Multi-Period Portfolio Optimization

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Introduction

At the heart of Asset, Wealth and Portfolio Management lies the classic Markowitz optimization strategy or simply called, **Single-Period Portfolio Optimization** which can be written in the following form,

$$\text{minimize } \frac{1}{2} w^T Var(\mathcal{R}) w$$

$$\ni \quad w^T E(\mathcal{R}) = E_p, w_i \geq 0, w^T \mathbf{1} = 1$$

where, w is the vector of weights that the portfolio gives to N assets which comprise our portfolio, $\mathcal{R} = (\mathcal{R}_i)_{i=1}^N$ are the random variable indicating the returns of each asset and E_p is the particular expected portfolio returns the Portfolio or Asset Manager desires. There is a trade-off between the expected volatility of the portfolio and the expected returns. A higher risk yields higher expected returns. The above form of the optimization problem tries to find the weights which minimize the variance for a given expected returns. An equivalent formulation would fix the portfolio volatility to a desired risk level and maximizing the returns. We can therefore also quantify the portfolio manager's risk seeking nature in a risk-aversion hyper-parameter γ with the following objective function,

$$\text{maximize } w^T E(\mathcal{R}) - \frac{\gamma}{2} w^T Var(\mathcal{R}) w$$

$$\ni \quad w_i \geq 0, w^T \mathbf{1} = 1$$

Observe that the above problem and the original problem is the same but with a different parameterization based on the risk-aversion hyper-parameter. The portfolio managers usually run this optimization daily or monthly or quarterly based on the current market scenario. But the above model is still far for reality as it does not takes into account several important factors like transaction cost, holding cost, turnover limits, leverage limits, etc. We can generalize the above Mean-Variance Portfolio Optimization Problem as shown below,

$$w_{t+1}^* = \text{argmax}_{w_{t+1},\ldots,w_{t+h}} E[\mathcal{U}(w_{t+1}, \ldots, w_{t+h}) | \mathcal{F}_t]$$

$$\ni \quad w_{t+1}, \ldots, w_{t+h} \in \Omega$$

where, Ω are the set of linear constraints on the portfolio weights, $\mathcal{U}$ is a utility function that

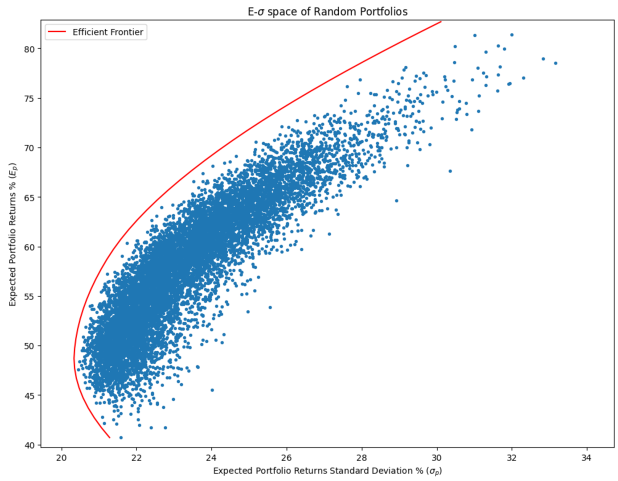

Figure 1: *Efficient Frontier of the Classic Markowitz Mean Variance Problem of a portfolio of* 8 *stocks from India NSE: TATAMOTORS, TATASTEEL, TATAPOWER, ONGC, CESC, EXIDEIND, SUZLON and NTPC based on last* 4 *year's returns.*

might take into consideration inter-temporal weight changes into consideration upto h timesteps ahead and $\mathcal{F}_t$ is the filtration until time t. Although the maximization is done over upto h time steps ahead the only weights we are interested in are just the next portfolio weights. This general problem formulation considers the various costs and constraints discussed earlier through Ω and $\mathcal{U}$. But it also gives us a particularly interesting flexibility that is incoporate a planning step upto h time steps ahead based on our belief about the future making the portfolio more closer to what a portfolio manager might desire. Some interesting implications of being able to formulate and optimize along with a future belief is that, it helps us plan the dynamics of the portfolio over a course of time like portfolio decarbonization, managing transaction costs, reacting to upcoming events, etc. This problem is called the **Multi-Period Portfolio Optimization**.

