Market Microstructure Modeling
In High-Frequency Cryptocurrency Markets

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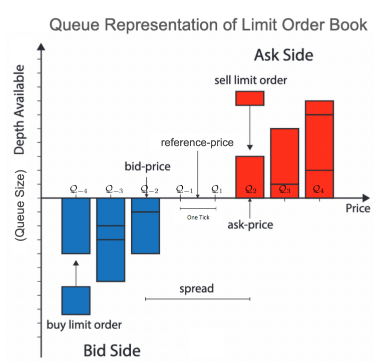
1 Introduction

A cryptocurrency exchange is an marketplace where buyers and sellers trade cryptocurrencies, eg. Bitcoin, Ethereum and derivative contracts. Trades occur only when both parties agree on a common price. With multiple buyers and sellers placing Limit Orders (LO), Cancellations (CO) and Market Orders (MO) at varying prices and quantities, the exchange employs a mechanism called Limit Order Book (LOB, denoted by $\mathcal{L}(t)$ at time t, shown in table 1) to match them and execute trades, usually using price-time priority. For each asset's $\mathcal{L}(t)$, the exchange imposes a tick-size ($\theta = 0.01$ USDT for table 1) and a lot-size ($\sigma = 10^{-5}$ BTC for table 1) parameter. To fully characterize $\mathcal{L}(t)$, we also need to specify a reference price (p_{ref}) between the best bid (denoted by $p_{\text{best bid}}$, bid 1 in table 1) and best ask (denoted by $p_{\text{best ask}}$, ask 1 in table 1). p_{ref} divides $\mathcal{L}(t)$ into the bid side $\mathcal{B}(t)$ and the ask side $\mathcal{A}(t)$. *Market microstructure* refers to the study of the process and outcomes of exchanging assets under explicit trading rules which involves aspects like liquidity and price discovery dynamics and is essential for designing effective trading strategies.

2 Description of the Problem

With tick-size discretization of price levels, $\mathcal{L}(t)$ is naturally interpreted as a set of $2K$ queues, where K is the number of queues we consider on both sides $\mathcal{B}(t)$ and $\mathcal{A}(t)$. $[Q_{-i} : i = 1, \ldots, K]$ denotes the queues belonging to $\mathcal{B}(t)$ and $[Q_i : i = 1, \ldots, K]$ denotes the same for $\mathcal{A}(t)$, where $Q_{\pm i}$ represents the limit at the distance $i - 0.5$ ticks to the right $(+i)$ or to the left $(-i)$ of p_{ref}. The number of outstanding LOs at Q_i is denoted by q_i. Let the arrival point process of LOs, MOs and COs at Q_i be denoted as $N_i^L(t), N_i^M(t)$ and $N_i^C(t)$ resp. We consider a constant order size at each queue. However, the order sizes at the different queues are allowed to be different. In practice, these sizes can be chosen as the average trade size (ATS_i) observed at each queue Q_i.



The $2K$-dimensional process $X(t) = (q_{-K}(t), \dots, q_{-1}(t), q_1(t), \dots, q_K(t))$ is then modeled as a continuous-time jump process in the countable state space $\Omega = \mathbb{N}^{2K}$, with jump size equal to one. For $q = (q_{-K}, \dots, q_{-1}, q_1, \dots, q_K) \in \Omega$, and e_i denoting the i^{th} canonical basis of Ω space, the components $\mathcal{Q}_{q,p}(t)$ of the infinitesimal generator matrix $\mathcal{Q}(t)$ of the process $X(t)$ are assumed to be of the following form,

$$Q_{q,p}(t) = \begin{cases} f_i(t,q) & p = q + e_i \\ g_i(t,q) & p = q - e_i \\ -\sum_{q \in \Omega, p \neq q} \mathcal{Q}_{q,p}(t) & p = q \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Now, the goal is the capture the dynamics of $\mathcal{L}(t)$ through $X(t)$ by assuming different parametric forms of $f_i(t,q)$ and $g_i(t,q)$ and estimating the parameters from data. For example,

- Time-Homogeneous State-Independent Poisson Process

  $$f_i(t,q) = \lambda_i^L, \;\; g_i(t,q) = \lambda_i^M + \lambda_i^C \tag{2}$$

  where $N_i^L \sim PP(\lambda_i^L), N_i^M \sim PP(\lambda_i^M)$ and $N_i^C \sim PP(\lambda_i^C)$. Here, we assume independence in the order flows at different Q_is, and at any given Q_i, conditional on the state of $\mathcal{L}(t)$, the arrival point processes of the 3 types of orders are taken to be independent.

- Time-Homogeneous State-Dependent Poisson Process

  $$f_i(t,q) = \lambda_i^L(q), \;\; g_i(t,q) = \lambda_i^M(q) + \lambda_i^C(q) \tag{3}$$

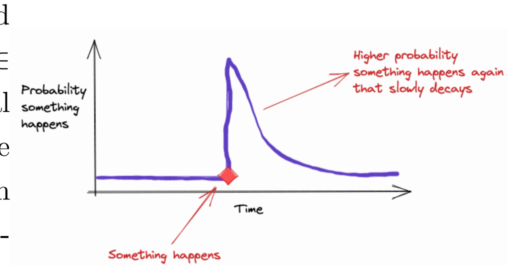
  with the same assumptions as above but here the intensity values depend on the queue sizes at each Q_i. Allowing us to construct signals from the state of $\mathcal{L}(t)$ that can affect the orderflow rates eg. order imbalance and bid-ask spread.

