}\int_0^{\infty} \frac{dt}{\left(\frac{\sqrt{5}}{\sqrt{7}}\right)^2 + t^2}$$

$$= \frac{2}{7} \times \sqrt{\frac{7}{5}}\left[\tan^{-1}\sqrt{\frac{7}{5}}t\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{35}}\ \frac{\pi}{2}$$

$$= \frac{\pi}{\sqrt{35}}$$

$$\therefore \int_0^{\pi} \frac{dx}{6-\cos x} = \frac{\pi}{\sqrt{35}}$$

Question 58

Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 4\sin x}$

Solution 58

We have,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 4\sin x}$$

Substitute $\cos x = \dfrac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\sin x = \frac{2\tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2\left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right) + 4\left(\frac{2\tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(1 + \tan^2 \frac{x}{2}\right)dx}{2\left(1 - \tan^2 \frac{x}{2}\right) + 8\tan \frac{x}{2}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 \frac{x}{2}}{\left(1 - \tan^2 \frac{x}{2}\right) + 8\tan \frac{x}{2}} \qquad \left[\because 1 + \tan^2 \frac{x}{2} = \sec^2 \frac{x}{2}\right]$$

Let $\tan \frac{x}{2} = t$

$$\frac{1}{2}\sec^2 \frac{x}{2} dx = dt$$

When $x = 0 \Rightarrow t = 0$

$$x = \frac{\pi}{2} \Rightarrow t = 0$$

$$= \frac{1}{2}\int_0^1 \frac{2\,dt}{\left(1 - t^2\right) + 4t}$$

$$= \int_0^1 \frac{dt}{1 - t^2 + 4t} \qquad \left[\text{Forming perfect square by adding and subtracting 4 in Denominator}\right]$$

$$= \int_0^1 \frac{dt}{1 + 4 - \left(t^2 - 4t + 4\right)}$$

$$= \int_0^1 \frac{dt}{5 - \left(t + 2\right)^2}$$

$$= \left[\frac{1}{2\sqrt{5}} \log \frac{\sqrt{5}+t+2}{\sqrt{5}-t-2}\right]_0^1$$

$$= \frac{1}{2\sqrt{5}}\left[\log \frac{\sqrt{5}+3}{\sqrt{5}-3} - \log \frac{\sqrt{5}+2}{\sqrt{5}-2}\right]$$

$$= \frac{1}{2\sqrt{5}} \log \frac{\left(\sqrt{5}+3\right)\left(\sqrt{5}-2\right)}{\left(\sqrt{5}-3\right)\left(\sqrt{5}+2\right)}$$

$$= \frac{1}{2\sqrt{5}} \log \left[\frac{5 - 2\sqrt{5} + 3\sqrt{5} - 6}{5 + 2\sqrt{5} - 3\sqrt{5} - 6}\right]$$

$$= \frac{1}{2\sqrt{5}} \log \left[\frac{\sqrt{5} - 1}{-\left(\sqrt{5} + 1\right)}\right]$$

$$= \frac{1}{2\sqrt{5}} \log \left(\frac{\sqrt{5} + 1}{\sqrt{5} - 1}\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 4\sin x} = \frac{1}{2\sqrt{5}} \log \left(\frac{\sqrt{5} + 1}{\sqrt{5} - 1}\right)$$

Question 59

Evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\operatorname{cosec} x \cot x}{1 + \operatorname{cosec}^2 x} dx$

Solution 59

We have,

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\operatorname{cosec} x \cot x}{1+\operatorname{cosec}^2 x}\,dx$$

Let $\operatorname{cosec} x = t$

$- \operatorname{cosec} x \cot x = dt$

When $x = \frac{\pi}{6} \Rightarrow t = 2$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$= -\int_2^1 \frac{dt}{1+t^2} \quad = -\left[\tan^{-1} t\right]_2^1$$

$$= -\left[\tan^{-1} 1 - \tan^{-1} 2\right]$$

$$= \tan^{-1} 2 - \tan^{-1} 1$$

$$= \tan^{-1}\left(\frac{2-1}{1+2}\right) \qquad \left[\because \tan^{-1} A - \tan^{-1} B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$$

$$= \tan^{-1}\left(\frac{1}{3}\right)$$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\operatorname{cosec} x \cot x}{1+\operatorname{cosec}^2 x}\,dx \quad = \tan^{-1}\left(\frac{1}{3}\right)$$

Question 60

Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{4\cos x + 2\sin x}$

Solution 60

Put $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$; $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{4\cos x + 2\sin x} = \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}dx}{4\left(1-\tan^2\frac{x}{2}\right)+2\tan\frac{x}{2}} = \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}dx}{4-4\tan^2\frac{x}{2}+2\tan\frac{x}{2}}$$

Let $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

Now, $x = 0 \Rightarrow t = 0$ *and* $x = \frac{\pi}{2} \Rightarrow t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}dx}{4-4\tan^2\frac{x}{2}+2\tan\frac{x}{2}} = \int_0^1 \frac{2dt}{4-4t^2+2t} = \frac{2}{4}\int \frac{dt}{1-t^2+\frac{1}{2}t}$$

Forming perfect square by adding and subtracting $\frac{1}{4}$

$$\frac{2}{4}\int_0^1 \frac{dt}{1-t^2+\frac{1}{2}t} = \frac{1}{2}\int_0^1 \frac{dt}{\frac{5}{4}-\left(t-\frac{1}{4}\right)^2} = \frac{1}{2}\cdot\frac{1}{2}\sqrt{\frac{4}{5}}\log\left(\frac{\sqrt{\frac{5}{4}}+t-\frac{1}{4}}{\sqrt{\frac{5}{4}}-t+\frac{1}{4}}\right)_0^1 \qquad \left[\because \int\frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{x+a}{x-a}\right)\right]$$

$$= \frac{1}{2\sqrt{5}}\left\{\log\left(\frac{\sqrt{\frac{5}{4}}+\frac{3}{4}}{\sqrt{\frac{5}{4}}-\frac{3}{4}}\right) - \log\left(\frac{\sqrt{\frac{5}{4}}-\frac{1}{4}}{\sqrt{\frac{5}{4}}+\frac{1}{4}}\right)\right\}$$

$$= \frac{1}{2\sqrt{5}}\log\left(\frac{\left(2\sqrt{5}+3\right)\left(2\sqrt{5}+1\right)}{\left(2\sqrt{5}-3\right)\left(2\sqrt{5}-1\right)}\right)$$

$$= \frac{1}{\sqrt{5}}\log\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)$$

Question 61

Evaluate $\int_0^4 x\,dx$

Solution 61

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + ----+f(a+(n-1)h)\right], \text{ where } h = \frac{b-a}{n}$$

$$\int_0^4 x dx$$

Here, $a = 0,\ b = 4$ and $f(x) = x$

$$h = \frac{4}{n} \Rightarrow nh = 4$$

Thus, we have,

$$\Rightarrow I = \int_0^4 (x) dx$$

$$\Rightarrow I = \lim_{h\to 0} h\left[f(0) + f(h) + f(2h) + ----f((n-1)h)\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[0 + (h) + (2h) + ----+((n-1)h)\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[h(1+2+3+----+(n-1))\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[h\left(\frac{n(n-1)}{2}\right)\right] \qquad \left[\because h \to 0\ \&\ h = \frac{4}{n} \Rightarrow n \to \infty\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{4}{n}\left[\frac{4}{n}\frac{(n^2-1)}{2}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{16}{2}\left(1 - \frac{1}{n^2}\right) = 8$$

Question 62

Evaluate $\int_0^2 (2x^2 + 3)dx$

Solution 62

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here $a = 0$, $b = 2$ and $f(x) = 2x^2 + 3$

$$\therefore h = \frac{2}{n} \Rightarrow nh = 2$$

Thus, we have,

$$I = \int_0^2 (2x^2+3)\,dx$$

$$= \lim_{h\to 0} h\left[f(0)+f(0+h)+f(0+2h)+----f(0+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[(3)+\{2(h)^2+3\}+\{2(2h)^2+3\}+-----+\{2((n-1)h)^2+3\}\right]$$

$$= \lim_{h\to 0} h\left[3n+2h^2\left(1^2+2^2+3^2+--(n-1)^2\right)\right]$$

$$\because h = \frac{2}{n} \text{ \& if } h\to 0 \Rightarrow n\to\infty$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[3n+\frac{8}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} 6+\frac{8}{3n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= 6+\frac{16}{3} = \frac{34}{3}$$

$$\therefore \int_2^3 (2x^2+3)\,dx = \frac{34}{3}$$

Question 63

Evaluate $\int_1^4 (x^2+x)\,dx$

Solution 63

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here $a = 1$, $b = 4$ and $f(x) = x^2 + x$

$$h = \frac{3}{n} \Rightarrow nh = 3$$

Thus, we have,

$$I = \int_1^4 \left(x^2 + x\right) dx$$

$$= \lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h)+----f(1+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[\left(1^2+1\right)+\left\{(1+h)^2+(1+h)\right\}+\left\{(1+2h)^2+(1+2h)\right\}+----\right]$$

$$= \lim_{h\to 0} h\left[\left(1^2+(1+h)^2+(1+2h)^2+\ldots\right)+\left(1+(1+h)+(1+2h)+\ldots\right)\right]$$

$$= \lim_{h\to 0} h\left[\left(n+2h(1+2+3+\ldots)+h^2\left(1+(2)^2+(3)^2+\ldots\right)\right)+\left(n+h(1+2+3+\ldots)\right)\right]$$

$$= \lim_{h\to 0} h\left[2n+3h(1+2+3+\ldots+(n-1))+h^2\left(1+(2)^2+(3)^2+\ldots(n-1)^2\right)\right]$$

$$\because h = \frac{3}{n} \text{ \& if } h\to 0 \Rightarrow n\to\infty$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[2n+\frac{9}{n}\frac{n(n-1)}{2}+\frac{9}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} 6+\frac{27}{2n^2}n^2\left(1-\frac{1}{n}\right)+\frac{27}{6n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= 6+\frac{27}{2}+\frac{27}{3}$$

$$\therefore \int_1^4 \left(x^2+x\right)dx = \frac{57}{2}$$

Question 64

Evaluate $\int_{-1}^{1} e^{2x}\,dx$

Solution 64

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\ldots+f\big(a+(n-1)h\big)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=-1, b=1, f(x)=e^{2x}$ and $h=\dfrac{1-(-1)}{n}=\dfrac{2}{n}$

