CSE400: Fundamentals of Probability in Computing
Lecture 8: Gaussian, Uniform, Exponential, and Gamma Random Variables

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1 Continuous Random Variables: Moments

1.1 Central Moments

The n-th order central moment is defined as the expectation of the random variable centered around its mean.

$$[cite_start]E[(x-\mu_X)^n] = \int_{-\infty}^{\infty} (x-\mu_X)^n f_X(x)dx \quad \text{[cite: 20, 51]} \tag{1}$$

Specific Moments:

[cite$_start$]

- **Zeroth Moment ($n = 0$):** $E[(x-\mu_X)^0] = E[1] = 1$[cite: 55]. [cite$_start$]
- **First Moment ($n = 1$):** $E[(x-\mu_X)] = E[x] - \mu_x = 0$[cite: 56, 58].
- **Second Moment ($n = 2$):** Defined as the Variance (σ_X^2).

$$[cite_start]\sigma_X^2 = E[(x-\mu_X)^2] = E[x^2] - \mu_X^2 \quad \text{[cite: 59, 61]} \tag{2}$$

1.2 Higher Order Moments (Shape)

- **Skewness ($n = 3$):** A measure of the symmetry of the Probability Density Function (PDF).

$$[cite_start]C_s = \frac{E[(X-\mu_X)^3]}{\sigma_X^3} \quad \text{[cite: 81]} \tag{3}$$

  [cite$_start$]
  - If C_s is positive (+), the PDF is Right Skewed[cite: 82]. [cite$_start$]
  - If C_s is negative (-), the PDF is Left Skewed[cite: 83].

- **Kurtosis ($n = 4$):** A measure of the "tailedness" or peak of the distribution.

$$[cite_start]C_k = \frac{E[(X-\mu_X)^4]}{\sigma^4} \quad \text{[cite: 86, 341]} \tag{4}$$

[cite$_start]Alargevalueindicatestherandomvariablewillhavealargepeaknearthemean[cite: 88, 89]$.

2 Theorems on Expectation

2.1 Linearity of Expectation

For any constants a and b:

$$[cite_start]E[ax+b] = aE[x] + b \quad \text{[cite: 94]} \tag{5}$$

2.2 Sum of Functions

For a sum of several functions $g(x) = g_1(x) + g_2(x) + \cdots + g_N(x)$:

$$[cite_start]E\left[\sum_{k=1}^{N} g_k(x)\right] = \sum_{k=1}^{N} E[g_k(x)] \quad \text{[cite: 97]} \tag{6}$$

3 Gaussian Random Variable

3.1 Definition

A Gaussian random variable $X \sim \mathcal{N}(m, \sigma^2)$ is defined by the PDF:

$$[cite_start]f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) \quad \text{[cite: 119]} \tag{7}$$

Where m is the mean and σ^2 is the variance. [cite$_s tart]If$m=0$and\sigma^2$ = 1, it is a Standard Normal Distribution[cite: 121].

3.2 Standard Functions

Several standard integrals are used to evaluate Gaussian probabilities:

- **Error Function:** [cite$_s tart$]erf(x) = $2\frac{}{\sqrt{\pi}\int_0^x \exp(-t^2)dt}$[cite: 166].
- **Complementary Error Function:** [cite$_s tart$]erfc(x) = 1 - erf(x) = $2\frac{}{\sqrt{\pi}\int_x^\infty \exp(-t^2)dt}$[cite: 167].
- **Φ-function (CDF of Standard Normal):** Represents the area under the left tail.

$$[cite_start]\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt \quad \text{[cite: 175, 250]} \tag{8}$$

- **Q-function (Gaussian Tail Function):** Represents the area under the right tail.

$$[cite_start]Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \quad \text{[cite: 176, 254]} \tag{9}$$

3.3 Key Identities and Relations

[cite$_s tart$]

- $Q(x) = 1 - \Phi(x)$[cite: 214]. [cite$_s tart$]
- $\Phi(-x) = 1 - \Phi(x)$[cite: 305].
- The CDF of a general Gaussian variable can be expressed as:

$$[cite_start]F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right) = 1 - Q\left(\frac{x-m}{\sigma}\right) \quad \text{[cite: 211, 214]} \tag{10}$$

- Tail probabilities for a general Gaussian variable:

$$[cite_start]Pr(X > x) = Q\left(\frac{x-m}{\sigma}\right) \quad \text{[cite: 205]} \tag{11}$$

4 Worked Example 1: Gaussian Probabilities

Problem: A random variable X has a PDF given by:

$$f_X(x) = \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(x+3)^2}{8}\right)$$

[cite$_s tart]FindprobabilitiesintermsofQ - functions[cite : 262]$.