- Hawkes Process

  $$f_i(t,q) = \mu_i^1(q) + \sum_{T \in \{L,M,C\}} \sum_{\substack{j=-K \\ j \neq 0}}^{K} \int_{-\infty}^{t} \varphi_{ij}^{1,T}(t-u) dN_j^T(u) \tag{4}$$

  $$g_i(t,q) = \mu_i^2(q) + \sum_{T \in \{L,M,C\}} \sum_{\substack{j=-K \\ j \neq 0}}^{K} \int_{-\infty}^{t} \varphi_{ij}^{2,T}(t-u) dN_j^T(u) \tag{5}$$

where μ_i^1, μ_i^2 are exogenous or base intensities for an upward and downward jump at Q_i and $\varphi_{ij}^{l,T}$ $\forall T \in \{L, M, C\}, l \in \{1, 2\}$ and $i, j \in \{-K, \ldots, -1, 1, \ldots, K\}$ is a positive causal kernel (i.e. $Supp(\varphi) \subset \mathbb{R}^+$) which increases or decreases the intensity of an upward or downward jump at Q_i based on arrival of LOs, MOs and COs at any queue Q_j, thereby capturing mutually and self-exciting behavior between queues.



A simple and natural choice for all the kernels above is a right-sided exponential function,

$$\varphi_{ij}^{l,T}(t) = \alpha_{ij}^{l,T} e^{-\beta_{ij}^{l,T} t} 1_{\mathbb{R}^+}(t) \tag{6}$$

for all $T \in \{L, M, C\}, l \in \{1, 2\}, i, j \in \{-K, \ldots, -1, 1, \ldots, K\}$. Another reason for the choice of this kernel is that the resulting process is Markovian and shows a good level of tractability.

3 Literature Review

There has been several models suggested in the market microstructure literature for modeling the queue dynamics of $\mathcal{L}(t)$ using point process models. We will briefly review some of them here.

3.1 Poisson Process and Variants

Poisson Process modeling assumes that the order arrivals in each Q_i and of each type are *independent* of each other. Usually, the Poisson Process is classified as a 'zero-intelligence model' i.e. a model which uses no prior information about the financial market. However, there are several ways of adding priors to make the model more representative of empirical observations.

Zero-Intelligence Models: In [CST10], the authors develop a model for $\mathcal{L}(t)$ by assuming Poisson arrival processes for LOs, MOs and COs. It assumes the arrival rates for the LOs depend inversely on the distance from p_{ref} and that of COs depend on the number of outstanding shares at a Q_i and MOs arrival rate is considered constant. Specifically, the authors chose the following intensity values for equation 2,

$$\lambda_i^L = \frac{k}{|i|^\alpha}, \lambda_i^M = \mu, \lambda_i^C = \theta(|i|) \cdot q_i \tag{7}$$

The expression for λ_i^C can be understood as if there are q_i outstanding orders each of which can get canceled at a rate of $\theta(i)$ then the overall cancellation rate is $\theta(i) \cdot q_i$. The authors show, using Laplace Transform, the probabilities for quantities of interest like direction of price move, making the spread, order execution probabilities and filling time conditional on the current state of $\mathcal{L}(t)$ can be calculated analytically. $X(t)$, under the above framwork, is shown to be an ergodic Markov process. Further in [CdL13], the authors show that with a Poisson arrival queuing system for $\mathcal{L}(t)$, the price dynamics of the asset can be thought of as a sum of IID random variables which under CLT, forms a diffusion process. Finally, [AJ13] shows a stronger result for the price dynamics converging to a Brownian Motion for the same setup for $\mathcal{L}(t)$.

Variable Order Intensity Poisson Models: Moving away from the zero-intelligence approach, one way of adding priors to the Poisson model is to have non-constant order intensities. [HLR14] studies a 'queue-reactive' model where the order arrival rates in each Q_i is allowed to vary and is conditional on the current queue-size q_i as stated in equation 3. It also states that under 2 conditions on $\mathcal{Q}$ in equation 1, the process $X(t)$ in ergodic. The conditions are *negative individual drift* ($\exists\ C_{bound} \in \mathbb{Z}^+$ and $\delta > 0$ s.t. $\forall\ i, q_i \in \Omega$, if $q_i > C_{bound}$, then $f_i(q) - g_i(q) < -\delta$, which intuitively means queue size of a limit q_i tend to decrease when it becomes too large) and *bounded incoming flow* ($\exists\ H > 0$ s.t. $\forall\ q \in \Omega,\ \sum_{i \in [-K,\ldots,-1,1,\ldots,K]} f_i(q) \leq H$, which ensures there is no explosion in the system i.e. order arrival rate always stays bounded). [LA18] extends the above model and proposes a non-Markovian order flow dynamics. It still uses Poisson order arrivals, but by considering the order flow intensities to be dependent on not only the current state but also the previous order flow history. They further proposed that in case of a queue depletion, the new LOs not only depends on the side of the cleared queue but also on the past removal events.
Limitation: The core issue with Poisson Process models is the assumption that all orders are independent. This fails to explain the duration between orders being auto-correlated leading to volatility clustering effect. However, due to suitable behaviour under scaling limits and explainability they are quite popular.