$$\therefore \qquad I=\int_{-1}^{1} e^{2x}dx$$

$$\Rightarrow \qquad I=\lim_{h\to 0} h\left[f(-1)+f(-1+h)+f(-1+2h)+\ldots+f\big(-1+(n-1)h\big)\right]$$

$$\Rightarrow \qquad I=\lim_{h\to 0} h\left[e^{-2}+e^{-2+2h}+e^{-2+4h}+\ldots+e^{-2+2(n-1)h}\right]$$

$$\Rightarrow \qquad I=\lim_{h\to 0} he^{-2}\left[1+e^{2h}+e^{4h}+\ldots+e^{2(n-1)h}\right]$$

$$\Rightarrow \qquad I=\lim_{h\to 0} he^{-2}\left[\left\{\frac{\left(e^{2h}\right)^n-1}{e^{2h}-1}\right\}\right] \qquad \left[\text{Using: } a+ar+\ldots+ar^{n-1}=a\left(\frac{r^n-1}{r-1}\right)\right]$$

$$\Rightarrow \qquad I=\lim_{h\to 0}\frac{e^{-2}}{2}\left\{\frac{e^4-1}{\left(\frac{e^{2h}-1}{2h}\right)}\right\} \qquad \left[\because h=\frac{2}{n}\Rightarrow nh=2\right]$$

$$\Rightarrow \qquad I=\frac{e^{-2}}{2}\left(\frac{e^4-1}{1}\right)=\frac{e^2-e^{-2}}{2} \qquad \left[\because \lim_{h\to 0}\frac{e^h-1}{h}=1\right]$$

Question 65

Evaluate $\int_2^3 e^{-x}dx$

Solution 65

We have,

$$\int_a^b f(x) = dx \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\ldots+f(a+(n-1)h)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=2, b=3$ and $f(x)=e^{-x}$

$\therefore \quad h=\frac{1}{n} \Rightarrow nh=1$

Thus, we have

$$I=\int_2^3 e^{-x}dx$$

$$\Rightarrow \quad I=\lim_{h\to 0} h\left[f(2)+f(2+h)+f(2+2h)+\ldots+f(2+(n-1)h)\right]$$

$$\Rightarrow \quad I=\lim_{h\to 0} h\left[e^{-2}+e^{-(2+h)}+e^{-(2+2h)}+\ldots+e^{-(2+(n-1)h)}\right]$$

$$\Rightarrow \quad I=\lim_{h\to 0} he^{-2}\left[1+e^{-h}+e^{-(2h)}+e^{-(3h)}+\ldots+e^{-(n-1)h}\right]$$

$$\Rightarrow \quad I=\lim_{h\to 0} he^{-2}\left\{\frac{\left(e^{-h}\right)^{n}-1}{e^{-h}-1}\right\}$$

$$\Rightarrow \quad I=\lim_{h\to 0} he^{-2}\left[\frac{e^{-nh}-1}{e^{-h}-1}\right]=-e^{-2}\lim_{h\to 0}\frac{e^{-1}-1}{\left(\frac{e^{-h}-1}{-h}\right)} \qquad [\because nh=1]$$

$$\Rightarrow \quad I=-e^{-2}\left(e^{-1}-1\right)=e^{-2}-e^{-3}$$

Question 66

Evaluate $\int_1^3 \left(2x^2+5x\right)dx$

Solution 66

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + ---- f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here, $a = 1,\ b = 3$ and $f(x) = 2x^2 + 5x$

$$I = \lim_{h\to 0} h\left[f(1) + f(1+h) + f(1+2h) + ---- f(a+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[(2+5) + \left\{2(1+h)^2 + 5(1+h) + \left\{2(1+2h)^2 + 5(1+2h)\right\} + ----\right.\right]$$

$$= \lim_{h\to 0} h\left[7n + 9h(1+2+3+--) + 2h^2(1+2^2+3^2+---)\right]$$

$$\because h = \frac{2}{n} \text{ \& if } h \to 0 \Rightarrow n \to \infty$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[7n + \frac{18}{n}\frac{n(n-1)}{2} + \frac{8}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} 14 + \frac{18}{n^2}n^2\left(1 - \frac{1}{n}\right) + \frac{8}{3n^3}n^3\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)$$

$$= 14 + 18 + \frac{8}{3} \quad = \frac{112}{3}$$

Question 67

Evaluate $\int_1^3 (x^2 + 3x)dx$

Solution 67

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

where $h = \dfrac{b-a}{n}$

Here, $a = 1,\ b = 3$ and $f(x) = x^2 + 3x$

$$I = \lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h)+----f(a+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[(1+3)+\left\{1(1+h)^2+3(1+h)+\left\{1(1+2h)^2+3(1+2h)\right\}+----\right.\right]$$

$$= \lim_{h\to 0} h\left[4n+5h(1+2+3+--)+h^2\left(1+2^2+3^2+---\right)\right]$$

$$\because h = \frac{2}{n} \text{ \& if } h \to 0 \Rightarrow n \to \infty$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[4n+\frac{10}{n}\frac{n(n-1)}{2}+\frac{4}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} 8+\frac{10}{n^2}n^2\left(1-\frac{1}{n}\right)+\frac{4}{3n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= 8+10+\frac{8}{3} \quad = \frac{62}{3}$$

Question 68

Evaluate $\int_0^2 \left(x^2+2\right)dx$

Solution 68

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+----f(a+(n-1)h)\right]$$

where $h = \dfrac{b-a}{n}$

Here, $a = 0,\ b = 2$ and $f(x) = x^2+2$

$\therefore \quad h = \dfrac{2}{n} \Rightarrow nh = 2$

Thus, we have,

$$I = \int_0^2 \left(x^2+2\right)dx$$

$$= \lim_{h\to 0} h\left[f(0)+f(0+h)+f(2h)+----f(0+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[2+\left(h^2+2\right)+\left\{(2h)^2+2\right\}+----\right]$$

$$= \lim_{h\to 0} h\left[2h+h^2\left(1+2^2+3^2+----(n-1)^2\right]\right.$$

$\because h = \dfrac{2}{n}$ & if $h \to 0 \Rightarrow n \to \infty$

$$= \lim_{n\to\infty} \frac{2}{n}\left[2n+\frac{4}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} 4+\frac{4}{3n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= 4+\frac{8}{3} = \frac{20}{3}$$

$$\therefore \quad \int_0^2 \left(x^2+2\right)dx = \frac{20}{3}$$

Question 69

Evaluate $\int_0^3 \left(x^2+1\right)dx$

Solution 69

We have,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + ---- f(a+(n-1)h)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

Here, $a = 0$, $b = 3$ and $f(x) = x^2 + 1$

$$\therefore \quad h = \frac{3}{n} \Rightarrow nh = 3$$

Thus, we have,

$$I = \int_0^3 (x^2+1)dx$$

$$= \lim_{h\to 0} h\left[f(0) + f(0+h) + f(2h) + ---- f(0+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[1 + (h^2+1) + \{(2h)^2 + 1\} + ----\right]$$

$$= \lim_{h\to 0} h\left[n + h^2\left(1 + 2^2 + 3^2 + ---- (n-1)^2\right]\right.$$

$$\because h = \frac{3}{n} \text{ \& if } h \to 0 \Rightarrow n \to \infty$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[n + \frac{9}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty} 3 + \frac{9}{2n^3}n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= 3 + 9 = 12$$

Chapter 20 - Definite Integrals Exercise Ex. 20VSAQ

Question 1

Write the value of $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$.

Solution 1

We have,

$$\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$

We know that $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2x}{2} dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \cos 2x \, dx$$

$$= \frac{1}{2} [x]_0^{\frac{\pi}{2}} - \left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] - \frac{1}{2}\left[\sin 2\frac{\pi}{2} - \sin 0\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}\right] - \frac{1}{2}[\sin \pi] \qquad [\because \sin \pi = 0]$$

$$= \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{\pi}{4}$$

Question 2

Write the value of $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$.

Solution 2

We have,

$$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$$

We know that $\cos^2 x = \dfrac{1+\cos 2x}{2}$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(1+\cos 2x\right) dx$$

$$= \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} dx + \int_0^{\frac{\pi}{2}} \cos 2x\, dx\right]$$

$$= \frac{1}{2}\left[x\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}\right] + \frac{1}{2}\left[\sin \pi - \sin 0\right]$$

$$= \frac{\pi}{4} + \frac{1}{2} \times 0$$

$$= \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \frac{\pi}{4}$$

Question 3

Write the value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$.

Solution 3

We have,

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$$

We know that $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$$\therefore \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx$$

$$= \frac{1}{2} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (1-\cos 2x)\, dx$$

$$= \frac{1}{2}\left[\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} dx - \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos 2x\, dx\right]$$

$$= \frac{1}{2}[x]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] - \frac{1}{2}\left[\frac{\sin\pi}{2} - \frac{\sin(-\pi)}{2}\right]$$

$$= \frac{1}{2}[\pi] \quad = \frac{\pi}{2}$$

$$\therefore \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{\pi}{2}$$

Question 4

Write the value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\, dx$.

Solution 4

We have,

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\, dx$$

We know that $\cos^2 x = \dfrac{1+\cos 2x}{2}$

$$= \frac{1}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2x)\, dx$$

$$= \frac{1}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} dx + \frac{1}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos 2x\, dx$$

$$= \frac{1}{2}\left[x\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] + \frac{1}{2}\left[\frac{\sin \pi}{2} - \frac{\sin(-\pi)}{2}\right]$$

$$= \frac{\pi}{2}$$

$$\therefore \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\, dx = \frac{\pi}{2}$$

Question 5

Write the value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^3 x\, dx$.

Solution 5

We have,

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \, dx$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{3\sin x - \sin 3x}{4} dx \qquad \left[\because \sin^3 x = \frac{3\sin x - \sin 3x}{4}\right]$$

$$= \frac{1}{4}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (3\sin x - \sin 3x)\, dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[-3\cos\frac{\pi}{2} + \frac{\cos\frac{3\pi}{2}}{3} + 3\cos\left(\frac{-\pi}{2}\right) - \frac{\cos\frac{-3\pi}{2}}{3}\right]$$

$$= \frac{1}{4}[0+0+0-0]$$

$$\therefore \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \, dx = 0$$

Question 6

Write the value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x\cos^2 x \, dx$.

