FUZZY GRAPH STRUCTURES

INTRODUCTION

Fuzzy Mathematics is the branch of mathematics including fuzzy set theory and fuzzy logic that deals with partial inclusion of elements in a set on a spectrum. As opposed to simple binary ”yes” or “no” (0 or 1) inclusion. Looking back to the history of sciences.it seems that fuzzy sets were bound to appear at some point in the 20th century. It started in 1965 after the publication of lotfi asker Zadeh’s seminal work fuzzy sets. Linguistics is an example of a field that utilize fuzzy set theory.

In 1975, Rosenfeld considered fuzzy relations on fuzzy sets. He developed the theory of fuzzy graphs. Bang and Yeh during the same time introduced various connectedness concepts in fuzzy graphs. Inexact information is used in expressing or describing human behaviors and mental process. The information depends upon a person subjectively and it is difficult to process objectively. Fuzzy information can be analyzed by using a fuzzy graph. Fuzzy graph is an expression of fuzzy relation and thus the fuzzy graph is frequently expressed in fuzzy matrix. N.R.Santhi Maheswari and C.Sekar in 2016 on strongly edge irregular fuzzy graph. O.T.Manjusha, in 2016, connected domination in fuzzy graphs. K.R. Sandeep Narayan and M.S.Sunitha in 2012, Connectively in fuzzy graphs and its complement. In 1965, Lofi A. Zadeh introduced the concept of a fuzzy subset of a set as the method for representing the phenomena of uncertainty in real life situations. Azriel rosenfeld introduced fuzzy graphs in 1975. Alison northup introduced semiregular graph is also called as (2,k)– regular graph and studied some properties of (2,k)-regular graph.

N.R.Santhi Maheswari and C.Sekar introduced d_2 of a vertex in graph and also discussed some properties on d_2 of a vertex in in graph and introduced (r,2,k)–regular graph and also discussed some properties of (r,2,k)– regular graph.in this paper we define d_2 degree of a vertex in fuzzy graphs and total d_2 –degree of a vertex in fuzzy graphs and (2,k)–regular fuzzy graphs are introduced.(2,k)–regular fuzzy graphs and totally (2,k)-regular fuzzy graphs are compared through various examples. Sampathkumar introduced the notion of graph structures. Graph structures are the generalization of graphs and widely useful in the study of some structures, like graphs, signed graphs, semigraphs, edge-colored graphs, and edge-labeled graphs. Dinesh introduced the concept of fuzzy-graph structures and described some related concepts. Fuzzy-graph structures are more useful than graph structures because they deal with the uncertainty and ambiguity of many realworld phenomena. Ramakrishnan and Dinesh worked on generalized fuzzy-graph structures. Harinath and Lavanya discussed fuzzy graph structures for wheel, helm, and star graphs.

MAXIMUL PRODUCT OF FUZZY GRAPH STRUCTURES

DEFINITION - 1

A graph structure (GS g = (V, $R_1,\ldots,R_n$) consists of a nonempty set V with relations $R_1,R_2,\ldots,R_n$ on set V which are mutually disjoint such that each relation R_i, $1\leq i\leq n$ is symmetric and ir-reflexive.

Graph structure $G* = (V, R_1,...,Rn)$ can be represented just like a graph where each edge is labeled as R_i, $1\leq i \leq n$.

DEFINITION - 2

Let σ be the fuzzy set on set V and $\mu_1,\mu_2,...,\mu n$ be fuzzy sets on $R_1,R_2,...,R_n$, respectively. If $0 \leq \mu_i(vu) \leq \sigma(v)\wedge\sigma(u)$ $\forall u,v \in V$, $i = 1,2,...,n$, then $G = (\sigma,\mu_1,\mu_2,...,\mu_n)$ is called fuzzy graph structure(FGS) of graph structure G^*.

If $vu \in supp(\mu_i)$, then vu is named as μ_i-edge of FGS G.

DEFINITION - 3

Let $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ be two fuzzy-graph structures with underlying crisp graph structures $G_1^*=(V_1,R_1',R_2',...,R_n')$ and $G_2^*=(V_2,R_1'',R_2'',...,R_n'')$,respectively. $G_1*G_2=(\sigma,\mu_1,\mu_2,...\mu_n)$ is called maximal fuzzy graph structure with underlying crisp graph structure $G^*=(V,R_1,R_2,...,R_n)$, where, $V=V_1\times V_2$ and $R_i=\{(u_1,v_1)(u_2,v_2)/u_1 = u_2,v_1v_2 \in R_i'' \text{ or } v_1=v_2,u_1u_2 \in R_i'\}$. Fuzzy vertex set σ and fuzzy relations μ_i in maximal product $G_1*G_2 =$
$(\sigma,\mu_1,\mu_2,...\mu_n)$ are defined as: $\sigma = \sigma_1*\sigma_2$,

$\sigma(u,v) = \sigma_1(u)\vee\sigma_2(v)$, for all $(u,v) \in V = V_1\times V_2$,

and $\mu_i = \mu_i' *\mu_i''$,

$$\mu_i((u_1,v_1)(u_2,v_2)) = \begin{cases} \sigma_1(u_1) \vee \mu''_i(v_1v_2), & u_1 = u_2, v_1v_2 \in R_i'', \\ \sigma_2(v_1) \vee \mu'_i(u_1u_2), & v_1 = v_2, u_1u_2 \in R_i', \end{cases} \quad i = 1,2,...,n.$$

DEFINITION - 4

A fuzzy-graph structure $G = (\sigma,\mu_1,\mu_2,...\mu_n)$ is μ_i-strong if $\mu_i(v_1v_2) = \sigma(v_1)\wedge\sigma(v_2)$, for all $v_1v_2 \in R_i$, $i \in\{1,2,...,n\}$. If G is μ_i-strong $\forall i \in\{1,2,...,n\}$, then G is called strong fuzzy-graph structure.

THEOREM - 1

Maximal product of two strong fuzzy-graph structures is also a strong fuzzy-graph structure.

