Dynamic Programming

- Planning by dynamic programming assumes full knowledge of the MDP

For Prediction / Evaluation
* Input : MDP $\langle S, A, P, R, \gamma \rangle$ and policy π
* Output : Value function v_π

For Control
* Input : MDP $\langle S, A, P, R, \gamma \rangle$
* Output : Optimal value function v_* and optimal policy π_*

& NO	Problem	Bellman Equation	Algorithm
1.	Prediction	Bellman Equation Expectation	Iterative Policy Evaluation
2.	Control	Bellman Expectation + Greedy Policy Improvement	Policy Iteration
3.	Control	Bellman Optimality Equation	Value Iteration

Bellman Expectation Equation

$$V_{k+1}(s) = E_\pi \left[R_{t+1} + \gamma V_k(S_{t+1}) \middle| S_t = s \right]$$

Example

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$V_1(6)$

Example : Grid World

1 G	2	3	4
5	6 R	7	8
9	10	11	12
13	14	15	16 G

Action ↔ ↕

Reward is -1 for all transition

A bot is required to transverse a grid of 4×4 dimensions to reach its goal (1 or 16) Each step is associated with a reward of -1. There are 2 terminal states here : 1 and 16 and 14 non-terminal states given by [2, 3 ... 16] Consider a random policy for which at every state, the probability of every action {up, down, left, right} is equal to 0.25.

Initializing V_0 for the Random Policy to all 0s

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

Let's calculate V_2 for all the states of 6:

$$V_1(6) = \sum_{a \in \{u,d,l,r\}} \pi(a|6) \sum_{s',r'} p(s',r|6,a)\left[r + \gamma V_0(s')\right]$$

$$= \sum_{a \in \{u,d,l,r\}} \underbrace{\pi(a|6)}_{0.25} \sum_{s'} p(s'|6,a)\left[\underbrace{r}_{=-1} + \underbrace{\gamma V_0(s')}_{=0 \,\forall s'}\right]$$

$$= 0.25 \times \{-p(2|6,u) - p(10|6,d) - p(5|6,l) - p(7|6,r)\}$$

$$= 0.25 * \{ -1 - 1 - 1 - 1 \}$$

$$= -1$$

$$\Rightarrow V_1(6) = -1$$

Similarly for all non-terminal states

$$\boxed{V_1(s) = -1}$$

For terminal states $p(s'|s,a) = 0$

Hence $V_k(1) = V_k(16) = 0$ for all k.

So V_1 for the Random policy is given by

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

For $V_2(s)$, Assume $\gamma = 1$

$$V_2(6) = \sum_{a \in \{u,d,l,r\}} \underbrace{\pi(a|6)}_{=0.25 \ \forall a} \sum_{s'} p(s'|6,a)\left[r + \gamma V_1(s')\right]$$

$r = -1$; $V_1(s') = -1$ if $s' \neq$ terminal, 0 if $s' =$ terminal

$$= 0.25 * \{ p(2|6,u)[-1-\gamma] + p(10|6,d)[-1-\gamma] + p(5|6,l)[-1-\gamma] + p(7|6,r)[-1-\gamma] \}$$

$= 0.25 * \{-2-2-2-2\}$

$= -2$

1 (T)	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16 (T)

All the states marked in Red in above are identical for the purpose of calculating the value function.
Hence for all these states

$$\boxed{V_2(s) = -2}$$

For all the remaining states i.e 2, 5, 12 and 15 V_2 can be calculated as follows:

$$V_2(2) = \sum_{a \in \{u,d,l,r\}} \underbrace{\pi(a|2)}_{=0.25} \sum_{s'} p(s'|2,a)\left[\underbrace{r}_{=-1} + \gamma \underbrace{V_1(s')}_{\substack{=-1 \\ =0}}\right]$$

$$= 0.25 * \{ p(2|2,u)[-1-\gamma] + p(6|2,d)[-1-\gamma] + p(1|2,l)[-1-\gamma*0] - p(3|2,r)[-1-\gamma] \}$$

$$= 0.25 * \{-2-2-1-2\}$$

$$= -1.75$$

$$\Rightarrow \boxed{V_2(2) = -1.75}$$

V_2 for the random policy

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

If we repeat this step several times we get V_∞

∴ Note : check PPT for the solution & diagrams.

1. Problem.

Frozen lake Solution

1. Using Value Iteration

given state Representation

S	F	F	F
F	H	F	H
F	F	F	H
H	F	F	G

S = State
F = Frozen
H = Hole
G = Goal
$\gamma = 0.9$

we get V_{π}

lution.