3.2 Hawkes Process

The Hawkes Process captures the inter-dependence between the Q_is in $\mathcal{L}(t)$ which the Poisson process discussed in section 3.1 fails capture due to the *independence* assumption. Hawkes process leads to significant improvements in terms of accounting for the volatility clustering effect and the market microstructure noise. The order flow is endogenously excited in its modeling. A multi-dimensional Hawkes process can have cross excitations terms between different dimensions (eg. a 2D Hawkes Process of [ask vol., bid vol.] can have 4 excitation terms: ask→ask, bid→bid, ask→bid and bid→ask) captured by the kernel function 6 in the order arrival rates as in equation 4 and 5.
n-dimensional Hawkes Process: In [Tok11], the author creates a two-agent based model where MOs and LOs are each modelled as 1D Hawkes Processes. COs are modeled to be Poisson arrivals and the price for LOs and COs are sampled from a probability distribution. Upon comparison of the improvement with using Hawkes Processes against Poisson Processes, the former with 3 excitations: LOs and MOs self-excitation and MOs exciting future LOs gives better fit to the empirical data. [BJM14] divides the events in $\mathcal{L}(t)$ into 2 categories - those which change p_{ref} and those which do not. The authors used as 8-dimensional Hawkes process with orders changing the mid price being modelled by one dimension, and for events which don't are modeled by 3 dimensions (MOs, LOs and COs). This is done for both bid and ask thereby giving us a total of 8 dimensions. [Kir16] proposed an alternative, non-parametric way of estimating the Hawkes Process and showed its application in LOB data. [FZ14] suggested and alternate strategy to fit the Hawkes Process by using a generalized method of moments to fit the first 4 moments of various quantities of interest in the Hawkes Process which is claimed to be faster than the traditional MLE methods.

4 Data Collection and Processing Methodology

High-Frequency historical data for various cryptocurrencies from popular exchanges are not freely available and is associated with a high-cost to obtain such data from third-party vendors. Hence, we resort to listening to the public data feed and storing them in a locally hosted database. Below, we will look at the nature of the data collected on two cryptocurrency pairs BTCUSDT and ETHUSDT from Binance Exchange,

Depth Data: Contains the snapshots of the limit orderbook upto the best 10 limits (shown in table 1) on both the bid and the ask side in regular intervals of $\approx$ 100ms, with a timestamp (in millisecond resolution) for when it was obtained from the exchange's server. However, during periods of high volatility the data can be delayed and hence providing limit orderbook snapshots at a larger interval than 100ms. It provides information about LOs and COs, but is interval-censored.

Timestamp	**Bids - Price (Volume)**									
	1	2	3	4	5	6	7	8	9	10
22:14:39.514	95436.13 (2.32778)	95436.02 (0.00018)	95436.01 (0.00091)	95436.00 (0.01214)	95435.99 (0.00018)	95435.98 (0.00018)	95435.97 (0.00018)	95435.86 (0.00018)	95435.85 (0.00012)	95435.80 (0.00012)
22:14:39.628	95439.61 (2.22165)	95439.01 (0.00006)	95438.75 (0.17964)	95438.52 (0.90935)	95438.51 (0.53943)	95438.16 (0.00072)	95438.00 (0.01180)	95437.32 (0.34380)	95437.08 (0.18230)	95436.75 (0.00006)
22:14:39.717	95450.30 (1.43311)	95450.26 (0.09666)	95450.10 (1.30676)	95450.09 (0.05371)	95447.99 (0.18920)	95447.36 (0.09660)	95447.10 (0.35094)	95445.79 (0.43878)	95445.68 (0.16764)	95444.14 (0.90920)
22:14:39.814	95450.30 (4.24832)	95450.29 (0.00012)	95450.11 (0.49400)	95450.09 (0.00012)	95450.00 (0.01192)	95449.42 (0.00012)	95447.95 (0.00019)	95447.60 (0.00018)	95447.36 (0.00012)	95447.10 (0.00072)

