AI1110:ASSIGNMENT

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Contents

***Abstract*—This manual provides solutions for the assignment - RANDOM NUMBERS**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/RishiManoj11045/
    AI1110_Assignments/blob/main/
    RandomNumbers/codes/uni_rand.c
wget https://github.com/RishiManoj11045/
    AI1110_Assignments/blob/main/
    RandomNumbers/codes/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/RishiManoj11045/
    AI1110_Assignments/blob/main/
    RandomNumbers/codes/uniform_cdf.py
```

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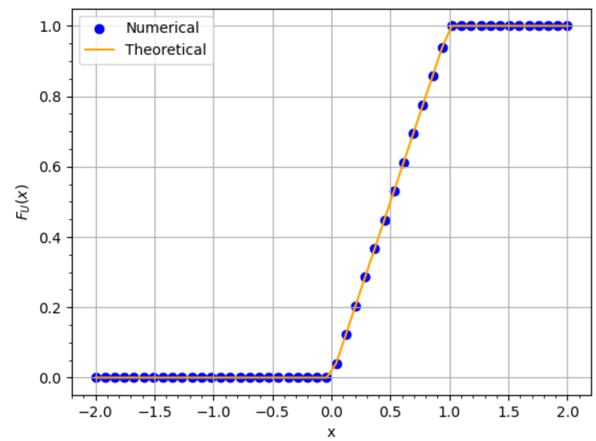

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: As we know
Probability density function of U:

$$P_U(x) = \begin{cases} 1 & \text{for } x \in (0,1) \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{x} \Pr(U = a)\, da \tag{1.3}$$

$$= \int_{-\infty}^{0} (0) da + \int_{0}^{x} (1) da \tag{1.4}$$

$$= x \tag{1.5}$$

$$\therefore F_U(x) = x \tag{1.6}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \tag{1.7}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.8}$$

Write a C program to find the mean and variance of U.
Solution: The following code gives mean and

variance.

```
wget https://github.com/RishiManoj11045/
    AI1110_Assignments/blob/main/
    RandomNumbers/codes/uni_mean-var.c
```

$$Mean = 0.50007$$
$$Variance = 0.083301$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

Solution:

1) k = 1 : E[U] is the mean
2) k = 2 : E[U^2]
   where, var[U] = E[U^2] - (E[U])^2

i) Verifying Mean of sample of 10^6 Uniform Random Variable

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.10}$$
$$\tag{1.11}$$

$$As, F_U(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \in (0,1) \\ 1 & \text{for } x > 1 \end{cases} \tag{1.12}$$

$$E[U] = 0 + \int_0^1 x d(x) + 0 \tag{1.13}$$

$$E[U] = \left.\frac{x^2}{2}\right|_0^1 \tag{1.14}$$

$$E[U] = 0.5 \tag{1.15}$$

ii) Verifying Variance of sample of 10^6 Uniform Random Variable

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.16}$$

$$As, F_U(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \in (0,1) \\ 1 & \text{for } x > 1 \end{cases}$$

$$E\left[U^2\right] = 0 + \int_0^1 x^2 d(x) + 0 \tag{1.17}$$

$$E\left[U^2\right] = \left.\frac{x^3}{3}\right|_0^1 \tag{1.18}$$

$$E\left[U^2\right] = \frac{1}{3} = 0.33 \tag{1.19}$$
$$\tag{1.20}$$

Thus,

$$Var[U] = E\left[U^2\right] - (E[U])^2 \tag{1.21}$$

$$Var[U] = (\frac{1}{3}) - (\frac{1}{2})^2 \tag{1.22}$$

$$Var[U] = 0.0833 \tag{1.23}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.
Solution: Download the following files and execute the C program.

```
wget https://github.com/RishiManoj11045/
    AI1110_Assignments/blob/main/
    RandomNumbers/codes/gau_rand.c
wget https://github.com/RishiManoj11045/
    AI1110_Assignments/blob/main/
    RandomNumbers/codes/coeffs.h
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2, using the code below

```
wget https://github.com/RishiManoj11045/
    AI1110_Assignments/blob/main/
    RandomNumbers/codes/gaussian_cdf.py
```