Solution 6

We have,

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x\cos^2 x \, dx$$

We know that,

$\int_{-a}^{a} f(x)dx = 0$ when $f(x)$ is an odd function

Here,

$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x\cos^2 x \, dx$ is an odd function

$$\therefore \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x\cos^2 x \, dx = 0$$

Question 7

Write the value of $\int_0^{\frac{\pi}{4}} \tan^2 x\, dx$.

Solution 7

We have,

$$\int_0^{\frac{\pi}{4}} \tan^2 x\, dx$$

Adding and subtracting 1

$$\therefore \int_0^{\frac{\pi}{4}} \left(1 + \tan^2 x - 1\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\sec^2 x - 1\right) dx \qquad \left[\because 1 + \tan^2 x = \sec^2 x\right]$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x\, dx - \int_0^{\frac{\pi}{4}} dx$$

$$= \left[\tan x\right]_0^{\frac{\pi}{4}} - \left[x\right]_0^{\frac{\pi}{4}}$$

$$= \left[\tan \frac{\pi}{4} - \tan 0\right] - \left[\frac{\pi}{4} - 0\right]$$

$$= \left[1 - 0 - \frac{\pi}{4} + 0\right]$$

$$\therefore \int_0^{\frac{\pi}{4}} \tan^2 x\, dx = 1 - \frac{\pi}{4}$$

Question 8

Write the value of $\int_0^1 \frac{1}{x^2 + 1} dx$.

Solution 8

We have,

$$\int_0^1 \frac{1}{x^2+1}dx$$

We know that $\int \frac{1dx}{x^2+1} = \tan^{-1}x$

$$\int_0^1 \frac{1}{x^2+1} = \left[\tan^{-1}x\right]_0^1$$

$$= \left[\tan^{-1}1 - \tan^{-1}0\right]$$

$$= \left[\tan^{-1}\left(\tan\frac{\pi}{4}\right) - 0\right] \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$\therefore \int_0^1 \frac{1}{x^2+1}dx = \frac{\pi}{4}$$

Question 9

Write the value of $\int_{-2}^{1} \frac{|x|}{x}dx$.

Solution 9

$$I = \int_{-2}^{1} \frac{|x|}{x}dx$$

$$\because \frac{|x|}{x} \geq 0 \text{ if } x \geq 0$$

$$\therefore I = \int_{-2}^{0} \frac{-x}{x}dx + \int_0^1 \frac{x}{x}dx$$

$$= \int_{-2}^{0} -dx + \int_0^1 dx$$

$$= \left[-x\right]_{-2}^{0} + \left[x\right]_0^1$$

$$= -2 + 1$$

$$= -1$$

$$\therefore \int_{-2}^{1} \frac{|x|}{x}dx = -1$$

Question 10

Write the value of $\int_0^{\infty} e^{-x}dx$.

Solution 10

We have,

$$\int_0^\infty e^{-x}dx$$

$$\left[-e^{-x}\right]_0^\infty$$

$$= \left[-e^{-\infty} + e^0\right] \qquad \left[\because e^0 = 1\right]$$

$$= \left[0 + 1\right]$$

$$= 1$$

$$\therefore \int_0^\infty e^{-x}dx = 1$$

Question 11

Write the value of $\int_0^4 \frac{1}{\sqrt{16 - x^2}}dx$.

Solution 11

We have,

$$\int_0^4 \frac{1}{\sqrt{16 - x^2}}dx$$

We know that $\frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$

$$\therefore \int_0^4 \frac{1}{\sqrt{16 - x^2}}\,dx$$

$$= \left[\sin^{-1}\frac{x}{4}\right]_0^4$$

$$= \left[\sin^{-1}1 - \sin^{-1}0\right] \qquad \left[\because \sin\frac{\pi}{2} = 1\right]$$

$$= \left[\sin^{-1}\left(\sin\frac{\pi}{2}\right) - \sin^{-1}(\sin 0)\right]$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^4 \frac{1}{\sqrt{16 - x^2}} = \frac{\pi}{2}$$

Question 12

Write the value of $\int_0^3 \frac{1}{x^2 + 9}dx$.

Solution 12

We have,

$$\int_0^3 \frac{1}{x^2+(3)^2}dx$$

We know that $\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$$\therefore \int_0^3 \frac{1}{x^2+(3)^2}dx$$

$$= \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$$

$$= \frac{1}{3}\left[\tan^{-1}1 - \tan^{-1}0\right]$$

$$= \frac{1}{3}\left[\tan^{-1}\left(\tan\frac{\pi}{4}\right) - \tan^{-1}(\tan 0)\right]$$

$$= \frac{1}{3}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{12}$$

$$\therefore \int_0^3 \frac{1}{x^2+9}dx = \frac{\pi}{12}$$

Question 13

Write the value of $\int_0^{\frac{\pi}{2}} \sqrt{1-\cos 2x}\,dx$.

Solution 13

We have,

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\cos 2x}\,dx$$

We know that $1-\cos 2x = 2\sin^2 x$

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{1-\cos 2x}\,dx$$

$$\sqrt{2}\int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= \sqrt{2}\left[-\cos x\right]_0^{\frac{\pi}{2}}$$

$$= -\sqrt{2}\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -\sqrt{2}\left[0-1\right]$$

$$= \sqrt{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{1-\cos 2x}\,dx = \sqrt{2}$$

Question 14

Write the value of $\int_0^{\frac{\pi}{2}} \log\tan x\,dx$.

Solution 14

We have,

$$I = \int_0^{\frac{\pi}{2}} \log\tan x\,dx \qquad \left[\because \int_0^b f(x)\,dx = \int_0^b f(b-x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \log\tan\left(\frac{\pi}{2}-x\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log\cot x\,dx$$

$$= -\int_0^{\frac{\pi}{2}} \log\tan x\,dx = -I$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

Hence,

$$\int_0^{\frac{\pi}{2}} \log\tan x\,dx = 0$$

Question 15

Write the value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{3+5\cos x}{3+5\sin x}\right)dx$.

Solution 15

We know that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore I = \int_0^{\frac{\pi}{2}} \log\left(\frac{3+5\cos x}{3+5\sin x}\right)dx \qquad --(i)$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{3+5\cos\left(\frac{\pi}{2}-x\right)}{3+5\sin\left(\frac{\pi}{2}-x\right)}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{3+5\sin x}{3+5\cos x}\right)dx$$

$$= -\int_0^{\frac{\pi}{2}} \log\left(\frac{3+5\cos x}{3+5\sin x}\right)dx$$

$-I$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

$$\therefore \int_0^{\frac{\pi}{2}} \log\left(\frac{3+5\cos x}{3+5\sin x}\right)dx = 0$$

Question 16

Write the value of $\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}dx, n \in N,$

Solution 16

We know that $\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx \qquad --(1)$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^n \left(\frac{\pi}{2} - x\right)}{\sin^n \left(\frac{\pi}{2} - x\right) + \cos^n \left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x} dx \qquad (2)$$

Adding (1) and (2)

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^n x + \sin^n x}{\cos^n x + \sin^n x} dx$$

$$= \int_0^{\frac{\pi}{2}} dx$$

$$= [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4}$$

Hence,

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4}$$

Question 17

Write the value of $\int_0^{\pi} \cos^5 x \, dx$.

Solution 17

We have,

$$I = \int_0^{\pi} \cos^5 x \, dx = \int_0^{\pi} \cos^4 x \times \cos x \, dx$$

$$= \int_0^{\pi} \left(1 - \sin^2 x\right)^2 \cos x \, dx$$

Let $\sin x = t$

$\Rightarrow \cos x \, dx = dt$

When $x = 0 \Rightarrow t = 0$

$x = \pi \Rightarrow t = 0$

$$\therefore \; I = \int_0^{0} \left(1 - t^2\right)^2 dt = 0$$

$$\therefore \int_0^{\pi} \cos^5 x \, dx = 0$$

Question 18

Write the value of $\int_0^2 [x] dx$.

Solution 18

We have,

$$\int_0^2 [x] dx$$

$$= \int_0^1 [x] dx + \int_1^2 [x] dx$$

$$= \int_0^1 0 \, dx + \int_1^2 dx$$

$$= 0 + [x]_1^2$$

$$= 1$$

$$\therefore \int_0^2 [x] dx = 1$$

Question 19

Write the value of $\int_0^{1.5} [x] dx$.

Solution 19

We have,

$$\int_0^{1.5} [x]dx$$

$$= \int_0^1 [x]dx + \int_1^{1.5} [x]dx$$

$$= \int_0^1 0\,dx + \int_1^{1.5} 1\,dx$$

$$= 0 + [x]_1^{1.5}$$

$$= [1.5 - 1]$$

$$= 0.5$$

$$\therefore \int_0^{1.5} [x]dx = 0.5$$

Question 20

Write the value of $\int_0^1 \{x\}dx$, where $\{x\}$ denotes the fractional part of x.

Solution 20

We have,

$$\int_0^1 \{x\}dx$$

$\{x\} = x$, as $\{x\}$ denotes the fractional part of x

$$\therefore \int_0^1 \{x\}dx = \int_0^1 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

Hence,

$$\int_0^1 \{x\}dx = \frac{1}{2}$$

Question 21

Write the value of $\int_0^1 e^{\{x\}}dx$.

Solution 21

$$\int_0^1 e^{\{x\}}\,dx$$

Since $\{x\} = x$, as $\{x\}$ denotes the fractional part of x.

$$\therefore \int_0^1 e^{\{x\}}\,dx = \int_0^1 e^x\,dx$$

$$= \left[e^x\right]_0^1$$

$$= e - 1$$

Hence,

$$\int_0^1 e^{\{x\}}\,dx = e - 1$$

Question 22

Write the value of $\int_0^2 x\left[x\right]dx$.

Solution 22

We have,

$$\int_0^2 x\left[x\right]dx$$

$$= \int_0^1 x\left[x\right]dx + \int_1^2 x\left[x\right]dx$$

$$= \int_0^1 0 \times x\,dx + \int_1^2 x\,dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2}\left[4 - 1\right]$$

$$= \frac{3}{2}$$

$$\therefore \int_0^2 x\left[x\right]dx = \frac{3}{2}$$

Question 23

Write the value of $\int_0^1 2^{x-[x]}\,dx$.

