Lecture 8: Gaussian RVs, Moments, and Engineering Modeling

Engineering Applications of Probability

2026

Topic: Moments of a Random Variable

Definitions

The n-th order **central moment** of a random variable X measures the deviation from its mean μ_X.

For **Continuous** RVs:

$$E[(X - \mu_X)^n] = \int_{-\infty}^{\infty} (x - \mu_X)^n f_X(x)\, dx$$

For **Discrete** RVs:

$$E[(X - \mu_X)^n] = \sum_k (x_k - \mu_X)^n p_X(x_k)$$

Key Moments and Interpretations

- **Zeroth Moment ($n = 0$):** $E[(X - \mu_X)^0] = E[1] = 1$.
- **First Moment ($n = 1$):** $E[X - \mu_X] = E[X] - \mu_X = 0$.
- **Second Moment ($n = 2$): Variance** (σ_X^2).

$$\sigma_X^2 = E[(X - \mu_X)^2] = E[X^2] - \mu_X^2$$

- **Third Moment ($n = 3$): Skewness**. Measures the symmetry of the PDF.

$$C_s = \frac{E[(X - \mu_X)^3]}{\sigma_X^3}$$

  - $C_s > 0$: Right Skewed (tail to the right).
  - $C_s < 0$: Left Skewed (tail to the left).

- **Fourth Moment ($n = 4$): Kurtosis**. Measures the "peakedness" or tail weight.

$$C_k = \frac{E[(X - \mu_X)^4]}{\sigma_X^4}$$

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Topic: Gaussian Random Variable

Definition

A Gaussian (Normal) random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ has the PDF:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Standard Forms and Special Functions

To calculate probabilities, we map X to the **Standard Normal** $Z \sim \mathcal{N}(0,1)$ using $Z = \frac{X-\mu}{\sigma}$.

- **Φ-Function (CDF):** Area under the left tail.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt$$

- **Q-Function (Tail Function):** Area under the right tail.

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt$$

- **Relationships:** $Q(x) = 1 - \Phi(x)$ and $\Phi(-x) = 1 - \Phi(x)$.

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Topic: Gaussian Probability Example

Problem: Given $f_X(x) = \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(x+3)^2}{8}\right)$, find $Pr(X > 4)$.

Solution:

1. **Identify Parameters:** Comparing to standard form, $\mu = -3$ and $2\sigma^2 = 8 \implies \sigma = 2$.
2. **Standardize:**

$$Pr(X > 4) = Q\left(\frac{4-(-3)}{2}\right) = Q\left(\frac{7}{2}\right) = Q(3.5)$$

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Topic: Engineering Modeling and Applications

Gaussian Modeling

In engineering, measurements are often modeled as:

$$X = \text{True Value} + \text{Gaussian Noise}$$

$$X = \mu + \sigma Z, \quad Z \sim \mathcal{N}(0,1)$$

Specific Applications

- **Electronics:** Modeling thermal noise voltage in circuits.
- **Networking:** Jitter (packet delay variation) analysis.
- **Image Processing:** Denoising camera sensor "grain."

Density Estimation

When parameters μ and σ are unknown, we estimate them from raw data samples:

- $\hat{\mu} = \frac{1}{n} \sum x_i$ (Sample Mean)
- $\hat{\sigma}^2 = \frac{1}{n-1} \sum (x_i - \hat{\mu})^2$ (Sample Variance)