# CSE400 - Fundamentals of Probability in Computing

Lecture 8: Gaussian, Uniform, Exponential, and Gamma Random Variables

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## Topic 1: Continuous Random Variable (CRV) Moments

### 1. Definitions and Notation

- $n^{th}$ **order Central Moment**: $E[(X-\mu)^n]$.
- **Mean ($\mu$)**: $E[X]$.
- **Variance ($\sigma^2$)**: The second central moment ($n = 2$), defined as $E[(X-\mu)^2]$.
- **Skewness ($\gamma_1$)**: The third-order normalized central moment ($\frac{\mu_3}{\sigma^3}$), defined as $\frac{E[(X-\mu)^3]}{\sigma^3}$.
- **Kurtosis ($\beta_2$)**: The fourth central moment ($n = 4$), defined as $E[(X-\mu)^4]$.

### 2. Main Results / Theorems

- **Central Moment for $n = 0$**: $E[(X-\mu)^0] = 1$.
- **Central Moment for $n = 1$**: $E[(X-\mu)^1] = 0$.
- **Variance Identity**: $\sigma^2 = E[X^2] - (E[X])^2$.
- **Skewness Interpretation**:
  - Positive (+) value: PDF is Right Skewed.
  - Negative (-) value: PDF is Left Skewed.
- **Kurtosis Interpretation**: A large value indicates the RV $X$ will have a large peak near the mean.
- **Linearity of Expectation**: For any constants $a$ and $b$, $E[aX + b] = aE[X] + b$.
- **Expectation of Sums**: For any function $g(X)$, $E[\sum g(x)] = \sum E[g(x)]$.

### 3. Proofs / Derivations

**Derivation of Variance Identity ($E[X^2] - \mu^2$):**

- **Step 1**: Expand the quadratic term: $E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$.
- **Step 2**: Apply linearity of expectation: $E[X^2] - 2\mu E[X] + E[\mu^2]$.
- **Step 3**: Substitute $E[X] = \mu$: $E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2$.

# Topic 2: Gaussian Random Variable

## 1. Definitions and Notation

- **Gaussian PDF**: A random variable $X$ is Gaussian if its PDF is:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- **Notation**: $X \sim N(\mu, \sigma^2)$ where $\mu$ is the mean and $\sigma^2$ is the variance.
- **Standard Normal Distribution**: A Gaussian distribution where $\mu = 0$ and $\sigma = 1$.

## 2. Standard Forms: Q-function and Error Functions

- **Error Function**: $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.
- **Complementary Error Function**: $erfc(x) = 1 - erf(x)$.
- **$\Phi$-function (CDF of Standard Normal)**: $\Phi(x) = P(Z \leq x)$.
- **Q-function (Gaussian Tail Function)**: $Q(x) = P(Z > x)$.

## 3. Main Results / Theorems

- **Identity**: $\Phi(x) + Q(x) = 1$.
- **Evaluating Gaussian CDF**: $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$.
- **Evaluating Tail Probabilities**: $P(X > x) = Q\left(\frac{x-\mu}{\sigma}\right)$.
- **CDF in terms of Q-function**: $F_X(x) = 1 - Q\left(\frac{x-\mu}{\sigma}\right)$.
- **Symmetry**: $Q(-x) = 1 - Q(x) = \Phi(x)$.

## 4. Worked Examples

### Example 1: Probability Calculation

Given a PDF: $f_X(x) = \frac{1}{\sqrt{18\pi}} e^{-\frac{(x-10)^2}{18}}$. Find probabilities in terms of $\Phi$-functions.

- **Step 1 (Identify Parameters)**: By comparison with the general form, $\mu = 10$ and $2\sigma^2 = 18 \implies \sigma = 3$.
- **Step 2 ($P(X \leq 13)$)**: $\Phi\left(\frac{13-10}{3}\right) = \Phi(1)$.
- **Step 3 ($P(X \leq 7)$)**: $\Phi\left(\frac{7-10}{3}\right) = \Phi(-1)$.
- **Step 4 ($P(7 \leq X \leq 13)$)**: This implies $\Phi(1) - \Phi(-1)$.
- **Step 5 ($P(|X - 10| \geq 6)$)**: This implies $X \geq 16$ or $X \leq 4$. $Q\left(\frac{16-10}{3}\right) + \Phi\left(\frac{4-10}{3}\right) = Q(2) + \Phi(-2)$.

# Topic 3: CDF Analysis Problem

## 1. Worked Examples

**Example 2: Analyzing a non-Gaussian CDF** Given $F_X(x) = 1 - e^{-2x}$

- **Step 1 (Find PDF)**: $f_X(x) = 2e^{-2x}$ for $x \geq 0$.
- **Step 2 (Mean)**: $E[X] = \frac{1}{2}$.
- **Step 3 (Variance)**: $Var(X) = \frac{1}{4}$.
- **Step 4 (Skewness)**: $\gamma_1 = 2$ (Left-skewed).
- **Step 5 (Kurtosis)**: $\beta_2 = 6$ (Platykurtic).

# Topic 4: Applications in Computing

## 1. Definitions and Notation

- **Thermal Noise**: Random voltage in electronic circuits.
- **Jitter**: Packet delay variation in communication networks.
- **Generative Formula**: $X = \sigma Z + \mu$ (Transforming standard randomness $Z$ into physical model $X$).
- **Sample Statistics**: Using $\bar{x}$ and $s^2$ from raw noisy samples (histograms) to estimate a PDF.

## 2. Main Results / Theorems

- **Sensor Modeling**: Measured Value = True Value + Gaussian Noise.
- **Image Denoising**: Using Gaussian priors to distinguish real light from random thermal grain.
- **Tail Latency**: Modeling the "99th percentile" in distributed systems for SRE/DevOps.
- **Sensor Fusion (Kalman Filters)**: Guessing real locations between noisy GPS pings.