Week-01

* Joint PMF :-

We have two discrete R.V.s X and Y and both have some range

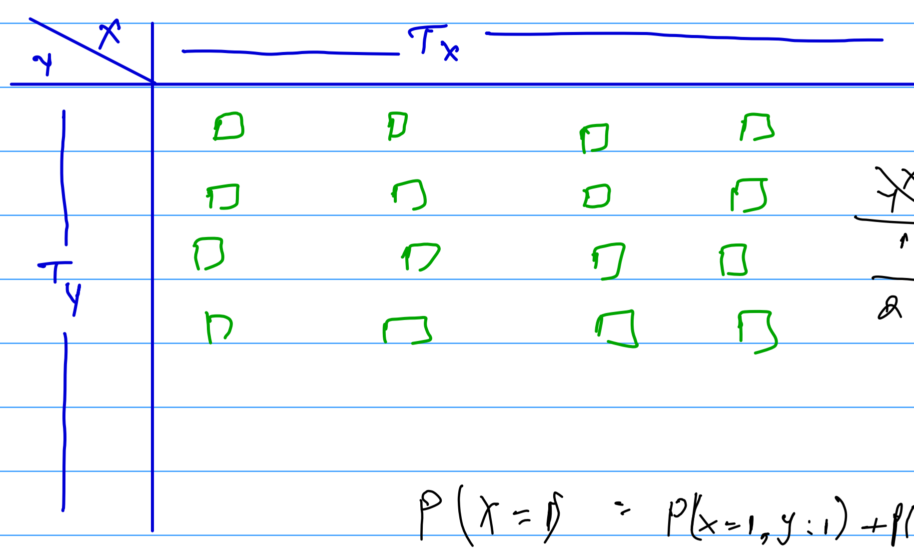
$$X \rightarrow T_X \checkmark \quad \{ \quad \} \quad \{x, y\}$$
$$Y \rightarrow T_Y \checkmark \quad \{ \quad \}$$

Then the joint PMF is defined as

$$\underset{\text{function}}{f_{XY}} : \underbrace{T_X \times T_Y}_{\text{Domain / input}} \longrightarrow \underbrace{[0, 1]}_{\text{Codomain / output}}$$

i.e., $f_{XY}(t_1, t_2) = P(X = t_1, Y = t_2)$, $t_1 \in T_X$
$t_2 \in T_Y$

For Joint PMF, We draw Joint PMF Table, i.e.,



$X \rightarrow \{1, 2, 3\}$
$Y \rightarrow \{1, 2\}$

Y \ X	1	2	3
1	0	□	□
2	□	□	□

$P(X=1) = P(X=1, Y=1) + P(X=1, Y=2)$

Properties of Joint PMF Table

1. Each entry on the table has value between 0 and 1 Inclusive
2. Sum of all entries in the table should be equal to 1.

X, Y → Joint

Y ✓ X

* Marginal PMF

Marginal PMF or will say Individual PMF.

Suppose we are given with Joint PMF, from there we can calculate the Individual PMFs of x and y using the table. So, these individual PMFs are called Marginal PMF

$f_{xy}(t_1, t_2)$

i.e., $f_x(t_1) = P(x = t_1) = \sum_{t_2 \in T_y} f_{xy}(t_1, t_2)$

e.g., If $x \to \{0, 1, 2\}$
$y \to \{0, 1, 2\}$

$$f_x(1) = P(x=1) = f_{xy}(1,0) + f_{xy}(1,1) + f_{xy}(1,2)$$

Also, If we have Joint PMF Table, then for finding PMF of x we will add all rows of x.

e.g.,

	$f_x(0)$	$f_x(1)$	$f_x(2)$	$f_x(3)$
y \ x	0	1	2	3
0	1/16	1/16	1/8	1/16
1	1/8	1/4	0	0
2	0	0	1/16	1/4

$\to f_y(1)$

$f_x(1) = 5/16$ $\qquad$ $f_x(3) = 5/16$

So, from here, we will get two table.

