AN OPERATIONS ON INTERVAL-VALUED INTUITIONISTIC FUZZY MATRICES

Dissertation submitted to Sri Vidya Mandir Arts & Science College (A) Katteri-636 902, Uthangarai, Tamil Nadu in partial fulfillment of the requirements for the award of degree of

MASTER OF SCIENCE IN MATHEMATICS

By

RISHIRAGUL S
Reg.No:21PMAT046

Under the guidance of

Dr. C. RAGAVAN, M.Sc., M.Phil., B.Ed., Ph.D.,

Assistant Professor of Mathematics

PG & RESEARCH DEPARTMENT OF MATHEMATICS

SRI VIDYA MANDIR ARTS & SCIENCE COLLEGE (AUTONOMOUS)
(Accredited by NAAC with ‘A’ Grade [3.27])
(Recognised 2(f) &12(B) under UGC Act of 1956)
Vignesh Nagar, Katteri(Vill),
Uthangarai(TK), Krishnagiri(Dt)-636 902

MAY-2023

BONAFIDE CERTIFICATE

This is to certify that the dissertation entitled **“AN OPERATIONS ON INTERVAL-VALUED INTUITIONISTIC FUZZY MATRICES”** submitted in partial fulfillment of the requirements for the award of degree of **MASTER OF SCIENCE IN MATHEMATICS** is a record of the original research work done by **RISHIRAGUL S (Reg.No:21PMAT046)** during the period of his study in the PG & Research Department of Mathematics, Sri Vidya Mandir Arts & Science College(Autonomous), Uthangarai, under my guidance and the project has not found the basis for the award of any degree/diploma /associate ship/fellowship or other similar title to any other candidate of any university.

Counter signed by

Head of the Department **Signature of the guide**

Internal Examiner **External Examiner**

DECLARATION

I declare that the project work entitled **"AN OPERATIONS ON INTERVAL-VALUED INTUITIONISTIC FUZZY MATRICES"** submitted for the award of the Degree of Master of Science in Mathematics is a record of original and independent research work done by me during 2021 – 2023 under the supervision of **Dr. C. RAGAVAN, M.Sc., M.Phil., B.Ed., Ph.D.,** Assistant Professor of Mathematics, Sri Vidya Mandir Arts & Science College(Autonomous), Uthangarai and it has not previously formed the basis for the award of any Degree, Diploma, Associate ship or Fellowship of other similar title to any candidate of any university.

Signature of the Candidate

Place:

Date:

ACKNOWLEDGEMENT

First and foremost, I thank God, Almighty for being with me in every part of my life and who supported me throughout the completion of this project successfully. On the submission of this project, I gratefully remember my parents, their care, their concern and their key roles in moulding me and my future.

I owe my heartfelt thanks to our Founder **Mr. V. CHANDRASEKARAN,** Sri Vidya Mandir Higher Educational Trust, Uthangarai. Former Member Syndicate of Periyar University, Salem.

I sincerely thank **Mr. M. ARUN KUMAR,** Secretary, Sri Vidya Mandir Higher Educational Trust, Uthangarai, for his support.

I express my sincere thanks to **Dr. N. GUNASEKARAN, M.A., M.Phil., Ph.D.,** Principal, Sri Vidya Mandir Arts & Science College (Autonomous), Uthangarai, who helped me in the completion of this work.

I express my sincere thanks to **Dr. C. RAGAVAN, M.Sc., M.Phil., B.Ed., Ph.D.,** Associate Professor & Head, PG Department of Mathematics, who encourage me in the completion of this work.

I express my profound thanks to my guide **Dr. C. RAGAVAN, M.Sc., M.Phil., B.Ed., Ph.D.,** Assistant Professor of Mathematics, Sri Vidya Mandir Arts & Science College (Autonomous), Uthangarai, for his timely advice, continuous encouragement and leading me in the right way in completing this project. It gives me pleasure to mention that his inspirations and creative thoughts are spread over my entire project.

Last but away from least, I thank all the special people, perhaps their name may not be mentioned here by mistakes, for their help in completing this project and my friends who are always being with me in all my success.

RISHIRAGUL S

CONTENT

CHAPTER	TITLE	PAGE NO
I	INTRODUCTION	1
II	DEFINITIONS AND PRELIMINARIES	2
III	OPERATIONS ON INTUITIONISTIC FUZZY MATRICES	5
IV	OPERATIONS ON INTER-VALUED INTUITIONISTIC FUZZY MATRICES	10
	NEED OF INTER-VALUED INTUITIONISTIC FUZZY MATRICES	30
	PROPERTIES OF INTER-VALUED INTUITIONISTIC FUZZY MATRICES	34
V	CONCLUSION	42
VI	REFERENCES	43

AN OPERATIONS ON INTERVAL -VALUED INTUITIONISTIC FUZZY MATRICES

CHAPTER I

INTRODUCTION

Matrices play important roles in various areas in science and engineering. The classical matrix theory can't solve the problems involving various types of uncertainties. That type of problems are solved by using fuzzy matrix. Later much works have been done by many researchers. Fuzzy matrix deals with only membership values. These matrices can't deal non membership values. Intuitionistic fuzzy matrices (IFMs) introduced first time by Khan, Shyamal and Pal. Several properties on IFMs have been studied in. But, practically it is difficult to measure the membership or non membership value as a point. So, we consider the membership value as an interval and also in the case of non membership values, it is not selected as a point, it can be considered as an interval. Here, we introduce the interval valued intuitionistic fuzzy matrices (IVIFMs) and introduce some basic operators on IVIFMs. The interval-valued intuitionistic fuzzy determinant (IVIFD) is also defined. A real life problem on IVIFM is presented. Interpretation of some of the operators are given with the help of this example.

In this paper, the interval-valued intuitionistic fuzzy matrix (IVIFM) is introduced. The interval valued intuitionistic fuzzy determinant is also defined. Some fundamental operations are also presented. The need of IVIFM is explain by an example. Also, we define an operation on the intuitionistic fuzzy matrices called the Gödel implication operator as an extension to the definition of this operator in the case of ordinary fuzzy matrices due to Sanchez and Hashimoto.

Using this operator, we prove several important results for intuitionistic fuzzy matrices. Particularly, some properties concerning pre-orders, sub-inverses, and regularity. We concentrate our discussion on the reflexive and transitive matrices. This studying enables us to give a largest sub-inverse and a largest generalized inverse for a reflexive and transitive intuitionistic fuzzy matrix.

CHAPTER II

DEFINITIONS AND PRELIMINARIES

In this section, we first define the intuitionistic fuzzy matrix (IFM) based on the definition of intuitionistic fuzzy sets introduced by Atanassov. The intuitionistic fuzzy matrices are introduced by Pal, Khan and Shyamal.

DEFINITION 1

Intuitionistic fuzzy matrix (IFM)

An intuitionistic fuzzy matrix (IFM) A of order $m \times n$ is defined as $A = \left[x_{ij}, \langle a_{ij\mu}, a_{ij\vartheta} \rangle\right]_{m\times n}$, where $a_{ij\mu}$ and $a_{ij\vartheta}$ are called membership and non membership values of x_{ij} in A, which maintaining the condition $0 \leq a_{ij\mu} + a_{ij\vartheta} \leq 1$.For simplicity, we write $A = \left[x_{ij\mu}, a_{ij\mu}\right]_{m\times n}$ or simply $\left[a_{ij}\right]_{m\times n}$ where $a_{ij} = \langle a_{ij\mu}, a_{ij\vartheta} \rangle$.

Using the concept of intuitionistic fuzzy sets and interval valued fuzzy sets, we define interval-valued intuitionistic fuzzy matrices as follows:

DEFINITION 2

Interval-valued intuitionistic fuzzy matrix (IVIFM)

An interval valued intuitionistic fuzzy matrix (IVIFM) A of order $m \times n$ is defined as $A = \left[x_{ij}, \langle a_{ij\mu}, a_{ij\vartheta} \rangle\right]_{m\times n}$ where $a_{ij\mu}$ and $a_{ij\vartheta}$ are both the subsets of [0,1] which are denoted by $a_{ij\mu} = \left[a_{ij\mu L}, a_{ij\mu U}\right]$ and $a_{ij\vartheta} = \left[a_{ij\nu L}, a_{ij\vartheta U}\right]$ which maintaining the condition $a_{ij\mu U} + a_{ij\nu U} \leq 1$ for i = 1,2,⋯ ,m and j = 1,2,⋯ ,n.

DEFINITION 3

Interval-valued intuitionistic fuzzy determinant (IVIFD)

An interval valued intuitionistic fuzzy determinant (IVIFD) function $f : M \longrightarrow F$ is a function on the set M (of all $n \times n$ IVIFMs) to the set F, where F is the set of elements of the form $\langle [a_{\mu L}, a_{\mu U}], [a_{\vartheta L}, a_{\vartheta U}] \rangle$, maintaining the condition $0 \leq a_{\mu U} + a_{\vartheta U} \leq 1$, $0 \leq a_{\mu L} \leq a_{\mu U} \leq$

1 and $0 \leq a_{\vartheta L} \leq a_{\vartheta U} \leq 1$ and $0 \leq a_{ij\vartheta L} \leq a_{ij\vartheta U} \leq 1$ such that $A \subset M$ then $f(A)$ or $|A|$ or $det(A)$ belongs to F and is given by

$$|A| = \sum_{\sigma \epsilon S_n} \prod_{i=1}^{n} \langle [a_{i\sigma(i)\mu L}, a_{i\sigma(i)\mu U}], [a_{i\sigma(i)\vartheta L}, a_{i\sigma(i)\vartheta U}] \rangle$$

And S_n denotes the symmetric group of all permutations of the symbols$\{1,2, \dots, n\}$.

DEFINITION 4

The adjoint IVIFM of an IVIFM

The adjoint IVIFM of an IVIFM A of order$n \times n$, is denoted by $adj.A$ and is defined by$adj.A = [A_{ji}]$, where $A_{ji}$the determinant of the IVIFM A of order is $(n-1) \times (n-1)$ formed by suppressing row j and column i of the IVIFM A. In other words, A_{ji} can be written in the form

$$\sum_{\sigma \epsilon S_{n_i n_j}} \prod_{t \epsilon n_j} \langle [a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}], [a_{t\sigma(t)\vartheta L}, a_{t\sigma(t)\vartheta U}] \rangle$$

Where, $n_j = \{1,2, \dots, n\} \setminus \{j\}$ and $S_{n_i n_j}$ is the set of all permutations of set n_j over the setn_j.

Depending on the values of diagonal elements, the unit IVIFM are classified into two types:

(i) a − unit IVIFM and

(ii) r − unit IVIFM.

DEFINITION 5

Acceptance unit IVIFM (a-unit IVIFM)

A square IVIFM is a-unit IVIFM if all diagonal elements are $\langle [0,0], [1,1] \rangle$ and all remaining elements are $\langle [0,0], [1,1] \rangle$ and it is denoted by $I_{\langle [0,0],[1,1] \rangle}$.

DEFINITION 6

Rejection unit IVIFM (r-unit IVIFM)

A square IVIFM is a r-unit IVIFM if all diagonal elements are $\langle [0,0], [1,1] \rangle$ and all remaining elements are $\langle [1,1], [0,0] \rangle$ and it is denoted by $I_{\langle [0,0],[1,1] \rangle}$.

Similarly, three types of null IVIFMs are defined on its elements.

DEFINITION 7

Complete null IVIFM (c-null IVIFM)

An IVIFM is a c-null IVIFM if all the elements are $\langle [0,0], [1,1] \rangle$.

DEFINITION 8

Acceptance null IVIFM (a-null IVIFM)

An IVIFM is a a-null IVIFM if all the elements are $\langle [0,0], [1,1] \rangle$.

DEFINITION 9

Rejection null IVIFM (r-null IVIFM)

An IVIFM is a r-null IVIFM if all the elements are $\langle [1,1], [0,0] \rangle$.

CHAPTER III

OPERATIONS ON INTUITIONISTIC FUZZY MATRICES

DEFINITION

In this section we recall the notion of an intuitionistic fuzzy matrix and we define some operations on intuitionistic fuzzy matrices. As it is well known, a fuzzy matrix A is a function from the Cartesian product $X \times Y$ to the unit interval [0,1], where X and Y are finite. If $|X| = m$, $|Y| = n$, then the number $A(x_i , y_j) = a_{ij}$ for $i = 1,2, \dots , m$ and $j = 1,2, \dots , n$ is called the degree of membership of the element $A(x_i , y_j)$ in the fuzzy matrix A. Thus in briefly, a fuzzy matrix takes its elements from the interval [0,1] and we denote it by $A = a_{ij}[a_{ij}]_{m\times n}$. Now, we extend this definition to intuitionistic fuzzy matrices as follows.

Let $A' = [a'_{ij}]_{m\times n}$ and $A'' = [a''_{ij}]_{m\times n}$ be two fuzzy matrices such that $a'_{ij} + a''_{ij} \leq 1$ for every $i \leq m$, and $j \leq n$. The pair $\langle A', A''\rangle$ is called an intuitionistic fuzzy matrix and is denoted by A and then we may write $A = [a_{ij} = \langle a'_{ij}, a''_{ij}\rangle]_{m\times n}$.

As an example of an intuitionistic fuzzy matrix, we put the identity intuitionistic fuzzy matrix $I_n = [\delta_{ij} = \langle \delta'_{ij}, \delta''_{ij}\rangle]$ in the form

$$I_n = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle & \dots & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle & \dots & \langle 0,1\rangle \\ \dots & \dots & & \dots \\ \langle 0,1\rangle & \langle 0,1\rangle & \dots & \langle 1,0\rangle \end{bmatrix}_{n\times n}$$

i.e,

$$\delta'_{ij} = \begin{cases} 1 & if\ i = j, \\ 0 & if\ i \neq j. \end{cases}, \ \delta''_{ij} = \begin{cases} 0 & if\ i = j, \\ 1 & if\ i \neq j. \end{cases}.$$

We see in above Definition that the intuitionistic fuzzy matrix is a pair of fuzzy matrices which represent a membership and a non-membership function, respectively. Thus, an intuitionistic fuzzy matrix takes its elements from the set $F = \{a =< a', a'' > : a', a'' \in [0,1], a' + a'' \leq 1\}$. When $a'_{ij} + a''_{ij} = 1$ for every $i \leq m$ and $j \leq n$, the intuitionistic fuzzy matrix A is reduced to be an ordinary fuzzy matrix. The existence of the membership degree interval $[a'_{ij}, a''_{ij} = 1 - a'_{ij}]$ is always possible thanks to the condition $a'_{ij} + a''_{ij} \leq 1$ which an intuitionistic fuzzy matrix $A = [a_{ij} = \langle a'_{ij}, a''_{ij} \rangle]$ should fulfill. The number $\pi_{ij} = 1 - a'_{ij} - a''_{ij}$ is called an index of an element a_{ij} in the intuitionistic fuzzy matrix A. It is also described as an index or degree of hesitation whether a_{ij} is in the intuitionistic fuzzy matrix A or not. The larger of the intuitionistic indices π_{ij} the higher is the value of non-determinancy or uncertainty.

Now, we define some operations on the set F. For $a, b \in F$, we have:

$$a \vee b = \langle a', a'' \rangle \vee \langle b', b'' \rangle = \langle a' \vee \ b', a'' \wedge \ b'' \rangle$$

$$a \wedge b = \langle a', a'' \rangle \wedge \langle b', b'' \rangle = \langle a' \wedge \ b', a'' \wedge \ b'' \rangle$$

$a \leq b$ if and only if $a' \leq b'$ and $a'' \geq b''$where $a' \vee b' = max(a', b')$ and $a' \wedge b' = min(a', b')$.

