Fall 2022

UNIVERSITY OF MARYLAND 1856

FINAL PROJECT

Control of Robotic Systems

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Course code:
ENPM667

Contents

1 Problem Statement

Consider a crane that moves along a one-dimensional track. It behaves as a frictionless cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m1 and m2, and the lengths of the cables are l1 and l2, respectively. The following figure depicts the crane and associated variables used throughout this project.

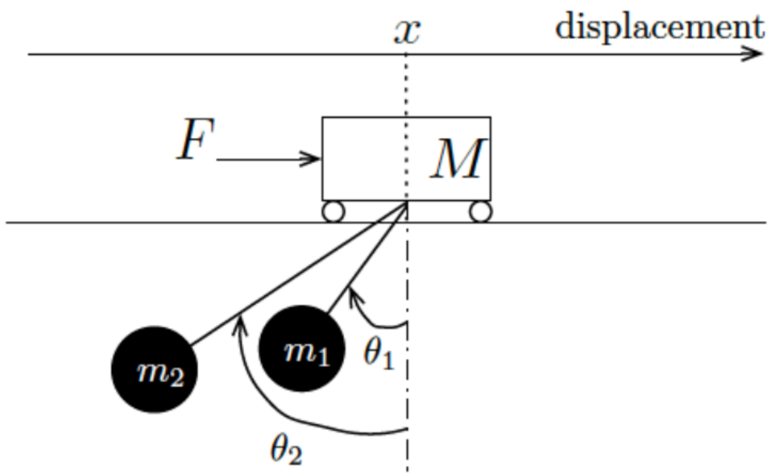

Figure 1: Dual pendulum on cart

A) Obtain the equations of motion for the system and the corresponding nonlinear state-space representation.
B) Obtain the linearized system around the equilibrium point specified by x = θ and θ1 = θ2 = 0. Write the state-space representation of the linearized system.
C) Obtain conditions on M,m1,m2,l1,l2 for which the linearized system is controllable.
D) Choose M=1000Kg, m1=m2=100Kg, l1=20m and l2=10m. Check that the system is controllable and obtain an LQR controller. Simulate the resulting response to initial conditions when the controller is applied to the linearized system and also to the original nonlinear system. Adjust the parameters of the LQR cost until you obtain a suitable response. Use Lyapunov's indirect method to certify the stability (locally or globally) of the closed-loop system.
E) Suppose that you can select the following output vectors:x(t),(θ1(t),θ2(t)),(x(t),θ2(t) or (x(t),θ1(t),θ2(t)). Determine for which output vectors the linearized system is observable.
F) Obtain your "best" Luenberger observer for each one of the output vectors for which the system is observable and simulate its response to initial conditions and unit step input. The simulation should be done for the observer applied to both the linearized system and the original nonlinear system.
G) Design an output feedback controller for your choice of the "smallest" output vector. Use the LQG method and apply the resulting output feedback controller to the original nonlinear system. Obtain your best design and illustrate its performance in simulation. How would you reconfigure your controller to asymptotically track a constant reference on x? Will your design reject constant force disturbances applied on the cart?

2 Equations of Motion

2.1 Velocity of each moving body

The position of the mass m1 and m2, in the x y plane can be given by the following equations,

P1= (x$-l_1\sin(\theta_1)$, $-l_1\cos(\theta_1)$)
P2= (x$-l_2\sin(\theta_2)$, $-l_1\cos(\theta_2)$)

Differentiating them with respect to time gives the velocity of that mass,

$$V_{1,x} = \frac{d}{dt}(x - l_1 sin\theta_1) = \dot{x} - l_1 cos\theta_1 \dot{\theta_1},(2.1)\ V_{1,y} = \frac{d}{dt}(l_1\cos\theta_1) = -l_1\sin\theta_1\dot{\theta_1}(2.2)$$
$$V_{2,x} = \frac{d}{dt}(x - l_2\sin\theta_2) = \dot{x}\text{-}l_2\cos\theta_2\dot{\theta_1}(2.3)$$
$$V_{2,y} = \frac{d}{dt}(l_2\cos\theta_2) = -l_1\sin\theta_2\dot{\theta_2}(2.4)$$

The velocity of the cart can be taken as

$$\text{V} = \dot{x}(2.5)$$

2.2 Kinetic and potential energy of the system

The kinetic energy of the system can be calculated by using velocities derived in an earlier section.

$$\text{K}=\tfrac{1}{2}MV^2 + \tfrac{1}{2}m_1(V_{1x}^2 + V_{1y}^2 + \tfrac{1}{2}m_2(V_{2x}^2 + V_{2y}^2)$$
$$\text{K}=\tfrac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \tfrac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2 l_2^2\dot{\theta}_2^2 - m_1 l_1\dot{x}\dot{\theta}_1 cos\theta_1 - m_2 l_2\dot{x}\dot{\theta}_2 cos\theta_2,(2.6)$$

Since the cart is moving on a flat surface, its potential energy will not change. We can find the potential energy of hanging masses by considering a flat plane as a reference. Equations of potential energy can be written as follows,

$$U = -m_1 l_1 cos\theta_1 g - m_2 l_2 cos\theta_2 g,(2.7)$$

2.3 Lagrangian

Lagrangian for this system can be defined as

$$L = K - U$$

The following three equations will give the equations of motion.

$$\frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = F$$
$$\frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{\theta}_1}) - \frac{\partial L}{\partial \theta_1} = 0$$
$$\frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{\theta}_2}) - \frac{\partial L}{\partial \theta_2} = 0$$

