A General Introduction to Game Theory: An Interdisciplinary Approach*

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Abstract. Submissions to Problem Set 2 for COMPSCI/ECON 206 Computational Microeconomics, 2023 Spring Term (Seven Week - Second) instructed by Prof. Luyao Zhang at Duke Kunshan University.

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1 Part I: Self-Introduction (2 points)

Jiaolun Zhou is a computer science student at Duke Kunshan University. He is currently working in Professor Mingjun Huang's Sensing and Interaction Lab. His research interests include brain waves and game interaction. In addition to game programming, he is also interested in economics and social sciences and has plans to combine these disciplines with computer science in the future.

2 Part II: Reflections on Game Theory (5 points)

Game theory was first proposed in 1944 in The publication of the book "Theory of Games and Economic Behavior" (Neumann and Morgenstern [1], Neumann and Morgenstern, 1947).

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In 1950, Nash proposed the theory of equilibrium in non-cooperative games (Nash Jr [2], Nash Jr, 1950).

In 1994, In 1994, Nash made the distinction between cooperative and non-cooperative games and proposed the Nash equilibrium.(Nash Jr [2], Nash Jr, 1950)

In 1994, Reinhard Selten first perfected Nash equilibrium by analyzing dynamic strategic interaction.(Selten [3], Selten, 1965)

In 1994, John C. Halsanyi pointed out Nash equilibrium in incomplete information. (Harsanyi [4], Harsanyi, 1967)

In 2005, Robert J.Aumann proposed the application of Nash equilibrium in social sciences(Schelling [5], Schelling, 1958)

In 2005, Thomas C. Elling analyzed the existence of Nash equilibria in infinite-round games.(Schelling [5], Schelling, 1958)

3 Part III: Bayesian Nash Equilibrium: Definition, Theorem, and Proof (3 points)

According to the text book, the following is the concept of Bayesian gamesZamir [6], Zamir, 2020:

Definition 1. *(textbook page.170) (Bayesian game: information sets) A Bayesian game is a tuple (N, G, P, I) where:*
- *N is a set of agents;*
- *G is a set of games with N agents each such that if $g, g' \in G$ then for each agent $i \in N$ the strategy space in g is identical to the strategy space in g';*
- *$P \in \prod(G)$ is a common prior over games, where $\prod(G)$ is the set of all probability distributions over G; and*
- *$I = (I_1, \cdots, I_N)$ is a tuple of partitions of G, one for each agent.*

This is the definition of a Bayesian game, and from this, we can see that the difference between a Bayesian game and a normal Nash equilibrium is the way in which information is processed. This is embodied in standard game theory, where all players have complete information, and each player knows the strategy and payment information of the other players. In Bayesian games, players have uncertainty about other players' information. Players rely on priori and belief to make assumptions about other players' behavior and choose how to respond.

Secondly, the concept of the solution is different in Bayesian games. We find that there is no optimal solution in the information set of Bayesian games, but in this set of information, each player takes the correct behavior and has the correct belief about the other players' behavior.

Epistemic types

Definition 2. *(textbook page.170) (Bayesian game: types) A Bayesian game is a tuple (N, A, Θ, p, u) where:*
- *N is a set of agents;*

· $A = A_1 \times \cdots \times A_n$, *where* A_i *is the set of actions available to player* i;
· $\Theta = \Theta_1 \times \cdots \Theta_n$, *where* Θ_i *is the type space of player* i;
· $p : \Theta \mapsto [0, 1]$ *is a common prior over types; and*
· $u = (u_1, \cdots, u_n)$, *where* $u_i : A \times \Theta \mapsto \mathbb{R}$ *is the utility function for player* i.

The above is the definition of Epistemic types in Bayesian games, and it reveals the solution for Bayesian games. In a Bayesian game, the type of each player can be represented by a set of random variables, including the actions that the player can perform, the type space of the player, the prior, and a formula that covers all of these variables.

Strategies and equilibria In a Bayesian game, there are three meaningful notions of expected utility, which is ex post, ex interim and ex ante.

Definition 3. *(textbook page.171-172) (Ex post expected utility) Agent* i'*s ex post expected utility in a Bayesian game* (N, A, Θ, p, u)*, where the agents' strategies are given by* s *and the agent' types are given by* θ*, is defined as*

$$EU_i(s, \theta) = \sum_{a \in A} (\prod_{j \in N} s_j(a_j | \theta_j)) u_i(a, \theta).$$

Definition 4. *(textbook page.172) (Ex interim expected utility) Agent* i'*s ex interim expected utility in a Bayesian game* (N, A, Θ, p, u)*, where* i'*s type is* θ_i *and where the agents' strategies are given by the mixed-strategy profile* s*, is defined as*

