Lecture – 3

Qubits and Linear Algebra

→ In the previous Lecture notes you have seen a two level system (TLS) formed using spins.

↳ These TLS's are called qubits.

↳ Classical Computer stores Information in bits which binary Integers 0 & 1. Simarly quantum has qubits

↳ Isolating 2-D subspace from the Hilbert Space Creates a qubit which can done in several ways.

↳ If you Isolate 3 levels $|0\rangle, |1\rangle, |2\rangle$ then it is called a qutrit

↳ A d-level system is Called Qudit.

→ We are mostly Interested in qubit based quantum Computing.

→ Bloch Sphere can represent one qubit but doesn't work for multiple qubit | System.

→ This lecture discusses about the Complex linear algebra to describe and play with the qubit.

Linear Algebra for Quantum Mechanics

→ Basic elements are state vectors and linear operators

↳ State vectors denote the quantum states
Matrices (linear operators are transformations on those states

↳ state vectors can written in a linear combination using the basis vectors

ex:- $|\psi\rangle = a|0\rangle + b|1\rangle$ (or) $|\psi\rangle = c|+\rangle + d|-\rangle$

↳ You can change the basis by applying matrix transformation

$$\left.\begin{array}{l} |\psi\rangle \rightarrow \text{ket Notation} \\ \langle\psi| \rightarrow \text{Bra Notation} \end{array}\right\} \longrightarrow \text{Dirac notation}$$

$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$\langle\psi| = \begin{pmatrix} a^* & b^* \end{pmatrix}$

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

If a matrix $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

$H|1\rangle =$ Do the Calculation to check $= |-\rangle$

$|+\rangle, |-\rangle$ are called the Hadamard basis.

↳ There are many other basis and Infact you can form your own.

So $|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$, $\langle\psi| = (\alpha_1^*, \alpha_2^*, \ldots\ldots) = |\psi\rangle^\dagger$

→ If $|\psi\rangle$ is normalized then $\langle\psi|\psi\rangle = \sum_{j=1}^{N} |\alpha_j|^2 = 1$ (Inner product)
which lies on the bloch sphere.

↳ for two orthogonal vectors (one's opposite sides on the bloch sphere) $\langle\psi|\psi\rangle = 0$

→ The basis vectors should be orthonormal. which is they should be orthogonal to each other and normalized

→ Any operator $\hat{O}$ can also be represented according to the basis chosen

Outer product:-

$$|\psi\rangle\langle\phi| = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} \beta_1^*, & \beta_2^*, & \cdots & \beta_n^* \end{bmatrix} = \begin{bmatrix} \alpha_1\beta_1^* & \cdots & \alpha_1\beta_n^* \\ \vdots & \ddots & \vdots \\ \alpha_n\beta_1^* & \cdots & \alpha_n\beta_n^* \end{bmatrix}$$

$$|0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|0\rangle\langle 1| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ; \quad |1\rangle\langle 0| = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So any 2×2 matrices that you want to write in $|0\rangle, |1\rangle$ basis will be a linear combination of above outer products.

$$a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$$

$$\hat{O} = \sum_{ij} a_{ij} |i\rangle\langle j| \Rightarrow a_{ij} = \langle i|\hat{O}|j\rangle$$

a_{ij} are matrix elements & $|i\rangle, |j\rangle$ are same basis as $\hat{O}$

Hermitian Conjugation

$(AB)^\dagger = B^\dagger A^\dagger$ $\quad (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$ $\quad (|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi|$

$(a\hat{O})^\dagger = a^* O^\dagger$ $\quad |\psi\rangle^\dagger = \langle\psi|$

If $\hat{O} = O^\dagger$ then O is called Hermitian matrix

→ Every physicaly measurable quantity (observable) has a hermitian matrix associated to it.

→ If a matrix $A = A^\dagger$ then it is hermitian.

↳ Hermitian matrices have spectral decomposition. which means they can be decomposed into their eigenvalues and eigenvectors. linear combination

↳ In other words they are diagonalizable

↳ And their eigenvalues are real.

$$A = A^\dagger \Rightarrow \sum a_{ii} |i\rangle\langle i| = \sum a_{ii}^* |i\rangle\langle i|$$

$$\Rightarrow a_{ii} = a_{ii}^* \qquad a_{ii} \text{ are real.}$$

→ So every observable has a hermitian matrix associated

↳ Which can be spectral decomposed into eigenvalues & corresponding eigenvectors

↳ When you measure an observable the state will collapse into one of the eigenvectors output its eigenvalue

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle \longrightarrow$ This Called expectation value

→ Expectation value is the Average value of the observable measurement.