Solution 23

We have,

$$\int_0^1 2^{x-[x]}dx$$

$$= \int_0^1 e^{(x-[x])\log 2}dx$$

$$= \left[\frac{e^{x\log 2}}{\log 2}\right]_0^1$$

$$= \frac{e^{\log 2}}{\log 2} - \frac{1}{\log 2}$$

$$= \frac{1}{\log_e 2}\left[2-1\right]$$

$$= \frac{1}{\log_e 2}$$

$$\therefore \int_0^1 2^{x-[x]}dx = \frac{1}{\log_e 2}$$

Question 24

Write the value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right)d\theta.$

Solution 24

We have,

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right)d\theta$$

$$f(\theta) = \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right)$$

$$f(-\theta) = \log\left(\frac{a-\sin(-\theta)}{a+\sin(-\theta)}\right)$$

$$= \log\frac{a+\sin\theta}{a-\sin\theta}$$

$$= -\log\left(\frac{a-\sin\theta}{a+\sin\theta}\right)$$

$$= -f(\theta)$$

$\because f(\theta)$ is odd function

$$\therefore \int_{-a}^{a} f(x)dx = 0$$

$$\therefore \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right)d\theta = 0$$

Question 25

Write the value of $\int_{-1}^{1} x\,|x|\,dx$.

Solution 25

We have,

$\int_{-1}^{1} x\,|x|\,dx$

Here, $f(x) = x\,|x|$

$f(-x) = -x\,|x| \quad = -f(x)$

$\because f(x)$ is odd function

$\therefore \int_{-a}^{a} f(x)\,dx = 0 \qquad$ when $f(x)$ is odd

$\therefore \int_{-1}^{1} x\,|x|\,dx = 0$

Question 26

Write the value of $\int_{1}^{2} \log_e [x]\,dx$.

Solution 26

We have,

$\int_{1}^{2} \log_e [x]\,dx$

$= \int_{1}^{2} \log_e (1)\,dx = 0 \qquad [\because \log_e 1 = 0]$

$\therefore \int_{1}^{2} \log_e [x]\,dx = 0$

Question 27

Write the value of $\int_{a}^{b} \frac{f(x)}{f(x) + f(a+b-x)}\,dx$.

Solution 27

$$I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx \qquad \text{(i)}$$

$$I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}dx \qquad \text{(ii)}$$

Add (i) & (ii)

$$2I = \int_a^b \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}dx$$
$$= \int_a^b dx$$
$$= [x]_a^b$$
$$= b - a$$

$$I = \frac{b-a}{2}$$

Question 28 n

Evaluate: $\int_0^1 \frac{1}{1+x^2}dx$

Solution 28 n

We have,

$$\int_0^1 \frac{1}{1+x^2}dx$$

$$= \left[\tan^{-1} x\right]_0^1$$
$$= \left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$
$$= \left[\tan^{-1}\left(\tan\frac{\pi}{4}\right) - \tan^{-1}(\tan 0)\right]$$
$$= \frac{\pi}{4}$$

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4}$$

Question 29

Evaluate: $\int_0^1 \frac{1}{1+x^2}dx$

Solution 29

We have,

$$\int_0^1 \frac{1}{1+x^2}dx$$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \left[\tan^{-1}\left(\tan\frac{\pi}{4}\right) - \tan^{-1}(\tan 0)\right]$$

$$= \frac{\pi}{4}$$

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4}$$

Question 30

Evaluate $\int_2^3 \frac{1}{x}dx$

Solution 30

$$\text{Let} \quad I = \int_2^3 \frac{1}{x}dx$$

$$= [\ln x]_2^3$$

$$= \ln 3 - \ln 2$$

$$= \ln\left(\frac{3}{2}\right)$$

Question 31

Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$

Solution 31

Since

$$\int \sqrt{a^2-x^2}\,dx = a\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2} + C$$

Therefore,

$$\int_0^2 \sqrt{4-x^2}\,dx = \int_0^2 \sqrt{2^2-x^2}\,dx$$

$$= \left[2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{2^2-x^2}\right]_0^2$$

$$= \left[2\sin^{-1}\left(\frac{2}{2}\right) + \frac{1}{2}\cdot 2\sqrt{2^2-2^2} - \left(2\sin^{-1}\left(\frac{0}{2}\right) + \frac{1}{2}\cdot 0\sqrt{2^2-0^2}\right)\right]$$

$$= 2\frac{\pi}{2}$$

$$= \pi$$

Question 32

Write the value of $\int_0^1 \frac{2x}{1+x^2}dx$

Solution 32

Consider that

$$1+x^2=t$$
$$2xdx=dt$$

We can put

$$x=0, t=1$$
$$x=1, t=2$$

Therefore,

$$I=\int_1^2 \frac{2x}{1+x^2}dx$$
$$=\int_1^2 \frac{dt}{t}$$
$$=[\log t]_1^2$$
$$=\log 2-\log 1$$
$$=\log 2-0$$
$$=\log 2$$

Question 33

Write the value of the integral $\int_0^{\sqrt{2}}[x^2]dx$.

Solution 33

We have,

$$\int_0^{\sqrt{2}}[x^2]dx$$

$$=\int_0^1[x^2]dx+\int_1^{\sqrt{2}}[x^2]dx$$
$$=0+\int_1^{\sqrt{2}}dx$$
$$=[x]_1^{\sqrt{2}}$$
$$=\sqrt{2}-1$$

$$\therefore \int_0^{\sqrt{2}}[x^2]dx=\sqrt{2}-1$$

Question 34

If $\int_0^1(3x^2+2x+k)dx=0$, find the value of k.

Solution 34

If $\int_0^1 \left(3x^2 + 2x + k\right)dx = 0$

$\Rightarrow \left[x^3 + x^2 + kx\right]_0^1 = 0$

$\Rightarrow 2 + k = 0$

$\therefore k = -2$

Question 35

Write the value of the integral $\int_0^{\frac{\pi}{4}} \sin\{x\}dx$, where $\{\ \}$ denotes the fractional part function.

Solution 35

We have,

$\int_0^{\frac{\pi}{4}} \sin\{x\}dx$

$= \left[-\cos\right]_0^{\frac{\pi}{4}}$

$= -\left[\cos\frac{\pi}{4} - \cos 0\right]$

$= -\left[\frac{1}{\sqrt{2}} - 1\right]$

$= 1 - \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{2} - 1}{\sqrt{2}}$

$\therefore \int_0^{\frac{\pi}{4}} \sin\{x\}dx = \frac{\sqrt{2} - 1}{\sqrt{2}}$

Question 36

Write the coefficient a, b, c of which the value of the integral $\int_{-3}^{3} \left(ax^2 + bx + c\right)dx$ is independent.

Solution 36

We have,

$$\int_{-3}^{3}\left(ax^2+bx+c\right)dx$$

$\because$ the value is independent.

$$\therefore f(-x) = -f(x)$$
$$\Rightarrow ax^2+bx+c = -\left(ax^2+bx+c\right)$$
$$\Rightarrow 2\left(ax^2+c\right) = 0$$

$\therefore$ b is the required coefficient.

Question 37

If $\int_0^a 3x^2 dx = 8$, write the value of a.

Solution 37

Given that $\int_0^a 3x^2 dx = 8$

$$\Rightarrow \left[\frac{3x^3}{3}\right]_0^a = 8$$
$$\Rightarrow a^3 - 0 = 8$$
$$\Rightarrow a^3 = 8$$
$$\Rightarrow a = 2$$

Question 38

If $f(x)=\int_0^x t\sin t\,dt$, then write the value of $f'(x)$.

Solution 38

$$f(x)=\int_0^x t\sin t\ dt$$

By the Newton-Leibnitz formula,

$$\frac{d}{dx}f(x) = x\sin x$$

Question 39

Evaluate : $\int_2^4 \frac{x}{x^2+1}dx$

Solution 39

$$\int_2^4 \frac{x}{x^2+1}dx$$

Substitute $x^2+1=t$

$\Rightarrow 2xdx = dt$

The integral becomes

$$\frac{1}{2}\int_5^{17}\frac{dt}{t} = \frac{1}{2}[\ln t]_5^{17} = \frac{1}{2}\ln\left(\frac{17}{5}\right)$$

Question 40

Evaluate: $\int_0^{\frac{\pi}{2}} e^x(\sin x - \cos x)dx$

Note: The lower limit is incorrect in textbook. Consider the lower limit as '0'.

Solution 40

$$\int_0^{\frac{\pi}{2}} e^x(\sin x - \cos x)dx$$

Let $f(x) = -\cos x$

$\Rightarrow f'(x) = \sin x$

Hence, the integral becomes

$$\int_0^{\frac{\pi}{2}} e^x\left(f(x)+f'(x)\right)dx$$

$$= \left[e^x(-\cos x)\right]_0^{\frac{\pi}{2}}$$

$$= 0-(-1) = 1$$

Question 41

Evaluate : $\int_e^{e^2}\frac{1}{x\log x}dx$

Solution 41

$$\int_e^{e^2}\frac{1}{x\log x}dx$$

Substitute $\log x = t$

$\Rightarrow \frac{1}{x}dx = dt$

The definite integral becomes

$$\int_1^2 \frac{\mathrm{dt}}{\mathrm{t}} = [\ln t]_1^2 = \ln 2 - \ln 1 = \ln 2$$

Question 42

If $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$, find the value of a.