Methodology

In most Portfolio optimization problem, we are usually interested in utility function of the separable form below,

$$E[\mathcal{U}(w_{t+1}, \ldots, w_{t+h}) | \mathcal{F}_t] = \sum_{s=t+1}^{t+h} \{g_s(w_s) + h_s(w_{s-1}, w_s)\}$$

where g_s and h_s are convex functions. This also makes the optimization problem more tractable. In order to obtain a tractable objective function, we assume that the utility function is separable in time. While $g_s(x_s)$ only depends on the current portfolio $w_s, h_s(w_{s-1}, w_s)$ is a convex function that depends on both the current portfolio w_s and the previous portfolio w_{s-1}. Here w_s represents the weight vector of our portfolio at time s. Therefore, $g_s(w_s)$ is the static part of the objective function whereas the dynamic part is modeled by the coupling function $h_s(w_{s-1}, w_s)$. Similarly, we split the set of constraints as $\Omega = \Omega^{(g)} \cap \Omega^{(h)}$ where $\Omega^{(g)} = \bigcap_{s=t+1}^{t+h} \Omega_s$ and Ω_s corresponds to the constraints that only relies on w_s and not on the other variables $w_u (u \neq s)$. Therefore, the problem becomes,

$$\begin{aligned} w_{t+1}^{\star} = \quad & \arg\min_{w} \{g(w) + h(w)\} \\ & \text{s.t. } x \in \Omega^{(g)} \cap \Omega^{(h)} \end{aligned}$$

where, $w = (w_{t+1}, \ldots, w_{t+h})$ i.e. the collection of weight vectors from time $t+1$ to $t+h$.

We could discuss what the goal is when writing the objective function as $f(w) = g(w) + h(w)$. Indeed, the problem is equivalent to the traditional nonlinear optimization problem $w_{t+1}^{\star} = \arg\min f(w)$ s.t. $w \in \Omega$. In fact, the underlying idea is to separate the coupling and non-coupling parts. Therefore, we notice that the problem is the overlapping of two problems,

$$\begin{cases} w_{t+1}^{\star} = \arg\min g(w) \quad \text{s.t.} \quad x \in \Omega^{(g)} \\ w_{t+1}^{\star} = \arg\min h(w) \quad \text{s.t.} \quad x \in \Omega^{(h)} \end{cases}$$

The first problem is static and corresponds to a traditional single-period optimization problem since it is equivalent to,

$$w_{t+1}^{\star} = \arg\min g_{t+1}(x_{t+1}) \quad \text{s.t.} \quad w_{t+1} \in \Omega_{t+1}^{(g)}$$

The second problem is a dynamic feedback problem. Knowing the optimal solution at time $t+2$, it modifies the solution at time $t+1$ because of the feedback effects. In asset allocation, $h(w)$ is generally a penalty function and not really an objective function. In what follows, we extensively use the previous breakdown to find the numerical solution.

Common Objective Functions

In this section, we consider different objective functions that are used in portfolio management.

Single-Period Optimization Problem When h is equal to 1 , the problem reduces to,

$$\begin{aligned} & w_{t+1}^{\star} = \arg\min_{w} \{g_{t+1}(w_{t+1}) + h_{t+1}(w_t, w_{t+1})\} \\ & \text{s.t.} \quad w_{t+1} \in \Omega \end{aligned}$$

Mean-Variance Optimization In the mean-variance optimization problem, the objective function $g_s(w_s)$ is equal to ,

$$g_s(w_s) = \frac{1}{2} w_s^{\top} \Sigma_s w_s - \gamma w_s^{\top} \mu_s$$

where Σ_s is the covariance matrix and μ_s is the vector of expected returns. The parameter γ is a coefficient that controls the trade-off between the portfolio's volatility and its expected return. Let $\boldsymbol{\mathcal{R}}_s = (\boldsymbol{\mathcal{R}}_{1,s}, \ldots, \boldsymbol{\mathcal{R}}_{n,s})$ be the vector of asset returns at time s. Since we have,

$$\mathbb{E}[\boldsymbol{\mathcal{R}}_s \mid \mathcal{F}_t] = \mathbb{E}[\boldsymbol{\mathcal{R}}_t \mid \mathcal{F}_t] = \mu_t \quad \text{for } s \geq t$$

and,

$$\operatorname{var}(\boldsymbol{\mathcal{R}}_s \mid \mathcal{F}_t) = \operatorname{var}(\boldsymbol{\mathcal{R}}_t \mid \mathcal{F}_t) = \Sigma_t \quad \text{for } s \geq t$$

we obtain,

$$g(w) = \sum_{s=t+1}^{t+h} \left\{ \frac{1}{2} w_s^{\top} \Sigma_t w_s - \gamma w_s^{\top} \mu_t \right\}$$