We may write 0 instead of the element $\langle 0,1 \rangle \in F$ and 1 instead of the element $\langle 0,1 \rangle$. It is noted that $a \vee 0 = 0 \vee a = a$ and $a \vee 1 = 1 \vee a = a$, for any $a \in F$.

Basic operations on intuitionistic fuzzy matrices are extensions of the respective operations on fuzzy matrices. As a result, operations on fuzzy matrices are particular cases of the ones on intuitionistic fuzzy matrices which are defined in the following way.

DEFINITION 1

Let $A = [a_{ij} = \langle a'_{ij}, a''_{ij} \rangle]_{m \times n}$, $B = [b_{ij} = \langle b'_{ij}, b''_{ij} \rangle]_{m \times n}$ and $C = [c_{ij} = \langle c'_{ij}, c''_{ij} \rangle]_{n \times l}$ be three intuitionistic fuzzy matrices. We define the following operations:

$A \vee B = [a_{ij} \vee b_{ij}]_{m \times n}$,

$A \wedge B = [a_{ij} \wedge b_{ij}]_{m \times n}$,

$A^t = [a_{ij} = \langle a'_{ij}, a''_{ij} \rangle]$ (the transpose of A),

$$\mathrm{D} = \mathrm{AC} = \left[d_{ij} = \langle d'_{ij}, d''_{ij} \rangle = \langle \bigvee_{k=1}^{n} \left(a'_{ik} \wedge c'_{kj} \right), \bigwedge_{k=1}^{n} \left(a''_{ik} \wedge c''_{kj} \right) \rangle \right],$$

$A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for every $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

DEFINITION 2

For $a = \langle a', a'' \rangle, b = \langle b', b'' \rangle \epsilon\ F$, we define $a \rhd b$ as:

$$a \rhd b = \begin{cases} < 1,0 > & if a' \leq b' \\ < b', 0 > & if\ a' > b', a'' \geq b'' \\ < b', b'' > & if\ a' > b', a'' < b'' \end{cases}$$

This definition is an extension of the definition of Hashimoto for the ordinary fuzzy matrices which corresponds to Sanchez α operator. We recall the definition of this operation in the ordinary fuzzy case which is as follows:

$$a \rhd b = \begin{cases} 1 & if\ a \leq b \\ b & if\ a > b \end{cases}$$

for every $a, b \epsilon [0,1]$

However, the operator $\rhd$ is the Godel implication operator which is well known in many branches of fuzzy mathematics. Its properties in ordinary fuzzy case were examined by some authors. From the definition of the operation $\rhd$ on the set F, it is noted that

$a \rhd b \geq b, a \rhd 1 = 1$ and $0 \rhd a = 1$ for every $a, b \in F$. Moreover,$a \rhd b = b \lhd a$.

The $min\!-\!\rhd$ composition of two intuitionistic fuzzy matrices $A = \left[a_{ij} = \langle a'_{ij}, a''_{ij} \rangle \right]_{m \times n}$ and $C = \left[c_{ij} = \langle c'_{ij}, c''_{ij} \rangle \right]_{n \times l}$ is denoted by $D = A \rhd C$ and is defined as:

$$d_{ij} = \bigwedge\nolimits_{k=1}^{n} (a_{ik} \rhd c_{kj})$$

Reflexivity and Transitivity of Intuitionistic Fuzzy Matrices

In this section, we examine some properties of the operations defined above. Also, we examine in briefly, some properties of intuitionistic fuzzy matrices representing intuitionistic fuzzy pre-orders using the operations $\rhd$ ($\lhd$). We concentrate our discussions on the reflexive and transitive intuitionistic fuzzy matrices. Now, let us point out some useful properties of the operations $\lhd$ ($\rhd$).

LEMMA 1

For $a = \langle a', a'' \rangle, b = \langle b', b'' \rangle, c = \langle c', c'' \rangle \in$ F, we have $(a \rhd b) \lhd C = a \rhd (b \lhd c) = b \lhd (a \wedge c)$.

Proof

Based on the definition of the operation $\rhd$ on the set F, we have the following three cases, each of them has also three subcases. The cases are:

Case (1). If $a' \leq b'$.

Case (2). If $a' \leq b'$ and $a'' \leq b''$.

Case (3). If $a' \leq b'$ and $a'' \leq b''$.

The subcases of each case are:

1. If $c' \leq b'$.
2. If $c' > b'$ and $c'' \geq b''$.
3. If $c' > b'$ and $c'' < b''$.

We prove one case, namely, Case (2) and the proofs of the other two cases are similar. To do that, suppose $a' > b'$ and $a'' > b''$.

1. If $c' \leq b'$, then

$(\langle a', a'' \rangle \rhd \langle b', b'' \rangle) \lhd \langle c', c'' \rangle = \langle b', 0 \rangle \lhd \langle c', c'' \rangle = \langle 1,0 \rangle$

And

$\langle a', a'' \rangle \rhd (\langle b', b'' \rangle \rhd \langle c', c'' \rangle) = \langle a', a'' \rangle \rhd \langle 1,0 \rangle = \langle 1,0 \rangle$.

Since

$a' \wedge c' \leq b'$, we get

$\langle b', b'' \rangle \lhd (\langle a', a'' \rangle \rhd \langle c', c'' \rangle) = \langle 1,0 \rangle$.

2. If $c' \leq b'$and If $c'' \leq b''$, then

$(\langle a', a'' \rangle \rhd \langle b', b'' \rangle) \lhd \langle c', c'' \rangle = \langle b', 0 \rangle \lhd \langle c', c'' \rangle = \langle b', 0 \rangle$

And

$\langle a', a'' \rangle \rhd (\langle b', b'' \rangle \rhd \langle c', c'' \rangle) = \langle a', a'' \rangle \rhd \langle b', b'' \rangle = \langle b', 0 \rangle$.

Since

$a' \wedge c' > b'$and $a'' \wedge c'' \geq b''$ we get

$\langle b', b'' \rangle \lhd (\langle a', a'' \rangle \wedge \langle c', c'' \rangle) = \langle b', 0 \rangle$.

3. If $c' > b'$and If $c'' \geq b''$, then

$(\langle a', a'' \rangle \rhd \langle b', b'' \rangle) \lhd \langle c', c'' \rangle = \langle b', 0 \rangle \lhd \langle c', c'' \rangle = \langle b', 0 \rangle$

And

$\langle a', a'' \rangle \rhd (\langle b', b'' \rangle \rhd \langle c', c'' \rangle) = \langle a', a'' \rangle \rhd \langle b', b'' \rangle = \langle b', 0 \rangle$.

Last,

$a' \wedge c' > b'$and $a'' \vee c'' \geq b''$ imply

$\langle b', b'' \rangle \lhd (\langle a', a'' \rangle \wedge \langle c', c'' \rangle) = \langle b', 0 \rangle$.

Hence from all the above cases we conclude

$(a \rhd b) \lhd c = a \rhd (b \lhd c) = b \lhd (a \wedge c)$ for every $a, b, c \in F$.

From this lemma we can write $a \rhd b \lhd c$ instead of $(a \rhd b) \lhd c$ or $a \rhd (b \lhd c)$. That is, we may remove parentheses. Also, we note that this lemma is equivalent to the relationship

$$(a \wedge c) \rhd b = a \rhd (c \rhd b) = c \rhd (a \rhd b)$$

CHAPTER IV

SOME OPERATIONS ON IVIFM

Let

$A = [\langle[a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}]\rangle]$

and

$B = [\langle[b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}]\rangle]$

be two Inter-valued Intuitionistic Fuzzy Matrix. Then,

(i) $\langle[a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}]\rangle + \langle[b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}]\rangle =$
$\langle[max(a_{ij\mu L}, b_{ij\mu L}), max(a_{ij\mu U}, b_{ij\mu U})], [min(a_{ij\nu L}, b_{ij\nu L}), min(a_{ij\nu U}, b_{ij\nu U})]\rangle$.

(ii) $\langle[a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}]\rangle \cdot \langle[b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}]\rangle =$
$\langle[min(a_{ij\mu L}, b_{ij\mu L}), min(a_{ij\mu U}, b_{ij\mu U})], [max(a_{ij\nu L}, b_{ij\nu L}), max(a_{ij\nu U}, b_{ij\nu U})]\rangle$.

(iii) $A + B =$
$[\langle[max\{a_{ij\mu L}, b_{ij\mu L}\}, max\{a_{ij\mu U}, b_{ij\mu U}\}], [min\{a_{ij\nu L}, b_{ij\nu L}\}, min\{a_{ij\nu U}, b_{ij\nu U}\}]\rangle]$.

(iv) $A \cdot B =$
$[\langle[min\{a_{ij\mu L}, b_{ij\mu L}\}, min\{a_{ij\mu U}, b_{ij\mu U}\}], [max\{a_{ij\nu L}, b_{ij\nu L}\}, max\{a_{ij\nu U}, b_{ij\nu U}\}]\rangle]$.

(v) $\bar{A} = [\langle[a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}]\rangle]$. (complement of A)

(vi) $A^T = [\langle[a_{ji\mu L}, a_{ji\mu U}], [a_{ji\nu L}, a_{ji\nu U}]\rangle]_{n\times m}$. (transpose of A)

(vii) $A \oplus B = [\langle[a_{ij\mu L} + b_{ij\mu L} - a_{ij\mu L} \cdot b_{ij\mu L}, a_{ij\mu U} + b_{ij\mu U} - a_{ij\mu U} \cdot b_{ij\mu U}], [a_{ij\nu L} + b_{ij\nu L} - a_{ij\nu L} \cdot b_{ij\nu L}, a_{ij\nu U} + b_{ij\nu U} - a_{ij\nu U} \cdot b_{ij\nu U}]\rangle]$.

(viii) $A \odot B = [\langle[a_{ij\mu L} \cdot b_{ij\mu L}, a_{ij\mu U} \cdot b_{ij\mu U}], [a_{ij\nu L} + b_{ij\nu L} - a_{ij\nu L} \cdot b_{ij\nu L}, a_{ij\nu U} + b_{ij\nu U} - a_{ij\nu U} \cdot b_{ij\nu U}]\rangle]$.

(ix) $\mathrm{A@B} = \left[< [\frac{a_{ij\mu L} + b_{ij\mu L}}{2}, \frac{a_{ij\mu U} + b_{ij\mu U}}{2}], [\frac{a_{ij\nu L} + b_{ij\nu L}}{2}, \frac{a_{ij\nu U} + b_{ij\nu U}}{2}] >\right]$.

(x) $\mathrm{A\$B} = \left[[\sqrt{a_{ij\mu L}.\, b_{ij\mu L}}, \sqrt{a_{ij\mu U}.\, b_{ij\mu U}}], [\sqrt{a_{ij\nu L}.\, b_{ij\nu L}}, \sqrt{a_{ij\nu U}.\, b_{ij\nu U}}] >\right]$.

(xi) $\mathrm{A\#B} = \left[< [\frac{2a_{ij\mu L}.b_{ij\mu L}}{a_{ij\mu L} + b_{ij\mu L}}, \frac{2a_{ij\mu U}.b_{ij\mu U}}{a_{ij\mu U} + a_{ij\mu U}}], \left[\frac{2a_{ij\nu L}.b_{ij\nu L}}{a_{ij\nu L} + b_{ij\nu L}}, \frac{2a_{ij\nu U}.b_{ij\nu U}}{a_{ij\nu U} + b_{ij\nu U}}\right] >\right]$.

(xii) $\mathrm{A} * \mathrm{B} = \left[< \left[\frac{a_{ij\mu L} + b_{ij\mu L}}{2(a_{ij\mu L}.b_{ij\mu L} + 1)}, \frac{a_{ij\mu U} + b_{ij\mu U}}{2(a_{ij\mu U}.b_{ij\mu U} + 1)}\right],\right.$

$\left.\left[\frac{a_{ij\nu L} + b_{ij\nu L}}{2(a_{ij\nu L}.b_{ij\nu L} + 1)}, \frac{a_{ij\nu U} + b_{ij\nu U}}{2(a_{ij\nu U}.b_{ij\nu U} + 1)}\right] >\right]$

(xiii) $A \leq B$ iff $a_{ij\mu L} \leq b_{ij\mu L}$,$a_{ij\mu U} \leq b_{ij\mu U}$,$a_{ij\nu L} \geq b_{ij\nu L}$ and $a_{ij\nu U} \geq b_{ij\nu U}$.

(xiv) $A = B$ iff $A \leq B$ and $B \leq A$.

In the following section, we consider two matrix to solve and satisfy the above properties.

EXAMPLE FOR OPERATIONS ON IVIFM

EXAMPLE 1

If A and B are two inter-valued intuitionistic fuzzy matrix then prove that $A + B$ is also an inter-valued intuitionistic fuzzy matrix.

PROOF

Let A =

$$\begin{bmatrix} < [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] > & < [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] > & < [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] > \\ < [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] > & < [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] > & < [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] > \\ < [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] > & < [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] > & < [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] > \end{bmatrix}$$

and B =

$$\begin{bmatrix} < [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] > & < [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] > & < [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] > \\ < [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] > & < [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] > & < [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] > \\ < [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] > & < [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] > & < [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] > \end{bmatrix}$$

Then,

A + B

$$= \begin{bmatrix} < [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] > & < [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] > & < [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] > \\ < [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] > & < [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] > & < [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] > \\ < [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] > & < [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] > & < [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] > \end{bmatrix}$$

$$+ \begin{bmatrix} < [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] > & < [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] > & < [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] > \\ < [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] > & < [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] > & < [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] > \\ < [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] > & < [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] > & < [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] > \end{bmatrix}$$

$$\mathrm{A}+\mathrm{B}=\begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} \\ A_{21}+B_{21} & A_{22}+B_{22} & A_{23}+B_{23} \\ A_{31}+B_{31} & A_{32}+B_{32} & A_{33}+B_{33} \end{bmatrix}$$

Where,

$A_{11}=\langle[a_{11\mu L},a_{11\mu U}],[a_{11\nu L},a_{11\nu U}]\rangle$ and $B_{11}=\langle[b_{11\mu L},b_{11\mu U}],[b_{11\nu L},b_{11\nu U}]\rangle$

$A_{12}=\langle[a_{12\mu L},a_{12\mu U}],[a_{12\nu L},a_{12\nu U}]\rangle$ and $B_{12}=\langle[b_{12\mu L},b_{12\mu U}],[b_{12\nu L},b_{12\nu U}]\rangle$

$A_{13}=\langle[a_{13\mu L},a_{13\mu U}],[a_{13\nu L},a_{13\nu U}]\rangle$ and $B_{13}=\langle[b_{13\mu L},b_{13\mu U}],[b_{13\nu L},b_{13\nu U}]\rangle$

$A_{21}=\langle[a_{21\mu L},a_{21\mu U}],[a_{21\nu L},a_{21\nu U}]\rangle$ and $B_{21}=\langle[b_{21\mu L},b_{21\mu U}],[b_{21\nu L},b_{21\nu U}]\rangle$

$A_{22}=\langle[a_{22\mu L},a_{22\mu U}],[a_{22\nu L},a_{22\nu U}]\rangle$ and $B_{22}=\langle[b_{22\mu L},b_{22\mu U}],[b_{22\nu L},b_{22\nu U}]\rangle$

