Assignment Question 2

AI1110: Probability and Random Variables

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It is known that 10% of certain articles manufactured are defective.What is the probability that in a random sample space of 12 such articles, 9 are defective?

Solution:

Let X_i be a random variable corresponding to i^{th} article such that

$$X_i = \begin{cases} 1 & \text{if the article is defective} \\ 0 & \text{if the article is not defective} \end{cases} \tag{1}$$

$X_1, X_2, ..., X_{12}$ is a sequence of independent and identically distributed random variables.This sequence forms a Binomial Distribution with mean μ and variance σ^2

For this Binomial Distribution, n = 12 and p = 0.1. The mean and standard deviation of the Binomial distribution are

$$\mu = np \tag{2}$$
$$\mu = 12 \times 0.1$$
$$\mu = 1.2 \tag{3}$$
$$\sigma = \sqrt{np(1-p)} \tag{4}$$
$$\sigma = \sqrt{12 \times 0.1 \times 0.9}$$
$$\sigma = 1.04 \tag{5}$$

Let $S_n = \sum_{i=1}^{n} X_i$

$$\text{Standardized sample mean} = \frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}}$$
$$= \frac{S_n - \mu n}{\sigma\sqrt{n}} \tag{6}$$

According to Central Limit Theorem
as $n \to \infty$,$\frac{S_n - \mu n}{\sigma\sqrt{n}} \to N(0,1)$
here N(0,1) denotes a standard normal distribution with mean 0 and variance 1.

Proof of Central Limit Theorem

Let $X_1, X_2, .., X_n$ be independent and identically distributed Random Variables with mean μ and variance σ^2.
Let

$$S_n = \sum_{i=1}^{n} X_i \tag{7}$$
$$Z_i = \frac{X_i - \mu}{\sigma} \tag{8}$$
$$E(Z_i) = E(\frac{X_i - \mu}{\sigma}) \tag{9}$$
$$= \frac{1}{\sigma} E(X_i - \mu)$$
$$= \frac{1}{\sigma}(E(X_i) - E(\mu))$$
$$= \frac{1}{\sigma}(\mu - \mu)$$
$$E(Z_i) = 0 \tag{10}$$
$$Var(Z_i) = Var(\frac{X_i - \mu}{\sigma}) \tag{11}$$
$$= \frac{1}{\sigma^2} Var(X_i - \mu)$$
$$= \frac{1}{\sigma^2} Var(X_i)$$
$$= \frac{1}{\sigma^2}\sigma^2$$
$$Var(Z_i) = 1 \tag{12}$$

Let M_{Z_i} be Moment Generating Function of Z_i.

$$M_{Z_i}(t) = E(e^{tZ_i}) \tag{13}$$
$$M_{Z_i}(0) = E(1) = 1 \tag{14}$$
$$E(Z_i) = M'_{Z_i}(0) \tag{15}$$
$$M'_{Z_i}(0) = 0 \text{ From eq(10)} \tag{16}$$
$$Var(Z_i) = E({Z_i}^2) - E(Z_i)^2 \tag{17}$$
$$E({Z_i}^2) = Var(Z_i) + E(Z_i)^2$$
$$= 1 - 0 = 1$$
$$M''_{Z_i}(0) = E({Z_i}^2) = 1 \tag{18}$$

Consider $Y \sim N(0,1)$

$$M_Y(t) = e^{\frac{t^2}{2}} \tag{19}$$

Let $T_n = \sum_{i=1}^{n} Z_i$

$$M_{\frac{T_n}{\sqrt{n}}}(t) = E(e^{t\frac{T_n}{\sqrt{n}}}) \tag{20}$$
$$= E(e^{\frac{t}{\sqrt{n}}\sum_{i=1}^{n} Z_i})$$
$$= E(e^{\frac{t}{\sqrt{n}}Z_1}.e^{\frac{t}{\sqrt{n}}Z_2}...e^{\frac{t}{\sqrt{n}}Z_n})$$
$$= E(e^{\frac{t}{\sqrt{n}}Z_1}).E(e^{\frac{t}{\sqrt{n}}Z_2})...E(e^{\frac{t}{\sqrt{n}}Z_n})$$
$$= M_{Z_1}(\frac{t}{\sqrt{n}}).M_{Z_2}(\frac{t}{\sqrt{n}})...M_{Z_n}(\frac{t}{\sqrt{n}})$$
$$= [M_{Z_i}(\frac{t}{\sqrt{n}})]^n \tag{21}$$

As $n \to \infty$, $\lim_{n\to\infty} M_{\frac{T_n}{\sqrt{n}}}(t)$ takes the form 1^{∞}
Applying natural logarithm to the eq(21)

$$\ln(M_{\frac{T_n}{\sqrt{n}}}(t)) = n\ln(M_{Z_i}(\frac{t}{\sqrt{n}}))$$
$$\lim_{n\to\infty}\ln(M_{\frac{T_n}{\sqrt{n}}}(t)) = \lim_{n\to\infty} n\ln(M_{Z_i}(\frac{t}{\sqrt{n}})) \tag{22}$$

Let $u = \frac{1}{\sqrt{n}}$
Then as $n \to \infty$, $u \to 0$
Substituting in eq(22) we get

$$\lim_{n\to\infty}\ln(M_{\frac{T_n}{\sqrt{n}}}(t)) = \lim_{u\to 0}\frac{1}{u^2}\ln(M_{Z_i}(tu))$$
$$= \lim_{u\to 0}\frac{1}{2u}\frac{tM'_{Z_i}(tu)}{M_{Z_i}(tu)}$$
$$= \frac{t}{2}\frac{1}{M_{Z_i}(0)}\lim_{u\to 0}\frac{M'_{Z_i}(tu)}{u}$$

[From eq(16)]

$$= \frac{t}{2}\lim_{u\to 0}\frac{tM''_{Z_i}(tu)}{1}$$
$$= \frac{t^2}{2}M''_{Z_i}(0)$$

[From eq(18)]

$$M_{\frac{T_n}{\sqrt{n}}}(t) = \frac{t^2}{2} \tag{23}$$

Therefore

$$M_{\frac{T_n}{\sqrt{n}}}(t) = M_Y(t) \tag{24}$$

According to Uniqueness Theorem of Moment Generating Function,
If two random variables have the same MGF then they have the same distribution.
Therefore,
as $n \to \infty$ $\frac{T_n}{\sqrt{n}}$ and Y have the same distribution.
From eq(7) and eq(8)

$$\frac{T_n}{\sqrt{n}} = \frac{S_n - n\mu}{\sigma\sqrt{n}} \tag{25}$$

Therefore
as $n \to \infty$,$\frac{S_n - \mu n}{\sigma\sqrt{n}} \to N(0,1)$
Hence Proved.

Let, E be the event that exactly 9 articles are defective.
For the event E

$S_n = 9$[as exactly 9 articles are defective]

Standardized sample mean for the event E is

$$z = \frac{S_n - np}{\sqrt{np(1-p)}} \tag{26}$$
$$z = \frac{9 - 1.2}{1.04}$$
$$z = 7.5 \tag{27}$$

From the Standard Normal Distribution table,
The probability of an event having standardized sample mean value of 7.5 is very low and is equal to $4.338751580235112 \times 10^{-13}$
Therefore, $\Pr(E) = 4.338751580235112 \times 10^{-13}$