1st

x	0	1	2	3
P(x=x)				

← Marginal PMF of x

2nd

y	0	1	2	3
P(y=y)				

← marginal PMF of y

X , Y

NOTE :- Given Joint PMF, we can construct Marginal PMF, but vice-versa is not true, As there will be many possibilities.

* Given the joint PMF, Marginal is unique

* <u>Conditional PMF</u> :-

From Conditional probability of 1 R.V., we know that

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

So, If we have two R.V. X and Y, we can write it as

$X \rightarrow C$, $Y \rightarrow D$

$$P(X \in C \mid Y \in D) = \frac{P(X \in C, Y \in D)}{P(Y \in D)} \quad \text{where } C, D \in R$$

So, If we are given an event A, and we have to find Conditional PMF of X, then

$$P_{X|A}(x_i) = P(X=x_i|A) = \frac{P(X=x_i \text{ and } A)}{P(A)}$$

* Example

I rolled a die. Let X be the number observed. Find the Conditional PMF of X given that we know the observed number was less than 5.

So, event $A = \{X<5\}$, where $P(A) = \frac{4}{6}$

$$P(X|X<5) = P_{X|A}(1) + P_{X|A}(2) + P_{X|A}(3) + P_{X|A}(4)$$

$$P_{X|A}(1) = P(X=1|X<5) = \frac{P(X=1 \text{ and } X<5)}{P(X<5)}$$

$$= \frac{P(X=1)}{P(X<5)} = \frac{1/6}{4/6} = \boxed{\frac{1}{4}}$$

Similarly

$$P_{X|A}(2) = P_{X|A}(3) = P_{X|A}(4) = \frac{1}{4}$$

$$P(X|X<5) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \boxed{1}$$

i.e., $\boxed{P_{X|A}(5) = P_{X|A}(6) = 0}$

Now, If we have two R.V.s X and Y, then

Conditional PMF of Y given $X = t_1$

$$F_{Y|X=t_1}(t_2) = P(Y = t_2 \mid X = t_1) \leftarrow \text{Conditional}$$

$$= \frac{P(Y = t_2 \cap X = t_1)}{P(X = t_1)} \begin{array}{l} \leftarrow \text{Joint PMF} \\ \leftarrow \text{Marginal PMF} \end{array}$$

$f_{XY}(t_1, t_2) = P(X = t_1, Y = t_2)$

$F_X(t_1) =$

$$= \frac{P(Y = t_2, X = t_1)}{P(X = t_1)} \begin{array}{l} \leftarrow \text{Joint PMF} \\ \leftarrow \text{Marginal} \end{array}$$

So, From here we get an important Information that is

$$\boxed{\text{Joint PMF} = \text{Marginal PMF} \times \text{Conditional PMF}}$$

$$\underbrace{F_{XY}(x, y)}_{\text{Joint}} = \underbrace{\overset{\text{Marginal}}{F_X(x)} \times \overset{\text{Conditional}}{F_{Y|X=x}(y)}} = \underbrace{\overset{\text{Marginal}}{F_Y(y)} \times F_{X|Y=y}(x)} \rightarrow \text{conditional}$$

Joint
Marginal } — Two R.V.s
conditional

* More than two variables :

* Joint PMF :

x, y

$X_1, X_2, X_3, \ldots, X_n$ are discrete R.V.s defined in same probability space

and Let Range of x_i is T_{x_i}

Then, Joint PMF is denoted by

$P(x_1 = t_1, x_2 = t_2, x_3 = t_3)$

$f_{x_1 x_2 x_3}(t_1, t_2, t_3) =$

$$\boxed{f_{x_1, x_2, \ldots x_n}(t_1, t_2, \ldots t_n) = P(x_1 = t_1, x_2 = t_2, \ldots x_n = t_n), \; t_i \in T_{x_i}}$$

This one will also be written in tables.