PROOF:

Let $G_1= (\sigma_1, \mu_1', \mu_2',\ldots, \mu_n')$ and $G_2 = (\sigma_2, \mu_1'',\mu_2'',\ldots,\mu_n'')$ be two strong fuzzy-graph structures. then, $\mu_I'(v_1v_2) = \sigma_1(v_1)\wedge\sigma_1(v_2)$ for any $v_1v_2 \in R_i'$ and $\mu_i''(u_1u_2) = \sigma_2(u_1)\wedge\sigma_2(u_2)$ for any $u_1u_2 \in R_i''$, i = 1,2,...,n. Then, proceeding according to the definition of maximal product,

Case (i): $u_1 = u_2$ and $v_1v_2 \in R_I''$. Then,

$$\begin{aligned}\mu_I((u_1,v_1)(u_2,v_2)) &= \sigma_1(u_1)\vee\mu_I''(v_1v_2)=\sigma_1(u_1)\vee[\sigma_2(v_1)\wedge\sigma_2(v_2)]\\ &= [\sigma_1(u_1)\vee\sigma_2(v_1)]\wedge[\sigma_1(u_1)\wedge\sigma_2(v_2)]\\ &= [\sigma(u_1,v_1)\wedge\sigma(u_2,v_2)].\end{aligned}$$

Case (ii): $v_1 = v_2$ and $u_1u_2 \in R_I'$. Then,

$$\begin{aligned}\mu_I((u_1,v_1)(u_2,v_2)) &= \sigma_2(v_1)\vee\mu_I'(u_1u_2)=\sigma_2(v_1)\vee[\sigma_1(u_1)\wedge\sigma_1(u_2)]\\ &= [\sigma_1(u_1)\vee\sigma_2(v_1)]\wedge[\sigma_1(u_2)\wedge\sigma_2(v_1)]\\ &= [\sigma(u_1,v_1)\wedge\sigma(u_2,v_2)].\end{aligned}$$

Thus, $\mu_I((u_1,v_1)(u_2,v_2)) = \sigma(u_1,v_1)\wedge\sigma(u_2,v_2)$ for all edges of maximal product.

Hence $G = G_1 * G_2 = (\sigma,\mu_1,\mu_2,...\mu_n)$ is a strong fuzzy graph structure.

REMARK - 1

Converse of Theorem 1 above may not be true. That is, maximal product $G = G_1*G_2$ may be a strong fuzzy-graph structure, when G_1 and G_2 are not strong fuzzy-graph structures.

THEOREM - 2

The maximal product of two connected fuzzy-graph structures is a connected fuzzy graph structure.

PROOF:

Let $G_1 = (\sigma_1,\mu_1',\mu_2',\ldots,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',\ldots,\mu_n'')$ be two connected fuzzy-graph structures with underlying crisp graph structures $G_1^* = (V_1,R_1',R_2',\ldots,R_n')$ and $G_2^*= (V_2,R_1'',R_2'',\ldots,R_n'')$, respectively. Let $V_1 = \{u_1,u_2,\ldots,u_m\}$ and $V_2 = \{v_1,v_2,\ldots,v_n\}$. Then $\mu_i'^{\infty}(u_iu_j) > 0$ for all $u_i,u_j \in V_1$ and $\mu_i'^{\infty}(v_iv_j) > 0$ for all $v_i,v_i \in V_2$. The maximal product of $G_1 = (\sigma_1,\mu_i',\mu_2',\ldots,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'', \mu_2'',\ldots, \mu_n'')$ is written as $G = (\sigma,\mu_1,\mu_2,\ldots,\mu_n)$. Now consider 'm' subgraphs of G with the vertex sets $\{u_iv_1,u_iv_2,\ldots, u_iv_n\}$for i = 1,2,...,m. Each of these subgraphs of G is connected since u, is are the same and G_2 is connected, each v_i is adjacent to at least one of the vertices in V_2. Since G_1 is connected, each u_i is also adjacent to at least one of the vertices in V_1. Therefore, there exists one edge between any pair of the above 'm' subgraphs. Thus, we have and $\mu_i^{\infty}((u_iv_j)(u_kv_l))>$ for all $(u_i,v_j)(u_k,v_l) \in R_i$. Hence, G is a connected fuzzy-graph structure.

REMARK - 2

The maximal product of two complete fuzzy-graph structures is not a complete fuzzy-graph structure due to the absence of case $u_1u_2 \in R_i'$ and $v_1v_2 \in R_i''$ in the maximal-product definition. Since every complete fuzzy-graph structure is strong, according to Theorem 1, the maximal product of two complete fuzzy-graph structures is a strong fuzzy-graph structure.

DEFINITION - 5

The degree of a vertex in maximal product G_1*G_2 of two fuzzy-graph structures $G_1 = (\sigma_1,\mu_1',\mu_2',\ldots,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',\ldots,\mu_n'')$ is given by:

$$d_{G_1*G_2}(u_i,v_j) = \sum_{u_iu_j \in R_i',\, v_j=v_1} \mu_i'(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_iv_l \in R_i''',\, u_i=u_k} \mu''_i(v_iv_l) \vee \sigma_1(u_i)$$

μ_i – degree of a vertex of maximal product G_1*G_2 is given by

$$\mu_i - d_{G_1*G_2}(u_i,v_j) = \sum_{u_iu_j \in R_i',\, v_j=v_1} \mu'_i(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_iv_l \in R_i''',\, u_i=u_k} \mu''_i(v_iv_l) \vee \sigma_1(u_i).$$

THEOREM - 3

If $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ are two fuzzy-graph structures, such that $\sigma_1 \leq \mu_i''$,i = 1,2,...,n,then degree of any vertex I maximal product G_1*G_2 is given by

$d_{G_1*G_2}(u_i,v_i)= d_{G_1}(u_i)\sigma_2(v_j) +d_{G_2}(v_j)$.

PROOF:

Let $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ be two fuzzy-graph structures, such that $\sigma_1 \leq \mu_i''$,i = 1,2,...,n, Then $\mu_i' \leq \sigma_2$, i = 1,2,...,n. Then, degree of any vertex in G_1*G_2 (maximal product) is given by:

$$d_{G_1*G_2}(u_i,v_j) = \sum_{u_iu_j \in R_i' ,v_j=v_1} \mu'_i(u_iu_k) \vee \sigma_2(v_j) +\sum_{v_iv_l \in R_i''' ,u_i=u_k} \mu''_i(v_iv_l) \vee \sigma_1(u_i)$$

$$= \sum_{u_iu_j \in R_i' ,v_j=v_1} \sigma_2(v_j) + \sum_{v_iv_l \in R_i''' ,u_i=u_k} \mu''_i(v_iv_l)$$

$d_{G_1*G_2}(u_i,v_j) = d_{G_1*}(u_i)\sigma_2(v_j) +dG_2(v_j)$.

THEOREM - 4

If $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ are two fuzzy graph structures such that $\sigma_1 \leq \mu_i''$, i = 1,2,...,n and σ_2 is a constant function of value 'c', then degree of any vertex in G_1*G_2 (maximal product) is given by: $d_{G_1*G_2}(u_i,v_j) = d_{G_1*}(u_i)c+d_{G_2}(v_j)$.