Timestamp	**Asks - Price (Volume)**									
	1	2	3	4	5	6	7	8	9	10
22:14:39.514	95436.14 (0.07081)	95436.15 (0.00018)	95436.17 (0.00030)	95436.18 (0.00155)	95436.19 (0.00030)	95436.20 (0.00192)	95436.28 (0.00018)	95436.29 (0.00018)	95436.30 (0.00012)	95436.50 (0.00012)
22:14:39.628	95439.62 (0.00149)	95439.99 (0.00012)	95440.00 (0.00114)	95440.91 (0.00006)	95441.31 (0.00012)	95441.99 (0.00012)	95442.31 (0.00044)	95442.32 (0.13118)	95442.80 (0.00017)	95443.23 (0.00019)
22:14:39.717	95450.31 (0.00060)	95450.64 (0.00006)	95450.65 (0.13676)	95451.51 (0.10000)	95451.80 (0.00007)	95452.21 (0.08000)	95452.41 (0.04000)	95452.87 (0.10200)	95455.29 (0.00175)	95458.53 (0.09331)
22:14:39.814	95450.31 (0.36814)	95450.64 (0.00036)	95450.65 (0.00018)	95450.87 (0.00030)	95450.88 (0.00096)	95451.31 (0.00018)	95451.41 (0.00024)	95451.50 (0.00040)	95451.51 (0.00072)	95451.80 (0.00012)

Table 1: Few rows from the BTCUSDT depth data with limit orderbook snapshots on 09-02-2025.

Trades Data: Contains the real-time (in millisecond resolution) data of executed MOs (shown in table 2) which includes the trade information like price, quantity, indicator for if the market maker was buyer i.e. if the trade was executed by an incoming MO lifting an outstanding LO belonging to $\mathcal{B}(t)$ and a timestamp for when the trade was completed in the exchange's matching engine.

Timestamp	**Price**	**Quantity**	**MM Buy**
	95436.14	(0.01767)	0
	95436.14	(0.03116)	0
22:14:39.604	95436.14	(0.00590)	0
	95436.14	(0.00047)	0
	95436.14	(0.01453)	0
...	...	...	...
	95450.30	(0.00006)	0
22:14:39.709	95450.30	(0.00006)	0
	95450.30	(0.00006)	0
	95450.30	(0.00006)	0
...	...	...	...
22:14:39.723	95450.30	(0.01654)	1

Table 2: Trades data showing MOs that happened between the orderbook snapshots in table 1.

4.1 Intermediate LOB and Trades Reconstruction

Due to the interval nature of the depth data, multiple (sometimes over 10000s) new LOs and COs can arrive after a LOB snapshot which can only be partially or completely known after the next snapshot is observed. Since the trades data provide real-time MOs, we can reconstruct the

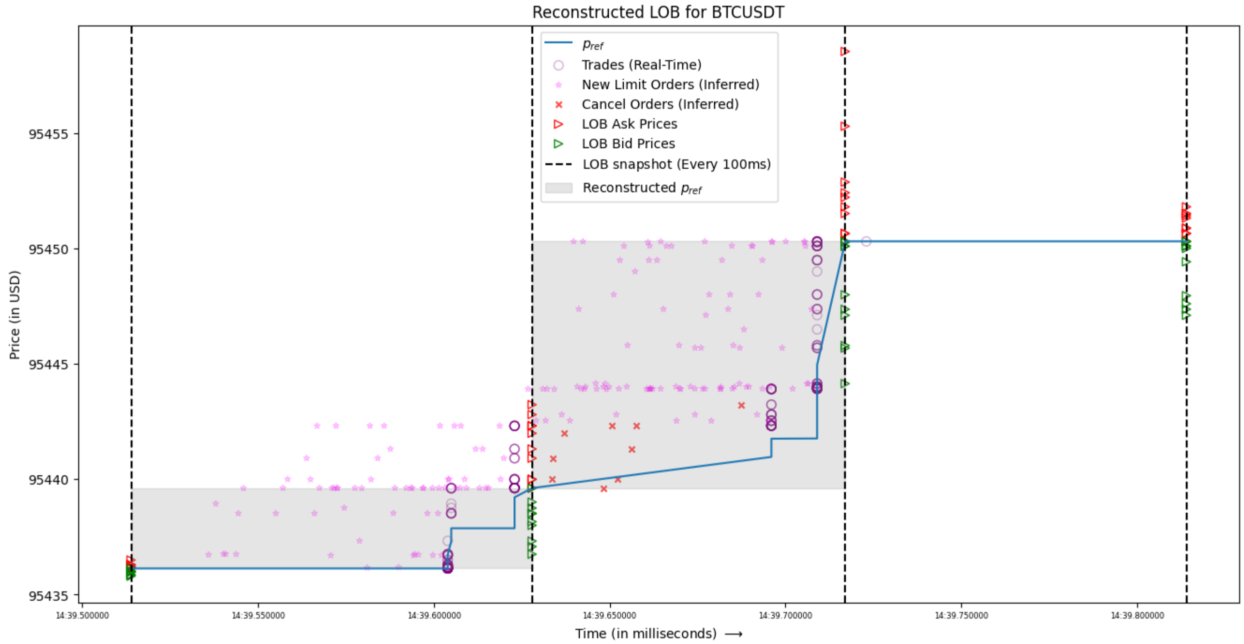


Figure 1: The reconstructed LOB illustrated for BTCUSDT on 09-02-2025 between snapshots in table 1 using trades in table 2. The new limit orders and cancel orders are inferred at the time of trades and the exact times are unknown. A small jitter is added to them to indicate that they occurred between the previous LOB snapshot and the observed trade.

intermediate LOB states to some extent giving us an enhanced higher frequency depth data as illustrated in figure 1.

