

Lecture Scribe: Gaussian Random Variables

L8_S1_A

 PDF

L8_Gaussi...Estimation

 IPYNB

Purpose is to generate exam-oriented lecture notes (lecture scribe) strictly from the provided lecture PPT and my notes. The scribe must serve as a precise reference for exams.

Scope is to use only the content explicitly present in the provided material. Do not add external explanations, background knowledge, interpretations or examples. If a definition, assumption, example, derivation, or proof is not present in the material, do not include it.

Structural constraint :

Follow the exact structure and order of the lecture PPT. Preserve topic sequence and hierarchy.

Example Lecture Scribe (structure only)

Topic:

Sample Topic Title

Definition:

A formal definition as stated in the lecture.

Assumptions:

List of assumptions used.

Key Results:

- Statement of a theorem or result.

Proof:

Step 1: First step as presented.

Step 2: Second step as presented.

Conclusion

Example:

A short worked example exactly as shown in the lecture.

Instruction

Using the same structure as the example above, generate the lecture scribe for the provided lecture PPT and notes. The scribe must:

- Be structured exactly like the example
- Include definitions, assumptions, results, proofs, derivations, and worked examples only if they appear in the provided material
- Use simple, clear language while preserving correctness
- Remain fully aligned with the lecture content and ordering

What a lecture scribe is?

A lecture scribe is a faithful, exam-ready reconstruction of what was taught in class, written so that a student can later rely on it as reference material while studying.

A correct lecture scribe should allow a student to:

- Recall definitions and notation
- Reconstruct proofs and derivations
- Follow example solutions step by step
- Understand logical dependencies between concepts

A lecture scribe should answer the question:

"If I study only this document, can I reliably revise what was taught?"

What it's not.

A lecture scribe is not:

- A creative explanation
- A simplified tutorial
- A textbook rewrite
- A solution manual for unseen problems
- A place to add intuition not discussed in class

Everything included must be traceable to the provided context



Here are the exam-oriented lecture notes based on the provided material.

Lecture 8: Gaussian, Uniform, Exponential, and Gamma Random Variables

Topic: Moments of a Random Variable

Definition: The n -th order central moment of a random variable X is defined as the expected value of the n -th power of the deviation of X from its mean μ_X .

$$E[(X - \mu_X)^n] = \int_{-\infty}^{\infty} (x - \mu_X)^n f_X(x) dx$$

For discrete random variables:

$$E[(X - \mu_X)^n] = \sum_k (x_k - \mu_X)^n p_X(x_k)$$

Key Results:

- **Zeroth Moment ($n = 0$):** $E[(X - \mu_X)^0] = E[1] = 1$.
- **First Moment ($n = 1$):** $E[X - \mu_X] = E[X] - \mu_X = 0$.

- **Second Moment ($n = 2$):** Variance.

$$\sigma_X^2 = E[(X - \mu_X)^2] = E[X^2] - \mu_X^2$$

- **Third Moment ($n = 3$):** Skewness.

- **Measure:** Symmetry of the PDF.
- **Coefficient of Skewness (C_s):**

$$C_s = \frac{E[(X - \mu_X)^3]}{\sigma_X^3}$$

- **Interpretation:**

- Positive (+): PDF is Right Skewed.
- Negative (-): PDF is Left Skewed.

- **Fourth Moment ($n = 4$):** Kurtosis.

- **Coefficient of Kurtosis (C_k):**

$$C_k = \frac{E[(X - \mu_X)^4]}{\sigma_X^4}$$

- **Interpretation:** A large value implies the random variable will have a large peak near the mean.

Topic: Linear Transformations of Random Variables

Theorem: For any constants a and b :

$$E[aX + b] = aE[X] + b$$

Key Results: For a sum of several functions $g_k(X)$:

$$E \left[\sum_{k=1}^N g_k(X) \right] = \sum_{k=1}^N E[g_k(X)]$$

Topic: Gaussian Random Variable

Definition: A Gaussian random variable is a continuous random variable whose Probability Density Function (PDF) is given by:

$$f_X(x) = \frac{1}{2\pi\sigma^2} \exp \left(-\frac{(x-m)^2}{2\sigma^2} \right)$$

Notation:

$X \sim \mathcal{N}(m, \sigma^2)$ or $X \sim \mathcal{N}(\mu, \sigma^2)$.

Standard Normal Distribution: If $m = 0$ and $\sigma^2 = 1$, then $X \sim \mathcal{N}(0, 1)$.

Topic: Standard Forms and Special Functions

Definitions:

1. **Error Function (erf):**

$$erf(x) = \frac{2}{\pi} \int_0^x \exp(-t^2) dt$$

2. **Complementary Error Function ($erfc$):**

$$erfc(x) = 1 - erf(x) = \frac{2}{\pi} \int_x^\infty \exp(-t^2) dt$$

3. **Φ -Function (CDF of Standard Normal):**

$$\Phi(x) = \frac{1}{2\pi} \int_{-\infty}^x \exp\left(-\frac{t^2}{2}\right) dt$$

- Represents the area under the left tail.

4. Q-Function (Gaussian Tail Function):

$$Q(x) = \frac{1}{2\pi} \int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt$$

- Represents the area under the right tail.