PROOF:

If $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ are two fuzzy graph structures such that $\sigma_1 \leq \mu_i''$, i = 1,2,...,n and σ_2 is a constant function of value 'c'. Moreover, $\sigma_1 \leq \mu_i'$ implies $\mu_i' \leq \sigma_2$, i = 1,2,...,n. Then degree of any vertex in G_1*G_2 (maximal product) is given by

$$d_{G_1*G_2}(u_i,v_j) = \sum_{u_iu_k \in R_i' ,v_j=v_l} \mu'_i(u_iu_k) \vee \sigma_2(v_j) +\sum_{v_iv_l \in R_j''' ,u_i=u_k} \mu''_j(v_iv_l) \vee \sigma_1(u_i)$$

$$= \sum_{u_iu_k \in R_i' ,v_j=v_l} \sigma_2(v_j) + \sum_{v_iv_l \in R_j''' ,u_i=u_k} \mu''_j(v_iv_l)$$

$$= d_{G_1*}(u_i)c +dG_2(v_j).$$

THEOREM - 5

If $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ are two fuzzy-graph structures such that $\sigma_2 \le \mu_i'$, i = 1,2,...,n, then degree of any vertex in maximal product G_1*G_2 is given by: $d_{G_1*G_2}(u_i,v_j) = d_{G_2*}(v_j)\sigma_1(u_i) + d_{G_1}(u_i)$.

PROOF :

Let $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ be two fuzzy-graph structures such that $\sigma_2 \le \mu_i'$, then $\mu_i'' \le \sigma_1$, i = 1,2,...,n. Then, degree of any vertex in G_1*G_2 (maximal product) is given by:

$$d_{G_1*G_2}(u_i,v_j) = \sum_{u_iu_k \in R_i', v_j=v_l} \mu'_i(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_iv_l \in R_j''', u_i=u_k} \mu''_j(v_iv_l) \vee \sigma_1(u_i)$$

$$= \sum_{u_iu_k \in R_i', v_j=v_l} \mu'_j(u_iu_k) + \sum_{v_jv_l \in R_j''', u_i=u_k} \sigma_1(u_i)$$

$$= d_{G_1}(u_i) + d_{G_2*}(v_j)\sigma_1(u_i).$$

THEOREM - 6

If $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ are two fuzzy-graph structures such that $\sigma_2 \le \mu_i'$, i = 1,2,...,n and σ_1 is a constant function of value 'c', then degree of any vertex in G_1*G_2 (maximal product) is given by: $d_{G_1*G_2}(u_i,v_j) = d_{G_1}(u_i) + d_{G_2*}(v_j)c$.

PROOF:

Let $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$be two fuzzy-graph structures such that $\sigma_2 \le \mu_i'$,i = 1,2,...,n and σ_1 is a constant function of value 'c'. Moreover, $\sigma_2 \le \mu_i'$ implies $\mu_i'' \le \sigma_1$, i = 1,2,...,n. Then, degree of any vertex in G_1*G_2 (maximal product) is given by:

$$d_{G_1*G_2}(u_i,v_j) = \sum_{u_iu_k \in R_i', v_j=v_l} \mu'_i(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_iv_l \in R_j''', u_i=u_k} \mu_j''(v_Jv_l) \vee \sigma_1(u_i)$$

$$= \sum_{u_iu_k \in R_i', v_j=v_l} \mu'_j(u_iu_k) + \sum_{v_jv_l \in R_j''', u_i=u_k} \sigma_1(u_i)$$

$$= d_{G_1}(u_i) + d_{G_2*}(v_j)c.$$

THEOREM - 7

If $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ are two fuzzy-graph structures such that $\mu_i'' \leq \sigma_1$ and $\mu_i' \leq \sigma_2$, $i = 1,2,...,n$, then degree of any vertex in G_1*G_2 (maximal product) is given by: $d_{G1*G2}(ui,vj) = d_{G1*}(u_i)\ \sigma_2(v_j) + d_{G2*}(v_j)\ \sigma_1(u_i)$.

PROOF:

Let $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ be two fuzzy-graph structures such that $\mu_i'' \leq \sigma_1$ and $\mu_i' \leq \sigma_2$, $i = 1,2,...,n$, then degree of any vertex in G_1*G_2 (maximal product) is given by:

$$d_{G1*G2}(u_i,v_j) = \sum_{u_iu_k \in R_i',v_j=v_l} \mu'_i(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_iv_l \in R_j''',u_i=u_k} \mu''_j(v_jv_l) \vee \sigma_1(u_i)$$

$$= \sum_{u_iu_k \in R_{i'},v_j=v_l} \sigma_2(v_j) + \sum_{v_jv_l \in R_j''',u_i=u_k} \sigma_1(u_i)$$

$$= d_{G1*}(u_i)\ \sigma_2(v_j) + d_{G2*}(v_j)\ \sigma_1(u_i).$$

THEOREM - 8

If $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ are two fuzzy-graph structures, such that $\mu_i'' \geq \sigma_1$, $i = 1,2,...,n$, then total degree of any vertex in $G_1 *G_2$ (maximal product) is given by: $td_{G1*G2}(ui,vj) = d_{G1*}(u_i)\ \sigma_2(v_j) + td_{G2}(vj)$.

PROOF:

If $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',...,\mu_n'')$ are two fuzzy-graph structures, such that $\mu_i'' \geq \sigma_1$, $\mu_i' \leq \sigma_2, i = 1,2,...,n$, Then, total degree of any vertex in $G_1 *G_2$ (maximal product) is given by:

$$td_{G1*G2}(u_i,v_j) = \sum_{u_iu_k \in R_i',v_j=v_l} \mu_i'(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_iv_l \in R_j''',u_i=u_k} \mu''_j(v_jv_l) \vee \sigma_1(u_i) + \sigma(u_i, v_j)$$

$$= \sum_{u_iu_k \in R_i',v_j=v_l} \sigma_2(v_j) + \sum_{v_jv_l \in R_j''',u_i=u_k} \mu''_j(v_jv_l) + [\ \sigma_1(u_i) \vee \sigma(u_i, v_j)\ ]$$

$$= d_{G1*}(u_i)\ \sigma_2(v_j) + [\ d_{G2*}(v_j)\ \sigma_2(v_j)\ ]$$

$$= d_{G1*}(u_i)\ \sigma_2(v_j) + td_{G2}(v_j).$$

THEOREM - 9

If $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ and $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ are two fuzzy-graph structures, such that $\mu_i'' \geq \sigma_1$, i = 1,2,...,n, and σ_2 is a constant function of value 'c', then the total degree of any vertex in $G_1 * G_2$ (maximal product) is given by: $td_{G_1 * G_2}(u_i, v_j) = td_{G_2 *}(v_j) + d_{G_1 *}(u_i)c$.