$A_{23}=\langle[a_{23\mu L},a_{23\mu U}],[a_{23\nu L},a_{23\nu U}]\rangle$ and $B_{23}=\langle[b_{23\mu L},b_{23\mu U}],[b_{23\nu L},b_{23\nu U}]\rangle$

$A_{31}=\langle[a_{31\mu L},a_{31\mu U}],[a_{31\nu L},a_{31\nu U}]\rangle$ and $B_{31}=\langle[b_{31\mu L},b_{31\mu U}],[b_{31\nu L},b_{31\nu U}]\rangle$

$A_{32}=\langle[a_{32\mu L},a_{32\mu U}],[a_{32\nu L},a_{32\nu U}]\rangle$ and $B_{32}=\langle[b_{32\mu L},b_{32\mu U}],[b_{32\nu L},b_{32\nu U}]\rangle$

$A_{33}=\langle[a_{33\mu L},a_{33\mu U}],[a_{33\nu L},a_{33\nu U}]\rangle$ and $B_{33}=\langle[b_{33\mu L},b_{33\mu U}],[b_{33\nu L},b_{33\nu U}]\rangle$

$$\mathrm{A}+\mathrm{B}=\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}$$

Where,

$X_{11}=[<[\max\{a_{11\mu L},b_{11\mu L}\},\max\{a_{11\mu U},b_{11\mu U}\}],[\min\{a_{11\nu L},b_{11\nu L}\},\min\{a_{11\nu U},b_{11\nu U}\}]\ >]$

$X_{12}=[<[\max\{a_{12\mu L},b_{12\mu L}\},\max\{a_{12\mu U},b_{12\mu U}\}],[\min\{a_{12\nu L},b_{12\nu L}\},\min\{a_{12\nu U},b_{12\nu U}\}]\ >]$

$X_{13}=[<[\max\{a_{13\mu L},b_{13\mu L}\},\max\{a_{13\mu U},b_{13\mu U}\}],[\min\{a_{13\nu L},b_{13\nu L}\},\min\{a_{13\nu U},b_{13\nu U}\}]\ >]$

$X_{21}=[<[\max\{a_{21\mu L},b_{21\mu L}\},\max\{a_{21\mu U},b_{21\mu U}\}],[\min\{a_{21\nu L},b_{21\nu L}\},\min\{a_{21\nu U},b_{21\nu U}\}]\ >]$

$$X_{22} = [< [\max\{a_{22\mu L}, b_{22\mu L}\}, \max\{a_{22\mu U}, b_{22\mu U}\}], [\min\{a_{22\nu L}, b_{22\nu L}\}, \min\{a_{22\nu U}, b_{22\nu U}\}] >]$$

$$X_{23} = [< [\max\{a_{23\mu L}, b_{23\mu L}\}, \max\{a_{23\mu U}, b_{23\mu U}\}], [\min\{a_{23\nu L}, b_{23\nu L}\}, \min\{a_{23\nu U}, b_{23\nu U}\}] >]$$

$$X_{31} = [< [\max\{a_{31\mu L}, b_{31\mu L}\}, \max\{a_{31\mu U}, b_{31\mu U}\}], [\min\{a_{31\nu L}, b_{31\nu L}\}, \min\{a_{31\nu U}, b_{31\nu U}\}] >]$$

$$X_{32} = [< [\max\{a_{32\mu L}, b_{32\mu L}\}, \max\{a_{32\mu U}, b_{32\mu U}\}], [\min\{a_{32\nu L}, b_{32\nu L}\}, \min\{a_{32\nu U}, b_{32\nu U}\}] >]$$

$$X_{33} = [< [\max\{a_{33\mu L}, b_{33\mu L}\}, \max\{a_{33\mu U}, b_{33\mu U}\}], [\min\{a_{33\nu L}, b_{33\nu L}\}, \min\{a_{33\nu U}, b_{33\nu U}\}] >]$$

EXAMPLE 2

If A and B are two inter-valued intuitionistic fuzzy matrix then prove that $A \cdot B$ is also an inter-valued intuitionistic fuzzy matrix.

PROOF

Let A =

$$\begin{bmatrix} < [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] > & < [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] > & < [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] > \\ < [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] > & < [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] > & < [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] > \\ < [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] > & < [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] > & < [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] > \end{bmatrix}$$

and B =

$$\begin{bmatrix} < [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] > & < [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] > & < [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] > \\ < [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] > & < [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] > & < [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] > \\ < [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] > & < [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] > & < [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] > \end{bmatrix}$$

Then

$A \cdot B$

$$= \begin{bmatrix} < [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] > & < [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] > & < [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] > \\ < [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] > & < [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] > & < [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] > \\ < [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] > & < [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] > & < [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] > \end{bmatrix}$$

$$\cdot \begin{bmatrix} < [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] > & < [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] > & < [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] > \\ < [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] > & < [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] > & < [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] > \\ < [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] > & < [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] > & < [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] > \end{bmatrix}$$

$$\mathrm{A}\cdot\mathrm{B}=\begin{bmatrix}A_{11}\cdot B_{11} & A_{12}\cdot B_{12} & A_{13}\cdot B_{13}\\ A_{21}\cdot B_{21} & A_{22}\cdot B_{22} & A_{23}\cdot B_{23}\\ A_{31}\cdot B_{31} & A_{32}\cdot B_{32} & A_{33}\cdot B_{33}\end{bmatrix}$$

Where,

$A_{11}=\langle[a_{11\mu L},a_{11\mu U}],[a_{11\nu L},a_{11\nu U}]\rangle$ and $B_{11}=\langle[b_{11\mu L},b_{11\mu U}],[b_{11\nu L},b_{11\nu U}]\rangle$

$A_{12}=\langle[a_{12\mu L},a_{12\mu U}],[a_{12\nu L},a_{12\nu U}]\rangle$ and $B_{12}=\langle[b_{12\mu L},b_{12\mu U}],[b_{12\nu L},b_{12\nu U}]\rangle$

$A_{13}=\langle[a_{13\mu L},a_{13\mu U}],[a_{13\nu L},a_{13\nu U}]\rangle$ and $B_{13}=\langle[b_{13\mu L},b_{13\mu U}],[b_{13\nu L},b_{13\nu U}]\rangle$

$A_{21}=\langle[a_{21\mu L},a_{21\mu U}],[a_{21\nu L},a_{21\nu U}]\rangle$ and $B_{21}=\langle[b_{21\mu L},b_{21\mu U}],[b_{21\nu L},b_{21\nu U}]\rangle$

$A_{22}=\langle[a_{22\mu L},a_{22\mu U}],[a_{22\nu L},a_{22\nu U}]\rangle$ and $B_{22}=\langle[b_{22\mu L},b_{22\mu U}],[b_{22\nu L},b_{22\nu U}]\rangle$

$A_{23}=\langle[a_{23\mu L},a_{23\mu U}],[a_{23\nu L},a_{23\nu U}]\rangle$ and $B_{23}=\langle[b_{23\mu L},b_{23\mu U}],[b_{23\nu L},b_{23\nu U}]\rangle$

$A_{31}=\langle[a_{31\mu L},a_{31\mu U}],[a_{31\nu L},a_{31\nu U}]\rangle$ and $B_{31}=\langle[b_{31\mu L},b_{31\mu U}],[b_{31\nu L},b_{31\nu U}]\rangle$

$A_{32}=\langle[a_{32\mu L},a_{32\mu U}],[a_{32\nu L},a_{32\nu U}]\rangle$ and $B_{32}=\langle[b_{32\mu L},b_{32\mu U}],[b_{32\nu L},b_{32\nu U}]\rangle$

$A_{33}=\langle[a_{33\mu L},a_{33\mu U}],[a_{33\nu L},a_{33\nu U}]\rangle$ and $B_{33}=\langle[b_{33\mu L},b_{33\mu U}],[b_{33\nu L},b_{33\nu U}]\rangle$

$$\mathrm{A}\cdot\mathrm{B}=\begin{bmatrix}X_{11} & X_{12} & X_{13}\\ X_{21} & X_{22} & X_{23}\\ X_{31} & X_{32} & X_{33}\end{bmatrix}$$

Where,

$X_{11}=[<[\min\{a_{11\mu L},b_{11\mu L}\},\min\{a_{11\mu U},b_{11\mu U}\}],[\max\{a_{11\nu L},b_{11\nu L}\},\max\{a_{11\nu U},b_{11\nu U}\}]\ >]$

$X_{12}=[<[\min\{a_{12\mu L},b_{12\mu L}\},\min\{a_{12\mu U},b_{12\mu U}\}],[\max\{a_{12\nu L},b_{12\nu L}\},\max\{a_{12\nu U},b_{12\nu U}\}]\ >]$

$X_{13}=[<[\min\{a_{13\mu L},b_{13\mu L}\},\min\{a_{13\mu U},b_{13\mu U}\}],[\max\{a_{13\nu L},b_{13\nu L}\},\max\{a_{13\nu U},b_{13\nu U}\}]\ >]$

$X_{21}=[<[\min\{a_{21\mu L},b_{21\mu L}\},\min\{a_{21\mu U},b_{21\mu U}\}],[\max\{a_{21\nu L},b_{21\nu L}\},\max\{a_{21\nu U},b_{21\nu U}\}]\ >]$

$X_{22} = [< [\min\{a_{22\mu L}, b_{22\mu L}\}, \min\{a_{22\mu U}, b_{22\mu U}\}], [\max\{a_{22\nu L}, b_{22\nu L}\}, \max\{a_{22\nu U}, b_{22\nu U}\}] >]$

$X_{23} = [< [\min\{a_{23\mu L}, b_{23\mu L}\}, \min\{a_{23\mu U}, b_{23\mu U}\}], [\max\{a_{23\nu L}, b_{23\nu L}\}, \max\{a_{23\nu U}, b_{23\nu U}\}] >]$

$X_{31} = [< [\min\{a_{31\mu L}, b_{31\mu L}\}, \min\{a_{31\mu U}, b_{31\mu U}\}], [\max\{a_{31\nu L}, b_{31\nu L}\}, \max\{a_{31\nu U}, b_{31\nu U}\}] >]$

$X_{32} = [< [\min\{a_{32\mu L}, b_{32\mu L}\}, \min\{a_{32\mu U}, b_{32\mu U}\}], [\max\{a_{32\nu L}, b_{32\nu L}\}, \max\{a_{32\nu U}, b_{32\nu U}\}] >]$

$X_{33} = [< [\min\{a_{33\mu L}, b_{33\mu L}\}, \min\{a_{33\mu U}, b_{33\mu U}\}], [\max\{a_{33\nu L}, b_{33\nu L}\}, \max\{a_{33\nu U}, b_{33\nu U}\}] >]$

EXAMPLE 3

If A and B are two inter-valued intuitionistic fuzzy matrix then prove that A⊕B is also an inter-valued intuitionistic fuzzy matrix.

PROOF

Let A =

$$\begin{bmatrix} < [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] > & < [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] > & < [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] > \\ < [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] > & < [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] > & < [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] > \\ < [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] > & < [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] > & < [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] > \end{bmatrix}$$

and B =

$$\begin{bmatrix} < [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] > & < [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] > & < [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] > \\ < [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] > & < [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] > & < [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] > \\ < [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] > & < [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] > & < [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] > \end{bmatrix}$$

Then

A⊕B

$$= \begin{bmatrix} < [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] > & < [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] > & < [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] > \\ < [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] > & < [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] > & < [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] > \\ < [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] > & < [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] > & < [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] > \end{bmatrix} \oplus$$

$$\begin{bmatrix} < [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] > & < [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] > & < [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] > \\ < [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] > & < [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] > & < [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] > \\ < [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] > & < [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] > & < [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] > \end{bmatrix}$$

$$A \oplus B = \begin{bmatrix} A_{11} \oplus B_{11} & A_{12} \oplus B_{12} & A_{13} \oplus B_{13} \\ A_{21} \oplus B_{21} & A_{22} \oplus B_{22} & A_{23} \oplus B_{23} \\ A_{31} \oplus B_{31} & A_{32} \oplus B_{32} & A_{33} \oplus B_{33} \end{bmatrix}$$

Where,

$A_{11} = \langle [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] \rangle$ and $B_{11} = \langle [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] \rangle$

$A_{12} = \langle [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] \rangle$ and $B_{12} = \langle [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] \rangle$

$A_{13} = \langle [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] \rangle$ and $B_{13} = \langle [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] \rangle$

$A_{21} = \langle [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] \rangle$ and $B_{21} = \langle [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] \rangle$

$A_{22} = \langle [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] \rangle$ and $B_{22} = \langle [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] \rangle$

$A_{23} = \langle [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] \rangle$ and $B_{23} = \langle [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] \rangle$

$A_{31} = \langle [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] \rangle$ and $B_{31} = \langle [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] \rangle$

$A_{32} = \langle [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] \rangle$ and $B_{32} = \langle [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] \rangle$

$A_{33} = \langle [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] \rangle$ and $B_{33} = \langle [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] \rangle$

$$A \oplus B = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}$$

Where,

$$\boldsymbol{X_{11}} = [< [a_{11\mu L} + b_{11\mu L} - a_{11\mu L} \cdot b_{11\mu L}, a_{11\mu U} + b_{11\mu U} - a_{11\mu U} \cdot b_{11\mu U}],$$
$$[a_{11\nu L} + b_{11\nu L} - a_{11\nu L} \cdot b_{11\nu L}, a_{11\nu U} + b_{11\nu U} - a_{11\nu U} \cdot b_{11\nu U}] >]$$

$$\boldsymbol{X_{12}} = [< [a_{12\mu L} + b_{12\mu L} - a_{12\mu L} \cdot b_{12\mu L}, a_{12\mu U} + b_{12\mu U} - a_{12\mu U} \cdot b_{12\mu U}],$$
$$[a_{12\nu L} + b_{12\nu L} - a_{12\nu L} \cdot b_{12\nu L}, a_{12\nu U} + b_{12\nu U} - a_{12\nu U} \cdot b_{12\nu U}] >]$$

$$\boldsymbol{X_{13}} = [< [a_{13\mu L} + b_{13\mu L} - a_{13\mu L} \cdot b_{13\mu L}, a_{13\mu U} + b_{13\mu U} - a_{13\mu U} \cdot b_{13\mu U}],$$
$$[a_{13\nu L} + b_{13\nu L} - a_{13\nu L} \cdot b_{13\nu L}, a_{13\nu U} + b_{13\nu U} - a_{13\nu U} \cdot b_{13\nu U}] >]$$

$$\boldsymbol{X_{21}} = [< [a_{21\mu L} + b_{21\mu L} - a_{21\mu L} \cdot b_{21\mu L}, a_{21\mu U} + b_{21\mu U} - a_{21\mu U} \cdot b_{21\mu U}],$$
$$[a_{21\nu L} + b_{21\nu L} - a_{21\nu L} \cdot b_{21\nu L}, a_{21\nu U} + b_{21\nu U} - a_{21\nu U} \cdot b_{21\nu U}] >]$$

$$\boldsymbol{X_{22}} = [< [a_{22\mu L} + b_{22\mu L} - a_{22\mu L} \cdot b_{22\mu L}, a_{22\mu U} + b_{22\mu U} - a_{22\mu U} \cdot b_{22\mu U}],$$
$$[a_{22\nu L} + b_{22\nu L} - a_{22\nu L} \cdot b_{22\nu L}, a_{22\nu U} + b_{22\nu U} - a_{22\nu U} \cdot b_{22\nu U}] >]$$

$$\boldsymbol{X_{23}} = [< [a_{23\mu L} + b_{23\mu L} - a_{23\mu L} \cdot b_{23\mu L}, a_{23\mu U} + b_{23\mu U} - a_{23\mu U} \cdot b_{23\mu U}],$$
$$[a_{23\nu L} + b_{23\nu L} - a_{23\nu L} \cdot b_{23\nu L}, a_{23\nu U} + b_{23\nu U} - a_{23\nu U} \cdot b_{23\nu U}] >]$$

$$\boldsymbol{X_{31}} = [< [a_{31\mu L} + b_{31\mu L} - a_{31\mu L} \cdot b_{31\mu L}, a_{31\mu U} + b_{31\mu U} - a_{31\mu U} \cdot b_{31\mu U}],$$
$$[a_{31\nu L} + b_{31\nu L} - a_{31\nu L} \cdot b_{31\nu L}, a_{31\nu U} + b_{31\nu U} - a_{31\nu U} \cdot b_{31\nu U}] >]$$

$$\boldsymbol{X_{32}} = [< [a_{32\mu L} + b_{32\mu L} - a_{32\mu L} \cdot b_{32\mu L}, a_{32\mu U} + b_{32\mu U} - a_{32\mu U} \cdot b_{32\mu U}],$$
$$[a_{32\nu L} + b_{32\nu L} - a_{32\nu L} \cdot b_{32\nu L}, a_{32\nu U} + b_{32\nu U} - a_{32\nu U} \cdot b_{32\nu U}] >]$$

$$\boldsymbol{X_{33}} = [< [a_{33\mu L} + b_{33\mu L} - a_{33\mu L} \cdot b_{33\mu L}, a_{33\mu U} + b_{33\mu U} - a_{33\mu U} \cdot b_{33\mu U}],$$
$$[a_{33\nu L} + b_{33\nu L} - a_{33\nu L} \cdot b_{33\nu L}, a_{33\nu U} + b_{33\nu U} - a_{33\nu U} \cdot b_{33\nu U}] >]$$

EXAMPLE 4

If A and B are two inter-valued intuitionistic fuzzy matrix then prove that $A \odot B$ is also an inter-valued intuitionistic fuzzy matrix.

