

Assignment Question 2

AI1110: Probability and Random Variables

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It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample space of 12 such articles, 9 are defective?

Solution:

Let X_i be a random variable corresponding to i^{th} article such that

$$X_i = \begin{cases} 1 & \text{if the article is defective} \\ 0 & \text{if the article is not defective} \end{cases} \quad (1)$$

X_1, X_2, \dots, X_{12} is a sequence of independent and identically distributed random variables. This sequence forms a Binomial Distribution with mean μ and variance σ^2

For this Binomial Distribution, $n = 12$ and $p = 0.1$. The mean and standard deviation of the Binomial distribution are

$$\mu = np \quad (2)$$

$$\mu = 12 \times 0.1$$

$$\mu = 1.2 \quad (3)$$

$$\sigma = \sqrt{np(1-p)} \quad (4)$$

$$\sigma = \sqrt{12 \times 0.1 \times 0.9}$$

$$\sigma = 1.04 \quad (5)$$

$$\text{Let } S_n = \sum_{i=1}^n X_i$$

$$\begin{aligned} \text{Standardized sample mean} &= \frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} \\ &= \frac{S_n - \mu n}{\sigma \sqrt{n}} \end{aligned} \quad (6)$$

According to Central Limit Theorem

as $n \rightarrow \infty$, $\frac{S_n - \mu n}{\sigma \sqrt{n}} \rightarrow N(0, 1)$

here $N(0,1)$ denotes a standard normal distribution with mean 0 and variance 1.

Proof of Central Limit Theorem

Let X_1, X_2, \dots, X_n be independent and identically

distributed Random Variables with mean μ and variance σ^2 .

Let

$$S_n = \sum_{i=1}^n X_i \quad (7)$$

$$Z_i = \frac{X_i - \mu}{\sigma} \quad (8)$$

$$E(Z_i) = E\left(\frac{X_i - \mu}{\sigma}\right) \quad (9)$$

$$= \frac{1}{\sigma} E(X_i - \mu)$$

$$= \frac{1}{\sigma} (E(X_i) - E(\mu))$$

$$= \frac{1}{\sigma} (\mu - \mu)$$

$$E(Z_i) = 0 \quad (10)$$

$$\text{Var}(Z_i) = \text{Var}\left(\frac{X_i - \mu}{\sigma}\right) \quad (11)$$

$$= \frac{1}{\sigma^2} \text{Var}(X_i - \mu)$$

$$= \frac{1}{\sigma^2} \text{Var}(X_i)$$

$$= \frac{1}{\sigma^2} \sigma^2$$

$$\text{Var}(Z_i) = 1 \quad (12)$$

Let M_{Z_i} be Moment Generating Function of Z_i .

