Lecture 4: Joint Probability and Conditional Probability

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1 Engineering Applications

[cite$_start]Probabilitytheoryisfundamentaltovariousengineeringsystems[cite: 19]$.

Speech Recognition: Using templates and vocabulary (e.g., Hello, Yes, No) to map speech signals $x(t)$ to words $x(w)$ amidst noise and interference[cite: 26, 39, 59].

Radar Systems: Detecting signals (S_i) amidst noise (W_i). [cite$_start]Thesystemmustdistinguishbetweenhypothes$ (noise only) and H_1 (signal + noise) to minimize false alarms and missed detections[cite: 64, 67, 77]. [cite$_start]$

Communication Networks: Managing packet arrival, QoS, and delay in networks (WiFi, 4G, 5G)[cite: 87, 101].

2 Introduction to Probability Theory

2.1 Definitions

- **Experiment (E):** A procedure performed that produces some result (e.g., Tossing a coin five times, E_5)[cite: 148]. [cite$_start]$
- **Outcome (ξ):** A possible result of an experiment (e.g., $\xi_1 = HHTHT$)[cite: 156]. [cite$_start]$
- **Event:** A certain set of outcomes of an experiment (e.g., Event C = all outcomes consisting of an even number of heads)[cite: 168, 170]. [cite$_start]$
- **Sample Space (S):** The collection or set of "all possible" distinct outcomes of an experiment[cite: 184]. Outcomes in S must be:
  - **Mutually Exclusive:** You can get one outcome or another, but not both (e.g., Heads or Tails)[cite: 186]. [cite$_start]$
  - **Collectively Exhaustive:** No other outcomes are possible[cite: 187].

2.2 Types of Sample Spaces

- **Discrete:** Finite set of outcomes (e.g., Flipping a coin, rolling a die)[cite: 209]. [cite$_start]$
- **Countably Infinite:** (e.g., Flipping a coin until a tails occurs)[cite: 218]. [cite$_start]$
- **Continuous:** Uncountably infinite (e.g., Random number generator in interval $[0, 1)$)[cite: 219].

3 Axioms and Propositions

3.1 Axioms of Probability

[cite$_start]Probabilityisafunctionproducinganumericalquantitymeasuringlikelihood[cite: 235]$.

Axiom 1: For any event A, $0 \leq Pr(A) \leq 1$[cite: 238]. [cite$_start$]

Axiom 2: If S is the sample space, $Pr(S) = 1$[cite: 239]. [cite$_start$]

Axiom 3: If $A \cap B = \emptyset$ (mutually exclusive), then $Pr(A \cup B) = Pr(A) + Pr(B)$[cite: 240].

General Axiom 3: For an infinite number of mutually exclusive sets A_i where $A_i \cap A_j = \emptyset$ for all $i \neq j$:

$$[cite_start]Pr(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} Pr(A_i) \quad \text{[cite: 245]} \tag{1}$$

3.2 Corollaries and Propositions

- **Corollary 2.1:** For a finite number (M) of mutually exclusive sets:

$$[cite_start]Pr(\bigcup_{i=1}^{M} A_i) = \sum_{i=1}^{M} Pr(A_i) \quad \text{[cite: 259]} \tag{2}$$

  [cite$_start$]

- **Proposition 2.1 (Complement):** $Pr(A^c) = 1 - Pr(A)$[cite: 271]. [cite$_start$]
- **Proposition 2.2 (Subset):** If $A \subset B$, then $Pr(A) \leq Pr(B)$[cite: 278].
- **Proposition 2.3 (Union of 2 Sets):** For any sets A and B:

$$[cite_start]Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) \quad \text{[cite: 287]} \tag{3}$$

- **Proposition 2.4 (Union of M Sets - Inclusion-Exclusion):**

$$\begin{gathered} Pr(A_1 \cup A_2 \cup \cdots \cup A_M) = \sum_{i=1}^{M} Pr(A_i) - \sum_{i_1 < i_2} Pr(A_{i_1} A_{i_2}) + \dots \\ +(-1)[cite_start]^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} Pr(A_{i_1} \dots A_{i_r}) + \cdots + (-1)^{M+1} Pr(A_1 \dots A_M) \quad \text{[cite: 302, 304]} \end{gathered} \tag{4}$$

4 Assigning Probabilities

- **Classical Approach:** Based on atomic outcomes (e.g., Coin flipping $Pr(H) = 1/2$, Dice rolling)[cite: 310].
- **Relative Frequency Approach:** Requires repeating an event n times. Let $n_{A,B}$ be the occurrences:

$$[cite_start]Pr(A, B) = \lim_{n \to \infty} \frac{n_{A,B}}{n} \quad \text{[cite: 400]} \tag{5}$$

  [cite$_start]Drawback: Requiresinfiniterepetition; manyphenomenaarenotrepeatable[cite :$ 326].

5 Joint Probability

5.1 Definitions

- **Motivation:** Events are not always mutually exclusive. [cite$_s tart] We need the probability of the intersecti$
353].$[cite_s tart]$
- **Notation:** $Pr(A, B)$ or $Pr(A \cap B)$[cite: 361]. [cite$_s tart$]
- **Multiple Events:** Denoted $Pr(A_1, A_2, \ldots, A_M)$[cite: 370].