We will get the following equations after partial differentiation,

$$\ddot{x} = \frac{1}{M + m_1 + m_2}[m_1 l_1\ddot{\theta}_1 cos\theta_1 + m_2 l_2\ddot{\theta}_2 cos\theta_2 - m_1 l_1\dot{\theta}_1^2 sin\theta_1 - m_2 l_2\dot{\theta}_2^2 sin\theta_2 + F],(2.8)$$

$$\ddot{\theta}_1 = \frac{\ddot{x}cos\theta_1}{l_1} - \frac{g sin\theta_1}{l_1},(2.9)$$
$$\ddot{\theta}_2 = \frac{\ddot{x}cos\theta_2}{l_2} - \frac{g sin\theta_2}{l_2},(2.10)$$

Taking $x, \dot{x}, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2$ as system states, we can write state space form of the nonlinear system as follows,

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{-m_1 g sin\theta_1 cos\theta_2 - m_1 g sin\theta_2 cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 sin\theta_2 + F}{M + m_1 + m_2 - m_1 cos^2\theta_1 - m_2 cos^2\theta_2} \\ \dot{\theta}_1 \\ \frac{-m_1 g sin\theta_1 cos\theta_2 - m_1 g sin\theta_2 cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 sin\theta_2 + F}{(M + m_1 + m_2 - m_1 cos^2\theta_1 - m_2 cos^2\theta_2) l_1} - \frac{g sin\theta_1}{l_1} \\ \dot{\theta}_2 \\ \frac{-m_1 g sin\theta_1 cos\theta_2 - m_1 g sin\theta_2 cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 sin\theta_2 + F}{(M + m_1 + m_2 - m_1 cos^2\theta_1 - m_2 cos^2\theta_2) l_2} - \frac{g sin\theta_2}{l_2} \end{bmatrix}, (2.11)$$

3 Linear State Space Representation of System

3.1 Linearization of system

We can achieve linearization of the system from the values we have obtained by considering the conditions of the system as:

$$sin(\theta) \approx \theta$$

$$cos(\theta) \approx 1$$

$$sin^2(\theta) \approx 0$$

$$cos^2(\theta) \approx 1$$

After taking this approximation, we will get the following equations,

$$\ddot{x} = \frac{1}{M}(-m_1 g \theta_1 - m_2 g \theta_2 + F)$$

$$\ddot{\theta}_1 = \frac{1}{M l_1}(-m_1 g \theta_1 - m_2 g \theta_2 - M g \theta_1 + F), (3.1)$$

$$\ddot{\theta}_1 = \frac{1}{M l_2}(-m_1 g \theta_1 - m_2 g \theta_2 - M g \theta_2 + F)$$

3.2 Linear State Space Representation

The system of equations represented in Eq 3.1 can be represented in a state space form. Taking $x, \dot{x}, \theta_1, \dot{\theta}_1, \theta_2 and \dot{\theta}_2$ as state variables, state space representation can be written as shown below,

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-g m_1}{M} & 0 & \frac{-g m_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(-M-m_1)}{M l_1} & 0 & \frac{-g m_2}{M l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-g m_1}{M l_2} & 0 & \frac{-g(-M-m_2)}{M l_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M l_1} \\ 0 \\ \frac{1}{M l_2} \end{bmatrix} u, (3.2)$$

$$y = CX + DU, (3.3)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D = 0$$

Which is similar to $\dot{X} = AX + Bu$. Input force on cart F is represented as u.

3.3 Controllability

We can check the controllability of the derived system by checking the rank of the controllability matrix. The controllability matrix can be defined as below,

$$R = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix}$$

The determinant of this matrix should not be equal to zero for the system to be controllable,

$$Det(R) = -\frac{g^6 l_1^2 - 2g^{6l_1l_2} + g^6 l_2^2}{M^6 l_1^6 l_2^6} = -\frac{g^6(l_1 - l_2)^2}{M^6 l_1^6 l_2^6}$$

Which gives us the condition that $l1 \neq 0, l2 \neq 0$ and $l1 \neq l2$.

4 Part-D LQR

Putting M = 1000K g , $m_1 = m_2$ = 100K g , l_1 = 20m and l_2 = 10m in Eq 3.2 , we get

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.98 & 0 & 0.98 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.539 & 0 & 0.049 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.098 & 0 & 1.078 & 0 \end{bmatrix}, (4.1)$$

$$B = \begin{bmatrix} 0 \\ 0.001 \\ 0 \\ 0.00005 \\ 0 \\ 0.0001 \end{bmatrix}, (4.2)$$

The controllability condition needs to be checked to verify if the system is controllable or not. Using the code given below, we can check the controllability condition.
The code above, when run, gives the rank of matrix 6, which shows that the matrix is of full rank and thus the system is controllable.
Different values of Q and R shall be taken to control the system in an optimum way and using the graphs we can decide which value of Q and R gives good results and stabilizes the system as soon as possible. Q can be decided on how much approximate time the system should take to stabilize the particular state and what is the error band within which it should work.

$$Q_i = \frac{1}{(time) * (error)^2}$$

After several experiments, we have finalized the values of Q and R,

$$Q = \begin{bmatrix} \frac{1}{12*1^2} & 0 & 10 & 0 & 0 & 0 \\ 0 & \frac{1}{12*0.1^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8*0.02^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{12*0.01^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8*0.02^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{12*0.01^2} \end{bmatrix}, (4.3)$$

The LQR controller design uses a cost function which then gives rise to the Riccati equation on minimization. The cost function is given by:

$$J = \int_0^\infty (x^T Q x + u^T R u) dt$$

And the Riccatti equation is written as:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

