Lecture - 2

The Advent of Spin & Intro to qubit Math

→ In 1920 Stern Gerlach experiment atoms were seen deflecting when passed through a magnetic field.

→ The valence e^-'s in the atoms were acting like bar magnets with magnetic dipole moment

↳ They were either going up or down in Z-direction

↳ If you rotate the machine setup then they go up or down in X-direction.

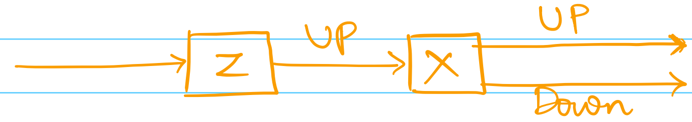
→ So this up in Z-direction is called spin up & down in Z-direction is called spin down

→ The e^-'s had angular momentum.

↳ So placing the machine in any angle, e^-'s are deflected up or down

↳ The spin is quantized. $-\frac{1}{2}, +\frac{1}{2}$ (only two directions)

→ But if you pass the same e^- that went up in the first SG exp. through a second one it will always go up.



→ So now not just x, y, z, if you measure spin along some random axis, The probability for Up & down will be $\cos^2(\theta/2)$ and $\sin^2(\theta/2)$

→ It's time to give all these spin properties a mathematical and visual representation.

$$\psi(\theta) = \cos\frac{\theta}{2}\,(ZUP) + \sin\frac{\theta}{2}\,(ZDOWN)$$

→ here $\cos\frac{\theta}{2}$ & $\sin\frac{\theta}{2}$ are probability Amplitudes for up & down

↳ $\cos^2(\theta/2)$ & $\sin^2(\theta/2)$ are respective Probabilities.

→ As the X-axis is 90° ($\frac{\pi}{2}$ angle) to the Z axis

$$XUP = \cos\frac{90^\circ}{2}\,ZUP + \sin\frac{90^\circ}{2}\,ZDOWN$$

$$XDOWN = \cos\frac{90^\circ}{2}\,ZUP - \sin\frac{90^\circ}{2}\,ZDOWN$$

→ Even Y-axis is 90° to Z But as Y & X Cannot be the Same, we take the help of complex numbers.

$$YUP = (ZUP + i\,ZDOWN)/\sqrt{2}$$

$$YDOWN = (ZUP - i\,ZDOWN)/\sqrt{2}$$

So a General State Can be represented using

$$\psi = \alpha\,ZUP + \beta\,ZDOWN$$

→ Where α, β are Complex numbers (probability amplitudes)

→ As we know from Linear Algebra, If you consider ZUP & ZDOWN your basis, then any vector Can be written as

$$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \, ZUP + \beta \, ZDOWN$$

→ You can change the basis if you want.

→ But the UP & DOWN in any basis are orthonormal and are orthogonal vectors of unit length.

$ZUP = |0\rangle$ → Ground State

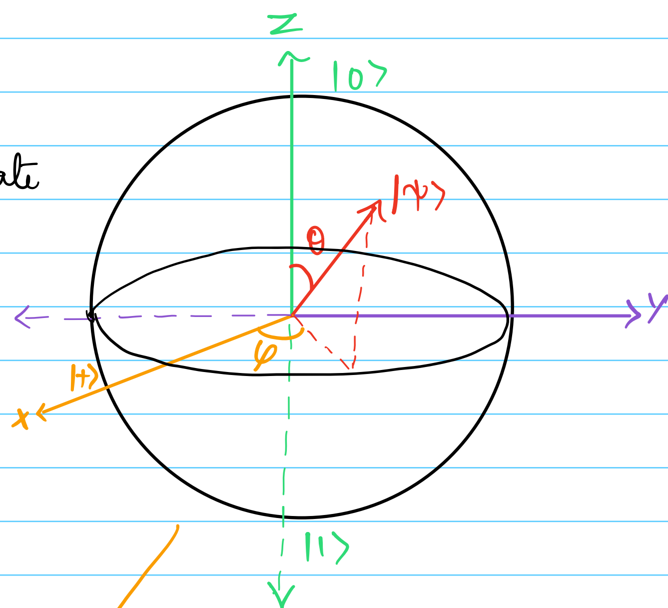
$ZDOWN = |1\rangle$ → First Excited State

$XUP = |+\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$

$XDOWN = |-\rangle = \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$

$YUP = |i\rangle = \dfrac{|0\rangle + i|1\rangle}{\sqrt{2}}$

$YDOWN = |-i\rangle = \dfrac{|0\rangle - i|1\rangle}{\sqrt{2}}$



→ This is called bloch sphere representation invented by Felix Bloch.