5.2 Worked Examples

Example 1: Card Deck

Problem: Consider a 52-card deck. $A = \{Red\}$, $B = \{NumberCard\}$, $C = \{Heart\}$. [cite$_s tart] Find joint probabilities [cite : 408]$.
Solution:

$Pr(A) = 26/52 = 1/2$ (13 Hearts + 13 Diamonds)[cite: 440]. [cite$_s tart$]

$Pr(B) = 40/52 = 5/26$ (10 number cards per suit)[cite: 441]. [cite$_s tart$]

$Pr(C) = 13/52 = 1/4$[cite: 442]. [cite$_s tart$]

$Pr(A, B) = 20/52 = 5/13$ (10 red number cards in Hearts + 10 in Diamonds)[cite: 444]. [cite$_s tart$]

$Pr(A, C) = 13/52 = 1/4$ (All hearts are red)[cite: 445]. [cite$_s tart$]

$Pr(B, C) = 10/52$ (10 number cards that are hearts)[cite: 446].

Example 2: Costume Party

Problem: Alex has 4 tops (3 T-shirts, 1 Cape) and 6 bottoms (2 Pants, 4 Boxers). She selects one top and one bottom randomly. [cite$_s tart] What is the probability of the outfit Cape, Polka - dot$
$454, 459]$.
Solution:

Analyze Tops: Total = 4. $Pr(Cape) = 1/4$[cite: 479]. [cite$_s tart$]

Analyze Bottoms: Total = 6. $Pr(Boxers) = 4/6 = 2/3$[cite: 491].

Joint Probability: Since selections are independent:

$$[cite_s tart]Pr(Cape, Boxers) = Pr(Cape) \times Pr(Boxers) = \frac{1}{4} \times \frac{4}{6} = \frac{1}{6} \quad \text{[cite: 494]} \tag{6}$$

6 Conditional Probability

6.1 Definitions

- **Definition:** The probability of A conditioned on knowing B occurred:

$$[cite_s tart]Pr(A|B) = \frac{Pr(A, B)}{Pr(B)} \quad \text{where } Pr(B) > 0 \quad \text{[cite: 530]} \tag{7}$$

- **Product Rule:**

$$[cite_s tart]Pr(A, B) = Pr(A|B)Pr(B) = Pr(B|A)Pr(A) \quad \text{[cite: 540]} \tag{8}$$

- **Chain Rule (M events):**

$$[cite_s tart]Pr(A_1, \ldots, A_M) = Pr(A_M|A_1 \ldots A_{M-1}) \times \cdots \times Pr(A_2|A_1)Pr(A_1) \quad \text{[cite: 566]} \tag{9}$$

6.2 Worked Examples

Example 3: Cards Without Replacement

Problem: Select two cards. The first is not returned. [cite$_s tart$]$Find$Pr(B—A)$where$A=$FirstisS$
576].
Solution:

Initial State: 52 cards, 13 Spades[cite: 604]. [cite$_s tart$]

Event A Occurs: 1 Spade removed[cite: 605]. [cite$_s tart$]

Remaining State: 51 cards total, 12 Spades left[cite: 626].

Calculation:

$$[cite_s tart]Pr(B|A) = \frac{12}{51} \quad \text{[cite: 629]} \tag{10}$$

Example 4: Game of Poker (Flush)

Problem: What is the probability of being dealt a flush (5 cards of same suit) in Spades? [cite$_s tart$]$Whataboutanysuit?[cite: 638]$.
Solution:

Spade Flush: Let A_i be the event the i^{th} card is a spade. Using the chain rule:

$$[cite_s tart]P(SpadeFlush) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} \quad \text{[cite: 670]} \tag{11}$$

Any Flush: Since suits are mutually exclusive:

$$[cite_s tart]P(AnyFlush) = 4 \times P(SpadeFlush) \quad \text{[cite: 685]} \tag{12}$$

Example 5: The Missing Key

Problem: Bob is 80% certain the key is in the jacket (K). Left pocket (L) has 40%, Right pocket (R) has 40%. [cite$_s tart$]$IfthekeyisnotintheLeftpocket$(L^c), what is the probability it is in the Right?[cite: 693].
Solution:

- **Given:** $P(L) = 0.4$, $P(R) = 0.4$, $P(K) = 0.8$[cite: 706]. [cite$_s tart$]
- **Goal:** Find $P(R|L^c)$[cite: 709].
- **Derivation:**

$$P(R|L^c) = \frac{P(R \cap L^c)}{P(L^c)} \tag{13}$$

[cite$_s tart$]$Since$R$implies$L^c (key cannot be in both pockets), $P(R \cap L^c) = P(R)$[cite: 722].

$$[cite_s tart]P(R|L^c) = \frac{P(R)}{1 - P(L)} = \frac{0.4}{1 - 0.4} = \frac{0.4}{0.6} = \frac{2}{3} \quad \text{[cite: 721]} \tag{14}$$