PROOF

Let A =

$$\begin{bmatrix} <[a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}]> & <[a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}]> & <[a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}]> \\ <[a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}]> & <[a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}]> & <[a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}]> \\ <[a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}]> & <[a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}]> & <[a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}]> \end{bmatrix}$$

and B =

$$\begin{bmatrix} <[b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}]> & <[b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}]> & <[b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}]> \\ <[b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}]> & <[b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}]> & <[b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}]> \\ <[b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}]> & <[b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}]> & <[b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}]> \end{bmatrix}$$

Then

A $\odot$ B

$$= \begin{bmatrix} <[a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}]> & <[a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}]> & <[a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}]> \\ <[a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}]> & <[a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}]> & <[a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}]> \\ <[a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}]> & <[a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}]> & <[a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}]> \end{bmatrix}$$

$$\odot \begin{bmatrix} <[b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}]> & <[b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}]> & <[b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}]> \\ <[b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}]> & <[b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}]> & <[b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}]> \\ <[b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}]> & <[b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}]> & <[b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}]> \end{bmatrix}$$

$$\mathrm{A} \odot \mathrm{B} = \begin{bmatrix} A_{11} \odot B_{11} & A_{12} \odot B_{12} & A_{13} \odot B_{13} \\ A_{21} \odot B_{21} & A_{22} \odot B_{22} & A_{23} \odot B_{23} \\ A_{31} \odot B_{31} & A_{32} \odot B_{32} & A_{33} \odot B_{33} \end{bmatrix}$$

Where,

$A_{11} = \langle [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] \rangle$ and $B_{11} = \langle [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] \rangle$

$A_{12} = \langle [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] \rangle$ and $B_{12} = \langle [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] \rangle$

$A_{13} = \langle [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] \rangle$ and $B_{13} = \langle [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] \rangle$

$A_{21} = \langle [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] \rangle$ and $B_{21} = \langle [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] \rangle$

$A_{22} = \langle [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] \rangle$ and $B_{22} = \langle [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] \rangle$

$A_{23} = \langle [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] \rangle$ and $B_{23} = \langle [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] \rangle$

$A_{31} = \langle [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] \rangle$ and $B_{31} = \langle [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] \rangle$

$A_{32} = \langle [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] \rangle$ and $B_{32} = \langle [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] \rangle$

$A_{33} = \langle [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] \rangle$ and $B_{33} = \langle [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] \rangle$

$$\mathrm{A} \odot \mathrm{B} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}$$

Where,

$$\boldsymbol{X_{11}} = [< [a_{11\mu L}.b_{11\mu L}, a_{11\mu U}.b_{ij\mu U}],$$
$$[a_{11\nu L} + b_{11\nu L} - a_{11\nu L}.b_{11\nu L}, a_{11\nu U} + b_{11\nu U} - a_{11\nu U}.b_{11\nu U}] >]$$

$$\boldsymbol{X_{12}} = [< [a_{12\mu L}.b_{12\mu L}, a_{12\mu U}.b_{ij\mu U}],$$
$$[a_{12\nu L} + b_{12\nu L} - a_{12\nu L}.b_{12\nu L}, a_{12\nu U} + b_{12\nu U} - a_{12\nu U}.b_{12\nu U}] >]$$

$$\boldsymbol{X_{13}} = [< [a_{13\mu L}.b_{13\mu L}, a_{13\mu U}.b_{ij\mu U}],$$
$$[a_{13\nu L} + b_{13\nu L} - a_{13\nu L}.b_{13\nu L}, a_{13\nu U} + b_{13\nu U} - a_{13\nu U}.b_{13\nu U}] >]$$

$$\boldsymbol{X_{21}} = [< [a_{21\mu L}.b_{21\mu L}, a_{21\mu U}.b_{ij\mu U}],$$
$$[a_{21\nu L} + b_{21\nu L} - a_{21\nu L}.b_{21\nu L}, a_{21\nu U} + b_{21\nu U} - a_{21\nu U}.b_{21\nu U}] >]$$

$$\boldsymbol{X_{22}} = [< [a_{22\mu L}.b_{22\mu L}, a_{22\mu U}.b_{ij\mu U}],$$
$$[a_{22\nu L} + b_{22\nu L} - a_{22\nu L}.b_{22\nu L}, a_{22\nu U} + b_{22\nu U} - a_{22\nu U}.b_{22\nu U}] >]$$

$$\boldsymbol{X_{23}} = [< [a_{23\mu L}.b_{23\mu L}, a_{23\mu U}.b_{ij\mu U}],$$
$$[a_{23\nu L} + b_{23\nu L} - a_{23\nu L}.b_{23\nu L}, a_{23\nu U} + b_{23\nu U} - a_{23\nu U}.b_{23\nu U}] >]$$

$$X_{31} = [< [a_{31\mu L}.b_{31\mu L}, a_{31\mu U}.b_{ij\mu U}],$$
$$[a_{31\nu L} + b_{31\nu L} - a_{31\nu L}.b_{31\nu L}, a_{31\nu U} + b_{31\nu U} - a_{31\nu U}.b_{31\nu U}] >]$$

$$X_{32} = [< [a_{32\mu L}.b_{32\mu L}, a_{32\mu U}.b_{ij\mu U}],$$
$$[a_{32\nu L} + b_{32\nu L} - a_{32\nu L}.b_{32\nu L}, a_{32\nu U} + b_{32\nu U} - a_{32\nu U}.b_{32\nu U}] >]$$

$$X_{33} = [< [a_{33\mu L}.b_{33\mu L}, a_{33\mu U}.b_{ij\mu U}],$$
$$[a_{33\nu L} + b_{33\nu L} - a_{33\nu L}.b_{33\nu L}, a_{33\nu U} + b_{33\nu U} - a_{33\nu U}.b_{33\nu U}] >]$$

EXAMPLE 5

If A and B are two inter-valued intuitionistic fuzzy matrix then prove that $A@B$ is also an inter-valued intuitionistic fuzzy matrix.

PROOF

Let A =

$$\begin{bmatrix} < [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] > & < [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] > & < [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] > \\ < [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] > & < [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] > & < [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] > \\ < [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] > & < [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] > & < [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] > \end{bmatrix}$$

and B =

$$\begin{bmatrix} < [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] > & < [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] > & < [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] > \\ < [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] > & < [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] > & < [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] > \\ < [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] > & < [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] > & < [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] > \end{bmatrix}$$

Then

A@B

$$= \begin{bmatrix} < [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] > & < [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] > & < [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] > \\ < [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] > & < [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] > & < [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] > \\ < [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] > & < [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] > & < [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] > \end{bmatrix} @$$

$$\begin{bmatrix} < [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] > & < [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] > & < [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] > \\ < [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] > & < [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] > & < [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] > \\ < [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] > & < [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] > & < [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] > \end{bmatrix}$$

$$\text{A@B} = \begin{bmatrix} A_{11}@B_{11} & A_{12}@B_{12} & A_{13}@B_{13} \\ A_{21}@B_{21} & A_{22}@B_{22} & A_{23}@B_{23} \\ A_{31}@B_{31} & A_{32}@B_{32} & A_{33}@B_{33} \end{bmatrix}$$

Where,

$A_{11} = \langle [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] \rangle$ and $B_{11} = \langle [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] \rangle$

$A_{12} = \langle [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] \rangle$ and $B_{12} = \langle [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] \rangle$

$A_{13} = \langle [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] \rangle$ and $B_{13} = \langle [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] \rangle$

$A_{21} = \langle [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] \rangle$ and $B_{21} = \langle [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] \rangle$

$A_{22} = \langle [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] \rangle$ and $B_{22} = \langle [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] \rangle$

$A_{23} = \langle [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] \rangle$ and $B_{23} = \langle [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] \rangle$

$A_{31} = \langle [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] \rangle$ and $B_{31} = \langle [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] \rangle$

$A_{32} = \langle [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] \rangle$ and $B_{32} = \langle [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] \rangle$

$A_{33} = \langle [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] \rangle$ and $B_{33} = \langle [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] \rangle$

$$\text{A@B} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}$$

Where,

$$\boldsymbol{X_{11}} = \left[< [\frac{a_{11\mu L} + b_{11\mu L}}{2}, \frac{a_{11\mu U} + b_{11\mu U}}{2}], [\frac{a_{11\nu L} + b_{11\nu L}}{2}, \frac{a_{11\nu U} + b_{11\nu U}}{2}] > \right]$$

$$\boldsymbol{X_{12}} = \left[< [\frac{a_{12\mu L} + b_{12\mu L}}{2}, \frac{a_{12\mu U} + b_{12\mu U}}{2}], [\frac{a_{12\nu L} + b_{12\nu L}}{2}, \frac{a_{12\nu U} + b_{12\nu U}}{2}] > \right]$$

$$\boldsymbol{X_{13}} = \left[< [\frac{a_{13\mu L} + b_{13\mu L}}{2}, \frac{a_{13\mu U} + b_{13\mu U}}{2}], [\frac{a_{13\nu L} + b_{13\nu L}}{2}, \frac{a_{13\nu U} + b_{13\nu U}}{2}] >\right]$$

$$\boldsymbol{X_{21}} = \left[< [\frac{a_{21\mu L} + b_{21\mu L}}{2}, \frac{a_{21\mu U} + b_{21\mu U}}{2}], [\frac{a_{21\nu L} + b_{21\nu L}}{2}, \frac{a_{21\nu U} + b_{21\nu U}}{2}] >\right]$$

$$\boldsymbol{X_{22}} = \left[< [\frac{a_{22\mu L} + b_{22\mu L}}{2}, \frac{a_{22\mu U} + b_{22\mu U}}{2}], [\frac{a_{22\nu L} + b_{22\nu L}}{2}, \frac{a_{22\nu U} + b_{22\nu U}}{2}] >\right]$$

$$\boldsymbol{X_{23}} = \left[< [\frac{a_{23\mu L} + b_{23\mu L}}{2}, \frac{a_{23\mu U} + b_{23\mu U}}{2}], [\frac{a_{23\nu L} + b_{23\nu L}}{2}, \frac{a_{23\nu U} + b_{23\nu U}}{2}] >\right]$$

$$\boldsymbol{X_{31}} = \left[< [\frac{a_{31\mu L} + b_{31\mu L}}{2}, \frac{a_{31\mu U} + b_{31\mu U}}{2}], [\frac{a_{31\nu L} + b_{31\nu L}}{2}, \frac{a_{31\nu U} + b_{31\nu U}}{2}] >\right]$$

$$\boldsymbol{X_{32}} = \left[< [\frac{a_{32\mu L} + b_{32\mu L}}{2}, \frac{a_{32\mu U} + b_{32\mu U}}{2}], [\frac{a_{32\nu L} + b_{32\nu L}}{2}, \frac{a_{32\nu U} + b_{32\nu U}}{2}] >\right]$$

$$\boldsymbol{X_{33}} = \left[< [\frac{a_{33\mu L} + b_{33\mu L}}{2}, \frac{a_{33\mu U} + b_{33\mu U}}{2}], [\frac{a_{33\nu L} + b_{33\nu L}}{2}, \frac{a_{33\nu U} + b_{33\nu U}}{2}] >\right]$$

EXAMPLE 6

If A and B are two inter-valued intuitionistic fuzzy matrix then prove that $A\$B$ is also an inter-valued intuitionistic fuzzy matrix.

