(1) $S.E = \|x_1-\mu\|_2^2 + \|x_2-\mu\|_2^2 + \ldots + \|x_n-\mu\|_2^2$

To minimize S.E, differentiate S.E w.r.t μ

$$\frac{d(SE)}{d(\mu)} = \frac{d\left(\sum_{i=1}^{d}(x_i-\mu)^2\right)}{d(\mu)} = -2(x_1-\mu) + -2(x_2-\mu) + \ldots -2(x_n-\mu) = 0$$

$$(x_1-\mu)+(x_2-\mu)+\ldots(x_n-\mu)$$

$$x_1+x_2+x_3+\ldots x_n = n\mu$$

$$\mu = \frac{x_1+x_2+\ldots x_n}{n}$$

$$\boxed{\mu = \bar{x}}$$

(2) <u>Proove the following</u>

a) $\|x\|_2 \le \|x\|_1 \le \sqrt{d}\,\|x\|_2$

$$\|x\|_2 = \sqrt{\sum x_i^2} \le \sqrt{\left(\sum |x_i|\right)^2}$$

$$\|x\|_2 \le \sum |x_i|$$

$$\therefore \boxed{\|x\|_2 \le \|x\|_1}$$

From Cauchy Schwarz Inequality,

$$\left|\sum_1^n a_i b_i\right|^2 \leq \sum_1^n |a_i|^2 \cdot \sum_1^n |b_i|^2$$

$$\|x\|_1^2 = \left|\sum |x_i|\right|^2 = \left|\sum x_i \cdot 1\right|^2$$

$$\left|\sum |x_i| \cdot 1\right|^2 \leq \sum |x_i|^2 \cdot \sum 1^2$$

Take sq root on Both sides

$$\sum |x_i| \leq \sum |x_i|^2 \cdot \sqrt{d}$$

$$\therefore \boxed{\|x\|_1 \leq \sqrt{d} \cdot \|x\|_2}$$

Thus, $\boxed{\|x\|_2 \leq \|x\|_1 \leq \sqrt{d}\, \|x\|_2}$

b) $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{d}\, \|x\|_\infty$

$$\|x\|_\infty = \max |x_i|$$

$$= \max \sqrt{x_i^2} \leq \sqrt{\sum x_i^2}$$

$$\therefore \boxed{\|x\|_\infty \leq \|x\|_2}$$

Consider, $x_1^2 + x_2^2 + \dots + x_d^2 \leq \underbrace{\max(x)^2 + \max(x)^2 + \dots}_{d \text{ times}}$

$$\sum_1^d x_i^2 \leq d \cdot \max(x^2)$$

$$\|x\|_2^2 \leq d\, \|x\|_\infty^2$$

$$\therefore \boxed{\|x\| \leq \sqrt{d}\ \|x\|_\infty}$$

Thus, $\boxed{\|x\|_\infty \leq \|x\|_2 \leq \sqrt{d}\ \|x\|_\infty}$

c) $\|x\|_\infty \leq \|x\|_1 \leq d\|x\|_\infty$

We have already proven that $\|x\|_\infty \leq \|x\|_2$ and
$\|x\|_2 \leq \|x\|_1$

$\boxed{\|x\|_\infty \leq \|x\|_1}$ By using

We can say that

$$|x_1| + |x_2| + \dots + |x_d| \leq \underbrace{|\max(x_i)| + \dots + |\max(x_i)|}_{d \text{ terms}}$$

$$\sum |x_i| \leq d \cdot \max(x_i)$$

$$\boxed{\|x\|_1 \leq d\ \|x\|_\infty}$$

Thus, $\|x\|_\infty \leq \|x\|_1 \leq d \cdot \|x\|_\infty$