Key Identities:

- $Q(x) = 1 - \Phi(x)$.
 - $\Phi(-x) = 1 - \Phi(x)$.
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Topic: Evaluating Gaussian Probabilities

Key Results: To evaluate the Cumulative Distribution Function (CDF) and probabilities of a Gaussian Random Variable $X \sim \mathcal{N}(m, \sigma^2)$:

1. CDF Evaluation:

$$F_X(x) = \Phi\left(\frac{x - m}{\sigma}\right)$$

$$F_X(x) = 1 - Q\left(\frac{x - m}{\sigma}\right)$$

2. **Tail Probability ($Pr(X > x)$):** The Q-function is more natural for evaluating these probabilities.

$$Pr(X > x) = Q\left(\frac{x - m}{\sigma}\right)$$

3. **Interval Probability** ($x_1 \leq X \leq x_2$):

$$Pr(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1)$$

Topic: Gaussian Probability Example

Problem Statement: A random variable X has a PDF given by:

$$f_X(x) = \frac{1}{8\pi} \exp\left(-\frac{(x+3)^2}{8}\right)$$

. Find the following probabilities in terms of Q-functions:

1. $Pr(X \leq 0)$
2. $Pr(X > 4)$
3. $Pr(|X + 3| < 2)$
4. $Pr(|X - 2| > 1).$

Solution: Step 1: Identify Parameters Comparing with the standard Gaussian PDF form:

$$f_X(x) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

- Mean (m) = -3.
- Variance ($2\sigma^2 = 8 \Rightarrow \sigma^2 = 4 \Rightarrow \sigma = 2$.

Step 2: Solve Probability 1 ($Pr(X \leq 0)$)

$$Pr(X \leq 0) = F_X(0) = \Phi\left(\frac{0 - (-3)}{2}\right) = \Phi\left(\frac{3}{2}\right)$$

Using identity $Q(x) = 1 - \Phi(x)$:

$$Pr(X \leq 0) = 1 - Q(1.5)$$

Step 3: Solve Probability 2 ($Pr(X > 4)$)

$$Pr(X > 4) = Q\left(\frac{4 - (-3)}{2}\right) = Q\left(\frac{7}{2}\right) = Q(3.5)$$

Step 4: Solve Probability 3 ($Pr(|X + 3| < 2)$) Inequality:

$$-2 < X + 3 < 2 \Rightarrow -5 < X < -1.$$

$$Pr(-5 < X < -1) = F_X(-1) - F_X(-5) = \Phi(1) - \Phi(-1)$$

. Using $\Phi(-1) = 1 - \Phi(1)$:

$$= \Phi(1) - (1 - \Phi(1)) = 2\Phi(1) - 1 = 1 - 2Q(1)$$

Step 5: Solve Probability 4 ($Pr(|X - 2| > 1)$) Inequality:

$$X - 2 > 1 \text{ or } X - 2 < -1 \Rightarrow X > 3 \text{ or } X < 1.$$

$$Pr(|X - 2| > 1) = F_X(1) + Pr(X > 3)$$

$$= \left(1 - Q\left(\frac{1 - (-3)}{2}\right)\right) + Q\left(\frac{3 - (-3)}{2}\right)$$

$$= 1 - Q(2) + Q(3)$$

Topic: Continuous Random Variable Analysis (CDF Example)

Problem Statement: Given the CDF:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

. Find Mean, Variance, Skewness, and Kurtosis.

Solution: Step 1: Find PDF

$$f_X(x) = \frac{d}{dx} F_X(x) = 2x, \quad 0 \leq x \leq 1$$

Step 2: Calculate Mean (μ_X)

$$\mu_X = \int_0^1 x(2x)dx = \int_0^1 2x^2 dx = \left[\frac{2x^3}{3} \right]_0^1 = \frac{2}{3}$$

Step 3: Calculate Variance (σ_X^2)

$$\sigma_X^2 = \int_0^1 x^2(2x)dx - \mu_X^2 = \int_0^1 2x^3 dx - \left(\frac{2}{3} \right)^2$$

$$= \left[\frac{2x^4}{4} \right]_0^1 - \frac{4}{9} = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

Step 4: Calculate Skewness (c_s)

$$c_s = \frac{E[(X - \mu)^3]}{\sigma^3} = -\frac{2}{5}$$

. Conclusion: Distribution is left-skewed.

Step 5: Calculate Kurtosis (c_k)

$$c_k = \frac{E[(X - \mu)^4]}{\sigma^4} = \frac{12}{5} = 2.4$$

. Conclusion: Platykurtic (lighter tails than Gaussian, where $c_k = 3$).
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Topic: Gaussian Modeling and Applications

Motivation: Real-world measurements are uncertain due to noise (e.g., thermal noise, sensor error). Repeated measurements form a "cloud" of uncertainty.

Gaussian Model: Noise is modeled using a Gaussian RV: $X \sim \mathcal{N}(\mu, \sigma^2)$. Generative Formula:

$$X = \sigma Z + \mu$$

where $Z \sim \mathcal{N}(0, 1)$.

Applications:

1. **Electronics:** Thermal noise voltage in circuits.
2. **Sensors:** Measurement error (Measured Value = True Value + Gaussian Noise).
3. **Networking:** Packet delay variation (jitter) fluctuating around a mean value.
 - Goal is to determine Typical Delay (μ) and Jitter (σ).
 - Packet delays are modeled as a single Gaussian random variable.
4. **Image Processing:** Denoising camera sensor "grain".

Density Estimation: In practice, μ and σ are unknown. We use sample statistics from raw, noisy samples (histograms) to estimate the distribution.

- $\hat{\mu}$ = Sample Mean
- $\hat{\sigma}$ = Sample Std Dev
- The estimated PDF smoothes the noise into a predictive tool.