Generic One-Shot Lecture Scribe Example (Structure Only)

Lecture Title: Joint Probability and Conditional Probability
Course: CSE400 - Fundamentals of Probability in Computing
Purpose: Exam-oriented lecture scribe for revision

1. Overview

This lecture introduces the fundamental concepts of probability theory, including experiments, sample spaces, and events. It establishes the core Axioms of Probability and derives key propositions. The lecture progresses to define and calculate Joint Probability using classical and relative frequency approaches. Finally, it covers Conditional Probability, introducing the Product Rule and Chain Rule, supported by engineering applications such as Speech Recognition, Radar Systems, and Communication Networks.

2. Definitions and Notation

Definition 2.1 (Experiment and Outcome):

- Experiment (E): A procedure performed that produces some result (e.g., Tossing a coin five times, E_5).
- Outcome (ξ): A possible result of an experiment (e.g., $\xi_1 = HHTHT$).

Definition 2.2 (Event): An event (denoted by any letter) is a certain set of outcomes of an experiment.

Definition 2.3 (Sample Space): The Sample Space (S) is the universal set or collection of "all possible" distinct outcomes of an experiment. The sample space can be Discrete, Countably infinite, or Continuous.

Definition 2.4 (Probability): Probability is a measure of the likelihood of various events, or a function of an event that produces a numerical quantity measuring the likelihood of that event.

Definition 2.5 (Joint Probability): The probability of the intersection of two events A and B (not mutually exclusive).

- Notation: Denoted as $Pr(A, B)$ or $Pr(A \cap B)$.

Definition 2.6 (Conditional Probability): The probability of event A conditioned on knowing event B occurred.

- Formula: $Pr(A|B) = \frac{Pr(A,B)}{Pr(B)}$ where $Pr(B) > 0$.

3. Assumptions and Conditions

The following assumptions are established regarding sample spaces and probability calculation:

Assumption A (Sample Space Nature): Outcomes in a sample space must be Mutually Exclusive (one can get heads or tails, not both) and Collectively Exhaustive (one cannot get anything other than the defined outcomes).

Assumption B (Classical Approach): Both events A and B can be expressed in terms of atomic outcomes, allowing for the calculation of probabilities based on common atomic outcomes.

Assumption C (Relative Frequency Approach): To get an exact measure of probability, an event must be repeated an infinite number of times ($n \to \infty$). This is noted as a drawback since many phenomena are not repeatable.

4. Main Result / Theorem

Theorem 4.1 (Axioms of Probability): These statements are taken as self-evident and require no proof:

- For any event A, $0 \leq Pr(A) \leq 1$.
- If S is the sample space, $Pr(S) = 1$.
- If $A \cap B = \emptyset$ (mutually exclusive), then $Pr(A \cup B) = Pr(A) + Pr(B)$.
- For an infinite number of mutually exclusive sets A_i, $Pr(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} Pr(A_i)$.

Theorem 4.2 (Propositions from Axioms):

- Proposition 2.1: $Pr(A^c) = 1 - Pr(A)$.
- Proposition 2.2: If $A \subset B$, then $Pr(A) \leq Pr(B)$.
- Proposition 2.3 (Inclusion-Exclusion): $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$.

Theorem 4.3 (Product and Chain Rules):

- Product Rule: $Pr(A, B) = Pr(A|B)Pr(B) = Pr(B|A)Pr(A)$.
- Chain Rule (M events):

$$Pr(A_1, A_2, ...A_M) = Pr(A_M|A_1, ...A_{M-1}) \times ... \times Pr(A_2|A_1) \times Pr(A_1).$$

5. Proof/ Derivation

Proof Sketch (Derivation of Missing Key Probability): In Example 5 (The Missing Key), we derive $Pr(R|L^c)$ (Probability key is in Right pocket given it is NOT in Left).

Step 1: State the initial probabilities.

$$Pr(L) = 0.4 \text{ (Left)}, Pr(R) = 0.4 \text{ (Right)}, Pr(K) = 0.8 \text{ (Key is in jacket)}.$$

Step 2: Apply the Conditional Probability Definition.

$$Pr(R|L^c) = \frac{Pr(R \cap L^c)}{Pr(L^c)}.$$

Step 3: Simplify the numerator. Since R implies L^c (the key cannot be in both pockets simultaneously), the intersection $Pr(R \cap L^c)$ simplifies to $Pr(R)$.

Step 4: Substitute and Solve.

- Numerator: $Pr(R) = 0.4$
- Denominator: $Pr(L^c) = 1 - Pr(L) = 1 - 0.4 = 0.6$
- Result: $\frac{0.4}{0.6} = \frac{2}{3}$.

6. Worked Example

Example 6.1 (Costume Party - Joint Probability): Alex selects one top and one bottom randomly.

- Tops: 3 T-shirts, 1 Cape (Total 4).
- Bottoms: 2 Pants, 4 Boxers (Total 6).

Goal: Find the probability of the outfit being the Cape and Polka-dot Boxers.

Solution:

- Analyze Tops: $Pr(Cape) = 1/4$.
- Analyze Bottoms: $Pr(Boxers) = 4/6 = 2/3$.

Calculate Joint Probability (Independence):
Total Probability $= Pr(Cape) \times Pr(Boxers)$

$$= (1/4) \times (4/6) = 1/6.$$

Example 6.2 (Cards Without Replacement - Conditional Probability): Two cards are selected at random without replacement.

- Event A: First card was a Spade.
- Event B: Second card was a Spade.

Goal: Find $Pr(B|A)$.

Solution:
Initial State: 52 cards, 13 Spades.
Event A Occurs: One Spade removed.
Remaining State:

- Total cards remaining $= 52 - 1 = 51$.
- Spades remaining $= 13 - 1 = 12$.

Calculation:

$$Pr(B|A) = \frac{\text{Remaining Spades}}{\text{Remaining Total}} = \frac{12}{51}.$$

7. Summary

This lecture established the key definitions, assumptions, and results required for exam preparation regarding probability theory. It covered the axioms of probability, the calculation of joint probabilities for mutually exclusive and non-exclusive events, and the rules of conditional probability including the Chain Rule. Engineering applications in speech recognition and radar systems were used to contextualize these mathematical concepts.