4.2 Estimation of Reference Price

As mentioned in section 1 and 2, the estimation of a relevant reference price p_{ref} is the basis for defining the limits in the order book. Indeed, p_{ref} provides the center point of the LOB and thus the positions of the K limits on both the bid and ask sides. In our framework, if we write p_i for the price level of the limit $Q_i, i = -K, \ldots, -1, 1, \ldots, K$, we must have,

$$p_{ref} = \frac{p_1 + p_{-1}}{2} \tag{8}$$

When the observed bid-ask spread $(p_1 - p_{-1})$ is equal to one tick, p_{ref} is obviously taken as the midprice (denoted by p_{mid}) and both Q_1 and Q_{-1} are non-empty. When it is larger than one tick, several choices are possible for p_{ref}. We build p_{ref} from the data in the following way: when the spread is odd (in tick unit), it is

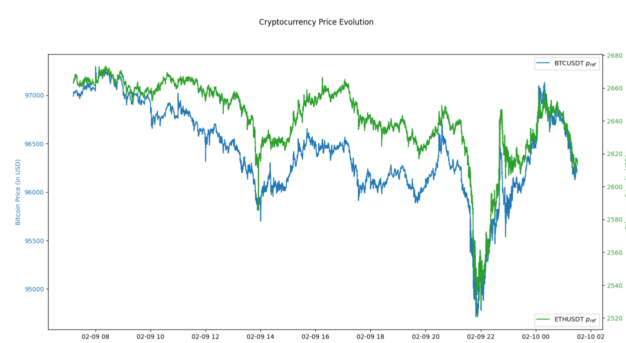


Figure 2: The reference price path for BTCUSDT and ETHUSDT from 09-02-2025 to 10-02-2025.

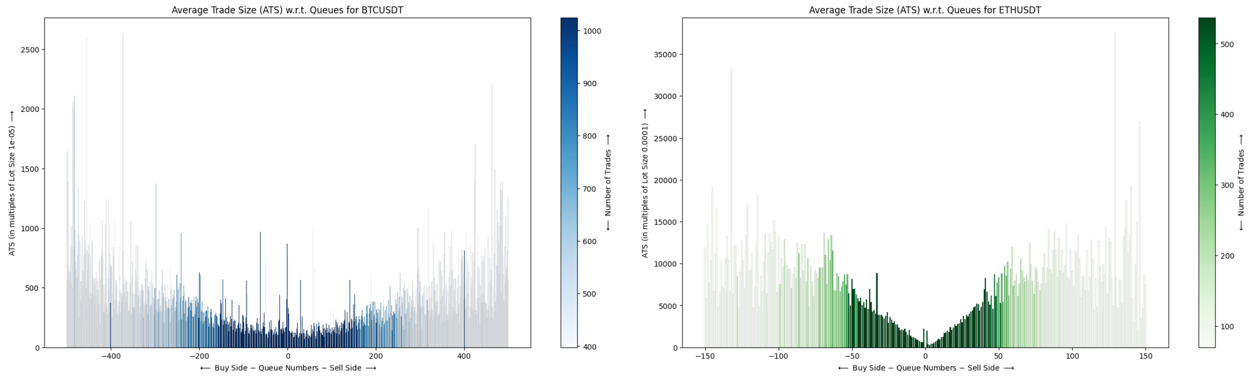


Figure 3: The Average Trade Size (ATS_i) for Q_i is computed from the trades data by taking the average of the order size q of the MOs received on Q_i. The ATS_i for the queues closer to the mid-price for both BTCUSDT ($|i| \leq 400$) and ETHUSDT ($|i| \leq 150$) are shown. ATS_i increases as $|i|$ increase but the number of trades or MOs (trading activity) decreases.

still natural to use p_{mid} as the LOB center,

$$p_{ref} = p_{mid} = \frac{p_{\text{best bid}} + p_{\text{best ask}}}{2}$$

When it is even, p_{mid} is no longer appropriate since it is now itself a possible position for order arrivals. In such case, we use either one of the following which is closest to the previous p_{ref},

$$p_{mid} + \frac{\text{tick size}}{2} \text{ or } p_{mid} - \frac{\text{tick size}}{2}$$

4.3 Data Preprocessing

The depth data and trades data as shown in table 1 and table 2 are not suitable to be used for modeling purposes in the raw format and hence we perform the following preprocessing steps,

- Convert all prices and volumes to integers after dividing by respective tick size and lot size of the cryptocurrency: bid price$_i$ $\rightarrow$ bid price$_i/\theta$, bid vol$_i$ $\rightarrow$ bid vol$_i/\sigma$ and similarly for asks.
- The best bid and best ask price is extracted from the depth data and is inferred from trades data if there is any change in the best bid or ask price. COs and new LOs inferred from trades are stored for later parts of the preprocessing.
- The reference price is estimated as described in section 4.2 for the combined depth and trades data. The reference price evolution path for both the cryptocurrencies is shown in figure 2.
- The prices in the depth data, trades data and reconstructed LOB data are converted to queue numbers based on the p_{ref} as described in section 2.
- The queue sizes for each of the queues in above data are approximated by taking the smallest integer greater that is larger than or equal to the volume available at the queue, divided by

the cryptocurrency's average trade size (ATS_i) at the corresponding queue. The ATS_i for each queue is shown in figure 3.