Tracking-error minimization We recall that the tracking error variance of the portfolio w_s with respect to the benchmark b_s is equal to,

$$\sigma^2(w_s \mid b_s) = (w_s - b_s)^{\top} \Sigma_s (w_s - b_s)$$

Therefore, we can show that the objective function $g_s(w_s)$ is equal to,

$$g_s(w_s) = \frac{1}{2} w_s^{\top} \Sigma_s w_s - w_s^{\top} \Sigma_s b_s$$

Finally, we obtain,

$$g(w) = \sum_{s=t+1}^{t+h} \left\{ \frac{1}{2} w_s^{\top} \Sigma_t w_s - w_s^{\top} \Sigma_t b_s \right\}$$

If we assume that we do not know the future composition of the benchmark at time $s > t$, above equation becomes becomes,

$$g(w) = \sum_{s=t+1}^{t+h} \left\{ \frac{1}{2} w_s^{\top} \Sigma_t w_s - w_s^{\top} \Sigma_t b_t \right\}$$

Portfolio optimization with a benchmark We can mix the two approaches. In this case, the investor would like to maximize the expected excess return of the portfolio with respect to the benchmark and control the level of the tracking error volatility. The multiperiod objective function becomes,

$$g(w) = \sum_{s=t+1}^{t+h} \left\{ \frac{1}{2} w_s^\top \Sigma_t w_s - w_s^\top \left(\Sigma_t b_t + \gamma \mu_t \right) \right\}$$

We notice that mean-variance, tracking-error and benchmark optimization problems can be cast into a quadratic programming problem,

$$g_s\left(w_s\right) = \frac{1}{2} w_s^\top Q_s w_s - w_s^\top R_s$$

where $Q_s = \Sigma_s$ and R_s is respectively equal to $\gamma\mu_s, \Sigma_s b_s$ and $\Sigma_s b_s + \gamma\mu_s$. In what follows, we use this notation and the term 'mean-variance' to name these three problems.

Other objective functions Perrin and Roncalli (2020, Table 1, page 29) reviewed the different objective functions used in portfolio optimization. It includes minimum variance, most diversified, risk budgeting or Kullback-Leibler portfolios.

Penalty Functions

Four regularization penalties are mainly used in portfolio management: ridge, lasso, log-barrier and entropy.

Ridge Penalization In the case of the ridge penalty, we have,

$$h_s\left(w_{s-1}, w_s\right) = \frac{\lambda_s}{2} \left\| w_s - w_{s-1} \right\|_2^2$$

where λ_s is the scalar penalty value. Quadratic transaction costs can be used instead of ridge penalty,

$$h_s\left(w_{s-1}, w_s\right) = \frac{1}{2} \left(w_s - w_{s-1}\right)^\top \Lambda_s \left(w_s - w_{s-1}\right)$$

where Λ_s is the Kyle's matrix for temporary price impact. We notice that the penalization with quadratic transaction costs generalizes the ridge penalty where $\Lambda_s = \lambda_s I_n$.

Lasso Penalization Instead of using the ℓ_2-norm, we can use the ℓ_1-norm,

$$h_s\left(w_{s-1}, w_s\right) = \lambda_s \left\| w_s - w_{s-1} \right\|_1$$

This regularization can be viewed as a turnover penalization problem.