This is used as the general equation to obtain the value of P, which then gives us the K, the controller gain in u = -Kx. This is done using:

$$-K = R^{-1}BP$$

The simulation results are shown below:-
Using the LQR function in MATLAB, we obtain the value of K. We also consider the initial conditions (in degrees) to be as:

$$X = \begin{bmatrix} 0 & 0 & 15 & 0 & 20 & 0 \end{bmatrix}$$

(a) Response to initial conditions:

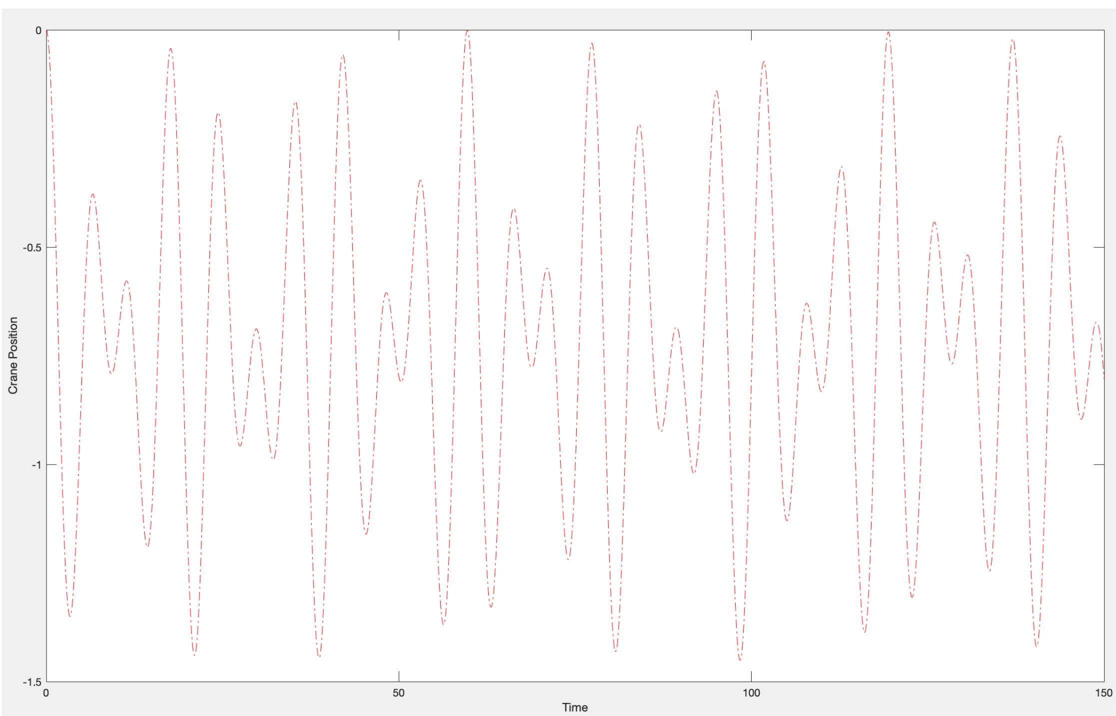

Figure 2: Crane Position

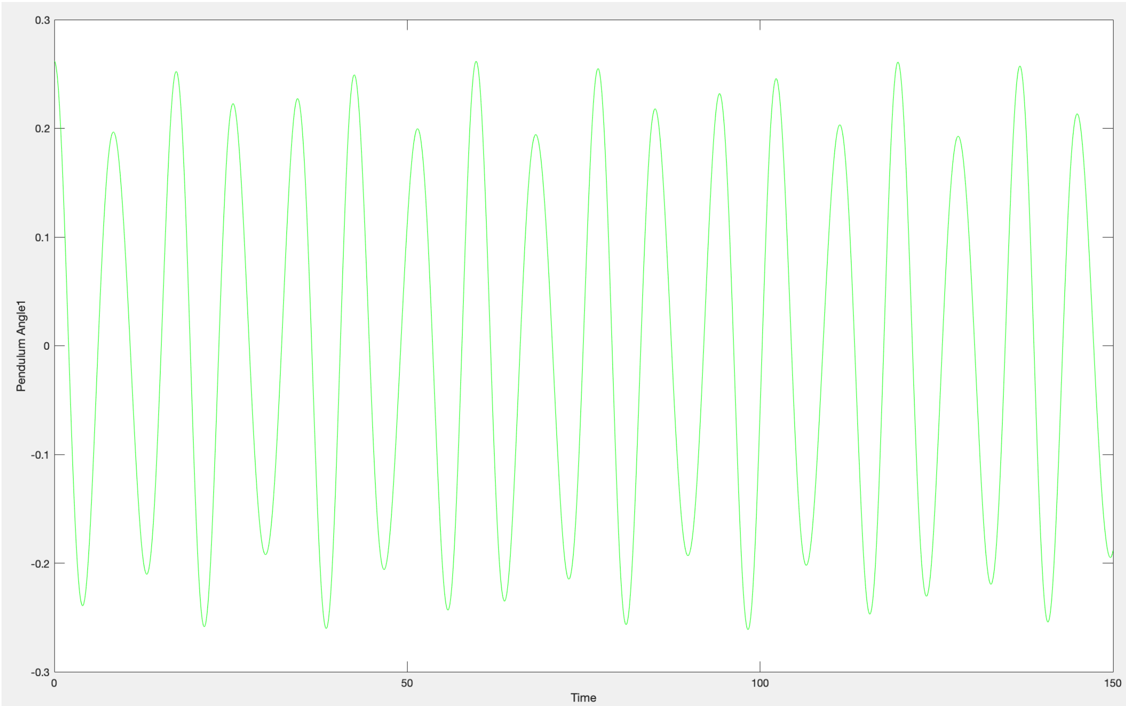

Figure 3: Pendulum Angle 1

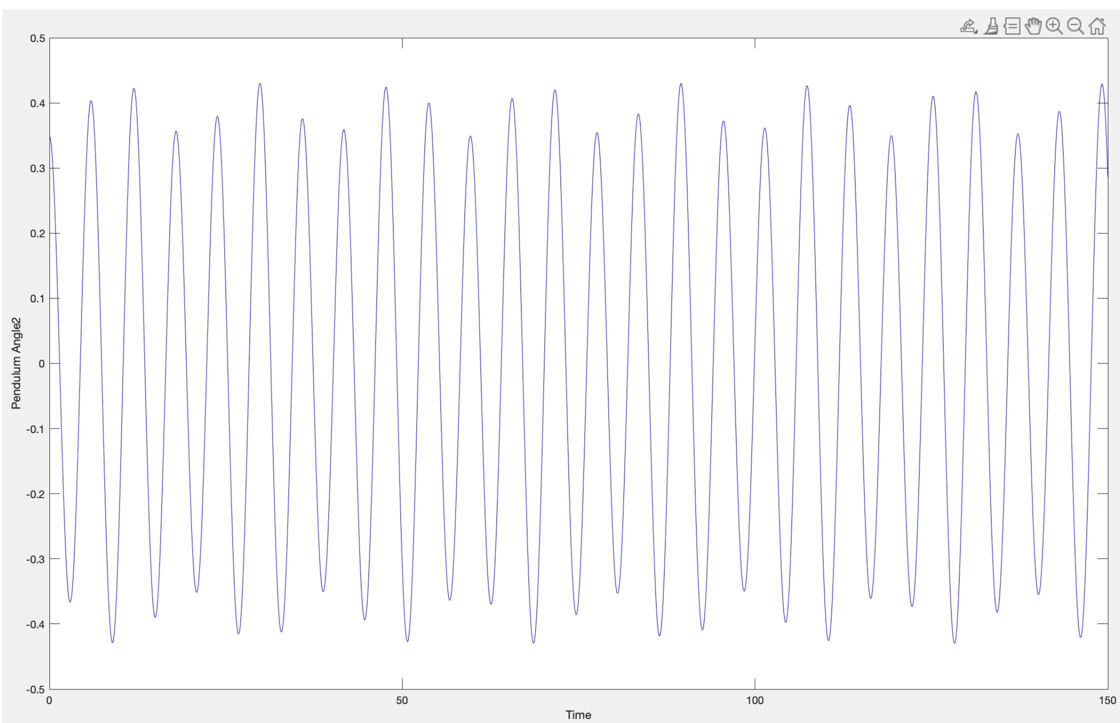

Figure 4: Pendulum Angle 2

4.1 Lyapunov's Indirect Stability

When we see the eigenvalues of $(A-BK)$, it should be in the left half plane for the system to be stable. Real parts of all the eigenvalues are negative, and thus with the help of Lyapunov's indirect stability criterion, we can say that the system is stable. The stability of the obtained system is checked using Lyapunov's indirect method. This gives information about the stability of the system using the eigenvalues for our given A matrix. It states that if a system has N eigenvalues and all the values have no positive real part, then the system is locally stable. Similarly, if any of the eigenvalues have a positive real part, we get that the system is unstable. In the case of our project, we have calculated the eigenvalues of A to be as follows:

$$\begin{bmatrix} -0.3511+1.0782i \\ -0.3511-1.0782i \\ -0.2141+0.0797i \\ -0.2141-0.0797i \\ -0.0579+0.7236i \\ -0.0579-0.7236i \end{bmatrix}$$

Since they do not have any positive real part and only negative real part, we can say that the system is locally stable.
(b) Response to LQR controller:

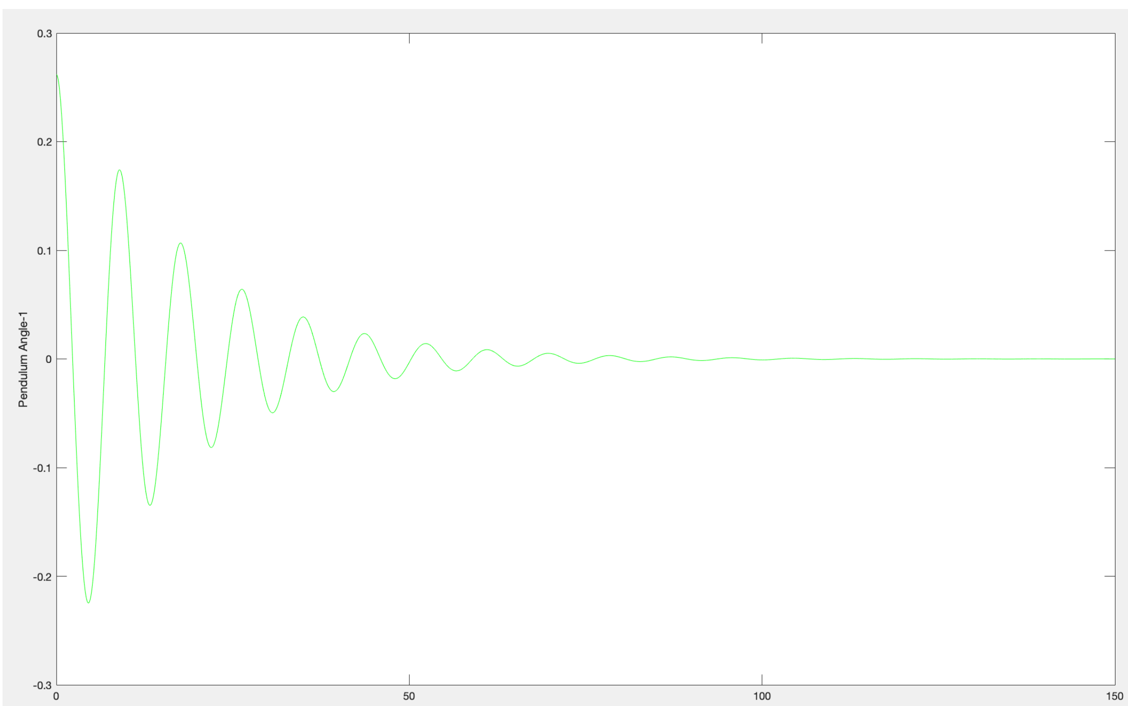

Figure 5: Pendulum Angle 1

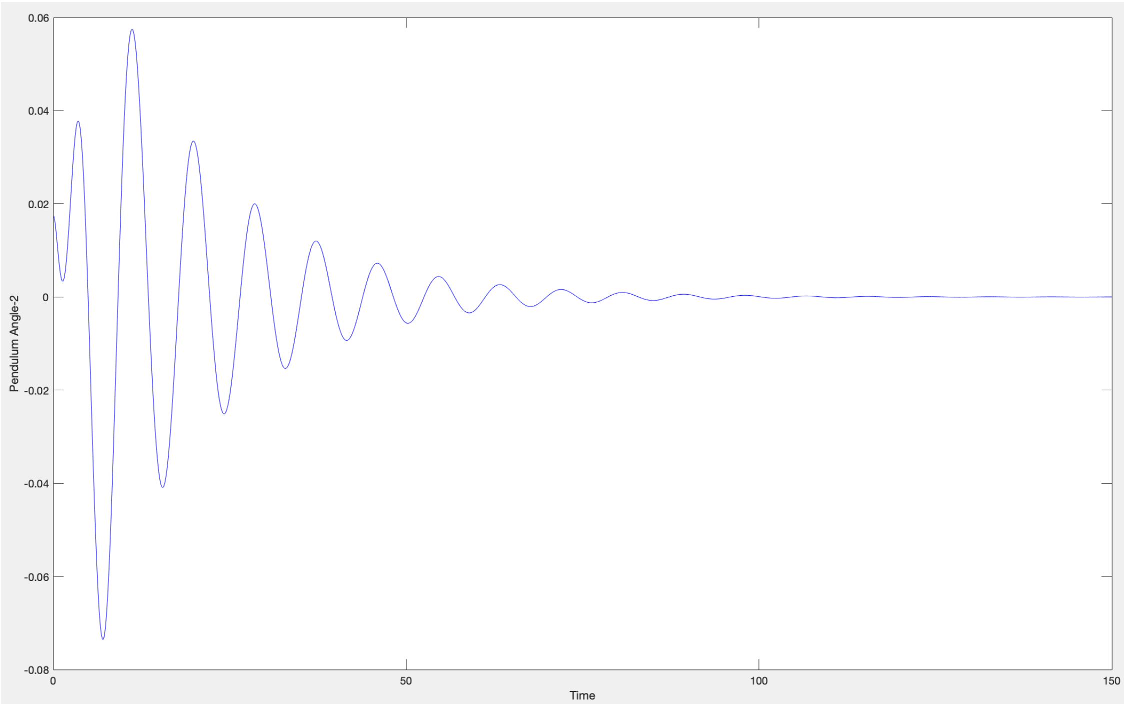

Figure 6: Pendulum Angle 2

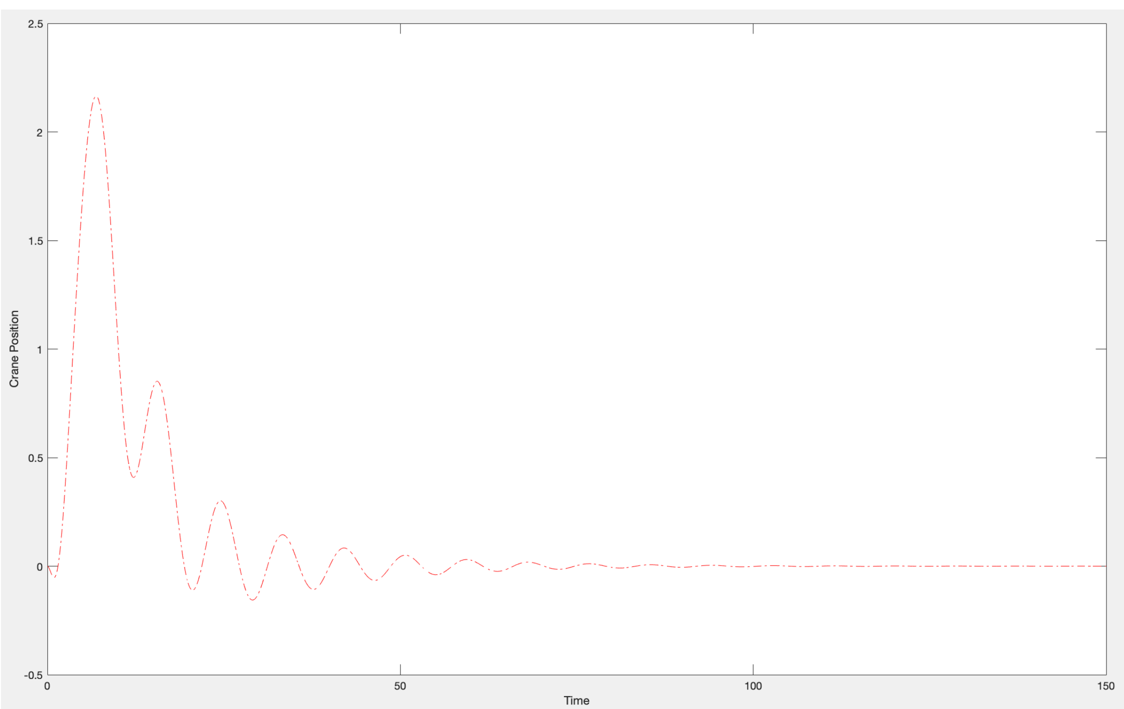

Figure 7: Crane Position

5 Part-E Observability

We have four sets of output values that need to be checked for the observability condition.
1.$x(t)$
2.$\theta_1(t), \theta_2(t)$
3.$x(t), \theta_2(t)$
4.$x(t), \theta_1(t), \theta_2(t)$
We will check the rank of the following matrix and it should be 6, for the system to be observable with those states.

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix} \; and \; rank(O) = 6$$

5.1 Observability for $x(t)$

We will take,

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

When we check the rank condition with code, we get the rank of O is 6 which means that we can observe the full system with the x(t) as our system output.

5.2 Observability for $\theta_1(t)$ and $\theta_2(t)$

We will take,

$$c = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

When we check the rank condition with code, we get the rank of O is 4, which means that we can not observe the full system with the $\theta_1(t)$ and $\theta_2(t)$ as our system output.

5.3 Observability for $x(t)$ and $\theta_2(t)$

We will take,

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

5.4 Observability for $x(t)$, $\theta_1(t)$ and $\theta_2(t)$

We will take,

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

When we check the rank condition with code module, we get the rank of O is 6, which means that we can observe the full system with the x(t), $\theta_1(t)$ and $\theta_2(t)$ as our system output.

Following the checking of the controllability and stability of the system that we have designed, we now obtain the values and simulation results of observability for the given scenarios.
The simulation results for the observability were obtained using MATLAB and are mentioned below for the various output vectors of x.

```
**************************************************************
clc
clear all
syms x1 x1_dot t1 t1_dot t2 t2_dot
%This is the final project submission for the subject ENPM-667 and group
%members are Rishikesh Jadhav(119256534) and Nishant Pandey(119247556)
M=1000;
m1=100;m2=100;
l1=20;l2=10;
g=10;
%% State Space form of the System X = A*x +B*u
A=[0 1 0 0 0 0
0 0 (-g*m1)/M 0 (-g*m2)/M 0
0 0 0 1 0 0
0 0 (-g*(m1+M))/(M*l1) 0 (-g*m2)/(M*l1) 0
0 0 0 0 0 1
0 0 (-g*m1)/(M*l2) 0 (-g*(m2+M))/(M*l2) 0 ];
Order_of_A=6;
%% Checks for Different Cases