Solution 42

$$\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$$

$$\Rightarrow \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^a = \frac{\pi}{8}$$

$$\Rightarrow \tan^{-1}\left(\frac{a}{2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan\frac{\pi}{4} = \frac{a}{2}$$

$$\Rightarrow a = 2$$

Question 43

Evaluate: $\int_0^{\frac{\pi}{4}} \sin 2x \ dx$

Solution 43

$$\int_0^{\frac{\pi}{4}} \sin 2x \ dx$$

$$= -\frac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{2}\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -\frac{1}{2}\left[-1\right] = \frac{1}{2}$$

Question 44

Evaluate: $\int_0^1 xe^{x^2}dx$

Solution 44

$$\int_0^1 xe^{x^2}dx$$

Substitute $x^2 = t$

$\Rightarrow 2xdx = dt$

The integral becomes $\frac{1}{2}\int_0^1 e^t dt$

$$= \frac{1}{2}\left[e^t\right]_0^1 = \frac{1}{2}(e-1)$$

Question 45

Evaluate: $\int_0^{\frac{\pi}{4}} \tan x \ dx$

Solution 45

$$\int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$\left[\ln(\sec x)\right]_0^{\frac{\pi}{4}} = \ln\left(\sec\frac{\pi}{4}\right) - \ln(\sec 0)$$

$$= \ln\sqrt{2} = \frac{1}{2}\ln 2$$

Chapter 20 - Definite Integrals Exercise MCQ

Question 1

Mark the correct alternative in each of the following:

$$\int_0^1 \sqrt{x(1-x)}\, dx \text{ equals}$$

(a) p/2
(b) p/4
(c) p/6
(d) p/8

Solution 1

Correct option: (d)

$$I = \int_0^1 \sqrt{x(1-x)}\, dx$$

$$I = \int_0^1 \sqrt{x - x^2}\, dx$$

$$I = \int_0^1 \sqrt{\frac{1}{4} + x - x^2 - \frac{1}{4}}\, dx$$

$$I = \int_0^1 \sqrt{\frac{1}{4} - \left(x^2 - x + \frac{1}{4}\right)}\, dx$$

$$I = \int_0^1 \sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}\, dx$$

$$I = \left[\frac{x - \frac{1}{2}}{2} \times \sqrt{x(1-x)} + \frac{1}{2} \times \frac{1}{4}\sin^{-1}(2x-1)\right]_0^1$$

$$I = 0 + \frac{1}{8}\left(\sin^{-1}(1) - \sin^{-1}(-1)\right)$$

$$I = \frac{1}{8}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$I = \frac{\pi}{8}$$

Question 2

$$\int_0^{\pi} \frac{1}{1 + \sin x}\, dx \text{ equal}$$

(a) 0
(b) 1/2
(c) 2

(d) 3/2

Solution 2

Correct option: (c)

$$I = \int_0^{\pi} \frac{1}{1+\sin x}\, dx$$

$$I = \int_0^{\pi} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\, dx$$

$$I = \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}\, dx$$

$$I = \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\, dx$$

$$I = \int_0^{\pi} \left(\sec^2 x - \tan x \sec x\right) dx$$

$$I = \left[\tan x - \sec x\right]_0^{\pi}$$

$$I = -\left(-1-1\right)$$

$$I = 2$$

Question 3

The value of $\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\, dx$ is

(a) $\frac{\pi^2}{4}$

(b) $\frac{\pi^2}{2}$

(c) $\frac{3\pi^2}{2}$

(d) $\frac{\pi^2}{3}$

Solution 3

Correct option: (a)

$$I=\int_0^{\pi}\frac{x\tan x}{\sec x+\cos x}\,dx\ldots\ldots\ldots\ldots\ldots(i)$$

$$I=\int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\cos(\pi-x)}\,dx$$

$$I=\int_0^{\pi}\frac{-(\pi-x)\tan x}{-\sec x-\cos x}\,dx$$

$$I=\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x+\cos x}\,dx\ldots\ldots\ldots\ldots\ldots(ii)$$

$$2I=\int_0^{\pi}\frac{x\tan x}{\sec x+\cos x}\,dx+\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x+\cos x}\,dx\quad(\because \text{Adding (i) and (ii)})$$

$$2I=\int_0^{\pi}\frac{\pi\tan x}{\sec x+\cos x}dx$$

$$2I=\int_0^{\pi}\frac{\pi\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\cos x}dx$$

$$2I=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2x}dx$$

Put $\cos x=t\Rightarrow-\sin x\,dx=dt$

$\Rightarrow\sin x\,dx=-dt$

x	0	π
t	1	-1

$$\Rightarrow 2I=\int_1^{-1}\frac{-\pi dt}{1+t^2}$$

$$I=\frac{-\pi}{2}\left[\tan^{-1}t\right]_1^{-1}$$

$$I=\frac{-\pi}{2}\left(\tan^{-1}[-1]-\tan^{-1}[1]\right)$$

$$I=\frac{-\pi}{2}\left(\frac{-\pi}{4}-\frac{\pi}{4}\right)$$

$$I=\frac{\pi^2}{4}$$

Question 4

The value of $\int_0^{2\pi}\sqrt{1+\sin\frac{x}{2}}\,dx$ is

(a) 0
(b) 2
(c) 8
(d) 4

Solution 4

Correct option: (c)

$$I = \int_0^{2\pi} \sqrt{1 + \sin\frac{x}{2}}\ dx$$

$$I = \int_0^{2\pi} \sqrt{\left(\cos\frac{x}{4} + \sin\frac{x}{4}\right)^2}\ dx$$

$$I = \int_0^{2\pi} \left(\cos\frac{x}{4} + \sin\frac{x}{4}\right) dx$$

$$I = \left[\frac{\sin\frac{x}{4}}{\frac{1}{4}} - \frac{\cos\frac{x}{4}}{\frac{1}{4}}\right]_0^{2\pi}$$

$$I = 4\left[\sin\frac{\pi}{2} - \left(\cos\frac{\pi}{2} - \cos 0\right)\right]$$

$$I = 4(1 - 0 + 1)$$

$$I = 8$$

Note: Answer not matching with back answer.

Question 5

The value of the integral $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\ dx$ is

(a) 0
(b) p/2
(c) p/4
(d) None of these

Solution 5

Correct option:(c)

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx ....(i)$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}} dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx ....(ii)$$

Adding (i) and (ii)

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$

$$I = \frac{1}{2} \int_0^{\pi/2} 1\, dx$$

$$I = \frac{1}{2} \left[x\right]_0^{\pi/2}$$

$$I = \frac{\pi}{4}$$

Note: In these type of sums integration can be calculated using formula $\frac{b-a}{2}$ where a is lower and b is upper limit.

Question 6

$$\int_0^{\infty} \frac{1}{1+e^x} dx \text{ equals}$$

(a) Log 2-1
(b) Log 2
(c) Log 4-1
(d) -log 2

Solution 6

Correct option: (b)

$$I = \int_0^{\infty} \frac{1}{1+e^x} dx$$

$$I = \int_0^{\infty} \frac{\frac{1}{e^x}}{\frac{1+e^x}{e^x}} dx$$

$$I = \int_0^{\infty} \frac{e^{-x}}{1+e^{-x}} dx$$

Put $1 + e^{-x} = t \Rightarrow -e^{-x}dx = dt$

$e^{-x}dx = -dt$

x	0	∞
t	2	1

$$I = \int_2^1 \frac{-dt}{t}$$

$$I = -\left[\log|t|\right]_2^1$$

$$I = -(\log 1 - \log 2)$$

$$I = \log 2$$

Question 7

$$\int_0^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}} \, dx \text{ equals}$$

(a) 2
(b) 1
(c) p/4
(d) p²/8

Solution 7

Correct option: (a)

$$I = \int_0^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

$$\Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$$

x	0	$\frac{\pi^2}{4}$
t	0	$\frac{\pi}{2}$

$$I = \int_0^{\pi/2} 2\sin t\,dt$$

$$I = 2\left[-\cos t\right]_0^{\pi/2}$$

$$I = -2(0-1)$$

$$I = 2$$

Question 8

$$\int_0^{\pi/2} \frac{\cos x}{(2+\sin x)(1+\sin x)}dx \text{ equals}$$

(a) $\log\left(\frac{2}{3}\right)$

(b) $\log\left(\frac{3}{2}\right)$

(c) $\log\left(\frac{3}{4}\right)$

(d) $\log\left(\frac{4}{3}\right)$

Solution 8

Correct option: (d)

$$I = \int_0^{\pi/2} \frac{\cos x}{(2+\sin x)(1+\sin x)} dx$$

Put $\sin x = t \Rightarrow \cos x dx = dt$

x	0	$\pi/2$
t	0	1

$$I = \int_0^1 \frac{dt}{(1+t)(2+t)}$$

$$I = \int_0^1 \frac{[(t+2)-(t+1)]dt}{(1+t)(2+t)}$$

$$I = \int_0^1 \left(\frac{1}{t+1} - \frac{1}{t+2}\right)dt$$

$$I = \left[\log|t+1| - \log|t+2|\right]_0^1$$

$$I = \log 2 - \log 1 - (\log 3 - \log 2)$$

$$I = 2\log 2 - \log 3$$

$$I = \log 4 - \log 3$$

$$I = \log\left(\frac{4}{3}\right)$$

Question 9

$\int_0^{\pi/2} \frac{1}{2+\cos x}$ dx equals

(a) $\frac{1}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(b) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(c) $\sqrt{3}\tan^{-1}(\sqrt{3})$

(d) $2\sqrt{3}\tan^{-1}(\sqrt{3})$

Solution 9

Correct option: (b)

$$I = \int_0^{\pi/2} \frac{1}{2+\cos x}\,dx$$

$$\text{Put } \tan\frac{x}{2} = t \Rightarrow x = 2\tan^{-1} t$$

$$\Rightarrow dx = \frac{2dt}{1+t^2}$$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

x	0	$\frac{\pi}{2}$
t	0	1

$$I = \int_0^1 \frac{\frac{2dt}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}$$

$$I = \int_0^1 \frac{2dt}{3+t^2}$$

$$I = \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_0^1$$

$$I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Question 10

$$\int_0^{\pi} \sqrt{\frac{1-x}{1+x}}\,dx =$$

(a) $\frac{\pi}{2}$

(b) $\frac{\pi}{2} - 1$

(c) $\frac{\pi}{2} + 1$

(d) $\pi + 1$

Solution 10

$$I = \int_0^{\pi} \sqrt{\frac{1-x}{1+x}}\,dx$$

$$I = \int_0^{\pi} \sqrt{\frac{1-x}{1+x} \times \frac{1-x}{1-x}}\,dx$$

$$I = \int_0^{\pi} \frac{1-x}{\sqrt{1-x^2}}\,dx$$

Consider,

$$1-x = A\frac{d\left(1-x^2\right)}{dx} + B$$

$$1-x = -2Ax + B$$

Comparing both sides,

$$-2A = -1 \Rightarrow A = \frac{1}{2}$$

And $B = 1$

$$\Rightarrow I = \int_0^{\pi} \left(\frac{\left(\frac{1}{2}\right)(-2x) + \frac{1}{2}}{\sqrt{1-x^2}} \right) dx$$

$$I = \frac{1}{2}\int_0^{\pi} \frac{-2x}{\sqrt{1-x^2}}\,dx + \int_0^{\pi} \frac{1}{2\sqrt{1-x^2}}\,dx$$

$$I = \left[\frac{1}{2}\left(2\sqrt{1-x^2}\right) + \frac{1}{2}\sin^{-1}x \right]_0^{\pi}$$

$$I = \frac{1}{2}\left(2\sqrt{1-\pi^2} - 2 + \sin^{-1}\pi - \sin^{-1}0\right)$$

$$I = \sqrt{1-\pi^2} - 1$$

Note: Answer not matching with back answer.