PROOF:

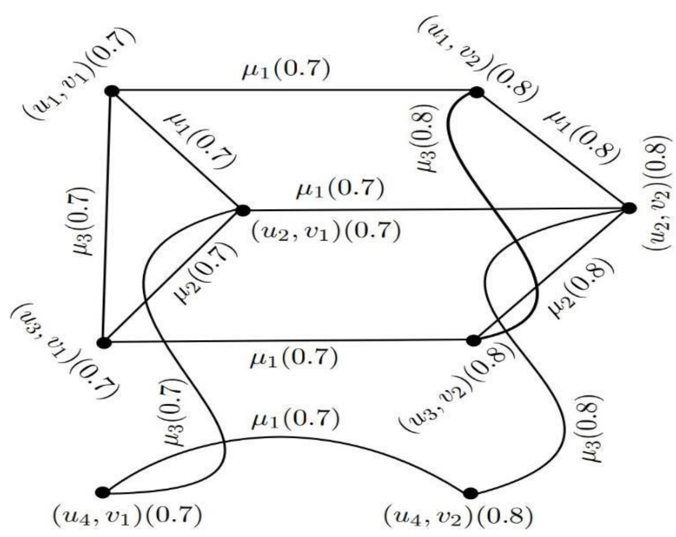
Let $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ and $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ are two fuzzy-graph structures, such that $\mu_i'' \geq \sigma_1$, i = 1,2,...,n, and σ_2 is a constant function of value 'c', Moreover $\mu_i'' \geq \sigma_1$ implies $\sigma_2 \geq \sigma_1$, $\mu_i' \geq \sigma_2$, i = 1,2,...,n, Then, the total degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$d_{G_1 * G_2}(u_i, v_j) = \sum_{u_i u_k \in R_i', v_j = v_l} \mu'_i(u_i u_k) \vee \sigma_2(v_j) + \sum_{v_j v_l \in R_j''', u_i = u_k} \mu''_j(v_j v_l) \vee \sigma_1(u_i) + \sigma(u_i, v_j)$$

$$= \sum_{u_i u_k \in R_i', v_j = v_l} \sigma_2(v_j) + \sum_{v_j v_l \in R_j''', u_i = u_k} \mu_j''(v_j v_l) + [\,\sigma_1(u_i) \vee \sigma_2(v_j)\,]$$

$$= d_{G_2}(v_j) + d_{G_1 *}(u_i) + \sigma_2(v_j)$$

$$= td_{G_2}(v_j) + d_{G_1 *}(u_i)c.$$



$G_1 * G_2 = (\sigma, \mu_1, \mu_2, \ldots, \mu_n)$

FIGURE 1: Maximal product of two fuzzy-graph structures (FGSs).

THEOREM 10

If $G_1 = (\sigma_1,\mu_1',\mu_2',\ldots,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',\ldots,\mu_n'')$ are two fuzzy-graph structures, such that $\mu_i'' \geq \sigma_2$, i = 1,2,...,n, then the total degree of any vertex in $G_1 * G_2$ (maximal product) is given by: $td_{G_1*G_2}(ui,vj) = d_{G_2*}(v_j)\,\sigma_1(u_i) + td_{G_1}(u_i)$.

PROOF:

Let $G_1 = (\sigma_1,\mu_1',\mu_2',\ldots,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',\ldots,\mu_n'')$ be two fuzzy-graph structures, such that $\mu_i' \geq \sigma_2$, then $\sigma_2 \geq \sigma_1$, $\mu_i'' \geq \sigma_1$, i = 1,2,...,n, Then, the total degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$d_{G_1*G_2}(u_i,v_j) = \sum_{u_iu_k \in R_i', v_j=v_l} \mu'_i(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_jv_l \in R_j''', u_i=u_k} \mu''_j(v_jv_l) \vee \sigma_1(u_i) + \sigma(u_i, v_j)$$

$$= \sum_{u_iu_k \in R_i', v_j=v_l} \mu_i'(u_iu_k) + \sum_{v_jv_l \in R_j''', u_i=u_k} \sigma_1(u_i) + [\,\sigma_1(u_i) \vee \sigma_2(v_j)\,]$$

$$= d_{G_1}(u_i) + d_{G_2*}(v_j)\sigma_1(u_i) + \sigma_1(u_i)$$

$$= td_{G_1}(u_i) + d_{G_2*}(v_j)\,\sigma_1(u_i).$$

THEOREM 11

If $G_1 = (\sigma_1,\mu_1',\mu_2',\ldots,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',\ldots,\mu_n'')$ are two fuzzy-graph structures, such that $\mu_i' \geq \sigma_2$, i = 1,2,...,n, and σ_1 is a constant function of value 'c', then the total degree of any vertex in $G_1 * G_2$ (maximal product) is given by: $td_{G_1*G_2}(ui,vj) = d_{G_2*}(v_j)\,c + td_{G_1}(u_i)$.

PROOF:

Let $G_1 = (\sigma_1,\mu_1',\mu_2',\ldots,\mu_n')$ and $G_2 = (\sigma_2,\mu_1'',\mu_2'',\ldots,\mu_n'')$ be two fuzzy-graph structures, such that $\mu_i' \geq \sigma_2$, i = 1,2,...,n, and $\sigma_1$is a constant function of value 'c'. Moreover , $\mu_i' \geq \sigma_2$ implies $\sigma_1 \geq \sigma_2$, $\mu_i'' \leq \sigma_2$, i = 1,2,...,n, Then, the total degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$td_{G_1*G_2}(u_i,v_j) = \sum_{u_iu_k \in R_i', v_j=v_l} \mu'_i(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_jv_l \in R_j''', u_i=u_k} \mu''_j(v_jv_l) \vee \sigma_1(u_i) + \sigma(u_i, v_j)$$

$$= \sum_{u_iu_k \in R_i', v_j=v_l} \mu_i'(u_iu_k) + \sum_{v_jv_l \in R_j''', u_i=u_k} \sigma_1(u_i) + [\,\sigma_1(u_i) \vee \sigma_2(v_j)\,]$$

$$= d_{G_1}(u_i) + d_{G_2*}(v_j)c + \sigma_1(u_i)$$

$$= td_{G_1}(u_i) + d_{G_2*}(v_j)\,c.$$

REMARK 3

The maximal product of two regular fuzzy-graph structures may not be a regular fuzzygraph structure.

THEOREM 12

If $G_1 = (\sigma_1, \mu_1', \mu_2', ..., \mu_n')$ is partially regular fuzzy-graph structure and $G_2 = (\sigma_2, \mu_1'', \mu_2'', ..., \mu_n'')$ is fuzzy-graph structure such that $\mu_i'' \geq \sigma_1$, i = 1,2,...,n and σ_2 is a constant function of value 'c', then maximal product $G_1 * G_2$ is regular if and only if G_2 is regular.