%Case 1 for x(t)
disp('Observability check for Case1-x(t)')
C1=[1 0 0 0 0 0];
%Check1 = rank([C1;C1 * A;C1 * (A ∧ 2);C1 * (A ∧ 3);C1 * (A ∧ 4);C1 * (A ∧ 5)])
Check1=Observability(C1,A)
disp('Observability check for Case1-x(t)')
if Check1==Order_of_A
disp('System is Observable for x(t)')
else
disp('System is not Observable for the expected Output')
end
%Case 2 for (t1,t2)
C2=[001000000010];

Check2=Observability(C2,A)

disp('Observability check for Case2-(t1,t2)')

if Check2==Order_of_A
disp('System is Observable for (t1,t2)')
else
disp('System is not Observable for the expected Output')
end

%Case 3 for (x,t2)

C3=[100000000010];
```

```
Check3=Observability(C3,A)

disp('Observability check for Case3-(x,t2)')

if Check3==Order_of_A
disp('System is Observable for (x,t2)')
else
disp('System is not Observable for the expected Output')
end

%Case 4 for (x,t1,t2)

C4=[100000001000000010];

Check4=Observability(C4,A)

disp('Observability check for Case4-(x,t1,t2)')

if Check4==Order_of_A
disp('System is Observable for (x,t1,t2)')
else
disp('System is not Observable for the expected Output')
end
```

****

The Output Of the Following code for the Observability check is

```
Observability check for Case1-x(t)

Check1 =

     6

Observability check for Case1-x(t)
System is Observable for x(t)

Check2 =

     4

Observability check for Case2-(t1,t2)
System is not Observable for the expected Output

Check3 =

     6

Observability check for Case3-(x,t2)
System is Observable for (x,t2)

Check4 =

     6

Observability check for Case4-(x,t1,t2)
System is Observable for (x,t1,t2)
```

Figure 8: Observability Check Output

5.5 Part-F Luenberger Observer

The Luenberger Observer is a state observer that provides estimates and calculates the error and corrects it with respect to the output values of the state.
It is given by the following general equation:

$$\dot{X} = AX + BU + LC(X - \widehat{X})$$

We now let Xe = X   and rearrange to get the final equation for the error to be:

$$\widehat{X}_e = (A - LC)(X_e)$$

Now we develop the 'best' possible Luenberger observer by simulating it using both the original nonlinear system as well as the linearized version. Conditions for the Luenberger observer are:
For the (A^T, C^T) to be stabilizable, we need them to be detectable and similarly, for controllability, they need to be observable.
Also, the matrix $(A - LC)^T$ must be stable for A-LC to be stable.
The MATLAB and Simulink results for the response to Luenberger Observer are:-