Bellman Equation for Value Iteration

Value function

$$V(s) = \max_{a} \sum_{s'} p(s'|s,a)\left[R(s,a,s') + \gamma V(s')\right]$$

given

G = 1.0 (terminal State)

H = 0 (game Over)

other state = 0

Initial Value table

S	0	0	0
0	H	0	H
0	0	0	H
H	0	0	G

Formula

~~Step 1~~: for Iteration Value Update

$$V(s) = \max_{a} \sum_{s'} p(s'|s,a)\left[R(s,a,s') + \gamma V(s')\right]$$

Initial $V(s) = 0$ $V(G) = 1$

Iteration 1:

$G = 0$

State adjacent to the goal update

1. (3,2) Best move Right to (3,3)

$$V_{(3,2)} = 0.9 \times 1.0 = 0.9$$

2. (2,3) Best move Down to (3,3)

$$V_{(2,3)} = 0.9 \times 1.0 = 0.9$$

3. (3,1) Best move Right to (3,2)

$$V_{(3,1)} = 0.9 \times 0.9 = 0.98$$

4. (2,2) Best move Right (2,3)

$$V_{(2,2)} = 0.9 \times 0.9 = 0.98$$

5. (2,1) Best move Right (2,2)

$$V_{(2,1)} = 0.9 \times 0.98 = 0.97$$

6. (1,2) Best move Down to (2,2)

$$V_{(1,2)} = 0.9 \times 0.98 = 0.97$$

Iteration 2:

Repeating the same steps, value propagate backward

1. (1,1) (avoiding hole):

$$V_{(1,0)} = 0.9 \times 0.97 = 0.96$$

2. (0,2)

$$V_{(0,2)} = 0.9 \times 0.97 = 0.96$$

3. (0,1)

$$V_{(0,1)} = 0.9 \times 0.96 = 0.95$$

4. (0,0) (Start state)

$$V_{(0,0)} = 0.9 \times 0.95 = 0.94$$

S	0.94	0.95	0.96	~~0.97~~
0.96	H	0.97	H	0.
0.97	0.98	0.99	H	
H	0.98	0.99		

Extracting the Optimal Policy

(0,0) → Right (Towards higher value)

0,1 → Right

0,2 → Right

0,3 → Down

1,3 → Down

2,3 → Right

3,2 → Right

3,3 → goal

final O/P

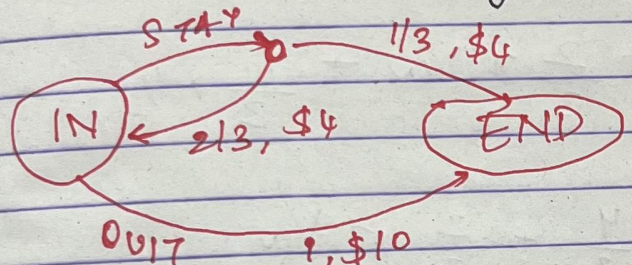
→	→	→	↓
↓	H	→	H
↓	→	→	H
H	↓	→	G

2. Problem

At any time the game is specified by two states: IN and END

When the game in the state IN, the agent can choose the action from the set $A = \{STAY, QUIT\}$

If QUIT is selected, then the game will go to state END and the agent recive a reward $10 with a probability of 1. If STAY is selected, the game will stay at IN state & the agent receive a reward $4 with a probability of 2/3 or transition to state END. and the agent recive a reward $4 with a probability of 1/3



STATE Transition Probabilities.

Action	Next State	Prob	Reward
QUIT	END	1.0	10
STAY	IN	2/3	4
STAY	END	1/3	4

$$V(s) = \max_a \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma V(s')\right]$$

γ = between 0 & 1 ~~= 0.5~~

= 0.9

$V_{(END)} = 0$

~~State = IN. given that the agent~~

For QUIT:

$$V_{quit}(IN) = 10 + 0.9 = 10$$

For STAY:

$$V_{STAY}(IN) = \left(\frac{2}{3} \times (4 + \gamma V_{(IN)})\right) + \left(\frac{1}{3} \times (4 + \gamma V_{(END)})\right)$$