PROOF

Let A =

$$\begin{bmatrix} < [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] > & < [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] > & < [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] > \\ < [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] > & < [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] > & < [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] > \\ < [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] > & < [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] > & < [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] > \end{bmatrix}$$

and B =

$$\begin{bmatrix} < [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] > & < [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] > & < [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] > \\ < [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] > & < [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] > & < [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] > \\ < [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] > & < [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] > & < [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] > \end{bmatrix}$$

Then

A$B

$$= \begin{bmatrix} <[a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}]> & <[a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}]> & <[a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}]> \\ <[a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}]> & <[a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}]> & <[a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}]> \\ <[a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}]> & <[a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}]> & <[a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}]> \end{bmatrix} \$$$

$$\begin{bmatrix} <[b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}]> & <[b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}]> & <[b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}]> \\ <[b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}]> & <[b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}]> & <[b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}]> \\ <[b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}]> & <[b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}]> & <[b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}]> \end{bmatrix}$$

$$\text{A\$B} = \begin{bmatrix} A_{11}\$B_{11} & A_{12}\$B_{12} & A_{13}\$B_{13} \\ A_{21}\$B_{21} & A_{22}\$B_{22} & A_{23}\$B_{23} \\ A_{31}\$B_{31} & A_{32}\$B_{32} & A_{33}\$B_{33} \end{bmatrix}$$

Where,

$A_{11} = \langle [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] \rangle$ and $B_{11} = \langle [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] \rangle$

$A_{12} = \langle [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] \rangle$ and $B_{12} = \langle [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] \rangle$

$A_{13} = \langle [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] \rangle$ and $B_{13} = \langle [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] \rangle$

$A_{21} = \langle [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] \rangle$ and $B_{21} = \langle [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] \rangle$

$A_{22} = \langle [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] \rangle$ and $B_{22} = \langle [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] \rangle$

$A_{23} = \langle [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] \rangle$ and $B_{23} = \langle [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] \rangle$

$A_{31} = \langle [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] \rangle$ and $B_{31} = \langle [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] \rangle$

$A_{32} = \langle [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] \rangle$ and $B_{32} = \langle [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] \rangle$

$A_{33} = \langle [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] \rangle$ and $B_{33} = \langle [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] \rangle$

$$\text{A\$B} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}$$

Where,

$$\boldsymbol{X_{11}} = \left[\left[\sqrt{a_{11\mu L}.b_{11\mu L}}, \sqrt{a_{11\mu U}.b_{11\mu U}}\right], [\sqrt{a_{11\nu L}.b_{11\nu L}}, \sqrt{a_{11\nu U}.b_{11\nu U}}] >\right]$$

$$\boldsymbol{X_{12}} = \left[\left[\sqrt{a_{12\mu L}.b_{12\mu L}}, \sqrt{a_{12\mu U}.b_{12\mu U}}\right], [\sqrt{a_{12\nu L}.b_{12\nu L}}, \sqrt{a_{12\nu U}.b_{12\nu U}}] >\right]$$

$$\boldsymbol{X_{13}} = \left[\left[\sqrt{a_{13\mu L}.b_{13\mu L}}, \sqrt{a_{13\mu U}.b_{13\mu U}}\right], [\sqrt{a_{13\nu L}.b_{13\nu L}}, \sqrt{a_{13\nu U}.b_{13\nu U}}] >\right]$$

$$\boldsymbol{X_{21}} = \left[\left[\sqrt{a_{21\mu L}.b_{21\mu L}}, \sqrt{a_{21\mu U}.b_{21\mu U}}\right], [\sqrt{a_{21\nu L}.b_{21\nu L}}, \sqrt{a_{21\nu U}.b_{21\nu U}}] >\right]$$

$$\boldsymbol{X_{22}} = \left[\left[\sqrt{a_{22\mu L}.b_{22\mu L}}, \sqrt{a_{22\mu U}.b_{22\mu U}}\right], [\sqrt{a_{22\nu L}.b_{22\nu L}}, \sqrt{a_{22\nu U}.b_{22\nu U}}] >\right]$$

$$\boldsymbol{X_{23}} = \left[\left[\sqrt{a_{23\mu L}.b_{23\mu L}}, \sqrt{a_{23\mu U}.b_{23\mu U}}\right], [\sqrt{a_{23\nu L}.b_{23\nu L}}, \sqrt{a_{23\nu U}.b_{23\nu U}}] >\right]$$

$$\boldsymbol{X_{31}} = \left[\left[\sqrt{a_{31\mu L}.b_{31\mu L}}, \sqrt{a_{31\mu U}.b_{31\mu U}}\right], [\sqrt{a_{31\nu L}.b_{31\nu L}}, \sqrt{a_{31\nu U}.b_{31\nu U}}] >\right]$$

$$\boldsymbol{X_{32}} = \left[\left[\sqrt{a_{32\mu L}.b_{32\mu L}}, \sqrt{a_{32\mu U}.b_{32\mu U}}\right], [\sqrt{a_{32\nu L}.b_{32\nu L}}, \sqrt{a_{32\nu U}.b_{32\nu U}}] >\right]$$

$$\boldsymbol{X_{33}} = \left[\left[\sqrt{a_{33\mu L}.b_{33\mu L}}, \sqrt{a_{33\mu U}.b_{33\mu U}}\right], [\sqrt{a_{33\nu L}.b_{33\nu L}}, \sqrt{a_{33\nu U}.b_{33\nu U}}] >\right]$$

EXAMPLE 7

If A and B are two inter-valued intuitionistic fuzzy matrix then prove that $A\#B$ is also an inter-valued intuitionistic fuzzy matrix.

PROOF

Let A =

$$\begin{bmatrix} <[a_{11\mu L},a_{11\mu U}],[a_{11\nu L},a_{11\nu U}]> & <[a_{12\mu L},a_{12\mu U}],[a_{12\nu L},a_{12\nu U}]> & <[a_{13\mu L},a_{13\mu U}],[a_{13\nu L},a_{13\nu U}]> \\ <[a_{21\mu L},a_{21\mu U}],[a_{21\nu L},a_{21\nu U}]> & <[a_{22\mu L},a_{22\mu U}],[a_{22\nu L},a_{22\nu U}]> & <[a_{23\mu L},a_{23\mu U}],[a_{23\nu L},a_{23\nu U}]> \\ <[a_{31\mu L},a_{31\mu U}],[a_{31\nu L},a_{31\nu U}]> & <[a_{32\mu L},a_{32\mu U}],[a_{32\nu L},a_{32\nu U}]> & <[a_{33\mu L},a_{33\mu U}],[a_{33\nu L},a_{33\nu U}]> \end{bmatrix}$$

and B =

$$\begin{bmatrix} <[b_{11\mu L},b_{11\mu U}],[b_{11\nu L},b_{11\nu U}]> & <[b_{12\mu L},b_{12\mu U}],[b_{12\nu L},b_{12\nu U}]> & <[b_{13\mu L},b_{13\mu U}],[b_{13\nu L},b_{13\nu U}]> \\ <[b_{21\mu L},b_{21\mu U}],[b_{21\nu L},b_{21\nu U}]> & <[b_{22\mu L},b_{22\mu U}],[b_{22\nu L},b_{22\nu U}]> & <[b_{23\mu L},b_{23\mu U}],[b_{23\nu L},b_{23\nu U}]> \\ <[b_{31\mu L},b_{31\mu U}],[b_{31\nu L},b_{31\nu U}]> & <[b_{32\mu L},b_{32\mu U}],[b_{32\nu L},b_{32\nu U}]> & <[b_{33\mu L},b_{33\mu U}],[b_{33\nu L},b_{33\nu U}]> \end{bmatrix}$$

Then

A#B

$$= \begin{bmatrix} <[a_{11\mu L},a_{11\mu U}],[a_{11\nu L},a_{11\nu U}]> & <[a_{12\mu L},a_{12\mu U}],[a_{12\nu L},a_{12\nu U}]> & <[a_{13\mu L},a_{13\mu U}],[a_{13\nu L},a_{13\nu U}]> \\ <[a_{21\mu L},a_{21\mu U}],[a_{21\nu L},a_{21\nu U}]> & <[a_{22\mu L},a_{22\mu U}],[a_{22\nu L},a_{22\nu U}]> & <[a_{23\mu L},a_{23\mu U}],[a_{23\nu L},a_{23\nu U}]> \\ <[a_{31\mu L},a_{31\mu U}],[a_{31\nu L},a_{31\nu U}]> & <[a_{32\mu L},a_{32\mu U}],[a_{32\nu L},a_{32\nu U}]> & <[a_{33\mu L},a_{33\mu U}],[a_{33\nu L},a_{33\nu U}]> \end{bmatrix} \#$$

$$\begin{bmatrix} <[b_{11\mu L},b_{11\mu U}],[b_{11\nu L},b_{11\nu U}]> & <[b_{12\mu L},b_{12\mu U}],[b_{12\nu L},b_{12\nu U}]> & <[b_{13\mu L},b_{13\mu U}],[b_{13\nu L},b_{13\nu U}]> \\ <[b_{21\mu L},b_{21\mu U}],[b_{21\nu L},b_{21\nu U}]> & <[b_{22\mu L},b_{22\mu U}],[b_{22\nu L},b_{22\nu U}]> & <[b_{23\mu L},b_{23\mu U}],[b_{23\nu L},b_{23\nu U}]> \\ <[b_{31\mu L},b_{31\mu U}],[b_{31\nu L},b_{31\nu U}]> & <[b_{32\mu L},b_{32\mu U}],[b_{32\nu L},b_{32\nu U}]> & <[b_{33\mu L},b_{33\mu U}],[b_{33\nu L},b_{33\nu U}]> \end{bmatrix}$$

$$\mathrm{A\#B} = \begin{bmatrix} A_{11}\#B_{11} & A_{12}\#B_{12} & A_{13}\#B_{13} \\ A_{21}\#B_{21} & A_{22}\#B_{22} & A_{23}\#B_{23} \\ A_{31}\#B_{31} & A_{32}\#B_{32} & A_{33}\#B_{33} \end{bmatrix}$$

Where,

$A_{11} = \langle [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] \rangle$ and $B_{11} = \langle [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] \rangle$

$A_{12} = \langle [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] \rangle$ and $B_{12} = \langle [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] \rangle$

$A_{13} = \langle [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] \rangle$ and $B_{13} = \langle [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] \rangle$

$A_{21} = \langle [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] \rangle$ and $B_{21} = \langle [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] \rangle$

$A_{22} = \langle [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] \rangle$ and $B_{22} = \langle [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] \rangle$

$A_{23} = \langle [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] \rangle$ and $B_{23} = \langle [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] \rangle$

$A_{31} = \langle [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] \rangle$ and $B_{31} = \langle [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] \rangle$

$A_{32} = \langle [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] \rangle$ and $B_{32} = \langle [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] \rangle$

$A_{33} = \langle [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] \rangle$ and $B_{33} = \langle [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] \rangle$

$$\mathrm{A\#B} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}$$

Where,

$$\boldsymbol{X_{11}} = \left[< [\frac{2a_{11\mu L}.b_{11\mu L}}{a_{11\mu L} + b_{11\mu L}}, \frac{2a_{11\mu U}.b_{11\mu U}}{a_{11\mu U} + a_{11\mu U}}], [\frac{2a_{11\nu L}.b_{11\nu L}}{a_{11\nu L} + b_{11\nu L}}, \frac{2a_{11\nu U}.b_{11\nu U}}{a_{11\nu U} + b_{11\nu U}}] > \right]$$

$$\boldsymbol{X_{12}} = \left[< [\frac{2a_{12\mu L}.b_{12\mu L}}{a_{12\mu L} + b_{12\mu L}}, \frac{2a_{12\mu U}.b_{12\mu U}}{a_{12\mu U} + a_{12\mu U}}], [\frac{2a_{12\nu L}.b_{12\nu L}}{a_{12\nu L} + b_{12\nu L}}, \frac{2a_{12\nu U}.b_{12\nu U}}{a_{12\nu U} + b_{12\nu U}}] > \right]$$

$$\boldsymbol{X_{13}} = \left[< [\frac{2a_{13\mu L}.b_{13\mu L}}{a_{13\mu L} + b_{13\mu L}}, \frac{2a_{13\mu U}.b_{13\mu U}}{a_{13\mu U} + a_{13\mu U}}], [\frac{2a_{13\nu L}.b_{13\nu L}}{a_{13\nu L} + b_{13\nu L}}, \frac{2a_{13\nu U}.b_{13\nu U}}{a_{13\nu U} + b_{13\nu U}}] > \right]$$

$$\boldsymbol{X_{21}} = \left[< [\frac{2a_{21\mu L}.b_{21\mu L}}{a_{21\mu L} + b_{21\mu L}}, \frac{2a_{21\mu U}.b_{21\mu U}}{a_{21\mu U} + a_{21\mu U}}], [\frac{2a_{21\nu L}.b_{21\nu L}}{a_{21\nu L} + b_{21\nu L}}, \frac{2a_{21\nu U}.b_{21\nu U}}{a_{21\nu U} + b_{21\nu U}}] > \right]$$

$$\boldsymbol{X_{22}} = \left[< [\frac{2a_{22\mu L}.b_{22\mu L}}{a_{22\mu L} + b_{22\mu L}}, \frac{2a_{22\mu U}.b_{22\mu U}}{a_{22\mu U} + a_{22\mu U}}], [\frac{2a_{22\nu L}.b_{22\nu L}}{a_{22\nu L} + b_{22\nu L}}, \frac{2a_{22\nu U}.b_{22\nu U}}{a_{22\nu U} + b_{22\nu U}}] > \right]$$

$$\boldsymbol{X_{23}} = \left[< [\frac{2a_{23\mu L}.b_{23\mu L}}{a_{23\mu L} + b_{23\mu L}}, \frac{2a_{23\mu U}.b_{23\mu U}}{a_{23\mu U} + a_{23\mu U}}], [\frac{2a_{23\nu L}.b_{23\nu L}}{a_{23\nu L} + b_{23\nu L}}, \frac{2a_{23\nu U}.b_{23\nu U}}{a_{23\nu U} + b_{23\nu U}}] > \right]$$

$$\boldsymbol{X_{31}} = \left[< [\frac{2a_{31\mu L}.b_{31\mu L}}{a_{31\mu L} + b_{31\mu L}}, \frac{2a_{31\mu U}.b_{31\mu U}}{a_{31\mu U} + a_{31\mu U}}], [\frac{2a_{31\nu L}.b_{31\nu L}}{a_{31\nu L} + b_{31\nu L}}, \frac{2a_{31\nu U}.b_{31\nu U}}{a_{31\nu U} + b_{31\nu U}}] > \right]$$

$$\boldsymbol{X_{32}} = \left[< [\frac{2a_{32\mu L}.b_{32\mu L}}{a_{32\mu L} + b_{32\mu L}}, \frac{2a_{32\mu U}.b_{32\mu U}}{a_{32\mu U} + a_{32\mu U}}], [\frac{2a_{32\nu L}.b_{32\nu L}}{a_{32\nu L} + b_{32\nu L}}, \frac{2a_{32\nu U}.b_{32\nu U}}{a_{32\nu U} + b_{32\nu U}}] > \right]$$

$$\boldsymbol{X_{33}} = \left[< [\frac{2a_{33\mu L}.b_{33\mu L}}{a_{33\mu L} + b_{33\mu L}}, \frac{2a_{33\mu U}.b_{33\mu U}}{a_{33\mu U} + a_{33\mu U}}], [\frac{2a_{33\nu L}.b_{33\nu L}}{a_{33\nu L} + b_{33\nu L}}, \frac{2a_{33\nu U}.b_{33\nu U}}{a_{33\nu U} + b_{33\nu U}}] > \right]$$

EXAMPLE 8

If A and B are two inter-valued intuitionistic fuzzy matrix then prove that $A\#B$ is also an inter-valued intuitionistic fuzzy matrix.