PROOF:

Let $G_1 = (\sigma_1, \mu_1', \mu_2', ..., \mu_n')$ be partially regular fuzzy-graph structure such that G_1^* is $r_1$regular and $G_2 = (\sigma_2, \mu_1'', \mu_2'', ..., \mu_n'')$ be fuzzy-graph structures such that $\mu_i'' \geq \sigma_2$,i = 1,2,...,n and σ_2 is a constant function of value 'c'. Moreover, suppose that $G_2 = (\sigma_2, \mu_1'', \mu_2'', ..., \mu_n'')$ is k-regular fuzzy-graph structure. Then, $d_{G_1 * G_2}(u_i, v_j) = d_{G_2}(v_j) + d_{G_1^*}(u_i)\sigma_2(v_j) = k + r_1 c$. This holds for all vertices of $V_1 \times V_2$. Hence, maximal product $(G_1 * G_2)$ is regular fuzzy-graph structure.

Conversely, suppose that maximal product $(G_1 * G_2)$ is a regular fuzzy-graph structure. Then, for any two vertices of $V_1 \times V_2$,

$$d_{G_1 * G_2}(u_1, v_1) = d_{G_1 * G_2}(u_2, v_2)$$

$$\Rightarrow d_{G_2}(v_1) + d_{G_1^*}(u_1)\, \sigma_2(v_1) = d_{G_2}(v_2) + d_{G_1^*}(u_2)\, \sigma_2(v_2)$$

$$\Rightarrow d_{G_2}(v_1) + r_1 c = d_{G_2}(v_2) + r_1 c$$

$$\Rightarrow d_{G_2}(v_1) = d_{G_2}(v_2).$$

This holds for all vertices of G_2. Thus, G_2 is regular fuzzy-graph structure.

THEOREM 13

If $G_2 = (\sigma_1, \mu_1'', \mu_2'', ..., \mu_n'')$ is partially regular fuzzy-graph structure and $G_1 = (\sigma_1, \mu_1', \mu_2', ..., \mu_n')$ is fuzzy-graph structure such that $\mu_i' \geq \sigma_2$, i = 1,2,...,n and σ_1 is a constant function of value 'c', then maximal product $G_1 * G_2$ is regular if and only if G_1 is regular.

PROOF:

Let $G_2 = (\sigma_2, \mu_1'', \mu_2'', ..., \mu_n'')$ be partially regular fuzzy-graph structure such that G_2^* is $r_2$regular and $G_1 = (\sigma_1, \mu_1', \mu_2', ..., \mu_n')$ be fuzzy-graph structures such that $\mu_i' \geq \sigma_1$,i = 1,2,...,n and σ_1 is a constant function of value 'c'. Moreover, suppose that $G_1 = (\sigma_1, \mu_1', \mu_2', ..., \mu_n')$ is k-regular fuzzy-graph structure. Then, $d_{G_1*G_2}(u_i,v_j) = d_{G_1}(u_i)+d_{G_{2*}}(v_j)\sigma_1(u_i) = k+r_2c$. This holds for all vertices of $V_1\times V_2$. Hence, maximal product (G_1*G_2) is regular fuzzy-graph structure.

Conversely, suppose that maximal product (G_1*G_2) is a regular fuzzy-graph structure. Then, for any two vertices of $V_1\times V_2$,

$$d_{G_1*G_2}(u_1,v_1) = d_{G_1*G_2}(u_2,v_2)$$

$$\Rightarrow d_{G_1}(u_1)+d_{G_{2*}}(v_1)\ \sigma_1(u_1) = d_{G_1}(u_2) + d_{G_{2*}}(v_2)\ \sigma_1(u_2)$$

$$\Rightarrow d_{G_1}(u_1)+r_2c= d_{G_2}(u_2) + r_2c$$

$$\Rightarrow d_{G_1}(u_1) = d_{G_1}(u_2).$$

This holds for all vertices of G_1. Thus, G_1 is regular fuzzy-graph structure.

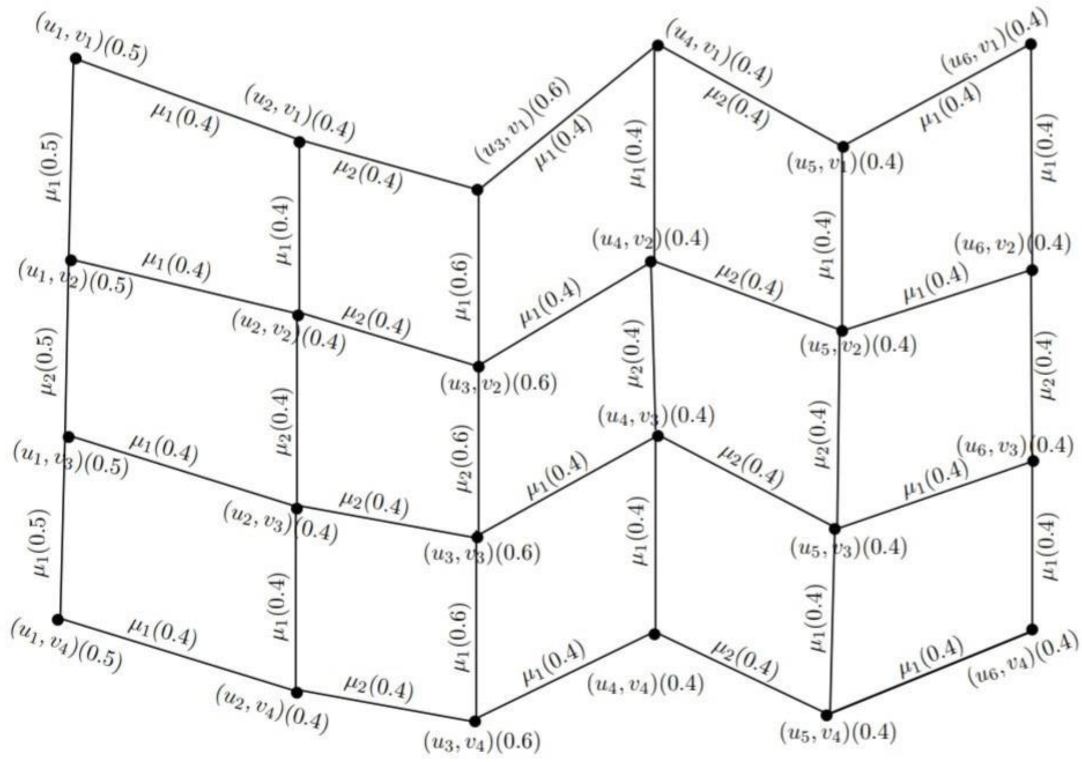


FIGURE 2: Maximum product of G_1 and G_2.

