QOSF Challenge - Task 2

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September 2020

1 Problem Statement

1.1 Part 1

Implement a circuit that returns $|01\rangle \, and \, |10\rangle$ with equal probability. Requirements of the circuit are as follows -

- The circuit should consist only of CNOT, R_x and R_y.
- Start from all parameters in parametric gates being equal to 0 or randomly chosen.
- You should find the right set of parameters using gradient descent (you can use more advanced optimization methods if you like).
- Simulations must be done with sampling (i.e. a limited number of measurements per iteration) and noise.

Compare the results for different numbers of measurements: 1, 10, 100, 1000.

1.2 Part 2 - Bonus Question

How to make sure you produce state $|01\rangle + |10\rangle$ and not $|01\rangle - |10\rangle$?
(Actually for more careful readers, the "correct" version of this question is posted below:
How to make sure you produce state $|01\rangle + |10\rangle$ and not any other combination of $|01\rangle + e^{i\phi}|10\rangle$ (for example $|01\rangle - |10\rangle$)?)

2 Working Strategy

Let, we have two qubits - q_0 and q_1. Each of these qubit is initialed to $|0\rangle$ state, as per the requirement of the question statement. I will now change the state of q_0 to a superposition state of $|0\rangle \, and \, |1\rangle$ such that probability of measuring each state is equal. To accomplish this task, i will be using R_x gate on q_0 (explained further in section 2.1).
Our next goal will be to convert q_1 from state $|0\rangle$ to state $|1\rangle$. This can be

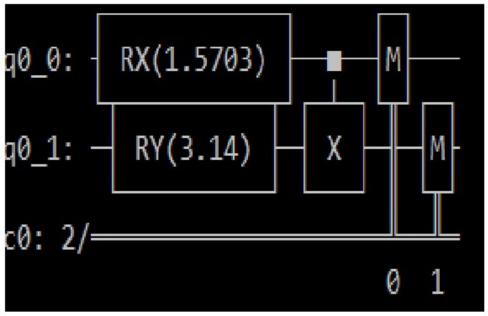

Figure 1: Quantum Circuit implementation in QISKIT simulation

simply done using R_y gate on q_1(explained in section 2.2).
Further we can apply CNOT gate considering $q_0(\phi)$ as controlled bit and $q_1(\phi)$ as the target bit. Finally the resulting qubits will be in superposition of $|01\rangle\ and\ |10\rangle$ with equal probabilities. Finally we can measure it to have the required output.The resultant circuit implemented using qiskit simulation will look like as in figure 1.

2.1 R_x gate on q_0

R_x gate is defined in equation 1.

$$R_x(\phi) = \begin{pmatrix} \cos\phi/2 & -\iota\sin(\phi/2) \\ -\iota\sin(\phi/2) & \cos\phi/2 \end{pmatrix} \tag{1}$$

Qubit q_0 in state $|0\rangle$ is given by equation 2

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

After applying R_x gate at some arbitrary angle ϕ on qubit q_0 is given by equation 3

$$q_0(\phi) = R_x(\phi)q_0 = \begin{pmatrix} \cos\phi/2 \\ -\iota\sin(\phi/2) \end{pmatrix} \tag{3}$$

The required condition of this new state are

- It should follow Born's normalization rule.

$$|\cos\phi/2|^2 + |-\iota\sin(\phi/2)|^2 = (\cos\phi/2)^2 + (\sin\phi/2)^2 = 1 \tag{4}$$

  Clearly, this is true for any value of ϕ

- Since the probability of $q_0(\phi)$ being in $|0\rangle$ or $|1\rangle$ state is equal, therefore

$$|\cos\phi/2|^2 = |-\iota\sin(\phi/2)|^2 \tag{5}$$

which can be rewritten as

$$|\cos\phi/2|^2 - |-\iota\sin(\phi/2)|^2 = (\cos\phi/2)^2 - (\sin\phi/2)^2 = 0 \tag{6}$$

We can formally define a function, which depends on ϕ as

$$f(\phi) = (\cos\phi/2)^2 - (\sin\phi/2)^2 \tag{7}$$

Our task is to find the value of ϕ for which $f(\phi) = 0$. We can find this using Gradient Decent optimisation technique. it will help in finding the minimum value of function. To remove negative values, we will apply Gradient Decent technique on $(f(\phi))^2 = 0$, using the initial value of $\phi = 0$, as per the requirement.

2.2 R_y gate on q_1

R_y gate is defined in equation 8.

$$R_y(\phi) = \begin{pmatrix} \cos\phi/2 & -\sin(\phi/2) \\ \sin(\phi/2) & \cos\phi/2 \end{pmatrix} \tag{8}$$

Qubit q_1 in state $|0\rangle$ is given by equation 2

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9}$$

After applying R_y gate at some arbitrary angle ϕ on qubit q_1 is given by equation 10

$$q_1(\phi) = R_y(\phi)q_1 = \begin{pmatrix} \cos\phi/2 \\ \sin(\phi/2) \end{pmatrix} \tag{10}$$

The required condition of this new state are

- It should follow Born's normalization rule.

$$|\cos\phi/2|^2 + |\sin(\phi/2)|^2 = (\cos\phi/2)^2 + (\sin\phi/2)^2 = 1 \tag{11}$$

  Clearly, this is true for any value of ϕ

- Since the probability of $q_1(\phi)$ being in $|0\rangle$ is 0 and in $|1\rangle$ state is 1, therefore

$$|\cos\phi/2|^2 = 0 \tag{12}$$

$$|\sin(\phi/2)|^2 = 1 \tag{13}$$

  We can formally define a function on either of these conditions, which depends on ϕ and apply the gradient descendent technique to it as weel to find the most optimal value of ϕ. To remove negative values, we will apply Gradient Decent technique on square of the function, using the initial value of $\phi = 0$, as per the requirement.