PROOF

Let A =

$$\begin{bmatrix} < [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] > & < [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] > & < [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] > \\ < [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] > & < [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] > & < [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] > \\ < [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] > & < [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] > & < [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] > \end{bmatrix}$$

and B =

$$\begin{bmatrix} < [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] > & < [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] > & < [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] > \\ < [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] > & < [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] > & < [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] > \\ < [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] > & < [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] > & < [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] > \end{bmatrix}$$

Then

A ∗ B

$$= \begin{bmatrix} < [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] > & < [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] > & < [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] > \\ < [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] > & < [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] > & < [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] > \\ < [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] > & < [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] > & < [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] > \end{bmatrix}$$

$$* \begin{bmatrix} < [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] > & < [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] > & < [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] > \\ < [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] > & < [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] > & < [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] > \\ < [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] > & < [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] > & < [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] > \end{bmatrix}$$

$$\mathrm{A} * \mathrm{B} = \begin{bmatrix} A_{11} * B_{11} & A_{12} * B_{12} & A_{13} * B_{13} \\ A_{21} * B_{21} & A_{22} * B_{22} & A_{23} * B_{23} \\ A_{31} * B_{31} & A_{32} * B_{32} & A_{33} * B_{33} \end{bmatrix}$$

Where,

$A_{11} = \langle [a_{11\mu L}, a_{11\mu U}], [a_{11\nu L}, a_{11\nu U}] \rangle$ and $B_{11} = \langle [b_{11\mu L}, b_{11\mu U}], [b_{11\nu L}, b_{11\nu U}] \rangle$

$A_{12} = \langle [a_{12\mu L}, a_{12\mu U}], [a_{12\nu L}, a_{12\nu U}] \rangle$ and $B_{12} = \langle [b_{12\mu L}, b_{12\mu U}], [b_{12\nu L}, b_{12\nu U}] \rangle$

$A_{13} = \langle [a_{13\mu L}, a_{13\mu U}], [a_{13\nu L}, a_{13\nu U}] \rangle$ and $B_{13} = \langle [b_{13\mu L}, b_{13\mu U}], [b_{13\nu L}, b_{13\nu U}] \rangle$

$A_{21} = \langle [a_{21\mu L}, a_{21\mu U}], [a_{21\nu L}, a_{21\nu U}] \rangle$ and $B_{21} = \langle [b_{21\mu L}, b_{21\mu U}], [b_{21\nu L}, b_{21\nu U}] \rangle$

$A_{22} = \langle [a_{22\mu L}, a_{22\mu U}], [a_{22\nu L}, a_{22\nu U}] \rangle$ and $B_{22} = \langle [b_{22\mu L}, b_{22\mu U}], [b_{22\nu L}, b_{22\nu U}] \rangle$

$A_{23} = \langle [a_{23\mu L}, a_{23\mu U}], [a_{23\nu L}, a_{23\nu U}] \rangle$ and $B_{23} = \langle [b_{23\mu L}, b_{23\mu U}], [b_{23\nu L}, b_{23\nu U}] \rangle$

$A_{31} = \langle [a_{31\mu L}, a_{31\mu U}], [a_{31\nu L}, a_{31\nu U}] \rangle$ and $B_{31} = \langle [b_{31\mu L}, b_{31\mu U}], [b_{31\nu L}, b_{31\nu U}] \rangle$

$A_{32} = \langle [a_{32\mu L}, a_{32\mu U}], [a_{32\nu L}, a_{32\nu U}] \rangle$ and $B_{32} = \langle [b_{32\mu L}, b_{32\mu U}], [b_{32\nu L}, b_{32\nu U}] \rangle$

$A_{33} = \langle [a_{33\mu L}, a_{33\mu U}], [a_{33\nu L}, a_{33\nu U}] \rangle$ and $B_{33} = \langle [b_{33\mu L}, b_{33\mu U}], [b_{33\nu L}, b_{33\nu U}] \rangle$

$$\mathrm{A} * \mathrm{B} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}$$

Where,

$$\boldsymbol{X_{11}} = \left[< \left[\frac{a_{11\mu L} + b_{11\mu L}}{2\left(a_{11\mu L}.b_{11\mu L} + 1\right)}, \frac{a_{11\mu U} + b_{11\mu U}}{2\left(a_{11\mu U}.b_{11\mu U} + 1\right)} \right], \right.$$
$$\left. \left[\frac{a_{11\nu L} + b_{11\nu L}}{2(a_{11\nu L}.b_{11\nu L} + 1)}, \frac{a_{11\nu U} + b_{11\nu U}}{2(a_{11\nu U}.b_{11\nu U} + 1)} \right] > \right]$$

$$\boldsymbol{X_{12}} = \left[< \left[\frac{a_{12\mu L} + b_{12\mu L}}{2\left(a_{12\mu L}.b_{12\mu L} + 1\right)}, \frac{a_{12\mu U} + b_{12\mu U}}{2\left(a_{12\mu U}.b_{12\mu U} + 1\right)} \right], \right.$$
$$\left. \left[\frac{a_{12\nu L} + b_{12\nu L}}{2(a_{12\nu L}.b_{12\nu L} + 1)}, \frac{a_{12\nu U} + b_{12\nu U}}{2(a_{12\nu U}.b_{12\nu U} + 1)} \right] > \right]$$

$$\boldsymbol{X_{13}} = \left[< \left[\frac{a_{13\mu L} + b_{13\mu L}}{2\left(a_{13\mu L}.b_{13\mu L} + 1\right)}, \frac{a_{13\mu U} + b_{13\mu U}}{2\left(a_{13\mu U}.b_{13\mu U} + 1\right)} \right], \right.$$

$$\left. \left[\frac{a_{13\nu L} + b_{13\nu L}}{2(a_{13\nu L}.b_{13\nu L} + 1)}, \frac{a_{13\nu U} + b_{13\nu U}}{2(a_{13\nu U}.b_{13\nu U} + 1)} \right] > \right]$$

$$\boldsymbol{X_{21}} = \left[< \left[\frac{a_{21\mu L} + b_{21\mu L}}{2\left(a_{21\mu L}.b_{21\mu L} + 1\right)}, \frac{a_{21\mu U} + b_{21\mu U}}{2\left(a_{21\mu U}.b_{21\mu U} + 1\right)} \right], \right.$$

$$\left. \left[\frac{a_{21\nu L} + b_{21\nu L}}{2(a_{21\nu L}.b_{21\nu L} + 1)}, \frac{a_{21\nu U} + b_{21\nu U}}{2(a_{21\nu U}.b_{21\nu U} + 1)} \right] > \right]$$

$$\boldsymbol{X_{22}} = \left[< \left[\frac{a_{22\mu L} + b_{22\mu L}}{2\left(a_{22\mu L}.b_{22\mu L} + 1\right)}, \frac{a_{22\mu U} + b_{22\mu U}}{2\left(a_{22\mu U}.b_{22\mu U} + 1\right)} \right], \right.$$

$$\left. \left[\frac{a_{22\nu L} + b_{22\nu L}}{2(a_{22\nu L}.b_{22\nu L} + 1)}, \frac{a_{22\nu U} + b_{22\nu U}}{2(a_{22\nu U}.b_{22\nu U} + 1)} \right] > \right]$$

$$\boldsymbol{X_{23}} = \left[< \left[\frac{a_{23\mu L} + b_{23\mu L}}{2\left(a_{23\mu L}.b_{23\mu L} + 1\right)}, \frac{a_{23\mu U} + b_{23\mu U}}{2\left(a_{23\mu U}.b_{23\mu U} + 1\right)} \right], \right.$$

$$\left. \left[\frac{a_{23\nu L} + b_{23\nu L}}{2(a_{23\nu L}.b_{23\nu L} + 1)}, \frac{a_{23\nu U} + b_{23\nu U}}{2(a_{23\nu U}.b_{23\nu U} + 1)} \right] > \right]$$

$$\boldsymbol{X_{31}} = \left[< \left[\frac{a_{31\mu L} + b_{31\mu L}}{2\left(a_{31\mu L}.b_{31\mu L} + 1\right)}, \frac{a_{31\mu U} + b_{31\mu U}}{2\left(a_{31\mu U}.b_{31\mu U} + 1\right)} \right], \right.$$

$$\left. \left[\frac{a_{31\nu L} + b_{31\nu L}}{2(a_{31\nu L}.b_{31\nu L} + 1)}, \frac{a_{31\nu U} + b_{31\nu U}}{2(a_{31\nu U}.b_{31\nu U} + 1)} \right] > \right]$$

$$\boldsymbol{X_{32}} = \left[< \left[\frac{a_{32\mu L} + b_{32\mu L}}{2\left(a_{32\mu L}.b_{32\mu L} + 1\right)}, \frac{a_{32\mu U} + b_{32\mu U}}{2\left(a_{32\mu U}.b_{32\mu U} + 1\right)} \right], \right.$$

$$\left. \left[\frac{a_{32\nu L} + b_{32\nu L}}{2(a_{32\nu L}.b_{32\nu L} + 1)}, \frac{a_{32\nu U} + b_{32\nu U}}{2(a_{32\nu U}.b_{32\nu U} + 1)} \right] > \right]$$

$$\boldsymbol{X_{33}} = \left[< \left[\frac{a_{33\mu L} + b_{33\mu L}}{2\left(a_{33\mu L}.b_{33\mu L} + 1\right)}, \frac{a_{33\mu U} + b_{33\mu U}}{2\left(a_{33\mu U}.b_{33\mu U} + 1\right)} \right], \right.$$

$$\left. \left[\frac{a_{33\nu L} + b_{33\nu L}}{2(a_{33\nu L}.b_{33\nu L} + 1)}, \frac{a_{33\nu U} + b_{33\nu U}}{2(a_{33\nu U}.b_{33\nu U} + 1)} \right] > \right]$$

NEED OF IVIFM

We consider a network consisting of six important cities (vertices) in a country. They are interconnected by roads (edges). The network is shown in Figure 1.

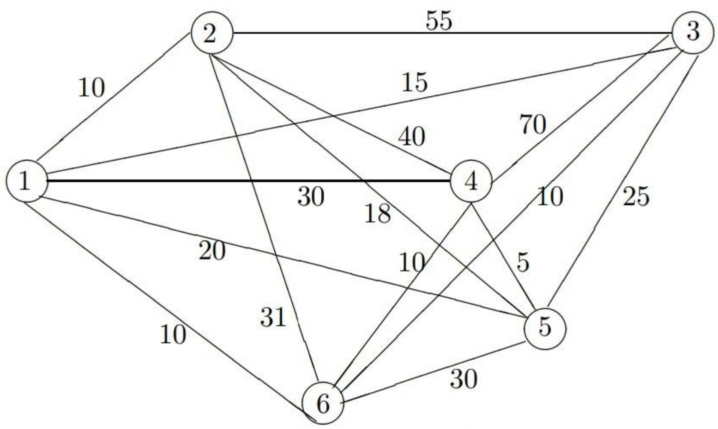


Figure 1: A network

The number adjacent to an edge represents the distance between the cities (vertices). The above network can be represented with the help of a classical matrix $A = [a_{ij}]$, $i, j = 1,2, \dots, n$, where, n is the total number of nodes. The ij-th element a_{ij} of A is defined as

$$a_{ij} = \begin{cases} 0, & \text{if } i = j \\ \infty, & \text{the vertices i and j are not directly connected by an edge} \\ w_{ij}, & w_{ij} \text{ is the distance of the road connecting i and j.} \end{cases}$$

Thus the adjacent matrix of the network of Figure 1 is

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & 10 & 15 & 30 & 20 & 10 \\ 10 & 0 & 55 & 40 & 18 & 30 \\ 15 & 55 & 0 & 70 & 25 & 10 \\ 30 & 40 & 70 & 0 & 5 & 10 \\ 20 & 18 & 25 & 5 & 0 & 30 \\ 10 & 31 & 10 & 10 & 30 & 0 \end{bmatrix} \end{array}$$

Since the distance between two vertices are known, precisely, so the above matrix is obviously a classical matrix. Generally, the distance between two cities are crisp value, so the corresponding matrix is crisp matrix.

Now, we consider the crowdness of the roads connecting cities. It is clear that the crowdness of a road obviously, is a fuzzy quantity. The amount of crowdness depends on the decision makers mentality, habits, natures, etc. i.e., completely depends on the decision maker. The measurement of crowdness as a point is a difficult task for the decision maker. So, here we consider the amount of crowdness as an interval instead of a point. Similarly, the loneliness is also considered as an interval. The crowdness and loneliness of a network can't be represented as a crisp matrix, it can be represented appropriately by a matrix which we designate by interval-valued intuitionistic fuzzy matrices (IVIFMs).

For illustration, we consider the crowdness and loneliness of the road (i,j) connecting the places i and j as follows:

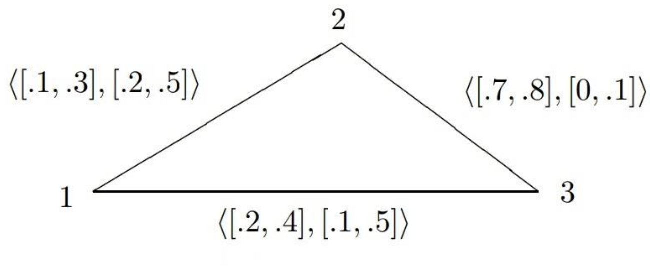
Roads Crowdness Lonelines	(1,2)(1,3)(1,4)(1,5)(1,6)(2,3)(2,4)(2,5) $[.1,.3][.2,.4][.3,.4][.2,.4][.3,.6][.7,.8][.3,.5][.3,.4]$ $[.2,.5][.1,.5][.5,.6][.4,.5][.2,.3][0,.1][.4,.5][.4,.6]$
Roads Crowdness Lonelines	(2,6)(3,4)(3,5)(3,6)(4,5)(4,6)(5,6) $[.2,.3][.5,.6][.3,.5][.3,.6][.4,.6][.2,.4][.3,.5]$ $[.4,.5][.2,.3][.2,.3][.2,.3][.3,.4][.3,.5][.2,.4]$

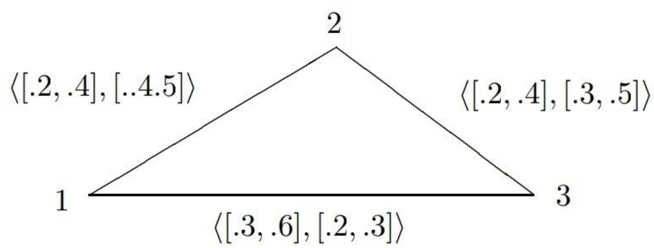
Table 1: The crowdness and loneliness of the network of Figure 1.

The matrix representation of the traffic crowdness and loneliness of the network of Figure 1 is shown in the following IVIFM.

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad & \quad 6 \quad \end{array} \\
\begin{bmatrix}
<[0,0],[1,1]> & <[.1,.3],[.2,.5]> & <[.2,.4],[.1,.5]> & <[.3,.4],[.5,.6]> & <[.2,.4],[.4,.5]> & <[.3,.6],[.2,.3]> \\
<[.1,.3],[.2,.5]> & <[0,0],[1,1]> & <[.7,.8],[0,.1]> & <[.3,.5],[.4,.5]> & <[.3,.4],[.4,.6]> & <[.2,.3],[.4,.5]> \\
<[.2,.4],[.1,.5]> & <[.7,.8],[0,.1]> & <[0,0],[1,1]> & <[.5,.6],[.2,.3]> & <[.3,.5],[.2,.3]> & <[.3,.6],[.2,.3]> \\
<[.3,.4],[.5,.6]> & <[.3,.5],[.4,.5]> & <[.5,.6],[.2,.3]> & <[0,0],[1,1]> & <[.4,.6],[.3,.4]> & <[.2,.4],[.3,.5]> \\
<[.2,.4],[.4,.5]> & <[.3,.4],[.4,.6]> & <[.3,.5],[.2,.3]> & <[.4,.6],[.3,.4]> & <[0,0],[1,1]> & <[.3,.5],[.2,.4]> \\
<[.3,.6],[.2,.3]> & <[.2,.3],[.4,.5]> & <[.3,.6],[.2,.3]> & <[.2,.4],[.3,.5]> & <[.3,.5],[.2,.4]> & <[0,0],[1,1]>
\end{bmatrix}
\end{array}$$

To explain the meaning of the operators defined earlier we consider two IVIFMs A and B. Let A and B represent respectively the crowdness and the loneliness of the network at two time instances t and t′. Now, the IVIFM $A + B$ represents the maximum amount of traffic crowdness and minimum amount of loneliness of the network between the time instances t and t′. $A.B$ represents the minimum amount of traffic crowdness and maximum amount of loneliness of the network. $\bar{A}$ matrix represents the loneliness and crowdness of the network. $A@B$, $A\$B$ and $A\#B$ reveals the arithmetic mean, geometric mean and harmonic mean of the crowdness and loneliness in between the two time instances t and t′ of the network.