Question 11

$$\int_0^{\pi} \frac{1}{a + b\cos x}\,dx =$$

(a) $\frac{\pi}{\sqrt{a^2 - b^2}}$

(b) $\frac{\pi}{\alpha b}$

(c) $\frac{\pi}{a^2 + b^2}$

(d) $(a+b)\,\pi$

Solution 11

Correct option: (a)

$$I = \int_0^{\pi} \frac{1}{a + b\cos x}\,dx$$

$$\text{Put } \tan\frac{x}{2} = t \Rightarrow x = 2\tan^{-1}t$$

$$dx = \frac{2dt}{1+t^2}$$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

x	0	π
t	0	∞

$$I = \int_0^{\infty} \frac{\frac{2dt}{1+t^2}}{a + b\left(\frac{1-t^2}{1+t^2}\right)}$$

$$I = \int_0^{\infty} \frac{2dt}{a + at^2 + b - bt^2}$$

$$I = 2\int_0^{\infty} \frac{dt}{a + b + (a-b)t^2}$$

$$I = 2\int_0^{\infty} \frac{dt}{\left(\sqrt{a+b}\right)^2 + \left(\sqrt{(a-b)}t\right)^2}$$

$$I = \frac{2}{\sqrt{(a+b)(a-b)}}\left[\tan^{-1}\left(\frac{t}{\sqrt{\frac{a+b}{a-b}}}\right)\right]_0^{\infty}$$

$$I = \frac{2}{\sqrt{(a+b)(a-b)}}\left(\frac{\pi}{2} - 0\right)$$

$$I = \frac{\pi}{\sqrt{(a^2-b^2)}}$$

Question 12

$$\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx \text{ is}$$

(a) p/3
(b) p/6
(c) p/12
(d) p/2

Solution 12

Correct option:(c)

$$I=\int_{\pi/6}^{\pi/3}\frac{1}{1+\sqrt{\cot x}}\,dx$$

$$I=\int_{\pi/6}^{\pi/3}\frac{1}{1+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}}\,dx$$

$$I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx.........(i)$$

$$I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}\,dx$$

$$I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx.....................(ii)$$

Adding (i) and (ii)

$$2I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx+\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$2I=\int_{\pi/6}^{\pi/3}dx$$

$$I=\frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$

$$I=\frac{\pi}{12}$$

Note: In these type of sums integration can be calculated using formula $\frac{b-a}{2}$ where a is lower and b is upper limit.

Question 13

Given that $\int_0^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}dx=\frac{\pi}{2(a+b)(b+c)(c+a)}$,

the value of $\int_0^{\infty}\frac{dx}{(x^2+4)(x^2+9)}$, is

(a) $\frac{\pi}{60}$

(b) $\frac{\pi}{20}$

(c) $\frac{\pi}{40}$

(d) $\frac{\pi}{80}$

Solution 13
Correct option: (a)

$$I = \int_0^{\infty} \frac{dx}{(x^2+4)(x^2+9)}$$

$$I = \frac{1}{5}\int_0^{\infty} \frac{5dx}{(x^2+4)(x^2+9)}$$

$$I = \frac{1}{5}\int_0^{\infty} \frac{\left[x^2+9-\left(x^2+4\right)\right]dx}{(x^2+4)(x^2+9)}$$

$$I = \frac{1}{5}\left(\int_0^{\infty} \frac{1}{x^2+4}dx - \int_0^{\infty} \frac{1}{x^2+9}dx\right)$$

$$I = \frac{1}{5}\left(\frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{1}{3}\tan^{-1}\frac{x}{3}\right)_0^{\infty}$$

$$I = \frac{1}{5}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$I = \frac{\pi}{60}$$

Question 14

$$\int_1^e \log x \, dx =$$

(a) 1
(b) e-1
(c) e+1
(d) 0

Solution 14
Correct option: (a)

$$I = \int_1^e \log x \, dx$$

$$I = \int_1^e \log x\,(1)\, dx$$

$$I = \log x\int_1^e dx - \int_1^e \left(\frac{d(\log x)}{dx}\int dx\right)dx$$

$$I = \left[x\log x\right]_1^e - \int_1^e \left(\frac{1}{x}x\right)dx$$

$$I = e\log e - \log 1 - \left[x\right]_1^e$$

$$I = e - 0 - (e-1)$$

$$I = 1$$

Question 15

$\int_{1}^{\sqrt{3}} \frac{1}{1+x^2} dx$ is equal to

(a) $\frac{\pi}{12}$

(b) $\frac{\pi}{6}$

(c) $\frac{\pi}{4}$

(d) $\frac{\pi}{3}$

Solution 15

Correct option:(a)

$$I = \int_{1}^{\sqrt{3}} \frac{1}{1+x^2} dx$$

$$I = \tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)$$

$$I = \frac{\pi}{3} - \frac{\pi}{4}$$

$$I = \frac{\pi}{12}$$

Question 16

$\int_{0}^{3} \frac{3x+1}{x^2+9} dx =$

(a) $\frac{\pi}{12} + \log(2\sqrt{2})$

(b) $\frac{\pi}{2\infty} + \log(2\sqrt{2})$

(c) $\frac{\pi}{6} + \log(2\sqrt{2})$

(d) $\frac{\pi}{3} + \log(2\sqrt{2})$

Solution 16

Correct option:(a)

$$I = \int_0^3 \frac{3x+1}{x^2+9}dx$$

Consider,

$$3x + 1 = A\frac{d\left(x^2+9\right)}{dx} + B$$

$$3x + 1 = 2Ax + B$$

Comparing both sides,

$$2A = 3 \Rightarrow A = \frac{3}{2}$$

$$B = 1$$

$$I = \int_0^3 \frac{2 \times \frac{3}{2}x + 1}{x^2+9}dx$$

$$I = \frac{3}{2}\int_0^3 \frac{2x}{x^2+9}dx + \int_0^3 \frac{1}{x^2+9}dx$$

$$I = \left[\frac{3}{2}\log\left|x^2+9\right| + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$I = \frac{3}{2}\left(\log 18 - \log 9\right) + \frac{1}{3}\left(\tan^{-1}(1)\right)$$

$$I = \frac{3}{2}\left(\log 9 + \log 2 - \log 9\right) + \frac{1}{3}\frac{\pi}{4}$$

$$I = \log 2^{\frac{3}{2}} + \frac{\pi}{12}$$

$$I = \log 2\sqrt{2} + \frac{\pi}{12}$$

Question 17

The value of the integral $\int_0^\infty \frac{x}{(1+x)(1+x^2)}dx$ is

(a) $\frac{\pi}{2}$

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$

(d) $\frac{\pi}{3}$

Solution 17

Correct option:(b)

$$I=\int_0^{\infty}\frac{x}{(1+x)(1+x^2)}dx$$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\Rightarrow d\theta = \frac{1}{1+x^2}dx$$

x	0	∞
t	0	$\frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}}\frac{\tan\theta}{1+\tan\theta}d\theta$$

$$I = \int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\cos\theta+\sin\theta}d\theta \ldots\ldots\ldots\ldots(i)$$

$$I = \int_0^{\frac{\pi}{2}}\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)+\sin\left(\frac{\pi}{2}-\theta\right)}d\theta$$

$$I = \int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\cos\theta+\sin\theta}d\theta \ldots\ldots\ldots\ldots\ldots(ii)$$

adding (i) and (ii)

$$2I = \int_0^{\frac{\pi}{2}}d\theta$$

$$I = \frac{\pi}{4}$$

Question 18

$$\int_{-\pi/2}^{\pi/2}\sin\ |x|\ dx \text{ is equal to}$$

(a) 1
(b) 2
(c) -1
(d) -2

Solution 18

Correct option: (b)

$$I = \int_{-\pi/2}^{\pi/2} \sin |x| \, dx$$

$$I = \int_{-\pi/2}^{0} \sin(-x) dx + \int_{0}^{\pi/2} \sin x dx$$

$$I = -\int_{-\pi/2}^{0} \sin x dx + \int_{0}^{\pi/2} \sin x dx$$

$$I = -[-\cos x]_{-\pi/2}^{0} - [\cos x]_{0}^{\pi/2}$$

$$I = \cos 0 - \cos\left(\frac{-\pi}{2}\right) - \left(\cos\left(\frac{-\pi}{2}\right) - \cos 0\right)$$

$$I = 1 - 0 - (0 - 1)$$

$$I = 2$$

Question 19

$$\int_{0}^{\pi/2} \frac{1}{1 + \tan x} dx \text{ is equal to}$$

(a) $\frac{\pi}{4}$

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$

(d) π

Solution 19

Correct option: (a)

$$I = \int_{0}^{\pi/2} \frac{1}{1 + \tan x} dx$$

$$I = \int_{0}^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx ....(i)$$

$$I = \int_{0}^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)} dx$$

$$I = \int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx ....(ii)$$

Adding (i) and (ii)

$$2I = \int_{0}^{\pi/2} dx$$

$$I = \frac{\pi}{4}$$

Question 20

The value of $\int_0^{\pi/2} \cos x\, e^{\sin x}\, dx$ is

(a) 1
(b) e-1
(c) 0
(d) -1

Solution 20
Correct option: (b)

$$I = \int_0^{\pi/2} \cos x\, e^{\sin x}\, dx$$

Put $\sin x = t \Rightarrow \cos x dx = dt$

x	0	$\frac{\pi}{2}$
t	0	1

$$I = \int_0^1 e^t dt$$

$$I = e - e^0$$

$$I = e - 1$$

Question 21

If $\int_0^{\alpha} \frac{1}{1+4x^2} dx = \frac{\pi}{8}$, then α equals

(a) $\frac{\pi}{2}$

(b) $\frac{1}{2}$

(c) $\frac{\pi}{4}$

(d) 1

Solution 21
Correct option:(b)

$$\int_0^{\alpha} \frac{1}{1+4x^2} dx = \frac{\pi}{8}$$

$$\frac{1}{2}\left[\tan^{-1}(2x)\right]_0^{\alpha} = \frac{\pi}{8}$$

$$\tan^{-1}(2\alpha) = \frac{2\pi}{8}$$

$$2\alpha = \tan\frac{\pi}{4}$$

$$\alpha = \frac{1}{2}$$

Question 22

If $\int_0^1 f(x)\,dx = 1,\ \int_0^1 x\,f(x)\,dx = \alpha, \int_0^1 x^2 f\,(x)\,dx = \alpha^2$, then $\int_0^1 (\alpha - x)^2 f(x)\,dx$ equals

