# Lecture 8: Gaussian, Uniform, Exponential, and Gamma Random Variables

CSE 400: Fundamentals of Probability in Computing

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## 1 Definitions and Notation

**Continuous Random Variable (CRV):** Defined by its Probability Density Function (PDF) and Cumulative Distribution Function (CDF).

**Gaussian Random Variable:** A random variable $X$ is Gaussian if its PDF is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

**Notation:** $X \sim \mathcal{N}(m, \sigma^2)$, where $m$ is the mean ($\mu_X$) and $\sigma^2$ is the variance.
**Standard Normal Distribution:** A Gaussian distribution where $m = 0$ and $\sigma^2 = 1$.
$n^{th}$ **order Central Moment:** Defined as $E[(X - \mu_X)^n]$.
**Skewness ($C_s$):** A measure of the symmetry of the PDF, defined as:

$$C_s = \frac{E[(X-\mu_X)^3]}{\sigma_X^3}$$

**Kurtosis ($C_k$):** A measure of the "peakiness" of the PDF, defined as:

$$C_k = E[(X-\mu_X)^4]$$

**Error Function ($erf(x)$):** Defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)dt$$

**Complementary Error Function ($erfc(x)$):** Defined as:

$$erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2)dt$$

## 2 Assumptions / Conditions

- **PDF Normalization:** For any valid PDF, the total area under the curve must equal 1: $\int_{-\infty}^{\infty} f_X(x)dx = 1$.
- **Constants in Expectation:** When calculating expectations, $a$ and $b$ are treated as constants.
- **Linearity of Expectation:** Assumes that the expectation of a sum of functions is equal to the sum of their individual expectations.

## 3 Main Results / Theorems

**Theorem on Linear Transformation of Expectation:** For any constant $a$ and $b$,

$$E[ax + b] = aE[X] + b$$

**Theorem on Sum of Functions:** For any function $g(x)$ that is a sum of several other functions $g_k(x)$, the expectation is:

$$E\left[\sum_{k=1}^{N} g_k(x)\right] = \sum_{k=1}^{N} E[g_k(x)]$$

**Symmetry and Skewness:**

- If $C_s > 0$, the PDF is right-skewed.
- If $C_s < 0$, the PDF is left-skewed.

**Kurtosis Significance:** A large value of Kurtosis indicates the random variable $X$ will have a large peak near the mean.

## 4 Proofs / Derivations

### Derivation of Central Moments ($n = 0, 1, 2$)

**For $n = 0$:**

$$E[(X - \mu_X)^0] = E[1] = 1$$

**For $n = 1$:**

1. **Expand the expectation:** $E[X - \mu_X] = E[X] - \mu_X$.
2. **Substitute $\mu_X$ for $E[X]$:** $\mu_X - \mu_X = 0$.
3. **Result:** The first central moment is always 0.

**For $n = 2$ (Variance $\sigma_X^2$):**

1. **Start with definition:** $\sigma_X^2 = E[(X - \mu_X)^2]$.
2. **Expand the square:** $E[X^2 - 2\mu_X X + \mu_X^2]$.
3. **Apply linearity:** $E[X^2] - 2\mu_X E[X] + \mu_X^2$.
4. **Substitute $E[X] = \mu_X$:** $E[X^2] - 2\mu_X^2 + \mu_X^2$.
5. **Simplify:** $\sigma_X^2 = E[X^2] - \mu_X^2$.

## 5 Worked Examples

**Example 1: Gaussian PDF and CDF Parameters**
Consider a Gaussian Random Variable with mean $m = 3$ and standard deviation $\sigma = 2$.

**PDF:**

$$f_X(x) = \frac{1}{\sqrt{2\pi(2^2)}} \exp\left(-\frac{(x-3)^2}{2(2^2)}\right) = \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(x-3)^2}{8}\right)$$

**CDF:** $F_X(x) = Pr(X \leq x)$, represented as an S-shaped curve centered at $x = 3$.

**Example 2: Linearity of Expectation with Constants**
If $Y = aX + b$, find $E[Y]$.

1. $E[Y] = E[aX + b]$.
2. Using the theorem $E[ax + b] = aE[X] + b$, the result is $a\mu_X + b$.