Portfolio constraints

Linear constraints If the constraints are linear, we have,

$$w \in \Omega \Leftrightarrow \begin{cases} Aw = B \\ Cw \leq D \\ \underline{w} \leq w \leq \bar{w} \end{cases}$$

It follows that $\Omega = \Omega^{(h)}$ and $\Omega^{(g)} = \left\{ w \in \mathbb{R}^{nh} \right\}$. In the case where constraints are separable, we obtain $\Omega = \Omega^{(g)}$ where:

$$w_s \in \Omega_s \Leftrightarrow \begin{cases} A_s w_s = B_s \\ C_s w_s \leq D_s \\ \underline{w}_s \leq w_s \leq \bar{w}_s \end{cases}$$

Turnover constraint The turnover constraint is defined as,

$$\Omega^{(h)} = \left\{ \forall s = t+1, \ldots, t+h : \left\| w_s - w_{s-1} \right\|_1 \leq \tau_s \right\}$$

where τ_s is the turnover limit at time s. In the single-period optimization problem, imposing a turnover constraint is equivalent to add a lasso penalization. Therefore, we have a relationship between τ_s and λ_s. In the multi-period optimization problem, we lose the strict equivalence.

Other constraints We can specify other constraints such as asset class limits, sector limits, number of active bets, etc.

Numerical Algorithms

For solving the Multi-Period Portfolio Optimization Problem under various choice of convex functions g and h, we can resort to one of the many convex optimization algorithms available which includes **Alternating Direction Method of Multipliers** which uses Augmented Langrange Equations and optimizes the separable functions individually each in an alternate manner, **Block Coordinate Decent** which updates one variable at a time out of the several inputs to the optimization problem and **Quadratic Programming** when our objective function includes quadratic form of the variance matrix, etc. Standard libraries are available for solving such convex optimization problems and I resort to **cvpxy** and **cvxportfolio** Python library for performing these optimizations.

Real-Life Data Application

Data

I collected daily OHLCV data from **yfinance** for the last 4 years of 8 stocks: TATAMOTORS, TATASTEEL, TATAPOWER, ONGC, CESC, EXIDEIND, SUZLON and NTPC listed on NSE India Exchange as shown in figure 2. The choice of these stocks are somewhat

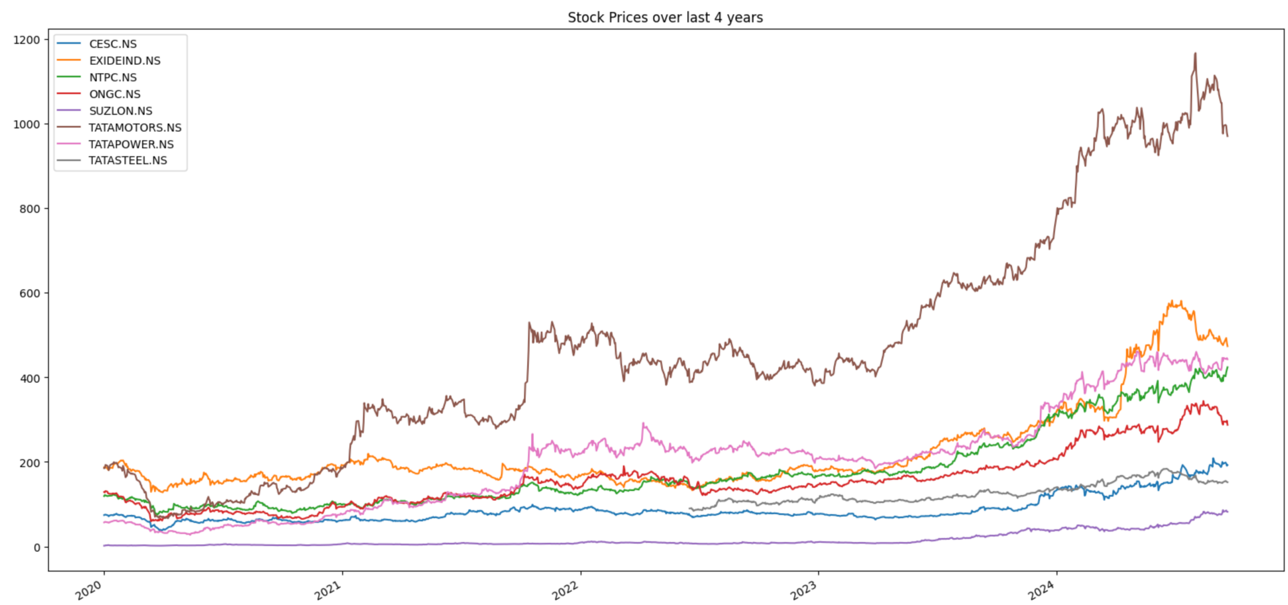

Figure 2: *The stock prices evolution across the last 4 years from 01-01-2020 to 19-09-2024.*