(a) $4a^2$
(b) 0
(c) $2a^2$
(d) None of these

Solution 22

Correct option: (b)

$$I = \int_0^1 (\alpha - x)^2 f(x)\,dx$$

$$I = \int_0^1 (\alpha^2 - 2\alpha x + x^2) f(x)\,dx$$

$$I = \int_0^1 (\alpha^2 f(x) - 2\alpha x f(x) + x^2 f(x))dx$$

$$I = \alpha^2 \int_0^1 f(x)dx - 2\alpha \int_0^1 x f(x)dx + \int_0^1 x^2 f(x)dx$$

$$I = \alpha^2 \times 1 - 2\alpha \times \alpha + \alpha^2$$

$$I = \alpha^2 - 2\alpha^2 + \alpha^2$$

$$I = 0$$

Question 23

The value of $\int_{-\pi}^{\pi} \sin^3 x \cos^2 x\ dx$ is

(a) $\dfrac{\pi^4}{2}$

(b) $\dfrac{\pi^4}{4}$

(c) 0

(d) none of these

Solution 23

Correct option: (c)

$$I = \int_{-\pi}^{\pi} \sin^3 x \cos^2 x \, dx$$

Here,

$f(x) = \sin^3 x \cos^2 x$

$\Rightarrow f(-x) = -\sin^3 x \cos^2 x$

$\Rightarrow f(-x) = -f(x)$

Function is odd.

$$\Rightarrow \int_{-\pi}^{\pi} \sin^3 x \cos^2 x \, dx = 0$$

Question 24

$\int_{\pi/6}^{\pi/3} \frac{1}{\sin 2x} dx$ is equal to

(a) $\log_e 3$

(b) $\log_e \sqrt{3}$

(c) $\frac{1}{2}\log(-1)$

(d) $\log(-1)$

Solution 24

Correct option: (b)

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{\sin 2x} dx$$

$$I = \int_{\pi/6}^{\pi/3} \operatorname{cosec} 2x \, dx$$

$$I = \frac{1}{2}\left[\log_e |\tan x|\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$I = \frac{1}{2}\left(\log_e \sqrt{3} - \log_e\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$I = \frac{1}{2} \times 2\log_e \sqrt{3}$$

$$I = \log\sqrt{3}$$

Question 25

$\int_{-1}^{1} |1 - x| dx$ is equal to

(a) -2

(b) 2

(c) 0

(d) 4

Solution 25

Correct option: (b)

$$I = \int_{-1}^{1} |1 - x| dx$$

$$I = \int_{-1}^{0} (1 - x) dx + \int_{0}^{1} (1 - x) dx$$

$$I = \left[x - \frac{x^2}{2}\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{1}$$

$$I = 1 + \frac{1}{2} + 1 - \frac{1}{2}$$

$$I = 2$$

Question 26

The derivative of $f(x) = \int_{x^2}^{x^3} \frac{1}{\log_e t} dt, (x > 0)$, is

(a) $\frac{1}{3 \ 1n \ x}$

(b) $\frac{1}{3 \ln x} - \frac{1}{2 \ln x}$ (c)

(c) $(\ln x)^{-1} \ x(x - 1)$

(d) $\frac{3x^2}{\ln x}$

Solution 26

Correct option:(c)

$$f(x) = \int_{x^2}^{x^3} \frac{1}{\log_e t} dt$$

$$f(x) = \left[\log(\log_e t)\right]_{x^2}^{x^3}$$

$$f(x) = \log(\log_e x^3) - \log(\log_e x^2)$$

$$\Rightarrow f'(x) = \frac{1}{\log_e x^3} \times 3x^2 - \frac{1}{\log_e x^2} \times 2x$$

$$\Rightarrow f'(x) = \frac{1}{3\log_e x} \times 3x^2 - \frac{1}{2\log_e x} \times 2x$$

$$\Rightarrow f'(x) = \frac{1}{\log_e x} \times (x - 1) = (\log x)^{-1} \times (x - 1)$$

Question 27

If $I_{10} = \int_0^{\pi/2} x^{10} \sin x \, dx$, then the value of $I_{10} + 90I_8$ is

(a) $9\left(\frac{\pi}{2}\right)^9$

(b) $10\left(\frac{\pi}{2}\right)^9$

(c) $\left(\frac{\pi}{2}\right)^9$

(d) $9\left(\frac{\pi}{2}\right)^8$

Solution 27

Correct option: (b)

$$I_{10} = \int_0^{\pi/2} x^{10} \sin x \, dx$$

$$I_{10} = x^{10}\int_0^{\pi/2} \sin x dx - \int_0^{\pi/2}\left(\frac{dx^{10}}{dx}\int \sin x dx\right)dx$$

$$I_{10} = \left(-x^{10}\cos x\right)_0^{\pi/2} - \int_0^{\pi/2}\left(10x^9\left(-\cos x\right)dx\right)$$

$$I_{10} = -[0] + 10\int_0^{\pi/2} x^9 \cos x dx$$

$$I_{10} = \left[10x^9 \sin x\right]_0^{\pi/2} - 10\int_0^{\pi/2}\left(\frac{dx^9}{dx}\int \cos x dx\right)dx$$

$$I_{10} = 10\left(\frac{\pi}{2}\right)^9 - 10\int_0^{\pi/2}\left(9x^8 \sin x\right)dx$$

$$I_{10} = 10\left(\frac{\pi}{2}\right)^9 - 90I_8$$

$$I_{10} + 90I_8 = 10\left(\frac{\pi}{2}\right)^9$$

Question 28

$$\int_0^1 \frac{x}{(1-x)^{54}} dx =$$

(a) $\frac{15}{14}$

(b) $\frac{3}{16}$

(c) $-\frac{3}{16}$

(d) $-\frac{16}{3}$

Solution 28

Correct option: (d)

$$I = \int_0^1 \frac{x}{(1-x)^{\frac{5}{4}}}\,dx$$

Put, $1 - x = t \Rightarrow x = 1 - t$

$\Rightarrow dx = -dt$

x	0	1
t	1	0

$$I = \int_1^0 \frac{(1-t)(-dt)}{t^{\frac{5}{4}}}$$

$$I = \int_0^1 \frac{1-t}{t^{\frac{5}{4}}}dt$$

$$I = \int_0^1 \left(t^{-\frac{5}{4}} - t^{\frac{-1}{4}}\right)dt$$

$$I = \left[\frac{t^{\frac{-1}{4}}}{\frac{-1}{4}} - \frac{t^{\frac{3}{4}}}{\frac{3}{4}}\right]_0^1$$

$$I = -4 - \frac{4}{3}$$

$$I = \frac{-16}{3}$$

Note: Question is modified.

Question 29

$\lim\limits_{n\to\infty}\left\{\frac{1}{2n+1} + \frac{1}{2n+2} + \ldots + \frac{1}{2n+n}\right\}$ is equal to

(a) In $\left(\frac{1}{3}\right)$

(b) In $\left(\frac{2}{3}\right)$

(c) In $\left(\frac{3}{2}\right)$

(d) In $\left(\frac{4}{3}\right)$

Solution 29

Correct option: (c)

$$\lim_{n\to\infty}\left\{\frac{1}{2n+1}+\frac{1}{2n+2}+\dots+\frac{1}{2n+n}\right\}$$

$$\lim_{n\to\infty}\sum_{r=1}^{r=n}\frac{1}{2n+r}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{r=n}\frac{1}{2+\frac{r}{n}}$$

Let, $\frac{r}{n}=y$

$r=1, n\to\infty \Rightarrow y\to 0$

$r=n \Rightarrow y\to 1$

$$=\int_0^1\frac{1}{2+y}dy$$

$$=\left[\log(2+y)\right]_0^1$$

$$=\log 3-\log 2$$

$$=\log\left(\frac{3}{2}\right)$$

Question 30

The value of the integral $\int_{-2}^{2}\left|1-x^2\right|dx$ is

(a) 4
(b) 2
(c) -2
(d) 0

Solution 30
Correct option:(a)

$$I=\int_{-2}^{2}\left|1-x^2\right|dx$$

$$I=\int_{-2}^{-1}-\left(1-x^2\right)dx+\int_{-1}^{1}\left(1-x^2\right)dx+\int_{1}^{2}-\left(1-x^2\right)dx$$

$$I=\int_{-2}^{-1}\left(-1+x^2\right)dx+\int_{-1}^{1}\left(1-x^2\right)dx+\int_{1}^{2}\left(-1+x^2\right)dx$$

$$I=\left[-x+\frac{x^3}{3}\right]_{-2}^{-1}+\left[x-\frac{x^3}{3}\right]_{-1}^{1}+\left[-x+\frac{x^3}{3}\right]_{1}^{2}$$

$$I=\left(-1+2+\frac{1}{3}-\frac{8}{3}\right)+\left(1+1-\left(\frac{1}{3}+\frac{1}{3}\right)\right)+\left[-(2-1)+\frac{8}{3}-\frac{1}{3}\right]$$

$$I=4$$

Question 31

$\int_0^{\pi/2}\frac{1}{1+\cot^3 x}dx$ is equal to

(a) 0
(b) 1

(c) p/2
(d) p/4

Solution 31
Correct option:(d)

$$I = \int_0^{\pi/2} \frac{1}{1+\cot^3 x} dx \ldots\ldots\ldots\ldots(i)$$

$$I = \int_0^{\pi/2} \frac{1}{1+\cot^3\left(\frac{\pi}{2}-x\right)} dx$$

$$I = \int_0^{\pi/2} \frac{1}{1+\tan^3 x} dx \ldots\ldots\ldots\ldots(ii)$$

Adding (i) and (ii)

$$2I = \int_0^{\pi/2} \frac{1}{1+\cot^3 x} dx + \int_0^{\pi/2} \frac{1}{1+\tan^3 x} dx$$

$$2I = \int_0^{\pi/2} \frac{1+\tan^3 x+1+\cot^3 x}{\left(1+\cot^3 x\right)\left(1+\tan^3 x\right)} dx$$

$$2I = \int_0^{\pi/2} \frac{2+\tan^3 x+\cot^3 x}{\left(1+\cot^3 x\right)\left(1+\tan^3 x\right)} dx$$

$$2I = \int_0^{\pi/2} \frac{2+\tan^3 x+\cot^3 x}{1+\tan^3 x+\cot^3 x+\cot^3 x\tan^3 x} dx$$

$$2I = \int_0^{\pi/2} \frac{2+\tan^3 x+\cot^3 x}{1+\tan^3 x+\cot^3 x+1} dx$$

$$2I = \int_0^{\pi/2} dx$$

$$I = \frac{\pi}{4}$$

Question 32

$$\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x} dx \text{ is equal to}$$