* Example : Tossing a coin thrice

Let $x_i = 1$ if ith toss is head and $x_i = 0$ if ith toss is tails.
So, $i = 1, 2, 3$

t_1	t_2	t_3	$f_{x_1 x_2 x_3}(t_1, t_2, t_3)$
0	0	0	1/8 ✓
0	0	1	1/8
0	1	0	1/8
0	1	1	1/8
1	0	0	1/8
1	0	1	1/8
1	1	0	1/8
1	1	1	1/8

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

Joint PMF table

* Marginal PMF

The PMF of individual R.V. $x_1, x_2, \dots, x_n$

$$f_{x_1}(t) = P(x_1 = t) = \sum_{t_2'\in T_{x_2}, t_3'\in T_{x_3} \dots t_n'\in T_{x_n}} f_{x_1 \dots x_n}(t_1, t_2', t_3', \dots t_n')$$

t = success (Head), t' = failure (tails)

t_1	t_2	t_3	$f_{x_1x_2x_3}(t_1, t_2, t_3)$
0	0	0	1/8
0	0	1	1/8
0	1	0	1/8
0	1	1	1/8
1	0	0	1/8
1	0	1	1/8
1	1	0	1/8
1	1	1	1/8

$$f_{x_1}(0) = f_{xyz}(0,0,0) + f_{xyz}(0,1,0) + f_{xyz}(0,0,1) + f_{xyz}(0,1,1)$$
$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{4}{8} = \boxed{\frac{1}{2}}$$

$$f_{x_1}(1) = f_{xyz}(1,0,0) + f_{xyz}(1,1,0) + f_{xyz}(1,0,1) + f_{xyz}(1,1,1)$$
$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \boxed{\frac{1}{2}}$$

* Marginalisation : (Sum over everything that you don't want)

$x_1, x_2, x_3 \to$ Joint

Suppose $X_1, X_2, X_3 \sim f_{x_1x_2x_3}$, $X_i \in T_{x_i}$

	t_1	t_2	t_3	$f_{x_1x_2x_3}(t_1, t_2, t_3)$
$P(x_1=0, x_2=0) \leftarrow$	0	0	0	1/8
	0	0	1	1/8
$P(x_1=0, x_2=1) \rightarrow$	0	1	0	1/8
	0	1	1	1/8
$P(x_1=1, x_2=0) \rightarrow$	1	0	0	1/8
	1	0	1	1/8
$P(x_1=1, x_2=1) \rightarrow$	1	1	0	1/8
	1	1	1	1/8

Suppose I want $f_{x_1,x_2}(t_1, t_2)$, i.e., joint PMF of x_1, x_2

$$f_{x_1x_2}(t_1, t_2) = P(x_1 = t_1, x_2 = t_2)$$
$$= \sum_{t_3' \in T_{x_3}} f_{x_1x_2x_3}(t_1, t_2, t_3')$$

$t_1 \to \{0, 1\}$
$t_2 = \{0, 1\}$
$(0,0), (0,1)$
$(1,0), (1,1)$

$(0,0,0)$, $f_{x_1x_2}(0,0) = f_{x_1x_2x_3}(0,0,0) + f_{x_1x_2x_3}(0,0,1)$

$x_1 \backslash x_2$	0	1
0	1/4	1/4
1	1/4	1/4

* Conditional PMF :-

$$(X_1 | X_2 = t_2) \sim f_{X_1|X_2=t_2}(t_1) = \frac{f_{X_1X_2}(t_1, t_2)}{f_{X_2}(t_2)}$$

$$(X_1, X_2 | X_3 = t_3) \sim f_{X_1X_2|X_3=t_3}(t_1, t_2) = \frac{f_{X_1X_2X_3}(t_1, t_2, t_3)}{f_{X_3}(t_3)}$$

→ Joint

← marginal

↑ conditional

$$(X_1 | X_2 = t_2, X_3 = t_3) \sim f_{X_1|X_2=t_2, X_3=t_3}(t_1) = \frac{f_{X_1X_2X_3}(t_1, t_2, t_3)}{f_{X_2X_3}(t_2, t_3)}$$

→ Joint

← Marginal

Week-1 ends