Solution:

1. **Identify parameters:** Comparing to the standard form $\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-m)^2}{2\sigma^2})$, we get: [cite$_s tart]m = -3, \quad \sigma^2 = 4 \Rightarrow \sigma = 2$ [cite: 287]

2. **Calculate $Pr(X \leq 0)$:** [cite$_s tart]Pr(X \leq 0) = F_X(0) = \Phi\left(\frac{0-(-3)}{2}\right) = \Phi(1.5) = 1 - Q(1.5)$ [cite: 297]

3. **Calculate $Pr(X > 4)$:** [cite$_s tart]Pr(X > 4) = Q\left(\frac{4-(-3)}{2}\right) = Q(3.5)$ [cite: 298]

4. **Calculate $Pr(|X+3| < 2)$:**

$$|X+3|[cite_start] < 2 \Rightarrow -2 < X+3 < 2 \Rightarrow -5 < X < -1 \quad \text{[cite: 303]}$$

$$Pr(-5 < X < -1) = F_X(-1) - F_X(-5) = \Phi(1) - \Phi(-1)$$

Using $\Phi(-1) = 1 - \Phi(1)$: [cite$_s tart$]$= \Phi(1) - (1 - \Phi(1)) = 2\Phi(1) - 1 = 1 - 2Q(1)$ [cite: 307]

5. **Calculate $Pr(|X-2| > 1)$:**

$$|X-2| > [cite_start]1 \Rightarrow X < 1 \text{ or } X > 3 \quad \text{[cite: 308]}$$

$$Pr = F_X(1) + Pr(X > 3)$$

$$F_X(1) = \Phi\left(\frac{1-(-3)}{2}\right) = \Phi(2) = 1 - Q(2)$$

$$Pr(X > 3) = Q\left(\frac{3-(-3)}{2}\right) = Q(3)$$

[cite$_s tart$]Total $= 1 - Q(2) + Q(3)$ [cite: 309]

5 Worked Example 2: CDF Analysis

Problem: The CDF of a random variable is given by:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

[cite$_s tart]FindtheMean, Variance, Skewness, andKurtosis[cite : 322, 323]$.

Solution:

1. **Find PDF:** [cite$_s tart]f_X(x) = \frac{d}{dx}F_X(x) = 2x, \quad 0 \leq x \leq 1$ [cite: 333]

2. **Calculate Mean (μ_X):** [cite$_s tart]\mu_X = \int_0^1 x(2x)dx = \int_0^1 2x^2 dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$ [cite: 333]

3. **Calculate Variance (σ_X^2):**

$$\sigma_X^2 = \int_0^1 x^2(2x)dx - \mu_X^2 = \int_0^1 2x^3 dx - \left(\frac{2}{3}\right)^2$$

[cite$_s tart$]$= \left[\frac{2x^4}{4}\right]_0^1 - \frac{4}{9} = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$ [cite: 340]

4. **Skewness and Kurtosis:** [cite$_s tart]C_s = \frac{E[(X-\mu)^3]}{\sigma^3} = -\frac{2\sqrt{2}}{5}$ (Distribution is left-skewed) [cite: 341, 343][$cite_s tart]C_k$ $\frac{E[(X-\mu)^4]}{\sigma^4} = \frac{12}{5}$ (Platykurtic) [cite: 341, 344]

6 Applications

Gaussian Random Variables are used to model:

[cite$_{s}tart$]

- **Thermal Noise:** Noise voltage in electronic circuits[cite: 314]. [cite$_{s}tart$]
- **Measurement Error:** Sensor readings where Measured Value = True Value + Gaussian noise[cite: 315]. [cite$_{s}tart$]
- **Jitter:** Packet delay variation in communication networks[cite: 316]. [cite$_{s}tart$]
- **Density Estimation:** Using sample statistics (histogram) to estimate the distribution parameters $\hat{\mu}$ and $\hat{\sigma}$[cite: 397, 404].