$V_{END} = 0$

$$= \frac{2}{3} \times 4 + \frac{2}{3}\gamma V_{IN} + \frac{1}{3} \times 4$$

$$= \frac{8}{3} + \frac{4}{3} + \frac{2}{3}\gamma V_{IN}$$

$$V_{STAY}(IN) = 4 + \frac{2}{3}\gamma V_{IN}$$

∴ the agent selects STAY with probability 0.5 and QUIT with 0.5, we take weighted sum

$$V(IN) = 0.5 \times \left(4 + \frac{2}{3}\gamma V_{IN}\right) + 0.5 \times 10$$

$$V_{IN} = 0.5 \times 0.4 + 0.5 \times \frac{2}{3}\gamma V_{IN} + 5$$

$$= 2 + \frac{1}{3}\gamma V_{IN} + 5$$

$$= 7 + \frac{1}{3}\gamma V_{IN} + 5$$

$$V_{IN} - \frac{1}{3}\gamma V_{IN} = 7$$

$$\left(1 - \frac{1}{3}\gamma\right) V_{IN} = 7$$

$$V_{IN} = \frac{7}{1 - \frac{1}{3}\gamma}$$

$\boxed{\gamma = 0.9}$

$$V_{IN} = \frac{7}{1 - \frac{1}{3} \times 0.9} = \boxed{10}$$

$V_{(IN)} = 0$

Iteration 1 = $V_{IN} = 0.5 \times (4 + 0.6 \times 0) + 0.5 \times 10$

$\Rightarrow V_{IN} = 7$

Iteration 2 = $V_{IN} = 0.5 \times (4 + 0.6 \times 7) + 0.5 \times 10$

$= 9.1$

Iteration 3 = $V_{IN} = 0.5 \times (4 + 0.6 \times 9.1) + 0.5 \times 10$

$= 9.73$

4 $V_{IN} = 0.5 \times (4 + 0.6 \times 9.73) + 0.5 \times 10$

$= 9.919$

5 $V_{IN} = 9.9757$

6 $V_{IN} = 9.9927$

7 $V_{IN} = 9.9978$

8 $V_{IN} = 9.999$

3. Problem

Consider a finite, episodic and undiscounted Markov Decision Process (MDP) with states A and B apart from the terminal state. Assume the following two episodes are observed when a Markov Monte Carlo (MC) Evaluation is being carried out. Assume discount factor $\gamma = 1$

Episode 1: $(A, +3) \rightarrow (A, +2) \rightarrow (B, -4) \rightarrow (A, +4)$
$\rightarrow (B, -3)$

Episode 2: $(B, -2) \rightarrow (A, +3) \rightarrow (B, -3)$

For example, a sample such as $(B, -2) \rightarrow (A, +3) \rightarrow (B, -3)$ means that the episode starts at B then goes to A, then goes to B again and then terminates. On the way, the agent gets rewards of -2, $+3$ and -3.

i. Estimate the state value of both A & B using First Visit Monte Carlo Evaluation

ii. Estimate the state value of both A & B using Every Visit Monte Carlo Evaluation.

iii Construct a Markov model that best explains the observations given in the Equation.

~~Hint~~ :

Given

Episode 1: $(A, +3) \rightarrow (A, +2) \rightarrow (B, -4) \rightarrow (A, +4) \rightarrow (B, -3)$

Episode 2 $(B, -2) \rightarrow (A, +3) \rightarrow (B, -3)$

$\gamma = 1$

Step (1) First visit

Compute Returns for each Episode

$$G_t = R_{t+1} + R_{t+2} + \dots + R_T$$

Episode 1:

Time	State	Reward	Return (G)
1	A	+3	$3+2+(-4)+4+(-3) = 2$
2	A	+2	$2+(-4)+4+(-3) = -1$
3	B	−4	$-4+4+(-3) = -3$
4	A	+4	$4+(-3) = 1$
5	B	−3	-3

Episode 2:

Time	State	Reward	G
1	B	−2	$-2+3+(-3) = -2$
2	A	3	$3+(-3) = 0$
3	B	−3	-3

(ii) Compute first visit Estimate

State A : First visit in Episode 1 $G = 2$

First visit in Episode 2 $G = 0$

Estimate $V(A) = \frac{2+0}{2} = 1.0$

State B : 1st visit in Episode 1 $G = -3$

1st visit in Episode 2 $G = -2$

Estimate $V(B) = \frac{-3+(-2)}{2} = -2.5$

(iii) Constructing the Markov Model

from observed episodes :-

$P(A \to A) = 0.25$, $P(A \to B) = 0.75$

$P(B \to A) = 0.5$, $P(B \to B) = 0.5$

Transition	
$A \to A$	1
$A \to B$	2
$B \to A$	2
$B \to B$	1

Total Transition A : $A \to A + A \to B = 1 + 3 = 4$

B : $B \to A + B \to B = 2 + 1 = 3$

$P(A \to A) = 1/4 = 0.25$

$P(A \to B) = 3/4 = 0.75$

$P(B \to A) = 2/3 = 0.67$

$P(B \to B) = 1/3 = 0.3$

Reward $[R(A) \approx 1]$

$[R(B) \approx -2.5]$

∴ The MDP can be represented as

$P(A|A) = 0.25$

$P(B|A) = 0.75$

$P(A|B) = 0.5$

$P(B|B) = 0.5$