THEOREM 14

If $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_1 = (\sigma_1,\mu_1'',\mu_2'',...,\mu_n'')$ are two partially regular fuzzy-graph structures such that $\mu_i'' \leq \sigma_1$, $\mu_i \leq \sigma_2$, i = 1,2,...,n and σ_2 is a constant function of value 'c', then maximal product $G_1{*}G_2$ is regular if and only if σ_1 is a constant function.

PROOF:

Let $G_1 = (\sigma_1,\mu_1',\mu_2',...,\mu_n')$ and $G_1 = (\sigma_1,\mu_1'',\mu_2'',...,\mu_n'')$ are two partially regular fuzzygraph structures such that $\mu_i'' \leq \sigma_1$, $\mu_i \leq \sigma_2$, i=1,2,...,n and σ_2 is a constant function of value 'c'. Moreover, G_1^* is r_1−regular and G_2^* is r_2−regular. Furthermore, suppose that σ_1 is a constant function of value k. Then, $d_{G_1*G_2}(u_i, v_j) = d_{G_2*}(v_j)\sigma_1(u_i) + d_{G_1*}(u_i)\sigma_2(v_j) = r_2k + r_1c$. This holds for all vertices of $V_1 \times V_2$. Hence, maximal product $(G_1{*}G_2)$ is a regular fuzzy-graph structure. Conversely, suppose that maximal product $(G_1{*}G_2)$ is a regular fuzzy-graph structure. Then, for any two vertices of $V_1 \times V_2$,

$$d_{G_1*G_2}(u_1, v_1) = d_{G_1*G_2}(u_2, v_2)$$

$$\Rightarrow d_{G_2*}(v_1)\, \sigma_1(u_1) + d_{G_1*}(u_1)\, \sigma_2(v_1) = d_{G_2*}(v_2)\, \sigma_1(u_2) + d_{G_1*}(u_2)\, \sigma_2(v_2)$$

$$\Rightarrow r_2\sigma_1(u_1) + r_1c = r_2\sigma_1(u_2) + r_1c$$

$$\Rightarrow \sigma_1(u_1) = \sigma_1(u_2).$$

This holds for all vertices of G_1 Thus, σ_1 is a constant function.

REMARK 4

$G_1{*}G_2$ (maximal product) of two full regular fuzzy-graph structures G_1 and G_2 is not always full regular. Moreover, if G_1 and G_2 are two regular fuzzy-graph structures on complete graph structures G_1^* and G_2^*, respectively, then $G_1{*}G_2$ (maximal product) is a partially regular fuzzy-graph structure on a complete underlying graph structure $G_1^*{*}G_2^*$.

APPLICATIONS

Identification of most controversial issues among countries: Now a days, the world has become a global village, due to which all countries are related to each other. Relationships among different countries are not of the same nature. Some countries have good relationships with each other; for example, Pakistan and China have a very good relationship for many years. However, some countries do not have good relationships due to which global peace is at stake. The reasons for bad relationships are some controversial issues between those countries. There are many controversial issues that are permanently dangerous to global peace, including line of control issues, counterterrorism activities, Proliferation of nuclear weapons, power struggles, religious issues, and occupations of other countries.

Different countries have many issues with each other, but, in particular time period, one issue is the most controversial that needs the attention of peace-loving organizations to be solved, so that war fare activities among those countries can be contained. For example, India and Pakistan have many issues with each other, including the issue of Kashmir, water issues, religious issues, terrorism, power struggles, and line of control issues. In different time periods, the India–Pakistan relationship has been disturbed due to different issues. Nowadays, the most controversial issue between Pakistan and India is the line of control issue, which needs to be promptly solved.

We can use a fuzzy-graph structure to highlight the most controversial issue between any two countries in a particular time period, and can also tell the severity level of the issue at that time with the help of the membership function. The fuzzy-graph structure of the most controversial issues can be very helpful for peace-keeping organizations and the United Nations to maintain global peace. Consider a set S of eight powerful countries:

S = {North Korea, America, South Korea, Pakistan, Iran, Russia, India, Afghanistan}.

Let σ be a fuzzy set on S, defined in Table 1.

TABLE 1. Fuzzy set σ of countries.

COUNTRY	DEGREE OF MEMBERSHIP
North Korea	0.8
America	0.9
South Korea	0.7
Pakistan	0.8
Iran	0.7
Russia	0.9
India	0.8

Afghanistan	0.5

In Table 1, we describe the degrees of membership in a set of powerful countries. In Tables 2–10, we have mentioned the membership values of controversial issues between each pair of countries. Membership value of each pair of countries is according to $\mu(v_1v_2) \leq \sigma(v_1) \wedge \sigma(v_2)$, for all $v_1, v_2 \in S$. By using these membership values, we show the severity level of each controversial issue between each pair of countries.

TABLE 2. Fuzzy set of controversial issues between America and other countries.

Type of Controversial Issues	**(America, North Korea)**	**(America, South Korea)**	**(America, Pakistan)**	**(America, Iran)**
Proliferation of nuclear weapons	0.8	0.5	0.7	0.7
Counter terrorism activities	0.6	0.4	0.8	0.6
Line of control issues	0.1	0.1	0.3	0.2
To be more powerful	0.7	0.2	0.4	0.6
Religious issue	0.5	0.3	0.6	0.5
To occupy other country	0.3	0.5	0.6	0.5

TABLE 3. Fuzzy set of controversial issues between Russia and other countries.

Type of Controversial Issues	**(America, North Korea)**	**(America, South Korea)**	**(America, Pakistan)**	**(America, Iran)**
Proliferation of nuclear weapons	0.8	0.5	0.2	0.3
Counter terrorism activities	0.6	0.6	0.2	0.4
Line of control issues	0.2	0.6	0.1	0.5
To be more powerful	0.6	0.5	0.2	0.9
Religious issue	0.5	0.5	0.4	0.3

To occupy other country	0.6	0.5	0.5	0.4

TABLE 4. Fuzzy set of controversial issues between North Korea and other countries

Type of Controversial Issues	**(North Korea, America)**	**(North Korea, South Korea)**	**(North Korea, Pakistan)**
Proliferation of nuclear weapons	0.8	0.5	0.8
Counter terrorism activities	0.6	0.2	0.4
Line of control issues	0.1	0.6	0.3
To be more powerful	0.7	0.7	0.4
Religious issue	0.5	0.5	0.6
To occupy other country	0.3	0.6	0.4

TABLE 5. Fuzzy set of controversial issues between India and other countries.