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} | \theta_i) \sum_{a \in A} (\prod_{j \in N} s_j(a_j | \theta_j)) u_i(a, \theta_{-i}, \theta_i),$$

or equivalently as

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} | \theta_i) EU_i(s, (\theta_i, \theta_{-i}))$$

Definition 5. *(textbook page.172-173) (Ex ante expected utility) Agent* i'*s ex ante expected utility in a Bayesian game* (N, A, Θ, p, u)*, where the agents' strategies are given by the mixed-strategy profile* s*, is defined as*

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} (\prod_{j \in N} s_j(a_j | \theta_j)) u_i(a, \theta),$$

or equivalently as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) EU_i(s, \theta),$$

or again equivalently as

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s, \theta_i),$$

In Bayesian games, Ex post represents the state of the game after certain information has been disclosed. Previously we have explained the concept of information sets in Bayesian games, where players' knowledge of other players is incomplete. All players need other players to make judgments about the information they have intentionally revealed or implied by game mechanics.

For example, in an auction game, Ex post is the judgment of all players after they know what everyone has bid. This has a significant impact on the player's final bid because assuming that the player doesn't know what other players are bidding for, he or she will rely on his or her own assumptions, which may rely on faulty or subjective evidence.

Definition 6. *(textbook page.173-174) (Best response in a Bayesian game) The set of agent i's best responses to mixed-strategy profile s_{-i} are given by*

$$BR_i(s_{-i}) = \underset{s_i' \in S_i}{argmax} EU_i(s_i', s_{-i})$$

Bayes–Nash equilibrium Once we understand the concepts of information sets, priors, and beliefs, we can understand Bayesian equilibria.

Definition 7. *(textbook page.174) (Bayes–Nash equilibrium) A Bayes–Nash equilibrium is a mixed-strategy profile s that satisfies $\forall i \; s_i \in BR_i(s_{-i})$*

Ex post equilibrium

Definition 8. *(textbook page.174) (Ex post equilibrium) An ex post equilibrium is a mixed-strategy profile s that satisfies $\forall \theta, \forall i, s_i \in argmax_{s_i' \in S_i} EU_i(s_i', s_{-i})$*

The ex post utilities is stronger than Bayes-Nash equilibrium on defining an equilibrium concept.The reason is that Nash equilibrium only considers the combination of strategies under a priori information and ignores the influence of a posteriori information.

3.1 Twenty lectures on algorithmic game theory

The Challenge of Revenue Maximization: Bayesian Analysis Considering the follow model for classical approach is to use average-case or Bayesian analysis.

Definition 9. *(textbook page.174-175)*

- *A single-parameter environment. We assume that there is a constant M such that $x_i \leq M$ for every i and feasible solution $(x_1, \cdots, x_n) \in X$.*
- *Independent distributions $F_1, \cdots, F_n$ with positive and continuous density functions $f_1, \cdots f_n$, We assume that the private valuation v_i of of participant i is drawn from the distribution F_i. We also assume that the support of every distribution F_i belongs to $[0, v_{max}]$ for some $v_{max} < \infty$.*

The Challenge of Revenue Maximization: One Bidder and One Item, Revisited Based on the above model, it is easy to construct a single-bidder and single-item auctions.

Definition 10. *(textbook page.175) The expected revenue of a posted price r is simply*

$$\underbrace{r}_{revenue\ of\ a\ sale} \cdot \underbrace{(1 - F(r))}_{probability of a sale}$$

This definition defines that given a distribution F and the best price r and an optimal posted price is called a monopoly price of the distribution of F.

Characterization of Optimal DSIC Mechanisms: Preliminaries

Definition 11. *(textbook page.175) The expected revenue of a DSIC mechanism (x, p) is, by definition,*

$$E_{v\sim F}[\sum_{i=1}^{n} p_i(v)],$$

where the expectation is with respect to the distribution $F = F_1 \times \cdots \times F_n$ over agents' valuations.

Characterization of Optimal DSIC Mechanisms: Virtual Valuations The following formula for expected revenue uses the important concept of virtual valuations.

Definition 12. *(textbook page.176) For an agent i with valuation distribution F_i and valuation v_i , her virtual valuation is defined as*

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

The player's judgment depends only on his own set of information, which is the v in the formula, but not on the other player's set.

Characterization of Optimal DSIC Mechanisms: Expected Revenue Equals Expected Virtual Welfare

Lemma 1. *For every single-parameter environment with valuation distributions $F_1, \cdots F_n$, every DSIC mechanism (x, p), every agent i, and every value v_{-i} of the valuations of the other agents,*

$$E_{v_i\sim F_i}[p_i(v)] = E_{v_i\sim F_i}[\phi_i(v_i) \cdot x_i(v)] \quad (3.2.1)$$

This formula tells us that player revenue is the same as virtual revenue.
With the above formula, there comes an important theorem.