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

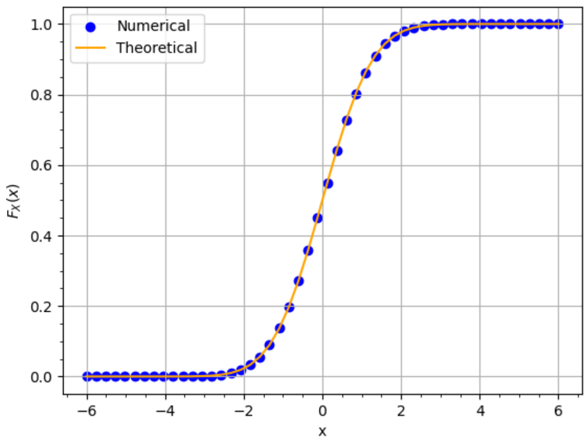

Fig. 2.2: The CDF of X

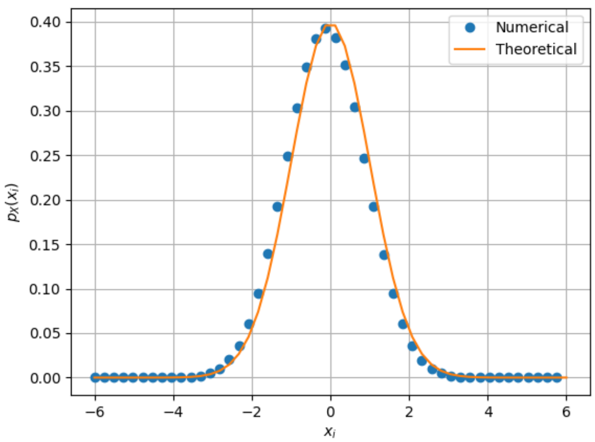

Fig. 2.3: The PDF of X

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/RishiManoj11045/
    AI1110_Assignments/blob/main/
    RandomNumbers/codes/gaussian_pdf.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution: The following code gives mean and variance.

```
wget https://github.com/RishiManoj11045/
    AI1110_Assignments/blob/main/
    RandomNumbers/codes/gau_mean−var.c
```

.

$$Mean = 0.000326$$
$$Variance = 1.000907$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.
Solution:

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)$$

a) k = 1 : E[U] is the mean
b) k = 2 : E[U^2]

where, var[U] = E[U^2] - (E[U])^2

i) Verifying Mean by using (2.3):

$$E[U] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.4)$$

$$E[U] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.5)$$

$$E[U] = \int_{-\infty}^{0} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx) \quad (2.6)$$

$$+ \int_{0}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx) \quad (2.7)$$

$$E[U] = \int_{0}^{\infty} (-x) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx) \quad (2.8)$$

$$+ \int_{0}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx) \quad (2.9)$$

$$E[U] = 0 \quad (2.10)$$

ii) Verifying Variance by using (2.3):

$$E[U] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{2.11}$$

$$E[U] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.12}$$

$$E[U] = 2\int_{0}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.13}$$

$$E[U] = \sqrt{\frac{2}{\pi}} \left[x \exp\left(-\frac{x^2}{2}\right) - \int_{0}^{\infty} \exp\left(-\frac{x^2}{2}\right)\right] \tag{2.14}$$

$$E[U] = \sqrt{\frac{2}{\pi}} \left[0 + \sqrt{\frac{\pi}{2}}\right] \tag{2.15}$$

$$E[U] = 1 \tag{2.16}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: The CDF of X is plotted in Fig. 3.2 using the code below

```
wget https://github.com/RishiManoj11045/
    AI1110_Assignments/blob/main/
    RandomNumbers/codes/v_cdf.py
```

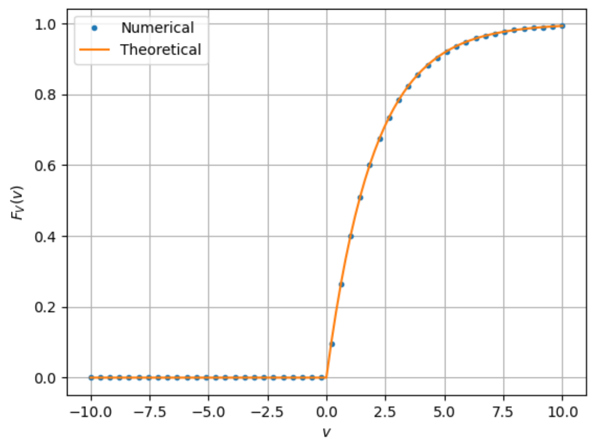

Fig. 3.2: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$\begin{aligned} F_V(x) &= \Pr(V \le x) && (3.2)\\ &= \Pr(-2\ln(1-U) \le x) && (3.3)\\ &= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) && (3.4)\\ & && (3.5) \end{aligned}$$

From eq.1.3,

$$\begin{aligned} F_V(x) &= F_U\left\{1 - \exp\left(-\frac{x}{2}\right)\right\} && (3.6)\\ &= 1 - \exp\left(-\frac{x}{2}\right) && (3.7) \end{aligned}$$