Assignment 6
Papoullis Textbook Chapter 5 Example 33

Rishi Manoj - CS21BTECH11045

May 31, 2022

Outline

Question

A person writes n letters and addresses n envelopes. Then one letter is randomly placed into each envelope. What is the probability that at least one letter will reach its correct destination?

Solution

Let X_k represents the event that there are exactly k coincidences among the n envelopes. All the events corresponding to the values of k starting from 0 to n are mutually exclusive and exhaustive. So, by theorem of total probability,

$$p_n(0) + p_n(1) + p_n(2) + \cdots + p_n(n) = 1 \tag{1}$$

$$p_n(k) \triangleq P(X_k) \tag{2}$$

Let P be the probability of generating k coincidences with the chosen k letters and Pr be the required probability that atleast one letter reaches its correct destination.

Probability P

Number of ways of drawing k letters from a group of n is $\binom{n}{k}$. Probability of generating k coincidences with the chosen k letters is,

$$P = \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)\cdots\left(\frac{1}{n-k+1}\right) \tag{3}$$

Probability $p_n(k)$

Probability of no coincidences with remaining $n-k$ letters is given by $p_{n-k}(0)$. So,

$$p_n(k) = \binom{n}{k}\left(\frac{1}{n(n-1)\cdots(n-k+1)}\right)p_{n-k}(0) \quad (4)$$

$$= \frac{p_{n-k}(0)}{k} \quad (5)$$

Solving

We know that,

$$p_n(n) = \frac{1}{n!} \tag{6}$$

Substituiting eq.(5) in eq.(1) along with the above one gives,

$$p_n(0) + \frac{p_{n-1}(0)}{1!} + \cdots + \frac{p_1(0)}{(n-1)!} + \frac{1}{n!} = 1 \tag{7}$$

From this we can get the following,

$$p_1(0) = 0 \tag{8}$$

$$p_2(0) = \frac{1}{2} \tag{9}$$

$$p_3(0) = \frac{1}{6} \tag{10}$$

Moment Generating Function

We define the moment generating function in the following way to get an explicit expression for $p_n(0)$,

$$\phi(z) = \Sigma_{n=0}^{\infty} p_n(0) z^n \tag{11}$$

$$e^z \phi(z) = \left(\Sigma_{k=0}^{\infty} \frac{z^k}{k!}\right) \left(\Sigma_{n=0}^{\infty} p_n(0) z^n\right) \tag{12}$$

$$= 1 + z + z^2 + \cdots + z^n + \cdots \tag{13}$$

$$= \frac{1}{1-z} \tag{14}$$

Solving for $p_n(0)$

Using the above equations, we get,

$$\phi(z) = \frac{e^{-z}}{1-z} \tag{15}$$

$$= \Sigma_{n=0}^{\infty} \left(\Sigma_{k=0}^{n} \frac{(-1)^k}{k!} \right) z^n \tag{16}$$

$$p_n(0) = \Sigma_{k=0}^{n} \frac{(-1)^k}{k!} \rightarrow \frac{1}{e} \tag{17}$$

$$= 0.378 \tag{18}$$

Using eq.(5) we get,

$$p_n(k) = \frac{1}{k!} \Sigma_{m=0}^{n-k} \frac{(-1)^m}{m!} \tag{19}$$

Probability Pr

The required probability is given by,

$$Pr = 1 - p_n(0) \tag{20}$$

$$= 1 - \Sigma_{k=0}^{n} \frac{(-1)^k}{k!} \to 0.632 \tag{21}$$