(a) p
(b) p/2
(c) p/3
(d) p/4

Solution 32
Correct option: (d)

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx ....(i)$$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx ....(ii)$$

Adding (i) and (ii)

$$2I = \int_0^{\pi/2} dx$$

$$I = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$$

Question 33

$$\int_0^1 \frac{d}{dx}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\} dx \text{ is equal to}$$

(a) 0
(b) p
(c) p/2
(d) p/4

Solution 33

Correct option:(c)

$$I = \int_0^1 \frac{d}{dx}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\} dx$$

$$I = \left[\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right]_0^1$$

$$I = \sin^{-1} 1 - \sin^{-1} 0$$

$$I = \frac{\pi}{2}$$

Note: Answer not matching with back answer.

Question 34

$$\int_0^{\pi/2} x \sin x \, dx \text{ is equal to}$$

(a) p/4
(b) p/2
(c) p
(d) 1

Solution 34

Correct option:(d)

$$I = \int_0^{\pi/2} x \sin x \, dx$$

$$I = x\int_0^{\pi/2} \sin x dx - \int_0^{\pi/2}\left(\frac{dx}{dx}\int \sin x dx\right)dx$$

$$I = \left[-x\cos x\right]_0^{\pi/2} - \int_0^{\pi/2} -\cos x dx$$

$$I = \left[\sin x\right]_0^{\pi/2}$$

$$I = 1$$

Note: Answer not matching with back answer.

Question 35

$$\int_0^{\pi/2} \sin 2x \log \tan x \, dx \text{ is equal to}$$

(a) p
(b) p/2
(c) 0
(d) 2p

Solution 35

Correct option: (c)

$$I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx.....(i)$$

$$I = \int_0^{\pi/2} \sin\left[2\left(\frac{\pi}{2} - x\right)\right] \log \tan \left(\frac{\pi}{2} - x\right) dx$$

$$I = \int_0^{\pi/2} \sin 2x \log \cot x \, dx.............(ii)$$

$$2I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx + \int_0^{\pi/2} \sin 2x \log \cot x \, dx$$

$$2I = \int_0^{\pi/2} \sin 2x \left(\log \tan x + \log \cot x\right) dx$$

$$2I = \int_0^{\pi/2} \sin 2x \left(\log\left(\tan x \cot x\right)\right) dx$$

$$2I = \int_0^{\pi/2} \sin 2x \left(\log\left(1\right)\right) dx = 0$$

Question 36

$$\text{The value of } \int_0^{\pi} \frac{1}{5 + 3\cos x} dx \text{ is}$$

(a) p/4
(b) p/8
(c) p/2
(d) 0

Solution 36

Correct option: (a)

$$I = \int_0^{\pi} \frac{1}{5+3\cos x}\,dx$$

$$I = \int_0^{\pi} \frac{1}{5+3\left(\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}\,dx$$

$$I = \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{8+2\tan^2\frac{x}{2}}\,dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}\,dx = dt$

x	0	π
t	0	∞

$$I = \int_0^{\infty} \frac{dt}{4+t^2}$$

$$I = \frac{1}{2}\left[\tan^{-1}\left(\frac{t}{2}\right)\right]_0^{\infty}$$

$$I = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}$$

Question 37

$$\int_0^{\infty} \log\left(x+\frac{1}{x}\right)\frac{1}{1+x^2}\,dx =$$

(a) p In 2
(b) -p In 2
(c) 0
(d) $-\frac{\pi}{2}\text{In } 2$

Solution 37

Correct option:(d)

$$I=\int_0^{\infty}\log\left(x+\frac{1}{x}\right)\frac{1}{1+x^2}dx$$

Put $x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta$

x	0	∞
t	0	$\frac{\pi}{2}$

$$I=\int_0^{\frac{\pi}{2}}\log\left(\tan\theta+\frac{1}{\tan\theta}\right)\frac{1}{1+\tan^2\theta}\sec^2\theta d\theta$$

$$I=\int_0^{\frac{\pi}{2}}\log\left(\frac{\sec^2\theta}{\tan\theta}\right)\frac{1}{\sec^2\theta}\sec^2\theta d\theta$$

$$I=\int_0^{\frac{\pi}{2}}\log\left(\frac{1}{\sin\theta\cos\theta}\right)d\theta$$

$$I=-\int_0^{\frac{\pi}{2}}\log(\sin\theta\cos\theta)d\theta$$

$$I=-\int_0^{\frac{\pi}{2}}(\log\sin\theta+\log\cos\theta)d\theta$$

Let, $I_1=\int_0^{\frac{\pi}{2}}(\log\sin\theta)d\theta$

$$\Rightarrow I_1=\int_0^{\frac{\pi}{2}}\left(\log\sin\left(\frac{\pi}{2}-\theta\right)\right)d\theta$$

$$\Rightarrow I_1=\int_0^{\frac{\pi}{2}}(\log\cos\theta)d\theta$$

Adding I and I_1

$$2I=\int_0^{\frac{\pi}{2}}(\log\cos\theta+\log\sin\theta)d\theta$$

$$2I=\int_0^{\frac{\pi}{2}}\left[\log(\sin\theta\cos\theta)\right]d\theta$$

$$2I=\int_0^{\frac{\pi}{2}}\left[\log(\sin2\theta)-\log2\right]d\theta$$

$$I=\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\log(\sin2\theta)\,d\theta-\frac{\pi}{2}\log2\right)$$

$2\theta=t\Rightarrow 2d\theta=dt$

$$I = \left(\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[\log(\sin t)\,dt - \frac{\pi}{2}\log 2\right]\right)$$

$$I = \left(\frac{I}{2} - \frac{\pi}{2}\log 2\right)$$

$$I = -\frac{\pi}{2}\log 2$$

Now,

$$-\int_0^{\frac{\pi}{2}}(\log\sin\theta + \log\cos\theta)d\theta$$

$$= -2\int_0^{\frac{\pi}{2}}(\log\sin\theta)d\theta$$

$$= -2I$$

$$= -2\left(\frac{-\pi}{2}\log 2\right) = \pi\log 2$$

NOTE: Answer is not matching with back answer.

Question 38

$\int_0^{2\alpha} f(x)\,dx$ is equal to

(a) $2\int_0^{\alpha} f(x)\,dx$

(b) 0

(c) $\int_0^{\alpha} f(x)\,dx + \int_0^{\alpha} f(2\alpha - x)dx$

(d) $\int_0^{\alpha} f(x)\,dx + \int_0^{2\alpha} f(2\alpha - x)dx$

Solution 38

Correct option: (c)

$$\int_0^{2\alpha} f(x)\,dx = \int_0^{\alpha} f(x)\,dx + \int_0^{\alpha} f(2\alpha - x)dx$$

Property of definite integrals.

Question 39

If $f(a+b-x) = f(x)$, then $\int_a^b x\, f(x)\, dx$ is equal to

(a) $\frac{a+b}{2}\int_a^b f(b-x)\, dx$

(b) $\frac{a+b}{2}\int_a^b f(b+x)\, dx$

(c) $\frac{b-a}{2}\int_a^b f(x)\, dx$

(d) $\frac{a+b}{2}\int_a^b f(x)\, dx$

Solution 39

Correct option: (d)

$$I = \int_a^b x\, f(x)\, dx \ldots\ldots(i)$$

$$I = \int_a^b (a+b-x)\, f(a+b-x)\, dx \ldots\ldots(ii)$$

Adding (i) and (ii)

$$2I = \int_a^b x\, f(x)\, dx + \int_a^b (a+b-x)\, f(a+b-x)\, dx$$

$$2I = \int_a^b (a+b) f(x)\, dx$$

$$I = \frac{a+b}{2}\int_a^b f(x)\, dx$$

Question 40

The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$, is

(a) 1
(b) 0
(c) -1
(d) p/4

Solution 40

Correct option: (b)

$$I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx \ldots\ldots\ldots\ldots(i)$$

$$I = \int_0^1 \tan^{-1}\left(\frac{2(1-x)-1}{1+1-x-(1-x)^2}\right) dx$$

$$I = \int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right) dx \ldots\ldots\ldots(ii)$$

Adding (i) and (ii)

$$2I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx + \int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right) dx$$

$$2I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx - \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$$

$$I = 0$$

Question 41

The value of $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ is

(a) 2

(b) $\frac{3}{4}$

(c) 0

(d) -2

Solution 41

Correct option: (c)

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx \ldots\ldots\ldots(i)$$

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right) dx$$

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx \ldots\ldots\ldots\ldots(ii)$$

Adding (i) and (ii)

$$2I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx + \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx$$

$$2I = \int_0^{\pi/2} \log(1)\, dx$$

$$I = 0$$

Question 42

The value of $\int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1)\, dx$, is

(a) 0

(b) 2
(c) p
(d) 1

Solution 42
Correct option: (c)

$$I = \int_{-\pi/2}^{\pi/2} (x^3 + x \cos x + \tan^5 x + 1)\, dx$$

$$I = \left(\frac{x^4}{4}\right)_{-\pi/2}^{\pi/2} + (x \sin x - \sin x)_{-\pi/2}^{\pi/2} + (x)_{-\pi/2}^{\pi/2}$$

$$+ \int_{-\pi/2}^{\pi/2} \tan^3 x (\sec^2 x - 1)\, dx$$

$$I = \pi$$