Type of Controversial Issues	**(India, North Korea)**	**(India, America)**	**(India, South Korea)**	**(India, Russia)**
Proliferation of nuclear weapons	0.8	0.2	0.5	0.7
Counter terrorism activities	0.6	0.3	0.4	0.6
Line of control issues	0.1	0.2	0.3	0.4
To be more powerful	0.6	0.8	0.6	0.6
Religious issue	0.2	0.5	0.3	0.5
To occupy other country	0.3	0.3	0.4	0.4

TABLE 6. Fuzzy set of controversial issues between Pakistan and other countries.

Type of Controversial Issues	**(Pakistan, Iran)**	**(Pakistan, Russia)**	**(Pakistan, India)**	**(Pakistan, Afghanistan)**
Proliferation of nuclear weapons	0.3	0.8	0.5	0.3
Counter terrorism activities	0.2	0.6	0.4	0.2
Line of control issues	0.5	0.5	0.8	0.5
To be more powerful	0.6	0.4	0.7	0.1
Religious issue	0.7	0.5	0.6	0.1
To occupy other country	0.3	0.3	0.6	0.1

TABLE 7. Fuzzy set of controversial issues between Afghanistan and other countries.

Type of Controversial Issues	**(Afghanistan, South Korea)**	**(Afghanistan, Pakistan)**	**(Afghanistan, America)**
Proliferation of nuclear weapons	0.1	0.1	0.1
Counter terrorism activities	0.1	0.2	0.1
Line of control issues	0.2	0.5	0.0
To be more powerful	0.1	0.2	0.0
Religious issue	0.5	0.2	0.4
To occupy other country	0.0	0.1	0.5

TABLE 8. Fuzzy set of controversial issues between Iran and other countries.

Type of Controversial Issues	**(Iran, Russia)**	**(Iran, India)**	**(Iran, Afghanistan)**	**(Iran, North Korea)**
Proliferation of nuclear weapons	0.7	0.6	0.4	0.1
Counter terrorism activities	0.2	0.1	0.0	0.1
Line of control issues	0.1	0.4	0.2	0.0
To be more powerful	0.1	0.1	0.0	0.0
Religious issue	0.2	0.7	0.3	0.5
To occupy other country	0.0	0.2	0.0	0.0

TABLE 9. Fuzzy set of controversial issues between South Korea and other countries.

Type of Controversial Issues	**(South Korea, Iran)**	**(South Korea, Russia)**	**(South Korea, North Korea)**
Proliferation of nuclear weapons	0.1	0.2	0.6
Counter terrorism activities	0.0	0.3	0.4
Line of control issues	0.0	0.2	0.6
To be more powerful	0.1	0.2	0.7
Religious issue	0.5	0.1	0.4
To occupy other country	0.3	0.5	0.6

TABLE 10. Fuzzy set of controversial issues between some countries.

Type of Controversial Issues	**(North Korea, Iran)**	**(Afghanistan, India)**	**(South Korea, Pakistan)**
Proliferation of nuclear weapons	0.1	0.1	0.5
Counter terrorism activities	0.0	0.2	0.4
Line of control issues	0.0	0.5	0.0
To be more powerful	0.4	0.1	0.4
Religious issue	0.5	0.3	0.6
To occupy other country	0.3	0.0	0.2

On set S, many relations can be defined. Let us define following relations on S: R_1 = Proliferation of nuclear weapons, R_2 = Counter terrorism activities, R_3 = Line of control issues, R_4 = To be more powerful , R_5 = Religious issues, R_6 = To occupy other country, such that $(S,R_1,R_2,R_3,R_4,R_5,R_6)$ is a graph structure. Each element in the relation depicts a particular kind of most controversial issues among those two countries. As $(S, R_1, R_2, R_3, R_4, R_5, R_6)$ is the graph structure, therefore a pair of countries can appear in just one relation. Hence, it would be considered an element of that relation, for which its membership value is comparatively high than those of other relations. Using the given data above, elements in relations are paired with their membership values, resulting sets are the fuzzy sets on R_1, R_2, R_3, R_4, R_5, R_6, respectively. These fuzzy sets are μ_1, μ_2, μ_3, μ_4, μ_5, μ_6, respectively. Let

R_1 = {(America, North Korea), (America, Iran), (North Korea, Russia), (Pakistan, North Korea)},

R_2= {(Pakistan, America)},

R_3= {(Pakistan, India), (Afghanistan, India)},

R_4= {(America, Russia), (South Korea, North Korea)},

R_5 = {(Iran, India), (Afghanistan, South Korea)}, R_6

= {(America, Afghanistan)}.

And the corresponding fuzzy sets are:

μ_1= {((America, North Korea), 0.8), ((America, Iran), 0.7), ((North Korea, Russia), 0.8), ((Pakistan, North Korea), 0.8)}, μ_2=

{((Pakistan, America), 0.8)}

μ_3= {((Pakistan, India), 0.8), ((Afghanistan, India), 0.5)}, μ_4=

{(America, Russia), 0.9), ((South Korea, North Korea), 0.7)},

μ_5= {((Iran, India), 0.7), ((Afghanistan, South Korea), 0.5)}, μ_6 =

{((America, Afghanistan), 0.5)}.

Clearly, (σ, μ_1, μ_2, μ_3, μ_4, μ_5, μ_6) is a fuzzy-graph structure and is shown in Figure 15.

In the FGS shown in Figure15, each edge depicts the most controversial issue in the corresponding countries. For example: the most controversial issue between America and North Korea is the proliferation of nuclear weapons, and its membership value is 0.8. It can be noted that vertex America hasthehighestvertexdegreefortherelationproliferationofnuclearweapons. This means that America has the proliferation of nuclear weapons as a controversial issue with other countries. Moreover, according to this fuzzy-graph structure, Pakistan and India have the line of control issue as the most controversial issue at this time with membership value 0.8. A FGS of all countries can be very helpful for United Nations and other organizations to maintain global peace. It would highlight those controversial issues that needed to be promptly solved.

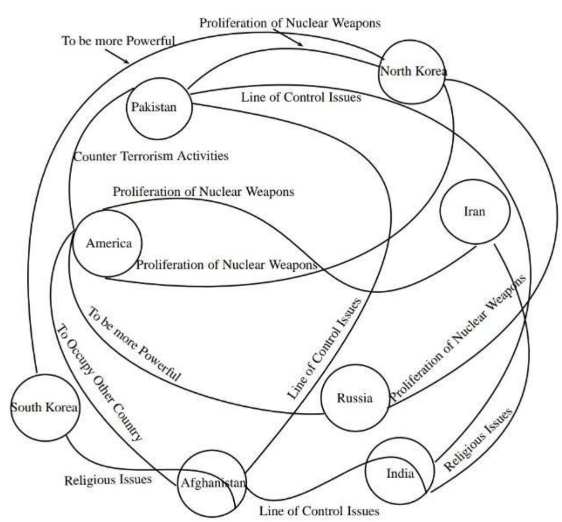


FIGURE 3. Fuzzy graph structure depicting most controversial issues among different countries.