→ It is a 3D Unit Sphere. All pure states lie on the Sphere Surface.

The State $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$ in Z-Basis.

$\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$ → Here $|\alpha|^2 + |\beta|^2 = 1$ (always)

→ Quantum states can be represented with orthonormal basis with linear combination of two orthogonal states whose probabilities add up to 1. (Norm = 1)

→ You can transform from one basis to another using matrix transformations

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{Z\text{-Basis}} \longrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} \alpha+\beta \\ \alpha-\beta \end{bmatrix}_{X\text{-Basis}}$$

→ Not just basis change any rotation by angle θ, φ in the above bloch sphere can be represented using special matrices called Unitary Matrices

↳ If a matrix is unitary then $U^{\dagger}U = UU^{\dagger} = I$

↳ Unitary matrices cannot translate (extend or reduce) the vectors, they can only rotate.

↳ So the eigenvalues of all Unitary matrices can be given by the form $e^{i2\pi\theta}$

$$\boxed{U|\psi\rangle = e^{i2\pi\theta}|\psi\rangle}$$

→ Inner Product is another property which is very important to know about. It tells how much two different states overlap.

→ If $|\psi\rangle$ & $|\phi\rangle$ are 2 different states, $\langle\psi|\phi\rangle$ is their inner product

→ $\langle\psi|$ is called bra notation where $|\psi\rangle$ is ket notation

If $\underbrace{|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{\text{Column vector}}$ then $\langle\psi| = |\psi\rangle^{\dagger} = (|\psi\rangle^{*})^{T} = \underbrace{(\alpha^{*}, \beta^{*})}_{\text{Complex conjugated row vector}}$

→ If state $|\psi\rangle = e^{i\phi}(\alpha|0\rangle + e^{i\varphi}\beta|1\rangle)$

↳ Here $e^{i\phi}$ doesn't matter because it is Global phase but $e^{i\varphi}$ matters is relative phase

→ $|e^{i\phi}\alpha|^2 = e^{-i\phi}\alpha^* \cdot e^{i\phi}\alpha = \alpha^*\alpha$

↳ So the Global phase doesn't change the probability

→ If we multiply α & β with different phases then we would have observable Consequences.

→ In $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$, $0<\theta<\pi$, $-\pi<\varphi<\pi$

↳ Every point on Bloch sphere is a State

↳ opposite points on the block sphere are orthogonal states.

Unitary Matrices

→ Lets get back to talking about Unitary matrices

↳ A special set of Unitary matrices are Called pauli matrices.

→ There are 4 pauli matrices :- $\{I, X, Y, Z\}$

↳ $I \rightarrow$ Identity $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $X \longrightarrow$ pauli-X $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$Y \longrightarrow$ pauli Y $= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $Z \rightarrow$ pauli-Z $= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Pauli Matrices properties

$$X(\alpha|0\rangle + \beta|1\rangle) = \beta|0\rangle + \alpha|1\rangle$$

$$Y(\alpha|0\rangle + \beta|1\rangle) = \beta|0\rangle - \alpha|1\rangle$$

$$Z(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$$

·	X	Y	Z
X	I	iZ	−iY
Y	−iZ	I	iX
Z	iY	−iX	I

$$XX = YY = ZZ = I$$

$$\sigma_i \sigma_j = i\varepsilon_{ijk}\sigma_k$$

$$i,j,k \in \{x,y,z\}$$

→ X, Y, Z can also be denoted as $\sigma_x, \sigma_y, \sigma_z$

$\varepsilon_{ijk} \rightarrow$ levi civita symbol

$$Tr(X) = Tr(Y) = Tr(Z) = 0$$

→ Pauli Matrices form basis for any 2×2 Matrix

$$\hat{O} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\hat{I} + b\hat{X} + c\hat{Y} + d\hat{Z}$$

$$a = \frac{1}{2}Tr\{\hat{O}\} \quad b = \frac{1}{2}Tr\{\hat{X}\hat{O}\} \quad c = \frac{1}{2}Tr\{\hat{Y}\hat{O}\} \quad d = \frac{1}{2}Tr\{\hat{Z}\hat{O}\}$$

→ So we formed a two level system using the spin property. This TLS is called a qubit. There are many physical realizations for a qubit.

↳ photon polarization states is another example.

→ So we talked about spin & qubits, Bloch Sphere representation, Basis states and Unitaries.

→ This Notes talked about Qubits & their Rotation using Unitary matrices.

↳ Next one's will dive deep into these and will also talk about measurement; the most controversial QM postulate