CSE400: Fundamentals of Probability in Computing

Lecture 4: Joint Probability and Conditional Probability

Scribe: Fagun Rathod - AU2440111

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1 Introduction to Probability Theory

1.1 Definitions and Notation

- **Experiment (E):** A procedure we perform that produces some result.
  Example: Tossing a coin five times (E_5).
- **Outcome (ξ):** A possible result of an experiment.
  Example: $\xi_1 = HHTHT$ is one possible outcome of E_5.
- **Event (Any Letter):** A certain set of outcomes of an experiment.
  Example: Event C (all outcomes consisting of an even number of heads) within experiment E_5.
- **Sample Space (S):** The collection or set of "all possible" distinct outcomes of an experiment. It is the universal set of outcomes and can be discrete, countably infinite, or continuous.
- **Mutually Exclusive:** Outcomes where you can get one result or another, but not both (e.g., heads or tails).
- **Collectively Exhaustive:** Outcomes where you cannot get anything other than the defined set (e.g., nothing other than heads or tails).
- **Probability:** A function of an event that produces a numerical quantity measuring the likelihood of that event.

1.2 Assumptions / Conditions

- **Classical Approach:** Probability is assigned to various outcomes and events based on a finite sample space.
- **Relative Frequency Approach:** To get an exact measure, the event must be repeatable an infinite number of times.

1.3 Main Results / Theorems (Axioms and Propositions)

- **Axiom 1:** For any event A, $0 \leq \Pr(A) \leq 1$.
- **Axiom 2:** If S is the sample space, $\Pr(S) = 1$.
- **Axiom 3:** If $A \cap B = \emptyset$, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

- **Proposition 2.1:** $\Pr(A^c) = 1 - \Pr(A)$.
- **Proposition 2.2:** If $A \subset B$, then $\Pr(A) \leq \Pr(B)$.
- **Proposition 2.3:** $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.

1.4 Worked Examples

Example 2.6 (Coin flipping): Compute $\Pr(H)$ and $\Pr(T)$.
Example 2.7 (Dice rolling): Compute Pr(even number is rolled).
Example 2.8 (Pair of dice): Compute $\Pr(A)$ where A is the event of the sum equaling five.
Relative Frequency Table: As n increases from 1000 to 10,000, n_A/n converges toward approximately 0.110.

2 Joint Probability

2.1 Definitions and Notation

- **Joint Probability:** The probability of the intersection of two or more events that are not mutually exclusive.
- **Notation:** Denoted as $\Pr(A, B)$ or $\Pr(A \cap B)$.
- **Multiple Events:** Denoted as $\Pr(A_1, A_2, \ldots, A_M)$.

2.2 Proofs / Derivations (Calculation Approaches)

1. **Step 1 (Classical):** Express events A and B in terms of atomic outcomes.
2. **Step 2 (Classical):** Identify atomic outcomes common to both events.
3. **Step 3 (Classical):** Calculate the probabilities of these common outcomes.
4. **Relative Frequency Formula:** $\Pr(A, B) = \lim_{n \to \infty} \frac{n_{A,B}}{n}$.

2.3 Worked Examples

Example 1: Card Deck
Let A = Red card, B = Number card (Ace included), C = Heart card.

- $\Pr(A) = 26/52 = 1/2$.
- $\Pr(B) = 40/52 = 10/13$.
- $\Pr(C) = 13/52 = 1/4$.
- $\Pr(A, B) = 20/52 = 5/13$.
- $\Pr(A, C) = 13/52 = 1/4$.
- $\Pr(B, C) = 10/52 = 5/26$.

Example 2: Costume Party
Alex has 4 tops (3 t-shirts, 1 cape) and 6 bottoms (2 pants, 4 boxers).

- Step 1: $\Pr(\text{Cape}) = 1/4$.
- Step 2: $\Pr(\text{Boxers}) = 4/6 = 2/3$.
- Step 3: $\Pr(\text{Cape}, \text{Boxers}) = \frac{1}{4} \times \frac{4}{6} = \frac{1}{6}$.

3 Conditional Probability

3.1 Definitions and Notation

Conditional Probability: The probability of event A occurring given that event B has already occurred.
Notation: $\Pr(A|B)$.

3.2 Main Results / Theorems

- **Definition Formula:** $\Pr(A|B) = \frac{\Pr(A,B)}{\Pr(B)}$ where $\Pr(B) > 0$.
- **Product Rule:** $\Pr(A, B) = \Pr(A|B)\Pr(B) = \Pr(B|A)\Pr(A)$.
- **Chain Rule:** $\Pr(A_1, \ldots, A_M) = \Pr(A_M|A_1, \ldots, A_{M-1}) \ldots \Pr(A_2|A_1)\Pr(A_1)$.

3.3 Worked Examples

Example 3: Cards Without Replacement
Select two cards; A is first card is Spade, B is second card is Spade.

- Initial: 13 Spades in 52 cards.
- After A: 12 Spades and 51 total cards remain.
- Result: $\Pr(B|A) = 12/51$.

Example 4: Poker Flush
Probability of a flush in Spades (5 cards):

$$\frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48}$$

Probability of any flush: $4 \times \Pr(\text{Spade Flush})$.

Example 5: Missing Key
Key in jacket ($\Pr(K) = 0.8$), Left Pocket ($\Pr(L) = 0.4$), Right Pocket ($\Pr(R) = 0.4$). Search of Left Pocket fails (L^c).

$$\Pr(R|L^c) = \frac{\Pr(R \cap L^c)}{\Pr(L^c)} = \frac{\Pr(R)}{1 - \Pr(L)} = \frac{0.4}{1 - 0.4} = \frac{0.4}{0.6} = \frac{2}{3}$$