- We set the value of K to be largest queue number i for which both ATS_i and ATS_{-i} is available from the trades data.

	BTCUSDT	**ETHUSDT**
K	1624	183

The queue representation of the $\mathcal{L}(t)$ and the LO, MO and CO event times at each queue along with the number of LO, MO and CO events are computed and stored for the purpose of modeling. For example as seen in figure 4, let us say that we obtained the LOB snapshots $\mathcal{L}(t_{k-1}), \mathcal{L}(t_k)$ and $\mathcal{L}(t_{k+1})$. We observe a CO update on queue Q_1 at t_k which leads to a queue size depletion by $q_{k-1} - q_k$. Then, a LO update on Q_1 at t_{k+1} increases the queue size by $q_{k+1} - q_k$. Hence, we will store $(\Delta t_{k+1}, q_{k+1} - q_k)$ and $(\Delta t_{k-1}, q_k - q_{k-1})$ in the set of LO and CO events respectively for Q_i. We can define the LO, MO and CO event sets for Q_i as,

$$\mathcal{L}_i = \{(\Delta t_k^{i,L}, \Delta n_k^{i,L}\}_{k=1}^{K_i^L}, \;\; \mathcal{M}_i = \{(\Delta t_k^{i,M}, \Delta n_k^{i,M})\}_{k=1}^{K_i^M} \text{ and } \mathcal{C}_i = \{(\Delta t_k^{i,C}, \Delta n_k^{i,C})\}_{k=1}^{K_i^C} \qquad (9)$$

where $\Delta t_k^{i,L} = t_k^{i,L} - t_{k-1}^{i,L}$ is the time difference between two updates of N_i^L, $\Delta n_k^{i,L} = n_k^{i,L} - n_{k-1}^{i,L}$ is the number of LO events observed between $t_{k-1}^{i,L}$ and $t_k^{i,L}$ for N_i^L and K_i^L is the total number of LO updates on Q_i and similarly we have the other terms for MOs (N_i^M) and COs (N_i^C). An important fact to note here is that we only know the exact arrival times of MOs but not that of LOs and COs, which are *interval censored* providing us only the aggregated counts within a time interval.

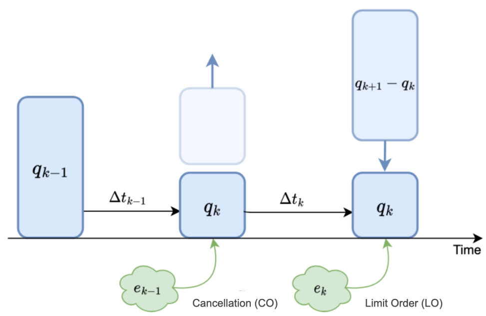


Figure 4: The diagram illustrates sequential updates in the size of Q_1 over time.

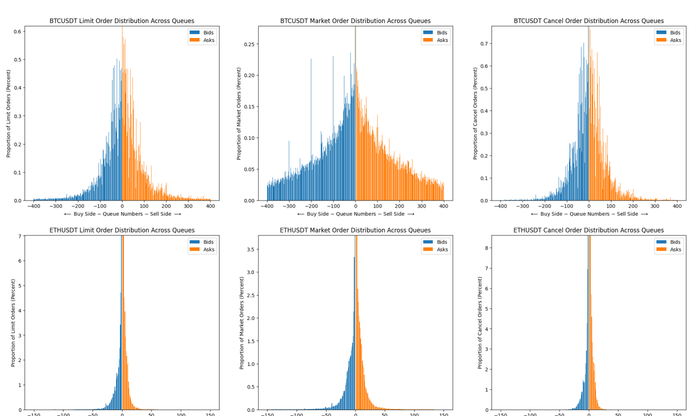


Figure 5: The proportion of all LOs, MOs and COs distributed across Q_is for BTCUSDT and ETHUSDT.

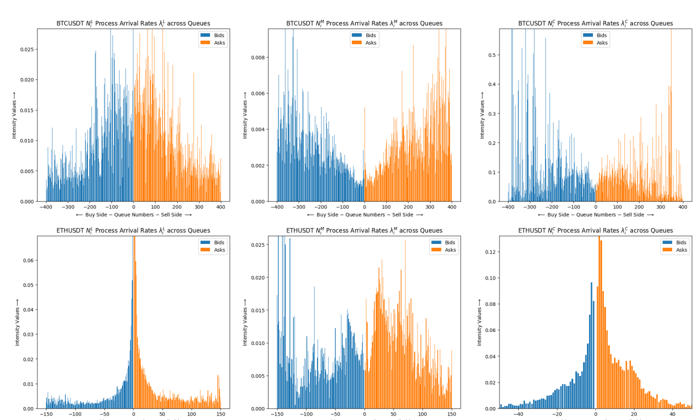


Figure 6: The estimated rate parameters of the LO, MO and CO Poisson Processes for each queue Q_i.