To illustrate the operators $A.B$, $A + B$ and $|A|$, we consider a network consisting three vertices and three edges. The crowdness and loneliness of the network are observed at two different time instances t and t′. The matrices A_t and $A_{t'}$ represent the status of the network at t (Figure 2) and at t′ (Figure 3). The number adjacent to the sides represents the crowdness and loneliness of the roads at two different instances of the same network. A_t and $A_{t'}$ be the matrix representation of crowdness and loneliness at time t and t′ respectively, Let $A_t = \begin{bmatrix} <[0,0],[1,1]> & <[.1,.3],[.2,.5]> & <[.2,.4],[.1,.5]> \\ <[.1,.3],[.2,.5]> & <[0,0],[1,1]> & <[.7,.8],[0,.1]> \\ <[.2,.4],[.1,.5]> & <[.7,.8],[0,.1]> & <[0,0],[1,1]> \end{bmatrix}$

and $A_{t'} = \begin{bmatrix} <[0,0],[1,1]> & <[.2,.4],[.4,.5]> & <[.3,.6],[.2,.3]> \\ <[.2,.4],[.1,.5]> & <[0,0],[1,1]> & <[.2,.4],[.3,.5]> \\ <[.3,.6],[.2,.3]> & <[.2,.4],[.3,.5]> & <[0,0],[1,1]> \end{bmatrix}$

So, $A_t \cdot A_{t'} = \begin{bmatrix} <[0,0],[1,1]> & <[.1,.3],[.4,.5]> & <[.2,.4],[.2,.5]> \\ <[.1,.3],[.4,.5]> & <[0,0],[1,1]> & <[.2,.4],[.3,.5]> \\ <[.2,.4],[.2,.5]> & <[.2,.4],[.3,.5]> & <[0,0],[1,1]> \end{bmatrix}$

and, $A_t + A_{t'} = \begin{bmatrix} <[0,0],[1,1]> & <[.2,.4],[.2,.5]> & <[.3,.6],[.1,.3]> \\ <[.2,.4],[.2,.5]> & <[0,0],[1,1]> & <[.7,.8],[0\ .1]> \\ <[.3,.6],[.1,.3]> & <[.7,.8],[0\ .1]> & <[0,0],[1,1]> \end{bmatrix}$

$|A_t| = \langle[0,0],[1,1]\rangle\{\langle[0,0],[1,1]\rangle\langle[0,0],[1,1]\rangle + \langle[.7,.8],[0,.1]\rangle\langle[.7,.8],[0,1]\rangle\} +$
$\langle[.1,.3],[.2,.5]\rangle\{\langle[.7,.8],[0,.1]\rangle\langle[.2,.4],[.1,.5]\rangle + \langle[.1,.3],[.2,.5]\rangle\langle[0,0],[1,1]\rangle\} +$
$\langle[.2,.4],[.1,.5]\rangle\{\langle[.1,.3],[.2,.5]\rangle\langle[.7,.8],[0,.1]\rangle + \langle[0,0],[1,1]\rangle\langle[.2,.4],[.1,.5]\rangle\}$

$=\langle[0,0],[1,1]\rangle\{\langle[0,0],[1,1]\rangle + \langle[.7,.8],[0,.1]\rangle\} + \langle[.1,.3],[.2,.5]\rangle\{\langle[.2,.4],[.1,.5]\rangle +$
$\langle[0,0],[1,1]\rangle\} + \langle[.2,.4],[.1,.5]\rangle\{\langle[.1,.3],[.2,.5]\rangle + \langle[0,0],[1,1]\rangle\}$

$= \langle[0,0],[1,1]\rangle\langle[.7,.8],[0,.1]\rangle + \langle[.1,.3],[.2,.5]\rangle\langle[.2,.4],[.1,.5]\rangle +$
$\langle[.2,.4],[.1,.5]\rangle\langle[.1,.3],[.2,.5]\rangle$

$= \langle[0,0],[1,1]\rangle + \langle[.1,.3],[.2,.5]\rangle + \langle[.1,.3],[.2,.5]\rangle$

$= \langle[.1,.3],[.2,.5]\rangle$

It may be noted that if the ij-th element of the IVIFM At is $\langle[0,0],[1,1]\rangle$ then it indicates that the road (i,j) is fully lonely (not crowd), but, if it is $\langle[1,1],[0,0]\rangle$ then the road (i,j) is fully crowd or blocked.

PROPERTIES OF IVIFMs

In this section some properties of IVIFMs are presented.

IVIFMs satisfy the commutative and associative properties over the operators $+, \cdot, \oplus$ and $\odot$. The operator '$\cdot$' is distributed over '$+$' in left and right but the left and right distribution laws do not hold for the operators $\oplus$ and $\odot$.

(1) $A + B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $A \cdot B = B \cdot A$

(4) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

(5) (i) $A \cdot (B + C) = A \cdot B + A \cdot C$

(ii) $(B + C) \cdot A = B \cdot A + C \cdot A$

(6) $A \oplus B = B \oplus A$

(7) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

(8) $A \odot B = B \odot A$

(9) $A \odot (B \odot C) = (A \odot B) \odot C$

(10) (i) $A \odot (B \oplus C) \neq (A \odot B) \oplus (A \odot C)$

(ii) $(B \oplus C) \odot A \neq (B \odot A) \oplus (C \odot A)$

Proof of (i):

Let $A = \left[\langle\left[a_{ij\mu L}, a_{ij\mu U}\right], \left[a_{ij\nu L}, a_{ij\nu U}\right]\rangle\right]$

$B = \left[\langle\left[b_{ij\mu L}, b_{ij\mu U}\right], \left[b_{ij\nu L}, b_{ij\nu U}\right]\rangle\right]$ and

$C = \left[\langle\left[c_{ij\mu L}, c_{ij\mu U}\right], \left[c_{ij\nu L}, c_{ij\nu U}\right]\rangle\right]$. So,

$$B \oplus C = [\langle [b_{ij\mu L} + c_{ij\mu L} - b_{ij\mu L} \cdot c_{ij\mu L}, b_{ij\mu U} + c_{ij\mu U} - b_{ij\mu U} \cdot c_{ij\mu U}], \\ [b_{ij\nu L} \cdot c_{ij\nu L}, b_{ij\nu U} \cdot c_{ij\nu U}] \rangle]$$

And

$$A \odot (B \oplus C) = [\langle [a_{ij\mu L}(b_{ij\mu L} + c_{ij\mu L} - b_{ij\mu L} \cdot c_{ij\mu L}), a_{ij\mu U}(b_{ij\mu U} + c_{ij\mu U} - b_{ij\mu U} \cdot c_{ij\mu U})], \\ [a_{ij\nu L} + b_{ij\nu L} \cdot c_{ij\nu L} - a_{ij\nu L} \cdot b_{ij\nu L} \cdot c_{ij\nu L}, a_{ij\mu U} + b_{ij\mu U} \cdot c_{ij\mu U} - a_{ij\mu U} \cdot b_{ij\mu U} \\ \cdot c_{ij\mu U}] \rangle]$$

$$A \odot B = [\langle [a_{ij\mu L} \cdot b_{ij\mu L}, a_{ij\mu U} \cdot b_{ij\mu U}], \\ [a_{ij\nu L} + b_{ij\nu L} - a_{ij\nu L} \cdot b_{ij\nu L}, a_{ij\nu U} + b_{ij\nu U} - a_{ij\nu U} \cdot b_{ij\nu U}] \rangle]$$

$$A \odot C = [\langle [a_{ij\mu L} \cdot c_{ij\mu L}, a_{ij\mu U} \cdot c_{ij\mu U}], \\ [a_{ij\nu L} + c_{ij\nu L} - a_{ij\nu L} \cdot c_{ij\nu L}, a_{ij\nu U} + c_{ij\nu U} - a_{ij\nu U} \cdot c_{ij\nu U}] \rangle]$$

Now,

$(A \odot B) \oplus (A \odot C) = [\langle [a_{ij\mu L}(b_{ij\mu L} + c_{ij\mu L}) - a^2_{\mathrm{ij\mu L}} \cdot b_{ij\mu L} \cdot c_{ij\mu L}, a_{ij\mu U}(b_{ij\mu U} + c_{ij\mu U}) - a^2_{\mathrm{ij\mu U}} \cdot b_{ij\mu U} \cdot c_{ij\mu U}], [(a_{ij\nu L} + b_{ij\nu L} - a_{ij\nu L} \cdot b_{ij\nu L}) \cdot (a_{ij\nu L} + c_{ij\nu L} - a_{ij\nu L} \cdot c_{ij\nu L}), (a_{ij\nu U} + b_{ij\nu U} - a_{ij\nu U} \cdot b_{ij\nu U}) \cdot (a_{ij\nu U} + c_{ij\nu U} - a_{ij\nu U} \cdot c_{ij\nu U})] \rangle]$.

So,

$$A \odot (B \oplus C) \neq (A \odot B) \oplus (A \odot C)$$

PROPERTY 1

Let A be an IVIFM of any order then, $A + A = A$.

Proof:

Let $A = [\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}] \rangle]$

Then,

$$A + A = [\langle [max(a_{ij\mu L}, a_{ij\mu L}), max(a_{ij\mu U}, a_{ij\mu U})], [min(a_{ij\nu L}, a_{ij\nu L}), min(a_{ij\nu U}, a_{ij\nu U})]\rangle]$$

$$= [\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}]\rangle]$$

$$= A$$

PROPERTY 2

If A be an IVIFM of any order then,$A + I_{\langle[0,0],[0,0]\rangle} \geq A$ where, $I_{\langle[0,0],[0,0]\rangle}$ is the null IVIFM of same order.

Proof:

Let $A = [\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}]\rangle]$

And $I = \langle[0,0], [0,0]\rangle$.

Then,

$$A + I = [\langle [max(a_{ij\mu L}, 0), max(a_{ij\mu U}, 0)], [min(a_{ij\nu L}, 0), min(a_{ij\nu U}, 0)]\rangle]$$

$$= [\langle [a_{ij\mu L}, a_{ij\mu U}], [0,0]\rangle]$$

Therefore, $A + I \geq A$.

Some more properties on determinant and adjoint of IVIFM are presented below.

PROPERTY 3

Like classical matrices the determinant value of an IVIFM and its transpose are equal. If A be a square IVIFM then $|A| = |A^T|$.

Proof:

Let $A = [\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}]\rangle]$.

Then, $A^T = B = [\langle [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}]\rangle] = [\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}]\rangle]$.

Now,

$$|B| = \sum_{\sigma \epsilon S_n} \langle [b_{1\sigma(1)\mu L}, b_{1\sigma(1)\mu U}], [b_{1\sigma(1)\nu L}, b_{1\sigma(1)\nu U}] \rangle \langle [b_{2\sigma(2)\mu L}, b_{2\sigma(2)\mu U}],$$

$$[b_{2\sigma(2)\nu L}, b_{2\sigma(2)\nu U}] \rangle \dots \langle [b_{n\sigma(n)\mu L}, b_{n\sigma(n)\mu U}], [b_{n\sigma(n)\nu L}, b_{n\sigma(n)\nu U}] \rangle$$

$$= \sum_{\sigma \epsilon S_n} \langle [a_{\sigma(1)1\mu L}, a_{\sigma(1)1\mu U}], [a_{\sigma(1)1\nu L}, a_{\sigma(1)1\nu U}] \rangle \langle [a_{\sigma(2)2\mu L}, a_{\sigma(2)2\mu U}],$$

$$[a_{\sigma(2)2\nu L}, a_{\sigma(2)2\nu U}] \rangle \dots \langle [a_{\sigma(n)n\mu L}, a_{\sigma(n)n\mu U}], [a_{\sigma(n)n\nu L}, a_{\sigma(n)n\nu U}] \rangle$$

Let φ be the permutation of $\{1,2, \dots, n\}$ such that $\varphi\sigma = 1$, the identity permutation. Then $\varphi = \sigma^{-1}$. As σ runs over the whole set of permutations, so does φ.

Let $\sigma(i) = j, i = \sigma^{-1}(j) = \varphi(j)$.

Therefore,

$a_{\sigma(i)i\mu L} = a_{j\varphi(j)\mu L}$, $a_{\sigma(i)i\mu U} = a_{j\varphi(j)\mu U}$, $a_{\sigma(i)i\nu L} = a_{j\varphi(j)\nu L}$, $a_{\sigma(i)i\nu U} = a_{j\varphi(j)\nu U}$ for all i, j. As i runs over the set $\{1,2, \dots, n\}$, j does so.

$$\langle [a_{\sigma(1)1\mu L}, a_{\sigma(1)1\mu U}], [a_{\sigma(1)1\nu L}, a_{\sigma(1)1\nu U}] \rangle \langle [a_{\sigma(2)2\mu L}, a_{\sigma(2)2\mu U}],$$

$$[a_{\sigma(2)2\nu L}, a_{\sigma(2)2\nu U}] \rangle \dots \langle [a_{\sigma(n)n\mu L}, a_{\sigma(n)n\mu U}], [a_{\sigma(n)n\nu L}, a_{\sigma(n)n\nu U}] \rangle$$

$$= \langle a_{1\varphi(1)\mu L}, a_{j\varphi(1)\mu U} \rangle, \langle a_{1\varphi(1)\nu L}, a_{1\varphi(1)\nu L} \rangle, \langle a_{2\varphi(2)\mu L} a_{2\varphi(2)\mu U} \rangle,$$

$$\langle a_{2\varphi(2)\nu L}, a_{2\varphi(2)\nu L} \rangle \dots \langle a_{n\varphi(n)\mu L}, a_{n\varphi(n)\mu U} \rangle, \langle a_{n\varphi(n)\nu L}, a_{n\varphi(n)\nu L} \rangle$$

Therefore,

$$|B| = \sum_{\sigma \epsilon S_n} \langle [a_{\sigma(1)1\mu L}, a_{\sigma(1)1\mu U}], [a_{\sigma(1)1\nu L}, a_{\sigma(1)1\nu U}] \rangle \langle [a_{\sigma(2)2\mu L}, a_{\sigma(2)2\mu U}],$$

$$[a_{\sigma(2)2\nu L}, a_{\sigma(2)2\nu U}] \rangle \dots \langle [a_{\sigma(n)n\mu L}, a_{\sigma(n)n\mu U}], [a_{\sigma(n)n\nu L}, a_{\sigma(n)n\nu U}] \rangle$$

$$\sum_{\varphi \epsilon S_n} \langle a_{1\varphi(1)\mu L}, a_{j\varphi(1)\mu U} \rangle, \langle a_{1\varphi(1)\nu L}, a_{1\varphi(1)\nu L} \rangle, \langle a_{2\varphi(2)\mu L} a_{2\varphi(2)\mu U} \rangle,$$

$$\langle a_{2\varphi(2)\nu L}, a_{2\varphi(2)\nu L} \rangle \dots \langle a_{n\varphi(n)\mu L}, a_{n\varphi(n)\mu U} \rangle, \langle a_{n\varphi(n)\nu L}, a_{n\varphi(n)\nu L} \rangle$$

$$= |A|$$

PROPERTY 4

If A and B be two square IVIFMs and $A \leq B$, then, $adj.A \leq adj.B$.

