Assignment 1

AI1110: Probability and Random Variables

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12.13.1.15: Question:
Consider the experiment of throwing a die,if a multiple of 3 comes up,throw the die again and if any other number comes,toss a coin.Repeat this experiment till a coin is tossed.Find the conditional probability of the event 'the coin shows a tail',given that 'at least one die shows a 3'.

Answer: $\frac{1}{2}$.

Solution:
Given that a die is thrown and if the outcome is a multiple of 3 i.e.,3 or 6 then another die is thrown, else a coin is tossed.The experiment is repeated till a coin is tossed.
Let k be the outcome of j^{th} die roll.
And X_j be a random variable such that

$$X_j = \begin{cases} 1 & k \in \{3, 6\} \\ 0 & k \in \{1, 2, 4, 5\} \end{cases} \tag{1}$$

$$\Pr\left(X_j = i\right) = \begin{cases} \frac{1}{3} & i = 1 \\ \frac{2}{3} & i = 0 \end{cases} \tag{2}$$

Let Y be a random variable for the coin toss then

$$Y = \begin{cases} 1 & tail \\ 0 & head \end{cases} \tag{3}$$

$$\Pr(Y = i) = \begin{cases} \frac{1}{2} & i = 1 \\ \frac{1}{2} & i = 0 \end{cases} \tag{4}$$

Let Z be a random variable which represents the number of times 3 has occured in the die rolls.
Then $Z \in \{0, 1, 2, ..., \infty\}$
Need to Find, Conditional Probability of the event 'the coin shows a tail',given that 'at least one die shows a 3', i.e., $\Pr(Y = 1|Z \geq 1)$

$$\Pr(Y = 1|Z \geq 1) = \frac{\Pr(Y = 1, Z \geq 1)}{\Pr(Z \geq 1)} \tag{5}$$

The event of rolling the die is a markov process because the outcome of the n^{th} die roll depends only on the outcome of the $n - 1^{\text{th}}$ die roll i.e.,

$$\Pr(X_n = i_n|X_0 = i_0, ..., X_{n-1} = i_{n-1}) = \Pr(X_n = i_n|X_{n-1} = i_{n-1}) \tag{6}$$

$$\Pr(X_n = 1|X_{n-1} = 1) = \frac{1}{3} \tag{7}$$

$$\Pr(X_n = 0|X_{n-1} = 1) = \frac{2}{3} \tag{8}$$

$$\Pr(X_n = 1|X_{n-1} = 0) = 0 \tag{9}$$

$$\Pr(X_n = 0|X_{n-1} = 0) = 0 \tag{10}$$

If we are tossing the coin after n die rolls then

$$\Pr(Y = 1|X_n = 0) = \Pr(Y = 0|X_n = 0) = \frac{1}{2} \tag{11}$$

$$\Pr(Y = 1|X_n = 1) = \Pr(Y = 0|X_n = 1) = 0 \tag{12}$$

The outcome of coin toss is depending only on the last die roll (since we are tossing the coin only when $X_n = 0$) and this is independent of number of die rolls and also the number of times we get 3 before the last die roll.
Therefore,

$$\Pr(Y = 1, Z \geq 1) = \Pr(Y = 1).\Pr(Z \geq 1) \tag{13}$$

Substituting eq(13) in eq(5), we get

$$\Pr(Y = 1|Z \geq 1) = \frac{\Pr(Y = 1) . \Pr(Z \geq 1)}{\Pr(Z \geq 1)} \tag{14}$$

$$\Pr(Y = 1|Z \geq 1) = \Pr(Y = 1) \tag{15}$$

$$\Pr(Y = 1|Z \geq 1) = \frac{1}{2} \tag{16}$$

Hence,
Probability of the event 'the coin shows a tail',given that 'at least one die shows a 3' is $\frac{1}{2}$.

The below is the Markov Chain Diagram for this experiment.
States are labelled with numbers and they are defined as

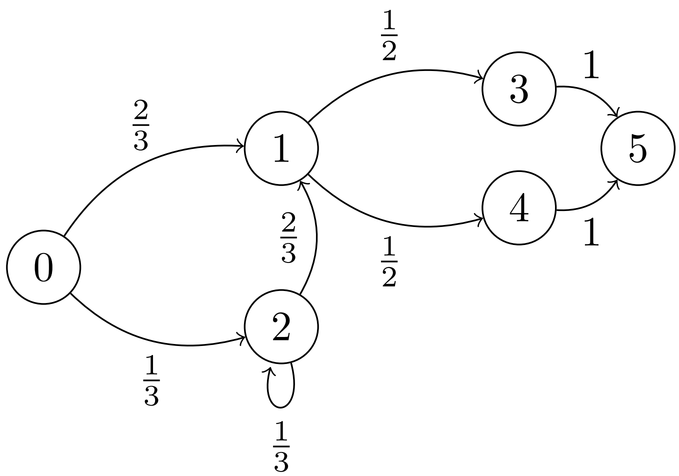

Fig. 0. Markov Chain Diagram

State 0 : Initial state , a die is to be rolled
State 1 : Die is rolled and the outcome is not a multiple of 3 , a coin is to be tossed next.
State 2 : Die is rolled and the outcome is multiple of 3, the die is to be rolled again.
State 3 : Coin is tossed and outcome is Head.
State 4 : Coin is tossed and outcome is Tail.
State 5 : Terminal state and end of experiment.