3 Bonus Question

As per the question we need to ensure that the resulting output has equal probabilities of getting $|01\rangle \, and \, |10\rangle$ and not of $|01\rangle \, and - |10\rangle$.More generalized form can be that we need to make sure that we do not produce a state which is a superposition of $|01\rangle + e^{i\phi} |10\rangle$ with equal probabilities.
The problem that we need to tackle here is to avoid phase change in the qubit. And if a phase change is there, due to noisy channel, then we need to detect and correct the error without disturbing the superposition. This problem wants the detection on one qubit only. Let's assume that phase change, if occurs, will be possible only on right qubit. Qubit $|q_1 q_0\rangle$ can be written as tensor product of both of them, as given below.

$$|q_1 q_0\rangle = |q_1\rangle \otimes |q_0\rangle \tag{14}$$

We will now protect the phase of qubit q_1. The way to do this is to encode it and use some error detection and correction technique. First let us understand phase-shift and phase-flip operator.

3.1 Phase Shift

This is a family of single-qubit gates that leave the basis state $|0\rangle$ unchanged and map $|1\rangle$ to $e^{i\phi} |1\rangle$. The probability of measuring a state is unchanged after applying this gate, however it modifies the phase of the quantum state. This is equivalent to tracing a horizontal circle (a line of latitude) on the Bloch sphere by ϕ radians.

$$R_\pi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \tag{15}$$

$$R_\phi |1\rangle = e^{\phi} |1\rangle \tag{16}$$

$$R_\phi |0\rangle = |0\rangle \tag{17}$$

3.2 Phase Flip

For a single-qubit system, the various possible states of qubit are $|0\rangle \, and \, |1\rangle$. The extreme phase of $|1\rangle$ is $-|1\rangle$. Negative sign for $|0\rangle$ have no significance. The Pauli-Z gate acts on a single qubit. It equates to a rotation around the Z-axis of the Bloch sphere by π radians. Thus, it is a special case of a phase shift gate. It is also denoted by R_π.

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{18}$$

$$Z |1\rangle = - |1\rangle \tag{19}$$

$$Z |0\rangle = |0\rangle \tag{20}$$

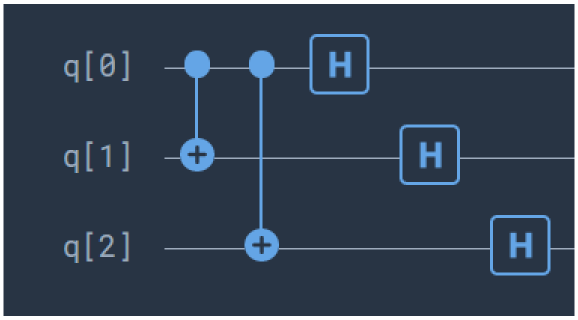

Figure 2: Encoding Qubit q_1 at sender's side, designed in Quantum Inspire

3.2.1 Error Detection and Correction

First, we will discuss the simpler case of preventing phase shift error. For that we will encode the single qubit q_1 into three qubits, at the sender side itself. Then we will apply Hadamard Gate to it, to convert it to $|+\rangle, |-\rangle$ basis. This can be achieved as per the circuit in fig. 2, wherein $|q_1\rangle = q[0]$ which has then been encoded in three qubits $q[0], q[1] and q[2]$

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{21}$$

$$H|1\rangle = |-\rangle \tag{22}$$

$$H|0\rangle = |+\rangle \tag{23}$$

The dot and the enclosed plus sign, connected by a vertical line, is a CNOT gate. It acts on two qubit, qubit at solid dot act as control qubit and qubit at other side is target qubit.

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{24}$$

At the receiver side, the circuit is shown in fig. 3. Qubit $q[3] and q[4]$ are called ancillary bits. They are compared with other logical qubits and can be measured to determine, whether the error has occured.

Qubit with Error	q[4]	q[3]
q[0]	1	1
q[1]	0	1
q[2]	1	0
No error	0	0

Similar procedure can be implemented for phase shift.

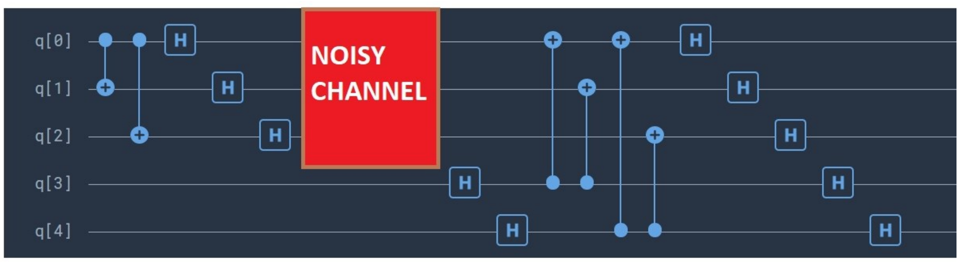

Figure 3: Circuit to detect and correct phase flip error , designed in Quantum Inspire