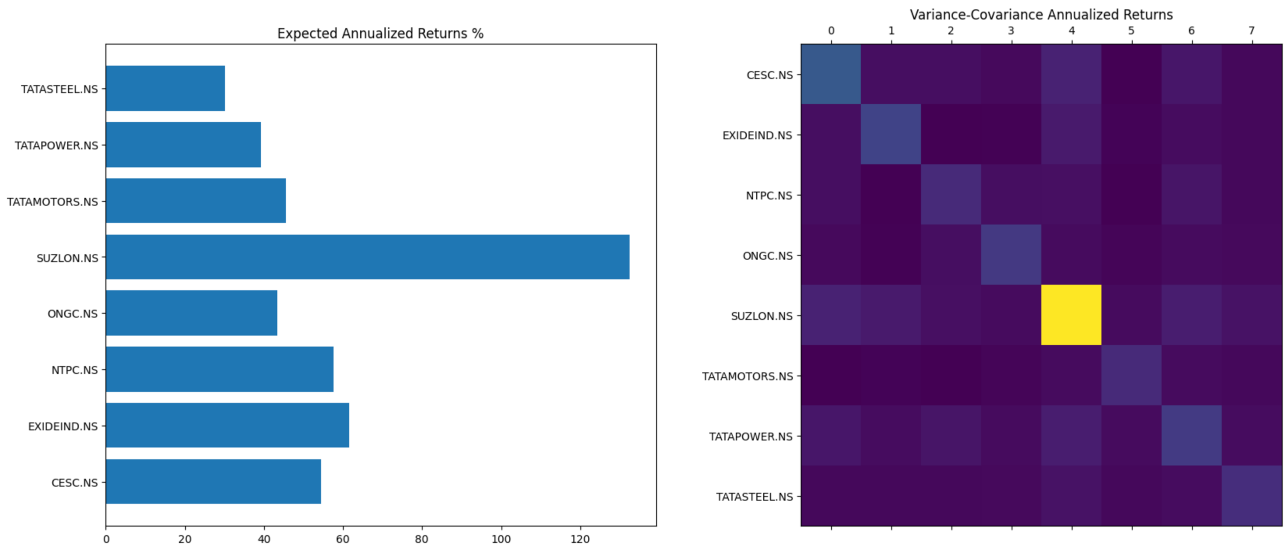

Figure 3: *The mean and variance-covariance matrix of the returns of the stock prices based on the past 4 year's Opening Price Data.*

arbitrary for the purpose of contructing a portfolio optimal for personal investment.

For performing the portfolio optimization, we transform the prices into returns,

$$R_{t+1} = \frac{P_{t+1}}{P_t} - 1$$

and compute the mean of the annualized (252 trading days per year) returns and the annualized variance-covariance matrix of the returns of all the stocks shown in 3.

Classic Mean-Variance Portfolio Optimization

Based on the original Single-Period Portfolio Optimization Problem by Markowitz, I solve for the efficient frontier for this portfolio to find the most efficient set of weights leading to an efficient portfolio shown in figure 1 along with portfolios of random weights in the $E - \sigma$ space which shows how the final optimized portfolio weights are optimal. Below, I have provided the least riskiest (least volatility) portfolio weights and expected returns,

Stock Name	Portfolio Weight
CESC	6.62%
EXIDEIND	14.86%
NTPC	23.47%
ONGC	13.03%
SUZLON	0%
TATAMOTORS	26.54%
TATAPOWER	0%
TATASTEEL	15.45%
Expected Returns	**Standard Deviation**
48.69%	20.33%