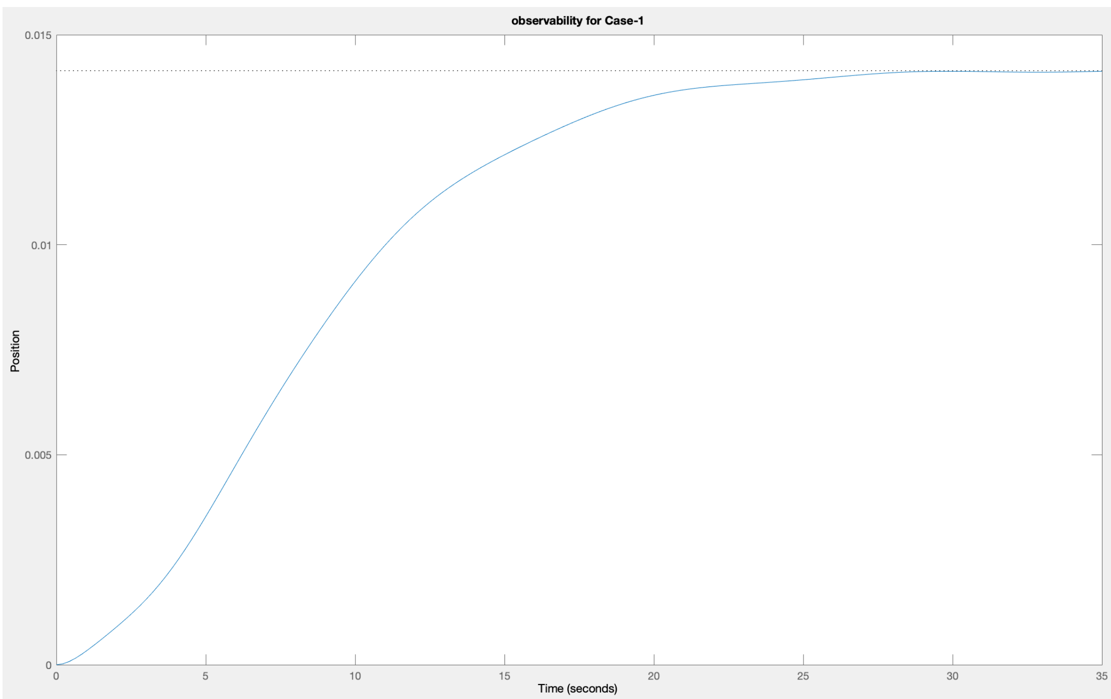

Figure 9: Observability for Case-1 Output

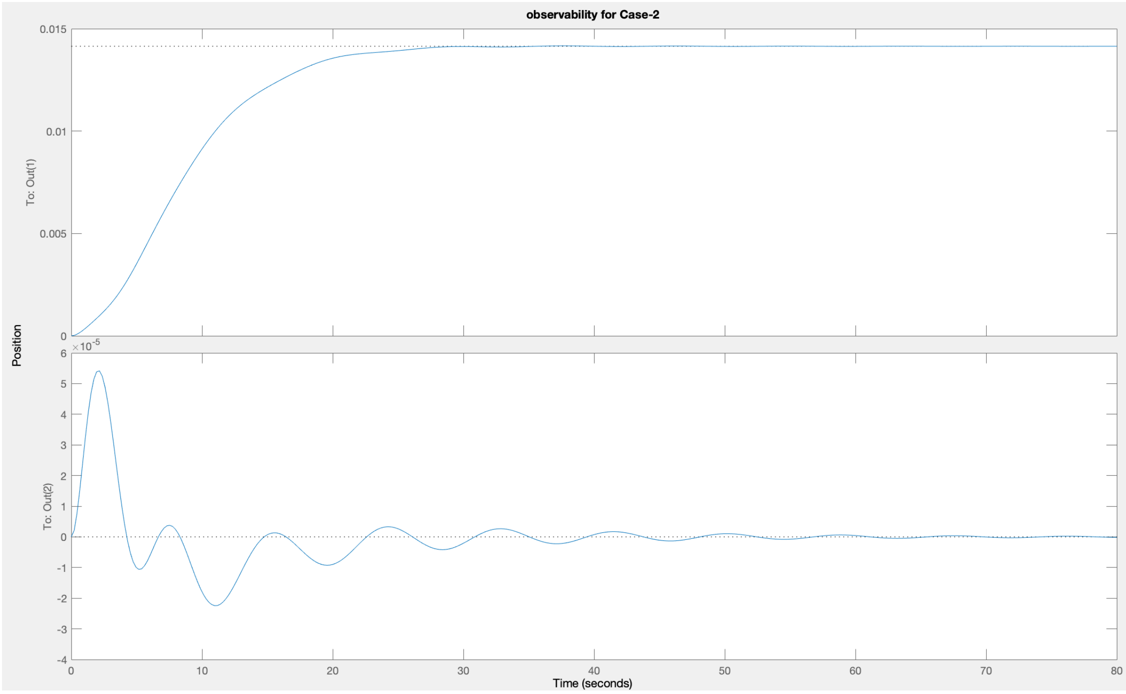

Figure 10: Observability for Case-2 Output

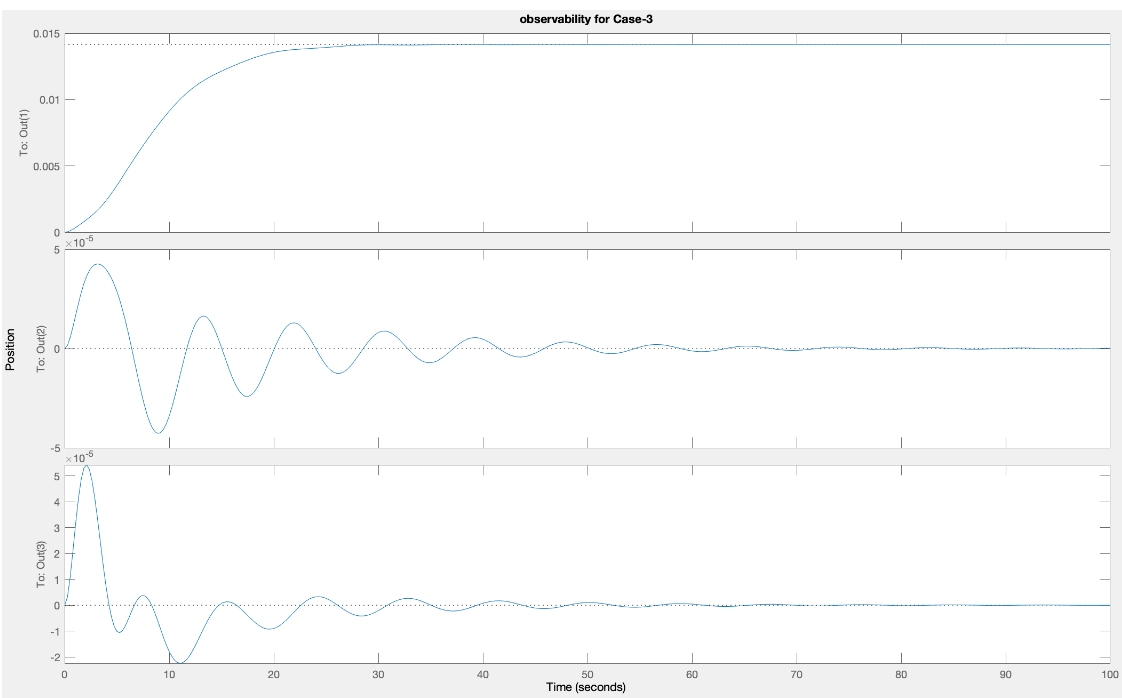

Figure 11: Observability for Case-3 Output

5.6 Part-G LQG Control

The LQG feedback controller is defined as the Linear Quadratic Gaussian which deals with a linear system driven by 'Gaussian white noise. It is simply a combination of the LQR (Linear Quadratic Regulator) controller and the Kalman filter. LQG controller