5 Limit Orderbook Modeling

In this section, we will explore a basic model for $\mathcal{L}(t)$ and price movement based on time-homogeneous state-independent Poisson Process and equation 2 with constant intensity values for each Q_i. In this model, we assume independence between the three type of order flows and also the order flows at different Q_is. Hence, we can write this model as,

$$N_i^L \sim PP(\lambda_i^L), N_i^M \sim PP(\lambda_i^M) \text{ and } N_i^C \sim PP(\lambda_i^C) \ \forall \ i = -K, \dots, -1, 1 \dots, K \tag{10}$$

5.1 Parameter Estimation

From section 4.3, we have obtained the order event times and counts for each Q_i from the processed data i.e. $\mathcal{L}_i, \mathcal{M}_i$ and $\mathcal{C}_i$. For the LO Poisson Process N_i^L, we know,

$$N_i^L(t_k^{i,L}) - N_i^L(t_{k-1}^{i,L}) \sim Pois(\lambda_i^L(t_k^{i,L} - t_{k-1}^{i,L})) \ \forall \ k = 1, \dots, K_i^L \tag{11}$$

and same for MO and CO Poisson Processes with their respective parameters. Using, the independent increment property and the independence assumption between queues and order flows we can compute the likelihood function as,

$$\begin{aligned} l &= \mathcal{L}(\{\lambda_i^L\}_{i=-K,i\neq 0}^K, \{\lambda_i^M\}_{i=-K,i\neq 0}^K, \{\lambda_i^C\}_{i=-K,i\neq 0}^K | \mathcal{L}, \mathcal{M}, \mathcal{C}) \\ &= \prod_{T\in\{L,M,C\}} \prod_{\substack{i=-K \\ i\neq 0}}^{K} \prod_{k=1}^{K_i^T} \frac{\exp(-\lambda_i^T \Delta t_k^{i,T}) \cdot (\lambda_i^T \Delta t_k^{i,T})^{\Delta n_k^{i,T}}}{\Delta n_k^{i,T}!} \\ &= H \cdot \prod_{T\in\{L,M,C\}} \prod_{\substack{i=-K \\ i\neq 0}}^{K} \exp(-\lambda_i^T \sum_{k=1}^{K_i^T} \Delta t_k^{i,T}) \cdot (\lambda_i^T)^{\sum_{k=1}^{K_i^T} \Delta n_k^{i,T}} \end{aligned} \tag{12}$$

where $\mathcal{L} = \bigcup_{\substack{i=-K \\ i\neq 0}}^{K} \mathcal{L}_i, \mathcal{M} = \bigcup_{\substack{i=-K \\ i\neq 0}}^{K} \mathcal{M}_i, \mathcal{C} = \bigcup_{\substack{i=-K \\ i\neq 0}}^{K} \mathcal{C}_i$ and H is a term independent of the parameters which we are interested in estimating. Now, we can compute the log-likelihood as,

$$\log(l) = \log(H) + \sum_{T\in\{L,M,C\}} \sum_{\substack{i=-K \\ i\neq 0}}^{K} \{-\lambda_i^T \sum_{k=1}^{K_i^T} \Delta t_k^{i,T} + (\sum_{k=1}^{K_i^T} \Delta n_k^{i,T}) \cdot \log \lambda_i^T\} \tag{13}$$

Setting the derivative of the log-likelihood to 0 w.r.t each parameter we get the MLE estimators,

$$\frac{\partial \log(l)}{\partial \lambda_i^T} = 0 \implies \hat{\lambda}_i^T = \frac{\sum_{k=1}^{K_i^T} \Delta n_k^{i,T}}{\sum_{k=1}^{K_i^T} \Delta t_k^{i,T}} \ \forall T \in \{L, M, C\}, i = -K, \dots, -1, 1, \dots, K \tag{14}$$

The estimated intensity values for each order and queue is shown in figure 6.

6 Future Plan of Study

Improve Data Quality: Building a high-frequency market data collection pipeline is inevitable for the study of market microstructure models. Currently, we have obtained a 100ms interval data for LOB snapshots and real-time market orders data. Although, LOB reconstruction helps recover some part of the intermediate states, it still mostly gives a partial view of the $\mathcal{L}(t)$, LOs and COs. Hence, we plan to improve the data collection strategy to obtain LOB snapshots at a higher frequency with the goal of creating a near real-time LOB updates database.

Improve Model: Cryptocurrency Markets as observed from the estimated intensity values behave differently from Stock Markets and hence to capture the complex dynamics of $\mathcal{L}(t)$, we will move towards better models involving non-homogeneous Poisson Processes and Hawkes Processes which we discussed in section 2 and 3. We plan to explore the impact of various orderbook based signals on these models. Order execution probabilities, price prediction and filling time are some of quantities of interest we would be exploring as a part of the model.