→ As long as our basis is fixed state vectors Can be written as column vectors & operators as Matrices

↳ An orthonormal basis for an N-dimensional space has N vectors that Satisfy

$$\langle i | j \rangle = \delta_{ij} \qquad \sum_{j=1}^{N} |j\rangle\langle j| = I$$

$$\mathrm{Tr}(\hat{O}) = \sum \langle j | O | j \rangle = \sum_j a_{jj}$$

Tr → Trace is sum of diagonal elements

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \qquad \mathrm{Tr}(ABC) = BCA = CAB$$

$$\mathrm{Tr}(|\psi\rangle\langle\phi|) = \langle\phi|\psi\rangle$$

→ Hermitian operator's come from a larger class Called Normal matrices

$A^\dagger A = AA^\dagger$ then A is Normal.

$\hat{A}$ is orthogonally diagonalizable If it is Normal.

↳ This means, for every Normal matrix we Can find a orthonormal basis

$H|\psi\rangle = E|\psi\rangle$ $\quad$ $H = \begin{bmatrix} E_1 & & & \\ & E_2 & & \\ & & \ddots & \\ & & & E_n \end{bmatrix}$

→ $\hat{O} = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$

↳ If all λ_i's are unique then its non degenerate

Hermitian matrices are sub class in Normal matrices

Example :- Pauli − X, Y, Z

→ If two normal operators A & B commute i.e

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0$$

then A & $\hat{B}$ simultaneously diagonalizable
means they can be diagonalized using same
basis

Unitary operators :- $U^\dagger U = U U^\dagger = I$ are normal.

→ Every unitary U has a Hermitian H such that

$$U = \exp(iH)$$

→ Unitary operator means Applying a basis change

Any 2×2 Unitary can be written as

$$\cos\frac{\theta}{2} I + i \sin\frac{\theta}{2} \hat{n}\cdot\vec{\sigma}$$

$$\hat{n} = (n_x, n_y, n_z) \qquad \vec{\sigma} = (\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z)$$

Hilbert Schmidt Inner product :- On $(A,B) = Tr\{A^{\dagger}B\}$

→ Its the Inner product on operator space

↳ The space of operators is denoted as $\mathcal{B}(\mathcal{H})$

↳ Linear transformation on operator space is called Super operator.

Polar Decomposition :- $\hat{O} = \hat{U}\hat{A} = \hat{B}U$

$A = \sqrt{O^{\dagger}O}$ $\quad$ $B = \sqrt{\hat{O}\hat{O}^{\dagger}}$

Singular Value Decomposition :-

$\hat{O} = \hat{U}\hat{D}\hat{V}$ $\quad$ $\hat{U}, \hat{V}$ are unitaries

$\hat{D}$ is Diagonal Matrix

→ If no elements in $\hat{D}$ are zero then $\hat{O}$ is invertible.

Tensor Product

→ For combining single qubit system to form multi qubit systems we need mathematical tool called tensor product.

If $|\psi\rangle \in \mathcal{H}_A$ $|\phi\rangle \in \mathcal{H}_B$ then $|\psi\rangle \otimes |\phi\rangle$ is called tensor product

$$|\psi\rangle \otimes |\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A \otimes B \begin{bmatrix} a\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & b\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ c\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & d\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} a & 2a & b & 2b \\ 3a & 4a & 3b & 4b \\ c & 2c & d & 2d \\ 3c & 4c & 3d & 4d \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{2\times1} \quad |\phi\rangle = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}_{2\times1} \quad |\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix}_{4\times1}$$

$$(|\psi\rangle \otimes |\phi\rangle)(|a\rangle \otimes |b\rangle) = |\psi a\rangle \otimes |\phi b\rangle$$

$$(\langle\psi'| \otimes \langle\phi'|)(|\psi\rangle \otimes |\phi\rangle) = \langle\psi'|\psi\rangle\langle\phi'|\phi\rangle$$

→ If $|i\rangle$ is basis for $\mathcal{H}_A$ & $|j\rangle$ is for $\mathcal{H}_B$

↳ Then $|i\rangle \otimes |j\rangle$ is basis for $\mathcal{H}_A \otimes \mathcal{H}_B$

$|i\rangle \otimes |j\rangle$ can be written as $|ij\rangle$

Example :-

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$
$$|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$$

$$|\psi\rangle \otimes |\phi\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

Pauli Matrices :-

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Z \otimes Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Z \otimes I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

→ Now as the necessary Math is done, we will go through Quantum Mechanics postulates formally in the next one.

To Be Continued .....