Single-Period Portfolio Optimization Strategy

To bring the portfolio weights closer to real-life, I adapt the portfolio weights to the generalized portfolio optimization problem of the form,

$$\text{maximize } w^T E(\mathcal{R}) - \frac{\gamma}{2} w^T Var(\mathcal{R}) w - t(w_t, w_{t+1})$$

$$\ni \quad w_i \geq 0, w^T \mathbf{1} = 1$$

where, w the weight vector also has a cash component (assumed to be the last element of w) and $t(w_t, w_{t+1})$ accounts for a fixed cost of 0.1 INR per share traded as well as some additional market impact term given by,

$$\sigma \frac{|w_{t+1} - w_t|^{3/2}}{V^{1/2}}$$

where, $|w_{t+1} - w_t|$ is the dollar trade amount, the number V is the total market volume traded for the asset in the time period, expressed in dollar value, so $|w_{t+1} - w_t|^{3/2}/V^{1/2}$ has units of dollars. The number σ the corresponding price volatility (standard deviation) over recent time periods, in dollars. According to a standard rule of thumb, trading one day's volume moves the price by about one day's volatility. This 3/2 power transaction cost model is widely known and employed by practitioners. All in all the above form represents the market impact term. Estimates $\hat{V}$ and $\hat{\sigma}$ are used from the past data while applying this price impact term to the object function. I set the risk aversion parameter $\gamma = 5$ and to backtest this strategy across the last 3 years where the starting date is 29-12-2020 i.e. the first mean and variance estimates for the returns are based on at least 1 year's data because we have data from 01-01-2020 to 19-09-2024. The backtesting results are shown in figure 4.

Multi-Period Portfolio Optimization Strategy

We solve the same optimization problem as in the Single-Period Portfolio Optimization but this time with 5 planning steps over which we also try to optimize the portfolio weights. This results in a superior set of weights which reduces the transaction costs significantly resulting in higher expected returns at comparable volatility. The backtesting results are shown in figure 5.

Strategy	Returns	Volatity	Sharpe Ratio
SPO	15.6%	11.4%	1.13
MPO	25.2%	15.6%	1.44

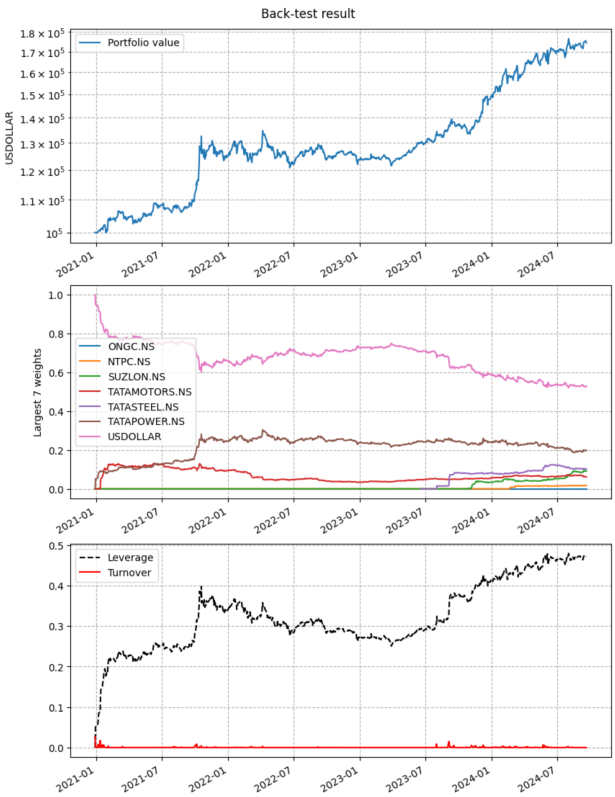

Figure 4: *Backtesting Result for Single-Period Portfolio Optimization over last 3 years.*

A comparison of the efficient frontier of the SPO strategy and MPO strategy for portfolio weights across various values of γ is shown in figure 6 shows that the MPO strategy outperforms the SPO at all risks. Although this behaviour is solely due to the fact that we added a transactional cost associated with number of shares traded. If this condition is removed (non-real life scenario) there is no need of planning a dynamics for the evolution of the portfolio weights and thus SPO and MPO would be the same.