is used for both linear time-invariant systems and linear time-varying systems. the LQG controller itself is a dynamic system that controls systems that have the same dimension in their state. The state space representation for the LQG controller is given as

$$\dot{\widehat{X}} = AX + BU + Bw$$

$$Y = CX + v$$

Where w is the process noise and v is the measurement noise. The response of the LQG system to the step source that is obtained from Simulink is:

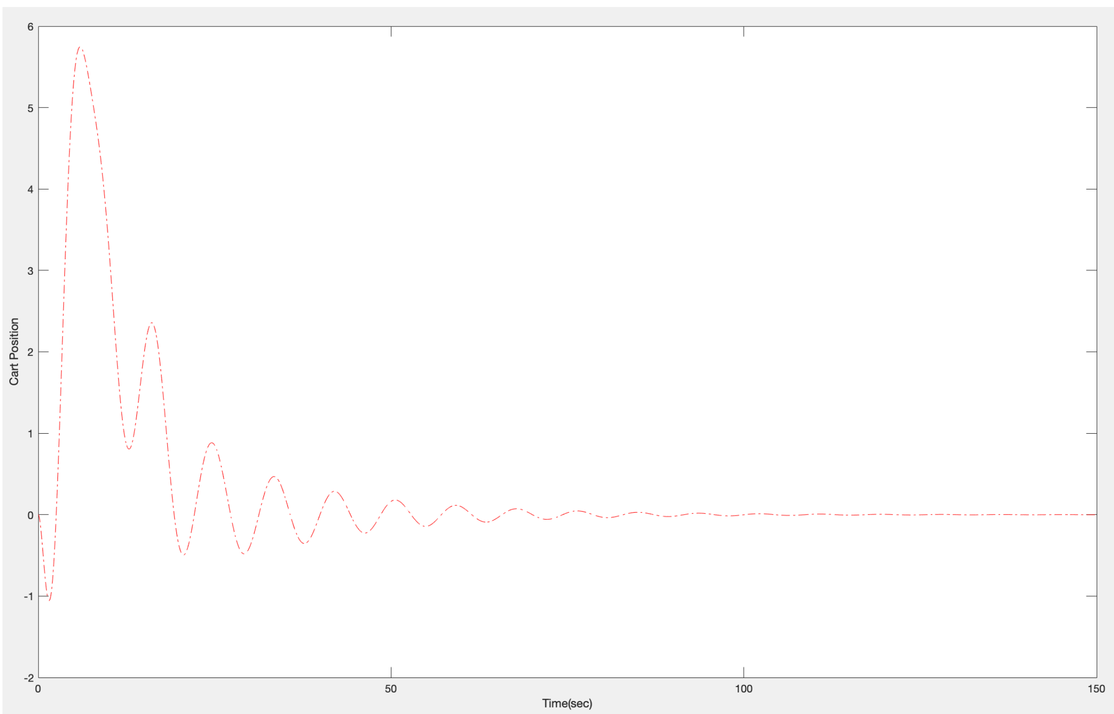

Figure 12: LQG response Output