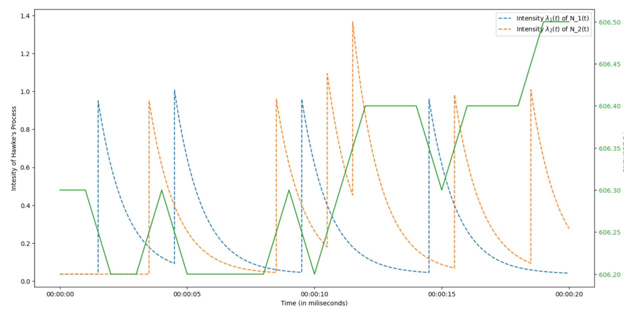


Figure 7: p_{ref} is modeled as $N_1(t) - N_2(t)$, where N_1 and N_2 are upward and downward jump process respectively using Hawkes intensity value corresponding to a mean-reverting behaviour i.e. an upward jump excites a downward jump and vice-versa.

Build Market Making Strategies: Market Makers (MM) are a class of market participants who engage in high-frequency trading via placing simultaneous bid and ask LOs in the LOB with the aim of making profit from the spread when both the LOs are executed. But when only one side of the order is executed, the MM takes a non-zero position in that asset called inventory size thereby exposing itself to the risk of making losses due to adverse price movement. Hence, by exploiting the trade off between the order execution speed and profit per trade, we will build market making strategies based on the LOB models.

Glossary

ask side The set of all outstanding sell limit orders in LOB. 1

best ask The best ask price is the lowest price at which there is a sell limit order in the LOB. 1

best bid The best bid price is the highest price at which there is a buy limit order in the LOB. 1

bid-ask spread Difference between the best ask and best bid, $p_{\text{best ask}} - p_{\text{best bid}}$, at time t. It often provides idea about liquidity in the market i.e. selling or buying an asset using MOs. 2

bid side The set of all outstanding buy limit orders in LOB. 1

cryptocurrency exchange Similar to a stock market (eg. Bombay Stock Exchange) but for cryptocurrencies which are highly secure digital currency. Some of the popular cryptocurrency exchanges are Binance, Deribit and Bybit. 1

filling time The expected time for a LO to be executed by an incoming MO. 3

inventory size The amount or size of assets which a trading strategy holds. It is always associated with a risk that the price of the asset might move adversely incurring losses. 10

Limit Order Book Collection of all outstanding limit orders in a market at a particular time. 1

Limit Orders A buy (resp. sell) limit order submitted at time t with price p and size ω is a commitment to buy (resp. sell) up to ω units of an asset at a price no more (resp. no less) than p. This does not execute immediately and stay *active* in the limit order book till it is matched or *canceled*. 1

liquidity Refers to the trade-off between the price at which an asset can be sold and how quickly it can be sold. High liquidity refers to large number of outstanding LOs near or at the best ask and best bid. 1

lot-size It is the unit of quantity or size of an order for a LOB. It is the smallest possible interval between consecutive order sizes. It is rare to see fractional lot-sizes in stock markets, but it is common in cryptocurrency markets. 1

making the spread Computing probability that 2 orders, one placed at the bid price and one at ask price, are both executed before p_{ref} moves, given that they are not canceled. If the probability of both the buy and sell LO execution before price moves is high, almost a risk-free profit can be generated from the bid-ask spread. 3

market microstructure noise It is a deviation from fundamental value that is induced by the characteristics of the market under consideration, eg. bid-ask bounce, the discreteness of price change, latency, and asymmetric information of traders. 4

Market Orders Orders that are immediately matched to a limit order resulting in a trade. 1

order execution probabilities The probability of a limit buy (resp. sell) order getting filled or executed when posting at a given price level δ tick-size away from p_{ref}, conditional on the arrival of an market order. Naturally, it is decreasing function of δ. It can be seen from table 1 that the limit orders at bid 10 (resp. ask 10) are far less likely to get executed than bid 1 (resp. ask 1) because of the way orders are matched. 3

order imbalance This is a measure of the buy versus sell pressure on an asset and it contains predictive power on both the arrival rates of market orders, and the direction and size of future price movements. It is measured as ratio of the quoted size imbalance to the total quoted size in LOB at time t $\rho(t) = \frac{\sum_{k=1}^{10} \text{ask vol}_k - \sum_{k=1}^{10} \text{bid vol}_k}{\sum_{k=1}^{10} \text{ask vol}_k + \sum_{k=1}^{10} \text{bid vol}_k}$, where ask vol_k and bid vol_k are the volumes at the k^{th} price level on the ask and bid side resp. 2

price discovery It is the process of determining the price of an asset in the marketplace through the interaction of buyers and sellers. 1

price-time priority It is an algorithm for matching newly arrived market orders with active limit orders stated as follows,

- For active buy orders, priority is given to the active orders with the highest price.
- For active sell orders, priority is given to the active orders with the lowest price.
- Ties are broken by selecting the active order with the earliest submission time.

Hence, faster an order reaches the exchange, better are its chances of execution. 1

tick-size It is the unit of price of an order for a LOB. It is the smallest possible interval between consecutive price levels. 1

trading strategies algorithms that observe LOB changes and trades to place MOs or LOs. 1

volatility clustering effect Refers to an observation about financial markets: large changes tend to be followed by large changes, and small changes tend to be followed by small changes. 4

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