In table 2, I have provided the weights allocated

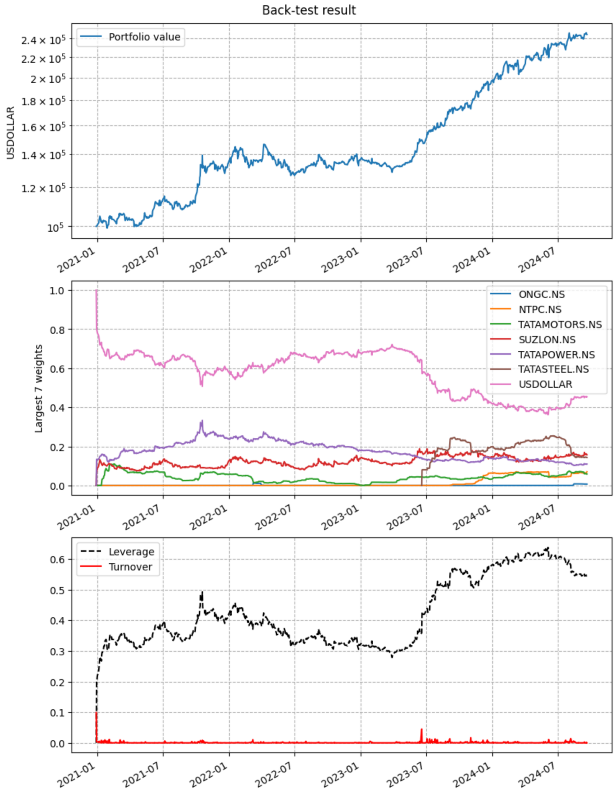

Figure 5: *Backtesting Result for Multi-Period Portfolio Optimization over last 3 years.*

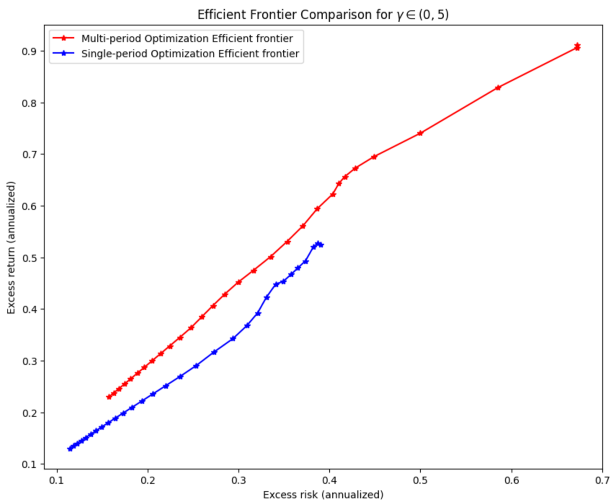

Figure 6: *Efficient Frontier of the SPO vs MPO strategy for obtaining Portfolio weights.*

by the MPO strategy on the 8 stocks which can be compared with that of the SPO strategy in table 1.

Conclusion

Multi-Period Portfolio Optimization opens an avenue for combining the best of two worlds i.e. predictive modelling and portfolio optimization together to create a perfect portfolio suiting the investor's need. Online Machine Learning and Reinforcement Learning techniques can be used in tandem of Multi-Period optimization to steer the portfolio dynamics in a particular way based on estimates of future events. The portfolio can be gradually driven to meet decarbonization goals, transactional cost optimization and events like important announcements on particular dates with anticipated jumps, etc. Some interesting follow up exploration can be to use Monte-Carlo Simulations and Prior Belief about the future to simulate multiple trajectories of the portfolio and come up with a much sound MPO strategy satisfying the investor's risk appetite. The relevant code for this work can be found at `https://github.com/RishiDarkDevil/Multi-Period-Portfolio-Optimization`.

Stock Name	Portfolio Weight
CESC	0.00%
EXIDEIND	0.00%
NTPC	20.46%
ONGC	0.00%
SUZLON	26.39%
TATAMOTORS	21.65%
TATAPOWER	31.46%
TATASTEEL	0.00%
Expected Returns	**Standard Deviation**
68.97%	28.00%

Table 1: *Allocation of Portfolio Weights for 19-09-2024 based on the last 4 years data using the SPO strategy.*

Stock Name	Portfolio Weight
CESC	0.00%
EXIDEIND	0.00%
NTPC	20.57%
ONGC	0.00%
SUZLON	29.59%
TATAMOTORS	13.12%
TATAPOWER	24.53%
TATASTEEL	12.17%
Expected Returns	**Standard Deviation**
70.34%	28.77%

Table 2: *Allocation of Portfolio Weights for 19-09-2024 based on the last 4 years data using the MPO strategy.*