Proof:

Let,

$C = [\langle[c_{ij\mu L}, c_{ij\mu U}], [c_{ij\nu L}, c_{ij\nu U}]\rangle] = adj.A$,

$D = [\langle[d_{ij\mu L}, d_{ij\mu U}], [d_{ij\nu L}, d_{ij\nu U}]\rangle] = adj.B$.

where,

$$\langle[c_{ij\mu L}, c_{ij\mu U}], [c_{ij\nu L}, c_{ij\nu U}]\rangle = \sum_{\sigma \epsilon S_{n_i n_j}} \prod_{t \in n_j} \langle[a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}]\rangle, \langle[a_{t\sigma(t)\nu L}, a_{t\sigma(t)\nu L}]\rangle$$

And

$$\langle[d_{ij\mu L}, d_{ij\mu U}], [d_{ij\nu L}, d_{ij\nu U}]\rangle = \sum_{\sigma \epsilon S_{n_i n_j}} \prod_{t \in n_j} \langle[b_{t\sigma(t)\mu L}, b_{t\sigma(t)\mu U}]\rangle, \langle[b_{t\sigma(t)\nu L}, b_{t\sigma(t)\nu L}]\rangle$$

It is clear that

$$\langle[c_{ij\mu L}, c_{ij\mu U}], [c_{ij\nu L}, c_{ij\nu U}]\rangle \leq \langle[d_{ij\mu L}, d_{ij\mu U}], [d_{ij\nu L}, d_{ij\nu U}]\rangle$$

Since,

$a_{t\sigma(t)\mu L} \leq b_{t\sigma(t)\mu L}$, $a_{t\sigma(t)\mu U} \leq b_{t\sigma(t)\mu U}$, $a_{t\sigma(t)\nu L} \leq b_{t\sigma(t)\nu L}$, $a_{t\sigma(t)\nu U} \leq b_{t\sigma(t)\nu U}$

for all $t \neq j$,$\sigma(t) \neq \sigma(j)$.

Therefore,

$C \leq D$, i.e., $adj.A \leq adj.B$

PROPERTY 5

For a square IVIFM A, $adj.(A^T) = (adj.A)^T$.

Proof:

Let $B = adj.A$, $C = adj.A^T$.

Therefore,

$$\langle [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}] \rangle = \sum_{\sigma \epsilon S_{n_i n_j}} \prod_{t \in n_j} \langle [a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}] \rangle, \langle [a_{t\sigma(t)\nu L}, a_{t\sigma(t)\nu L}] \rangle$$

and

$$\langle [c_{ij\mu L}, c_{ij\mu U}], [c_{ij\nu L}, c_{ij\nu U}] \rangle = \sum_{\sigma \epsilon S_{n_i n_j}} \prod_{t \in n_j} \langle [a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}] \rangle, \langle [a_{t\sigma(t)\nu L}, a_{t\sigma(t)\nu L}] \rangle$$

$$= \langle [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}] \rangle$$

Therefore, $adj.(A^T) = (adj.A)^T$. The following result is not valid for classical matrices, though it is true for IVIFM.

PROPERTY 6

For an IVIFM A, $|A| = |adj.A|$.

Proof:

$$adj.A = \langle [A_{ij\mu L}, A_{ij\mu U}], [A_{ij\nu L}, A_{ij\nu U}] \rangle$$

where, $\langle [A_{ij\mu L}, A_{ij\mu U}], [A_{ij\nu L}, A_{ij\nu U}] \rangle$ is the co-factor of the $\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}] \rangle$ in the IVIFM A.

Therefore,

$$|adj.A| = \sum_{\sigma\epsilon S_n} \langle [A_{1\sigma(1)\mu L}, A_{1\sigma(1)\mu U}], [A_{1\sigma(1)\nu L}, A_{1\sigma(1)\nu U}] \rangle \langle [A_{2\sigma(2)\mu L}, A_{2\sigma(2)\mu U}], [A_{2\sigma(2)\nu L}, A_{2\sigma(2)\nu U}] \rangle \dots \langle [A_{n\sigma(n)\mu L}, A_{n\sigma(n)\mu U}], [A_{n\sigma(n)\nu L}, A_{n\sigma(n)\nu U}] \rangle$$

$$= \sum_{\sigma\epsilon S_n} \prod_{i=1}^{n} < [A_{i\sigma(i)\mu L}, A_{i\sigma(i)\mu U}], [A_{i\sigma(i)\nu L}, A_{i\sigma(i)\nu U}] >$$

$$= \sum_{\sigma\epsilon S_n} \left[\prod_{i=1}^{n} \left(\sum_{\theta\epsilon S_{n_i n_{\sigma(i)}}} \prod_{t\in n_i} \langle [a_{t\theta(t)\mu L}, a_{t\theta(t)\mu U}], [a_{t\theta(t)\nu L}, a_{t\theta(t)\nu U}] \rangle \right) \right]$$

$$= \sum_{\sigma\epsilon S_n} \left[\left(\prod_{t\in n_1} \langle [a_{t\theta_1(t)\mu L}, a_{t\theta_1(t)\mu U}], [a_{t\theta_1(t)\nu L}, a_{t\theta_1(t)\nu U}] \rangle \right) \left(\prod_{t\in n_2} \langle [a_{t\theta_2(t)\mu L}, a_{t\theta_2(t)\mu U}], [a_{t\theta_2(t)\nu L}, a_{t\theta_2(t)\nu U}] \rangle \right) \dots \left(\prod_{t\in n_n} \langle [a_{t\theta_n(t)\mu L}, a_{t\theta_n(t)\mu U}], [a_{t\theta_n(t)\nu L}, a_{t\theta_n(t)\nu U}] \rangle \right) \right]$$

for some $\theta_1 \in S_{n_1 n_{\sigma(1)}}, \theta_2 \in S_{n_2 n_{\sigma(2)}}, \dots, \theta_n \in S_{n_1 n_{\sigma(n)}}$

$$= \sum_{\sigma\epsilon S_n} [(\langle [a_{2\theta_1(2)\mu L}, a_{2\theta_1(2)\mu U}], [a_{2\theta_1(2)\nu L}, a_{2\theta_1(2)\nu U}] \rangle \langle [a_{2\theta_1(2)\mu L}, a_{2\theta_1(2)\mu U}],$$
$$[a_{3\theta_1(3)\nu L}, a_{3\theta_1(3)\nu U}] \rangle \dots \langle [a_{n\theta_1(n)\mu L}, a_{2\theta_1(2)\mu U}], [a_{n\theta_1(n)\nu L}, a_{n\theta_1(n)\nu U}] \rangle)(\langle [a_{1\theta_2(1)\mu L}, a_{1\theta_2(1)\mu U}],$$
$$[a_{1\theta_2(1)\nu L}, a_{1\theta_2(1)\nu U}] \rangle \langle [a_{3\theta_2(3)\mu L}, a_{3\theta_2(3)\mu U}], [a_{3\theta_2(3)\nu L}, a_{3\theta_2(3)\nu U}] \rangle \dots \langle [a_{n\theta_2(n)\mu L}, a_{n\theta_2(n)\mu U}],$$
$$[a_{n\theta_2(n)\nu L}, a_{n\theta_2(n)\nu U}] \rangle) \dots \dots \dots (\langle [a_{1\theta_n(1)\mu L}, a_{1\theta_n(1)\mu U}], \ [a_{1\theta_n(1)\nu L}, a_{1\theta_n(1)\nu U}] \rangle \langle [a_{2\theta_n(2)\mu L}, a_{2\theta_n(2)\mu U}],$$
$$[a_{2\theta_n(2)\nu L}, a_{2\theta_n(2)\nu U}] \rangle \dots \langle [a_{(n-1)\theta_n(n-1)\mu L}, a_{(n-1)\theta_n(n-1)\mu U}],$$
$$[a_{(n-1)\theta_n(n-1)\nu L}, a_{(n-1)\theta_n(n-1)\nu U}] \rangle)]$$

$$= \sum_{\sigma \epsilon S_n} \Big[\Big(\langle [a_{1\theta_2(1)\mu L}, a_{1\theta_2(1)\mu U}], [a_{1\theta_2(1)\nu L}, a_{1\theta_2(1)\nu U}] \rangle \langle [a_{1\theta_3(1)\mu L}, a_{1\theta_3(1)\mu U}],$$

$$[a_{1\theta_3(1)\nu L}, a_{1\theta_3(1)\nu U}] \rangle \dots \langle [a_{1\theta_n(1)\mu L}, a_{1\theta_n(1)\mu U}], [a_{1\theta_n(1)\nu L}, a_{1\theta_n(1)\nu U}] \rangle \Big) \Big(\langle [a_{2\theta_1(2)\mu L}, a_{2\theta_1(2)\mu U}],$$

$$[a_{2\theta_1(2)\nu L}, a_{2\theta_1(2)\nu U}] \rangle \langle [a_{2\theta_3(2)\mu L}, a_{2\theta_3(2)\mu U}], [a_{2\theta_3(2)\nu L}, a_{2\theta_3(2)\nu U}] \rangle \dots \langle [a_{2\theta_n(2)\mu L}, a_{2\theta_n(2)\mu U}],$$

$$[a_{2\theta_n(2)\nu L}, a_{2\theta_n(2)\nu U}] \rangle \Big) \dots \dots \Big(\langle [a_{n\theta_1(n)\mu L}, a_{n\theta_1(n)\mu U}], [a_{n\theta_1(n)\nu L}, a_{n\theta_1(n)\nu U}] \rangle \langle [a_{n\theta_2(n)\mu L}, a_{n\theta_2(n)\mu U}],$$

$$[a_{n\theta_2(n)\nu L}, a_{n\theta_2(n)\nu U}] \rangle \dots \langle [a_{n\theta_{(n-1)} n\mu L}, a_{n\theta_{(n-1)} n\mu U}], [a_{n\theta_{(n-1)} n\nu L}, a_{n\theta_{(n-1)} n\nu U}] \rangle \Big) \Big]$$

$$= \sum_{\sigma \epsilon S_n} \Big(\langle [a_{1\theta_{f_1}(1)\mu L}, a_{1\theta_{f_1}(1)\mu U}], [a_{1\theta_{f_1}(1)\nu L}, a_{1\theta_{f_1}(1)\nu U}] \rangle \dots \langle [a_{2\theta_{f_2}(2)\mu L}, a_{2\theta_{f_2}(2)\mu U}],$$

$$[a_{2\theta_{f_2}(2)\nu L}, a_{2\theta_{f_2}(2)\nu U}] \rangle \langle [a_{n\theta_{f_n}(n)\mu L}, a_{n\theta_{f_n}(n)\mu U}], [a_{n\theta_{f_n}(n)\nu L}, a_{n\theta_{f_n}(n)\nu U}] \rangle \Big)$$

$f_{\hat{\theta}} \epsilon \{1,2,\dots,n\} \setminus \{\hat{\theta}\}, \hat{\theta} = 1,2,\dots,n.$

But since

$$\langle [a_{\hat{\theta}\theta_{f_{\hat{\theta}}}(\hat{\theta})\mu L}, a_{\hat{\theta}\theta_{f_{\hat{\theta}}}(\hat{\theta})\mu U}] \rangle \langle [a_{\hat{\theta}\theta_{f_{\hat{\theta}}}(\hat{\theta})VL}, a_{\hat{\theta}\theta_{f_{\hat{\theta}}}(\hat{\theta})VU}] \rangle$$

$$= \langle [a_{n\sigma(n)\mu L}, a_{n\sigma(n)\mu U}], [a_{n\sigma(n)\nu L}, a_{n\sigma(n)\nu U}] \rangle$$

Therefore,

$$|adj. A| = \sum_{\sigma \epsilon S_n} \langle [A_{1\sigma(1)\mu L}, A_{1\sigma(1)\mu U}], [A_{1\sigma(1)\nu L}, A_{1\sigma(1)\nu U}] \rangle \langle [A_{2\sigma(2)\mu L}, A_{2\sigma(2)\mu U}],$$

$$[A_{2\sigma(2)\nu L}, A_{2\sigma(2)\nu U}] \rangle \dots \langle [A_{n\sigma(n)\mu L}, A_{n\sigma(n)\mu U}], [A_{n\sigma(n)\nu L}, A_{n\sigma(n)\nu U}] \rangle$$

$$= |A|.$$

CHAPTER V

CONCLUSION

In this paper, the interval-valued intuitionistic fuzzy matrix (IVIFM) is introduced. The interval valued intuitionistic fuzzy determinant is also defined. Some fundamental operations are also presented. The need of IVIFM is explain by an example. Also, we define an operation on the intuitionistic fuzzy matrices called the Gödel implication operator as an extension to the definition of this operator in the case of ordinary fuzzy matrices due to Sanchez and Hashimoto. Using this operator, we prove several important results for intuitionistic fuzzy matrices. Particularly, some properties concerning pre-orders, sub-inverses, and regularity. We concentrate our discussion on the reflexive and transitive matrices. This studying enables us to give a largest sub-inverse and a largest generalized inverse for a reflexive and transitive intuitionistic fuzzy matrix.

Fuzzy matrix deals with only membership values. These matrices can't deal non membership values. Intuitionistic fuzzy matrices (IFMs) introduced first time by Khan, Shyamal and Pal. Several properties on IFMs have been studied in. But, practically it is difficult to measure the membership or non membership value as a point. So, we consider the membership value as an interval and also in the case of non membership values, it is not selected as a point, it can be considered as an interval. Here, we introduce the interval valued intuitionistic fuzzy matrices (IVIFMs) and introduce some basic operators on IVIFMs. The interval-valued intuitionistic fuzzy determinant (IVIFD) is also defined. A real life problem on IVIFM is presented. Interpretation of some of the operators are given with the help of this example.

CHAPTER VI

REFERENCES

[1] K.Atanassov, Intuitionistic Fuzzy Sets: Theory and Applications, Physica-Verlag, 1999.

[2] K.Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1986) 87-96.

[3] K.Atanassov, Operations over interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, 64 (1994) 159-174.

[4] H.Hashimoto, Canonical form of a transitive matrix, Fuzzy Sets and Systems, 11 (1983) 157-162.

[5] R.Hemasinha, N.R.Pal and J.C.Bezdek, Iterates of fuzzy circulant matrices, Fuzzy Sets and Systems, 60 (1993) 199-206.

[6] S.K.Khan and M.Pal, Some operations on intuitionistic fuzzy matrices, Acta Ciencia Indica, XXXII M (2) (2006) 515-524.

[7] K.H.Kim and F.W.Roush, Generalised fuzzy matrices, Fuzzy Sets and Systems, 4 (1980) 293- 315. 1143 [8] W.Kolodziejczyk, Canonical form of a strongly transitive fuzzy matrix, Fuzzy Sets and Systems, 22(1987) 292-302.

[9] J.B.Kim, Determinant theory for fuzzy and boolean matrices, Congressus Numerantium, (1988) 273-276.

[10] M.Pal, Intuitionistic fuzzy determinant, V.U.J.Physical Sciences, 7 (2001) 87-93.

[11] M.Pal, S.K.Khan and A.K.Shyamal, Intuitionistic fuzzy matrices, Notes on Intuitionistic Fuzzy Sets, 8(2) (2002) 51-62.

[12] M.Z.Ragab and E.G.Emam, The determinant and adjoint of a square fuzzy matrix, Fuzzy Sets and Systems, 61 (1994) 297-307.

[13] M.G.Ragab and E.G.Emam, On the min-max composition of fuzzy matrices, Fuzzy Sets and Systems, 75 (1995) 83-92.