Theorem 1. *(Exp. Revenue Equals Exp. Virtual Welfare)*
For every single-parameter environment with valuation distributions $F_1, \cdots, F_n$ and every DSIC mechanism (x, p),

$$\underbrace{E_{v\sim F}}_{expected\ revenue} = \underbrace{E_{v\sim F}[\sum_{i=1}^{n} \phi_i(v_i) \cdot x_i(v)]}_{expected\ virtual\ welfare} \quad (3.2.2)$$

Here is the proof for the above theorem.

Proof. Taking the expectation, with respect to $v_{-i} \sim F$ of both sides (1.1) of we obtain

$$E_{v\sim F}[p_i(v)] = E_{v\sim F}[\phi_i(v_i) \cdot x_i(v)]$$

Applying the linearity of expectation (twice) then gives

$$\begin{aligned} E_{v\sim F}[\sum_{i=1}^{n} p_i(v)] &= \sum_{i=1}^{n} E_{v\sim F}[p_i(v)] \\ &= \sum_{i=1}^{n} E_{v\sim F}[\phi_i(v_i) \cdot x_i(v)] \\ &= E_{v\sim F}[\sum_{i=1}^{n} \phi_i(v_i) \times x_i(v)], \end{aligned}$$

as desired.

It is not difficult to find in the above proof that maximizing expected revenue over the space of DSIC mechanisms reduces to maximizing expected virtual welfare over the same space.

Characterization of Optimal DSIC Mechanisms: Regular Distributions The following definition defines a sufficient condition for monotonicity.

Definition 13. *(textbook page.177) A distribution F is regular if the corresponding virtual valuation function $v - \frac{1-F(v)}{f(v)}$ is non-decreasing.*

Theorem 2. *(Virtual Welfare Maximizers Are Optimal)*
For every single-parameter environment and regular distributions $F_1, \cdots, F_n$, the corresponding virtual welfare maximizer is a DSIC mechanism with the maximum-possible expected revenue.

Proof of Lemma 1 Here is proof of lemma 1.

Proof. (textbook page.180) **Step 1:** Fix an agnet i By Myerson's payment formula, we can write the expected (over $v_i \sim F_i$) payment by i for a given value of v_{-i} as

$$E_{v_i \sim F_i}[p_i(v)] = \int_0^{v_{max}} p_i(v) f_i(v_i) dv_i$$
$$= \int_0^{v_{max}} [\int_0^{v_i} z \cdot x_i'(z, v_{-i}) dz] f_i(v_i) dv_i.$$

This step is to rewrite the expected payment in terms of the allocation rule.
Step 2: Reversing the order of integration in

$$\int_0^{v_{max}} [\int_0^{v_i} z \cdot x_i'(z, v_{-i} dz] f_i(v_i) dv_i$$

yields

$$\int_0^{v_{max}} [\int_0^{v_i} f_i(v_i) dv_i] z \cdot z \cdot x_i'(z, v_{-i}) dz,$$

which simplifies to

$$\int_0^{v_{max}} (1 - F_i(z)) \cdot z \cdot x_i'(z, v_{-i}) dz,$$

suggesting that we're on the right track.
Step 3: Continuing simplifications:

$$\int_0^{v_{max}} \underbrace{(1 - F_i(z)) \cdot z}_{g(z)} \cdot \underbrace{x_i'(z, v_{-i} dz)}_{h'(z)}$$

$$\underbrace{= (1 - F_i(z)) \cdot z \cdot x_i(z, v_{-i}|_0^{v_{max}})}_{=0-0} - \int_0^{v_{max}} \underbrace{z - \frac{1 - F_i(z)}{f_i(z)}}_{\phi_i(z)} x_i(z, v_{-i}) f_i(z) dz.$$

$$= \int_0^{v_{max}} \underbrace{z - \frac{1 - F_i(z)}{f_i(z)}}_{\phi_i(z)} x_i(z, v_{-i}) f_i(z) dz \quad (3.1.3)$$

Step 4: interpreting (3.1.3) as an expected value, with z drawn from the distribution F_i. Recalling

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

which is the virtual valuations, the expectation is $E_{v_i \sim F_i}[\phi_i(v_i) \cdot x_i(v)]$. Summarizing, we have

$$E_{v_i \sim F_i}[p_i(v)] = E_{v_i \sim F_i}[\phi_i(v_i) \cdot x_i(v)],$$

as desired.

4 Part IV: Game Theory Glossary Tables (5 points)

Table 1: Table captions should be placed above the tables.

Glossary	**Definition**	**Sou**
Zero-sum game	The sum of each player's returns is constant (usually zero)	Neumann and
Non-zero-sum game	In a game, each party's gains and losses do not have to sum to zero	Neumann and
Mixed Nash equilibrium	A combination of strategies by various players with mix of strategies	Nash
Dynamic game	Players can adjust them based on previous decisions and progress in the game	Nash
Subgame	In a dynamic game, it forms part of a complete game	Nash

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