$$M_{Z_i}(t) = E(e^{tZ_i}) \quad (13)$$

$$M_{Z_i}(0) = E(1) = 1 \quad (14)$$

$$E(Z_i) = M'_{Z_i}(0) \quad (15)$$

$$M'_{Z_i}(0) = 0 \text{ From eq(10)} \quad (16)$$

$$\text{Var}(Z_i) = E(Z_i^2) - E(Z_i)^2 \quad (17)$$

$$E(Z_i^2) = \text{Var}(Z_i) + E(Z_i)^2$$

$$= 1 - 0 = 1$$

$$M''_{Z_i}(0) = E(Z_i^2) = 1 \quad (18)$$

Consider $Y \sim N(0, 1)$, Then

$$M_Y(t) = e^{\frac{t^2}{2}} \quad (19)$$

Let $T_n = \sum_{i=1}^n Z_i$

$$\begin{aligned}
 M_{\frac{T_n}{\sqrt{n}}}(t) &= E(e^{t \frac{T_n}{\sqrt{n}}}) \\
 &= E(e^{\frac{t}{\sqrt{n}} \sum_{i=1}^n Z_i}) \\
 &= E(e^{\frac{t}{\sqrt{n}} Z_1} \cdot e^{\frac{t}{\sqrt{n}} Z_2} \dots e^{\frac{t}{\sqrt{n}} Z_n}) \\
 &= E(e^{\frac{t}{\sqrt{n}} Z_1}) \cdot E(e^{\frac{t}{\sqrt{n}} Z_2}) \dots E(e^{\frac{t}{\sqrt{n}} Z_n}) \\
 &= M_{Z_1}(\frac{t}{\sqrt{n}}) \cdot M_{Z_2}(\frac{t}{\sqrt{n}}) \dots M_{Z_n}(\frac{t}{\sqrt{n}}) \\
 &= [M_{Z_i}(\frac{t}{\sqrt{n}})]^n
 \end{aligned} \tag{21}$$

As $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} M_{\frac{T_n}{\sqrt{n}}}(t)$ takes the form 1^∞
Applying natural logarithm to the eq(21)

$$\begin{aligned}
 \ln(M_{\frac{T_n}{\sqrt{n}}}(t)) &= n \ln(M_{Z_i}(\frac{t}{\sqrt{n}})) \\
 \lim_{n \rightarrow \infty} \ln(M_{\frac{T_n}{\sqrt{n}}}(t)) &= \lim_{n \rightarrow \infty} n \ln(M_{Z_i}(\frac{t}{\sqrt{n}}))
 \end{aligned} \tag{22}$$

Let $u = \frac{1}{\sqrt{n}}$

Then as $n \rightarrow \infty$, $u \rightarrow 0$

Substituting in eq(22) we get

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \ln(M_{\frac{T_n}{\sqrt{n}}}(t)) &= \lim_{u \rightarrow 0} \frac{1}{u^2} \ln(M_{Z_i}(tu)) \\
 &= \lim_{u \rightarrow 0} \frac{1}{2u} \frac{t M'_{Z_i}(tu)}{M_{Z_i}(tu)} \\
 &= \frac{t}{2} \frac{1}{M_{Z_i}(0)} \lim_{u \rightarrow 0} \frac{M'_{Z_i}(tu)}{u}
 \end{aligned}$$

[From eq(16)]

$$\begin{aligned}
 &= \frac{t}{2} \lim_{u \rightarrow 0} \frac{t M''_{Z_i}(tu)}{1} \\
 &= \frac{t^2}{2} M''_{Z_i}(0)
 \end{aligned}$$

[From eq(18)]

$$M_{\frac{T_n}{\sqrt{n}}}(t) = e^{\frac{t^2}{2}} \tag{23}$$

Therefore

$$M_{\frac{T_n}{\sqrt{n}}}(t) = M_Y(t) \tag{24}$$

According to Uniqueness Theorem of Moment Generating Function,

If two random variables have the same MGF then they have the same distribution.

Therefore,

as $n \rightarrow \infty$ $\frac{T_n}{\sqrt{n}}$ and Y have the same distribution.

From eq(7) and eq(8)

$$\frac{T_n}{\sqrt{n}} = \frac{S_n - n\mu}{\sigma \sqrt{n}} \tag{25}$$

Therefore

as $n \rightarrow \infty$, $\frac{S_n - n\mu}{\sigma \sqrt{n}} \rightarrow N(0, 1)$

Hence Proved.

Let, E be the event that exactly 9 articles are defective.

For the event E

$S_n = 9$ [as exactly 9 articles are defective]

Z-Score for the event E is

$$z = \frac{S_n - np}{\sqrt{np(1-p)}} \tag{26}$$

$$z = \frac{9 - 1.2}{1.04}$$

$$z = 7.5 \tag{27}$$

From the Standard Normal Distribution table,

The probability of an event having z-score value of 7.5 is very low and is equal to $4.338751580235112 \times 10^{-13}$

Therefore, $\Pr(E) = 4.338751580235112 \times 10^{-13}$