The general procedure used in this application is shown in the following flowchart (Figure 4).

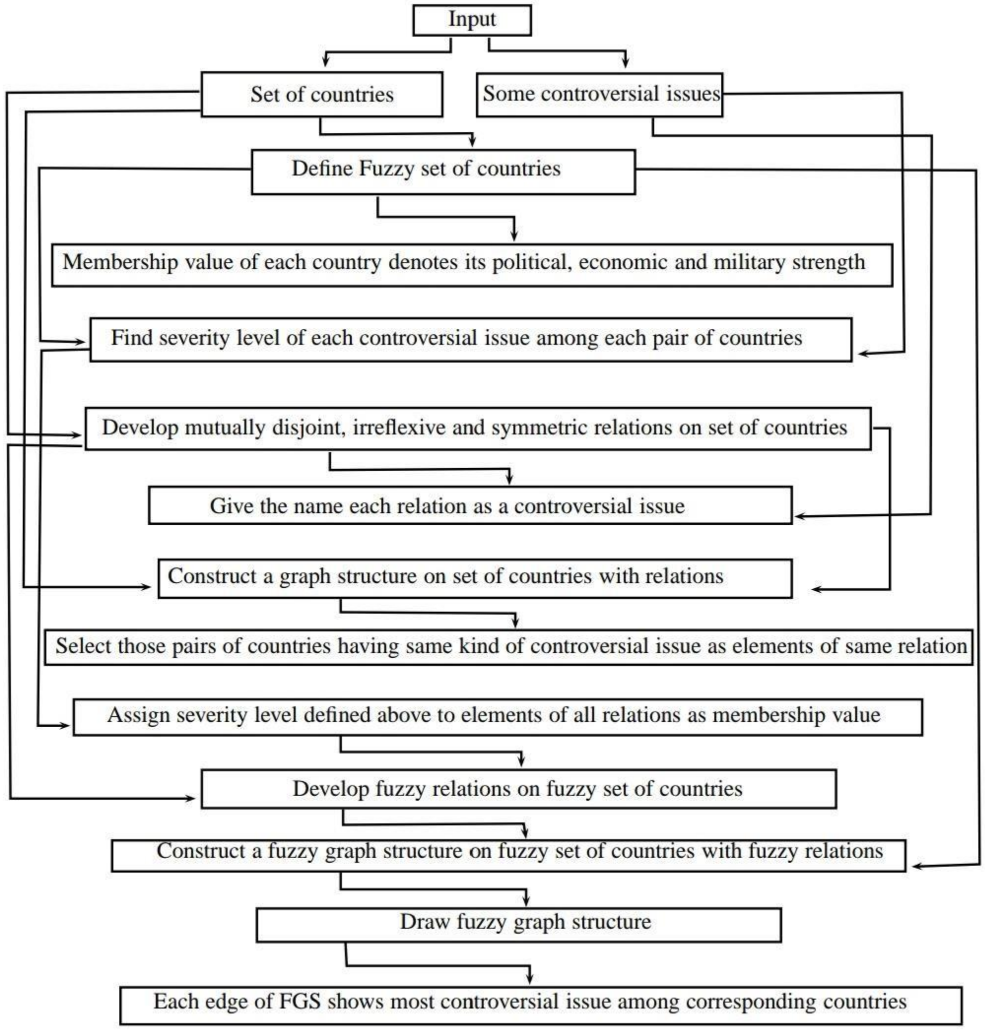


FIGURE 4. Flow chat

CONCLUSION

Graph theory has many applications in solving various problems of several domains, including networking, communication, data mining, clustering, image capturing, image segmentation, planning, and scheduling. However, in some situations, certain aspects of a graphtheoretical system may be uncertain. Use of fuzzy-graphical methods in dealing with ambiguity and vague notions is very natural. Fuzzy-graph theory has a large number of applications in modeling various real-time systems where the level of information inherent in the system varies with different levels of precision. In this internship activities, we have presented a new framework to handle fuzzy information by combining fuzzy sets with graph structures. We have introduced many basic notions concerning fuzzy-graph structures including maximal product of two fuzzygraph structures and regular fuzzy-graph structures, and investigated a few related properties. We have also defined the degree and total degree of vertex in the maximal product of fuzzy-graph structures, and discussed some properties with examples.

REFERENCES

1. Zadeh, L.A. Fuzzy sets. Inf. Control 1965, 8, 338–353. [CrossRef]
2. Rosenfeld, A. Fuzzy groups. J. Math. Anal. Appl. 1971, 35, 512–517. [CrossRef]
3. Kauffman, A. Introduction a la Theorie des Sous-Emsembles Flous; Masson et Cie: Paris, French, 1973.
4. Zadeh, L.A. Similarity relations and fuzzy orderings. Inf. Sci. 1971, 3, 177–200. [CrossRef]
5. Rosenfeld, A. Fuzzy graphs. In Fuzzy Sets and their Applications; Zadeh, L.A., Fu, K.S., Shimur, M., Eds.; Academic Press: New York, NY, USA, 1975; pp. 77–95.
6. Bhattacharya, P. Some remarks on fuzzy graphs. Pattern Recognit. Lett. 1987, 6, 297–302. [CrossRef]
7. Sunitha, M.S.; Vijayakumar, A. Complement of a fuzzy graph. IndianJ.PureAppl. Math. 2002, 33, 1451–1464.
8. Sunitha, M.S.; Vijayakumar, A. A characterization of fuzzy trees. Inf. Sci. 1999, 113, 293–300. [CrossRef]
9. Mordeson, J.N.; Nair, P.S. Fuzzy Graphs and Fuzzy Hypergraphs; Springer: Heidelberg, Germany, 2000; ISBN 978-3-7908-1854-3.
10. Bhutani, K.R.; Battou, A. On M-strong fuzzy graphs. Inf. Sci. 2003, 155, 103–109. [CrossRef]
11. Mathew, S.; Sunitha, M.S. Types of arcs in a fuzzy graph. Inf. Sci. 2009, 179, 1760–1768. [CrossRef]
12. Mordeson, J.N.; Chang-Shyh, P. Operations on fuzzy graphs. Inf. Sci. 1994, 79, 159–170. [CrossRef].